Center of flexure of beams of triangular cross-section

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CENTER OF FLEXURE OF BEAMS
OF TRIANGULAR CROSS-SECTION

by

Robert Nichols Goss

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Applied Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

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Dean of Graduate College

Iowa State College

1950

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INTRODUCTION

General Background of the Problem

One of the central problems of classical elasticity is the flexure problem of Saint-Venant. In this problem one considers an isotropic cantilever beam in the form of a right cylinder of finite length and uniform cross-section, the free end of which supports a transverse load, or, more precisely, is acted upon by forces statically equivalent to a single transverse force called the load. The weight of the beam itself is neglected, and it is assumed that, apart from the load at the free end and the forces necessary to maintain rigidity at the other, no external force acts on the beam. Under these assumptions, consideration of the forces acting on any normal cross-section shows that the normal components of these forces are equivalent to a couple and that the tangential components are statically equivalent to the terminal load. It is quite evident that in addition to the simple bending effect of the forces comprising the load, the beam is in general subject to a twist which can be measured about any longitudinal axis. The general
flexure problem thus involves the two simpler problems of torsion and pure bending.

In the problem as formulated by Saint-Venant, the load was restricted to act from the centroid of the terminal cross-section parallel to a principal axis (6, p. 329)\(^1\) and the twist was measured about the centroidal axis. An obvious generalization is to remove this restriction, allowing the load to act in any direction from an arbitrary point in the end section. For any particular choice of the load point \(P\) there is an axis parallel to the generators of the beam, intersecting the section in a point \(Q\), about which the local twist vanishes. Conversely, the position of \(Q\) can first be chosen arbitrarily, its choice determining \(P\). The selection of \(Q\), which can be made in any convenient manner, is often dictated by physical considerations. When \(Q\) is taken to be the centroid, the point \(P\) is called the "center of flexure".

---

\(^1\) Numbers in parentheses will be used to refer both to the literature cited at the end of the paper and to equations constituting part of the text. Whenever there is a possibility of confusion, the references to the literature will be associated with an author's name or a page citation.
Elastic Centers

Several names have been introduced by various authors to designate different points in the terminal section which are related to the flexure problem and which in some cases have been confused with the center of flexure. When the local twist vanishes at the load point itself, the point is called the "center of shear" (8). The determination of this point has been discussed by Schwalbe (9) and Trefftz (25). The term "flexural center" has been employed by a few writers; some, for example, Southwell (19, p. 29) have defined it as the point at which the load can be applied without causing rotation and have correctly used it in this sense as the center of shear, while others have made the name synonymous with center of flexure. Timoshenko (24, p. 301) gives still another meaning to this expression, although his apparent intent is to make it the same as center of flexure.

Again, if a twisting couple only is applied to the free end of the beam, the point which undergoes no displacement is called the "center of twist" (20). That this point coincides with the center of shear when the fixed end is "approximately" clamped is proved by Weinstein (26), while Stevenson (20) indicates its close relationship to the
center of flexure. Timoshenko (24, p. 301) gives to the term "center of twist" the same meaning as he does to "flexural center". Finally, the load point which results in minimum strain energy of the beam is called the "center of least strain". It is obtainable from the center of flexure by setting the elastic constant equal to zero (20).

Although the positions of all these points are ordinarily close to one another in a given section and although each has certain special features of interest, it is desirable for the sake of uniformity in treatment and ready comparison of results that one be selected as a standard. The center of flexure as defined above is to be preferred for this role for three main reasons. In the first place, when the local twist is taken to be zero at the centroid, the mean value of the twist over the entire section is zero, provided the centroid is at the origin, so that the problem may be approached from two somewhat different points of view. In the second place, the relative positions of the centroid, the center of flexure, and the load point in a given section provide an immediate physical picture of the direction and magnitude of the action of the beam under the load. Finally, when the center of flexure is known for a cross-
section, the flexure solution for an arbitrary load point may be obtained by superposing a simple Saint-Venant torsion solution with a known twist upon the solution for the load point at the center of flexure, thus permitting the exploitation of a considerable fund of known results in the torsion problem.

Flexure Solutions for Uniaxially Symmetric and Asymmetric Sections

There are two aspects of the flexure problem which have received attention in the literature. The first is the determination of a "flexure function", that is, a harmonic function satisfying the equations of elasticity which are operative in the flexure problem together with the boundary conditions for the particular cross-section, from which the stress components can be obtained by differentiation. When the section has more than one axis of symmetry and the load acts from the centroid, or when in the case of uniaxially symmetric sections the load acts along the axis of symmetry, the twisting effect is not present. In all other cases this effect adds considerably to the difficulties encountered in obtaining a solution.

The first to obtain results involving the twist were Young, Elderton and Pearson (30), who found solutions
for a number of sections related to the circular sector. Although their results are in no sense complete and are in some cases incorrect, the work merits special consideration if only for the fact that it stood for many years as the sole example of systematic research on uniaxial sections. Another early result was that of Timoshenko (23), who purported to obtain a solution for the isosceles triangle, but in reality found only that the center of flexure coincides with the centroid in an equilateral triangle when Poisson's ratio is one-half.

The circular section with an eccentric circular hole was first suggested by Love (6, p. 340) in an early edition of his treatise as a soluble problem, and the solution was given by Seth (14, 15) in 1936. The first solution for a rectilinear cross-section with uniaxial symmetry was obtained by Seth (11) in 1934, the figure being the isosceles right triangle. Later (12), using a method of great generality, he found flexure functions for the isosceles right triangle with loads acting both along and perpendicular to the axis of symmetry, the isosceles triangle containing an angle of 120°, the equilateral triangle, and the right triangle with an angle of 60°. These results were shortly generalized by him into the three cases, discussed below, for which the
center of flexure is obtained in this thesis (13). One of these cases, that of the general triangle with Poisson's ratio equal to one-half, is the first completely asymmetric section for which the flexure solution was obtained.

