Physical systems with time varying parameters

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PHYSICAL SYSTEMS WITH TIME VARYING PARAMETERS

by

Robert Landels Doty

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
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DOCTOR OF PHILOSOPHY

Major Subject: Electrical Engineering

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1955
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I. INTRODUCTION

A. General Discussion

In recent years there has been an increasing interest shown in the properties of variable or dynamic physical systems in that it has been found in many instances that the behavior of such systems constitutes an improvement over the behavior of corresponding fixed systems. The term physical systems is used here to mean any electro-mechanical system, and a variable or dynamic system is any physical system in which one or more of the parameters of that system such as resistance, inductance, mass, gain etc. is not fixed but varies according to some law.

Generally speaking, in variable systems the parameters may be functions of the input or output of the system, or the parameters may be made to vary as independent functions of time. In the first case, the differential equations of the system will be non-linear, while in the second case the system differential equations will be linear differential equations with nonconstant coefficients. It is the latter case which will be the subject of this paper, and such systems will be designated as linear time varying systems. A linear time varying system then may be defined as any linear physical system in which one or more of the parameters is caused to vary independently as a function of time. Without much loss in generality, it will also be assumed that the parameters
are all single-valued continuous functions of time.

Parametric excitation, for energy conversion, amplitude and frequency modulators, chopper amplifiers and sampled data servo systems are all examples of time varying systems. In fact, the entire class of fixed systems may be considered as a limiting case of time varying systems.

Since time varying systems are characterized by linear differential equations with variable coefficients, as one would suspect, the first attempts to analyze and synthesize such systems were those that made use of the extensive theory of such differential equations. Carson (2), an early pioneer in this field, used the classical differential equation approach to obtain by Picard's method approximate solutions to certain types of time varying circuits. Later, Neufeld (8), using an extension of Heaviside's method, and Pipes (9), using matrix methods were able to obtain solutions in the time domain to the differential equations of a time varying system.

Of course, solutions to the differential equations provide a complete description of the behavior of a system for any input, but this information is quite often in such a form that it cannot be conveniently used in design techniques. Other methods such as the response of the system to certain special inputs have been found to be much more useful. Examples of such are the step function response of a system, the frequency response (or response to a sinusoidal input), and the impulsive response or weighting function which is the response of a system to a unit impulse. All of the above response functions are related, and the
response to any arbitrary input may be determined from them by the use 
of the superposition theorem which is a common denominator for all 
linear systems.

B. The Transform Approach to the Problem

In recent years, the most common technique used in either the 
analysis or synthesis of fixed physical systems has been that in which 
the differential equations of the system are transformed either by the 
Fourier or LaPlace transformation into algebraic equations. That this 
is not generally a simplification for the case of a variable parameter 
system is shown below. For example, if \( v(t) \) is the output of the time 
varying system shown in Fig. 1, and if \( u(t) \) is the input, then the two 
are related by the system differential equation

\[
\begin{align*}
A_n(t) \frac{d^n v}{dt^n} + A_{n-1}(t) \frac{d^{n-1} v}{dt^{n-1}} + \ldots + A_1(t) \frac{dv}{dt} + A_0(t) v &= \\
B_m(t) \frac{d^m u}{dt^m} + B_{m-1}(t) \frac{d^{m-1} u}{dt^{m-1}} + \ldots + B_1(t) \frac{du}{dt} + B_0(t) u
\end{align*}
\]

or more simply, in operator form

\[
L(p,t) v(t) = K(p,t) u(t). 
\]

Where \( L(p,t) \) and \( K(p,t) \) are the polynomial differential operators

\[
\begin{align*}
L(p,t) &= A_n(t) p^n + A_{n-1}(t) p^{n-1} + \ldots + A_1(t) p + A_0(t) \\
K(p,t) &= B_m(t) p^m + B_{m-1}(t) p^{m-1} + \ldots + B_1(t) p + B_0(t) \\
p &= \frac{d}{dt}
\]
\]
Fig. 1. Block Diagram of a Time Varying System.
For a given system, the a's and b's would be known functions of time.

If one were to attempt to take the Laplace transform of the system equation directly, one would be faced with the necessity of transforming terms of the form

\[ a_i(t) \frac{d^i v}{d t^i} \]  \hspace{1cm} (4)

Now, if \( a_i(t) \) is written as a power series, then the task is that of transforming terms of the type

\[ a_{ir} t^r \frac{d^i v}{d t^i} \]  \hspace{1cm} (5)

where \( a_{ir} \) is a constant.

From the theory of Laplace transformations it is known that

\[ \mathcal{L} \left[ a_{ir} t^r f(t) \right] = (-1)^r a_{ir} \frac{d^r F(s)}{d s^r} \]  \hspace{1cm} (6)

where

\[ F(s) = \mathcal{L} \left[ f(t) \right] \]

while

\[ \mathcal{L} \left[ \frac{d^i v}{d t^i} \right] = s^i V(s) - \sum_{n=0}^{i-1} s^{i-n-1} \gamma^{(n)}(0) \]  \hspace{1cm} (7)

so that it may be seen that the direct Laplace transform of a differential equation whose coefficients are polynomials or which can be expanded in a power series results in another differential equation which in general may be more difficult than the original.
A similar situation exists for the case where the parameters of the time varying system vary periodically with a common period, \( T \).

The system differential equations then are differential equations having periodic coefficients. Thus a typical term of the differential equation expressed as a Fourier series of its coefficient would be

\[
A_r(t) \frac{d^r \nu}{dt^r} = \sum_{n=-\infty}^{\infty} a_{rn} e^{in \omega_0 t} \frac{d^r \nu}{dt^r}
\]  

(8)

where \( \omega_0 = \frac{2\pi}{T} \), and \( a_{rn} \) is a constant. The Laplace transform of such a term then is the infinite sum

\[
\mathcal{L} \left[ A_r(t) \frac{d^r \nu}{dt^r} \right] = \sum_{n=-\infty}^{\infty} a_{rn} \mathcal{L} \left[ e^{in \omega_0 t} \frac{d^r \nu}{dt^r} \right]
\]

(9)

or

\[
\mathcal{L} \left[ A_r(t) \frac{d^r \nu}{dt^r} \right] = \sum_{n=-\infty}^{\infty} a_{rn} \left[ \frac{1}{(s-jn \omega_0)} \frac{1}{(s-jn \omega_0)} - \sum_{m=0}^{r-1} \frac{1}{(s-jn \omega_0)} \nu \left( \frac{m}{\omega_0} \right) \right],
\]

(10)

Thus it may be seen that transforming an equation with periodic coefficients does not have much to offer in the way of simplification.

That the direct transformation of the differential equation does not generally simplify its solution has been demonstrated above. However, Zadeh (11) has recently developed a transform theory which is an extension of the Laplace transform theory for fixed systems and which allows one to draw analogies between circuit theory for fixed systems and that for time varying systems. Several other writers in this field using Zadeh's methods are Miller (7), Darlington (3), and Belvitch (1).
It is the concept of system function as proposed by Zadeh that forms the basis for this paper which will be concerned in particular with periodically time varying systems and with feedback systems having periodically varying loop gain. It is recognized that the system function approach is not yet one of very great practicality for explicitly solving time varying systems, but it is felt that it is of considerable value in describing such systems in terms of properties such as admittance, impedance, gain, etc. which are well known for fixed systems (12).
II. INVESTIGATION

A. General Discussion

As has been pointed out earlier, the physical systems being considered in this paper are those that are often described as linear time varying systems, which is to say, linear systems in which one or more of the parameters (resistance, inductance, mass, gain, etc.) vary independently with time. The differential equation or equations of such a system are then characterized as linear differential equations with variable coefficients. In this paper only systems in which the coefficients of the differential equations are single valued continuous functions of time will be considered. Linear fixed systems (which result in linear differential equations with constant coefficients) may be considered as special cases of the general time varying problem.

In approaching the problem of time varying systems, one is tempted to seek for a method which will make use of the considerable amount of information developed for fixed systems, and to extend, if possible, some of the concepts useful for the analysis and synthesis of fixed systems to the variable system case. Such a method has been proposed by Zadeh through a consideration of the impulsive response of time varying networks and which is repeated here in the interest of completeness.
B. Fixed System Analysis

Let us consider first of all a fixed system as shown in Fig. 2. Where \( u(t) \) is the input and \( v(t) \) is the output. A set of linear constant coefficient differential equations then may be written which will completely describe the behavior of the system, and by proper elimination of the internal dependent variables of the system, finally a single differential equation known as the fundamental system differential equation (5) may be obtained and represented as

\[
L(p) \cdot v(t) = K(p) \cdot u(t) \quad (11)
\]

where \( L(p) \) and \( K(p) \) are the polynomial differential operators

\[
\begin{align*}
&\alpha_n p^n + \alpha_{n-1} p^{n-1} + \cdots + \alpha_1 p + \alpha_0 \\
&\beta_m p^m + \beta_{m-1} p^{m-1} + \cdots + \beta_1 p + \beta_0
\end{align*}
\]

respectively.

