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The approximate solution of linear differential equations by the use of functionals

Charles Joseph Thorne
Iowa State College

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UMI®
THE APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

BY THE USE OF FUNCTIONALS

by

Charles J. Thorne

A Thesis Submitted to the Graduate Faculty for the Degree of

DOCTOR OF PHILOSOPHY

Major Subject Applied Mathematics

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1941
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I. INTRODUCTION

In any discussion which attempts to view the entire field of differential equations as a whole, one is forced to admit that the solutions to but relatively few systems of differential equations can be found in closed form. Even when an attempt is made to narrow the field to single differential equations the number which can be directly solved is but a small fraction of the total number. Moreover many practical problems are in this unsolved class.

Fortunately, systems of differential equations for which no closed form of solution is known, can be solved by approximate methods. The approximation may involve a slight change in one or more differential equations in which the less significant parts are discarded. This involves weakening the system of differential equations to some extent. It may be possible that a solution can be obtained which becomes exact if an infinite number of terms are used. In such cases a finite number of terms is to represent the solution. In some cases a series of terms may be determined in such a manner as to represent an approximate solution to the
system.

Usually

\[ z = f(x_1, x_2, \ldots, x_m; a_1, a_2, \ldots, a_n) \]

with a number \( k \) of the a's to be determined for an approximation having the desired degree of accuracy. Often an approximate solution is assumed to be represented by a linear sum of expansion functions with constant coefficients to be determined, i.e. \( f(x, y) = \sum a_i \phi_i \). If the expansion functions form a complete set the approximate solution can sometimes be proven to become exact if an infinite number of expansion functions are used.

For the solution to a particular problem it is not necessary to know that an exact solution results if an infinite number of terms are used. It is important to obtain as simple an approximation as is possible to satisfactorily represent the exact solution.
II. REVIEW OF LITERATURE

From the Weierstrass theorem, we know that an arbitrary continuous function together with its derivatives can be approximated to any desired degree of accuracy by a polynomial. This theorem has been widely used by G. Frobenius to obtain solutions to differential equations.

The Ritz and Boussinesq methods will apply when the boundary conditions are homogeneous and the expansion functions can be chosen so that they individually satisfy the boundary conditions. The Ritz method involves minimizing a certain associated variational integral. The Boussinesq or "least square" method involves making the integral of the square of the difference between the exact solution and the approximation a minimum. M. Kravchuk has combined the Ritz and Boussinesq methods into a single more inclusive method. The Kravchuk method requires the integral of the difference between the exact solution and the approximation multiplied by various factors derived from the differential equation to be zero.

The Trefftz method can be used when expansion
functions can be found which satisfy the differential equation. The method then attempts to satisfy the boundary conditions using these functions.

Perturbation theory\textsuperscript{1} due to E. Schrödinger,\textsuperscript{7} and generalized perturbation theory developed by P. S. Epstein\textsuperscript{10} can be used when the solution of another similar problem with the same boundary conditions is known. The method of difference equations is sometimes used to obtain approximate solutions\textsuperscript{11,12}.

Recently G. Gross\textsuperscript{13,14}, working under the direction of J. Atanasoff, has developed a general method for obtaining approximate solutions under arbitrary linear boundary conditions. The author\textsuperscript{15} has applied this method to obtain a solution for the problem in plate theory of a square, clamped plate, with a central point load. We will here obtain a solution to the plate problem of a corner loaded, infinite plate on an elastic foundation, by the functional method.

Several investigators have obtained solutions for plates on elastic foundations. Hertz\textsuperscript{16}, in obtaining a general solution for the problem of an infinite plate on an elastic foundation, first used the assumption of foundation pressure being proportional to deflection, which has been included in almost all analyses since then. Happel\textsuperscript{17} obtained a solution by an infinite series each term of which approximately satisfied the boundary
conditions at the edge of the plate. By minimizing the potential energy he obtained values for the constants in his infinite series.

Westergaard\textsuperscript{15,19,20} presented in three papers approximate formulas for the maximum stresses and corrections to them. Murphy\textsuperscript{21} used the same method as Happel but considered the case when only part of the plate is supported by the foundation. Lightburn\textsuperscript{22} used the brittle material method to obtain some maximum stresses. Several experimental investigations have been carried out on earth subgrades by the Illinois Highway Department\textsuperscript{23}, United States Bureau of Public Roads\textsuperscript{24,25}, Iowa Engineering Experiment Station\textsuperscript{26}. 
III. INVESTIGATION

A. Functional Method.

1. General theory.

The general method as first enunciated by Gross is as follows: We desire a solution of the linear differential equation

\[ L(w) = f(x_1, x_2, \ldots, x_n) \]

and the \( b \) linear boundary conditions

\[ B_i(w) = 0 \quad \text{when} \quad \lambda_i = 0 \quad (i = 1, 2, \ldots, b) \]

where the boundary equations are

\[ \lambda_i = \lambda_i(x_1, x_2, \ldots, x_n) = 0 \]

The given differential operators

\[ L = L(x_1, x_2, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial^n}{\partial x_n}) \quad \text{and} \quad B_i = B_i(x_1, x_2, \ldots, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial^n}{\partial x_n}) \]

are linear.

It is assumed that \( w \) can be represented as a linear sum of expansion functions,

\[ w = \sum a_i \phi_i \]

where the \( a_i \)'s are constants to be determined and the
\( \phi \)'s are known expansion functions. A subscript, repeated in a term, has the usual meaning that the term is to be summed for all values of the subscript, e.g.

\[
\alpha_i \phi_i = \sum_{i=0}^{\infty} \alpha_i \phi_i
\]

unless the contrary is expressly stated. If the \( \phi \)'s constitute a complete set of functions, it is known that a wide range of functions can be expressed by such a sum. It is assumed that a finite number of terms of (5) will be an approximation \( w^* \) to the solution \( w \).

\[
(7) \quad w \cong w^* = \sum_{m=0}^{\kappa} a_m \phi_m
\]

Substitution of the finite expansion (7) for \( w \) in equations (1) and (2) gives

\[
\sum_{m=0}^{\kappa} (a_m \phi_m) \cong f;
\]

or

\[
(8) \quad a_m \phi_m \cong f;
\]

and

\[
(9) \quad a_m \phi(q) - q \cong 0 \quad \text{on the boundary},
\]

since the linear operators commute the \( a_m \)'s.

Suppose we now choose two functional families \( F_j \) and \( G_j \) which are linear. The \( F \)'s are defined in the region \( S \) of the plate; the \( G \)'s on \( s \), the boundary of the plate. An approximation defined by the functional families \( F_j \) and \( G_j \) and the functions of expansion \( \phi_m \),
is determined when we require that the coefficients, $a_m^k$, satisfy the equations:

\begin{align}
\sum_j \left[ a_m^k L_j(f_m) - f \right] &= 0 \quad (j = 0, 1, 2, \ldots, K-r), \\
\sum_j \left[ a_m^k b_j(f_m) - g \right] &= 0 \quad (j = K-r, \ldots, K).
\end{align}

We have thus reduced the problem of finding an approximate solution of the differential equation to the problem of finding a solution of an arbitrary number of linear equations in the $a_m^k$'s. The number of equations to be solved is arbitrary because it depends upon the number of terms in the assumed approximation function. The resultant linear equations in the $a_m^k$'s are then solved for the $a_m^k$'s and an approximation for $w$ results.

Thus a given set of functionals and functions define an approximation to the solution of the problem. The properties of these approximations, such as their limits of error, have not been entirely determined, partly because of the complete generality of the method.

The choice of the equations for the determination of the $a_m^k$'s is usually guided by the relative degrees of approximation desired for the differential equation (1) and the boundary conditions (2). In general, different choices of equations for the determination
We suppose we choose to apply the Ritz method.

and we have the same equations as are given by the

\[ \int_0^1 \phi \psi \, dx = \int_0^1 \phi (\psi) \, dx \]

(12)

to (8). The results

\[ \int_0^1 \psi \phi \, dx = \int_0^1 \psi \phi \, dx \]

(13)

different-galvanization approximation of the function.

depend on these conditions. Suppose we choose to satisfy the

tions so that each (and hence the sum) satisfies

homoegenous we may be able to choose expansion func-

e. Ritz method. If the boundary conditions are

particular example of the functional method.

methods for determining approximations to

It is easily seen that most of the standard

5. Relation to other methods.

6. Fourier series.

the same such approximations may be called Fourier's, e.

en for different approximations. If the e.'s remain

solved. In general, the resultant e.'s will be differ-

usually different equations in the e.'s are to be

end (11) with more terms in \( \phi \) and hence more, and

for an approximation with more terms, equations (10)

of the e.'s leads to different approximations to w.
satisfy the differential equation by employing the functional family

\[(14) \quad F_j[ \quad ] = \int \mathcal{L}(\phi_m)[ \quad ] dt.\]

There results

\[(15) \quad a_m^2 \int [\mathcal{L}(\phi_m)]^2 dt = \int f \mathcal{L}(\phi_m) dt\]

which are the Boussinesq or "least square" approximation equations.

c. Trefftz method. If our problem is of such a nature that the boundary conditions are \(w = f(s)\) on I and \(\frac{\partial w}{\partial y} = g(s)\) on II, we may be able to choose the expansion functions such that each satisfies \(\frac{\partial w}{\partial y} = g(s)\) on II. If we apply the functional family

\[(16) \quad F_j[ \quad ] = \int \left[ \left( \frac{J \phi_j}{J \phi} \right) \right] ds\]

to the second boundary condition there results

\[(17) \quad a_m^2 \int \phi_m \frac{J \phi}{J \phi} ds = \int f(s) \frac{J \phi}{J \phi} ds\]

These are the same equations as were determined by Trefftz.

d. other applications. Gross has included many other well known mathematical methods and procedures as special cases of the functional method. This includes:

1. Perturbation theory.
2. Taylor's series expansion.
3. Expansion in orthogonal functions.


5. Method of Kravchuk.

The author has applied the method to the problem of obtaining the solution to a clamped, square, plate with a central point load. The solution obtained was entirely satisfactory.

