Solutions by dual integral equations of mixed boundary value problems in elasticity

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SOLUTIONS BY DUAL INTEGRAL EQUATIONS OF MIXED BOUNDARY VALUE PROBLEMS IN ELASTICITY

BY

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I. INTRODUCTION

The problem of the determination of the stress and strain in a semi-infinite elastic medium with a plane boundary which is loaded in some way is well known, and its history dates back to Boussinesq\(^1\) (1) in 1885. In the work of Boussinesq and subsequent writers on the topic, usually the load is prescribed over the plane boundary. The solution for a point load provides a Green's function from which the solution for other loadings can be readily obtained.

If, however, the situation is varied so that the normal displacement is prescribed over a portion of the boundary and the load prescribed over the remaining part, one has a mixed boundary value problem which is rather more complicated. This dissertation is primarily concerned with some problems of this nature. Sometimes these are called "punch problems", since a rigid punch, driven into the surface, prescribes the normal displacement over the region of contact.

Until recently, comparatively few satisfactory

\(^1\) Numbers in parentheses following an author's name refer to entries under Literature Cited (Chapter VI); the specific page number is given when necessary.
solutions of problems of this type were available, since formidable difficulties are usually encountered when classical procedures are used, especially if expressions for the stresses are required in a form suitable for computation. In 1939, A. E. H. Love (5) managed to solve the problem when the punch has the form of a right circular cone whose axis is normal to the boundary. He was successful mainly because of his skill in guessing a combination of potentials which would satisfy the given boundary conditions.

In 1945, J. W. Harding and I. N. Sneddon (3) introduced a new method of treating such problems when there is symmetry about an axis normal to the boundary. By using Hankel transforms, they showed how to reduce the main part of such a problem to the task of solving a pair of dual integral equations. Using this approach, I. N. Sneddon (9 and 10) subsequently gave a detailed analysis of the stress and strain in the medium when the punch is a right circular cylinder or cone and the elastic medium is isotropic. His results for the cone agreed with those previously obtained by Love (5) and were more complete, since Love had not determined the stress at an interior point of the medium. In Chapter III of this thesis, an analysis similar to that of Sneddon will be
undertaken when the indenting surface is expressible in cylindrical coordinates in the form $z = \text{polynomial in } r$, thus including the cone and the cylinder as special cases.

In Chapter IV, the analogous two-dimensional plane strain problem is studied for a type of anisotropic substance known as orthotropic. Here Fourier transforms are used to obtain a pair of dual integral equations whose solution leads to a determination of the stress and strain components. By proceeding to the limit, one can then deduce the results for the isotropic case, and these can be compared with the expressions obtained by Sneddon (II). The case where the indenting agent is a rectangular block is somewhat exceptional and is therefore given separate consideration.

A noteworthy feature of the results of Chapters III and IV is that the stress and strain components at any point are obtainable in closed form in terms of simple functions. Even if this were not so, it will be evident that the present procedure is more direct and convenient than the earlier classical technique for handling the kind of problem under consideration.

The dual integral equations which arise are of the Titchmarsh type, but the parameters occurring therein lie outside the range of values for which the solutions
of Titchmarsh (13) and Busbridge (2) are known to be valid. This fact is not mentioned by Sneddon in the works cited above but it obviously requires careful investigation. Therefore, Chapter II of this thesis is devoted to a study of those dual equations which arise later on.

The treatment of the indentation problems will presuppose that the strains are small and that statical elastic conditions prevail throughout. But, as is pointed out by Sneddon (10), results obtained in this way can give useful information in certain dynamical situations, (e.g., in the theory of armor penetration), if the velocity of the indenting agent is small compared with the velocity of waves in the medium. Applications to some problems in the theory of soil mechanics are also feasible.
II. DUAL INTEGRAL EQUATIONS

A. Explicit Solution of the Titchmarsh Type

In some problems it is required to find a function which satisfies one integral equation for some range of a given parameter and a different integral equation for another range of the parameter. Titchmarsh calls such equations dual integral equations.

In the succeeding chapters, the dual equations that occur are all of the form

\[ \int_0^\infty t^s f(t) J_p(ut) \, dt = g(u), \quad 0 < u < 1, \]
\[ \int_0^\infty f(t) J_p(ut) \, dt = 0, \quad u > 1, \]

where \( f(t) \) is the function to be found and \( g(u) \) is a given well-behaved function defined over the interval \((0,1)\).

Dual equations of this type were first considered by E. C. Titchmarsh (14, p.337) and may therefore be referred to as the Titchmarsh type. Two explicit formulas for the solution are known for certain values of \( s \) and \( p \).

Titchmarsh (14, p.339) gives the solution in the following form,
valid for \( s > 0 \).

If \( s \) is negative or zero, this formula breaks down but another explicit solution has been derived by I. W. Busbridge (2) namely,

\[
f(x) = \frac{2^{-s/2} x^{-s}}{\Gamma(t+s/2)} \left\{ x^{1+s/2} \int_0^1 u^{p+\frac{s}{2}} (1-u^2)^{s/2} g(u) du \right. \\
+ \int_0^1 u^{p+1} (1-u^2)^{s/2} du \int_0^1 g(\nu u) (x \nu)^{2+s/2} J_{p+1+\frac{s}{2}}(x \nu) dv \right\}, \tag{2.3}
\]

and this is known to be valid provided \( g(u) \) is integrable in \((0,1)\) and

\[ s > -2, \quad -p-1 < s-1/2 < p+1. \tag{2.4} \]

In Chapter III the solution of the dual equations (2.1) will be required when \( s = -1 \) and \( p = 0 \). These values of \( s \) and \( p \) violate condition (2.4), so that the formula (2.3) cannot be expected to provide a solution. But when \( g(u) \) is a polynomial, it will be verified in the next section that the Busbridge formula (2.3) does, in fact, yield a solution.

In Chapter IV the case where \( s = -1 \) and \( p = -1/2 \)
will arise, and again (2.4) will be violated. This time, however, the Busbridge formula will definitely fail for an arbitrary polynomial \( g(u) \). Therefore, this case will be examined separately in Section C.

B. Verification of the Busbridge Formula for the Case \( s = -1, \, p = 0 \)

Consider the dual equations (2.1) when \( s = -1, \, p = 0 \) and \( g(u) = u^n \), where \( n \) is a positive integer or zero. Even though the condition (2.4) is violated, the Busbridge formula (2.3) gives a plausible result, viz.,

\[
f(x) = (2/n)^{1/2} x^{5/2} I_{n+1} \left\{ J_{-1/2}(x)/x \right. \\
+ \left. \int_0^1 v^{n+3/2} J_{1/2}(xv) \, dv \right\},
\]

where

\[
I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{\sqrt{n} \Gamma((n+1)/2)}{n \Gamma(n/2)}. \quad (2.6)
\]

In the integrand of (2.5) one may use the familiar result,

\[
v^{1/2} J_{1/2}(xv) = -\frac{1}{x} \frac{d}{dv} \left\{ v^{1/2} J_{-1/2}(vx) \right\},
\]
so that, after an integration by parts, (2.5) simplifies to,

\[ f(x) = 2(n+1)\pi^{-1}I_{n+1} \int_0^1 v^n \cos(xv) dv. \quad (2.7) \]

It will now be verified directly that this is a solution of the dual equations. First let \( 0 < u < 1 \) and substitute for \( f(x) \) from (2.7) into the left member of the first equation of the dual pair. This yields

\[
\int_0^\infty t^{-1} f(t) J_0(ut) \, dt = 2\pi^{-1}(n+1)I_{n+1} \int_0^\infty dt \int_0^1 v^n J_0(ut) \cos(vt) dv. \quad (2.8)
\]

The order of integration in this double integral can be inverted, since the integrand is a continuous function of \( (v,t) \) over the region \( 0 \leq v \leq 1, \quad 0 \leq t \), and the \( t \)-integral is uniformly convergent with respect to \( v \) over the interval \( 0 \leq v \leq 1 \). Remembering that

\[
\int_0^\infty J_0(ut) \cos vt \, dt = \begin{cases} (u^2-v^2)^{-1/2}, & u > v, \\ 0, & u < v, \end{cases}
\]

one obtains from (2.8)

\[
\int_0^\infty t^{-1} f(t) J_0(ut) \, dt = 2\pi^{-1}(n+1)I_n I_{n+1}u^n = u^n,
\]
upon using the value of $I_0$ given in (2.6).

Hence, the first of the dual integral equations is satisfied. The second is also satisfied, since, for $u$ greater than unity,

$$\int_0^\infty f(t) J_0(ut) dt = (\text{Const.}) \int_0^\infty dt \int_0^1 v^n t J_0(ut) \cos vt dv,$$

and the right hand side vanishes by virtue of the result (Magnus and Oberhettinger (7, p.47)),

$$\int_0^\infty t J_0(ut) \cos vt dt = R\{iv(u^2 - v^2)^{3/2}\}.$$

So the Busbridge formula still provides a solution of the dual equations when $s = -1$, $p = 0$ and $g(u) = u^n$. Obviously this will remain valid if $g(u)$ is taken to be a polynomial in $u$.

C. Solution of a Special Pair

This section is concerned solely with the special case of the dual integral equations where $s = -1$ and $p = -1/2$ with $\gamma u g(u) = \text{polynomial in } u$. This is the type that will occur in Chapter IV.

Suppose, temporarily, that $\gamma u g(u) = u^n$. The Busbridge formula, after some simplification, yields
as a "solution",
\[ f_n(t) = -Q_n \frac{t^{3/2}}{} \int_0^1 x^n J_1(tx) \, dx, \]  
\[ (2.9) \]
where
\[ Q_n = \sqrt{2/\pi} \frac{\Gamma \left( \left( n+1 \right) / 2 \right)}{\Gamma \left( n/2 \right)} . \]  
\[ (2.10) \]
By direct substitution of the expression (2.9) for \( f_n(t) \)
in (2.1), it is found that the second of the dual integral equations is satisfied, but not the first. In fact,
\[ \int_0^\infty t^{-1} f_n(t) J_{-1/2}(ut) \, dt \]
\[ = -\left( 2/\pi u \right)^{1/2} Q_n \int_0^1 x^n \, dx \int_0^\infty J_1(xt) \cos ut \, dt, \]
the change in order of integration being justified as before. But from Watson (17, p. 405),
\[ \int_0^\infty J_1(xt) \cos ut \, dt = \begin{cases} 1/x & \text{if } u < x, \\ -x & \text{if } u > x, \end{cases} \]
\[ \sqrt{u^2-x^2} \left( u + \sqrt{u^2-x^2} \right) \]
Hence,
\[ \int_0^\infty t^{-1} f_n(t) J_{-1/2}(ut) \, dt \]
\[ = -Q_n \left( 2/\pi u \right)^{1/2} \left\{ \int_0^u \frac{-x^{n+1}}{\sqrt{u^2-x^2} (u^2 + \sqrt{u^2-x^2})} \, dx + \int_0^1 x^{n-1} \, dx \right\} . \]
When the integrations are performed, this simplifies to the following,
\[ \int_{0}^{\infty} t^{-1} f_n(t) J_{-1/2}(ut) \, dt = Q_n(2/\mu)^{1/2} (u^n I_{n-1} - 1/n). \]

After substituting for \( Q_n \) and \( I_{n-1} \) from (2.10) and (2.6) respectively, one obtains,

\[ \int_{0}^{\infty} t^{-1} f_n(t) J_{-1/2}(ut) \, dt = u^{-1/2}(u^n - \gamma_n / \pi), \quad (2.11) \]

where

\[ \gamma_n = \Gamma((n+1)/2) / \Gamma((n+2)/2). \quad (2.12) \]

It is now evident from this that the Busbridge formula has failed to provide a solution. In particular, it should be mentioned that the right hand member of (2.11) vanishes when \( n = 0 \).

Now take

\[ g(u) = c_0 + c_1 u + c_2 u^2 + \cdots + c_m u^m, \quad (2.13) \]

where the \( c \)'s are constants.

