Normal forms for coupled Takens-Bogdanov systems

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Normal forms for coupled Takens-Bogdanov Systems

by

David Mumo Malonza

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

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Program of Study Committee:
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DEDICATION

To my dearest wife Naomi and our two children, Anthony and Rachel for their support and endurance.
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The set of systems of differential equations that are in normal form with respect to a particular linear part has the structure of a module of equivariants, and is best described by giving a Stanley decomposition of that module. Groebner basis methods are used to determine the Stanley decomposition of the ring of invariants, that arise in normal forms for systems with nilpotent linear part consisting of repeated $2 \times 2$ Jordan blocks. Then an efficient algorithm developed by Murdock, is used to produce a Stanley decomposition of the module of the equivariants from the the Stanley decomposition of the ring of invariants. A discussion of the phenomenon of asymptotic unfolding is included and is used to find the unfolding of single Takens-Bogdanov systems.
CHAPTER 1. Introduction

In this chapter we collect together material needed for later chapters for easy reference.

1.1 Introduction to Normal Form Theory

The basis for normal form theory, which can be found in [3], is the observation that the vector field

\[ \dot{x} = Ax + a_2(x) + \cdots + a_j(x) + \cdots \]  

is transformed into

\[ \dot{y} = Ay + a_2(y) + \cdots + a_{j-1}(y) + b_j + \cdots \]  

by a change of co-ordinates

\[ x = y + s_j(y), \]

where \( s_j \) is homogeneous of degree \( j \), and

\[ L_A s_j = a_j - b_j \]

with

\[ (L_A v) x = v'(x) Ax - Av(x). \]  

A normal form is computed by repeating such calculations for \( j = 1, \cdots, k \) up to some desired finite \( k \), reverting to the original notation after each calculation. At each stage
it is necessary to choose $b_j$ so that $a_j - b_j \in \text{im } L_N$; then $s_j$ exists. In order to proceed systematically, it is best to select a complement to $\text{im } L_N$ in each degree, and determine the $b_j$ by projecting $a_j$ into that complement. The problem, then, comes down to selecting a complement to $\text{im } L_N$. This is called the choice of a normal from style. For more details see Murdock [12].

1.2 Invariants and Equivariants

Let $\mathcal{P}_j(\mathbb{R}^n, \mathbb{R}^m)$ denote the vector space of homogeneous polynomials of degree $j$ on $\mathbb{R}^n$ with coefficients in $\mathbb{R}^m$. Let $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of all such polynomials of any degree and let $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R}^m)$ be the vector space of formal power series. If $m=1$, $\mathcal{P}_*(\mathbb{R}^n, \mathbb{R})$ becomes a ring of (scalar) formal power series on $\mathbb{R}^n$, where $\mathbb{R}$ denotes the set of real numbers. From the viewpoint of smooth vector fields, it is most natural to work with formal power series (Taylor series), but since in practice these must be truncated at some degree, it is sufficient to work with polynomials. Now, for any matrix $A$, let the lie operator

$$L_A : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$$

be as defined in equation (1.1.3) and the differential operator

$$D_{Ax} : \mathcal{P}_j(\mathbb{R}^n, \mathbb{R}) \to \mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$$

be defined by

$$(D_{Ax}f)(x) = f'(x)A(x) = (N(x) \nabla) f(x) \quad (1.2.1)$$

In addition, notice that

$$L_A(fv) = (D_Af)v + fL_Av. \quad (1.2.2)$$

Therefore, $L_A$ is not a module homomorphism of $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ into itself but is a linear mapping. Recall that with every vector field $a(x) = (a_1(x), \cdots, a_n(x))$ there is an
associated differential operator given by

\[ D_{a(x)} = a_1(x) \frac{\partial}{\partial x_1} + \cdots + a_n(x) \frac{\partial}{\partial x_n}, \]  

(1.2.3)

acting on the space \( \mathcal{P}_f(\mathbb{R}^n, \mathbb{R}) \) of smooth (scalar) functions. Furthermore if \( v \) is a vector field and \( f \) is a scalar field, then \( D_{v(x)}f \) is a scalar field called the derivation of \( f \) along (the flow of) \( v(x) \). We will write \( D_A \) for \( D_{Ax} \), the derivation along the linear vector field \( Ax \).