B. Corner Loaded, Infinite Plate Supported on an Elastic Foundation

1. The problem.

We wish to obtain the deflection of a corner loaded, infinite plate resting on an elastic foundation. The foundation is assumed to be of a type for which Hooke's law will satisfactorily represent the reaction of the foundation.

To solve this problem it is necessary to obtain an expression for the deflection \( w \) which will satisfy the differential equations and the boundary conditions which have been set up in thin plate theory. In rectangular coordinates, the differential equation is

\[
(18) \quad N \nabla^4 w = P - K w,
\]
or

\[ \nabla^4 W = -\frac{k}{N} W, \]

where

- \( w \) = deflection of the plate,
- \( N = \frac{E t}{1 - \nu^2} \)
- \( \nabla^2 = \frac{\partial^2}{\partial x^2} + 2\nu \frac{\partial^2}{\partial y^2} \)
- \( E \) = Young's modulus for material in plate,
- \( I \) = moment of inertia of unit area of cross section,
- \( \nu \) = Poisson's ratio for material in plate,
- \( k \) = Hooke's law constant for the foundation.

Here \( -kw \) represents the reaction of the foundation.

The load \( P \) is zero everywhere except at \((0,0)\).

If we choose our coordinate system so that the plate occupies the first quadrant, the boundary conditions are:

\[ M_x = -N \left[ \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right]_{y=0} = 0, \]

\[ M_y = -N \left[ \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right]_{x=0} = 0, \]

\[ R_x = -N \left[ \frac{\partial^2 W}{\partial y^2} + (1-\nu) \frac{\partial^2 W}{\partial x \partial y} \right]_{x=0} = 0, \]

\[ R_y = -N \left[ \frac{\partial^2 W}{\partial x^2} + (1-\nu) \frac{\partial^2 W}{\partial x \partial y} \right]_{y=0} = 0, \]

\[ P_c = 2(M_{xy})_c = -2N(1-\nu) \frac{\partial^2 W}{\partial x \partial y} \bigg|_{x=0, y=0} = P. \]
The foregoing equations represent respectively the moments which are zero at the edges $x = 0$ and $y = 0$, the reactions which are zero at the same two edges, and the corner load which is the load $P$ itself.


It is now necessary to decide upon the form of the expansion to be used to represent the deflection $w$. Consider the differential equation (19). For $e^{-\lambda(x+y)}$ to be a solution of this equation, we are led to the condition that

$$\Delta N \lambda^4 = -K.$$ 

Hence,

$$\lambda = \sqrt[4]{\frac{K}{4N}} (1 \pm i), \quad -\sqrt[4]{\frac{K}{4N}} (1 \pm i)$$

The second pair of values is excluded since the deflection must vanish at infinity. We are led to choose approximate solutions of the form

$$e^{-\lambda (x+y)} \quad e^{\pm i(x+y)}$$

The second factor is the sinusoidal periodic function of modulus one. We replace this factor by a polynomial, $U(x, y)$, retaining the characteristic exponential factor which is the essential feature of the solution towards infinity. Therefore,
\[(27) \quad W = \mathcal{E}^{-\frac{\vec{x} \cdot \vec{y}}{\lambda^2}} U(\vec{x}, \vec{y}), \quad \lambda = \sqrt{\frac{\kappa}{4N}} \]

where \(\vec{x} = \lambda x, \vec{y} = \lambda y\) and \(U(\vec{x}, \vec{y})\) is a polynomial of any desired number of terms with a corresponding number of arbitrary constants.

Because of the symmetry conditions and since

\[(28) \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2} = \lambda \frac{\partial^2}{\partial \vec{x}^2}, \quad \frac{\partial^2}{\partial y^2} = \lambda \frac{\partial^2}{\partial \vec{y}^2}, \]

we see that equations (20) and (21) express the same analytical condition

\[(29) \quad e^{-\frac{\vec{x} \cdot \vec{y}}{\lambda^2}} \left[ \frac{\partial^2 U}{\partial \vec{x}^2} - 2 \frac{\partial^2 U}{\partial \vec{y}^2} + \mathcal{U}(1 + \mathcal{V}) + \mathcal{V} \frac{\partial^2 U}{\partial \vec{y}^2} - 2 \mathcal{V} \frac{\partial^2 U}{\partial \vec{y}^2} \right] \sim 0.\]

We also see that (22) and (23) similarly express the condition

\[(30) \quad e^{-\frac{\vec{x} \cdot \vec{y}}{\lambda^2}} \left[ \frac{\partial^2 U}{\partial \vec{x}^2} - 2 \frac{\partial^2 U}{\partial \vec{y}^2} + (5 - \mathcal{V}) \frac{\partial^2 U}{\partial \vec{y}^2} + \mathcal{U}(1 + \mathcal{V}) + (2 - \mathcal{V}) \frac{\partial^2 U}{\partial \vec{y}^2} \right] \sim 0.\]

From equation (24) we obtain

\[(31) \quad -2N(1 - \mathcal{V}) \left[ \frac{\partial^2 U}{\partial \vec{x}^2} - \frac{\partial^2 U}{\partial \vec{y}^2} + \mathcal{U} \right]_{\vec{x} = 0} \approx \frac{P}{\lambda^2}.\]

From the differential equation (19) there results

\[(32) \quad e^{-\frac{\vec{x} \cdot \vec{y}}{\lambda^2}} \left[ \frac{\partial^2 U}{\partial \vec{x}^2} + 2 \frac{\partial^2 U}{\partial \vec{y}^2} + \frac{\partial^2 U}{\partial \vec{y}^2} - 4 \left( \frac{\partial U}{\partial \vec{y}^2} + \frac{\partial^2 U}{\partial \vec{y}^2} \right) \right.\]

\[+ 8 \left( \frac{\partial^2 U}{\partial \vec{y}^2} + \frac{\partial^2 U}{\partial \vec{y}^2} + \frac{\partial^2 U}{\partial \vec{y}^2} - \frac{\partial U}{\partial \vec{y}^2} - \frac{\partial U}{\partial \vec{y}^2} \right)\]

\[-4 \left( \frac{\partial^2 U}{\partial \vec{x}^2} + \frac{\partial^2 U}{\partial \vec{y}^2} \right) + 20 \mathcal{U} \right] \approx 0.\]
Since \( w(x,y) = w(y,x) \), choose

\[
U(x,y) = A_1 + A_2 (\xi + \eta) + A_3 (\xi^2 + \eta^2) + A_4 \xi \eta + A_5 (\xi^3 + \eta^3)
+ A_6 (\xi^2 \eta + \xi \eta^2) + A_7 (\xi^4 + \eta^4) + A_8 (\xi^3 \eta + \xi \eta^3)
\]

\[
= A_9 \xi^4 \eta^2 + A_{10} (\xi^6 + \eta^6) + A_{11} (\xi^4 \eta + \xi \eta^4)
+ A_{12} (\xi^2 \eta^2 + \xi \eta^5) + A_{13} (\xi^4 \eta + \xi \eta^4)
+ A_{14} \xi^3 \eta^3.
\]

The boundary conditions become from equation (29)

\[
\mathcal{E'} \left\{ A_1 (\nu - 3) + A_2 [\nu - 3 \nu + (\nu - 3) \eta] + A_3 (\eta - 10 + 2 \nu
+ (1 + \nu) \eta^3] + A_4 [2 \xi \eta] + A_5 [6 \nu \eta - 6 \nu \eta^2
+ (1 + \nu) \eta^3] + A_6 [2 \xi - 2 \eta - 2 \eta^2] + A_7 [12 \nu \eta - 8 \nu \eta^3
+ (1 + \nu) \eta^5] + A_8 (-2 \xi \eta^3] + A_9 [2 \xi \eta] + A_{10} [2 \nu \eta \eta^3
- 10 \nu \eta^5 + (1 + \nu) \eta^6] + A_{11} [2 \xi \eta^3] + A_{12} [2 \xi \eta^3]
\right\} = 0,
\]

from equation (30)

\[
\mathcal{E'} \left\{ A_1 (\nu - 3) + A_2 [\nu - 3 \nu + (\nu - 3) \eta] + A_3 (\eta - 10 + 2 \nu
+ (1 + \nu) \eta^3] + A_4 [2 \xi \eta] + A_5 [6 \nu \eta - 6 \nu \eta^2
+ (1 + \nu) \eta^3] + A_6 [2 \xi - 2 \eta - 2 \eta^2] + A_7 [12 \nu \eta - 8 \nu \eta^3
+ (1 + \nu) \eta^5] + A_8 (-2 \xi \eta^3] + A_9 [2 \xi \eta] + A_{10} [2 \nu \eta \eta^3
- 10 \nu \eta^5 + (1 + \nu) \eta^6] + A_{11} [2 \xi \eta^3] + A_{12} [2 \xi \eta^3]
\right\} = 0.
\]
from equation (31)

\[ A_1 - 2A_2 + A_4 = -\frac{P}{\varepsilon N(1-\nu)\lambda^2} \]

or

\[ A_1 - 2A_2 + A_4 = -\Theta \nu \]

In calculations

\[ \frac{P}{\varepsilon N(1-\nu)\lambda^2} \equiv \Theta \nu \]

