INVERSION OF EDDY-CURRENT DATA VIA CONJUGATE GRADIENTS

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A LINEARIZED MULTIFREQUENCY MODEL

In a companion paper, [1], we developed a rigorous, nonlinear model for
inverting eddy-current data by means of the conjugate gradient algorithm. In
this paper we will present some results obtained from the linearized version
of the rigorous model. In this version we assume that the electric field within
the flaw is simply the incident field that exists in the absence of the flaw.
Hence, if we make this assumption in the multifrequency model of [1] (which
is equation (7)(b) of [1]), we derive the linearized multifrequency model for
inversion

\[ B_1 = T_{1,1} \otimes \sigma_1 + \cdots + T_{1,N_z} \otimes \sigma_{N_z} \]

\[ \vdots \]

\[ B_{N_f} = T_{N_f,1} \otimes \sigma_1 + \cdots + T_{N_f,N_z} \otimes \sigma_{N_z}, \]

where \( N_f \) is the number of frequencies used. Each of the \( B \)'s is a \((N_x + 1) \times (N_y + 1)\)-dimensional data array, the subscript denoting the frequency at which
the data is taken. The operator \( \otimes \) denotes the two-dimensional Toeplitz opera-
tion [1].

The question arises as to the best number, \( N_f \), of frequencies to use, and
what is the optimum range of frequencies. Generally, this can be answered by
trial-and-error, with the following ideas as a guide. Least-squares methods,
such as the conjugate gradient algorithm, often work better with overdeter-
dined systems, because the variance of the error is reduced as the number of
equations increases, for a given number of unknowns. Hence, we would like \( N_f \)
to generally be much larger than \( N_z \). It is time consuming, however, to gen-
erate too much data, so there is a trade-off that can only be determined by
conducting numerical experiments with typical problems.

The same can be said in determining the frequency range, but we know in-
tuitively, if for no other reason, that we should use as broad a frequency
range as possible. We can be a little bit more definite here, and rely upon
the phenomenon of skin effect to guide us. If we want a resolution of \( \delta_z \) in
depth, then our upper limit of frequency should produce a skin depth that is
smaller than $\delta_2$, though, as we will see later, we have gotten good results with simulated data at lower frequencies.

THE CONJUGATE GRADIENT METHOD [2]

Let us write the complex vector-matrix equation (1) as the operator equation

$$Y = A \circ X,$$

where

$$Y = \begin{bmatrix} B_1 \\ \vdots \\ B_{N_f} \end{bmatrix},$$

$$X = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_{N_z} \end{bmatrix},$$

and

$$A \circ X = \begin{bmatrix} T_{1,1} & \cdots & T_{1,N_z} \\ \vdots & \ddots & \vdots \\ T_{N_f,1} & \cdots & T_{N_f,N_z} \end{bmatrix} \circ \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_{N_z} \end{bmatrix}.$$  \hspace{1cm} (5)

Keep in mind that each of the $B$’s and $\sigma$’s is a two-dimensional array, say of dimension $32 \times 32$, and each $T_{i,j}$ in (5) is a two-dimensional Toeplitz matrix, $T_{i,j}(l - L, m - M)$. $l, m$ index the ‘row’ in each of the two dimensions, while $L, M$ index the ‘column’ of each dimension.

We will need the adjoint operator, $A^*$, which corresponds to the conjugate transpose of the block-matrix in (5):

$$A^* \circ Y = \begin{bmatrix} T_{1,1}^H & \cdots & T_{1,N_z}^H \\ \vdots & \ddots & \vdots \\ T_{N_f,1}^H & \cdots & T_{N_f,N_z}^H \end{bmatrix} \circ \begin{bmatrix} B_1 \\ \vdots \\ B_{N_f} \end{bmatrix}.$$  \hspace{1cm} (6)

$T_{i,j}^H$ is the Hermitian transpose of the two-dimensional Toeplitz matrix $T_{i,j}$; i.e., $T_{i,j}^H(l - L, m - M) = T_{i,j}^*(L - l, M - m)$, where $*$ denotes the complex-conjugate.

The conjugate gradient algorithm starts with an initial guess, $X_0$, from which we compute $R_0 = Y - A \circ X_0$, $P_1 = Q_0 = A^* \circ R_0$. In addition, we have a convergence parameter, $\epsilon$. Then for $k = 1, \ldots$, if $Test = \|R_k\|/\|Y\| < \epsilon$, stop; $X_k$ is the optimal solution of (2). Otherwise, update $X_k$ by the following steps:

$$S_k = A \circ P_k$$

$$a_k = \frac{\|Q_{k-1}\|^2}{\|S_k\|^2}$$

$$X_k = X_{k-1} + a_k P_k$$

$$R_k = R_{k-1} - a_k S_k$$

$$Q_k = A^* \circ R_k$$

$$b_k = \frac{\|Q_k\|^2}{\|Q_{k-1}\|^2}$$

$$P_{k+1} = Q_k + b_k P_k.$$  \hspace{1cm} (7)
The convolution and correlation operations that are a part of $A$ and $A^*$ are evaluated by using the FFT. This, together with the fact that the storage requirements are reasonably modest, are the reasons why the conjugate gradient algorithm becomes attractive for large problems.