If \( f_n(t) \) is still defined by (2.9), it will now be shown that,

\[ f(t) = c_1 f_1(t) + c_2 f_2(t) + \cdots + c_m f_m(t), \quad (2.14) \]

is a solution of the dual equations under consideration provided there is a certain linear relation among the \( c \)'s, namely,

\[ -\sqrt{\pi} c_0 = c_1 \gamma_1 + c_2 \gamma_2 + \cdots + c_m \gamma_m, \quad (2.15) \]
For, the second of the dual equations clearly remains satisfied, and the first is also satisfied now, since it follows from (2.11) that

\[ \int_0^\infty t^{-1} \sum_{n=1}^{m} f_n(t) J_{-1/2}(ut) \, dt \]

\[ = u^{-1/2} \left\{ \sum_{n=1}^{m} c_n u^n - \sum_{n=1}^{m} c_n \gamma_n / \pi \right\} \]

\[ = u^{-1/2} \left\{ \sum_{n=1}^{m} c_n u^n + c_0 \right\} \]

\[ = g(u). \]

So, if \( s = -1, \ p = -1/2, \) and \( g(u) \) is given by (2.13) subject to the restriction (2.15), it is clear that \( f(t) \), as defined in (2.14), is a solution of the dual integral equations being considered. This rather special result will be sufficient for the purposes of Chapter IV.
III. INDENTATION OF A SEMI-INFINITE ISOTROPIC SOLID BY AN AXIALLY SYMMETRIC PUNCH

A. The Potential Function $\phi$

Consider a homogeneous, isotropic, elastic solid of Young's modulus $E$ and Poisson's ratio $\nu$. Suppose, in cylindrical coordinates $(r, \theta, z)$, that the $z$-axis is an axis of symmetry. Then it is shown by A. E. H. Love (6, p.274) that, in the absence of body forces, the equilibrium equations are satisfied if the stress components are derivable from a potential function $\phi(r, z)$ in the following way,

$$
\tau_{rr} = \sigma \nabla^2 \phi_z - \phi_{rz},
$$

$$
\tau_{\theta \theta} = \sigma \nabla^2 \phi_z - r^{-1} \phi_{rz},
$$

$$
\tau_{zz} = (2 - \sigma) \nabla^2 \phi_z - \phi_{zzz},
$$

$$
\tau_{rz} = \frac{2}{\sigma r} \left\{ (1 - \sigma) \nabla^2 \phi - \phi_{zz} \right\},
$$

where the subscripts on $\phi$, but not on $\tau$, denote partial derivatives. (The other two stress components vanish identically because of the assumed symmetry.) Of the six compatibility relations, two are automatically satisfied, and the remaining four are also satisfied if,
Under these conditions, the components of displacement will then be given by,

\[ 2G u_r = - \varphi_{rz} \]  

(3.3)

and

\[ 2G u_z = 2(1 - \sigma) \nabla^2 \varphi - \varphi_{zz}, \]

where

\[ 2G = E/(1 + \sigma). \]

B. Formulation of the Problem

Consider a semi-infinite, isotropic, elastic solid whose boundary, in cylindrical coordinates \((r, \theta, z)\), is the plane \(z = 0\), with the \(z\)-axis directed into the medium. Suppose that a rigid, lubricated punch is in contact with the elastic medium over the region \(r < a\), and that the equation of the surface of contact is,

\[ z = g(r) = A_0 + A_1 r + A_2 r^2 + \cdots + A_m r^m. \]  

(3.4)

In addition, it will be assumed that the classical small strain theory of elasticity is applicable, that the body forces are negligible, and that the portion of the boundary not in contact with the punch is free from any external load. Hence, the boundary conditions to be met are as
The main purpose of this chapter is to determine the stress and displacement at any point of the elastic solid under consideration.

C. Derivation of the Stress and Strain from a Hankel Transform of \( \varphi \)

In Section B, the boundary conditions (3.5) - (3.7) are, by virtue of (3.1), restrictions on \( \varphi \), so that if a biharmonic function \( \varphi \) can be found satisfying these restrictions, the stress and displacement components can be found from it by using (3.1). However, it will be found desirable later on to have the stress and displacement expressed in terms of a Hankel transform of \( \varphi \) instead of \( \varphi \) itself. Therefore, this section will be devoted to this task.

\[
\begin{align*}
\left[ \tau_{rz} \right]_{z=0} &= 0, \quad r > 0, \quad (3.5) \\
\left[ u_z \right]_{z=0} &= g(r), \quad 0 \leq r < a, \quad (3.6) \\
\left[ \mathcal{T} \right]_{z=0} &= 0, \quad r > a, \quad (3.7)
\end{align*}
\]
Throughout this chapter, Hankel transforms of order zero, with respect to \( r \), will be indicated by a bar placed above the appropriate function; Hankel transforms of order one, also with respect to \( r \), will be indicated by a double bar similarly placed.

Suppose \( \mathcal{O}(r,z) \) is the potential function which satisfies all the conditions of the problem on hand. The transforms of \( \mathcal{O} \) and of its Laplacian are given by

\[
\mathcal{O}(p,z) = \int_0^\infty r \mathcal{O}(r,z) J_0(pr) \, dr , \quad (3.8)
\]

\[
\nabla^2 \mathcal{O}(p,z) = \int_0^\infty rv^2 \mathcal{O}(r,z) J_0(pr) \, dr = \mathcal{O}_{zz} + \int_0^\infty (r \mathcal{O}_r)_r J_0(pr) \, dr , \quad (3.9)
\]

where the subscripts \( r \) and \( z \) designate partial derivatives. The integral on the right-hand side of (3.9) can be subjected to two integrations by parts. Assuming that the behavior of \( \mathcal{O}(r,z) \), as \( r \) tends to zero or to infinity, is such that the expression

\[
r \mathcal{O}_r J_0(pr) - r p \mathcal{O} J_1(pr)
\]

vanishes as \( r \) tends to zero or infinity, one obtains the
following result from (3.9):

\[ \nabla^2 \phi = \left( \frac{\partial^2}{\partial z^2} - p^2 \right) \phi. \]  

(3.10)

Since \( \phi \) is biharmonic it follows that

\[ (\partial^2 / \partial z^2 - p^2) \phi = 0. \]  

(3.11)

The conditions on \( \phi \) at infinity indicate that the solution of (3.11) is of the form

\[ \phi(p, z) = (A + Bz)e^{-pz}. \]  

(3.12)

It is now desirable to formulate the boundary conditions in terms of \( \phi \). To do this, one considers Hankel transforms, of order zero or unity, of the equations (3.1) and (3.3). Firstly,

\[ (r_{rr} + r_{QQ}) = (2\sigma - 1) \nabla^2 \phi_z + \phi_{zzz}. \]

Therefore,

\[ \frac{(r_{rr} + r_{QQ})}{2\sigma \phi_z - (2\sigma - 1)p^2 \phi_z}, \]

and the Hankel inversion theorem then gives

\[ r_{rr} + r_{QQ} = \int_0^\infty p \left\{ 2\sigma \phi_z + (1-2\sigma)p^2 \phi_z \right\} J_0(pr) \, dp. \]  

(3.13)

Now, if \( f(r) \) is a function with the property that
rf(r)\text{J}_1(pr)\) tends to zero as \(r\) tends to infinity or zero, then

\[
\hat{f}^{*}(r) = \int_{0}^{\infty} r f'(r) \text{J}_1(pr) \, dr
\]

\[
= - \int_{0}^{\infty} f(r) \frac{d}{dr} r\text{J}_1(pr) \, dr
\]

\[
= - p \int_{0}^{\infty} f(r) \text{J}_0(pr) \, dr
\]

\[
= - p \hat{f} .
\] (3.14)

Hence,

\[
\tau_{rz} = (1 - \sigma) \frac{2}{\partial r} \left( \nabla^2 \bar{\phi} \right) - \frac{\partial}{\partial r} \bar{\phi}_{zz}
\]

\[
= - (1 - \sigma) p \left( \nabla^2 \bar{\phi} \right) + p \bar{\phi}_{zz}
\]

\[
= \sigma p \bar{\phi}_{zz} + (1 - \sigma) p^3 \bar{\phi} .
\]

The inversion of this now yields

\[
\tau_{rz} = \int_{0}^{\infty} p^2 \left\{ \sigma \bar{\phi}_{zz} + (1 - \sigma) p^2 \bar{\phi} \right\} \text{J}_1(pr) \, dp .
\] (3.15)

So \(\tau_{rz}\) is obtainable directly from \(\bar{\phi}\).

In order to obtain similar expressions for the other stress components it is necessary to use one of the equilibrium equations, namely,
\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0.
\]

This can be written in the form

\[
\frac{\partial}{\partial r}(r^2 \tau_{rr}) = r(\tau_{rr} + \tau_{\theta\theta}) - r^2 \frac{\partial \tau_{rz}}{\partial z}. \quad (3.16)
\]

It now yields, with the help of (3.13) and (3.15),

\[
\frac{\partial}{\partial r}(r^2 \tau_{rr}) = r \int_0^\infty p \left\{ 2 \sigma \overline{\sigma} z + (1-2\sigma)p^2 \overline{\sigma} z \right\} J_0(pr) dp
\]

\[
- r^2 \int_0^\infty p^2 \left\{ \sigma \overline{\sigma} z + (1-\sigma)p^2 \overline{\sigma} z \right\} J_1(pr) dp.
\]

Integrating with respect to \( r \), interchanging the order of integration on the right (assumed justified), and noting that

\[
\int_0^r pr J_0(pr) \, dr = rJ_1(pr),
\]

\[
\int_0^r p^2 r^2 J_1(pr) \, dr = 2rJ_1(pr) - r^2 p J_0(pr),
\]

one obtains the relation

\[
\tau_{rr} = \int_0^\infty p \left\{ \sigma \overline{\sigma} z + (1-\sigma)p^2 \overline{\sigma} z \right\} J_0(pr) \, dp
\]

\[
- r^{-1} \int_0^\infty \overline{\sigma} z J_1(pr) \, dp. \quad (3.17)
\]
With the help of Equation (3.13) this now yields the following formula:

\[ \tau_{dd} = \int_0^{\infty} \sigma p \left( \bar{\zeta}_{zzz} - p^2 \bar{\zeta}_z \right) J_0(pr) \, dp + r^{-1} \int_0^{\infty} p^2 \bar{\zeta}_z J_1(pr) \, dp \quad (3.18) \]

Taking the zero order Hankel transform of the expression given for \( \tau_{zz} \) in (3.1) and then inverting, one obtains

\[ \tau_{zz} = \int_0^{\infty} \frac{1}{p} \left\{ (1-\sigma)\bar{\zeta}_{zzz} - (2-\sigma)p^2 \bar{\zeta}_z \right\} J_0(pr) \, dp. \quad (3.19) \]

Consider now the expressions given for the displacement in (3.3). Taking the first order transform of \( u_r \) and the zero order transform of \( u_z \) and then inverting, one obtains

\[ 2G u_r = \int_0^{\infty} p^2 \bar{\zeta}_z J_1(pr) \, dp, \quad (3.20) \]

\[ 2G u_z = \int_0^{\infty} p \left\{ (1-2\sigma)\bar{\zeta}_{zz} - 2(1-\sigma)p^2 \bar{\zeta} \right\} J_0(pr) \, dp. \quad (3.21) \]

The procedure in this section has been mainly formal heuristic. Several assumptions on the behavior of the potential function have been made without mathematical justification; moreover, many of the conditions used are
necessary but not sufficient. However, when final expressions for the various functions are obtained, it can be verified that the results do, in fact, satisfy the basic conditions stipulated in the problem. Also, when the potential function is obtained explicitly, one could test whether the assumptions made in this section on its behavior are justified or not; but this would not be worth-while unless one could establish the uniqueness of the solution. Unfortunately, this question of uniqueness seems to be awkward and difficult and will be left unanswered.

D. The Equivalent Dual Integral Equations

With \( \overline{\phi} \) chosen to be of the form (3.12), namely,

\[
\overline{\phi}(p, z) = (A + Bz)e^{-pz},
\]

where \( A \) and \( B \) are functions of \( p \), it remains only to study the boundary conditions (3.5) - (3.7).

By virtue of the relation (3.15) the boundary condition (3.5) is expressible in the form

\[
\int_{0}^{\infty} p^3(Ap - 2\sigma B) J_1(pr) \, dp = 0, \quad r > 0.
\]

Hence, the boundary condition (3.5) will be satisfied if

\[
Ap - 2\sigma B = 0,
\]
that is, if \( \overline{\phi} \) is of the form

\[
\overline{\phi}(p, z) = F(p) (2\sigma + pz)e^{-pz}.
\] (3.22)

Now set

\[
f(ap) = (ap)^R F(p) \quad \text{and} \quad R = r/a.
\]

Then, by virtue of the relations (3.21) and (3.19), the remaining two boundary conditions (3.6), (3.7) require that \( f \) satisfy the dual integral equations

\[
\int_0^\infty t^{-1} f(t) J_0(Rt) \, dt = h(R), \quad R < 1,
\]

\[
\int_0^\infty f(t) J_0(Rt) \, dt = 0, \quad R > 1,
\]

where

\[
h(R) = -G a^4 g(aR)/(1 - \sigma).
\] (3.24)

It should be remarked that, by the nature of cylindrical coordinates, \( R \) cannot be negative.

It is now apparent that the problem has been reduced to the task of solving the dual equations (3.23). Once \( f \) is known \( \overline{\phi} \) is known from (3.22), and then the stress and displacement components can be derived from the formulas developed in Section C of this chapter.

Now \( h(R) \) is a polynomial because \( g(r) \) is a polynomial. Hence, the dual equations (3.23) are of the type
considered in Section B of Chapter I. The solution is, by virtue of the formula (2.7),

\[ f(t) = \frac{-E_* b}{(1 - \sigma^2)} \left\{ A_0 f_0(t) + A_1 f_1(t) + A_2 a^2 f_2(t) \right. \]
\[ + \cdots + A_m a^m f_m(t) \right\}, \quad (3.25) \]

where

\[ f_n(t) = \sqrt{\pi} \gamma_n^{-1} \int_0^1 v^n \cos tv \, dv \quad (3.26) \]

and \( \gamma_n \) is the expression defined in (2.12).

It will be found convenient to display the formula (3.26) for the first few values of \( n \). So,

\[ f_0(t) = \sin t \]
\[ f_1(t) = \frac{\pi}{2} \left( \sin t - \frac{1 - \cos t}{t} \right) \]
\[ f_2(t) = 2 \sin t + \frac{4}{t} \cos t - \frac{4}{t^2} \sin t \quad (3.27) \]
\[ f_3(t) = \frac{3\pi}{4} \left( \sin t + \frac{3}{t} \cos t - \frac{6}{t^2} \sin t + \frac{6(1 - \cos t)}{t^3} \right) \]
\[ f_4(t) = \frac{8}{3} \left( \sin t + \frac{4}{t} \cos t - \frac{12}{t^2} \sin t - \frac{24}{t^3} \cos t + \frac{24}{t^4} \sin t \right). \]

Now split off the first term in the expression of each \( f_n(t) \) and designate the sum of the remaining terms by \( F_n(t) \); that is, let
\[ f_n(t) = \sqrt{n} \gamma_n^{-1} \sin t + F_n(t), \]
\[ F_n(t) = -\sqrt{n} \gamma_n^{-1} n \int_0^1 v^{n-1} \sin(tv) dv. \]  

Therefore,
\[ F_0(t) = 0, \]
\[ F_1(t) = -\frac{\pi}{2} \left( 1 - \cos t \right), \]
\[ F_2(t) = -\frac{4(1 - \cos t)}{t} + \frac{4(t - \sin t)}{t^2}, \]
\[ F_3(t) = -\frac{\pi}{4} \left( 1 - \cos t \right) + \frac{\pi}{2} \left( \frac{t - \sin t}{t^2} \right) \]
\[ + \frac{9\pi}{2} \left( \frac{1 - (t^2/2)}{t^3} \right) - \cos t, \]
\[ F_4(t) = -\frac{32}{3} \left( \frac{1 - \cos t}{t} + \frac{32(t - \sin t)}{t^2} \right) \]
\[ + 64 \left( 1 - \frac{(t^2/2)}{t^3} \right) - \frac{64(t - (t^3/3))}{t^4} - \sin t. \]

The reason for decomposing the F's in the above manner will become apparent in the next section when it is required to evaluate certain integrals explicitly.