Sections for which the Center of Flexure is Known

In the classical problem of Saint-Venant the solution was deemed to be complete when the flexure function was found. The more general problem, however, requires in addition to the flexure function the discovery of the center of flexure, or some alternative quantity which provides equivalent information, in order that the stresses and displacements may be completely related to the external load. Both parts of the problem may involve great mathematical difficulties. As a result one finds only a few sections for which the flexure solution has been carried to completion in the modern sense, especially as compared with the large number of sections for which the companion problem of torsion has been solved (4).

For sections possessing an axis of symmetry the center of flexure lies on the axis (18, p. 228); hence for biaxially symmetric sections the center of flexure is the centroid. In the case of uniaxially symmetric sections, we find that the first solution for the center of flexure
was claimed in 1936 by Shepherd (17) for his work on the
cardioid. Two years later a remarkable paper by
Stevenson (20) on a general method of solving the flexure
problem appeared; it contained, among others, the com-
plete solution for the circular sector, the loop of a
lemniscate of Bernoulli, and a correction of Shepherd's
results on the cardioid. Stevenson's method has also
been successfully applied to the circle with a radial
crack (27), to the circle with two radial slits on the
same axis (28), to the circular section with an ortho-
gonal keyway and the lenticular section (29), and to the
circle with an eccentric circular hole (22). The
cardioid and its generalizations have also been studied
by Morris (7), Sokolnikoff (18, pp. 242-253), Holl and
Rock (5), Ghosh (2), and again by Stevenson (21). The
center of flexure has heretofore been found for only one
completely asymmetrical section—one of the halves of
the loop of a lemniscate bisected by its axis (20).

In the case of triangular sections, the determination
of the center of flexure under the most general conditions
is very difficult. Only in the cases of the isosceles
right triangle (20) and the equilateral triangle when
Poisson's ratio is one-half (24, p. 300) has this point
been determined theoretically. In addition, an experimental
determination of the center of flexure of a narrow isosceles triangle has been made by Duncan, Ellis and Scruton (1) in connection with their studies on airfoils. As mentioned above, a general method of discovering the flexure function which involves a Schwarz-Christoffel transformation of the section in the complex plane and replacement of the boundary condition by another transformation to the same region of the complex plane has been given by Seth (12,16). The results of this method in the general case are attained in the form of a comparison of two integrals, the evaluation of which is too formidable to yield usable information. On specializing the geometry or the elastic properties of the beam, however, three triangular cases become tractable: the isosceles triangle, the right triangle with the load applied parallel to the hypotenuse, and the general triangle with Poisson's ratio equal to one-half. Flexure functions for these three cases have been obtained by Seth (13). In this thesis the center of flexure for each of these three cases, insofar as it is determinate, is found, and from the results certain conclusions are drawn concerning the general triangular case. In addition, a generalized concept of center of flexure is introduced for the equilateral triangle. The material of this dissertation is
embodied in two papers, the one already published (3) and the other to appear in a forthcoming issue of the Bulletin of the American Mathematical Society.
FORMULATION OF THE PROBLEM

The Modified Flexure Problem of Saint-Venant

The formulation of the modified problem of Saint-Venant, the problem with arbitrary load point, is based upon the classical assumptions of linearized stress-strain relations. These assumptions permit the superposition of the effects of independent loadings. Certain further hypotheses about the stress components are made in accordance with the "semi-inverse" method of solution. Moreover, the results are valid only insofar as they meet the tests of Saint-Venant's principle, that

![Diagram of a cantilever beam]

Figure 1. The Cantilever Beam.
is, that they hold only for portions of the beam which are not in the immediate neighborhood of the two ends.

The cross-section is, to begin with, arbitrary. Let the origin of cartesian coordinates be placed at a convenient point in the fixed end and let the z-axis be taken parallel to the longitudinal edges of the beam (Figure 1). Let the coordinates of the centroid of the terminal section be \((\overline{x},\overline{y},h)\) and let the moments and product of inertia referred to axes through the centroid parallel to the coordinate axes be denoted by \(I_x, I_y, F_{xy}\).

Of the six stress components, three, \(\tau_{xx}, \tau_{yy}, \tau_{xy}\), are assumed to vanish; the remaining three must satisfy the stress-equilibrium equations and the equations of compatibility. Considerations of equilibrium lead to the assumption for \(\tau_{zz}\) of the form

\[
\tau_{zz} = - E(h - z) \left[ A_1(x - \overline{x}) + A_2(y - \overline{y}) \right],
\]

where \(E\) is Young's modulus and \(A_1, A_2\) are constants to be determined. It is shown in textbooks on the theory of elasticity how the remaining two stress components can be constructed (18, pp. 219-221). We shall here set
(1) \[ \tau_{zx} = \mu a \left( \frac{\partial \phi}{\partial x} - y \right) + \mu \frac{\partial x}{\partial x} - \mu A_1 \left[ \frac{\sigma}{E} (x - \bar{x})^2 \right] \\
+ \left( 1 - \frac{\sigma}{2} \right) (y - \bar{y})^2 \] \\
- \mu A_2 \left[ \frac{\sigma}{E} (y - \bar{y})^2 + \left( 1 - \frac{\sigma}{2} \right) (x - \bar{x})^2 \right],

(2) \[ \tau_{yz} = \mu a \left( \frac{\partial \phi}{\partial y} + x \right) + \mu \frac{\partial y}{\partial y} - \mu A_1 (2 + \sigma) (x - \bar{x})(y - \bar{y}) \]

\[ - \mu A_2 \left[ \frac{\sigma}{E} (y - \bar{y})^2 + \left( 1 - \frac{\sigma}{2} \right) (x - \bar{x})^2 \right], \]

where \( \phi \) is the torsion function, \( \chi \) the flexure function, \( a \) is a constant, \( \sigma \) is Poisson's ratio, and \( \mu \) is the rigidity, connected with \( E \) and \( \sigma \) by the equation \( E = 2\mu(1 + \sigma) \). \( \phi \) and \( \chi \) are harmonic functions of \( x \) and \( y \).