An important function which characterizes the behavior of this system is the impulsive response, \( W(t) \), defined as the response at time \( t \) of the initially unexcited system to a unit impulse applied at zero time. \( W(t) \) then is obviously the solution of the differential equation

\[
L(p) \cdot W(t) = K(p) \cdot \delta(t) \quad (13)
\]

subject to homogeneous initial conditions where \( \delta(t) \) is the Dirac delta
Fig. 2. Block Diagram of a Linear Fixed System.
function. The response of the system to a unit impulse applied at any time \( \tau \), \( \tau < t \) then is

\[
L(p)W(t-\tau) = K(p)\delta(t-\tau).
\]

(14)

It should be noted here that \( W(t-\tau) \) is zero for all \( \tau > t \) since the effect cannot precede the cause.

Because of the linearity of the system, its response to an arbitrary train of impulses, \( \delta_\tau(t) \) where

\[
\delta_\tau(t) = \sum_{n=1}^{N} C_n \delta(t-\tau_n)
\]

(15)

is the sum

\[
U(t) = \sum_{n=1}^{N} C_n W(t-\tau_n).
\]

(16)

This principle of superposition allows one to find the output of the system for any given input, \( u(t) \). Consider the input \( u(t) \) as shown in Fig. 3, and divide the interval from \( a \) to \( t \) into \( n \) smaller intervals by the points \( a = t_0, t_1, t_2, \ldots, t_n = t \). Let \( \Delta t_k = (t_k - t_{k-1}) \) and let the point \( t_k' \) be some point between \( t_{k-1} \) and \( t_k \). Then each rectangle as shown in the figure may be represented approximately by the delta function \( u(t_k') \Delta t_k \delta(t - t_k') \) so that an approximation for \( u(t) \) is

\[
u(t) \approx \sum_{k=1}^{n} u(t_k') \Delta t_k \delta(t - t_k').
\]

(17)
Fig. 3. Approximation of an Arbitrary Function by a Sequence of Delta Functions.
Correspondingly, the output of the system may be approximated by

\[ \mathcal{U}(t) = \sum_{k=1}^{n} W(t - t_k^*) u(t_k^*) \Delta t_k^* \]  

(18)

since by definition, \( W(t - t_k^*) \) is the response of the system to \( \delta(t - t_k^*) \). Now in Eq. (17), if the value of \( n \) is increased beyond bound in such a fashion that the largest of the intervals \( \Delta t_k \) approaches zero, then the sum, Eq. (18) approaches the integral

\[ \mathcal{U}(t) = \int_{\tau}^{t} W(t - \tau) u(\tau) \, d\tau. \]  

(19)

Which gives \( v(t) \) at any time \( t \) provided the system was unexcited at \( t = a \). If the lower limit be changed from \( a \) to \( -\infty \) to account for the existence of \( u(t) \) prior to \( a \), or to account for any initial conditions on the system prior to the application of the input \( u(t) \) at \( a \), then the total output at any time \( t \) is given by

\[ \mathcal{U}(t) = \int_{-\infty}^{t} W(t - \tau) u(\tau) \, d\tau \]  

(20)

or since \( W(t - \tau) \) is zero for \( t < \tau \), by

\[ \mathcal{U}(t) = \int_{-\infty}^{\infty} W(t - \tau) u(\tau) \, d\tau. \]  

(21)

Another important response function of fixed systems is the frequency response function, \( H(j\omega) \), which may be defined as the response of the system to an input \( e^{j\omega t} \) divided by that input. Upon substituting
in Eq. (21) for \( v(t) \) and \( u(t) \), then

\[
H(j\omega) e^{j\omega t} = \int_{-\infty}^{\infty} W(t-\tau) e^{j\omega \tau} d\tau. \tag{22}
\]

If the variable of integration is changed by making \( \lambda = t - \tau \), then

Eq. (22) becomes

\[
H(j\omega) e^{j\omega t} = e^{j\omega t} \int_{-\infty}^{\infty} W(\lambda) e^{-j\omega \lambda} d\lambda. \tag{23}
\]

The frequency response function is thus seen to be the Fourier transform of the impulsive response.

A third important function useful for the analysis of fixed systems may be obtained from Eq. (19) for the case where \( u(t) \) is zero for \( t < 0 \). Then Eq. (19) becomes

\[
\overline{v}(t) = \int_{0}^{t} W(t-\tau) u(\tau) d\tau \tag{24}
\]

which may be seen to be the convolution integral of the functions \( W(t) \) and \( u(t) \) and from Laplace transform theory, if \( V(s) \) and \( U(s) \) are the Laplace transforms of \( v(t) \) and \( u(t) \) respectively while \( H(s) \) is the Laplace transform of \( W(t) \), then

\[
V(s) = H(s) U(s) \tag{25}
\]

or

\[
H(s) = \frac{V(s)}{U(s)}. \tag{26}
\]
The function $H(s)$ then may be seen to be the Laplace transform of the impulsive response of the system, and is called the transfer function of the system. From Eq. (26), it is seen that the transfer function determines the Laplace transform of the output for any arbitrary input, and therefore it must depend only upon the design of the system. For this reason, the transfer function has become the most widely used of the system functions, and most present day design is carried out by the use of the transfer function.

C. Time Varying Systems

Turning now to the linear time varying system problem, the first step in the analysis of such systems is to obtain the system differential equation relating the output, $v(t)$, and the input, $u(t)$. This presumably can be done by the proper elimination of the internal dependent variables in the set of differential equations describing the system. The system differential equation will be of the form

$$a_n(t) \frac{d^n v}{dt^n} + a_{n-1}(t) \frac{d^{n-1} v}{dt^{n-1}} + \ldots + a_1(t) \frac{d v}{dt} + a_0(t) v =$$

$$b_m(t) \frac{d^m u}{dt^m} + b_{m-1}(t) \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1(t) \frac{d u}{d t} + b_0(t) u$$

(1)

where the $a$'s and $b$'s are known single-valued continuous functions of the independent variable, time. Expressed in operator form the system
differential equation is
\[ L(p, t) V(t) = K(p, t) U(t) \] (2)

where \( L(p, t) \) and \( K(p, t) \) are the polynomial differential operators
\[
L(p, t) = a_n(t) p^n + a_{n-1}(t) p^{n-1} + \cdots + a_0(t) p + a_s(t)
\]
\[
K(p, t) = b_m(t) p^m + b_{m-1}(t) p^{m-1} + \cdots + b_0(t) p + b_s(t). \] (3)

As in the fixed case, an impulsive response, \( W(t) \), may be defined as the response of the initially unexcited system to a unit impulse applied at \( t = 0 \). Thus \( W(t) \) must satisfy the differential equation
\[ L(p, t) W(t) = K(p, t) \delta(t). \] (27)

For an impulse applied at any time \( \tau \), previous to the time of observation, however, the impulsive response must be of a different form. It must depend upon the answer to two questions. What is the state of the system at the time of observation, \( t \), and what time has elapsed since the impulse was applied. From this it can be seen that the impulsive response of a time varying system must be a function of two variables, \( t \) and \( t - \tau \). The \( t - \tau \) variable is often called the age variable of the system. Thus an impulsive response \( W(t, t - \tau) \) exists which satisfies the differential equation
\[ L(p, t) W(t, t - \tau) = K(p, t) \delta(t - \tau). \] (28)
That such a function exists and is unique follows from the continuity of the coefficients of the differential equation. Also since effect cannot precede cause, it must be noted that \( W(t, t - \tau) \) is zero for all \( \tau > t \).

The output of a time varying system to any arbitrary input may be determined from the impulsive response since the principle of superposition applies to any linear system, hence just as in the fixed system case

\[
U(t) = \int_{-\infty}^{t} W(t, t-\tau) u(\tau) \, d\tau. \tag{29}
\]

D. The System Functions for Variable Systems

Let us consider the case where \( u(t) = e^{st} \) where \( s \) is any parameter real, imaginary or complex. For such a case Eq. (29) becomes

\[
U(t) \bigg|_{u(t) = e^{st}} = \int_{-\infty}^{t} W(t, t-\tau) e^{s\tau} \, d\tau. \tag{30}
\]

Upon making the change in the variable of integration, \( \lambda = t - \tau \), Eq. (30) becomes

\[
U(t) \bigg|_{u(t) = e^{st}} = e^{st} \left[ \int_{0}^{\infty} e^{-s\lambda} W(t, \lambda) \, d\lambda \right]. \tag{31}
\]
The quantity inside the brackets may be recognized as the Laplace transform of \( W(t, \lambda) \) with respect to \( \lambda \), if such exists; and is given the special designation of \( H(s,t) \) and is called the system function.

The system function then may be defined in either of two ways, first as the Laplace transformation with respect to the age variable of the impulsive response \( W(t, \lambda) \) or secondly as the output of the system for an input \( e^{st} \) divided by \( e^{st} \). Therefore,

\[
H(s,t) = \int_0^\infty e^{-\lambda t} W(t, \lambda) \, d\lambda,
\]

and

\[
H(s,t) = \left. \frac{U(t)}{u(t)} \right|_{u(t) = e^{st}}
\]

From the definition it may be seen that the system function is analogous to the transfer function of a fixed system, and as will be shown later it has all the analogous properties of the transfer function of a fixed system.