E. Determination of Stress and Displacement

The expression given in Equation (3.22) for \( \tilde{\sigma} \) can be written
\( \overline{\varphi}(p, z) = (2\sigma + Zt)e^{-Zt}t^{-h}f(t), \quad (3.30) \)

where

\[
t = \alpha p, \quad Z = z/a, \quad (3.31)
\]

and \( f(t) \) is the solution of the dual equations obtained in the previous section. In view of Equations (3.25) and (3.28) it is clear that \( \overline{\varphi} \) can be written in the form

\[
\overline{\varphi} = K(2\sigma + Zt)e^{-Zt}t^{-h}\left\{ \sin t + \sum_{n=1}^{m} A_n e^n f_n(t) \right\}, \quad (3.32)
\]

where

\[
K = -\frac{Ea^h}{\pi(1 - \sigma^2)} \quad (3.33)
\]

and

\[
C = A_0 + \frac{(\pi/2)A_1 a}{2A_2 a^2} + \frac{(3\pi/4)A_3 a^3}{(8/3)A_4 a^4} + \cdots + \frac{1}{2\pi} \gamma_{n-1} A_m a^m. \quad (3.34)
\]

It will now be shown that, unless \( A_n \) vanishes for all positive values of \( n \), \( C \) must be zero. It should be observed that the exception occurs when the polynomial \( g(u) \) is a mere constant and that this corresponds to the problem of indentation by a right circular cylinder which has been solved by I. N. Sneddon (9). It will be assumed here that at least one of the \( A_n \)'s, other than \( A_0 \), does not vanish. From the expression given for \( \overline{\tau}_{zz} \) in Equation (3.19) it follows, after a substitution for \( \overline{\varphi} \) from Equations (3.32), that
\[ a^5 \left[ \tau_{zz} \right]_{z=0} = K \int_0^\infty \left\{ C \sin t + \sum_{n=1}^m A_n a^n F_n(t) \right\} J_0(Rt) \, dt. \]

Now, when \( R = 1 \), the integral
\[
\int_0^\infty \sin t \, J_0(Rt) \, dt
\]
diverges (see Watson (17, p. 405)); whereas it will be seen in the appendices that the integrals
\[
\int_0^\infty F_n(t) \, J_0(Rt) \, dt
\]
converge for all values of \( R \). Therefore, if infinite stress values are to be excluded, \( C \) must vanish. Hence,
\[
A_0 = -\left( \frac{\pi}{2} \right) A_1 a - 2A_2 a^2 - \left( \frac{4}{3} \right) \pi A_3 a^3 - \left( \frac{8}{3} \right) A_4 a^4
\]
\[- \cdots - \psi \pi \gamma m^{-1} A_m a^m. \quad (3.35)\]

Now \( A_1, A_2, A_3, \ldots \) are geometrical constants associated with the punch so that this last equation determines the depth of penetration at the center, if the radius of contact, \( a \), is known.

Since \( C \) vanishes, Equation (3.32) can be written in the form
\[
\overline{\sigma} = \sum_{n=1}^m A_n a^n \overline{\sigma}_n \quad (3.36)
\]
where
\[
\overline{\sigma}_n = K(2\sigma + Zt)e^{-2zt} \overline{F}_n(t). \quad (3.37)
\]
It is evident from Equations (3.15), (3.17), ..., (3.21), that the operations on $\bar{\mathcal{F}}$ which yield the stress and displacement are linear. Hence, using subscripts $i$ and $j$ to indicate generic components, one may write

$$
\tau_{ij}^{(n)} = \sum_{n=1}^{m} A_n a^i a^j
$$

(3.38)

and

$$
u_i^{(n)} = \sum_{n=1}^{m} A_n a^i u_i^{(n)}.
$$

where $\tau_{ij}^{(n)}$ and $u_i^{(n)}$ are derived from $\bar{\mathcal{F}}_n$ in accordance with the equations (3.15), (3.17), ..., (3.21), the only difference being the presence of the index $n$.

It is now convenient, for the sake of brevity, to introduce some new symbols. Let

$$
P_s(n) = \int_0^\infty t^s F_n(t) e^{-Zt} J_0(Rt) \, dt,
$$

(3.40)

$$
Q_s(n) = \int_0^\infty t^s F_n(t) e^{-Zt} J_1(Rt) \, dt.
$$

It should be observed that the $P$'s and $Q$'s so defined are functions of $(R, Z)$ but the functional arguments will be omitted unless they take on special values.

Since $\bar{\mathcal{F}}_n$ is given by Equation (3.37) the relations (3.15), (3.17) - (3.21) lead to the formulas
\[
\tau_{rr}^{(n)} = L \left\{ P_0^{(n)} - ZP_1^{(n)} - \frac{1-2\sigma}{R} Q_{-1}^{(n)} + \frac{Z}{R} Q_0^{(n)} \right\},
\]

\[
\tau_{\theta\theta}^{(n)} = L \left\{ 2 P_0^{(n)} + \frac{1-2\sigma}{R} Q_{-1}^{(n)} - \frac{Z}{R} Q_0^{(n)} \right\},
\]

\[
\tau_{zz}^{(n)} = L \left\{ P_0^{(n)} + Z P_1^{(n)} \right\},
\]

\[
\tau_{rz}^{(n)} = L L Z Q_1^{(n)},
\]

\[
u_r^{(n)} = H \left\{ Z Q_0^{(n)} - (1 - 2\sigma)Q_{-1}^{(n)} \right\},
\]

\[
u_z^{(n)} = H \left\{ Z P_0^{(n)} + 2(1 - \sigma)P_{-1}^{(n)} \right\},
\]

where \( L \) and \( H \) are given by

\[
L = -E/\left[\pi a(1 - \sigma^2)\right], \quad H = 1/[\pi(1 - \sigma)]. \tag{3.42}
\]

An inspection of the Equations (3.38) - (3.42) shows that the stress and displacement components can now be determined explicitly in terms of the \( P \)'s and \( Q \)'s. In order to obtain answers in a satisfactory form it is therefore necessary to evaluate the integrals \( P_s^{(n)}, Q_s^{(n)} \), for \( s = -1, 0, 1 \) and \( n = 1, 2, \ldots, m \). It will be seen that these integrals can be evaluated in closed form in terms of elementary functions of the position coordinates. Explicit answers will be displayed for values of \( n \) up to 4 and the method of extending the list will be clear.
It will be found convenient to introduce further functions \( D, a, \beta, \ldots \) of the position coordinates by the following definitions:

\[
D^2 = R^2 + Z^2, \quad D > 0,
\]
\[
q^2 = 1 + Z^2, \quad q > 1,
\]
\[
cot a = Z, \quad 0 < a < \pi/2, \quad Z > 0,
\]
\[
s^2 = (D^2 - 1)^2 + 4Z^2, \quad s > 0,
\]
\[
\tan 2\beta = 2Z/(D^2 - 1), \quad 0 < \beta < \pi/2,
\]
\[
\tan \gamma = \frac{q \sin a + s \sin \beta}{q \cos a + s \cos \beta}, \quad 0 < \gamma < \pi/2,
\]
\[
w = \left\{ \frac{q^2 + s^2 + 2qs \cos(a-\beta)}{D + Z} \right\}^{1/2}, \quad w > 0,
\]

where \( R = r/a \) and \( Z = z/a \).

It is clear that the coordinates \((r, \Theta, z)\) of any point of the medium uniquely determine the corresponding values of \( D, q, a, \ldots \) so that it will be satisfactory to present answers in terms of \( D, q, a, \ldots \) as well as \( R, Z \).

From the expressions for \( F_n(t) \) given in (3.29) it is evident that the \( P \)'s and \( Q \)'s defined in (3.40) are linear combinations of the integrals \( C_{nk}^p(R, Z) \) defined by
\[ C_{nk}^P(R,Z) = \int_0^\infty t^{-k} \left\{ 1 - \frac{t^2}{2L} + \frac{t^4}{4L^2} + \cdots + \frac{(-1)^{n+1} t^{2n-2}}{(2n-2)!} \right\} e^{-Zt} J_p(Rt) \, dt, \]

\[ S_{nk}^P(R,Z) = \int_0^\infty t^{-k} \left\{ t - \frac{t^3}{3L} + \frac{t^5}{5L^2} + \cdots + \frac{(-1)^{n+1} t^{2n-1}}{(2n-1)!} \right\} e^{-Zt} J_p(Rt) \, dt, \]

where \( p = 0 \) or \( 1 \) and, for each \( n \), the associated value of \( k \) will be such as to make the integrals convergent. These integrals are evaluated in the appendices. In Appendix A it is stipulated that \( Z \) be strictly positive but this restriction is removed in Appendix B. The range of values of \( n \) considered is from 0 to 4 but this could be extended if required.

Since the \( C's \) and \( S's \) are expressible in simple closed form (see Appendices A and B) so are the \( P's \) and \( Q's \); in fact, the actual linear combinations of the former which yield the latter are as follows:

\[ p_n^{(1)} = -(\pi/2) C_{1,1-n}^0, \]

\[ p_n^{(2)} = -4C_{1,1-n}^0 + 4S_{1,2-n}^0, \]

\[ p_n^{(3)} = -(9\pi/4)C_{1,1-n}^0 + (9\pi/2)S_{1,2-n}^0 + (9\pi/2)C_{2,3-n}^0, \]

\[ p_n^{(4)} = (32/3)C_{1,1-n}^0 + 32S_{1,2-n}^0 + 64C_{2,3-n}^0. \]
Equations (3.38), (3.39) and (3.41) show that the stress and displacement at any point \((r, \theta, z)\) of the medium are known explicitly once the \(P\)'s and \(Q\)'s have been computed. The determination of \(\tau^{(n)}_{ij}\) and \(u^{(n)}_j\) requires for each \(n\), the evaluation of six integrals. For example, if \(n = 2\), these six integrals are found to be as follows:

\[
\begin{align*}
P^{(2)}_1 &= 4 \left( s^{-1} \cos \beta - \gamma \right), \\
P^{(2)}_0 &= 4 \left( \gamma Z - s \sin \beta \right), \\
P^{(2)}_{-1} &= -\gamma (2Z - R + 2) - 4s \cos \beta + 3qs \sin(\alpha + \beta), \\
Q^{(2)}_1 &= 4R^{-1} \left\{ s \sin \beta - (q/s) \cos(\alpha - \beta) \right\}, \\
Q^{(2)}_0 &= 4R^{-1} \left\{ s \cos \beta - (qs/2) \sin(\alpha + \beta) - R^2 \gamma/2 \right\},
\end{align*}
\]
\[
Q_{-1}^{(2)} = \frac{2}{3R} \left\{ -2 - 3qs \cos(a+\beta) + 3R^2 Z_\gamma \\
+ s^3 \sin 3\beta - 3R^2 s \sin \beta \right\},
\]
where the symbols \(a, \gamma, \ldots\) have the meanings which have been assigned to them in the relations (3.43).

An alternative, and probably more efficient, method for evaluating the integrals \(P_s(n), Q_s(n)\) is described in Appendix C.

The quantity "\(a\)" , which is present in most of the equations, can be determined by equating the total load to the integral of \(\tau_{zz}\) \(z=0\) over the region \(r<a\).

F. Spherical Indentation

1. Quartic approximation

Consider now the case where the surface of contact is spherical, the radius being \(c\). Since the preceding theory assumes that all displacements are small, the equation of the sphere may be approximated by

\[
z = A_0 + A_2 r^2 + A_4 r^4, \quad (3.47)
\]
where \(A_2 = -1/2c\) and \(A_4 = -1/8c^3\). This may be called the quartic approximation. For many purposes sufficient accuracy would be obtained by taking \(A_4 = 0\), but the quartic approximation is chosen here to illustrate the procedure.
for more general problems as well as to achieve greater accuracy for the spherical case.

From Equations (3.38) and (3.39) it follows that the stress and displacement are given by

\[
\tau_{1j} = A_2 a^2 \tau^{(2)}_{1j} + A_4 a^4 \tau^{(4)}_{1j},
\]
\[
u_1 = A_2 a^2 \nu^{(2)}_1 + A_4 a^4 \nu^{(4)}_1,
\]

where \( \tau^{(2)}_{1j}, \tau^{(4)}_{1j}, \nu^{(2)}_1, \nu^{(4)}_1 \) can be determined explicitly from Equations (3.41), (3.45) and the list of integrals in the appendices. It remains to evaluate the quantities \( a \) and \( A_0 \) in terms of the total load and the other constants of the problem.