The differential equation (32) becomes

\[ \Theta^{(u+\bar{u})} \left\{ 24A_{17} + 120A_{16} \bar{u} + 24A_{16} \bar{y} + 360A_{13} \bar{u} \right\} 
\]

\[ + 120A_{14} \bar{u} \bar{y} + A_{16}(24 \bar{y}^2) + 8A_{24} + 2A_{6}(\bar{u} + \bar{y}) 
\]

\[ + 48A_{16}(\bar{u}^2 + \bar{y}^2) + 72A_{15} \bar{u} \bar{y} + 24A_{7} + 120A_{10} \bar{y} 
\]

\[ + 24A_{11} \bar{u} + 360A_{13} \bar{y} + 120A_{14} \bar{u} \bar{y} + A_{16}(24 \bar{u}^2) 
\]

\[ - 24A_{5} - 96A_{7} \bar{u} - 24A_{8} \bar{y} - 240A_{10} \bar{u} \bar{y} - 96A_{13} \bar{u} \bar{y} \]
-24A_{12} \tilde{y}^2 - 480A_{13} \tilde{z}^3 - 240A_{14} \tilde{z}^2 \tilde{y} - 96A_{15} \tilde{z}
-24A_{16} \tilde{y}^3 - 24A_5 - 96A_7 \tilde{y} - 24A_8 \tilde{z} - 240A_{10} \tilde{y}^2
-96A_{11} \tilde{y} \tilde{z} - 24A_{12} \tilde{z}^2 - 480A_{13} \tilde{y}^3 - 240A_{14} \tilde{z} \tilde{y}^2
-96A_{15} \tilde{y}^2 \tilde{z} - 24A_{16} \tilde{z}^3 - 8A_6 - 24A_8 \tilde{z} - 16A_9 \tilde{y}
-48A_{11} \tilde{y}^2 - 24A_{13} (\tilde{z}^2 + 2 \tilde{y} \tilde{z}) - 80A_{14} \tilde{y}^3

-32A_{13} (3 \tilde{y}^2 + \tilde{z}^3) - 72A_{16} \tilde{y}^3 \tilde{z} + 16A_3 + 48A_5 \tilde{y}
-32A_{16} (\tilde{y}^3 + 3 \tilde{z}^3) - 72A_{16} \tilde{y} \tilde{z}^2 - 8A_6 - 24A_8 \tilde{z}
-16A_9 \tilde{y} - 48A_{11} \tilde{y}^2 - 24A_{12} (2 \tilde{y} \tilde{z} + \tilde{z}^2) - 80A_{14} \tilde{z}^3
+16A_6 \tilde{y}^2 + 96A_7 \tilde{z}^2 + 48A_8 \tilde{y} \tilde{z} + 16A_9 \tilde{y}^2 + 160A_{10} \tilde{z}^3
+96A_{11} \tilde{z} \tilde{y}^2 + 16A_{12} (\tilde{z}^3 + 3 \tilde{z} \tilde{y}^2) + 240A_{13} \tilde{z}^4
+160A_{14} \tilde{z}^2 \tilde{y} + 16A_{15} (\tilde{z}^4 + 6 \tilde{z} \tilde{y} \tilde{z}^2) + 48A_{16} \tilde{z}^3 \tilde{y}
+16A_3 + 48A_{16} \tilde{z} + 16A_6 \tilde{z} + 96A_7 \tilde{y}^2 + 48A_8 \tilde{z} \tilde{y}
+16A_9 \tilde{z}^2 + 160A_{10} \tilde{z}^3 + 96A_{11} \tilde{z} \tilde{y}^2 + 16A_{12} (3 \tilde{z}^3 + \tilde{z}^3)
+240A_{13} \tilde{z}^4 + 160A_{14} \tilde{z} \tilde{y} \tilde{z} + 16A_{15} (6 \tilde{z} \tilde{y} \tilde{z}^2 + \tilde{z}^4) + 48A_{16} \tilde{z}^2 \tilde{y}
+8A_4 + 16A_6 (\tilde{y} + \tilde{y}^3) + 24A_8 (\tilde{z}^2 + \tilde{y}^3) + 32A_9 \tilde{z} \tilde{y}
+32A_9 (\tilde{y}^3 + \tilde{z}^3) + 48A_{12} (\tilde{y}^3 + \tilde{z} \tilde{y}^2) + 90A_{14} (\tilde{z}^3 + \tilde{y}^3)
+64A_{16} (\tilde{z}^3 + \tilde{z} \tilde{y}^2) + 72A_{16} \tilde{z} \tilde{y}^2 + 8A_2 - 16A_3 \tilde{y}
-8A_4 \tilde{y} - 24A_6 \tilde{z}^2 - 8A_4 (2 \tilde{z} \tilde{y} + \tilde{y}^3) - 32A_7 \tilde{z}^3 - 8A_8 (3 \tilde{z}^3 + \tilde{z} \tilde{y}^2)
-16A_9 \tilde{z} \tilde{y}^2 - 40A_{10} \tilde{y}^3 - 8A_{11} \tilde{y} (\tilde{y}^3 + 4 \tilde{z} \tilde{y} \tilde{z})
-8A_{13} (2 \tilde{z}^3 + 3 \tilde{z} \tilde{y} \tilde{z}) - 48A_{13} \tilde{z}^2 - 8A_{14} (\tilde{y} \tilde{y}^3 + 5 \tilde{z}^3)
-16A_{15} (\tilde{y}^3 + 2 \tilde{z} \tilde{y} \tilde{z}) - 24A_{16} \tilde{z}^2 \tilde{y}^2 - 8A_2 - 16A_3 \tilde{y}
-8A_4 \tilde{y} - 24A_6 \tilde{z}^2 - 8A_4 (2 \tilde{z} \tilde{y} + \tilde{y}^3) - 32A_7 \tilde{z}^3 - 8A_8 (3 \tilde{z}^3 + \tilde{z} \tilde{y}^2)
-8A_8 (3 \tilde{z} \tilde{y}^2 + \tilde{z}^3 - 16A_9 \tilde{z} \tilde{y}^2 - 40A_{10} \tilde{y} \tilde{z}^2 - 8A_{11} \tilde{y} (\tilde{y}^3 + 2 \tilde{z} \tilde{y} \tilde{z})
-8A_{12} (3 \tilde{z} \tilde{y}^2 + \tilde{z}^3) - 98A_{13} \tilde{y}^2 \tilde{z} \tilde{y} \tilde{z} - 8A_{16} \tilde{z}^2 (5 \tilde{z} \tilde{y}^2 + \tilde{z}^3)
a. integral type of functional. Because of the infinite character of the plate and from the condition that the area integral of the difference between the exact value and the approximation should be zero, we are led to apply the functionals

\[
\int_0^\infty \bar{y}^\beta \left[ \ldots \right] d\bar{y}
\]

to equations (34) and (35).

Since

\[
\int_0^\infty \bar{y}^m e^{-\bar{y}} d\bar{y} = m!
\]

we obtain from equation (34)

\[
A_1 (1+\nu) + A_2 [(\beta-1)(1+\nu)] + A_3 [2 + 2\nu - 4\nu(\beta+1)]
\]

\[
+ (1+\nu)(\beta+1)(\beta+2)] + A_4 [-2(\beta+1)] + A_5 [6(\beta+1)\nu]
\]

\[- 6(\beta+1)(\beta+2)\nu + (1+\nu) \left(\frac{\beta+2}{\beta!}\right) + A_6 [2(\beta+1) - 2(\beta+1)(\beta+2)]
\]

\+[A_7 \left(\frac{(\beta+2)!}{\beta!} - 8 \frac{(\beta+3)!}{\beta!} \right) + (1+\nu) \left(\frac{(\beta+4)!}{\beta!}\right) + A_8 \left[-2 \frac{(\beta+3)!}{\beta!}\right]
\]

\+[A_9 \left(\frac{(\beta+2)!}{\beta!}\right) + A_{10} \left[20\nu \frac{(\beta+3)!}{\beta!} - 10\nu \frac{(\beta+4)!}{\beta!} + (1+\nu) \left(\frac{(\beta+5)!}{\beta!}\right)\right]
\]

\+[A_{11} \left[-2 \frac{(\beta+4)!}{\beta!}\right] + A_{12} \left[2 \frac{(\beta+3)!}{\beta!}\right] + A_{13} \left[30\nu \frac{(\beta+4)!}{\beta!}\right]]
\]
\[-12 \sqrt{\frac{(3+5)}{\beta}} + (1+y) \frac{(\beta+6)}{\beta} \bigg] + A_{14} \left[ -2 \frac{(\beta+5)}{\beta} \right] + A_{15} \left[ \frac{2(\beta+4)}{\beta} \right] = 0,\]

and from equation (35)

\[
A_{1} (\nu-3) + A_{2} \left[ 9-3\nu+(\nu-3)(\beta+1) \right] + A_{3} \left[ -10 + 2\nu \right]
\]

\[
+ 4(2-\nu)(\beta+1) + (\nu-3)(\beta+1)(\beta+2) + A_{4} \left[ 2(\nu-2) + \nu(\beta+1) \right]
\]

\[
+ A_{6} \left[ \frac{6+6(\nu-2)(\beta+2) + (\nu-3)(\beta+3)}{\beta} \bigg] + A_{7} \left[ 12(\nu-2)(\beta+2) + \beta (2-\nu) \frac{(\beta+3)}{\beta} \bigg] + A_{8} \left[ 2(\nu-2)(\beta+1) + (\nu-3)(\beta+4) \bigg] + A_{9} \left[ 6(\beta+2) \bigg] + A_{10} \left[ 20(\nu-2)(\beta+3) + 10(2-\nu)(\beta+4) \bigg]
\]

\[
+ (\nu-3) \frac{(\beta+5)}{\beta} \bigg] + A_{11} \left[ 12(\nu-2) \frac{(\beta+2)}{\beta} + \beta (\nu-2) \frac{(\beta+3)}{\beta} \bigg]
\]

\[
+ (\nu-3) \frac{(\beta+6)}{\beta} \bigg] + A_{12} \left[ 6 \frac{(\beta+2)}{\beta} - 6 \frac{(\beta+3)}{\beta} \bigg]
\]

\[
+ A_{13} \left[ 30(\nu-2) \frac{(\beta+4)}{\beta} + 12(2-\nu) \frac{(\beta+5)}{\beta} + (\nu-3) \frac{(\beta+6)}{\beta} \bigg]
\]

\[
+ A_{14} \left[ 20(\nu-2) \frac{(\beta+3)}{\beta} + 10(2-\nu) \frac{(\beta+4)}{\beta} + (\nu-3) \frac{(\beta+5)}{\beta} \bigg]
\]

\[
+ A_{15} \left[ -6 \frac{(\beta+4)}{\beta} \bigg] + A_{16} \left[ 6 \frac{(\beta+3)}{\beta} \bigg] = 0
\]