The frequency response function for a time varying system may be obtained from the system function merely by letting \( s = jo \). That this is so follows from the fact that \( W(t, t - \tau) = 0 \) for \( \tau > t \), so that Eq. (29) may also be written

\[
U(t) = \int_{-\infty}^{\infty} W(t, t - \tau) u(\tau) \, d\tau.
\]
Now if \( u(t) = e^{j\omega t} \), and if \( \lambda \) is substituted for \( (t - \tau) \) in Eq. (34),

\[
H(j\omega, t) e^{j\omega t} = e^{j\omega t} \int_{-\infty}^{\infty} W(t, \lambda) e^{-j\omega \lambda} d\lambda ,
\]

and

\[
H(j\omega, t) = \int_{-\infty}^{\infty} W(t, \lambda) e^{-j\omega \lambda} d\lambda .
\]

Therefore, the frequency response function \( H(j\omega, t) \) may be seen to be the Fourier transform of the impulsive response \( W(t, \lambda) \) with respect to \( \lambda \). From the above, it is seen that the three system functions so useful in analyzing fixed systems have counterparts in the variable system case, but that the system function and frequency function of variable systems are no longer independent of time, and contain time as a parameter.

E. Finding the Time Domain Solution

from the System Function

Next it will be shown that the time domain response of a variable system to an arbitrary input, \( u(t) \), defined as zero for \( t < 0 \) may be obtained from the known system function, \( H(s, t) \), by means of the inversion integral. To do this it is again noted that

\[
U(t) = \int_{-\infty}^{t} W(t, t-\tau) u(\tau) d\tau .
\]
or

\[ u(t) = \int_0^\infty W(t, \lambda) u(t-\lambda) \, d\lambda. \]  

(37)

Since \( u(t) = 0 \) for \( t < 0 \), then \( u(t) \) and \( U(s) \) constitute a Laplace transform pair

\[ U(s) = \int_0^\infty e^{-st} u(t) \, dt \]

\[ u(t) = \frac{1}{2\pi i} \int_C e^{st} U(s) \, ds, \]

(38)

where \( C \) is a Bromwich contour that contains all the singularities of \( U(s) \). Therefore,

\[ u(t-\lambda) = \frac{1}{2\pi i} \int_C e^{s(t-\lambda)} U(s) \, ds, \]

(39)

and upon substituting for \( u(t-\lambda) \) in Eq. (37),

\[ u(t) = \int_0^\infty W(t, \lambda) \left[ \frac{1}{2\pi i} \int_C e^{s(t-\lambda)} U(s) \, ds \right] d\lambda. \]

(40)

Since both integrals are absolutely convergent, the order of integration may be changed, and

\[ u(t) = \frac{1}{2\pi i} \int_C e^{st} U(s) \left[ \int_0^\infty e^{-s\lambda} W(t, \lambda) \, d\lambda \right] ds. \]

(41)
The integral inside the brackets is $H(s,t)$ so that finally

$$v(t) = \frac{1}{2\pi j} \int_C e^{st} H(s,t) U(s) \, ds . \tag{42}$$

From Eq. (42) it is seen that once $H(s,t)$ is known, the procedure for determining a $v(t)$ corresponding to any $u(t)$ is exactly the same as in the case of a fixed system, that is, once the product $H(s,t)U(s)$ is known, the output in the time domain may be obtained by either referring to a table of Laplace transform pairs, or by carrying out the integration indicated by Eq. (42). In connection with the latter case, it is noted that

$$\frac{1}{2\pi j} \int_C e^{st} H(s,t) U(s) \, ds = \sum_{i=1}^n K_i(t) \tag{43}$$

where $K_i$ is the residue of $e^{st}H(s,t)U(s)$ at the $i$th singularity, $s_i$. Since $e^{st}$ is analytic everywhere for all finite $t$, the singularities must be those of $H(s,t)U(s)$. If $H(s,t)U(s)$ is the ratio of two polynomials

$$H(s,t) U(s) = \frac{P(s,t)}{Q(s,t)} \tag{44}$$

and has only simple poles at $s_i$, $i = 1, 2, \ldots, n$, which are the same as the zeros of $Q(s,t)$, then

$$K_i = \left. \frac{P(s,t)}{\partial Q(s,t)} \right|_{s = s_i} \tag{45}$$
From Eq. (45) it may be seen that in general both the residues and the poles of the function $H(s,t)U(s)$ are functions of time. Finally, from Eq. (45), the output of such a time varying system may be written as

$$U(t) = \sum_{i=1}^{n} \left( \frac{P(s,t)}{aO(s,t)} \right) e^{s_i t}.$$  \hfill (46)

The above discussion has shown that one method for carrying out the analysis of a time varying system is to find the system function for the system, and that once the system function has been determined the output for any arbitrary input may be found from a table of Laplace transforms. As one might suspect then the greatest effort in the analysis of such systems centers around the determination of $H(s,t)$ and of its properties.

F. The Differential Equation for the System Function

From Eq. (33), it was noted that $H(s,t)$ could be defined as the response of the system to an input, $e^{st}$, divided by that input so that obviously, in terms of the system differential equation

$$L(p,t) \left[ H(s,t)e^{st} \right] = K(p,t)e^{st}. \hfill (47)$$

Now both $L(p,t)$ and $K(p,t)$ are polynomial differential operators, and it is seen then that

$$\nabla K(p,t)e^{st} = e^{st}K(s,t). \hfill (48)$$
Also it may be seen that a typical term of $L(p, t) H(s, t)e^{st}$ would be

$$\sum \alpha_r(t) e^{rt} [H(s, t)e^{st}] = e^{st} \sum \alpha_r(t) (p+s)^r H(s, t). \quad (49)$$

Therefore, the differential equation that $H(s, t)$ must satisfy is, in operator form,

$$L(p + s, t) H(s, t) = K(s, t). \quad (60)$$

As has been pointed out by Zadeh (11), the steady state (or particular integral) solution of Eq. (60) is all that is required provided that the variability of the system does not undergo any transitions during the time of observation, furthermore, he has shown methods for approximating $H(s, t)$ by applying perturbation techniques to the differential equation (60).

To summarize then, the following properties of $H(s, t)$ should be kept in mind.

1. If the input and output of a linear time varying system are related by the differential equation

$$L(p, t) U(t) = K(p, t) U(t) \quad (2)$$

where $L(p, t)$ and $K(p, t)$ are polynomial differential operators, then the system response, $H(s, t)$, is the Laplace transform of the impulsive response, $W(t, \lambda)$, with respect to $\lambda$, where $\lambda$ is the elapsed time from the application of the impulse to the time of observation, $t$. 

That is

$$H(s,\tau) = \int_0^\infty e^{-s\lambda} W(\tau, \lambda) \, d\lambda,$$

(32)

where $W(t, \lambda) = 0$ for $\lambda < 0$.

2. $H(s, \tau)e^{st}$ has been shown to be the response of the time varying system to an input of the form $e^{st}$, that is

$$L\{\rho(t)\left[ H(s, \tau) e^{st} \right]\} = K(\rho, t)e^{st}$$

(47)

or $H(s, \tau)$ is the solution to the non-homogeneous differential equation

$$L(\rho + s, \tau) H(s, \tau) = K(s, \tau),$$

(50)

3. In solving the above differential equation for $H(s, \tau)$ only the particular integral need be considered if the time variation of the system has no transitions during the time interval of interest.

4. Since the process of determining an output for any given input is reduced to that of using a standard table of Laplace transforms once the system function is known, the major part of any analysis is centered about the determination of the system function.

G. Further Properties of $H(s, \tau)$

Before proceeding further with a discussion of the system function and referring back to Eq. (2)

$$L\{\rho(t)U(t)\} = K(\rho, t)U(t)$$

(2)
an explicit expression for \( v(t) \) may be obtained by the method of variation of parameters as follows. Let \( v_1(t) , v_2(t) , \ldots , v_n(t) \) be a set of linearly independent solutions to the \( n \)th order homogeneous differential equation

\[
L(P,t) \psi_i(t) = 0. \tag{51}
\]

Furthermore, let it be supposed that a set of functions \( V_1(t) , V_2(t) , \ldots , V_n(t) \) can be found such that

\[
U(t) = V_1(t) \psi_1(t) + V_2(t) \psi_2(t) + \cdots + V_n(t) \psi_n(t) \tag{52}
\]

subject to the further restrictions that

\[
\begin{align*}
V_1' \psi_1 + V_2' \psi_2 + \cdots + V_n' \psi_n & = 0 \\
V_1' \psi_1' + V_2' \psi_2' + \cdots + V_n' \psi_n' & = 0 \\
\vdots & \quad \vdots \\
V_1' \psi_1^{(n-2)} + V_2' \psi_2^{(n-2)} + \cdots + V_n' \psi_n^{(n-2)} & = 0. \tag{53}
\end{align*}
\]

Under these suppositions then the derivatives of \( v(t) \) may be written as
\[
\frac{d\nu}{dt} = V_1 \nu_1' + V_2 \nu_2' + \ldots + V_n \nu_n' = \sum_{i=1}^{n} V_i \nu_i' \\
\frac{d^2\nu}{dt^2} = V_1 \nu_1'' + V_2 \nu_2'' + \ldots + V_n \nu_n'' = \sum_{i=1}^{n} V_i \nu_i'' \\
\frac{d^{n-1}\nu}{dt^{n-1}} = V_1 \nu_1^{(n-1)} + V_2 \nu_2^{(n-1)} + \ldots + V_n \nu_n^{(n-1)} = \sum_{i=1}^{n} V_i \nu_i^{(n-1)} \quad (54)
\]

while
\[
\frac{d^n\nu}{dt^n} = V_1 \nu_1^{(n)} + V_2 \nu_2^{(n)} + \ldots + V_n \nu_n^{(n)} + V_1 \nu_1^{(n-1)} + V_2 \nu_2^{(n-1)} + \ldots + V_n \nu_n^{(n-1)} = \sum_{i=1}^{n} V_i \nu_i^{(n)} + V_i \nu_i^{(n-1)}.
\]