From (3.41) it follows that

\[
\tau^{(n)}_{zz} (R,0) = -\frac{E}{\pi a (1 - \sigma^2)} P_0^{(n)} (R,0),
\]

and, by using (3.45) and Appendix B, one finds that, for \( R < 1, \)

\[
P_0^{(2)} (R,0) = -4(1 - R^2)^{1/2},
\]

\[
P_0^{(4)} (R,0) = -(32/9) (1 + 2R^2) (1 - R^2)^{1/2}.
\]

Hence, for \( R < 1, \)

\[
\tau_{zz}(R,0) = -\frac{2Ea}{\pi (1 - \sigma^2)c} \left\{ 1 + \frac{2a^2}{9c^2 (1 + 2R^2)} \right\} (1 - R^2)^{1/2}.
\]
Substituting this in the relation

\[ W = - \int_{0}^{1} \tau_{zz}(R,0) 2\pi R a^{2} dR, \]

where \( W \) is the total load, one obtains, as the equation to determine \( a \),

\[ a^{3} \left( 1 + \frac{2a^{2}}{5c^{2}} \right) = \frac{3c}{4E} \left( 1 - \sigma^{2} \right) W. \]  \( (3.49) \)

If \( a \) is small compared with \( c \), this equation has the approximate solution

\[ a = \left\{ \frac{3c(1 - \sigma^{2}) W}{4E} \right\}^{1/3} - \frac{W(1 - \sigma^{2})}{10Ec}, \]  \( (3.50) \)

and so the circle of contact is known.

The maximum depth of penetration is \( A_{0} \) and is given by Equation (3.35), namely,

\[ A_{0} = -2A_{2}a^{2} - \frac{8}{3}A_{4}a^{4} \]

\[ = (a^{2}/c) + (a^{4}/3c^{2}). \]  \( (3.51) \)

If quasi-static conditions are assumed to prevail during the deformation, the work \( V \) done in indenting the medium is given by

\[ V = \int_{0}^{A_{0}} W dA_{0} = \int_{0}^{a} W \left( \frac{\partial A_{0}}{\partial a} \right) da. \]
The components of stress on the axis of symmetry are found to be

\[ \tau_{rr}(0,z) = \tau_{\theta\theta}(0,z) = \frac{E_\theta}{(1-\sigma^2)c} \left[ 2(1+\sigma)(z\cot^{-1}Z - 1) + \frac{1}{1+Z^2} \right. \\
+ \frac{2a^2}{9c^2} \left\{ \frac{3}{1+Z^2} - (4+2\sigma)(1-3Z^2+3Z^3\cot^{-1}Z) \right\} \left. \right] \]

(3.52)

\[ \tau_{zz}(0,z) = \frac{-E_\theta}{(1-\sigma^2)c} \left[ \frac{2}{1+Z^2} - \frac{4a^2}{9c^2} \left\{ 2 - \frac{3}{1+Z^2} \right. \\
+ 6Z^2(Z\cot^{-1}Z - 1) \right\} \right] \]

(3.53)

\[ \tau_{rz}(0,z) = 0. \]

It follows that the Equations (3.52) give the principal stresses and so the maximum shearing stress at a depth \( z \) below the origin is

\[ \tau_m(z) = \left| \tau_{rr}(0,z) - \tau_{zz}(0,z) \right|. \]

(3.53)

2. Paraboloidal approximation

To simplify the exposition, further discussion will be confined to the cruder approximation where the equation
of the sphere is taken as \( z = A_0 - r^2/2c \). Since this is the equation of a paraboloid of revolution it may be called the paraboloidal approximation. It is clear that the results here may be considered as a first approximation for a problem involving any axially symmetric punch which has finite curvature \( 1/c \) at the vertex.

Setting

\[
L' = \frac{E a}{2\pi(1 - \sigma^2)} \quad \text{and} \quad H' = -\frac{a^2}{2\pi(1 - \sigma)}
\]

one finds that the components of stress and displacement at any point of the medium are given by

\[
\tau_{RR}(R,Z) = L' \left\{ P_0 - ZP_1 - (1 - 2\sigma)R^{-1}Q_{1-1} + (Z/R)Q_0 \right\},
\]

\[
\tau_{QQ}(R,Z) = L' \left\{ 2\sigma P_0 + (1 - 2\sigma)R^{-1}Q_{1-1} - (Z/R)Q_0 \right\},
\]

\[
\tau_{ZZ}(R,Z) = L' (P_0 + ZP_1),
\]

\[
\tau_{TZ}(R,Z) = L'Z Q_1,
\]

\[
\tau_{uR}(R,Z) = H' \left\{ Z Q_0 - (1 - 2\sigma)Q_{1-1} \right\},
\]

\[
\tau_{uZ}(R,Z) = H' \left\{ Z P_0 + 2(1 - \sigma)P_{1-1} \right\},
\]

where the P's and Q's are given in Equations (3.46), the superscript (2) being ignored.
By making use of the lists of integrals in the appendices, it is found that the components of stress and displacement at points of the boundary are given by

\[
\begin{align*}
N \tau_{rr}(R,0) &= \begin{cases} 
-(1-R^2)^{1/2} + (1-2\sigma) \left[1-(1-R^2)^{3/2}\right] / (3R^2), \\
(1-2\sigma)/(3R^2),
\end{cases} \\
N \tau_{\theta\theta}(R,0) &= \begin{cases} 
-2\sigma(1-R^2)^{1/2} - (1-2\sigma) \left[1-(1-R^2)^{3/2}\right] / (3R^2), \\
-(1-2\sigma)/(3R^2),
\end{cases} \\
N \tau_{zz}(R,0) &= \begin{cases} 
-(1-R^2)^{1/2}, \\
0,
\end{cases}
\end{align*}
\]

\(N \tau_{r\theta}(R,0) = 0\),

\[
\frac{\pi c(1-\sigma)}{2a}(1-2\sigma) u_r(R,0) = \begin{cases} 
\left[1 - (1-R^2)^{3/2}\right] / (3R), \\
1/(3R),
\end{cases}
\]

\[
u_z(R,0) = \begin{cases} 
(a^2/2c)(2-R^2) \\
(a^2/\pi c) \left[(R^2 - 1)^{1/2} + (2 - R^2)\csc^{-1}R\right]
\end{cases}
\]

where

\[
N = \pi c(1-\sigma^2)/(2Ea)
\]

and, in each formula, the upper line on the right-hand side is valid for \(R \leq 1\) and the lower line for \(R \geq 1\). It is evident from inspection that each component of stress and displacement is continuous at \(R = 1\).
On the axis of symmetry

\[ N \tau_{xx}(0, z) = (1 + \sigma)(z \cot^{-1} z - 1) + 1/ \left[ 2(1+z^2) \right] , \]

\[ = N \tau_{yy}(0, z) , \]

(3.56)

\[ N \tau_{zz}(0, z) = -1/(1 + z^2) , \]

\[ \tau_{xz}(0, z) = 0. \]

Therefore, the Equations (3.56) give the principal stresses and so the maximum shearing stress at a depth \( z \) below the origin is

\[ \tau_m(z) = \frac{E a}{(1-\sigma^2)c} \left| \frac{3}{1 + 2z^2} + 2(1+\sigma)(z\cot^{-1} z -1) \right| , \]

and this is a maximum when \( z = z_0 = \cot \beta \), where

\[ 8(1 + \sigma)\beta = (10 + 4\sigma)\sin 2\beta - 3 \sin 4\beta. \]

If the Poisson's ratio \( \sigma \) is taken to be 0.3, one finds that \( \beta = 1.122... \) and \( \cot \beta = 0.481... \) so that the maximum shearing stress occurs at a depth \( z_0 = 0.481 \) a below the origin. This is the point at which the elastic medium is most likely to fail in shear. It is of interest to compare this maximum shearing value with stress values at the origin; the approximate results are

\[ \tau_m(z_0) = 3.08 \tau_m(0) = -0.618 \tau_{zz}(0,0). \]
IV. PLANE-STRAIN INDENTATION OF AN
ORTHOTROPIC MEDIUM

A. Statement of the Problem

Consider a semi-infinite elastic medium with a plane boundary. Using rectangular Cartesian coordinates \((x,y,z)\) with the origin on the boundary, choose the \(x\)-axis so that it is directed normally into the medium. In this chapter it is assumed that the medium is orthotropic, as defined by I. S. Sokolnikoff (12, p.62), with the coordinate planes as planes of elastic symmetry and that, in addition, the physical properties of the medium are unaltered in description when the \(y\) and \(z\) axes are interchanged.

The main problem in this chapter is to determine the stress in the medium when the plane boundary is indented with a rigid punch which is in the form of a long cylinder with generators parallel to the \(z\)-axis. All sections by planes perpendicular to the \(z\)-axis will then be identical in behavior so that the problem becomes two-dimensional, that is, one of plane-strain. It is, therefore, sufficient to confine attention to the plane \(z = 0\) and to consider the stress at the point \((x,y)\).
Further, it will be assumed that the punch is well lubricated, that it is in contact with the medium over the strip \(-a < y < a\), and that the curve of contact in the \((x, y)\) plane is symmetrical about the \(x\)-axis. More precisely, the curve of contact will be assumed to be of the polynomial form

\[
x = g(y) = \sum_{n=0}^{m} A_n y^n, \quad |y| < a.
\]

(4.1)

Contrary to expectation, this problem presents greater difficulty, even in the isotropic case, than the analogous three-dimensional problem discussed in Chapter III. This is largely due to the fact that the method of solution via dual integral equations fails in the apparently simple case where \(g(y)\) is a constant. This exceptional case is therefore separately considered in Section C and the results so obtained are used to complete the solution of the more general problem in Section D.

Throughout this chapter it will be assumed that the body forces in the medium are negligible and that there is no external load apart from that due to the punch. Also, the strains are supposed to be small so that the classical theory of elasticity may be used.
B. The Potential Function $\phi$

Plane-strain phenomena in an orthotropic medium of the type described above have been studied by R. H. Tripp and D. L. Holl (16) and this section is a summary of the first part of their paper with a slight modification in the notation.

Four elastic constants $E_x$, $E_y$, $\sigma_x$, $\sigma_y$, which may be called generalized Young's moduli and Poisson's ratios, are used to specify the elastic properties of such an orthotropic substance. They are connected by one relation $E_x/E_y = \sigma_y/\sigma_x$ and it is convenient to introduce two more elastic constants $k$ and $K$ defined by

$$k^2 = \frac{E_y}{E_x} = \frac{\sigma_x}{\sigma_y},$$

$$K^2(1 - \sigma_y^2) = k^2(1 - \sigma_y^2k^2).$$

For the sake of simplicity $E_y$, $\sigma_y$, will henceforth be written as $E$, $\sigma$, respectively without the subscript. Hence,

$$K^2(1 - \sigma^2) = k^2(1 - \sigma^2k^2). \quad (4.2)$$

Under plane-strain conditions, the equilibrium equations and the compatibility relations are satisfied if the stress components are derivable from a potential function
\( \phi \) in accordance with the equations

\[
\tau_{xx} = \frac{\partial^2 \phi}{\partial y^2},
\]

\[
\tau_{yy} = \frac{\partial^2 \phi}{\partial x^2},
\]

\[
\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y},
\]

(4.3)

where \( \phi(x, y) \) satisfies the differential equation

\[
\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \left(1 + K^2\right) \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + K^2 \frac{\partial^2 \phi}{\partial y^4} = 0.
\]

(4.4)

The displacement components \((u, v)\) are then given by the equations

\[
E(\frac{\partial u}{\partial x}) = k^2 \left(1 - \sigma^2 k^2\right) \tau_{xx} - k^2 \sigma (1 + \sigma) \tau_{yy},
\]

\[
E(\frac{\partial v}{\partial y}) = (1 - \sigma^2) \tau_{yy} - k^2 \sigma (1 + \sigma) \tau_{xx},
\]

(4.5)

\[
G \left\{ (\frac{\partial u}{\partial y}) + (\frac{\partial v}{\partial x}) \right\} = \tau_{xy},
\]

where \( G \) is a shear modulus defined by the relation

\[
\frac{E}{G} = k^2 \left(1 - \sigma^2 k^2 + \sigma^2 + \sigma^2 + k^2 \sigma^2 + k^2 \sigma^2\right).
\]

(4.6)

For the isotropic case \( E_x = E_y = E, \sigma_x = \sigma_y = \sigma \) and \( K = k = 1 \).
C. The Equivalent Dual Integral Equations

The problem on hand, as stated in Section A, is equivalent to the task of solving the differential Equation (4.4) subject to the boundary conditions

\[ \tau_{ij} \to 0 \text{ as } x^2 + y^2 \to \infty, \quad (4.7) \]

\[ [\tau_{xy}]_{x=0} = 0 \text{ identically in } y, \quad (4.8) \]

\[ [u]_{x=0} = g(y), \quad |y| < a; \quad (4.9) \]

\[ [\tau_{xx}]_{x=0} = 0, \quad |y| > a. \quad (4.10) \]

The procedure henceforth is mainly formal, but once a solution is obtained it can be verified. The main defect is that the uniqueness of the solution is not assured mathematically.

Suppose \( \varphi(x, y) \) is the desired solution and let

\[ \Phi(x, \beta) \text{ be its Fourier transform relative to } y, \text{ namely,} \]

\[ \Phi(x, \beta) = \int_{-\infty}^{\infty} \varphi(x, y) e^{i\beta y} dy. \]

By making the assumption, suggested by condition (4.7), that \( \frac{\partial^n \varphi}{\partial y^n} \to 0 \text{ as } |y| \to \infty \text{ for } n = 1, 2, 3, \) it follows that the differential Equation (4.4) for
φ(x, y) implies that the Fourier transform \( \mathcal{F}(x, p) \) satisfies the ordinary differential equation

\[
\left\{ D^4 - (1 + K^2) \beta D^2 + K^2 \beta^4 \right\} \mathcal{F} = 0,
\]

i.e.,

\[
(D^2 - \beta^2) (D^2 - \beta^2 K^2) \mathcal{F} = 0,
\]  
(4.11)

where \( D = d/dx \) and \( \beta \) is regarded as a parameter.