Observe that

\[ D_A : \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R}^n, \mathbb{R}). \]

A function \( f \) is called an invariant of (the flow of) \( Ax \) if \( \frac{\partial}{\partial t} f(e^{At}x)|_{t=0} = 0 \) or equivalently \( D_Af = 0 \) or \( f \in \ker D_A \). Since

\[ D_A(f + g) = D_Af + D_Ag \]  

(1.2.4)

\[ D_A(fg) = fD_Ag + gD_Af, \]  

(1.2.5)

it follows that, if \( f \) and \( g \) are invariants, then, so are \( f + g \) and \( fg \); that is \( \ker D_A \) is both a vector space over \( \mathbb{R} \), and also a subring of \( \mathcal{P}(\mathbb{R}^n, \mathbb{R}) \), known as the ring of invariants. Similarly a vector field \( v \) is called an equivariant of (the flow of) \( Ax \), if \( \frac{\partial}{\partial t} (e^{-At}v(e^{At}x))|_{t=0} = 0 \), that is \( L_Av = 0 \) or \( v \in \ker L_A \). It turns out that the set of differential equation that have a given linear part and are in normal form to all orders possesses the structure of a module over a ring as the following lemma found in [12] shows.

**Lemma 1.2.1.** For any matrix \( A \), the space of equivariants \( \ker L_A \) is a module over the ring of invariants \( \ker D_A \).
1.3 An Introduction to the \( \mathfrak{sl}(2) \) Representation Theory

The matrices
\[
\begin{align*}
x &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & z &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]

span a vector space (over \( \mathbb{R} \) or \( \mathbb{C} \)) of \( 2 \times 2 \) matrices with trace zero, called the \( 2 \times 2 \) special linear Lie algebra, denoted by \( \mathfrak{sl}(2) \). The name Lie algebra expresses the fact that this space of matrices is closed under the commutator brackets, defined as \([P, Q] = PQ - QP\).

In fact, \( x, y, \) and \( z \) satisfy the relations
\[
[ x, y ] = z, \quad [ z, x ] = 2x, \quad [ z, y ] = -2y,
\]
which implies the closure of \( \mathfrak{sl}(2) \) under the commutator bracket. It frequently happens that a finite dimensional vector space \( V \) is equipped with three operators \( \{ X, Y, Z \} \) satisfying the following properties:
\[
\]

In such a case we say that the set \( \{ X, Y, Z \} \) form an \( \mathfrak{sl}(2) \) triad or simply a triad. If \( \{ X, Y, Z \} \) is an \( \mathfrak{sl}(2) \) triad, the vector space spanned by these operators is closed under commutator bracket and is isomorphic to \( \mathfrak{sl}(2) \) (unless \( X = Y = Z = 0 \)).

Given a vector space \( V \) and a triad \( \{ X, Y, Z \} \), there exist two splitting that provide complements to the images of \( X \) and \( Y \), namely;
\[
V = \text{im } X \oplus \ker Y = \text{im } Y \oplus \ker X.
\]

The existence of these splitting is established in the following theorem found in [12].

**Theorem 1.3.1.** Suppose that \( V \) is a finite dimensional vector space and \( \{ X, Y, Z \} \) is a triad of linear operators on \( V \) satisfying
\[
\]
Then the following properties hold:

**P1.** $X$ and $Y$ are nilpotent.

**P2.** $Z$ is diagonalizable and has integer eigenvalues (called weights).

**P3.** $\ker X$ has basis consisting of weight vectors (eigenvectors of $Z$).

**P4.** Any basis $\{v_1, v_2, ..., v_s\}$ of $\ker X$ consisting of weight vectors can be taken as a set of chains tops for Jordan chains for $Y$; that is, each sequence $v_j, Yv_j, Y^2v_j, ...$ terminates with 0 and constitutes an (independent) Jordan chain for $Y$, so that the nonzero vectors of the form $Y^i v_j$ form a basis for $V$. In particular, it follows that

$$V = \ker X \oplus \text{im } Y.$$  

(The term chain tops suggests that $Y$ be viewed as mapping down the chains.)

**P5.** The vectors $Y^i v_j$ are also weight vectors, with weights given by

$$\text{wt}(Y^i v_j) = \text{wt}(v_j) - 2i.$$

**P6.** The length of the chain headed by $v_j$ is $\text{wt}(v_j) + 1$, implying that the bottom vector of each chain is $Y^{\text{wt}(v_j)} v_j$ and has weight $-\text{wt}(v_j)$.