For the differential equation (39) a similar functional family is used

\[
(43) \quad \int_{0}^{\infty} \left[ \gamma \right] \chi \gamma^\beta \beta d\gamma.
\]
When this is applied to (39) there results

\[ A_1(20) + A_2[20(2\alpha + \beta + 2) - 16] + A_3[32 \]

\[ -16(\alpha + \beta + 2) + 20 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) + A_4[8 \]

\[ -8(\alpha + \beta + 2) + 20(\alpha + 1)(\beta + 1)] + A_5[-98 + 4(\alpha + \beta + 2) \]

\[ -24 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \bigg] + 20 \left[ \frac{(\alpha + 3)!}{\alpha !} + \frac{(\beta + 3)!}{\beta !} \right] \bigg] 

+ A_6[-16 + 32(\alpha + \beta + 2) - 8 \left( \frac{(\alpha + 2)!}{\alpha !} + 4(\alpha + 1)(\beta + 1) + \frac{(\beta + 2)!}{\beta !} \right) \]

\[ + 20(\alpha + 1)(\beta + 1) \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \bigg] + A_7[98 \]

\[ -96(\alpha + \beta + 2) + 96 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \bigg] - 32 \left( \frac{(\alpha + 3)!}{\alpha !} + \frac{(\beta + 3)!}{\beta !} \right) \bigg] 

+ A_8[48(\alpha + \beta + 12) \]

\[ + 96(\alpha + 1)(\beta + 1) + 24 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \bigg] - 8 \left( \frac{(\alpha + 3)!}{\alpha !} + \frac{(\beta + 3)!}{\beta !} \right) \bigg] 

\[ + 3(\alpha + 2)! \left( \frac{1}{\alpha + 1} + 3(\alpha + 1) \left( \frac{(\beta + 2)!}{\beta !} + \frac{(\beta + 3)!}{\beta !} \right) \right) \]

\[ + 20(\alpha + 1)(\beta + 1) \left( \frac{(\beta + 3)!}{(\alpha + 1)!} + \frac{(\alpha + 3)!}{(\alpha + 1)!} \right) \bigg] + A_9[8 \]

\[ -16(\alpha + \beta + 2) + 16 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) + 2(\alpha + 1)(\beta + 1) \]

\[ -16(\alpha + 1)(\beta + 1) \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \]

\[ + A_{10}[120(\alpha + \beta + 2) - 240 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 2)!}{\beta !} \right) \]

\[ + 160 \left( \frac{(\alpha + 3)!}{\alpha !} + \frac{(\beta + 3)!}{\beta !} \right) - 40 \left( \frac{(\alpha + 2)!}{\alpha !} + \frac{(\beta + 4)!}{\beta !} \right) \]
\[
+ 20 \left\{ \frac{(a+5)!}{a!} + \frac{(a+6)!}{(a+1)!} \right\} + A_{11} \left\{ 24(a+\beta + 2) \
- 48 \left\{ \frac{(a+2)!}{a!} + 4(a+1)(\beta+1) + \frac{(\beta+2)!}{\beta!} \right\} \right\} + 32 \left\{ \frac{(a+3)!}{\alpha!} + 3 \frac{(a+2)!}{(a+1)!} + 3 \frac{(a+1)(\beta+2)!}{\beta!} + \frac{(\beta+3)!}{\beta!} \right\} \\
- 8 \left\{ \frac{(a+4)!}{\alpha!} + 4 \frac{(a+3)!}{\alpha!} + 4 \frac{(a+1)(\beta+3)!}{\beta!} + \frac{(\beta+4)!}{\beta!} \right\} \\
+ 20 \left\{ \frac{(a+4)!}{\alpha!} + \frac{(a+4)!}{\alpha!} + \frac{(a+1)(\beta+4)!}{\beta!} \right\} + A_{12} \left\{ 24(a+\beta + 2) \
- 48 \left\{ \frac{(a+2)!}{\alpha!} + 2(a+1)(\beta+1) + \frac{(\beta+2)!}{\beta!} \right\} \right\} + 16 \left\{ \frac{(a+3)!}{\alpha!} + 6 \frac{(a+2)!}{(a+1)!} + \frac{(\beta+3)!}{\beta!} \right\} \\
- 16 \left\{ \frac{(a+3)!}{\alpha!} + \frac{(\beta+3)!}{\beta!} + \frac{(a+1)(\beta+3)!}{\beta!} \right\} \\
+ 20 \left\{ \frac{(a+3)!}{\alpha!} + \frac{(\beta+3)!}{\beta!} + \frac{(a+1)(\beta+3)!}{\beta!} \right\} \right\} + A_{13} \left\{ 360 \left\{ \frac{(a+2)!}{\alpha!} + \frac{(\beta+2)!}{\beta!} \right\} - 480 \left\{ \frac{(a+3)!}{\alpha!} + \frac{(\beta+3)!}{\beta!} \right\} \right\} \\
+ 240 \left\{ \frac{(a+4)!}{\alpha!} + \frac{(\beta+4)!}{\beta!} \right\} - 48 \left\{ \frac{(a+5)!}{\alpha!} + \frac{(\beta+5)!}{\beta!} \right\} \\
+ 20 \left\{ \frac{(a+6)!}{\alpha!} + \frac{(\beta+6)!}{\beta!} \right\} \right\} + A_{14} \left\{ 290(a+1)(\beta+1) \
- 80 \left\{ \frac{(a+3)!}{\alpha!} + \frac{(a+2)!}{(a+1)!} + \frac{(a+1)(\beta+2)!}{\beta!} + \frac{(\beta+3)!}{\beta!} \right\} \\
+ 40 \left\{ \frac{(a+4)!}{\alpha!} + \frac{(a+3)!}{\alpha!} + \frac{(\beta+3)!}{\beta!} + \frac{(a+1)(\beta+3)!}{\beta!} \right\} \right\} \\
+ 40 \left\{ \frac{(a+4)!}{\alpha!} + \frac{(a+3)!}{\alpha!} + \frac{(a+1)(\beta+3)!}{\beta!} + \frac{(\beta+3)!}{\beta!} \right\} \right\} \\
\right\} 
\]
equations are:

The values of $a$ and $b$ are used in the resultant

Choose small integer values instead of exact values.

Any positive value of $a$ or $b$ may be chosen in any

\[
\begin{align*}
\mathbf{0} &= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) + \\
&= \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] + \left( \left[ \frac{i \mathbf{e}}{(2 + \rho)} \mathbf{1} \cdot \mathbf{0} \right] \right) +
\end{align*}
\]
From equation (41) for

\[ \beta = 0, \]
\[ A_1(1+y) + A_2(1-y) + A_3(4) + A_4(-2) + A_5(6) \]
\[ + A_6(-2) + A_7(24) + A_8(-12) + A_9(4) + A_{10}(240) + A_{11}(-240) \]
\[ + A_{12}(12) + A_{13}(720) + A_{14}(-240) + A_{15}(48) = 0; \]
\[ (45) \]

\[ \beta = 1, \]
\[ A_1(1+y) + A_3(8) + A_4(-4) + A_5(24) + A_6(-8) + A_7(120) \]
\[ + A_8(-48) + A_9(12) + A_{10}(720) + A_{11}(-240) + A_{12}(96) \]
\[ + A_{13}(5040) + A_{14}(-1440) + A_{15}(240) = 0; \]
\[ (46) \]

\[ \beta = 2, \]
\[ A_1(1+y) + A_2(1+y) + A_3(14+2y) + A_4(-6) + A_5(60+6y) \]
\[ + A_6(-18) + A_7(360+24y) + A_8(-120) + A_9(24) + A_{10}(2520+120y) \]
\[ + A_{11}(-720) + A_{12}(120) + A_{13}(2016+720y) + A_{14}(-5040) + A_{15}(720) = 0; \]
\[ (47) \]

\[ \beta = 3, \]
\[ A_1(1+y) + A_2(2+2y) + A_3(22+6y) + A_4(-8) + A_5(120+24y) \]
\[ + A_6(-32) + A_7(840+120y) + A_8(-240) + A_9(90) + A_{10}(6720) \]
\[ + 720y) + A_{11}(-1680) + A_{12}(240) + A_{13}(5040+60480y) \]
\[ + A_{14}(-13440) + A_{15}(1680) = 0 \]
\[ (48) \]

From equation (42) for

\[ \beta = 0, \]
\[ A_1(\nu-3) + A_2(6-2\nu) + A_3(-8) + A_4(1+y) + A_5(-24) \]
\[ + A_6(24) + A_7(-12) + A_8(-120) + A_9(72) + A_{10}(-24) \]
\[ + A_{11}(-720) + A_{12}(360) + A_{13}(-1944) + A_{14}(36) = 0; \]
\[ (49) \]

\[ \beta = 1, \]
\[ A_1(\nu-3) + A_2(3-\nu) + A_3(-12) + A_4(-12) + A_5(-18) + A_6(12) \]
\[ (50) \]

\[ (50) \]
\[ A_1 (V - 3) + A_2 (V - 3) + A_3 (-38 + 6V) + A_4 (16 - 2V) + A_5 (-162 + 24V) + A_6 (98 - 6V) + A_7 (-1080 + 120V) + A_8 (932 - 24V) + A_9 (-120) + A_{10} (-8160 + 720V) + A_{11} (2760 - 120V) + A_{12} (-600) + A_{13} (-70560 + 5090V) + A_{14} (21600 - 720V) + A_{15} (-5090) + A_{16} (720) = 0 \]