Let the load \( W \) be resolved into components \( W_x + \frac{F_{xy}}{F_x} W_y \) and \( W_y + \frac{F_{xy}}{F_y} W_x \) parallel to the \( x \)- and \( y \)-axes respectively. Then the conditions

\[ \int_S \tau_{zx} \, dx \, dy = W_x + \frac{F_{xy}}{F_x} W_y, \quad \int_S \tau_{yz} \, dx \, dy = W_y + \frac{F_{xy}}{F_y} W_x \]

result in the evaluation of the constants \( A_1, A_2 \) as \( \frac{\mu W_x}{E F_y} \).
and $\frac{\mu W_y}{E I_x}$, respectively.

The local twist at a point $(x,y)$ in any cross-section is (18, p. 220)

$$\frac{1}{E \mu} \left( \frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{zx}}{\partial y} \right).$$

On substituting the values of $\tau_{zx}$ and $\tau_{yz}$ from (1) and (2) in this expression, we easily find that the twist is

$$\alpha + \sigma \left[ A_3(x - \bar{x}) - A_1(y - \bar{y}) \right].$$

When the load acts from the center of flexure as defined above, this quantity is zero at the centroid of the section; hence the load point $(x_0, y_0, h)$ is the center of flexure provided $\alpha = 0$.

In the terminal section the moment about the z-axis of the load $W$ applied at $(x_0, y_0, h)$ must equal the resultant torsional couple due to the stress components $\tau_{yz}$ and $\tau_{zx}$. This fact is expressed by the equation

$$(3) \int \int_S \left( \alpha \tau_{yz} - \gamma \tau_{zx} \right) dx dy = x_0 (W_y + \frac{W_x F_{xy}}{I_y}) - y_0 (W_x + \frac{W_y F_{xy}}{I_x}).$$
Since this relation is valid for an arbitrary direction of the load, it must be an identity in the quantities \( W_x \) and \( W_y \). To find the coordinates \((x_0, y_0)\) of the center of flexure in the plane \( z = h \), we therefore substitute the values of \( \tau_{zx} \) and \( \tau_{yz} \) given by (1) and (2) in (3) and equate the coefficients of \( W_x \) and \( W_y \) separately in the resulting identity. If the direction of the load is not arbitrary, that is, if the quantities \( W_x \) and \( W_y \) bear a fixed ratio to one another, then (3) determines a line any point of which is a load point making the local twist zero at the centroid. This line has been called by Ghosh (2) a "line of flexure".

The Torsion and Flexure Functions

The formulation of the problem thusfar has been free from any reference to the shape of the cross-section. The particularization in this respect is made through an additional condition which must be satisfied by the stress components, namely, the boundary condition

\[
(4) \quad \tau_{zx} \cos(x,n) + \tau_{yz} \cos(y,n) = 0,
\]

in which \( n \) is the direction normal to the lateral surface.
of the beam. To satisfy (4) we have at our disposal the harmonic functions $\phi$ and $\chi$ which as yet have not been adjusted. It is the determination of these functions which constitutes the solution of the flexure problem in the first sense mentioned above.

Since $\alpha$ is to be zero, it would appear that the torsion function $\phi$ is not involved in the solution at all, but it turns out that $\chi$ itself depends upon this function when the loading is not symmetric. $\phi$ is the function from which the stresses and displacements can be found when the free end of the beam is subjected to a twisting couple only. It is known for many cross-sections (Higgins 4), having been found in particular for certain triangular sections by Seth (10). In the problem under consideration it will be found that the torsion function need not be explicitly known.

It will be convenient in what follows to consider the applied load as being made up of two additive constituents, $W_1$ with components $W_x$ and $\frac{W_y}{I}$, parallel to the coordinate axes, and $W_2$ with components $\frac{W_x}{I}x$ and $W_y$. This is justified by the principle of superposition whenever the direction of the load is arbitrary. If we now set $\chi = A_1\chi_1 + A_2\chi_2$, where $\chi_1$ and $\chi_2$ are harmonic,
and suppose first of all that \( \frac{\partial W}{\partial y} = 0 \), the stress components become, for the load \( W_x \),

\[
\tau_{zx} = \mu a \left( \frac{\partial \phi}{\partial x} - y \right) + A_1 \left[ \frac{\partial \chi_1}{\partial x} - \frac{\sigma}{2}(x - \bar{x})^2 
- (1 - \frac{\sigma}{E})(y - \bar{y})^2 \right],
\]

\[
\tau_{yz} = \mu a \left( \frac{\partial \phi}{\partial y} + x \right) + A_1 \left[ \frac{\partial \chi_1}{\partial y} 
- (2 + \sigma)(x - \bar{x})(y - \bar{y}) \right].
\]