If these expressions for the derivatives of \( \nu(t) \) be substituted into the differential equation, Eq. (2), then
\[
a_n(t) \left[ \sum_{i=1}^{n} V_i \nu_i^{(n)} + \sum_{i=1}^{n} V_i' \nu_i^{(n-1)} \right] + a_{n-1}(t) \left[ \sum_{i=1}^{n} V_i \nu_i^{(n-1)} \right] + \ldots + a_1(t) \left[ \sum_{i=1}^{n} V_i \nu_i' \right] + a_0(t) \left[ \sum_{i=1}^{n} V_i \nu_i \right] = K(p,t) u(t) \quad (55)
\]
or upon collecting terms
\[
a_n(t) \sum_{i=1}^{n} V_i' \nu_i^{(n-1)} + \sum_{i=1}^{n} V_i L(p,t) \nu_i = K(p,t) u(t) \quad (56)
\]
But, since each $v_i$ satisfies the homogeneous equation, Eq. (51), then

$$A_n(t) \sum_{i=1}^{n} V_i(t) \bar{v}_i^{(n-i)} = K(p, t) U(t).$$

Equation (57) in conjunction with the equations of (53) thus constitute $n$ equations in the $n$ unknown functions $V_i'(t)$ so that each may be determined by means of Cramer's rule to obtain

$$V_i' = \frac{\Delta n_1}{\Delta} \frac{K(p, t) U(t)}{A_n(t)}$$
$$V_2' = \frac{\Delta n_2}{\Delta} \frac{K(p, t) U(t)}{A_n(t)}$$
$$\vdots$$
$$V_n' = \frac{\Delta n_n}{\Delta} \frac{K(p, t) U(t)}{A_n(t)}.$$

Where $\Delta$ is the determinant

$$\begin{vmatrix}
V_1 & V_2 & \cdots & V_n \\
V_1' & V_2' & \cdots & V_n' \\
\vdots & \vdots & \ddots & \vdots \\
V_{(n-i)} & V_{(n-i)} & \cdots & V_{(n-i)}
\end{vmatrix}$$
and \( \Delta_{n,i} \) is the cofactor of the element occupying the \( n \)th row and \( i \)th column in \( \Delta \). The determinant \( \Delta \), (called the Wronskian of the linearly independent set of solutions, \( v_1 \)) is a well known function in the theory of linear differential equations. It has the property of being non-singular, or nonzero, at any time \( t \) for which the \( v_1 \) constitute a linearly independent set, and is also related to the differential equation through a relation known as Abel's identity in that

\[
\frac{d\Delta}{dt} = -\frac{A_{n-1}(t)}{A_n(t)} \Delta,
\]

and upon solving the differential equation

\[
\Delta = \Delta_0 e^{\int_{\tau}^{t} \frac{A_{n-1}(\tau)}{A_n(\tau)} d\tau}
\]

where \( \Delta_0 \) is the value of \( \Delta \) at \( t = 0 \).

Upon integrating the \( v_1 \)'s in Eq. (58) and substituting into Eq. (62) \( v(t) \) then may be written explicitly as

\[
U(t) = \sum_{i=1}^{n} v_i(t) \int_{a}^{t} \frac{\Delta_{n;i}(\tau)}{A_n(\tau) \Delta(\tau)} K(\rho, \tau) U(\tau) d\tau.
\]

If the order of summation and integration be changed, then

\[
U(t) = \int_{a}^{t} \frac{\sum_{i=1}^{n} v_i(t) \Delta_{n;i}(\tau)}{A_n(\tau) \Delta(\tau)} K(\rho, \tau) U(\tau) d\tau.
\]

The summation \( \sum_{i=1}^{n} v_1(t) \Delta_{n;i}(\tau) \) occurring in the numerator of the
Although we allow $H(p') (q') \leq \theta$ (a) to be written in an exact form,

\begin{equation}
\int_{\mathbb{R}} \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R}) = \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R})
\end{equation}

therefore.

Equation (66) allows one to write an exact expression for

\begin{equation}
\int_{\mathbb{R}} \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R}) = \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R})
\end{equation}

The partition integral of the differential equation may be obtained

\begin{equation}
\int_{\mathbb{R}} \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R}) = \mathbb{E} (\mathbb{R}, \mathbb{R}) \mathbb{P} (\mathbb{R}, \mathbb{R}) (\mathbb{R}, \mathbb{R})
\end{equation}

then

\begin{equation}
\frac{(\mathbb{R}) \wedge (\mathbb{R})^D \wedge (\mathbb{R})^{1D}}{\mathbb{R}, \mathbb{R}} = (\mathbb{R}, \mathbb{R})^D
\end{equation}

defined as

$\wedge$ is the partition differential. $\wedge$ is the determinant. $\wedge$ is the determinant. $\wedge$ is the determinant. $\wedge$ is the determinant. $\wedge$ is the determinant. $\wedge$ is the determinant. $\wedge$ is the determinant.
solutions of the homogeneous equations be known, and secondly the inte-
gration indicated must be carried out. One of the principle uses of
Eq. (67) will be its use as a means for determining other properties
of \( H(s,t) \).

Referring to Eq. (66) and noting that it is in the form of a
Volterra integral, the Volterra operator, \( H(p,t) \) may be defined as
follows

\[
\mathcal{U}(t) = H(p,t) \mathcal{U}(t)
\]  

(68)

where by \( H(p,t) \) is meant the operation indicated by Eq. (66). Note
also that in the expression for \( H(s,t) \) formal substitution of \( p \) for \( s \)
will result in the operator \( H(p,t) \).

In fixed network analysis, usually it is possible to simplify the
analysis problem by the use of certain network theorems so that the
overall transfer function of the system may be obtained by operations
on the transfer functions of the constituent parts of the network. Of
these theorems certain of them such as the superposition theorem, the
reciprocity theorem, Thevinin's theorem and Norton's theorem are based
solely upon the premise of linearity, and therefore they will apply to
linear time varying systems as well. In cascading two fixed systems,
the overall transfer function may be obtained merely by multiplying
the transfer functions of the individual systems together in any order.
That such a simple process will not yield the correct system function
for time varying systems is shown below.

Referring to Fig. 4, wherein two time varying systems are to be
Fig. 4. Block Diagram Representation of the Cascaded Connection of Two Time Varying Systems.
cascaded, the following differential equations apply. For the first
system
\[ L_1(p, t) w(t) = K_1(p, t) u(t) \]
and
\[ H_1(s, t) = e^{-st} \int_{-\infty}^{t} G_{11}(t, \tau) e^{st} K_1(s, \tau) d\tau . \]  \hspace{1cm} (69)

While for the second system
\[ L_2(p, t) u(t) = K_2(p, t) w(t) \]
and
\[ H_2(s, t) = e^{-st} \int_{-\infty}^{t} G_{12}(t, \tau) e^{st} K_2(s, \tau) d\tau . \]  \hspace{1cm} (70)

If an input \( \text{est} \) is applied at the input to the first system, the output of the first system will be \( H_1(s, t) \text{est} \) while the output of the second system will be \( H(s, t) \text{est} \) where \( H(s, t) \) is the desired overall system function.

Making use of Eq. (66) the expression for the output of the second system is then
\[ H(s, t) e^{-st} = \int_{-\infty}^{t} G_{12}(t, \tau) e^{-st} K_2(p, \tau) \left[ H_1(s, \tau) e^{st} \right] d\tau , \]  \hspace{1cm} (71)
or
\[ H(s, t) = e^{-st} \int_{-\infty}^{t} G_{12}(t, \tau) e^{st} K_2(p, \tau) H_1(s, \tau) d\tau . \]  \hspace{1cm} (72)
Note that the expression for $H(s,t)$ is the same as that indicated by means of the operation

$$H(s,t) = H_2(p+s,t) H_1(s,t)$$

(73)

where the $p$ in $H_2(p+s,t)$ operates on the $t$ in $H_1(s,t)$. Also note that the operation is not commutative in that $H(s,t)$ is not $H_1(p+s,t)$ operating on $H_2(s,t)$. However, if neither $H_2$ nor $H_1$ involves time, then the operators are those for fixed systems and the operation is commutative.

A general rule for finding the overall system function for the cascaded connection of two variable systems may now be stated in that the overall system function may be obtained by operating on the first of the system functions (going in the direction of signal flow) with the system function of the second where in the second, the variable $s$ is replaced by $p+s$. It must be understood that the $p$ operates only on the $t$ in the first system function and that the operation is not commutative. Also it must be noted that only the steady state solution to the differential equation so obtained is required unless transitions in the variability of the parameters take place during the period of observation.