From equation (44) for

\[ \alpha = 0, \beta = 0, \]
\[ A_1 (20) + A_2 (20) + A_3 (80) + A_4 (12) + A_5 (192) + A_6 (69) + A_7 (816) + A_8 (144) + A_9 (88) + A_{10} (4080) + A_{11} (624) + A_{12} (4,32) + A_{13} (24480) + A_{14} (3120) + A_{15} (829) + A_{16} (509) = 0. \]

\[ \alpha = 0, \beta = 1, \]
\[ A_1 (20) + A_2 (44) + A_3 (144) + A_4 (24) + A_5 (504) + A_6 (162) + A_7 (2448) + A_8 (480) + A_9 (232) + A_{10} (14280) + A_{11} (2280) + A_{12} (1464) + A_{13} (97920) + A_{14} (12960) + A_{15} (6912) + A_{16} (2160) = 0; \]

\[ \alpha = 1, \beta = 1, \]
\[ A_1 (20) + A_2 (64) + A_3 (208) + A_4 (56) + A_5 (816) + A_6 (368) + A_7 (4080) + A_8 (1440) + A_9 (600) + A_{10} (24480) + A_{11} (7200) + A_{12} (9704) + A_{13} (171360) + A_{14} (43200) + A_{15} (23520) + A_{16} (9216) = 0; \]
\[ a = 1, \beta = 2, \]
\[ A_1(20) + A_2(84) + A_3(312) + A_4(88) + A_5(1440) \]
\[ + A_6(698) + A_7(8208) + A_8(2928) + A_9(1176) \]
\[ + A_{10}(55320) + A_{11}(16440) + A_{12}(10488) + A_{13}(42984) \]
\[ + A_{14}(109440) + A_{15}(68320) + A_{16}(23040) = 0; \]

\[ a = 2, \beta = 2, \]
\[ A_1(20) + A_2(104) + A_3(416) + A_4(140) + A_5(2064) \]
\[ + A_6(136) + A_7(12336) + A_8(5664) + A_9(2312) + A_{10}(86160) \]
\[ + A_{11}(33936) + A_{12}(23088) + A_{13}(688320) + A_{14}(2373120) \]
\[ + A_{15}(1384932) + A_{16}(57672) = 0; \]

\[ a = 2, \beta = 3, \]
\[ A_1(20) + A_2(120) + A_3(66) + A_4(192) + A_5(3120) \]
\[ + A_6(1728) + A_7(20688) + A_8(9552) + A_9(3864) \]
\[ + A_{10}(158760) + A_{11}(62856) + A_{12}(42400) + A_{13}(1382400) \]
\[ + A_{14}(479040) + A_{15}(277248) + A_{16}(116488) = 0; \]

\[ a = 3, \beta = 3, \]
\[ A_1(20) + A_2(144) + A_3(704) + A_4(264) + A_5(4176) \]
\[ + A_6(2608) + A_7(29040) + A_8(15662) + A_9(6472) \]
\[ + A_{10}(231360) + A_{11}(108480) + A_{12}(73776) + A_{13}(2076480) \]
\[ + A_{14}(865920) + A_{15}(540980) + A_{16}(231552) = 0. \]

Equations (37), (45)-(59) are the sixteen equations in the sixteen coefficients which are to be used to solve for these coefficients.

b. derivative type of functional. Another type of functional is suggested by considering the system of differential equations (54), (55) and (39) as identities
The following set of thirty equations in the foregoing
increased number of terms due to \((69)\), there results the
resultant equations \((59)\), \((55)\) and \((56)\) with the
applying the fundamental families \((56)\) and \((61)\) to
\begin{align*}
(\delta f_x^+ z^2 + \delta f_z^+ \zeta) &+ \delta v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^2 v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^3 v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^4 v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^5 v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^6 v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^7 v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^8 v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^9 v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{10} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{11} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{12} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{13} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{14} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{15} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{16} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{17} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{18} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{19} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{20} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{21} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{22} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{23} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{24} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{25} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{26} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{27} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{28} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{29} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{30} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{31} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{32} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{33} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{34} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{35} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{36} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{37} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{38} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{39} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{40} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{41} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{42} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{43} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{44} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{45} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{46} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{47} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{48} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{49} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{50} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{51} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{52} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{53} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{54} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{55} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{56} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{57} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{58} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{59} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{60} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{61} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{62} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{63} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{64} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{65} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{66} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{67} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{68} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{69} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{70} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{71} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{72} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{73} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{74} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{75} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{76} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{77} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{78} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{79} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{80} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{81} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{82} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{83} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{84} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{85} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{86} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{87} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{88} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{89} v + \\
(\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{90} v &+ (\delta f_x^+ z^2 + \delta f_z^+ \zeta)^{91} v + \\
\end{align*}

Since

\[
(x' + \lambda)\Omega = (x' + \lambda)\Omega
\]

choose

\[
\Omega = \frac{5}{3}
\]

for the differential equation \((39)\)

and

\[
\Omega = \frac{5}{3}
\]

for conditions \((42)\) and \((44)\)

and \((46)\)

In the variables \(x\) and \(y\), corresponding functions are:
Applying (60) to the moment condition for

$\beta = 0$,

$$A_1 - 2A_2 + 2A_3 = 0;$$

$\beta = 1$,

$$(\nu + \nu)A_2 - 4VA_3 - 2A_4 + 6VA_5 + 2A_6 = 0;$$

$\beta = 2$,

$$(\nu + \nu)A_3 - 6VA_5 - 2A_6 + 12VA_7 + 2A_9 = 0;$$

$\beta = 3$,

$$(\nu + \nu)A_6 - 8VA_7 - 2A_8 + 20VA_{10} + 2A_{12} = 0;$$

$\beta = 4$,

$$(\nu + \nu)A_7 - 10VA_{10} - 2A_{11} + 36VA_{13} + 2A_{15} = 0;$$

$\beta = 5$,

$$(\nu + \nu)A_{10} - 12VA_{13} - 2A_{14} + 42VA_{17} + 2A_{19} = 0;$$

$\beta = 6$,

$$(\nu + \nu)A_{13} - 14VA_{17} - 2A_{18} + 56VA_{21} + 2A_{23} = 0;$$

Applying (60) to the reaction condition for

$\beta = 0$,

$$(\nu - 3)A_1 + (9 - 3\nu)A_2 + (2\nu - 10)A_3 + (2\nu - 4)A_4 + 6A_5 + (4 - 2\nu)A_6 = 0;$$

$\beta = 1$,

$$(\nu - 3)A_2 + (8 - 4\nu)A_3 + (6 - \nu)A_4 + (6\nu - 12)A_5 + (4\nu - 14)A_6 + (18 - 6\nu)A_8 = 0;$$

$\beta = 2$,

$$(\nu - 3)A_3 + (12 - 6\nu)A_5 + (6 - \nu)A_6 + (12\nu - 24)A_7 + (6\nu - 12)A_8 + 6A_9 + (2\nu - 12)A_{11} + 6A_{12} = 0;$$

$\beta = 3$,

$$(\nu - 3)A_6 + (16 - 8\nu)A_7 + (5 - \nu)A_8 + (20\nu - 40)A_{10} + (8\nu - 16)A_{11} - 6A_{12} + (30 - 20\nu)A_{14} + 6A_{16} = 0;$$
\[ \beta = 4, \quad (\nu - 3)A_9 + (2\nu - 10\nu)A_{10} + (5\nu)(A_{11}) + (30\nu - 60)A_{13} - 6A_{16} + (10\nu - 20)A_{14} + (60 - 30\nu)A_{18} + 6A_{20} = 0; \]

\[ \beta = 5, \quad (\nu - 3)A_9 + (2\nu - 12\nu)A_{13} + (5\nu)A_{14} + (42\nu - 84)A_{17} + (12\nu - 24)A_{18} - 6A_{16} + (84 - 42\nu)A_{22} + 6A_{24} = 0. \]

Applying (61) to the differential equation condition for \( \alpha = 0, \beta = 0, \)

\[ 5A_1 - 4A_2 + 8A_3 - 2A_4 - 12A_6 - 4A_7 + 12A_9 + 2A_{10} = 0; \]

\[ a = 1, \beta = 0, \]

\[ 5A_2 - 4A_3 + 2A_4 + 12A_6 - 4A_7 - 24A_9 - 12A_{11} - 4A_{13} + 30A_{14} + 6A_{15} + 6A_{17} = 0; \]

\[ a = 2, \beta = 0, \]

\[ 5A_3 - 6A_5 - 2A_6 + 2A_7 + 8A_9 - 12A_{11} - 60A_{16} - 12A_{11} + 90A_{13} + 17A_{15} = 0; \]

\[ a = 1, \beta = 1, \]

\[ 5A_4 - 8A_6 + 2A_7 + 8A_9 - 4A_{11} - 2A_{12} + 18A_{16} + 60A_{14} = 0; \]

\[ a = 3, \beta = 0, \]

\[ 5A_5 - 8A_7 - 2A_8 + 4A_{10} + 8A_{11} + 4A_{12} - 12A_{13} - 20A_{16} - 8A_{11} + 20A_{19} + 6A_{20} = 0; \]

\[ a = 2, \beta = 1, \]

\[ 5A_6 - 6A_8 - 4A_9 + 2A_{10} + 2A_{12} - 60A_{14} - 4A_{16} - 18A_{16} + 90A_{18} + 30A_{19} + 36A_{20} = 0; \]