On setting

\[
\chi_1 = (H_1 + \sigma \bar{y})\phi + \left[ \frac{\sigma}{2} \bar{x}^2 + (1 - \frac{\sigma}{2})\bar{y}^2 \right] x
+ (2 + \sigma)\bar{y}xy - 2\bar{y}xy - \frac{\sigma}{2} (x^2 - y^2)
- \frac{1}{3}(1 - \frac{\sigma}{2})(x^3 - 3xy^2) + \chi',
\]

where \( H_1 \) is an undetermined constant and \( \chi' \) is harmonic, the boundary condition (4) may be written in the form
in which the terms involving the torsion function \( \phi \) have been separated out. Since the boundary condition satisfied by \( \phi \) is (18, p. 123)

\[
(8) \quad (a + A_1H_1 + A_2f(x)) \left[ \left( \frac{d\phi}{dx} - y \right) \cos(x,n) + \left( \frac{d\phi}{dy} + x \right) \cos(y,n) \right] \\
+ A_1 \left\{ \left[ \frac{d\lambda'}{dx} + H_1y - x^2 \right] \cos(x,n) + \left[ \frac{d\lambda'}{dy} - H_1x - 2\sigma xy + 2(1 + \sigma)xy \right] \cos(y,n) \right\} = 0,
\]

the condition which must be satisfied by the function \( \lambda' \) is given when the expression in braces in (8) is equated to zero.

By a similar process we find that for the load \( W_a \) we may set

\[
\lambda_a = (H_a - \sigma \frac{x}{y})\phi + \left[ (1 - \frac{\sigma}{2})x^2 + \frac{\sigma^2}{2y^2} \right] y + (2 + \sigma)\frac{x}{y} \frac{x}{y} x \\
- 2\frac{xy}{y}(x^2 - y^2) + \frac{1}{3}(1 - \frac{\sigma}{2})(3x^2y - y^3) + \lambda''.
\]
and obtain for the boundary condition on \( \chi'' \),

\[
(10) \quad \left[ \frac{\partial \chi''}{\partial x} - 2rxy + 2(1 + \sigma)xy + H_\alpha y \right] \cos(x, n) \\
+ \left[ \frac{\partial \chi''}{\partial y} - y^2 - H_\alpha x \right] \cos(y, n) = 0.
\]

Since in equations (1) through (4), the effective equations in determining the center of flexure, the length of the beam is of no consequence, the problem from here on will be treated as two-dimensional and will be characterized by the shape of the end section.
SOLUTIONS FOR SPECIAL SECTIONS

The General Isosceles Triangle

Let the origin of coordinates be at the vertex common to the two equal sides of the isosceles triangle (hereafter simply called the vertex), and let the y-axis lie along the axis of symmetry (Figure 2). Since the center of flexure lies on this axis, \( x_0 = 0 \), and the problem admits of simplification to the extent that the direction of the load need not be arbitrary but can be prescribed. We shall therefore suppose that a load \( W_1 \) is applied from the point \((0, y_0)\), where \( W_1 \) is given above.

![Figure 2. The General Isosceles Triangle.](image)
Owing to the symmetry of the figure, \( F_{xy} = 0 \), so that \( W_1 = W_x \), that is, the load acts parallel to the \( x \)-axis.

The stress components \( \tau_{yz} \) and \( \tau_{xx} \) are now given by (5) and (6) with \( \chi_1 \) given by (7). Still to be determined is the constant \( H_1 \) and a harmonic function \( \chi' \) satisfying the condition

\[
(11) \quad \left[ \frac{\partial^2 \chi'}{\partial x^2} + H_1 y - x^2 \right] \cos(x, n) + \left[ \frac{\partial^2 \chi'}{\partial y} - H_1 x \
- 2xy + 2(1 + \sigma)xy \right] \cos(y, n) = 0
\]

on the sides of the triangle. If \( d \) denotes the altitude from the vertex and \( \beta \) the semi-vertical angle of the triangle, then the equations of the sides are

\[
y = d, \quad y = x \cot \beta,
\]

on which we have

\[
\cos(x, n) = 0, \cos \beta, -\cos \beta, \text{ and } \cos(y, n) = 1, -\sin \beta, -\sin \beta,
\]

respectively. On setting (Seth 13)

\[
\chi' = Axy + B(x^2 - 3xy^2),
\]
we find that (11) is satisfied on the three sides provided

\[ A = \frac{d[(1 + \sigma)\tan^2 \beta - \sigma](\tan^2 \beta + 1)}{3 \tan^2 \beta - 1}, \]

\[ B = \frac{(1 - 2\sigma)\tan^2 \beta}{3(3 \tan^2 \beta - 1)}, \]

\[ H_1 = \frac{d[(1 + \sigma)\tan^2 \beta - \sigma](\tan^2 \beta - 1)}{3 \tan^2 \beta - 1}, \]

and \( \beta \neq \pi/6 \).

Equation (3) for determining \( y_0 \) is, since \( x_0 \) and \( \bar{F}_{xy} \) are both zero,

\[ -W_{xy} y_0 = \int_0^d \int_{-y}^{y} \tan \beta \left( x \tau_{yz} - y \tau_{zx} \right) dx \, dy. \]

Substituting into this equation the values of \( \tau_{yz} \) and \( \tau_{zx} \), which are now completely known apart from the terms containing \( \phi \), and setting \( \alpha = 0 \), we obtain
\[ y_0 = -\frac{\mu}{EI_y} (H_1 + \sigma \overline{y}) \iint_S (x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}) \, dx \, dy \]

\[- \frac{\mu}{EI_y} \iint_S \left\{ (A - 2\overline{y}) x^2 - 6 \left[ B - \frac{1}{3}(1 - \frac{\sigma}{E}) \right] x^2 y - (A - 2\overline{y}) y^3 \right\} \, dx \, dy \]

\[- \left[ B - \frac{1}{3}(1 - \frac{\sigma}{E}) \right] (3x^2 y - 3y^3) - (1 - \frac{\sigma}{E}) y^2 y \]

\[- (2 + \sigma)(y - \overline{y}) x^2 + \frac{\sigma x^2 y}{E} + (1 - \frac{\sigma}{E})(y - \overline{y}) y^3 \right\} \, dx \, dy.\]