H. The Inverse System Function

The inverse of a system operator is often required in the analysis of systems especially those systems involving feedback. Referring to
Eq. (68), the output was written in terms of a system operator operating on the input.

\[ \mathcal{U}(t) = H(p,t) \mathcal{U}(t). \]  
(68)

As might be expected, the inverse system operator, \( H^{-1}(p,t) \), is that operator which when used to operate on the output yields the input.

\[ \mathcal{U}(t) = H^{-1}(p,t) \mathcal{U}(t). \]  
(74)

Now, \( u(t) \) may be obtained explicitly as a Volterra integral from the differential equation in the same manner that \( v(t) \) was obtained in that if a set of \( m \) functions, \( u_1, u_2, \ldots, u_m \) constituting a set of linearly independent solutions of the homogeneous equation

\[ K(p,t) u_i = 0 \]  
(75)

is known, then \( G_2(t,T) \) may be defined as

\[ G_2(t,T) = \frac{\Delta_2(t,T)}{b_m(T) \Delta(T)}. \]  
(76)

Where \( \Delta(T) \) is the Wronskian of the set of linearly independent solutions \( u_1, \Delta_2(t,T) \) is the determinant obtained from \( \Delta \) by replacing the \( u_1(m-1)(t) \) in the last row of \( \Delta \) by \( u_1(t) \), and \( b_m(T) \) is the non-vanishing coefficient of the \( p^m \) th term in \( K(p,t) \). Again by the method of variation of parameters the input for a known output may be written as
\[ u(t) = \int_{-\infty}^{t} G_2(t, \tau) L(p, \tau) V(\tau) \, d\tau. \]  \hspace{1cm} (77)

If \( u(t) = e^{st} \), then \( v(t) = H(s,t)e^{st} \) and Eq. (77) becomes

\[ e^{st} = \int_{-\infty}^{t} G_2(t, \tau) L(p, \tau)[H(s, \tau)e^{s\tau}] \, d\tau, \]  \hspace{1cm} (78)

or

\[ I = e^{-st} \int_{-\infty}^{t} G_2(t, \tau)e^{s\tau}L(p+s, \tau)H(s, \tau) \, d\tau. \]  \hspace{1cm} (79)

In operator form

\[ I = H^{-1}(p+s,t)H(s,t) \]  \hspace{1cm} (80)

so that Eq. (79) and Eq. (80) serve to define the inverse system function as

\[ H^{-1}(s,t) = e^{-st} \int_{-\infty}^{t} G_2(t, \tau)e^{s\tau}L(s, \tau) \, d\tau. \]  \hspace{1cm} (81)

From Eq. (81) a physical interpretation of the inverse system function may be made in that \( H^{-1}(s,t)e^{st} \) is the input to the system necessary to cause an output \( e^{st} \).

I. Feedback Systems

The inverse operator as has been stated is useful in analyzing
systems that involve feedback. Fig. 5 illustrates in block diagram form the general feedback problem wherein the total system has been broken up into a forward system and a feedback system. The output of the feedback system, w(t), is subtracted from the input to the total system, u(t), to form the error signal, x(t). x(t) is the input to the forward system. Let the differential equations below apply to the forward and feedback systems respectively.

\[ L_1(p, t) U(t) = K_1(p, t) X(t) \]  
\[ L_2(p, t) W(t) = K_2(p, t) V(t) \]

where

\[ X(t) = U(t) - W(t). \]

Furthermore, let the system functions of the forward system be \( H_1(s, t) \) and its inverse \( H_1^{-1}(s, t) \) while the system function of the feedback system is \( H_2(s, t) \) and its inverse \( H_2^{-1}(s, t) \). The problem then is to find the overall system function for the closed loop system in terms of the system functions or inverse system functions of the two constituent systems. To do this, let

\[ X(t) = H_1^{-1}(s, t) e^{st}. \]

Then the output of the forward system is \( e^{st} \), and the input to the closed loop system must be \( H^{-1}(s, t)e^{st} \). The input to the feedback system being \( e^{st} \), its output must be \( H_2(s, t)e^{st} \), and since this quantity
Fig. 5. Block Diagram Representation of a General Feedback System.
is subtracted from \( u(t) \) to form \( x(t) \) then

\[
\chi(t) = H'(s, t) e^{s \tau} = (H'(s, t) - H_2(s, t)) e^{s \tau}
\]

(86)

or

\[
H'(s, t) = H'(s, t) + H_2(s, t)
\]

(87)

and the overall system function, \( H(s, t) \), may be found from the differential equation

\[
H'(s, t) H(s, t) = 1.
\]

(88)

J. Stability

In designing feedback systems of either the fixed or time varying type, the question of stability is always present. The most convenient methods for determining the stability of fixed systems have been those that made use of the behavior of the transfer function in the complex \( s \) plane. Zadeh (13) has shown that much the same principles apply in determining the stability of a time varying system.

A stable system has been defined as a system in which every bounded input results in a bounded output. An unstable system therefore would be one in which a bounded input would result in an unbounded output. A necessary and sufficient condition that a system be stable is that the impulsive response of the system be integrable with respect to the age variable, \( \lambda \), for all \( t \), for from Eq. (37) it has been shown
that the output \( v(t) \) of a system may be written as

\[
U(t) = \int_0^\infty W(t,\lambda) u(t-\lambda) \, d\lambda. 
\]  

(37)

If \( u(t) \) is bounded, then

\[
|U(t)| \leq M < \infty.
\]  

(89)

Furthermore if \( W(t,\lambda) \) is integrable with respect to \( \lambda \) for all \( t \), then

\[
\int_0^\infty |W(t,\lambda)| \, d\lambda < \infty
\]  

(90)

and from Eq. (37)

\[
|U(t)| = \left| \int_0^\infty W(t,\lambda) u(t-\lambda) \, d\lambda \right| \leq M \int_0^\infty |W(t,\lambda)| \, d\lambda.
\]  

(91)

Hence a sufficient condition that \( v(t) \) be bounded for bounded \( u(t) \) is that Eq. (90) apply.

The necessity of \( W(t,\lambda) \) being integrable may be shown by assuming there is one value of \( t \), say \( t_0 \), for which

\[
\int_0^\infty |W(t_0,\lambda)| \, d\lambda = \infty,
\]  

(92)

then if \( u(t_0 - \lambda) \) is chosen to be 1 whenever \( W(t_0,\lambda) \) is positive and -1 whenever \( W(t_0,\lambda) \) is negative, then
and it can be seen that a bounded input produces an unbounded output.

Since $H(s,t)$ is the Laplace transform of $W(t,\lambda)$ with respect to $\lambda$, and since every integrable function has a Laplace transform that is analytic in the right half of the $s$ plane and on the imaginary axis, then it follows that the system function of a stable system is one that is bounded and analytic in the right half plane and on the imaginary axis of the $s$ plane for all $t$.

Zadeh has pointed out that the condition that a system function be bounded and analytic in the right half plane for all $t$ is a necessary but not sufficient condition for stability, in that there are some systems, although they are of limited practical significance, which fulfill the condition and yet their impulsive responses are not integrable.

Equation (46) gives the output of a system in terms of the singularities of the product $H(s,t)U(s)$, for the case where $H(s,t)U(s)$ is the ratio of two polynomials $P(s,t)$ and $Q(s,t)$, and $Q(s,t)$, the denominator, has distinct zeros

$$ U(t) = \sum_{i=1}^{n} \left( \frac{P(s,t)}{\partial Q(s,t) / \partial s} \right)_{s=s_i} \cdot $$

(46)

Inspection of this equation shows that in general the zeros of $Q(s,t)$, $s_i$, are functions of time, and will move around in the $s$ plane as time
varies. In employing the stability criterion it is necessary that such zeros do not cross over into the right half plane.

One other relation is often of value in discussing the stability of a composite system. If two stable systems are cascaded, then from the definition of stability the resultant overall system is stable. Therefore, the conclusion follows that if $H_1(s,t)$ and $H_2(s,t)$ represent system functions of stable systems, then

$$H(s,t) = H_1(p+s,t) H_2(s,t)$$  \hspace{1cm} (94)$$

or

$$H(s,t) = H_2(p+s,t) H_1(s,t)$$  \hspace{1cm} (95)$$

are also stable system functions. Conversely if $H(s,t)$ represents an unstable system function, then either $H_1$ or $H_2$ or both are unstable. The above rule is of value in considering feedback systems in that parts of the system function may be known to be stable, and hence need not be included in the stability analysis. For instance, if for a given feedback system

$$H(s,t) = H_2(p+s,t) H_1(s,t)$$  \hspace{1cm} (96)$$

where $H_1(s,t)$ is known to be stable, the question of stability will be resolved by investigating $H_2(s,t)$. 
K. Systems with Singly Periodically Varying Parameters

If the time varying parameters of a system vary periodically, all having the same period, T, then the fundamental differential equation of the system

\[ L(p,t)\varphi(t) = K(p,t)u(t) \tag{2} \]

is a linear differential equation with periodic coefficients. The system function, \( H(s,t) \), must satisfy the non-homogeneous differential equation

\[ L(p+s,t)H(s,t) = K(s,t), \tag{50} \]

and if one substitutes \( t+T \) for \( t \) in Eq. (50) then

\[ L(p+s,t+T)H(s,t+T) = K(s,t+T). \tag{97} \]

However, because of the periodicity of all of the coefficients of \( L(p,t) \) and \( K(p,t) \)

\[ L(p+s,t+T) = L(p+s,t) \tag{98} \]

\[ K(s,t+T) = K(s,t) \tag{99} \]

so that

\[ L(p+s,t)H(s,t+T) = K(s,t) \tag{100} \]

and it is seen that since \( H(s,t+T) \) satisfies the same differential
equation as $H(s, t)$, then by the uniqueness theorem

$$H(s, t+\tau) = H(s, t). \tag{101}$$

Thus, if the parameters of a linear system are caused to vary periodically with the same period, then the system function of the system is also a periodic function having the same period.