\[ a = 4, \beta = 0, \]
\[ 5A_7 - 10A_{17} - 2A_9 + 60A_{13} + 10A_{14} + 4A_{15} - 210A_{17} \]
\[ -30A_{18} - 10A_{19} - 6A_{20} + 420A_{21} + 6A_{25} + 30A_{22} = 0; \]
\[ a = 3, \beta = 1, \]
\[ 5A_8 - 8A_{11} - 9A_{12} + 40A_{16} + 16A_{17} + 12A_{18} - 120A_{19} \]
\[ -48A_{20} - 40A_{19} + 210A_{22} + 90A_{24} = 0; \]
\[ a = 2, \beta = 2, \]
\[ 5A_9 - 12A_{12} + 48A_{16} + 18A_{17} - 120A_{19} - 72A_{20} + 180A_{23} \]
\[ + 72A_{26} = 0; \]
\[ a = 5, \beta = 0, \]
\[ 5A_{10} - 12A_{13} - 2A_{14} + 84A_{17} + 12A_{18} + 4A_{19} - 336A_{21} \]
\[ -42A_{22} - 12A_{23} - 6A_{24} + 756A_{26} + 42A_{28} + 6A_{30} = 0; \]
\[ a = 4, \beta = 1, \]
\[ 5A_{11} - 10A_{16} - 4A_{15} + 20A_{14} + 12A_{20} + 60A_{18} - 210A_{23} \]
\[ -60A_{24} - 30A_{25} - 24A_{26} + 420A_{27} + 90A_{29} \]
\[ + 30A_{30} = 0; \]
\[ a = 3, \beta = 2, \]
\[ 5A_{12} - 8A_{16} - 6A_{17} + 40A_{19} + 48A_{20} - 120A_{23} - 120A_{24} \]
\[ -48A_{25} + 210A_{28} + 90A_{29} + 120A_{30} = 0; \]
\[ a = 6, \beta = 0, \]
\[ 5A_{13} - 14A_{17} - 2A_{18} + 112A_{21} + 14A_{22} + 4A_{23} - 504A_{26} \]
\[ -56A_{27} - 14A_{28} - 6A_{29} = 0; \]
\[ a = 5, \beta = 1, \]
\[ 5A_{14} - 12A_{18} - 4A_{19} + 84A_{22} + 24A_{23} + 12A_{24} \]
\[ -336A_{27} - 84A_{28} - 36A_{29} - 24A_{30} = 0; \]
\[ a = 4, \beta = 2, \]
\[ 5A_{15} - 6A_{20} - 10A_{19} + 60A_{23} + 30A_{24} + 24A_{26} - 210A_{28} \]
\[ -90A_{29} - 120A_{30} = 0; \]
\[ a = 3, \beta = 3, \]
\[ (91) \quad 5A_{16} - 16A_{20} + 80A_{24} + 32A_{28} - 240A_{32} - 160A_{36} = 0. \]

2. **Approximate solution with the convergence factor undetermined.**

It may happen that for some particular case a different choice of \( \lambda \), than is given in equation (27) would be more desirable. Applying the foregoing method of section 2a page 21, the differential equation (32) will be the same except for the factor 20 \( U \) which becomes \((4 - k_j)U\) where \( k_j = \frac{K}{N^2} \). The other equations (29)-(31) are unchanged.

There will result the same set of linear equations (45)-(52) from the moment, reaction equations and the corner condition (31). The linear equations which result from the new differential equation are:

\[ a = 0, \beta = 0, \]
\[ A_1(4 - k_j) + A_2(-8 - 2k_j) + A_3(16 - 4k_j) + A_4(-4 - k_j) + A_5(-12k_j) \]
\[ + A_6(-4k_j) + A_7(48 - 48k_j) + A_8(-48 - 12k_j) + A_9(24 - 4k_j) \]
\[ + A_{10}(240 - 240k_j) + A_{11}(-144 - 48k_j) + A_{12}(48 - 24k_j) \]
\[ + A_{13}(1440 - 1440k_j) + A_{14}(-720 - 240k_j) + A_{15}(288 - 96k_j) \]
\[ + A_{16}(-72 - 36k_j) = 0. \]

\[ a = 0, \beta = 1, \]
\[ A_1(4 - k_j) + A_2(-4 - 3k_j) + A_3(16 - 8k_j) + A_4(-8 - 2k_j) \]
\[ + A_5(24 - 30k_j) + A_6(-8 - 10k_j) + A_7(194 - 144k_j) + A_8(-96 - 36k_j) \]
\[ + A_9 (-90 - 12 K_i) + A_{10} (840 - 840 K_i) + A_{11} (-908 - 168 K_i) + A_{12} (120 - 84 K_i) + A_{13} (5760 - 5760 K_i) + A_{14} (-2400 - 960 K_i) + A_{15} (768 - 384 K_i) + A_{16} (-144 - 144 K_i) = 0 \]

\[ a = 1, \, \beta = 1. \]

\[ A_1 (4 - K_i) + A_2 (-4 K_i) + A_3 (16 - 12 K_i) + A_4 (-8 - 4 K_i) + A_5 (48 - 48 K_i) + A_6 (-16 - 24 K_i) + A_7 (240 - 240 K_i) \]

\[ (94) + A_8 (-96 - 96 K_i) + A_9 (24 - 36 K_i) + A_{10} (1440 - 1440 K_i) + A_{11} (-480 - 480 K_i) + A_{12} (96 - 288 K_i) + A_{13} (10880 - 10880 K_i) + A_{14} (-2880 - 2880 K_i) + A_{15} (480 - 1440 K_i) + A_{16} (-676 K_i) = 0 \]

\[ a = 1, \, \beta = 2. \]

\[ A_1 (4 - K_i) + A_2 (4 - 6 K_i) + A_3 (24 - 18 K_i) + A_4 (-8 - 6 K_i) + A_5 (96 - 84 K_i) + A_6 (-24 - 42 K_i) + A_7 (528 - 480 K_i) \]

\[ (95) + A_8 (-144 - 192 K_i) + A_9 (24 - 72 K_i) + A_{10} (3480 - 3240 K_i) + A_{11} (-840 - 1080 K_i) + A_{12} (120 - 648 K_i) + A_{13} (24640 - 25200 K_i) + A_{14} (-5760 - 7200 K_i) + A_{15} (720 - 3600 K_i) + A_{16} (-1440 K_i) = 0 \]

\[ a = 2, \, \beta = 2. \]

\[ A_1 (9 - K_i) + A_2 (8 - 6 K_i) + A_3 (32 - 24 K_i) + A_4 (-4 - 9 K_i) + A_5 (144 - 120 K_i) + A_6 (-16 - 72 K_i) + A_7 (816 - 720 K_i) \]

\[ (96) + A_8 (-96 - 360 K_i) + A_9 (8 - 144 K_i) + A_{10} (5520 - 5040 K_i) + A_{11} (-624 - 2160 K_i) + A_{12} (48 - 1440 K_i) + A_{13} (43200 - 40320 K_i) + A_{14} (-4560 - 15120 K_i) + A_{15} (192 - 8640 K_i) + A_{16} (72 - 3600 K_i) = 0 \]

\[ a = 2, \, \beta = 3. \]

\[ A_1 (9 - K_i) + A_2 (12 - 7 K_i) + A_3 (48 - 32 K_i) + A_4 (-12 K_i) + A_5 (240 - 180 K_i) + A_6 (-108 K_i) + A_7 (1488 - 1200 K_i) \]

\[ (97) + A_8 (-48 - 600 K_i) + A_9 (24 - 240 K_i) + A_{10} (10920 - 9240 K_i) + A_{11} (-504 - 3960 K_i) + A_{12} (168 - 2640 K_i) + A_{13} (9240 - 80640 K_i) + A_{14} (-4800 - 30240 K_i) + A_{15} (768 - 17280 K_i) + A_{16} (288 - 7200 K_i) = 0 \]
\[ a = 3, \beta = 3, \]
\[ \begin{align*}
A_1(4-K_1) + A_2(16-8K_1) + A_3(64-40K_1) + A_4(8-16K_1) + A_5(336-240K_1) + A_6(48-160K_1) + A_7(2160-1680K_1) + A_8(192-960K_1) + A_9(72-480K_1) + A_{10}(16320-13440K_1) + A_{11}(960-6720K_1) + A_{12}(576-4800K_1) + A_{13}(16320-120960K_1) + A_{14}(5760-53760K_1) + A_{15}(2880-33600K_1) + A_{16}(1152-14900K_1) = 0.
\end{align*} \]

Equations (92)-(98) and equations (37), (45)-(52) are sixteen linear equations in the sixteen unknown coefficients. These equations involve two parameters \( \nu \) and \( k \). Equations (92)-(98) do not contain \( \nu \) explicitly in the equations. Equations (37), (45)-(52) do not contain \( k \). Equations (37), (45)-(52) can be solved for the values of nine of the coefficients in terms of the other seven and this resultant seven solved for any particular value of \( \nu \). These values have been determined for \( \nu = .2, .25 \) and \( .3 \).