**P7.** The action of $X$ on the basis vectors is given by

$$X(Y^i v_j) = \text{pr}(Y^i v_j)(Y^{i-1} v_j),$$

where $\text{pr}(Y^i v_j)$ is the nonzero constant

$$\text{pr}(Y^i v_j) = \text{wt}(v_j) + \text{wt}(Y v_j) + \cdots + \text{wt}(Y^{i-1} v_j).$$

The constant $\text{pr}(Y^i v_j)$ will be called the pressure on $Y^i v_j$, because it is the sum of the weights of the vectors above $Y^i v_j$ in its Jordan chain.
P8. The number of chain tops of weight \( w \geq 0 \) equals \( m(w) - m(w + 2) \), where \( m(w) \) is the multiplicity of \( w \) as an eigenvalue of \( Z \).

This theorem may be used in the converse manner to construct \( sl(2) \) triads (for more details see [12]). That is, given the nilpotent matrix \( X \) in upper Jordan form, the first step is to create matrices \( Y \) and \( Z \) such that \( Y \) is a nilpotent matrix with the same structure as \( X \) but in modified lower Jordan form (the entries on the lower diagonal are not necessary 1's), and \( Z \) is diagonal. Examples of such \( sl(2) \) triads with \( X = N_{22} \) and \( X = N_4 \) are:

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix},
Y = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},
Z = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  

(1.3.1)

and

\[
X = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix},
Y = \begin{pmatrix}
0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 3 & 0
\end{pmatrix},
Z = \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix}
\]  

(1.3.2)

Having obtained the triad \( \{X, Y, Z\} \) we create two additional (induced) triads \( \{X, Y, Z\} \) and \( \{X, Y, Z\} \) as follows:

\[
X = D_Y, \quad Y = D_X \quad Z = D_Z
\]

\[
X = L_Y, \quad Y = L_X \quad Z = L_Z.
\]  

(1.3.3)
The first of these is a triad of differential operators and the second is a triad of Lie operators. For example, with $X = N_4$ we have

\[
X = D_y = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4},
\]
\[
y = D_x = 3x_2 \frac{\partial}{\partial x_1} + 4x_3 \frac{\partial}{\partial x_2} + 3x_4 \frac{\partial}{\partial x_3},
\]
\[
Z = D_z = 3x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_3} - 3x_4 \frac{\partial}{\partial x_4}.
\]

The $sl(2)$ normal form is defined by constructing matrices $Y$ and $Z$ given $X = N$ such that

\[
\]

and defining the induced triads $\{X, Y, Z\}$ and $\{X, Y, Z\}$ accordingly. The operators $\{X, Y, Z\}$ map each $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ to itself, so that $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)$ can be taken as the vector space in Theorem (1.3.1). It then follows that

\[
\mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) = \text{im } Y \oplus \ker X = \text{im } X \oplus \ker Y
\]

Observe that $\ker X$ is a subring of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$, the ring of invariants and $\ker X$ is a module over this subring, the $sl(2)$ normal form module according to Lemma 1.2.1.
1.4 Groebner Basis

Let \( k[x_1, \ldots, x_n] \) denote a polynomial ring over the field \( k \).

**Definition 1.4.1.** Let \( f, g \in k[x_1, \ldots, x_n] \) be non-zero polynomials. Let \( x^\gamma = \text{lcm}(\text{lm}(f), \text{lm}(g)) \), then the S-polynomials of \( f \) and \( g \) is the combination

\[
S(f, g) = \frac{x^\gamma}{\text{lt}(f)} f - \frac{x^\gamma}{\text{lt}(g)} g
\]

where \( \text{lm}(f) \) is the leading monomial of \( f \) and \( \text{lt}(f) \) the leading term of \( f \).

**Example:** Let

\[
\begin{align*}
f &= x^3y^2 - x^2y^3 + x \\
g &= x^4y + y^2
\end{align*}
\]

Then \( x^\gamma = x^4y^2 \) and

\[
S(f, g) = \frac{x^4y^2}{x^3y^2} f - \frac{x^4y^2}{x^4y} g = -x^3y^3 + x^2 - y^3
\]

**Definition 1.4.2.** Fix a monomial order and let \( G = \{g_1, \cdots, g_t\} \subseteq k[x_1, \cdots, x_n] \). Given \( f \in k[x_1, \cdots, x_n] \), we say that \( f \) reduces to zero modulo \( G \), written \( f \rightarrow_G 0 \), if \( f \) can be written in the form \( f = a_1g_1 + \cdots + a_tg_t, a_i \in k[x_1, \cdots, x_n] \).