Equation (67), the integral form of $H(s, t)$

$$H(s, t) = e^{st} \int_{-\infty}^{t} G(s, \tau) e^{-s\tau} k(s, \tau) d\tau \tag{67}$$

is of little practical value for determining the system function in the periodic case because of the difficulty in finding a set of linearly independent solutions to the homogeneous equation

$$L(\rho, t) \psi_i(t) = 0. \tag{51}$$

However, Eq. (67) may be used to determine certain properties of $H(s, t)$. From the Floquet theory for differential equations with periodic coefficients, a set of linearly independent solutions to the homogeneous equation, Eq. (51), exist such that the functional character of each of the linearly independent solutions is

$$\psi_i(t) = e^{\rho_i t} h_i(t) \tag{102}$$

where each $\rho_i$, called a characteristic exponent, is a constant depending upon the coefficients of the differential equation, and each $h_i(t)$ is a periodic function having the same period as the coefficients of the
differential equation. The problem of finding the characteristic exponents is a very difficult one, and the solutions for higher order than two are not generally known.

In finding an expression for $H(s,t)$ by means of Eq. (67) it is recalled that

$$G_i(t,\tau) = \frac{\Delta_i(t,\tau)}{\Delta_n(\tau) \Delta(\tau)}.$$  \hspace{1cm} (64)

Where $\Delta(\tau)$ is the Wronskian of the $n$ linearly independent solutions

$$\Delta(\tau) = \begin{vmatrix} U_1(\tau) & U_2(\tau) & \cdots & U_n(\tau) \\ U_1'(\tau) & U_2'(\tau) & \cdots & U_n'(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ U_1^{(n-1)}(\tau) & U_2^{(n-1)}(\tau) & \cdots & U_n^{(n-1)}(\tau) \end{vmatrix},$$  \hspace{1cm} (59)

$\Delta_1(t,\tau)$ is the determinant obtained from $\Delta(\tau)$ by replacing the $v_1^{(n-1)}(\tau)$ in the last row by $v_1(t)$, and $a_n(\tau)$ is the coefficient of the highest ordered term in $L(p,s,t)$.

A singularly simple notation for the derivatives of the set of solutions, $v_1(t)$ is that

$$\frac{d^n v_i}{dt^n} = \rho_i^e U_i(t) = \rho_i^e (p + \rho_i^e)^n h_i(t).$$  \hspace{1cm} (103)
Upon substituting the expressions of Eq. (103) into the Wronskian, and factoring $e^{P_1 t}$ from each of the columns, then

$$\Delta(t) = \sum_{i=1}^{n} P_i \tau \begin{vmatrix} h_1(t) & h_2(t) & \cdots & h_n(t) \\ (P+P_1)h_1(t) & (P+P_1)h_2(t) & \cdots & (P+P_1)h_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ (P+P_1)^{n-1}h_1(t) & (P+P_1)^{n-1}h_2(t) & \cdots & (P+P_1)^{n-1}h_n(t) \end{vmatrix}$$

or

$$\Delta(t) = \sum_{i=1}^{n} P_i \tau \Delta_a(t).$$

Since every element in the determinant $\Delta_a$ is periodic, then $\Delta_a$ is periodic with period $T$.

The determinant $\Delta_1(t, \tau)$ may be written as

$$\Delta_1(t, \tau) = \sum_{i=1}^{n} e^{P_i \tau} \Delta h_i(t) \Delta_{n_i}(\tau)$$

where $\Delta_{n_i}(\tau)$ is the cofactor of $e^{P_1 t}h_i(t)$ in $\Delta_1(t, \tau)$. But in the same manner as $\Delta(t)$, $\Delta_{n_i}(\tau)$ may be written as

$$\Delta_{n_i}(\tau) = e^{\sum_{j \neq i}^{n} P_j \tau - P_i \tau} \Delta a_{n_i}(\tau)$$
where $\Delta a_{n,1}$ is the determinant

$$
\Delta a_{n1} = \left| \begin{array}{cccc}
h_1(t) & \cdots & h_{i-1}(t) & h_{i+1}(t) & \cdots & h_n(t) \\
(p+P_m)h_i(t) & \cdots & (p+P_m)h_{i-1}(t) & (p+P_m)h_{i+1}(t) & \cdots & (p+P_m)h_n(t) \\
& \vdots & \ddots & \vdots & \ddots & \vdots \\
& & & (p+P_m)h_1(t) & \cdots & (p+P_m)h_{i-1}(t) & (p+P_m)h_{i+1}(t) & \cdots & (p+P_m)h_n(t) \\
\end{array} \right|
$$

By inspection it is seen that $\Delta a_{n,1}$ is also periodic with period $T$.

Upon substituting the above expressions for $\Delta(t)$ and $\Delta_1(t, T)$ in Eq. (67), then

$$
H(s, t) = \sum_{i=1}^{n} A_{i}(s) = \int_{-\infty}^{\infty} \frac{\sum_{i=1}^{n} \frac{e^{P_m s - P_m \tau}}{A_{n}(\tau) \Delta_{a_{n1}}(\tau)} e^{s \tau} k(s, \tau) d\tau}{e^{P_m s - P_m \tau} \Delta_{a_{n1}}(\tau)}
$$

or a typical term, $H_i(s, t)$ is

$$
H_i(s, t) = e^{P_m t} \int_{-\infty}^{t} e^{(s-P_m)\tau} \left( \frac{\Delta a_{n1}(\tau) k(s, \tau)}{A_{n}(\tau) \Delta_{a_{n1}}(\tau)} \right) d\tau
$$

That part of the integrand enclosed in parentheses is periodic with period $T$, and may be expressed as a Fourier series
\[
\left( \frac{\Delta a_{n_i}(\tau)}{\Delta a(\tau)} \right) = \frac{K(s,\tau)}{\Delta a(\tau)} = \sum_{r=-\infty}^{\infty} A_i r(s) e^{j\omega_0 \tau} \tag{111}
\]

where \( \omega_0 = \frac{2\pi}{T} \). So that Eq. (110) becomes

\[
H_i(s,\tau) = C \sum_{r=-\infty}^{\infty} A_i r(s) e^{j\omega_0 \tau} d\tau. \tag{112}
\]

Upon changing the order of integration and summation, then

\[
H_i(s,\tau) = h_i(\tau) e^{j\omega_0 \tau} \sum_{r=-\infty}^{\infty} \frac{A_i r(s)}{s - P_i + j\omega_0}. \tag{113}
\]

Finally then, the \( H_i \)'s may be summed to show that

\[
\hat{H}(s,\tau) = \sum_{i=1}^{n} h_i(\tau) \left( \sum_{r=-\infty}^{\infty} \frac{A_i r(s)}{s - P_i + j\omega_0} \right). \tag{114}
\]

This shows the general form for \( H(s,\tau) \), in that singularities for
\( H(s,\tau) \) occur in families all having the same real part equal to \( \text{Re}[P_i] \)
and having imaginary parts which are displaced from one another by some
integral multiple of the frequency of variation of the parameters of
the system. Stable periodic systems then will be those for which the
\( \text{Re}[P_i] = 0 \). It is interesting to note that as in the case of a fixed
system, the poles of \( H(s,\tau) \) for periodically varying systems are not
functions of time, but are fixed quantities.
One other interesting property of periodically varying systems may be seen from the Wronskian of the linearly independent solutions to the homogeneous equation. From Eq. (105), the Wronskian was written as

$$
\Delta(t) = \prod_{i=1}^{n} \Delta_i(t)
$$

(105)

where the determinant $\Delta$ is periodic with period $T$. Also from the Abel identity, Eq. (61)

$$
\Delta(t) = \Delta_0 e^{-\int_0^t \frac{a_{n-1}(\tau)}{a_n(\tau)} d\tau}
$$

(61)

where $\Delta_0$ is a constant, and $a_n(t)$ and $a_{n-1}(t)$ are the coefficients of the $n$th and the $(n-1)$th derivatives in the differential operator $L(p, t)$, respectively. Equating Eq. (105) and Eq. (61) then

$$
\Delta_0 e^{-\int_0^t \frac{a_{n-1}(\tau)}{a_n(\tau)} d\tau} = e^{-\sum_{i=1}^{n} P_i t} \Delta(t)
$$

(115)

and at $t+T$

$$
\Delta_0 e^{-\int_0^{t+T} \frac{a_{n-1}(\tau)}{a_n(\tau)} d\tau} = e^{-\sum_{i=1}^{n} P_i (t+T)} \Delta(t+T)
$$

(116)

The periodicity of $\Delta$ requires that

$$
\Delta_0 e^{-\int_0^{t+T} \frac{a_{n-1}(\tau)}{a_n(\tau)} d\tau} = e^{-\sum_{i=1}^{n} P_i T}
$$

(117)
or that

\[ \sum_{i=1}^{n} p_i \tau = - \int_{0}^{\tau} \frac{a_{n-1}(\tau)}{a_n(\tau)} d\tau. \] (118)

Upon substituting \( \tau = t + \lambda \) for the variable of integration and noting that Eq. (117) holds for all \( t \), in particular for \( t = 0 \), then

\[ \sum_{i=1}^{n} p_i = - \frac{1}{\tau} \int_{0}^{\tau} \frac{a_{n-1}(\lambda)}{a_n(\lambda)} d\lambda. \] (119)

Therefore, the sufficient condition for a system to be unstable is that the average value of the ratio of \( \frac{a_{n-1}(t)}{a_n(t)} \) be negative. Furthermore, since both \( a_n(t) \) and \( a_{n-1}(t) \) are real, the imaginary parts of the characteristic exponents must add to zero.