In view of the condition (4.7) one chooses a solution of (4.11) of the form

\[
\mathcal{F}(x, \beta) = A e^{-|\beta|x} + B e^{-K|\beta|x},
\]  
(4.12)

where \( A \) and \( B \) depend on \( \beta \).

Now, from the relation

\[
\tau_{xy} = -\partial^2 \phi / \partial x \partial y
\]

it follows that

\[
\tau_{xy} = i\beta (d\mathcal{F}/dx),
\]

where bars over a symbol indicate Fourier transforms.

The inversion of the last equation yields

\[
\tau_{xy} = (1/2\pi) \int_{-\infty}^{\infty} \beta |\beta| \left( Ae^{-|\beta|x} + KB e^{-K|\beta|x} \right) e^{-i\beta y} d\beta,
\]  
(4.13)
so that condition (4.8) will be satisfied if

$$A + KB = 0.$$  \hspace{1cm} (4.14)

Therefore, if one chooses \( \bar{\beta} \) to be of the form

$$\bar{\beta}(x, \beta) = B(\beta) \left( K e^{-|\beta|y} - e^{-K|\beta|x} \right), \hspace{1cm} (4.15)$$

then it remains only to determine \( B(\beta) \) so that the last two boundary conditions (4.9), (4.10) are also met. From the relation

$$\tau_{xx} = \frac{\partial^2 \bar{\beta}}{\partial y^2}$$

it follows, by taking the Fourier transform of each side with respect to \( y \), that

$$\tau_{xx} = -\beta^2 \bar{\beta}(x, \beta),$$

provided \( \left( \frac{\partial \bar{\beta}}{\partial y} - i\beta \bar{\beta} \right) e^{i\beta y} \) tends to zero as \( y \) tends to \( \pm \infty \). This latter condition will be assumed to hold. The inversion theorem applied to the last equation above now yields

$$\tau_{xx} = -(1/2\pi) \int_{-\infty}^{\infty} \beta B(\beta) \left( K e^{-|\beta|y} - e^{-K|\beta|x} \right) e^{-i\beta y} d\beta.$$  \hspace{1cm} (4.16)
Therefore, the condition (4.10) will be met if
\[ \beta^2 B(\beta) e^{i\beta y} \frac{d\beta}{i\beta} = 0, \quad |y| > a. \]  
\( (4.17) \)

In order to meet the remaining boundary condition it is desirable to express \( u \) in terms of \( \phi \). This is done by first seeking \( v \). Making similar assumptions as before on the behavior of \( \phi \) as \( y \) tends to infinity, one obtains from the second of Equations (4.5)

\[ -i\beta \bar{\phi}(x,\beta) = (1 - \sigma^2)\frac{d^2 \phi}{dx^2} + k^2 \sigma (1 + \sigma) \beta^2 \phi, \]

where bars indicate, as always in this chapter, Fourier transforms with respect to \( y \). Substituting for \( \phi \) from (4.15) into the last equation one obtains, after some simplification,

\[ E \bar{v}(x,\beta) = i\beta B(\beta) (1 + \sigma) \left( b e^{-|\beta||x|} - b' e^{-K|\beta||x|} \right), \]  
\( (4.18) \)

where

\[ b = K(1 - \sigma + k^2 \sigma), \]  
\( (4.19) \)

\[ b' = K^2(1 - \sigma) + k^2 \sigma. \]

The Fourier inversion theorem now yields

\[ E v = \frac{1}{2\pi} (1 + \sigma) \int_{-\infty}^{\infty} \beta B(\beta) \left( b e^{-|\beta||x|} - b' e^{-K|\beta||x|} \right) e^{-i\beta y} \, d\beta. \]  
\( (4.20) \)
In a similar way the third of Equations (4.5), after the Fourier transform of each side is taken, gives

\[ G \left\{ -i \beta \bar{u} + (d\bar{v}/dx) \right\} = i\beta (d\mathcal{O}/dx). \]  

(4.21)

Substituting in this equation for \( \bar{v} \) from Equation (4.18) and for \( \mathcal{O} \) from Equation (4.15) one obtains

\[ G \bar{u} = K|\beta|B(\beta) \left( c e^{-|\beta|x} + c' e^{-K|\beta|x} \right), \]  

(4.22)

where

\[ c = 1 + \frac{G}{E} \left( 1 + \sigma \right) \left( 1 - \sigma + k^2 \sigma \right), \]  

(4.23)

\[ c' = -1 + \frac{G}{E} \left( 1 + \sigma \right) k^2 \left( 1 - \sigma \right) + k^2 \sigma. \]  

The Fourier inversion theorem applied to Equation (4.22) now yields

\[ 2\pi G u = K \int_{-\infty}^{\infty} |\beta| B(\beta) \left( c e^{-|\beta|x} + c' e^{-K|\beta|x} \right) e^{-i\beta y} d\beta. \]  

(4.24)

Hence, the remaining boundary condition (4.9) will be satisfied if

\[ K(c + c') \int_{-\infty}^{\infty} |\beta| B(\beta) e^{-i\beta y} d\beta = 2\pi G g(y), \quad |y| < a. \]  

(4.25)

It is now clear that the desired form for \( \mathcal{O}(x, \beta) \) is given by Equation (4.15) where \( B(\beta) \) satisfies the pair of
dual integral Equations (4.25), (4.17). As there is symmetry about the x-axis in the problem of this chapter it follows from Equation (4.15) and the definition of the Fourier transform that \( B(\beta) \) is an even function, so that the dual equations may be written

\[
\int_0^\infty \beta B(\beta) \cos \gamma \beta \, d\beta = \frac{\pi g(y)}{K(c+c')}, \quad |y| < a,
\]

\[
\int_0^\infty \beta^2 B(\beta) \cos \gamma \beta \, d\beta = 0, \quad |y| > a,
\]

or,

\[
\int_0^\infty \beta^{3/2} B(\beta) \, J_{-1/2}(\gamma \beta) \, d\beta = \frac{\sqrt{2\pi} g(y)}{K(c+c') \sqrt{y}} , \quad |y| < a,
\]

\[
\int_0^\infty \beta^{5/2} B(\beta) \, J_{-1/2}(\gamma \beta) \, d\beta = 0, \quad |y| > a.
\]

With the substitutions

\[
t = a \beta , \quad Y = y/a,
\]

the dual Equations (4.26) become

\[
\int_0^\infty t^{-1} f(t) \, J_{-1/2}(Yt) \, dt = M \frac{g(aY)}{\sqrt{Y}} , \quad |Y| < 1,
\]

\[
\int_0^\infty f(t) \, J_{-1/2}(Yt) \, dt = 0, \quad |Y| > 1,
\]

where

\[
M = \frac{\sqrt{2\pi} g \gamma^2}{K(c+c')}
\]
f(t) = t^{5/2}B(t/a) . \tag{4.30}

So the mixed boundary value problem of this chapter is reduced to the task of solving the dual Equations (4.28) for f(t). When this is done one can then determine \( \phi \) by using Equations (4.15) and (4.30), and an application of the Fourier inversion theorem would then give \( \phi(x,y) \). However, it is more convenient to obtain the components of stress and displacement directly from f(t). To do this, first consider Equation (4.16). Putting

\[ X = x/a \tag{4.31} \]

and remembering that B(\( \beta \)) is an even function related to f(t) by Equation (4.30), one obtains from (4.16)

\[ \tau_{xx} = -\frac{1}{\pi a^3} \int_0^\infty t^{-1/2} f(t) (K e^{-Xt} - e^{-KXt}) \cos Yt \, dt. \tag{4.32} \]

In similar fashion one finds

\[ \tau_{yy} = \frac{K}{\pi a^3} \int_0^\infty t^{-1/2} f(t) (e^{-Xt} - K e^{-KXt}) \cos Yt \, dt, \tag{4.33} \]

\[ \tau_{xy} = -\frac{K}{\pi a^3} \int_0^\infty t^{-1/2} f(t) (e^{-Xt} - e^{-KXt}) \sin Yt \, dt, \tag{4.34} \]

\[ u = \frac{K}{\pi Ga^2} \int_0^\infty t^{-3/2} f(t) (c e^{-Xt} + c'e^{-KXt}) \cos Yt \, dt, \]

\[ v = \frac{(1+\nu)}{\pi E a^2} \int_0^\infty t^{-3/2} f(t) (b e^{-Xt} - b'e^{-KXt}) \sin Yt \, dt, \tag{4.35} \]
where \( b, b', c, c' \) have already been defined, namely,

\[
\begin{align*}
 b &= K(1 - \sigma + k^2 \sigma^-), \\
 b' &= K^2(1 - \sigma) + k^2 \sigma^- , \\
 c &= 1 - \frac{G}{E} (1 + \sigma) (1 - \sigma + k^2 \sigma^-), \\
 c' &= -1 + \frac{G}{E} (1 + \sigma) \left\{ K^2(1 - \sigma) + k^2 \sigma^- \right\} .
\end{align*}
\]

D. Indentation by a Rectangular Block

The object of this section is to solve the problem stated in Section A when \( g(y) \) is a constant. For this case it appears that the equivalent dual integral equations do not have a solution. Hence, the procedure in this section is rather indirect.

Suppose the boundary \( x = 0 \) is subjected to the loading

\[
q(y) = \begin{cases} 
\frac{W}{\pi \sqrt{a^2 - y^2}}, & |y| < a, \\
0, & |y| > a,
\end{cases}
\]

where \( W \) is clearly the total applied load. It will appear later that this loading produces a constant deflection under the load and so yields a solution to the problem of this section.
Now let \( \phi(x, y) \) be the solution of the differential
Equation (4.4) which satisfies the boundary conditions

\[
\tau_{ij} \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \quad (4.38)
\]

\[
\left[ \tau_{xy} \right]_{x=0} = 0 \quad \text{identically in} \quad y, \quad (4.39)
\]

\[
\left[ \tau_{xx} \right]_{x=0} = -q(y). \quad (4.40)
\]

The results of Section B show that, to meet conditions (4.38) and (4.39), the Fourier transform of \( \phi \) with respect to \( y \) should be of the form

\[
\hat{\phi}(x, \beta) = B(\beta) \left\{ K e^{-|\beta|x} - e^{-K|\beta|x} \right\}. \quad (4.41)
\]

To satisfy the remaining boundary condition (4.40) it is necessary to choose \( B(\beta) \) so that

\[
\int_{0}^{\infty} \beta^2 B(\beta) \cos \gamma \beta \, d\beta = \begin{cases} 
2W/ \left[ (K-1) \sqrt{a^2 - \gamma^2} \right], & |\gamma| < a, \\
0, & |\gamma| > a.
\end{cases}
\]

Upon inversion this gives

\[
\beta^2 B(\beta) = \frac{2W}{\pi(K-1)} \int_{0}^{a} \frac{\cos \beta \gamma \, d\gamma}{\sqrt{a^2 - \gamma^2}}
\]

\[
= \frac{W}{K-1} J_0(\beta a). \quad (4.42)
\]
From (4.41) and (4.42) it follows that

$$
\bar{\theta}(x, \beta) = \frac{W}{K-1} \int \frac{J_{0}(\beta a)}{B^{2}} \left\{ e^{-|\beta| x} - e^{-K|\beta| x} \right\} . \quad (4.43)
$$

Taking the Fourier transforms of Equations (4.3), substituting for \( \bar{\theta}(x, \beta) \) from (4.43) and then using the inversion theorem, one obtains

$$
\tau_{xx} = \frac{-W}{\pi(K-1)} \int_{0}^{\infty} J_{0}(\beta a) \left\{ e^{-\beta x} - (1/K) e^{-K\beta x} \right\} \cos \beta \, d\beta, \quad (4.44)
$$

$$
\tau_{yy} = \frac{W}{\pi(K-1)} \int_{0}^{\infty} J_{0}(\beta a) \left\{ e^{-\beta x} - K e^{-K\beta x} \right\} \cos \beta \, d\beta, \quad (4.45)
$$

$$
\tau_{xy} = \frac{W}{\pi(K-1)} \int_{0}^{\infty} J_{0}(\beta a) \left\{ -e^{-\beta x} + e^{-K\beta x} \right\} \sin \beta \, d\beta. \quad (4.46)
$$

It is convenient now to introduce the complex variables \( z \) and \( s \) defined by

$$
z = x + iy \quad s = Kx + iy \quad (4.47)
$$

Hence by making use of the integral (see Magnus and Oberhettinger (7, p.47)),

$$
\int_{0}^{\infty} e^{-at} J_{0}(\gamma t) \, dt = (\alpha^{2} + \gamma^{2})^{-1/2}, \quad \Re \alpha > |\gamma|, \quad (4.48)
$$
one obtains if $x$ is positive,

\[
\int_{0}^{\infty} e^{-\beta x} \cos \beta y J_0(\beta a) \, d\beta = \mathcal{R} \frac{1}{\sqrt{a^2 + z^2}},
\]

\[
\int_{0}^{\infty} e^{-K\beta x} \cos \beta y J_0(\beta a) \, d\beta = \mathcal{R} \frac{1}{\sqrt{a^2 + s^2}},
\]

\[
\int_{0}^{\infty} e^{-\beta x} \sin \beta y J_0(\beta a) \, d\beta = -\mathcal{I} \frac{1}{\sqrt{a^2 + z^2}},
\]

\[
\int_{0}^{\infty} e^{-K\beta x} \sin \beta y J_0(\beta a) \, d\beta = -\mathcal{I} \frac{1}{\sqrt{a^2 + s^2}}.
\]

Hence, at an interior point ($x$ positive), the stress components given in Equations (4.44) - (4.46) become

\[
\tau_{xx} = \frac{WK}{\pi(K-1)} \mathcal{R} \left\{ \frac{-1}{\sqrt{a^2 + z^2}} + \frac{1/K}{\sqrt{a^2 + s^2}} \right\},
\]

\[
\tau_{yy} = \frac{WK}{\pi(K-1)} \mathcal{R} \left\{ \frac{1}{\sqrt{a^2 + z^2}} - \frac{K}{\sqrt{a^2 + s^2}} \right\}, \quad (4.50)
\]

\[
\tau_{xy} = \frac{WK}{\pi(K-1)} \mathcal{I} \left\{ \frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{a^2 + s^2}} \right\}.
\]
The values of the stress components at points of the boundary can most easily be found directly from the Equations (4.44) - (4.46) by putting \( x = 0 \), although they can be obtained from Equations (4.50) by a continuity argument. In any case, there results

\[
\tau_{xx}(0,y) = \begin{cases} 
-(W/\pi)(a^2 - y^2), & |y| < a, \\
0, & |y| > a,
\end{cases}
\]

\[
\tau_{yy}(0,y) = K \tau_{xx}(0,y),
\]

\[
\tau_{xy}(0,y) = 0,
\]

agreeing with the boundary conditions (4.39), (4.30).