In this equation the terms involving \( \phi \) have been separated in order that a new quantity may be introduced which makes unnecessary their direct integration. This is the torsional rigidity, defined by the equation

(12) \[ D = \iint_S (x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} + x^2 + y^2) \, dx \, dy, \]

and discussed below. With the substitution of \( D \) and collection of similar terms, the above equation for \( y_0 \) reduces to

\[ y_0 = - \frac{D}{EI_y} (H_1 + \sigma \overline{y}) - \frac{\mu}{EI_y} \iint_S \left[ (A - H_1)x^2 \right. \]

\[- (A + H_1)y^2 + (1 - 2B - 2\sigma)x^2 y + 3By^3 \left.] \, dx \, dy. \]
For this section \( \bar{y} = \frac{d^4 \tan^3 \beta}{3} \) and \( \bar{y} = 2d^3 \); using these values and those of \( A, B, H_a, \) and \( E \) in terms of \( d, \beta, \mu, \) and \( \sigma \), we obtain after integration

\[
(13) \quad y = \frac{2d \left\{ \frac{5}{3} \left[ (1 + \sigma) \tan^3 \beta - \sigma \right] - 2(1 - 2\sigma) \right\}}{3(1 + \sigma)(3 \tan \beta - 1)} - \frac{3(1 + \sigma) \tan^4 \beta - 3 \tan^2 \beta + \sigma}{(1 + \sigma) \tan^3 \beta (3 \tan \beta - 1)} \mu d^2
\]

The special case of the equilateral triangle \( \beta = \pi/6 \) is treated in a subsequent section.

**Right Triangle with Load Acting Parallel to the Hypotenuse**

Let the vertices of the right triangle be at \( (0,0), (0,c), (-c/m,0) \) (Figure 3). The equations of the sides are

\[
x = 0, \quad y = 0, \quad y = mx + c,
\]

on which

\[
\cos(x,n) = 1, 0, -m/\sqrt{m^2+1}, \quad \cos(y,n) = 0, -1, 1/\sqrt{m^2+1},
\]
respectively.

Figure 5. The Right Triangle.

Since the figure is not symmetric, we shall assume that both constituents $W_1$ and $W_2$ of the load are present. Then $\tau_{yz}$ and $\tau_{zx}$ are given by (1) and (2), with

$$\chi = A_1\chi_1 + A_2\chi_2$$

to be determined so as to satisfy the boundary condition. We find that we are unable to satisfy (10) and (11) separately by any choice of $\chi' \text{ and } \chi''$, indicating that the direction of the load is not arbitrary. If, however, we set (Seth 13)
\[ \chi' = 0, \]
\[ \chi'' = \left[ \bar{y}(1 + \sigma) + \frac{1}{2} \sigma \right](y^2 - x^2), \]

we find that

\[ \chi = -\frac{1}{\beta} A_1\left[ 2\bar{y}(1 + \sigma) + \sigma \right](x^2 - y^2) + (A_1\sigma\bar{y} - A_2\sigma\bar{x}) \phi \]
\[ + A_1 \left\{ \left[ \frac{1}{2} \sigma \bar{x}^2 + \left( 1 - \frac{1}{2} \sigma \right)\bar{y}^2 \right] x + (2 + \sigma)\bar{x} \bar{y} y - 2\bar{y} x y \right. \]
\[ - \frac{1}{2} \sigma \bar{x}(x^2 - y^2) - \frac{1}{3}(1 - \frac{1}{2} \sigma)(x^2 - 3xy^2) \left\} \right. \]
\[ + A_2 \left\{ \left[ (1 - \frac{1}{2} \sigma)x^2 + \frac{1}{2} \sigma y^2 \right] y + (2 + \sigma)\bar{x} \bar{y} x - 2\bar{x} x y \right. \]
\[ + \frac{1}{2} \sigma \bar{y}(x^2 - y^2) + \frac{1}{3}(1 - \frac{1}{2} \sigma)(3x^2 y - y^3) \right\}. \]

This value of \( \chi \) used with (1) and (2) satisfies the boundary condition (4), provided \( H_1 = H_2 = 0 \) and \( A_1/A_2 = m \). The last condition becomes \( \bar{T}_x W_x/\bar{T}_y W_y = m \), and since for the section \( \bar{T}_x = c^4/56m, \bar{T}_y = c^4/56m^2, \bar{T}_{xy} = c^4/72m^2 \), we have \( W_y = mW_x \) and also

\[ W_y + \frac{\bar{T}_{xy} W_x}{\bar{T}_x} = m \left( W_x + \frac{\bar{T}_{xy} W_y}{\bar{T}_x} \right), \]
showing that the load must act in a direction parallel to the hypotenuse.

For this section equation (3) becomes

\[(14) \quad -\frac{3Wx}{2}(mx_0 - y_0) = \int_0^c \int_0^m (x\tau_{yz} - y\tau_{zx}) \, dx \, dy.\]

Using the now determined values of \(\tau_{yz}\) and \(\tau_{zx}\), we have for the integrand in the right member after some reduction

\[(15) \quad x\tau_{yz} - y\tau_{zx} = \sigma(A_1\overline{y} - A_3\overline{x})(x \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x}) + \sigma(A_1\overline{y} - A_3\overline{x})(x + y^2) + (1 - 2\sigma)(A_1\overline{x}^2y - A_3xy^2) + \left[2(1 + \sigma)(A_1\overline{x} + A_3\overline{y}) + 2A_2\sigma \right]xy.\]