We now state one of the main theorems of the Groebner basis method.

**Theorem 1.4.3.** (Buchberger's theorem) A basis \( G = \{g_1, \cdots, g_t\} \) for an ideal \( I \) is a Groebner basis if and only if \( S(g_i, g_j) \rightarrow_G 0 \) for all \( i \neq j \), that is, if and only if for all pairs \( i \neq j \) the remainder on division of \( S(g_i, g_j) \) by \( G \) is zero, see [1], [2].

**Corollary 1.4.4.** (Buchberger's FirstCriterion) Given a finite set \( G \subseteq k[x_1, \cdots, x_n] \), suppose that we have \( f, g \in G \) such that \( \gcd(\text{lm}(f), \text{lm}(g)) = 1 \), then \( S(f, g) \rightarrow_G 0 \), see [2], [7].

With this criterion certain S-polynomials are guaranteed to reduce to zero without doing any calculations. The following corollary can be found in [7].
Corollary 1.4.5. *(Buchberger’s Second Criterion)* Given a finite set \( G \subseteq k[x_1, \cdots, x_n] \), suppose that we have \( f_i, f_j, f_k \in G \) such that \( i < j < k \). If

\[
\begin{align*}
S(f_i, f_j) &= x^a S(f_i, f_k) + x^b S(f_j, f_k) \\
S(f_i, f_k) &\rightarrow_G 0 \\
S(f_j, f_k) &\rightarrow_G 0
\end{align*}
\]

then \( S(f_i, f_j) \rightarrow 0 \).

Definition 1.4.6. Let \( \varphi : k[y_1, \cdots, y_m] \to k[x_1, \cdots, x_n] \) be a ring homomorphism defined by

\[ \varphi : y_i \mapsto f_i \]

where \( f_i \in k[x_1, \cdots, x_n], 1 \leq i \leq m \).

Let \( h \in k[y_1, \cdots, y_m] \), say \( h(y_1, \cdots, y_m) = \sum_{\mu} C_{\mu} y_1^{\mu_1} \cdots y_m^{\mu_m} \).

\( C_{\mu} \in k, \mu = (\mu_1, \cdots, \mu_m) \in \mathbb{N}^m \) and only finitely many \( C_{\mu} \)'s are non-zero, then we have

\[ \varphi(h) = h(f_1, \cdots, f_m) \in k[x_1, \cdots, x_n]. \]

Recall that the kernel of \( \varphi \) is the ideal

\[ \text{ker } \varphi = \{ h \in k[y_1, \cdots, y_m] : \varphi(h) = 0 \} \]

that is \( h \in \text{ker } \varphi \) if and only if \( h(f_1, \cdots, f_m) = 0 \). The ker \( \varphi \) is often called the *ideal of relations* among the polynomials \( f_1, \cdots, f_m \). This ideal will play an important role in later chapters.

The following theorem, whose proof can be found in [1], provides an algorithm for computing the kernel of \( \varphi \) or more precisely the Groebner basis for the kernel of \( \varphi \).

Theorem 1.4.7. Let \( K = \langle y_1 - f_1, \cdots, y_m - f_m \rangle \subseteq k[y_1, \cdots, y_m, x_1, \cdots, x_n] \). Then \( \text{ker } \varphi = K \cap k[y_1, \cdots, y_m] \).
1.5 The Full Ring of Invariants - An Example

Let $\mathcal{R} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a subring of the ring of polynomials. Let $R_1, \ldots, R_s$ be subrings of $\mathcal{R}$ and let $f_1, \ldots, f_s \in \mathbb{R}[x_1, \ldots, x_n]$. If

$$\mathcal{R} = R_1 f_1 \oplus R_2 f_2 \oplus \cdots \oplus R_s f_s$$

(1.5.1)

then (1.5.1) is called a Stanley decomposition of $\mathcal{R}$ and every element of $\mathcal{R}$ can be written as

$$\sum_{i=1}^{s} g_i f_i$$

(1.5.2)

for $g_i \in R_i, i = 1, \ldots, s$. One major application of Theorem (1.3.1) is the calculation of $\ker \mathcal{X}$, the ring of invariants. Four steps are required to complete the calculation in any example, see [11], [12] for more details.