The determination of the displacement is more awkward. The method used in Section C, leading to Equations (4.35) will fail here since the Fourier inversion theorem will be found to be inapplicable. So one substitutes the values of the stress given by Equations (4.50) into the primary relations (4.5) and obtains

\[
\frac{\pi(K-1)E}{KW} \frac{\partial u}{\partial x} = \mathcal{R}(-a z_1 + ka's_1), \tag{4.51}
\]

\[
\frac{\pi(K-1)E}{KW} \frac{\partial v}{\partial y} = \mathcal{R}(yz_1 - y's_1), \tag{4.52}
\]

\[
\frac{\pi(K-1)G}{KW} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mathcal{J}(z_1 - s_1), \tag{4.53}
\]
where the newly introduced symbols are defined by

\[ z_1 = (a^2 + z^2)^{-1/2}, \quad s_1 = (a^2 + s^2)^{-1/2}, \]

\[ a = k^2 (1 - \sigma k^2) + k^2 \sigma (1 + \sigma), \]

\[ a' = k^2 (1 - \sigma k^2)/k^2 + k^2 \sigma (1 + \sigma), \]

\[ \gamma = (1 - \sigma + k^2 \sigma) (1 + \sigma), \]

\[ \gamma' = (1 + \sigma) \left\{ k(1 - \sigma) + (k^2 \sigma)/k \right\}. \]

Integration of Equations (4.51) and (4.52) yield

\[
\frac{\pi (K-1)E}{KW} u = R \left\{ -a \sinh^{-1}(z/a) + a' \sinh^{-1}(s/a) \right\} + F(y), \tag{4.55}
\]

\[
\frac{\pi (K-1)E}{KW} v = J \left\{ \gamma \sinh^{-1}(z/a) - \gamma' \sinh^{-1}(s/a) \right\} + G(x), \tag{4.56}
\]

where the inverse hyperbolic functions are to be interpreted as their principal values. By considerations of symmetry it is clear that \( v \) must vanish when \( y = 0 \) and this implies that \( G(x) \) is identically zero. When the expressions for \( u \) and \( v \) given in (4.55) and (4.56) are substituted into Equation (4.53) there results

\[
G/E \left[ F'(y) + J \left\{ (a + \gamma)z_1 - (a' + K\gamma')s_1 \right\} \right] = J (z_1 - s_1). \tag{4.57}
\]
From the relations (4.54), (4.2) and (4.6) one finds that

\[ a + \gamma = a' + K \gamma' = E/G, \]

so that the Equation (4.57) is valid if, and only if,

\[ F'(y) = 0. \]

Therefore, \( F(y) \) is a constant, say \( D \), and so the displacement components are given by Equations (4.55) and (4.56) with \( F(y) \) replaced by \( D \) and \( G(x) \) by zero. If one sets \( x = 0 \) in Equation (4.55) it is found that the quantity in the square brackets is purely imaginary for \( |y| < a \), showing that the deflection is constant under the load. Therefore, the problem solved in this section is precisely the one propounded at the outset.

The result for the isotropic case can be derived from those just obtained by letting \( k \) tend to unity and using l'Hospital's theorem. For example, the first of Equations (4.50) yields, in the isotropic case,

\[
\tau_{xx} = \frac{W}{\pi} R \left\{ \frac{-1}{\sqrt{a^2 + z^2}} - \frac{z x}{(a + z)^{3/2}} \right\}
\]

\[
= - \frac{W}{\pi} R \frac{a^2 + z^2 + zx}{(a + z)^{3/2}},
\]

agreeing with the result obtained by Sneddon (11, p.432), apart from an erroneous factor of \( \pi \) in his formula.
It should be observed that (4.55), (4.56) give infinite values for $u, \ v$ at infinity. This is a known defect of the classical theory for plane-strain deformations and is noted by S. Timoshenko (13, p.88) and D. L. Holl (4, p.48). The constant $D$ may be chosen so that an arbitrarily selected point on the surface is taken as a standard reference point relative to which the other displacements are measured and the formulas for the displacements are considered to be valid only over a certain finite range. If $D$ is chosen to be $(2\pi/K)(K-1)(1-\sigma^2)\log 2$ then the results for the isotropic case agree with those obtained by M. Sadowsky (8) by an entirely different method.

E. Symmetric Indentation of Polynomial Type

This section is devoted to the main problem of the chapter as stated in Section A, the equation of the indenting punch being

$$x = g(y) = \sum_{0}^{m} A_n |y|^n, \ |y| < a,$$

(4.58)

It should be observed that $A_1, A_2, \ldots$ are geometric constants of the punch whereas $A_0$ is not. It is now convenient to introduce two new constants $c_0$ and $c_1$ by the
definitions

\[ c_0 = -(1/\gamma_n) \sum_{i}^{m} A_n \gamma_n , \quad (4.59) \]

\[ c_1 = A_0 - c_0 , \quad (4.60) \]

where \( \gamma_n \) is given by Equation (2.12).

The stress and displacement in the medium are derived from \( g(y) \) by means of linear operations so that the problem being considered may be solved by superposing the solutions of two indentation problems, the first being that due to a punch whose equation is

\[ x = h(y) = c_0 + \sum_{i}^{m} A_n |y|^n, \quad |y| < a, \quad (4.61) \]

and the second that due to the penetration of a rectangular block to a depth \( c_1 \). The second problem has already been solved in Section C so that it remains to deal with the case where the curve of contact is given by Equation (4.61) with \( c_0 \) given by Equation (4.59).

From Section C it follows that the stress and displacement are obtainable from \( f(t) \) by the relations (4.31) - (4.36), where \( f(t) \) is the solution of the dual integral equations.
\[
\int_0^\infty f(t) J_{-1/2}(Yt) \, dt = M h(aY)/\sqrt{Y}, \quad |Y| < 1, \\
\int_0^\infty f(t) J_{-1/2}(Yt) \, dt = 0, \quad Y > 1,
\]

where

\[
M = \frac{\gamma / 2 \pi G a^2}{K(c + c')}.
\]  

The dual integral Equations (4.62) are exactly of the type for which an explicit solution was developed in Section C of Chapter II. Therefore, the solution for \( f(t) \) is

\[
f(t) = M \sum_{n=1}^m A_n a^n f_n(t),
\]

where

\[
f_n(t) = -Q_n t^{3/2} \int_0^1 u^n J_1(tu) \, du,
\]

\[
Q_n = \gamma / 2 \Gamma((n+1)/2)/\Gamma(n/2) = 2 \sqrt{\gamma_n/n}.
\]

The stress and displacement can now be obtained from Equations (4.31) - (4.36). In order to express the answers in simple form it is convenient to introduce some new quantities by the definitions
\[ Z = X + iY, \quad S = KX + iY, \quad (4.67) \]

\[ P_{mn} = \int_0^1 \int_0^\infty u^m J_1(ut) e^{-zt} dt, \quad (4.68) \]

\[ Q_{mn} = \int_0^1 \int_0^\infty u^m J_1(ut) e^{-st} dt, \quad (4.69) \]

\[ \gamma_n = \frac{\Gamma[(n+1)/2]}{\Gamma(n/2)}, \quad (4.70) \]

\[ L = \frac{2G}{a \sqrt{\pi K(c+c')}}. \]

This last expression for \( L \) becomes, after substitution for \( c+c' \) from (4.36),

\[ L = \frac{2E}{a \sqrt{\pi} (1 - \sigma^2)K(K-1)} . \quad (4.71) \]

The integrals (4.68), (4.69) are required only for \( m = 0 \) or 1. They can be evaluated by using the standard integrals (see Magnus and Oberhettinger (7, p.47))

\[ \int_0^\infty e^{-pt} J_1(ut) dt = u^{-1} \left( 1 - p/\sqrt{u^2 + p^2} \right), \quad (4.72) \]
\[ \int_0^{\infty} t e^{-pt} J_1(ut) \, dt = u (u^2 + p^2)^{-3/2}, \] (4.73)

these being valid for \( p > 0 \). Hence one obtains

\[ P_{on}(X,Y) = \int_0^1 u^{n-1} \left\{ 1 - Z(u^2 + Z^2)^{-1/2} \right\} du, \] (4.74)

\[ P_{1n}(X,Y) = \int_0^1 u^{n+1} (u^2 + Z^2)^{-3/2} du, \] (4.75)

\[ Q_{on}(X,Y) = \int_0^1 u^{n-1} \left\{ 1 - S(u^2 + S^2)^{-1/2} \right\} du, \] (4.76)

\[ Q_{1n}(X,Y) = \int_0^1 u^{n+1} (u^2 + S^2)^{-3/2} du. \] (4.77)

After substituting for \( f(t) \) from (4.64) - (4.66) into Equations (4.32) - (4.35) one now obtains

\[ T_{xx}^* = L R \sum_{i=1}^m A_n \gamma_i^n a^n(K P_{1n} - Q_{1n}), \]

\[ T_{yy}^* = -KL R \sum_{i=1}^m A_n \gamma_i^n a^n(P_{1n} - K Q_{1n}), \]

\[ T_{xy}^* = -KL \sum_{i=1}^m A_n \gamma_i^n a^n(P_{1n} - Q_{1n}) \] (4.78)

\[ u = R(-aKL/G) \sum_{i=1}^m A_n \gamma_i^n a^n(c P_{on} + c Q_{on}). \]
\[ v* = \left[ a(1 + \sigma)/E \right] \int \sum_{i=1}^{m} A_i n_i a (b y_{in} a (b - b'q_{on}) \]

the first three equations being restricted by the condition that \( X \) be positive in order to avoid divergent integrals. Asterisks have been used above on the symbols \( \tau \) and \( u \) as these are not the final components of stress and displacement. If, in the problem of the penetration of a rectangular block to a depth \( c_1 \), the various quantities be distinguished by the superscript \( (o) \), then the stress and displacement components for the main problem of this chapter are given by

\[ \tau_{ij} = \tau_{ij}^{(o)} + \tau_{ij}^{(o)}, \quad u_{ij} = u_{ij}^{(o)} + u_{ij}^{(o)}. \]  

(4.79)

Also, if \( W \) is the total load applied to the boundary,

\[ W = W^{(o)} + W^{(o)}. \]  

(4.80)

The expression for \( \tau_{xx}^{(o)} \) given in (4.78), after an integration by parts, may be combined with the expression for \( \tau_{xx}^{(o)} \) obtained from (4.50) to yield

\[ \tau_{xx} = \tau_{xx}^{(o)} + \tau_{xx}^{(o)} \\
= R \left[ \left( \frac{W_i}{a(n(K-1))} + L \sum_{i=1}^{m} A_i n_i a n \right) \left\{ \frac{-K}{1+Z^2} + \frac{1}{1+S^2} \right\} \right] \\
+ L R \sum_{i=1}^{m} A_i n_i a n \int_{0}^{n} \left( \frac{K}{u^2+Z^2} - \frac{L}{u^2+S^2} \right) du. \]  

(4.81)
Hence, \( \tau_{xx} \) is finite at the point \( X = 0, Y = 1 \), if and only if

\[
W_1 = -\pi(K - 1)L \sum_{i=1}^{m} A_n \gamma_n^a a^n. \tag{4.82}
\]

Therefore, from (4.81) and two similar equations, one obtains final expressions for the stress components, namely,

\[
\tau_{xx} = L \mathcal{R} \sum_{i=1}^{m} A_n \gamma_n^a n \int_{0}^{1} u^{n-1} \left( \frac{K}{\sqrt{u^2 + Z^2}} - \frac{1}{\sqrt{u^2 + S^2}} \right) du,
\]

\[
\tau_{yy} = -L \mathcal{K} \mathcal{R} \sum_{i=1}^{m} A_n \gamma_n^a n \int_{0}^{1} u^{n-1} \left( \frac{1}{\sqrt{u^2 + Z^2}} - \frac{K}{\sqrt{u^2 + S^2}} \right) du,
\]

\[
\tau_{xy} = -L \mathcal{K} \int \sum_{i=1}^{m} A_n \gamma_n^a n \int_{0}^{1} u^{n-1} \left( \frac{1}{\sqrt{u^2 + Z^2}} - \frac{1}{\sqrt{u^2 + S^2}} \right) du. \tag{4.83}
\]

These integrals are elementary so that the stress components are expressible in closed form in terms of simple functions. Also, an inspection of (4.83) shows immediately that the boundary conditions on \( \tau_{xx} \) and \( \tau_{xy} \) are satisfied.