With the introduction of the torsional rigidity \(D\), given by (12), the right member of (14) becomes

\[
\sigma(A_1\overline{y} - A_3\overline{x})D/\mu - \int_0^c \int_0^m \frac{Y_0 - Y}{m} \left\{ (1 - 2\sigma)(A_1\overline{x}^2y - A_3xy^2) + \left[2(1 + \sigma)(A_1\overline{x} + A_3\overline{y}) + 2A_2\sigma \right]xy \right\} \, dx \, dy.
\]

Inserting \(\overline{x} = -c/3m, \overline{y} = c/3\), and carrying out the
integration, we have finally

\[ \frac{m x - y}{\alpha} = \frac{4rDm(m^2 + 1)}{\mu \alpha^3 (1 + \sigma)} - \frac{6(3 + 4\sigma)}{5(1 + \sigma)}. \]

Since the direction of the load is not arbitrary, (16) is the equation of a line of flexure. When \( m = 1 \) the triangle is isosceles, and when \( \beta = \pi/4 \) in (15) and the triangle in Figure 2 is rotated through a counterclockwise angle of \( \pi/4 \), we obtain (16) on noting that \( \alpha = d\sqrt{2} \).

Any Triangle with \( \sigma = \frac{1}{2} \)

Let the equations of the sides of the triangle be \( y = d, y = m_1 x, y = m_2 x \) (Figure 4), where \( \pi > \arctan m_2 > \arctan m_1 > 0 \), and \( m_1, m_2 \) are finite. For the load \( W \), consisting of \( W_1 \) superposed upon \( W_2 \), the stress components are again given by (1) and (2), and again \( \chi = A_1 \chi_1 + A_2 \chi_2 \) must be determined so as to satisfy the boundary conditions.

On assuming \( \chi' \) and \( \chi'' \) to contain harmonic terms through the second degree, we find by the method of undetermined coefficients that
\[ \chi' = \frac{d}{2m_1m_2} \left[ (m_1 + m_2)(x^2 - y^2) + (m_1m_2 - 1)xy \right], \]

\[ \chi'' = -\frac{1}{2} \delta(x^2 - y^2) \]

Figure 4. The General Triangle.

satisfy (11) and (10) respectively, provided

\[ H_1 = -\frac{d}{2} \frac{m_1m_2 + 1}{m_1m_2}, \quad H_2 = 0. \]

For this section we observe that \( \overline{x} = d(m_1 + m_2)/3m_1m_2, \)
\( \overline{y} = 2d/5, \)

\[ \overline{x} = d^4(m_2 - m_1)/4m_1m_2, \]
coordinates of \( x_s \) and \( x_M \), we obtain for the coordinates of \( x_s \) and \( x_M \), and by substituting the above values of \( \Delta x_s \), \( \Delta x_M \) and \( \Delta x \), we obtain the equation

\[
\begin{align*}
\frac{dx}{dt} &= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt} \\
&= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt} \\
&= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt} \\
&= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt}
\end{align*}
\]

Once again making use of \( \Delta x \), we obtain the equation

\[
\begin{align*}
\frac{dx}{dt} &= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt} \\
&= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt} \\
&= \left( \frac{\Delta x_M}{x_M} \right) + \left( \frac{\Delta x}{x_M} \right) \frac{dx}{dt}
\end{align*}
\]

Proceeding as before, we find

\[
\begin{align*}
\Delta x_s &= \left( \frac{\Delta x}{x_M} \right) \frac{x_s}{x} \\
\Delta x_M &= \left( \frac{\Delta x}{x_M} \right) \frac{x_M}{x} \\
\Delta x &= \left( \frac{\Delta x}{x_M} \right) \frac{x}{x}
\end{align*}
\]
the center of flexure

\[
\begin{align*}
(x_0 &= \frac{d(m_1+m_2)}{3m_1m_2} - \frac{4(m_1+m_2)(2m_2^3 - 5m_2m_3 + 2m_3^3 - m_1m_2^2)}{3(m_2 - m_1)^3}) \frac{D}{\mu d^3} \\
y_0 &= \frac{2d}{3} - \frac{4m_1m_2(m_1^2 - 4m_2m_3 + m_1^2 - 2m_1m_2)}{3(m_2 - m_1)^3}) \frac{D}{\mu d^3}
\end{align*}
\]

(17)

If in (17) we set \( m_1 = m_2 = \cot \beta \), we obtain for the isosceles triangle

\[
\begin{align*}
x_0 &= 0, \\
y_0 &= \frac{2d}{3} - \frac{3 \tan \beta - 1}{3 \tan^2 \beta} \frac{D}{\mu d^3}
\end{align*}
\]

which is in agreement with (13) when \( \sigma \) is set equal to \( \frac{1}{2} \) in that equation. Again, setting in (17) \( m_1 = 1/m \), \( m_2 = -m \), we obtain

\[
y_0 = \frac{2d}{3} - \frac{4Dm}{3\mu d^3(m^2 + 1)}
\]

(18)

If the triangle in Figure 3 is rotated through a clockwise angle of \( \arctan m \) radians, the line of flexure (16) with \( \sigma = \frac{1}{2} \) and \( c = d/\sqrt{m^2 + 1} \) is transformed into (18). The mutual consistency of the results in the three cases is thus verified.
The section whose center of flexure is given by (17) is the second completely asymmetric section for which the problem has been solved.