The above test, although inconclusive for determining the stability of a system, is very easy to apply, and often may be used for determining the character of the characteristic exponents. For example, if \( L(p,t) \) is of order two, and the integral of Eq. (119) is positive, it may be seen that a set of characteristic exponents will be of the form

\[ P_i = - \frac{p_2}{2} + \mu \]
\[ P_i = - \frac{p_2}{2} - \mu \]

where \( \mu = \alpha + j\beta \), and \( p_0 \) is the value of the integral of Eq. (119).
III. EXAMPLES OF TIME VARYING SYSTEM ANALYSIS

A. General Discussion

The examples of varying system analysis shown below are chosen so as to illustrate the principles set forth in the preceding section. The cases considered are all those of periodically varying systems since this is the one of most practical interest, and where alternative methods are of interest they often are included. In carrying out the analysis, once the system function for the system has been obtained, it is assumed that the problem has been solved since the problem of finding the time domain solution once the system function and input are known is merely that of using Laplace transform tables with the additional requirement that the $t$ variable in $H(s,t)$ is carried along as a parameter.

B. The Analysis of an RC Circuit with Periodically Varying Capacitance

In Fig. 6 an RC circuit is shown in which the capacitance is a periodic function of time. The problem is to find the system function for the case where

$$C(t) = \frac{C_0}{1 + A \cos \omega t}, \quad A < 1.$$  \hspace{1cm} (120)
Input \( u(t) \)  \[ R \]  \[ C(t) \]  \[ \text{Output} \ v(t) \]

\[ C(t) = \frac{C_0}{1 + \alpha \cos \omega_0 t} \]
\[ \alpha < 1 \]

Fig. 6. Time Varying RC Circuit.
First, the system differential equation may be written by summing the currents at the output node so that if $g = R^{-1}$, then

$$\frac{d}{dt}(Cu) + g(u' - u) = 0 \tag{121}$$

or, letting $T = RC$,

$$\frac{d}{dt}u + \left(\frac{1}{T} + \frac{C'}{C}\right)u = \frac{1}{T}u. \tag{122}$$

Therefore,

$$L(p,t) = p + \left(\frac{1}{T} + \frac{C'}{C}\right) \tag{123}$$

$$K(p,t) = \frac{1}{T}. \tag{124}$$

The system function $H(s,t)$ then may be obtained from the particular integral of the differential equation

$$\left[p + s + \frac{1}{T} + \frac{C'}{C}\right]H = \frac{1}{T}. \tag{125}$$

An integrating factor, $\exp(st + \ln C + \int_0^t \frac{d\lambda}{T(\lambda)})$, will make the equation exact so that

$$\frac{d}{dt} \left( HCE^{(st + \int_0^t \frac{d\lambda}{T})} \right) = \frac{C}{T}e^{(st + \int_0^t \frac{d\lambda}{T})}. \tag{126}$$

Upon integrating both sides, then
\[ H(s,t) = \frac{e}{\tau} \int_{-\infty}^{\infty} e^{-(s \tau + \int \frac{d\lambda}{\tau})} \int e^{(s \tau + \int \frac{d\lambda}{\tau})} d\tau. \] (127)

Now \( T \) may be written as
\[ T = \frac{T_0}{1 + a \cos \omega_0 t}, \] (128)
and
\[ \int \frac{d\lambda}{T} = \frac{1}{T_0} (t + \frac{a}{\omega_0} \sin \omega_0 t) \] (129)
so that
\[ H(s,t) = \frac{e}{\tau} \cdot e^{-(s + \frac{1}{\tau}) t} \cdot e^{-(\frac{a}{\omega_0 \tau}) \sin \omega_0 t} \int_{-\infty}^{\infty} e^{(s + \frac{1}{\tau}) \tau} \cdot e^{-(\frac{a}{\omega_0 \tau}) \sin \omega_0 \tau} d\tau. \] (130)

Upon substituting \( \lambda = t - \tau \) for the variable of integration, then
\[ H(s,t) = \frac{1}{T_0} e^{-\frac{a}{\omega_0 \tau_0}} \sin \omega_0 t \int_{0}^{\infty} e^{-(s + \frac{1}{\tau}) \lambda} \cdot e^{-(\frac{a}{\omega_0 \tau_0}) \sin \omega_0 (\lambda - t)} \] (131)

To perform the integration indicated by Eq. (131), an expansion of \( \exp\left(\frac{a}{\omega_0 \tau_0} \sin(\lambda - t)\right) \) into a trigonometric series with Bessel function coefficients may be made (10), and the integration of the series carried out term by term. However, if the parameters of the system are such that \( \frac{a}{\omega_0 \tau_0} \ll 1 \), then the first two terms of the power series expansion for \( \exp\left(\frac{a}{\omega_0 \tau_0} \sin \omega_0 t\right) \) will lead to a good approximation for \( H(s,t) \). With
such an approximation, Eq. (131) becomes

$$H(s,t) = \frac{1}{t} \left(1 - \frac{a}{\omega_0 \tau_0} \sin \omega_0 t\right) \int_0^\infty e^{-s \frac{t}{t_0}} \left[1 - \frac{a}{\omega_0 \tau_0} \sin \omega_0 (\lambda - t)\right] d\lambda \quad (132)$$

or

$$H(s,t) = \frac{1}{t_0} \left(1 + a \cos \omega_0 t\right) \left(1 - \frac{a}{\omega_0 \tau_0} \sin \omega_0 t\right) \times$$

$$\left[\frac{1}{s + \frac{1}{t_0}} + \frac{\left(\frac{a}{\omega_0 \tau_0}\right)(s + \frac{1}{t_0}) \cos \omega_0 t}{(s + \frac{1}{t_0})^2 + \omega_0^2} - \frac{\left(\frac{a}{\tau_0}\right) \sin \omega_0 t}{(s + \frac{1}{t_0})^2 + \omega_0^2}\right]. \quad (133)$$

Two examples of practical circuits having the same configuration as Fig. 6 are the capacitance microphone and the capacitance mechanical modulator. In both cases, the input $u(t)$ is either a fixed voltage or one whose angular frequency is small compared to $\omega_0$. From Eq. (133), those terms in $v(t)$ which have the fundamental frequency, $\omega_0$, as well as the distorting terms which have frequencies other than $\omega_0$ may be obtained, and the design of the system thus may be carried out completely by considering the system function alone.

C. A System Containing an Amplifier with Periodically Varying Gain

In Fig. 7 a system is shown consisting of a fixed RC integrating network followed by an amplifier having a periodically varying gain, $K(t)$, and which in turn is followed by a fixed system having a transfer
\[ K(t) = K_0 (1 + a \cos \omega_0 t) \]

\[ T_i = R C \]

Fig. 7. Block Diagram of a System Containing an Amplifier with Periodically Varying Gain.
The problem is to find the system function for the composite system, and the method used will illustrate the use of the combinatorial procedure developed in part F of the preceding section.

The system function of that part of the system consisting of the RC network and the amplifier will be designated as $H_1(s, t)$, and by inspection it is

$$H_1(s, t) = \frac{K(t)}{1 + T_1 s}$$  \hspace{1cm} (135)$$

where $T_1 = RC$. Also the system differential equation of the second part of the system may be obtained from the transfer function as

$$\frac{d^2 u}{dt^2} + \frac{1}{T_2} \frac{du}{dt} = k_2 u.$$  \hspace{1cm} (136)$$

A set of linearly independent solutions to the homogeneous equation

$$\frac{d^2 u}{dt^2} + \frac{1}{T_2} \frac{du}{dt} = 0$$  \hspace{1cm} (137)$$

may now be obtained, and of the infinite number possible, let it be the set

$$U_1 = \tau_2,$$

$$U_2 = e^{-t/\tau_2}.$$  \hspace{1cm} (138)$$
Therefore the Wronskian of the solutions \( v_1 \) and \( v_2 \) is

\[
\Delta(t) = \begin{vmatrix} T_2 & e^{-t/T_2} \\ -1 & e^{-t/T_2} \end{vmatrix} = -e^{-t/T_2}.
\] (139)

Also the determinant \( \Delta_1(t, \tau) \) may be calculated as

\[
\Delta_1(t, \tau) = \begin{vmatrix} T_2 & e^{-\tau/T_2} \\ T_2 & e^{-t/T_2} \end{vmatrix} = T_2(e^{-\tau/T_2} - e^{-t/T_2}).
\] (140)

The function \( G_1(t, \tau) \) is then

\[
G_1(t, \tau) = \frac{T_2(e^{-\tau/T_2} - e^{-t/T_2})}{-e^{-\tau/T_2}}.
\] (141)

or

\[
G_1(t, \tau) = T_2(1 - e^{-t-\tau/T_2}).
\] (142)