It is important to determine the quantity \( a \) in terms of the total load \( W \) and the other constants of the problem. To do this, one first writes down
$$\tau_{xx}(0,Y) = L \sum_{i=1}^{m} A_n \chi^\prime_n \alpha_n \frac{1}{Y} \int \frac{(k-1)u^{n-1}}{\sqrt{(u^2 - Y^2)}} \, du, \quad 0 \leq Y < 1.$$ 

Hence,

$$W = -2a \int_0^1 \tau_{xx}(0,Y) \, dY$$

$$= -\pi a (K - 1) L \sum_{i=1}^{m} A_n \chi^\prime_n \alpha_n$$

(4.84)

$$= W_1,$$

the last step being obvious after consulting (4.82).

After substituting for \( L \) from Equation (4.71) one obtains from (4.84) the relation

$$W(1 - \sigma^2)K(K + 1) + 2\psi E \sum_{i=1}^{m} A_n \chi^\prime_n \alpha_n = 0$$

(4.85)

from which \( a \) can be determined.

The components of displacement are found by simple superposition of the results contained in Equations (4.78) and (4.55) - (4.56). To check the boundary condition on \( u \), one finds that, if \( X = 0 \) and \( 0 \leq Y < 1 \),

$$u(0,Y) = c_1 - \frac{aKL(c+c^\prime)}{c} \sum A_n \chi^\prime_n \alpha_n \left\{ \frac{1}{n \gamma} - Y \int_0^Y \frac{u^{n-1}}{\sqrt{(Y^2 - u^2)}} \right\}.$$
and, with the help of (4.23), (4.59), (4.70), simplifies to the relation

\[ u(0,Y) = c_1 + c_0 + \sum_{1}^{m} A_n a_n Y^n, \quad 0 \leq Y < 1. \]

The right-hand member of this equation is, by virtue of Equations (4.58), (4.60) precisely \( g(y) \) so that the boundary condition on \( u \) is verified.

It has now been verified that the final expressions for the stress and displacement do, in fact, satisfy all the boundary conditions. To complete the verification of the solution one can substitute the expressions (4.83) for the stress components into the equilibrium equations

\[
\left( \frac{\partial}{\partial x} \right) \tau_{xx} + \left( \frac{\partial}{\partial y} \right) \tau_{xy} = 0, \\
\left( \frac{\partial}{\partial x} \right) \tau_{xy} + \left( \frac{\partial}{\partial y} \right) \tau_{yy} = 0, \tag{4.86}
\]

and the compatibility relation (4.4). An inspection of (4.83) shows immediately that the equilibrium equations (4.86) are satisfied. Also, the compatibility relation (4.4) can be written in the form

\[
\left\{ \left( \frac{\partial^2}{\partial x^2} \right) + K \left( \frac{\partial^2}{\partial y^2} \right) \right\} \left( \tau_{xx} + \tau_{yy} \right) = 0,
\]

and this is clearly satisfied since

\[
\tau_{xx} + \tau_{yy} = L(K^2-1) \mathcal{R} \sum A_n a_n \frac{n}{u} \frac{n-1}{(u^2 + S^2)}.
\]
F. Deduction of the Isotropic Case

The results for the isotropic case can be easily obtained from the last section by letting \( K \) tend to unity in the various formulas. The quantities \( E, \nu \) will now be the unique Young's modulus and Poisson's ratio of the material. Therefore, letting \( K \to 1 \) in (4.83) and using l'Hopital's theorem, one obtains for the stress components in the isotropic case

\[
\tau_{xx} = \frac{E}{\gamma \pi (1-\nu^2)} \mathcal{R} \sum_{n=1}^{m} A_n \gamma_n a_n \int_0^1 \frac{u^{n-1} (u^2 + Z^2 + ZZ)}{(u+Z)^{3/2}} \, du,
\]

\[
\tau_{yy} = \frac{E}{\gamma \pi (1-\nu^2)} \mathcal{R} \sum_{n=1}^{m} A_n \gamma_n a_n \int_0^1 \frac{u^{n-1} (u^2 + Z^2 - ZZ)}{(u+Z)^{3/2}} \, du,
\]

\[
\tau_{xy} = \frac{-E}{\gamma \pi (1-\nu^2)} \mathcal{R} \sum_{n=1}^{m} A_n \gamma_n a_n \int_0^1 \frac{u^{n-1} ZZ}{(u+Z)^{3/2}} \, du.
\]

Also, the equation to determine the quantity \( a \) is now

\[
(1-\nu^2)W + \nu \pi E \sum_{n=1}^{m} A_n \gamma_n a_n = 0.
\]
This last result agrees with that given by Sneddon (11, p. 436) apart from his omission of E. However, Sneddon does not give final expressions for the stress but gives formulas for what corresponds to the $\tau_{ij}$ of the last section. Altogether his treatment of the isotropic case leaves much to be desired since he appears to assume that the Busbridge formula does give a proper solution of the dual equations and, moreover, he uses an artificial condition about "zero resultant force on the boundary". However, his answers, with minor corrections, can be used to obtain expressions for the stress components in agreement with those given above.
V. SUMMARY

The central topic of this dissertation is the solution of mixed boundary value problems of the Boussinesq type in elasticity by using the method of dual integral equations introduced by Harding and Sneddon (3). The main results may be separated into three parts.

Firstly, in Chapter II, the Busbridge formula for the solution of the dual equations is studied when the parameters have certain values which lie outside the ranges for which the formula was established. When \( s = -1, \ p = 0 \) and \( g(u) \) is a polynomial, it is shown that the formula is still valid. However, when \( s = -1, \ p = -1/2 \) and \( u g(u) \) is a polynomial, it is proved that the Busbridge formula definitely fails to provide a solution unless there is a certain linear relation among the coefficients of the polynomial. These facts have not hitherto been noted in the literature on the subject as far as the author is aware.

Secondly, in Chapter III, a solution is obtained for the problem of a semi-infinite isotropic elastic solid whose plane boundary is indented by an axially symmetric rigid punch in such a way that the equation...
of the surface of contact can be represented, in cylindrical coordinates, in the form \( z = g(r) \), where \( g(r) \) is a polynomial in \( r \). This problem has been treated exhaustively by Sneddon (9 and 10) for the cases where \( g(r) \) is a constant or a linear expression in \( r \), so that the present investigation in this thesis is an extension of Sneddon's work to more general polynomials. A noteworthy feature of the results is the fact that the stress and displacement at any point are obtainable explicitly as finite combinations of simple functions. As an illustration, spherical indentation is given special consideration.

Finally, in Chapter IV, the analogous two-dimensional (plane-strain) problem is solved for an orthotropic medium with a certain kind of elastic symmetry. Again the results are obtained in simple closed form suitable for computational purposes. Also, the final solution is fully verified. By letting the elastic constant \( K \) tend to unity one obtains results for the isotropic case which are in general agreement with formulas given by Sneddon. However, Sneddon's presentation is unsatisfactory in several respects, even though he obtains essentially correct answers, so that the method used in this thesis is not exactly similar to his.
VI. LITERATURE CITED


VII. ACKNOWLEDGMENT

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Appendix A

This appendix is devoted to the evaluation of the integrals $C_{nk}^p(R,Z), S_{nk}^p(R,Z)$ defined in (3.44). It will be stipulated here that $Z$ is to be strictly positive, but this restriction will be removed in Appendix B. Only those values of $p, n, k$ will be considered which arise in the main problem of Chapter III when $g(r)$ is a polynomial of degree no higher than four.

To evaluate the integrals in question one may start with the known result (see Watson (17, p. 384))

$$
\int_0^\infty e^{-at} J_p(bt) \, dt = \frac{\sqrt{(a^2 + b^2) - a}^p}{b^p \sqrt{(a^2 + b^2)}} , \quad (8.1)
$$

where $\Re(p) > -1$ and $\Re(a) > |J(b)|$. This yields, as a special case,

$$
\int_0^\infty e^{ikt - Zt} J_0(Rt) \, dt = \left\{ (Z - ik)^2 + R^2 \right\}^{-1/2} \quad (8.2)
$$

which is valid for $Z > 0, R \geq 0, k \geq 0$. When $R = 0$, a simple calculation shows that the right-hand member of
(8.2) must reduce to \((Z + 1k)/(Z^2 + k^2)\) which represents a point in the positive quadrant of the Argand plane. Hence, the radical in (8.2) is interpreted as \(s_k e^{-\beta_k}\), where

\[
s_k^4 = (D^2 - k^2)^2 + 4Z^2k^2, \quad s_k > 0,
\]

\[
\tan 2\beta_k = \frac{2Zk}{(D^2 - k^2)} , \quad 0 < \beta_k < \pi/2,
\]

so that

\[
\int_0^\infty e^{ikt} e^{-Zt} J_0(Rt) \, dt = s^{-1} e^{-i\beta_k} , \tag{8.3}
\]

which shows that the real and imaginary parts of the integral are both positive or zero for the range of the parameters \(R, k, Z\) specified above. (The notation set up in (3.43) is used throughout the appendices.) It follows that successive integrations of (8.2) with respect to \(k\), between 0 and \(k\), must yield a complex number in the positive quadrant of the Argand plane. This observation will serve as a guide in choosing the correct branches of the multi-valued functions which arise in the sequel.

Now the integrand in (8.2) is a continuous function of \((k,t)\) over the region \(t \geq 0, k \geq 0\), and the Cauchy criterion shows that the integral is uniformly convergent (in fact, absolutely-uniformly convergent) with respect
to \( k \) over any interval \((0, k)\). Hence, (8.2) may be inte-
grated with respect to \( k \) between 0 and \( k \) and the order of
integration inverted (see Titchmarsh (15, p.53)). This
gives
\[
\int_0^\infty t^{-1} (1 - e^{-ikt})e^{-zt} J_0(Rt) \, dt
\]
\[
= \sinh^{-1}(u/R) - \sinh^{-1}(Z/R),
\]
where \( u = Z - ik \). Since the argument in \( \sinh^{-1}(u/R) \) is
complex it is a multi-valued function, but the method of
choosing the correct branch has already been explained.

Repeated integration of (8.4) with respect to \( k \),
"under the integral sign" between 0 and \( k \), can be justified
as above, and, after setting \( k = 1 \), the separation of the
results into real and imaginary parts yields
\[
C_{10}^0 = (1/D) - (\cos \beta)/s, \\
C_{11}^0 = \log w, \\
C_{12}^0 = -Z \log w + \gamma + s \cos \beta - D, \\
C_{12}^0 = C_{12}^0 - (1/2D), \\
C_{22}^0 = (1/4) \left\{ (2Z^2 - R^2 - 2) \log w - 4\gamma Z + 3ZD \\
- 3qs \cos(a + B) \right\},
\]
\[ C_{24}^0 = \frac{(Z/12)(3R^2 - 2Z^2 + 6)\log w + (\gamma/12)(6Z^2 - 3R^2 - 2)}{+ \frac{D}{2} + \frac{(11/36)(s^3 \cos 3\beta - D^3)}{5R^2}s \cos \beta - D)/12,} \]

\[ S_{11}^0 = \frac{(R^2 + Z^2)^{1/2} - \gamma}{,} \]

\[ S_{12}^0 = \log w + \gamma Z - s \sin \beta , \]

\[ S_{13}^0 = -Z \log w - (\gamma/4)(2Z^2 - R^2 - 2) - D \]

\[ + \frac{(3/4)qs \sin(a + \beta)}{,} \]

\[ S_{23}^0 = S_{13}^0 - 1/(6D) , \]

\[ S_{24}^0 = \frac{(1/12)\log w (6Z^2 - 3R^2 - 2) + (\gamma Z/12)(2Z^2 - 3R^2 - 6)}{+ \frac{32D/4 - (11/36)s^3 \sin 3\beta + (5/12)R^2s \sin \beta}{,} \]

\[ S_{25}^0 = \frac{(Z/12)(2-2Z^2 +3R^2)\log w - (\gamma/24)(Z^4 - 6Z^2 +1)}{+ \frac{\gamma R^2(Z^2 - 1)/8 - \gamma R^4/64 + (11/144)s^3 q \sin(a + 3\beta)}{-(s/96)((1-3Z^2)\cos \beta - (Z^3 - 3Z)\sin \beta) + D/6}{-(11/64)R^2sq \sin(a + \beta) - 11D^3/36 + 5R^2D/12} .} \]