- Compute a finite set of invariants $I_1, \ldots, I_s$, called the basic invariants, which suffice to generate all invariants up to some given degree $j$.

- Compute a Groebner basis for the ideal of relations among the basic invariants.

- From the Groebner basis, determine a Stanley decomposition for the ring $\mathcal{R}$ of polynomials in the basic invariants.

- From the Stanley decomposition, set up a two-variable generating function called the table function (Hilbert function), and use it to test that $\mathcal{R}$ is in fact all of $\ker \mathcal{X}$. If it is not, then not all of the basic invariants have been found. In that case, return back to the first step and increase the value of $j$.

We observe that the operators $\{X, Y, Z\}$ map each $\mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$ for $i = 1, \ldots, j$ to itself, so that $\mathcal{P}_j(\mathbb{R}^n, \mathbb{R})$ can be taken to be the vector space in Theorem (1.3.1). Since $Z$, is diagonal and $Z = D_z$, the monomials in $(x_1, \ldots, x_n)$ are the eigenvectors of $Z$, that is, the weight vectors. As an example we will find the ring of invariants $\ker \mathcal{X}$ for the triad
\{X, Y, Z\} with X = N_4. The associated differential operators are as defined in equation (1.3.4). For step one, the basic invariants can be shown to be
\[\alpha = x_1\]
\[\beta = 2x_1^2 - 3x_1x_3\]
\[\gamma = 4x_2^3 - 9x_1x_2x_3 + 9x_1^2x_4\]
\[\delta = 9x_1^2x_4^2 - 3x_2^2x_3^2 - 18x_1x_2x_3x_4 + 6x_1x_3^3 + 8x_2^2x_4\]

For step two, the relation satisfied by this invariants is
\[\gamma^2 = 2\beta^3 + 9\alpha^2\delta\]
and there are no other relations. Thus, \(\gamma^2 - 2\beta^3 - 9\alpha^2\delta\) is the Groebner basis for the ideal of relations. Now consider the ring \(R = \mathbb{R}[\alpha, \beta, \gamma, \delta] \subset P_3(\mathbb{R}^n, \mathbb{R}^n)\) of polynomials in the known basic invariants. The representation of an element of \(R\) as a polynomial is not unique because of the relation above, but this equation itself can be used to restore the uniqueness by excluding \(\gamma^2\) (or any high power of \(\gamma\)). Thus, a Stanley decomposition of \(R\) is:
\[R = \mathbb{R}[\alpha, \beta, \delta] \oplus \mathbb{R}[\alpha, \beta, \delta] \gamma.\quad (1.5.3)\]

Another way to say this is that any polynomial in \(R\) can be written uniquely as
\[f(\alpha, \beta, \delta) + g(\alpha, \beta, \delta)\gamma,\quad (1.5.4)\]
where \(f\) and \(g\) are polynomials in three variables \(\alpha, \beta, \) and \(\delta\). The Stanley decomposition (1.5.4) can be abbreviated as \(f.1 + g.\gamma\); \(f\) and \(g\) will be referred to as coefficient functions, and 1 and \(\gamma\) as Stanley basis elements.

To generate the table function of the Stanley decomposition, we replace each term in (1.5.3) by a rational function \(P/Q\) in \(d\) and \(w\)(for "d=degree in \(x\)" and "w=weight") constructed as follows: for each basic invariant (\(\alpha, \beta, \) or \(\delta\)) appearing in a coefficient
function \((f\text{ or } g)\), the denominator will contain a factor \(1 - dpw^q\), where \(p\) and \(q\) are the degree and weight of the invariant; the numerator will be \(dpw^q\), where \(p\) and \(q\) are the degree and weight of the Stanley basis element of that term. When the rational functions \(P/Q\) from each term of the Stanley decomposition are summed up we obtain the table function \(T\) given by \(T = \sum_i P_i/Q_i\). Thus, for this example, the table function is:
\[
T = \frac{1 + d^3w^3}{(1 - dw^3)(1 - d^2w^2)(1 - d^4)}.
\]

The following lemma found in [12] gives a method to check that enough basic invariants have been found.