The Torsional Rigidity

Each of the formulas (13), (16) and (17) contains as an unknown factor the torsional rigidity $D$, which must be evaluated before the solutions can be considered complete. $D$ is the ratio of the applied torque to the twist per unit length, and is a measure of the resistance offered by the beam to twist. For all three cases we may, if desired, use Saint-Venant's well known approximation (24, pp. 235, 235),

$$D = \frac{\mu A^4}{4\pi^2 (I_x + I_y)} ,$$

where $A$ is the area of the section. In the third case this becomes

$$D = \frac{9\mu d^4(m_2 - m_1)^3}{16\pi^2 m_1 m_2(m_1^2 m_2^2 + m_1^2 - m_1 m_2 + m_2^2)} .$$

In the second case, that of the right triangle, we
may readily obtain a first approximation by the Rayleigh-Ritz method (18, pp. 304-309). A well known formula from the theory of elasticity gives $D$ as

\[(20) \quad D = 2\mu \int_S \Psi \, dx \, dy,\]

where $\Psi$ is a function which vanishes on the boundary of the section. The total energy of the beam under the load, that is, the sum of the strain energy and the potential energy of the applied torque, is, except for a constant factor, given by the integral

\[I = \int_S \left[ (\nabla \Psi)^2 - 4\Psi \right] dx \, dy,\]

where

\[(\nabla \Psi)^2 = \left( \frac{\partial \Psi}{\partial x} \right)^2 + \left( \frac{\partial \Psi}{\partial y} \right)^2.\]

Upon assuming for $\Psi$ the form

\[\Psi = kxy(y - mx - c),\]

we find

\[I = \int_0^c \int_0^\frac{y-c}{m} \left\{ k \left[ y^4 - 4mxy^3 - 2cy^3 + 4(m^2 + 1)x^2y + c^2y^2 + 4mxy^2 - 4cx^2y - 4mx^2y + c^2xy + 2mcy^3 + mx^4 \right] - 4kxy(y - mx - c) \right\} dx \, dy.\]
The value of $k$ which minimizes the total energy, and hence furnishes the best value for $\Psi$, must satisfy the condition

$$\frac{\partial I}{\partial k} = 0.$$ 

On making use of this condition and carrying out the integration of $\Psi$, we find that the desired value of $k$ is

$$k = \frac{3m}{c(m^2 + 1)}.$$ 

Substituting $\Psi$ with $k$ as given into (20) and performing the integration, we then obtain

$$D = \frac{\mu a}{20m(m^2 + 1)},$$

which, when substituted in (16) gives as the equation of the line of flexure

$$mx_o - y_o = -3c/5.$$ 

This is the equation of the broken line in Figure 3. It is to be noted that this result is independent of $c$.

For the isosceles-triangular section $D$ is known when the vertical angle is a right angle (Stevenson 20) and when the triangle is equilateral (24, p. 237), its values
being $0.10436 \mu \cdot d^4$ and $\mu \cdot d^4/15 \sqrt{3}$, respectively. Otherwise an approximation obtained by Duncan, Ellis, and Scruton (1) using the Rayleigh-Ritz method is known and in our notation is

$$D = \frac{2 \mu \cdot d^4 \tan \beta}{3(1 - \tan \beta)} \frac{\sqrt{10 - 4 \tan \beta}}{\sqrt{6 \tan^2 \beta + 10 + 4 \tan \beta}}$$

(22)

The Rayleigh-Ritz method furnishes no answer to the question of the accuracy of approximation, and we therefore cannot estimate the error in the coordinates of the center of flexure due to the use of the above values of $D$. As we shall see below, however, formula (13) with $D$ given by (22) leads to a result in the special case when $\beta = 0.041$ radian which agrees well with experimental findings. Moreover, it is easily verified that (22) reduces to the exact value of $D$ given above for the equilateral triangle when $\beta = \pi/6$ and that the coefficient of $\mu \cdot d^4$ approaches as a limit the fraction $5/48$ as $\beta$ approaches $\pi/4$, thus giving good agreement with the known value for the right triangle. The formula (22), therefore, seems to furnish a good approximation to $D$, at least when the vertical angle is not too much larger than a right angle.
On placing \( m = 1 \) and \( c = d \sqrt{2} \) in the right member of (21), we obtain \( D = 0.1 \mu d^4 \), which compares acceptably with the above value for the isosceles right triangle. Equation (19) on the other hand, being based upon a formula valid for any cross-section, does not give as close an approximation as the others which were developed for specific sections. The numerical coefficients of \( \mu d^4 \) as given by (19) for the right isosceles and equilateral triangles are 9.6 percent and 14 percent larger, respectively, than the known values given above.
CONCLUSIONS

Agreement with Known Results

Formula (13) for the center of flexure of the isosceles triangle contains as special cases the two previously known results for triangular sections. For the isosceles right triangle Stevenson (20) has found the value

\[ y_0 = d \left( 0.60000 - 0.00872 \frac{\sigma}{1+\sigma} \right). \]

Using \( \beta = \pi/4 \) and \( D = 0.10436 \alpha d^4 \) in (13), we find agreement with this value.

For the equilateral triangle when \( \sigma = \frac{1}{2} \), it is known that the center of flexure coincides with the centroid (24, p. 300). Formula (13) with \( \beta = \pi/6 \) and \( D = \nu d^4 / 15 \sqrt{3} \) gives \( y_0 = 2d/3 \), confirming this fact. The same result can also be obtained from (17).

It is to be expected that the results for the isosceles triangle would be comparable to those obtained for the circular sector when the angle is small, and this proves to be the case. The coordinate of the center of flexure for the latter section has been tabulated.
by Stevenson (20) for various central angles; for an angle of 0.05 π, for example, Stevenson gives the value

\[ y_o = a(0.77151 + 0.10286 \frac{\sigma}{1+\sigma}), \]

where a is the radius of the sector, while (13) and (22) together give

\[ y_o = a(0.78435 + 0.09823 \frac{\sigma}{1+\sigma}), \]

the correction for the difference in a and d having been made. Since the numerical value of \( \sigma/(1+\sigma) \) is between 0 and 1/3, the two coefficients of a are very close.