RECONSTRUCTIONS USING SIMULATED DATA

The simulated data that was used in performing the numerical experiments was produced by a model that is different from (1), which will be used for inversion. Because of this difference, there will be an 'error' in the data, the peak error being about 15% at the higher frequencies. Hence, in the tests that are to be described, we cannot expect perfect reconstructions of known flaws, even in the absence of random noise in the data.

The laboratory arrangement that was simulated consisted of a long current-carrying wire, to which was attached a sensor, over a workpiece. The sensor consisted of ten square turns from 0.10 inch to 0.55 inch, the workpiece was 0.11 inch thick, and the conductivity tensor was (mhos/m)

$$
\begin{bmatrix}
10^4 & 0 & 0 \\
0 & 10^4 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

This represents a graphite-epoxy composite material that is isotropic in-plane and essentially nonconducting through its thickness.

Data measurements were simulated at each point of a 32 X 32 grid in the sensor plane at a resolution of 0.1 inch. The thickness was discretized into four layers, each also a 32 X 32 grid. A test flaw, then, could be defined as a collection of 4096 'voxels' each with dimension 0.1 X 0.1 X 0.0275.

Test Set #1

A flaw in the shape of a cross was chosen as the standard test flaw. This test flaw was placed in the center of each of the four layers (one at a time) and an attempt was made to reconstruct the flaw from emf data. The conductivity was scaled so that flaw locations had a value of 1 and host material locations had a value of 0. For ease of reference, we will refer to a particular flaw using the word 'flaw' followed by digit(s) indicating the layer(s) that contain the test flaw. This first test set, then, involves flaw1, flaw2, flaw3 and flaw4. Data was simulated at 5 frequencies: 1, 3, 5, 7 and 9 MHz. The unconstrained version of the CG algorithm was used and the initial guess was no flaw, that is, all zeroes.

The number of unknowns in this problem is $32 \times 32 \times 4 = 4096$. Because the emf data is complex and our solution (conductivity) is real, each set of emf data provides $2 \times 32 \times 32 = 2048$ equations. For 5 sets of emf data and 4 layers, the overdeterminedness of the system is 2.5.

The purpose of this set of tests was to determine the ability of the CG algorithm to isolate the flaw to the proper layer and to determine the effect of depth on the convergence. Figure 1 is a plot of the convergence measure, $Test$, for the first 500 iterations. Notice that the deeper the flaw, the slower the convergence. Flaw3 and flaw4 are more difficult to isolate, and this is to be expected. The 'skin effect' phenomenon suggests that the results should be better if some of the data were taken at higher frequencies. With this in mind, more tests were performed.
The error in our solutions reported in the previous section was greatest in the layers immediately adjacent to the layer that contained the flaw. The worst case for layer discrimination, then, would seem to be having a flaw in layers 2 and 4. The error that would appear in layer 3 may be large enough to lead us to believe that there is a flaw in that layer also. We will call this flaw arrangement flaw24. First we tried to reconstruct flaw24 using data taken at the same five frequencies as above (i.e., at 1, 3, 5, 7 and 9 MHz). The results are shown in Figure 2.

At 9 MHz, the skin depth of the workpiece is 0.066 inches or 2.4 times the layer thickness. A second reconstruction was performed for flaw24 using data simulated over a wider frequency range. The frequencies used were 1, 5, 10, 20 and 30 MHz. At 30 MHz, the skin depth is 0.036 inches or 1.3 times the layer thickness. The results of this second reconstruction are displayed in Figure 3. As expected, the higher frequency data improves the solution.

Post-processing Using Classification Theory

Post-processing based on some classification theory results seems to 'clean up' the solutions. Using some of the solutions as a 'training set', 0.23 was determined to be the optimum value for partitioning the data into two classes, host material and flaw. If we filter our first test from Test Set #2 by assigning 0 to all variables less than 0.23, we obtain the solution shown in Figure 4.

![Convergence measure, 5 frequencies: 1, 3, 5, 7, 9 MHz](image)
Fig. 2 Reconstruction of flaw24, 5 frequencies: 1, 3, 5, 7, 9 MHz
Fig. 3 Reconstruction of flaw 24, 5 frequencies: 1, 5, 10, 20, 30 MHz
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REFERENCES

1. H. A. Sabbagh, "A Rigorous Model for Inverting Eddy-Current Data:, these Proceedings.