**Lemma 1.5.1.** Let \(\{X, Y, Z\}\) be a triad of \(n \times n\) matrices, let \(\{X, Y, Z\}\) be the induced triad, and suppose that \(I_1, \ldots, I_l\) is a finite set of polynomials in \(\ker X\). Let \(R\) be a subring of \(\mathbb{R}[I_1, \ldots, I_l]\); suppose that the relations among the \(I_1, \ldots, I_l\) have been found, and that the Stanley decomposition and its associated table function \(T(d, w)\) have been determined. Then \(R = \ker X \subset \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)\) if and only if
\[
\frac{\partial}{\partial w} wT \bigg|_{w=1} = \frac{1}{(1 - d)^n}.
\]

In the above example of \(N_4\), \(R = \ker X\), since
\[
\frac{\partial}{\partial w} wT \bigg|_{w=1} = \frac{1}{(1 - d)^4}.
\]

### 1.6 The Basic Isomorphism and Stanley Decomposition of \(\ker X\)

The goal of this section is to describe a procedure for obtaining a Stanley decomposition for \(\ker X\) given a Stanley decomposition for \(\ker X\), where \(X\) and \(X\) are defined as in equations (1.3.3).
Let $N_{r_1, r_2, \ldots, r_k}$ be an $n \times n$ block diagonal nilpotent matrix with upper Jordan blocks of sizes $r_1, r_2, \ldots, r_k$ with $r_1 + r_2 + \ldots + r_k = n$. Let $R_i = r_1 + r_2 + \ldots + r_i, i = 1, 2, \ldots, k,$ so that $R_1, R_2, \ldots, R_k$ are the row numbers of the bottom rows of the Jordan blocks.

Define a map
\[
\tilde{\varphi} : \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n, \mathbb{R}^n)
\]
by
\[
\tilde{\varphi}(v_1, \ldots, v_n) = (v_{R_1}, \ldots, v_{R_k}).
\]

Clearly $\tilde{\varphi}$ is a homomorphism of modules over $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$. Let $\varphi$ be the restriction of $\tilde{\varphi}$ to $\ker X$, hence we have the following Theorem.

**Theorem 1.6.1.** The image of $\varphi$ is $\ker X^r \oplus \ker X^{r_2} \oplus \ldots \oplus \ker X^{r_k}$ and the mapping $\varphi : \ker X \longrightarrow \ker X^r \oplus \ker X^{r_2} \oplus \ldots \oplus \ker X^{r_k}$ is an isomorphism of modules over the ring $\ker X$.

Since the proof is by an example it is worthy repeating it here.

**Proof.** Observe that if $f \in \ker X$ and $g \in \mathcal{P}(\mathbb{R}^n, \mathbb{R})$ then
\[
\mathcal{X}(fg) = f \mathcal{X}g.
\]

It follows that if $g \in \ker X^r$ (for any $r$) then, $fg \in \ker X$: that is $\ker X^r$ is a module over $\ker X$. The rest of the proof will be clear after considering the example

\[
N_{2\times 2} = \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix},
\]
in this case follows that $\varphi(v_1, \cdots, v_6) = (v_2, v_4, v_6)$ and if $v \in \ker X$ then $Xv_1 = 0$, $Xv_2 = v_1$, $Xv_3 = 0$, $Xv_4 = v_3$, $Xv_5 = 0$, $Xv_6 = v_5$. These conditions imply that; $X^2v_2 = 0, X^2v_4 = 0, X^2v_6 = 0$, so that, $\varphi(v) = (v_2, v_4, v_6) \in \ker X^2 \oplus \ker X^2 \oplus \ker X^2$, and also shows that $v \in \ker X$ can be reconstructed from $(v_2, v_4, v_6)$ by the reconstruction map $\varphi^{-1}(v_2, v_4, v_6) = \begin{pmatrix} Xv_2 \\ v_2 \\ Xv_4 \\ v_4 \\ Xv_6 \\ v_6 \end{pmatrix}$.

Thus $\varphi$ is invertible. Since it is a module homomorphism, it is an isomorphism. 