For the corresponding integrals involving the first order Bessel functions a similar procedure can be followed by starting with (8.1) with \( p = 1, a = Z - ik, b = R, \) namely,

\[ \int_0^\infty ikte^{-Zt}J_1(Rt) dt = 1 - a(a^2 + R^2)^{-1/2} . \]
After successive integrations with respect to \( k \), setting \( k = 1 \), and equating real and imaginary parts, there results

\[
R_{C10} = (q/s) \cos(a-\beta) - (Z/D), \quad R_{C11} = D - s \cos \beta ,
\]

\[
R_{C12} = (1/2)\left\{ q s \cos(a+\beta) + R^2 \log w - ZD + 1 \right\} ,
\]

\[
R_{C22} = R_{C12} - (1 - Z/D)/2 ,
\]

\[
R_{C23} = (1/6)\left\{ -s^3 \cos 3\beta - 3R^2 Z \log w + 3R^2 \gamma \\
+ 3R^2 s \cos \beta + D^3 - 3R^2 D - 3D \right\} ,
\]

\[
R_{C24} = (1/48)\left\{ 2s^3 q \cos(a+3\beta) - 15R^2 s q \cos(a+\beta) - 24s^2 \gamma Z \\
+ 3R^2 (4Z^2 - 4R^2) \log w - 2ZD^3 + 15R^2 ZD + 12ZD + 2 \right\} ,
\]

\[
R_{S01} = 1 - s \sin \beta , \quad R_{S11} = s \sin \beta - (Z/D) ,
\]

\[
R_{S12} = D - (q/2) s \sin(a+\beta) - R \gamma/2 ,
\]

\[
R_{S13} = (1/6)\left\{ s^3 \sin 3\beta + 3R^2 \gamma Z + 3R^2 \log w - 3R^2 s \sin \beta \\
- 3ZD + 1 \right\} ,
\]

\[
R_{S23} = R_{S13} - (1 - Z/D)/6 ,
\]

\[
R_{S24} = (1/48)\left\{ -2s^3 q \sin(a+3\beta) + 15R^2 s q \sin(a+\beta) + 8D^3 \\
- 24R^2 Z \log w + 3R^2 \gamma(R^2 - 4Z^2 + 4) - 24R^2 D - 8D \right\} ,
\]
\[ R S_{25}^1 = \left( s^5/120 \right) \sin 5\beta - (19/124)R^2 s^3 \sin 3\beta \]
\[ + (7/48)R^4 s \sin \beta + (R^2/48)(12Z^2 - 3R^2 - 4) \log w \]
\[ + (4Z^2 - 3R^2 - 12)R^2ZY/48 - D^2Z/24 \]
\[ + 5R^2DZ/16 + 2D/12 - (1/120). \]

Appendix B

In Appendix A it is stipulated that \( Z \) be strictly positive. This condition is used to justify the process of inverting the order of integration and to ensure the convergence of the various integrals. However, the formulas (8.5) - (8.8) remain plausible, save possibly for \( R = 0 \), when one puts \( Z = 0 \) in the integrand and lets \( Z \) tend to +0 in the right-hand members. The validity of the results so obtained can be established by using the following three lemmas (and similar ones involving the Bessel function of order unity):

Lemma 1. If \( k \geq 0 \) and \( R \neq 0 \), then
\[ \int_1^\infty \frac{e^{-kt}}{t} \, J_0(Rt) \, dt \]

converges uniformly for \( 0 \leq Z \).

Lemma 2. In Lemma 1, the condition \( R \neq 0 \) may be dropped if \( k \) exceeds unity.
Lemma 3. If \( k > 1/2 \) and \( R \neq 0 \), then
\[
\int_{1}^{\infty} t^{-k} \cos t e^{-Zt} J_0(Rt) \, dt
\]
converges uniformly for \( 0 \leq Z \).

Lemma 1 is proved by applying the Abel test for uniform convergence since \( t^{-k} e^{-Zt} \) is a non-increasing function of \( t \) for each \( Z (\geq 0) \) and \( \int_{1}^{\infty} J_0(Rt) \, dt \) is convergent (\( R \neq 0 \)). Lemmas 2 and 3 follow readily upon application of the Cauchy criterion for uniform convergence.

Now in the Equations (8.5) - (8.8), for a fixed value of \( R (\neq 0) \), the integrands are continuous functions of \((t,Z)\) over the region \( Z \geq 0, t \geq 0 \), and the integrals converge uniformly with respect to \( Z \) over the range \( Z \geq 0 \). Hence, the integrals are continuous functions of \( Z \) in this range, and, in particular, they are continuous at the point \(+0\). Therefore, the values of the desired integrals with \( Z = 0 \) are obtained from the above formulas by letting \( Z \) tend to \(+0\). In some cases the restriction \( R \neq 0 \) is unnecessary.

As \( Z \) tends to \(+0\), it is found from (3.43) that
\[
\alpha \to \pi/2, \quad q \to 1, \quad D \to R, \quad s \to |R^2 - 1|^{1/2},
\]
\[
\beta \to \pi/2 \text{ or } 0 \text{ according as } R \leq 1, \quad (8.9)
\]
\[
\gamma \to \pi/2 \text{ or } \sin^{-1}(1/R) \text{ according as } R \lesssim 1,
\]

\[
\log w \to \cosh^{-1}(1/R) \text{ or } 0 \text{ according as } R \lesssim 1.
\]

Hence, on letting \( Z \to +0 \) in (8.5) - (8.8), one obtains the following formulas, where the upper and lower lines on the right-hand side of each equation are associated respectively with \( 0 \leq R < 1 \) and \( R > 1 \),

\[
C^0_{11}(R,0) = \begin{cases} 
\cosh^{-1}(1/R), \\
0,
\end{cases}
\]

\[
C^0_{12}(R,0) = \begin{cases} 
\pi/2 - R \\
\sin^{-1}(1/R) + \sqrt{R^2 - 1} - R,
\end{cases}
\]

\[
C^0_{22}(R,0) = \begin{cases} 
\pi/2 - R - 1/(2R), \\
\sin^{-1}(1/R) + \sqrt{R^2 - 1} - R - 1/(2R),
\end{cases}
\]

\[
C^0_{23}(R,0) = \begin{cases} 
((1/4)[-2+R^2]\cosh^{-1}(1/R) + 3\sqrt{1-R^2}] \\
0,
\end{cases}
\]

\[
C^0_{24}(R,0) = \begin{cases} 
-\pi(2 + 3R^2)/24 + R/2 + R^3/9, \\
(1/36)[-(6+9R^2)\sin^{-1}(1/R) + 18R + 4R^3 \\
- (4R^2 + 11) \sqrt{R^2 - 1}],
\end{cases}
\]

\[
S^0_{11}(R,0) = \begin{cases} 
(1/R) - \pi/2, \\
(1/R) - \sin^{-1}(1/R),
\end{cases}
\]

\[
S^0_{12}(R,0) = \begin{cases} 
\cosh^{-1}(1/R) - \sqrt{1 - R^2}, \\
0,
\end{cases}
\]

\[
S^0_{13}(R,0) = \begin{cases} 
\pi(2 + R^2)/8 - R, \\
(1/4)[(2+R^2)\sin^{-1}(1/R) + 3\sqrt{R^2 - 1} - 4R],
\end{cases}
\]
\[ S_{23}(R,0) = S_{13} - \frac{1}{(6R)} , \]
\[ S_{24}(R,0) = \begin{cases} \frac{1}{36} \left[ -(6+9R^2) \cosh^{-1} \left( \frac{1}{R} \right) + (11+4R^2) \sqrt{1-R^2} \right], \\ 0, \end{cases} \]
\[ S_{25}(R,0) = \begin{cases} \frac{R(3+2R^2)}{18} - \frac{\pi}{2} \left[ (1/48)+(R^2/16)+R^4/128 \right], \\ \left( -\frac{1}{24} + \frac{R^2}{8} + \frac{R^4}{64} \right) \sin^{-1} \left( \frac{1}{R} \right) - \frac{\sqrt{(R^2-1)}}{96} + \frac{R^3}{9} \\ + \frac{11R^2(1-R^2)}{6} /144 - 11R^2 \sqrt{R^2-1}/64 + R/6, \end{cases} \]
\[ C_{11}(R,0) = \begin{cases} 1, \\ 1 - \sqrt{(1 - R^2)} \end{cases} \]
\[ C_{12}(R,0) = \begin{cases} \left[ 1 + R^2 \cosh^{-1} \left( \frac{1}{R} \right) - \sqrt{(1 - R^2)} \right] / (2R), \\ 1/(2R), \end{cases} \]
\[ C_{22}(R,0) = C_{12}(R,0) - 1/(2R), \]
\[ C_{23}(R,0) = \begin{cases} \frac{\pi}{4} - \frac{R^2}{3} - 1/2, \\ (R/2) \sin^{-1} \left( \frac{1}{R} \right) + \sqrt{(R^2-1)} (1+2R^2) / (6R) \end{cases} \]
\[ C_{24}(R,0) = \begin{cases} (1/48R) \left[ 2(1-R^2)^{3/2} - 2 + 15R^2 \sqrt{(1-R^2)} \\ - 3R^2 (4+R^2) \cosh^{-1} \left( \frac{1}{R} \right) \right], \\ - 1/(24R), \end{cases} \]
\[ S_{01}(R,0) = \begin{cases} 1 - \sqrt{(1 - R^2)} / R, \\ 1/R, \end{cases} \] (8.11)
\[ S_{12}(R,0) = \begin{cases} 1 - \pi R/4, \\ 1 - (R/2) \sin^{-1}(1/R) - \sqrt{1 - R^{-2}}/2, \end{cases} \]
\[ S_{13}(R,0) = \begin{cases} (1/6R) \left[ 1 - 3R^2 \sqrt{1 - R^2} + 3R^2 \cosh^{-1}(1/R) - (1 - R^2)^{3/2} \right], \\ 1/6R, \end{cases} \]
\[ S_{23}(R,0) = S_{13}(R,0) - 1/(6R), \]
\[ S_{24}(R,0) = \begin{cases} \pi (4 + R^2)/32 - R^2/3 - 1/6, \\ -1/6 - R^2/3 + (R/16)(4+R^2) \sin^{-1}(1/R) \\ + (5R/16) \sqrt{R^2-1} - (R^2)^{3/2}/(24R), \end{cases} \]
\[ S_{25}(R,0) = \begin{cases} (1-R^2)^{5/2}/120 R + \frac{19R(1-R^2)^{3/2}}{144} + \frac{7}{48} R^3 (1-R^2)^{1/2}, \\ -(R/48)(4+3R^2) \cosh^{-1}(1/R) - 1/(120R), \end{cases} \]
\[ -1/(120 R). \]

Some of the above integrals are divergent for \( R = 0 \), but it is found that the linear combinations which give the stress and displacement in the main problem of Chapter III are finite and continuous at the origin. Also, it should be observed that the above integrals are continuous at \( R = 1 \).
Appendix C

This appendix is devoted to the evaluation of the P's and Q's defined as certain integrals in (4.30). Using the expression for \( F_n(t) \) given in (3.28) one obtains from (4.30)

\[
P_s(n) = \Delta_n \int_0^1 \int_0^\infty dt \, \frac{\sin^{n-1}(-Z-tv)\, J_0(Rt)}{t^{1/2}} \, dv,
\]

\[
Q_s(n) = \Delta_n \int_0^1 \int_0^\infty dt \, \frac{\cos^{n-1}(Z-tv)\, J_1(Rt)}{t^{1/2}} \, dv,
\]

where

\[
\Delta_n = -\sqrt{n/\pi n}. \tag{8.13}
\]

Of course, the above formulas are valid only if the integrals converge. For each value of \( n = 1, 2, 3, \ldots \) it is required to determine the integrals in (8.12) for \( s = 1, 0, -1 \).

It will now be convenient to display some standard integrals given by Magnus and Oberhettinger (7, p.47), namely,

\[
\int_0^\infty e^{-at} J_0(bt) \, dt = (a^2 + b^2)^{-1/2}, \tag{8.14}
\]
\[ \int_{-\infty}^{\infty} e^{-at} J_0(bt) dt = a(a^2 + b^2)^{-3/2}, \quad (8.15) \]
\[ \int_{-\infty}^{\infty} e^{-at} J_1(bt) dt = b(a^2 + b^2)^{-3/2}, \quad (8.16) \]
\[ \int_{-\infty}^{-1} e^{-at} J_1(bt) dt = \left\{ \sqrt{a^2 + b^2} - a \right\} / b, \quad (8.17) \]
\[ \int_{-\infty}^{0} e^{-at} J_1(bt) dt = b^{-1} \left\{ 1 - a(a^2 + b^2)^{-1/2} \right\}, \quad (8.18) \]

these being valid if \( K(a) > |I(b)| \).

It will be assumed in this appendix that \( Z \) is strictly positive. (The deductions for \( Z = 0 \) can be made as in Appendix B.) It follows that the order of integration in (8.12) can be inverted so that, upon using (8.15), one obtains

\[ P_{1}^{(n)} = \Delta_n \int \frac{v^{n-1}(Z - iv) dv}{(Z - iv)^2 + R^2} \quad (8.19) \]

Similarly,

\[ P_{0}^{(n)} = \Delta_n \int \frac{v^{n-1} dv}{\sqrt{(Z - iv)^2 + R^2}} \quad (8.20) \]

Now \( P_{-1}^{(n)} \) cannot be obtained from (8.12) as the integral is divergent. However, one can write
\[ P_{-1} = -\Delta_n \int_0^\infty \int_0^1 t^{n-1} e^{-Zt - ivt} J_0(\nu t) \, dt \]

and, by using Equation (8.4) of Appendix A, one obtains

\[ P_{-1} = \Delta_n \int_0^1 v^{n-1} \left[ \sinh^{-1}(Z/R) - \sinh^{-1}\left(\frac{Z-iv}{R}\right) \right] \, dv. \quad (8.21) \]

The Q's can be treated in a similar fashion and, with the help of Equations (8.16) - (8.18), one obtains

\[ Q_1 = \Delta_n \int_0^1 \frac{R v^{n-1} \, dv}{\left\{(Z-iv)^2 + R^2\right\}^{3/2}}, \quad (8.22) \]

\[ Q_0 = \Delta_n /R \int_0^1 v^{n-1} \left[ 1 - \frac{Z - iv}{\sqrt{(Z - iv)^2 + R^2}} \right] \, dv, \]

\[ Q_{-1} = \Delta_n /R \int_0^1 v^{n-1} \left[ \sqrt{(z-iv)^2 + R^2} - (z-iv) \right] \, dv. \quad (8.24) \]

It is now clear that the integrals in (8.19)-(8.24) can, for each positive integer n, be evaluated in simple closed form involving only elementary functions.