The values of the coefficients for \( \nu = .2 \) are:

\[ A_1 = \frac{21661K_1 + 89797K_1^3 - 2552K_1 - 16147K_1^4 - 832236K_1^2 - 0.12429K_1^6}{D_1} \]

\[ A_2 = 12600 + 29.76K_1^2 + 2.3294K_1^4 + 2.5296K_1^6 + 1.68K_1^8 + 25.18K_1^2 - 4138K_1^6 \]

\[ A_3 = 103007 + 31731K_1 + 12.68K_1^3 + 16.103K_1^5 + 30.835K_1^7 + 16.933K_1^9 - 5.0925K_1^6 \]

\[ A_4 = 10.52017 + 4.2531K_1^2 + 4.1913K_1^3 + 3.217K_1^4 + 5.0660K_1^6 + 5.6368K_1^2 + 21.083K_1^6 \]
\[ A_i = \begin{align*}
& A_1 = 0.03293 + 0.3073K_i + 3.0615K_i^2 - 10.713K_i^3 - 0.330K_i^4 - 2.134K_i^5 - 13.092K_i^6 \\
& A_2 = 0.02704 + 0.3985K_i + 3.294K_i^2 + 9.679K_i^3 + 19.399K_i^4 + 3.682K_i^5 - 120.11K_i^6 \\
& A_3 = 0.00819 - 0.2140K_i + 7.486K_i^2 + 11.79K_i^3 - 3.117K_i^4 + 1.904K_i^5 \\
& A_4 = 0.00331 - 0.2336K_i + 3.426K_i^2 - 0.879K_i^3 - 0.810K_i^4 - 0.00001K_i^5 \\
& A_5 = 0.00211 + 0.20234K_i - 6.268K_i^2 + 2.244K_i^3 + 0.154K_i^4 - 1.783K_i^5 - 0.316K_i^6 \\
& A_6 = 0.00190 + 0.2425K_i - 0.32977K_i^2 - 15.791K_i^3 - 6.991K_i^4 + 14.320K_i^5 + 44.324K_i^6 \\
& D_i = -0.4647K_i + 0.8794K_i^2 + 3.293K_i^3 + 3.622K_i^4 + 26.232K_i^5 - 46.806K_i^6 \\
& \Theta_i = \frac{\Theta}{2N(1-\nu)\lambda^2} \\
\end{align*} \]

where \( \lambda \) is the undetermined convergence factor, \( k_i = \frac{\lambda}{N^2} \),

and \( \Theta_i = \frac{\Theta}{2N(1-\nu)\lambda^2} \).
The values of the coefficients for $\nu = .25$ are:

\[ A_1 = -0.60604792K - 0.10431K^2 + 0.6721K^3 + 1.8320K^4 + 2.6743K^5 - 0.2658K^6 \]  
\[ D_{16} \]

\[ A_5 = -0.00048667 - 0.0017231K - 0.00316K^2 + 0.018801K^3 + 0.02285K^4 - 0.002606K^5 \]  
\[ (100) D_{125} \]

\[ A_3 = 0 - 0.00046299 - 0.00186K^2 + 0.01731K^3 - 0.08158K^4 - 0.12346K^5 + 0.069868K^6 \]  
\[ D_{125} \]

\[ A_6 = 0 - 0.00047299 - 0.001924K - 0.016376K^2 - 0.008339K^3 - 0.019454K^4 - 0.007275K^5 - 0.009172K^6 \]  
\[ D_{126} \]

\[ A_7 = -0.0004291 + 0.024289K - 0.5276K^2 + 0.4907K^3 - 0.34114K^4 - 0.13936K^5 + 0.0169K^6 \]  
\[ D_{126} \]

\[ A_8 = -0.0004908 + 0.086399K - 0.50923K^2 + 2.0207K^3 - 3.6048K^4 - 5.1372K^5 + 0.094662K^6 \]  
\[ D_{125} \]

\[ A_9 = -0.0002629 + 0.068399K + 0.35600K^2 + 1.5247K^3 + 3.35/2K^4 + 3.3365K^5 + 0.0039309K^6 \]  
\[ D_{125} \]

\[ A_{10} = -0.00043875 - 0.43382K + 1.27896K^2 + 1.5944K^3 + 1.7161K^4 + 0.4918K^5 - 0.018249K^6 \]  
\[ D_{125} \]

\[ A_{11} = -0.00040843 - 0.72648K + 1.73793K^2 + 1.9337K^3 + 2.7652K^4 + 2.1235K^5 + 0.033630K^6 \]  
\[ D_{125} \]

\[ A_{12} = -0.00044726 - 0.003787K + 0.01133K^4 + 0.0063419K^5 + 0.0062417K^6 + 0.0012306K^7 + 0.0009341K^8 \]  
\[ D_{125} \]

\[ A_{13} = -0.00015634 + 0.03703K - 0.31433K^2 + 1.014K^3 - 1.6050K^4 - 1.6176K^5 + 0.018707K^6 \]  
\[ D_{125} \]

\[ A_{14} = -0.00049124 + 0.01782K - 0.09023K^2 - 3.104K^3 + 0.493K^4 + 3.361K^5 + 0.096969K^6 \]  
\[ D_{125} \]
where \( \lambda \) is the undetermined convergence factor, \( k = \frac{K}{N_2} \),

\[
D_{23} = -0.000102317 K - 0.00082609 K^2 - 0.040363 K^3 - 0.75117 K^4 - 0.13823 K^5 + 0.020513 K^6
\]

and \( \theta_v = \frac{P}{2N(1-V)K} \)

The values of the coefficients for \( V = 0.3 \) are:

\[
A_3 = 0.00006676 + 0.0006573 K - 0.0009667 K^2 - 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]

\[
B_3 = 0.000084108 K + 0.000010278 K^2 - 0.0012684 K^3 - 0.0012365 K^4 - 0.001939 K^5
\]

\[
(101)
A_3 = 0.00004062 + 0.0001553 K - 0.0009667 K^2 - 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]

\[
B_3 = 0.000064004 + 0.0004174 K + 0.0009667 K^2 - 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]

\[
A_3 = 0.000055512 + 0.00040252 K + 0.0009667 K^2 + 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]

\[
B_3 = 0.00007035 + 0.0002538 + K + 0.0002538 + K - 0.002244 K - 0.002244 K^2 + 0.003194 K^3 + 0.003194 K^4 + 0.003194 K^5
\]

\[
A_3 = 0.000055512 + 0.00040252 K + 0.0009667 K^2 + 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]

\[
B_3 = 0.00007035 + 0.0002538 + K + 0.0002538 + K - 0.002244 K - 0.002244 K^2 + 0.003194 K^3 + 0.003194 K^4 + 0.003194 K^5
\]

\[
A_3 = 0.000055512 + 0.00040252 K + 0.0009667 K^2 + 0.0013857 K^3 + 0.001039 K^4 + 0.0002048 K^5
\]
\[ A_8 = 0.0000598 + 0.0001656 K, 0.0020478 K^2, 0.006294 K^3, 1.0093 K^4, 0.0035262 K^5 \]
\[ A_9 = 0.000080280 + 0.003138 K^2, 0.017933 K^3, 0.037530 K^4, 2.3317 K^5, 0.001993 K^6 \]
\[ A_{10} = 0.000031761 - 0.0042410 K, 0.0029409 K^2 + 0.00466 K^3 - 0.10449 K^4 + 0.0740 K^5 + 0.01562 K^6 \]
\[ A_{11} = 0.000061238 - 0.001827 K - 0.2092 K^2, 0.01808 K^3 - 0.0184 K^4 + 0.048 K^5 + 0.007138 K^6 \]
\[ A_{12} = 0.00008812 - 0.0018 K^2, 0.0110 + 0.0203 K^2 + 0.0223 K^3 + 0.03562 K^4 + 0.0004 K^5 + 0.00154 K^6 \]
\[ A_{13} = 0.000018137 - 0.0001283 K^2, 0.001629 K^3 + 0.001470 K^4 - 0.0017164 K^5 + 0.00264 K^6 - 0.0003257 K^7 \]
\[ A_{14} = 0.000004784 + 0.00002638 K^2 - 0.0004658 K^3 + 0.00007321 K^4 + 0.000933 K^5 + 0.0000449 K^6 \]
\[ A_{15} = 0.00000028162 - 0.000000234 K^2 - 0.00002344 K^3 + 0.0000147 K^4 + 0.0000253 K^5 + 0.0000025 K^6 + 0.0000002 K^7 \]
\[ A_{16} = 0.0000002649 K^2 + 0.0000047 K^3 - 0.00000075 K^4 + 0.0000001 K^5 + 0.00000004 K^6 + 0.00000001 K^7 \]

where \( \lambda \) is the undetermined convergence factor, \( k = \frac{\lambda}{N\lambda^2} \),
\[ D_3 = 0.000018249 K, 0.000017832 K^2, 0.000207 K^3, 0.001235 K^4, 0.0073156 K^5, 0.000379 K^6 \]
and \( \phi = \frac{P}{2N(1-\phi)\lambda^2} \).

4. **Numerical solution of the linear equations arising from the use of the integral type of functional.**

The set of linear equations (37), (45)-(59) can be
solved as follows. Since the differential equation conditions and the corner condition, equations (53)-(59) and (37) do not contain ν explicitly, these equations may be solved for eight of the coefficients in terms of the other eight and θ. These results substituted in the other eight equations can be reduced to seven equations in seven unknowns each of which is linear in ν. By reducing these equations to a form in which ν appears only as the coefficient of one different unknown in each equation, the equations can be solved.