The integral form of the function \( H_2(s, t) \) may now be written as

\[
H_2(s, t) = e^{-st} \int_{-\infty}^{t} \frac{e^{s\tau}}{T_2(1 - e^{-t-\tau/T_2})} d\tau.
\] (143)

and from Eq. (73) the overall system function, \( H(s, t) \) is

\[
H(s, t) = H_2(p + s, t)H_1(s, t) = e^{-st} \int_{-\infty}^{t} \frac{e^{s\tau}}{T_2(1 - e^{-t-\tau/T_2})} e^{s\tau} \frac{K_2(\tau)}{1 + T_2} d\tau.
\] (144)
or

\[ H(s,t) = \frac{k_2 T_2}{1 + \pi s} e^{st} \left[ e^{\frac{s}{2T_2}} \frac{e^{(s+\frac{\pi}{2})t} - e^{\frac{s}{2T_2}}}{K(\tau)} - \frac{e^{(s+\frac{\pi}{2})t}}{K(\tau)} \right] d\tau. \] (145)

Now, if \( K(t) = K_0(1 + a \cos(\omega_0 t)) \), then

\[ H(s,t) = \frac{k_0 k_1 T_3}{1 + \pi s} e^{st} \left[ e^{\frac{s}{2T_2}} \int_{-\infty}^{t} (1 + a \cos \omega_0 \tau) d\tau - e^{\frac{s}{2T_2}} \int_{-\infty}^{t} \left( e^{(s+\frac{\pi}{2})\tau} (1 + a \cos \omega_0 \tau) d\tau \right) \right]. \] (146)

Upon integrating

\[ H(s,t) = \frac{k_0 k_1 T_3}{1 + \pi s} \left[ \frac{1}{s} + \frac{a[s \cos \omega_0 t + \omega_0 \sin \omega_0 t]}{s^2 + \omega_0^2} \right] - \frac{1}{s + \frac{\pi}{T_2}} - \frac{a[(s+\frac{\pi}{2}) \cos \omega_0 t + \omega_0 \sin \omega_0 t]}{(s + \frac{\pi}{2})^2 + \omega_0^2} \right] \] (147)

or

\[ H(s,t) = k_0 k_1 T_3 \left\{ \frac{1}{s(1+\pi s)(1+\pi^2 s)} + \frac{(\frac{\pi}{2})[s^2 + \frac{s}{T_2} - \omega_0^2]}{(1+\pi s)(s^2 + \omega_0^2)[(s+\frac{\pi}{2})^2 + \omega_0^2]} \right\} \left[ \frac{2s + \frac{\pi}{2}}{(1+\pi s)(s^2 + \omega_0^2)[(s+\frac{\pi}{2})^2 + \omega_0^2]} \right] \] (148)
H(s, t) can be seen to have all its poles located in the left half plane except for a pair on the imaginary axis. Because of these imaginary poles, H(s, t) is only conditionally stable, and any steady input will result in a sustained oscillation. A system similar to this in practice is the so-called chopper amplifier which is used extensively to provide driftless d-c amplifiers.

D. A Closed Loop Variable System

Figure 8 shows in block diagram form a closed loop system consisting of an amplifier having a periodically varying gain which in turn excites a fixed linear system having a transfer function

\[ G_2(s) = \frac{K_L T_L}{s(1 + T_L s)} \]  \hspace{1cm} (149)

This latter transfer function might represent a servo motor driving an inertia and friction load. Direct feedback is employed so that the output, v(t), is subtracted from the input u(t) to form the servo error signal, x(t).

In finding the overall system function, H(s, t), it will be recalled from Eq. (87) that \( H^{-1}(s, t) \), the inverse system function was obtained by

\[ H^{-1}(s, t) = H_1^{-1}(s, t) + H_2(s, t) \]  \hspace{1cm} (87)

where \( H_1^{-1}(s, t) \) is the inverse function of the forward system while \( H_2(s, t) \) is the system function of the feedback system. Since in this
Fig. 8. Block Diagram of a Closed Loop Variable System.

\[ K(t) = K_0 (1 + a \cos \omega_0 t) \]
example direct feedback is used, $H_2(s,t) = 1$, and

$$H_i^{-1}(s,t) = H_i^{-1}(s,t) + 1.$$  \hspace{1cm} (150)

The system differential equation for the forward system may be written by inspection as

$$\left( p^2 + \frac{1}{T} p \right) \mathcal{U} = K_k K(t) \mathcal{X}(t),$$  \hspace{1cm} (151)

Remembering that a physical interpretation of the inverse system function is as the input necessary to bring about an output $e^st$, divided by $e^st$, the inverse forward function may be obtained from

$$\left( p^2 + \frac{1}{T} p \right) e^{st} = K_k K(t) \left[ H_i^{-1}(s,t) e^{st} \right]$$  \hspace{1cm} (152)

or

$$H_i^{-1}(s,t) = \frac{S(s + \frac{1}{T})}{K_k K(t)}.$$  \hspace{1cm} (153)

Note that $H_i^{-1}(s,t)$ could also have been obtained by finding $H_i(s,t)$ and then finding the inverse by means of the differential equation

$$H_i(p + s, t) H_i^{-1}(s, t) = 1,$$  \hspace{1cm} (154)

but that method is considerably more complex than the method used here.

The overall inverse system function then is

$$H^{-1}(s, t) = \frac{S(s + \frac{1}{T}) + K_k K(t)}{K_k K(t)}.$$  \hspace{1cm} (155)
and in order to find $H(s,t)$ it is necessary to solve the differential equation

$$H^{-1}(P + s, t) H(s, t) = 1.$$  \(\text{(88)}\)

So that

$$\left[ (P + s)(P + s + \frac{1}{4}) + K_2 K(t) \right] H(s, t) = K_2 K(t)$$  \(\text{(156)}\)

or

$$\left[ P^2 + (2s + \frac{1}{2}) P + (S^2 + \frac{1}{2} s + K_2 K(t)) \right] H = K_2 K(t).$$  \(\text{(157)}\)

The above differential equation can be transformed into a Mathieu's equation by means of the transformation

$$H(s, t) = \gamma(s, t) e^{-\frac{T}{2}(2s + \frac{1}{2}) t}$$  \(\text{(158)}\)

whereby Eq. (157) becomes

$$\frac{d^2 y}{dt^2} + \left( -\frac{1}{4} \tau_1^2 + K_2 K(t) \right) y = K_2 K(t) e^{\frac{T}{2}(2s + \frac{1}{2}) t}$$  \(\text{(159)}\)

and since $K(t) = K_0(1 + a \cos \omega_0 t)$, then

$$\frac{d^2 y}{dt^2} + \left( K_2 K_0 - \frac{1}{4} \tau_1^2 + a K_2 K_0 \cos \omega_0 t \right) y = K_2 K_0 e^{\frac{T}{2}(2s + \frac{1}{2}) t} (1 + a \cos \omega_0 t).$$  \(\text{(160)}\)

One other substitution, $\omega_0 t = 2 \gamma$, makes Eq. (160)
\[
\frac{d^2 y}{d \zeta^2} + \left[ \frac{4(K_k K_0 - \frac{1}{4T_i^2})}{\omega_0^2} + \frac{4aK_k K_0}{\omega_0^2} \cos 2\zeta \right] y = \frac{4K_k K_0}{\omega_0^2} e^{(25 + \frac{1}{4}) \frac{3}{2}\omega_0} (1 + a \cos 2\zeta),
\]

which if

\[
\alpha = \frac{4}{\omega_0^2} (K_k K_0 - \frac{1}{4T_i^2}), \quad \beta = \frac{2aK_k K_0}{\omega_0^2}
\]

and

\[
f(\zeta) = \frac{4K_k K_0}{\omega_0^2} e^{(25 + \frac{1}{4}) \frac{3}{2}\omega_0} (1 + a \cos 2\zeta)
\]

becomes

\[
\frac{d^2 y}{d \zeta^2} + (\alpha + 2\beta \cos 2\zeta) y = f(\zeta).
\]

Eq. (164) is recognized as the standard form of the nonhomogeneous Mathieu equation (6). The parameters \(a\) and \(q\) in conjunction with a Mathieu stability chart will yield information on the stability of the solutions to Eq. (164), and hence on the stability of the system.
The original purpose in considering this problem was to attempt to extend some of the concepts and methods used in fixed system analysis to the time varying case, and in particular to periodically varying systems which employ feedback. This has been done to the extent that it has been shown that the general behavior of a time varying system depends upon the behavior of its system function in the $s$ plane.

The Green's function approach for finding the system function is useful in that the $s$ dependence is often easy to predict from the integral form of the system function. In the same manner, the concept and physical interpretation of the inverse system function is one that is very useful, especially in feedback systems.

For the periodic case, the stability of the system has been shown to depend upon the real parts of the characteristic exponents of the solutions to the homogeneous system equation. It has also been shown that whereas in the general time varying case the poles of the system function may be functions of time, in the periodic case the poles are fixed and occur in families having real parts all the same and whose imaginary parts differ by integral multiples of the angular frequency of the system.

Before the system function approach can be a complete tool for the analysis and design of time varying systems, much more development is
necessary in devising methods for determining the system function and its inverse, particularly for the periodic case. Modern computer techniques both analogue and digital can help in this respect, and once these difficulties are overcome, it is felt that many more uses of time varying systems will be found.
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VI. LITERATURE CITED