Experimental Verification

As a final check on the validity of (13), we may compare its prediction with the experimental results of Duncan, Ellis and Scruton (1) on the center of flexure of a steel prism whose cross-section was an isosceles triangle of semi-vertical angle nearly 0.041 radian in magnitude. The measurements, obtained for the single value \( \sigma = 0.27 \), were made on sections at various distances from the fixed end of the beam. As extreme values they found \( y_o = 0.857d \) and \( y_o = 0.814d \). Formula
(13) with \( \beta = 0.041 \) and \( \sigma = 0.27 \) gives \( y_0 = 0.841d \), a value which agrees well with the experimental results.

The Equilateral Triangle

Under the conditions of the flexure problem as usually formulated, the symmetry of the equilateral triangle with respect to its three concurrent altitudes requires that the center of flexure coincide with the centroid for any value of \( \sigma \). This has been verified when \( \sigma = \frac{1}{2} \). If, however, we leave \( \sigma \) undetermined, place \( \beta = \pi/6 \) in (13), and set \( D = \pi d^4/15\sqrt{3} \), we obtain as a limiting value

\[
y_0 = \frac{11 + 3\sigma}{15(1+\sigma)} \cdot d.
\]

This is a new result, showing that despite the triaxial symmetry the center of flexure does not always fall upon the centroid.

It is readily verified that a torsion function \( \phi \) which satisfies (9) on the boundary of the equilateral triangle is

\[
\phi = \frac{1}{2} d(x^3 - 3xy^2) + 2xy.
\]
From (5), (6), and (7) we have, using this value of $\phi$ and setting $\alpha = 0$,

$$\tau_{zx} = \frac{\mu W x}{6EI y} [(1 - 2)(x^2 - y^2) - 6x^2 + 2dy],$$

$$\tau_{yz} = \frac{\mu W x}{3EI y} (1 + 4)(d - y)x.$$ 

These satisfy the boundary condition (4) only on the side $y = d$. On the sides $y = \pm x\sqrt{3}$ the stresses satisfy the condition

$$\int_0^{\pi} [(\tau_{zy} \cos(x,n) + \tau_{yz} \cos(y,n)]ds = 0,$$

where the integral is taken along the side. Unless the standard flexure problem is thus relaxed to the extent of permitting a resultant instead of an exact boundary condition to be satisfied on two of the sides, (23) is anomalous. Under the generalized condition, however, the triangle still possesses a point which may be called the center of flexure. Its position is shifted from the centroid toward the side on which the exact condition is satisfied by an amount which depends on Poisson's ratio. If the triangle fails to be perfectly equilateral, as is
nearly always the case in practice, the exact conditions may be imposed, and the center of flexure will lie very near the centroid for all values of $\sigma$. This concept of the generalized center of flexure is a new result.

**Dependence of the Center of Flexure**

**Upon Poisson's Ratio**

Although the problem of finding the center of flexure for a general triangular section and arbitrary $\sigma$ is virtually intractable by methods now known, the results obtained in this thesis furnish valuable information on the location of this point.

For the general isosceles triangle, Table I gives values of $y_o/d$ calculated for selected values of $\beta$ from

**Table I. Coordinate of Center of Flexure for Values of the Semi-Vertical Angle.**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$y_o/d$</th>
<th>$\beta$</th>
<th>$y_o/d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{24}$</td>
<td>$.7812 + .1163 \frac{\sigma}{1+\sigma}$</td>
<td>$\frac{7\pi}{24}$</td>
<td>$.5726 - .0075 \frac{\sigma}{1+\sigma}$</td>
</tr>
<tr>
<td>$\frac{\pi}{12}$</td>
<td>$.7401 + .0493 \frac{\sigma}{1+\sigma}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$.5495 - .0055 \frac{\sigma}{1+\sigma}$</td>
</tr>
<tr>
<td>$\frac{\pi}{8}$</td>
<td>$.7060 + .0190 \frac{\sigma}{1+\sigma}$</td>
<td>$\frac{3\pi}{8}$</td>
<td>$.5336 - .0036 \frac{\sigma}{1+\sigma}$</td>
</tr>
<tr>
<td>$\frac{5\pi}{24}$</td>
<td>$.6438 - .0530 \frac{\sigma}{1+\sigma}$</td>
<td>$\frac{5\pi}{12}$</td>
<td>$.5175 - .0018 \frac{\sigma}{1+\sigma}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$.6000 - .0083 \frac{\sigma}{1+\sigma}$</td>
<td>$\frac{11\pi}{24}$</td>
<td>$.5090 - .0005 \frac{\sigma}{1+\sigma}$</td>
</tr>
</tbody>
</table>
formula (13). This information is displayed in graphical form in Figure 5 in such a way that the dependence upon \( \sigma \) is apparent. When the vertical angle is small, the position of the center of flexure is displaced from the centroid toward the base with increasing \( \sigma \), while the displacement is in the opposite direction when \( \beta \) is greater than \( \pi/6 \). The shift is most pronounced for small angles. For angles greater than \( \pi/4 \) the dependence of the center of flexure on \( \sigma \) is negligible, varying from 0.5 percent for \( \beta = \pi/4 \) to 0.04 percent for \( \beta = 11 \pi/24 \).

For the right triangular section it was found above that the equation of the line of flexure was completely independent of \( \sigma \). On the basis of these facts the conclusion seems justified therefore that for the completely general case the coordinates of the center of flexure differ but slightly from those given by (17), at least when none of the angles is very small. Subject to the approximations made and to the conditions mentioned, the solution for the general triangular section has therefore been obtained.
Figure 5. Dependence of Center of Flexure of the Isosceles Triangle upon Poisson's Ratio.
LITERATURE CITED


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