Remembering the facts that \( \bar{x} = \lambda \), \( x \), \( \bar{y} = \lambda \), \( y \), and \( \lambda_1 = \sqrt{\frac{E}{4N}} \), together with the solution of the above set of linear equations there results the following values for the unknown coefficients of our approximate solution (33).

\[
A_1 = \frac{-0.13477 + 0.01979y + 0.002011y^2 - 0.005162y^3 + 0.001727y^4 + 0.000238y^5}{Dy} \theta
\]

\[
A_2 = \frac{-0.32429 - 0.19573y + 0.056404y^2 - 0.037808y^3 + 0.0163y^4 + 0.000089y^5}{Dy} \theta
\]

\[
A_3 = \frac{-0.71909 - 0.07271y + 0.0517y^2 - 0.037843y^3 - 0.002019y^4 + 0.000078y^5 \theta}{Dy}
\]

\[
A_4 = \frac{-0.28010 - 0.01372y + 0.043813y^2 + 0.005375y^3 + 0.014643y^4 + 0.000034y^5 \theta}{Dy}
\]

\[
A_5 = \frac{-0.041761 + 0.09189y + 0.002017y^2 + 0.004379y^3 - 0.00167y^4 + 0.000285y^5 \theta}{Dy}
\]
where \( D_v = \frac{1}{2
\nu^2 + \nu - 1} \)

and \( \Theta_v = \frac{\mu}{2
\nu^2 + \nu - 1} \)

a. deflection. Equations (102) result in the following expressions for the deflection \( w \) with
\[ V = .15, \]
\[ \frac{\lambda}{\phi_0} = e^{-\lambda,(x+y)} \left[ -0.57190 + 13678 \lambda_1 (x+y) + 34208 \lambda_2 (x^3+y^3) 
+ 1.5519 \lambda_1 (x+y) - 12392 \lambda_3 (x^3+y^3) + 0.7642 \lambda_4 \right] (x^4+y^4) \]
\[ (103) \]

\[ V = .2, \]
\[ \frac{\lambda}{\phi_0} = e^{-\lambda,(x+y)} \left[ -0.01246 \lambda_1 (x+y) + 0.0992 \lambda_2 (x^3+y^3) 
- 0.08379 \lambda_1 x_1 y_2 + 0.06947 \lambda_2 (x^3+y^3) + 0.0833 \lambda_5 \right] (x^5+y^5) \]
\[ + 0.0148 \lambda_6 \right] (x^6+y^6) - 0.0037 \lambda_7 \]
\[ (104) \]

\[ V = .25, \]
\[ \frac{\lambda}{\phi_0} = e^{-\lambda,(x+y)} \left[ -0.0370 \lambda_1 (x+y) + 0.120 \lambda_2 (x^3+y^3) 
- 0.1719 \lambda_1 (x+y) - 0.085 \lambda_3 (x^3+y^3) + 0.0740 \lambda_3 (x^4+y^4) \right] \]
\[ (105) \]

\[ V = .3, \]
\[ \frac{\lambda}{\phi_0} = e^{-\lambda,(x+y)} \left[ -0.0370 \lambda_1 (x+y) + 0.120 \lambda_2 (x^3+y^3) 
- 0.1719 \lambda_1 (x+y) - 0.085 \lambda_3 (x^3+y^3) + 0.0740 \lambda_3 (x^4+y^4) \right] \]
\[ (106) \]
\[-0.05036 \lambda_y^2 (x^2 + y^2) + 0.01241 \lambda_x y (y^2 + x y) - 0.00723 \lambda_x x^2 \]
\[+ 0.01248 \lambda_x^5 (x^5 + y^5) - 0.006982 \lambda_x \lambda_y (x^2 + y^2) \]
\[+ 0.009843 \lambda_x^6 (x^6 + x^2 y^4) - 0.0069261 \lambda_x^4 \lambda_y \]
\[+ 0.0032681 \lambda_x^6 (x^6 + y^6) - 0.0001097 \lambda_x^6 (x^6 + x^2 y^4) \]
\[+ 0.0005870 \lambda_x^4 \lambda_y^3 \].

The deflection functions (103)-(106) are plotted along the edge in figures 1-4 and along the diagonal in figures 5-6.

b. moments. The moment functions may be obtained by the usual derivatives of the deflection \(w\).

\[(107) \quad M_x = -N \left[ \frac{y^2 \omega}{\frac{\partial^2 \omega}{\partial x^2}} + \frac{\partial^3 \omega}{\partial x^3} \right].\]

The moment expression for the deflections as given in equations (103)-(106) is plotted along the edges in figures 9-12 and along the diagonal in figures 13-16.

c. stress. The stress function may be obtained by applying the customary derivatives.

\[(108) \quad V_x = -N \left[ \frac{y^3 \omega}{\frac{\partial^3 \omega}{\partial x^3}} + \frac{\partial^3 \omega}{\partial x\partial y^2} \right].\]

The stress as found from solutions (103)-(106) is plotted along the edges in figures 17-20 and along the diagonal in figures 21-24. Along the diagonal the stress \(V_x\) will be \(\sqrt{2}\) times as large as the stress \(V_x\).

The maximum value of the stress occurs at the
Figure 1
The Deflection
(y = 0, \( \nu = .15 \))

Figure 2
The Deflection
(y = 0, \( \nu = .2 \))

Figure 3
The Deflection
(y = 0, \( \nu = .25 \))

Figure 4
The Deflection
(y = 0, \( \nu = .3 \))
Figure 5
The Deflection
\( y = x, \mu = .15 \)

Figure 6
The Deflection
\( y = x, \mu = .25 \)

Figure 7
The Deflection
\( y = x, \mu = .25 \)

Figure 8
The Deflection
\( y = x, \mu = .3 \)
Figure 9
The Moment
(y = 0, \( \nu = .15 \))

Figure 10
The Moment
(y = 0, \( \nu = .2 \))

Figure 11
The Moment
(y = 0, \( \nu = .23 \))

Figure 12
The Moment
(y = 0, \( \nu = .3 \))
Figure 17
The Stress
($y = 0, \nu = .15$)

Figure 18
The Stress
($y = 0, \nu = .2$)

Figure 19
The Stress
($y = 0, \nu = .25$)

Figure 20
The Stress
($y = 0, \nu = .3$)
Figure 21
The Stress
($y = x, \beta = .15$)

Figure 22
The Stress
($y = x, \beta = .2$)

Figure 23
The Stress
($y = x, \beta = .25$)

Figure 24
The Stress
($y = x, \beta = .3$)
origin and is equal to

\[
\left(106\right) \quad \sqrt{k} \int_{r=0}^{\infty} = -N^3 \left[ -2A_1 + 6A_2 - 8A_3 + 6A_4 - 2A_5 \right].
\]

5. Accuracy of results.

The numerical calculations in obtaining the solution for the set of linear equations (37), (45)-(59) as given in (102) can be checked by substituting in the original set of equations (37), (45)-(59) this shows an accuracy of five significant figures.

An indication of the error in the approximate solution can be obtained by substituting (33) with (102) in the system (39)-(42). Equation (31) is satisfied to five significant figures. Three error functions are determined to describe the degree of approximation to equations (29), (30) and (32). These error functions are obtained as follows: substitute the approximate solution (33) with (102) into one of equations (29), (30) or (32). There will result positive and negative values as coefficients of each of the powers of the variables; these should add to zero in the exact solution. Obtain these positive and negative values when \( x = 0 \) and \( y = 0 \), divide the largest of these two numerical values into the foregoing function obtained by substitution of (33)
into one of (29), (30), or (32). The corresponding error functions are plotted; for the moment condition (29) in figure 25, for the reaction condition (30), figure 26, for the differential equation (32), along the edge, figure 27, along the diagonal, figure 28. The value of Poisson's ratio used in the error graphs was \( \nu = .2 \). Other values of Poisson's ratio result in only slight changes in the error function graphs chiefly in the moment condition (29) at the origin. The error function corresponding to the moment condition (29) is slightly larger for \( \nu = .15 \) and slightly smaller for \( \nu = .25 \) and \( \nu = .3 \). For the reaction and differential equation conditions (30) and (32) the error function is almost independent of \( \nu \).

For the results given in (99), (100) and (101), the numerical values obtained for the solution of (37), (45)–(52) for the first nine coefficients as functions of \( \nu \) and the other seven coefficients, substituted back in those equations are found to check exactly. The results given in (99), (100), (101) were checked by substituting in equation (98). Equation (98) was satisfied to five significant figures.
Figure 25
The Moment
\((x = 0, \gamma = 2)\)

Figure 26
The Reaction
\((x = 0, \gamma = 2)\)

Figure 27
The Differential Equation
\((x = 0, \gamma = 2)\)

Figure 28
The Differential Equation
\((x = y, \gamma = 2)\)
IV. DISCUSSION

The identity functional (61) was applied to the original conditions (29), (30) and (32) with the solution obtained by the integral type of functional substituted into them. At points other than \( x = 0, y = 0, \triangledown = 0 \), the maximum limit of error in (29) is found to become smaller as \( x, y \) and \( \triangledown \) recede from zero values. In (30) and (32) the error is almost independent of \( \triangledown \) but becomes smaller as \( x, y \) increase.

When one parameter occurred linearly in the coefficients of each equation in the sets of equations that were solved, it was found useful to reduce the system to diagonal form before beginning the usual method of eliminating one unknown at a time. Diagonal form means having the parameter occur in the coefficient of only one unknown of each equation.

The effect of the convergence factor \( \lambda \), can not be seen until a particular plate problem is given so that, \( \triangledown \), Poisson's ratio, \( N \), the rigidity modulus, and the subgrade reaction constant, \( k \), are known. In plate theory problems this factor will be about \( 1/20 \) or \( 1/30 \), which will cause convergence to be more rapid. The deflection curve indicates an oscillatory wave form whose amplitude
quickly approaches zero.

Further investigations of the general method are indicated along these three lines. First a theoretical approach with the most desirable result being a simple way of finding the error in an approximate solution. Second an investigation to show how to make the most judicious choice of functions and functionals, possibly along the line of biorthogonal functions. This would result in a stable approximation, that is an approximation such that previous terms remain fixed when more terms or constants in the approximating function are added. Third the development of machine methods for the solution of sets of linear equations.
V. SUMMARY

1. The general functional method as it applies to solutions for differential equations is outlined.

2. The equations arrived at by the use of the Ritz, Boussinesq, and Trefftz methods are obtained by the functional method.

3. Other known applications of the functional method are listed.

4. The functional method is used to obtain an approximate solution involving seventeen constants for the deflection of an infinite, corner loaded plate resting on an elastic foundation.

5. The deflection is given for four values of Poisson's ratio and plotted for these values at $x = 0$, and $x = y$.

6. Shearing stresses and moments along the lines $x = 0$, $x = y$ are given and plotted for four values of Poisson's ratio.

7. A discussion of the accuracy of the results is included. An error function is plotted for those original conditions of the problem which are not satisfied within the accuracy of the numerical calculations.

8. An approximate solution of the system of differ-
9. An approximate solution of the system of differential equations involving thirty-one unknowns is obtained by means of the functional method and the corresponding set of linear equations is given but not solved. The solution for this set of equations would give a more accurate approximation to the problem.
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