Lemma 1.6.2. If $h \in P(\mathbb{R}^n, \mathbb{R})$ belong to the $\ker D_N^r$, then the vector polynomial $v_{(s,h)}$ defined by

$$v_{(s,h)} = \sum_{i=0}^{r_s-1} (D_N^i h) e_{r_s-i}$$

belong to $\ker L_N$. For instance if

$$N_{2,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $s \in \{1, 2\}$, $r_2 = 2$, $R_1 = 2$, $r_2 = 3$, $R_2 = 5$,

$$v_{(1,h)} = \begin{bmatrix} D_N h \\ h \\ 0 \\ 0 \end{bmatrix}, \text{ for } h \in \ker D_N^2$$
and

\[ v_{(2,h)} = \begin{bmatrix} 0 \\ 0 \\ D_N^2 h \\ D_N h \\ h \end{bmatrix}, \quad \text{for } h \in \ker D_N^2. \]

**Definition 1.6.3.** If \( J \) is a monomial ideal, the monomials belonging to \( J \) are called *nonstandard monomials*. The *standard monomials* with respect to this ideal are the monomials that do not belong to it.

The following Lemma found in [11] forms the basis for obtaining the Stanley decomposition of \( \ker X \).

**Lemma 1.6.4.** Let \( \mathcal{R} \) be any subring of \( \ker X \) generated by homogeneous polynomials \( I_1, \ldots, I_s \), in \( x = (x_1, \ldots, x_n) \) which are weight vectors for the triad \( \{X, Y, Z\} \), and let \( \mathcal{R}_{ik} \) be the vector subspace of \( \mathcal{R} \) consisting of polynomials homogeneous in \( x \) of degree \( i \) and weight \( k \). Let a Groebner basis for the relations of \( I_1, \ldots, I_s \) be selected. Then:

1. The standard monomials in \( I_1, \ldots, I_s \) (with respect to the given Groebner basis) having degree \( i \) (in \( x \)) and weight \( k \) form a basis for \( \mathcal{R}_{ik} \).

2. If \( \mathcal{R} = \ker X \), the standard monomials of degree \( i \) form a set of chain tops for the chains in \( \mathcal{P}(\mathcal{R}^n, \mathcal{R}) \).

According to this Lemma, the chain tops of \( \mathcal{P}(\mathcal{R}^n, \mathcal{R}) \) under the triad \( \{X, Y, Z\} \) may be taken to be the standard monomials in the basic invariants \( I_1, \ldots, I_s \) with respect to the given Stanley decomposition of \( \ker X \). The chains under the chain tops can be obtained by repeated application of \( Y \), and a vector space basis for \( \ker X \) can be obtained by computing the iterates down to depth \( r \).
Let $f$ be a standard monomial of degree $j$ (in $x$) and let $y^j f$ be a non-zero entry in the chain under $f$. We define $g \in P_i(\mathbb{R}^n, \mathbb{R})$ to be a replacement for $y^j f$ if $x^r g$ is a non-zero multiple of $f$.

**Lemma 1.6.5.** If a vector subspace $V \subset \ker x^r$ contains a replacement for every chain element to depth $r$, then $V = \ker x^r$, see [11].

**Lemma 1.6.6.** Let $f$ be a standard monomial. A replacement for $y^j f$ can be found by placing $r$ copies of $y$ arbitrarily in front of the various factors of $f$, as long as the result is not zero, see [11].

Recall that the maximum power of $y$ that can be applied to an invariant equals the weight ($\text{length} - 1$) of the invariant. By the above Lemma, think of each standard monomial as being written without powers, so that $I_2^2 I_3^2$ appears as $I_2 I_2 I_3 I_3$. Apply $y$ to the last factor until the power of $y$ equals its weight, then to the factor before that, and so on, stopping when the total number of factors of $y$ reaches $r - 1$ (for the construction of replacements for the chain elements under a standard monomial to depth $r$). Each replacement constructed in this manner contains two parts, a prefix, which is itself a standard monomial and contains no $y$ and a suffix, which begins with the first occurrence of $y$. It is clear that no basic invariant of weight zero ($\text{length}$ one) can appear in a suffix; we call a such basic invariant trivial.

The next step is to describe the set of prefixes that can occur with any given suffix. Let $S$ be a suffix and let $g$ be the standard monomial that results from deleting all occurrences of $y$ in $S$; we call $g$ a stripped suffix. Let $f$ be any other standard monomial. Then $fS$ occurs as a replacement (that is, $f$ is a prefix for $S$) precisely when the following two conditions are satisfied:

1. $fg$ is a standard monomial (so that $fg$ occurs as a chain top);
2. The factors $fg$ are correctly ordered, equivalently, the final factor of $f$ either precedes or equals the first factor of $g$. 
Let \( m_1, \ldots, m_p \) be the leading monomials of the Groebner basis for the basic invariants \( I_1, \ldots, I_s \). Given \( g \), the condition (1) for \( fg \) to be standard is that \( f \) not be divisible by any of the monomials \( m'_i = m_i / \gcd(m_i, g) \). Let the first basic invariant appearing in \( g \) be \( I_{i(g)} \). Then the condition (2) for \( fg \) to be correctly ordered, is that \( f \) not divisible by \( I_{i(g)-1}, \ldots, I_1 \) (ordering the basic invariants by \( I_i < I_j \) if \( j < i \)). Therefore the prefix monomials \( f \) associated with the given stripped suffix \( g \) are the standard monomials with respect to the (new) ideal \( (m'_1, \ldots, m'_p, I_{i(g)-1}, \ldots, I_1) \). Now let \( f \) be the prefix monomial associated with a given suffix \( S \). Then the collection of polynomials which are linear combination of such prefix monomials for a given suffix \( S \) is a ring, called the prefix ring for \( S \), which has a Stanley decomposition (defined by its standard prefix monomials). This Stanley decomposition will be denoted by \( P(S) \), the Stanley decomposition of the prefix ring for the suffix \( S \). We conclude this section by the following theorem found in [11].

**Theorem 1.6.7.** A Stanley decomposition for \( \ker \mathcal{X} \) is given by

\[
\ker \mathcal{X} = SD(\ker \mathcal{X}) \oplus \bigoplus_S P(S)S,
\]

where:

1. \( SD(\ker \mathcal{X}) \) is the Stanley decomposition of the invariant ring determined by a particular Groebner basis for the relations among the invariants;

2. the sum ranges over all suffixes \( S \) of depth \( \leq r \), suffixes being defined as in 1.6.6 using a selected ordering of the basic invariants; and

3. \( P(S) \) is the Stanley decomposition of the prefix ring for \( S \), as defined above, using as standard monomials those determined by the same Groebner basis used to obtain \( SD(\ker \mathcal{X}) \).
As an example we give a Stanley decomposition of the $sl(2)$ normal form for the real system of differential equations

$$\dot{x} = Nx + \cdots$$

(1.6.3)

with $x \in \mathbb{R}^4$ and $N = N_{22}$. The natural triad $\{X, Y, Z\}$ with $X = N$ is:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

and the associated differential operators are

$$X = D_Y = x_1 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_4},$$

$$Y = D_X = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3},$$

$$Z = D_Z = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_4}.$$ 

The first step is to compute the ring of invariants $\ker X$. There are three invariants, namely,

$$\alpha = x_1,$$

$$\beta = x_3,$$

$$\gamma = x_1 x_4 - x_2 x_3$$

(1.6.4)

of weights 1, 1, and 0 respectively. A table function arguments, shows that they are a complete set of basic invariants. Thus,

$$\ker X = \mathbb{R}[\alpha, \beta, \gamma].$$

(1.6.5)

The next step is a compute $\ker X^2$ as module over $\ker X$; this will be used twice since the two Jordan blocks in $N$ have size 2. The chains under $\alpha$, $\beta$ and $\gamma$ have length 2,
2, 1 respectively. Ordering the basic invariants by $\gamma < \beta < \alpha$, the standard monomials can be written $\gamma \beta \alpha^k$. There are three classes of monomials, those ending in $\alpha, \beta,$ and $\gamma$ respectively. Monomials of the first type produced the suffix $y_\alpha = x_2$, with prefix ring $\mathbb{R}[\alpha, \beta, \gamma]$; those of the second type yield suffix $y_\beta = x_4$ with prefix ring $\mathbb{R}[\beta, \gamma]$; and those of the third type have no suffix, since $y\gamma = 0$. Thus, according to Theorem (1.6.7)

$$\ker X^2 = \mathbb{R}[\alpha, \beta, \gamma] \oplus \mathbb{R}[\alpha, \beta, \gamma] x_2 + \oplus \mathbb{R}[\alpha, \beta, \gamma] x_4.$$  \hspace{1cm} (1.6.6)

Finally, according to Theorem (1.6.1) the normal form module of equivariants is, $\ker X \cong \ker X^2 \oplus \ker X^2$, and explicitly by Lemma 1.6.2,

$$\ker X = \mathbb{R}[\alpha, \beta, \gamma] v_{1,1} \oplus \mathbb{R}[\alpha, \beta, \gamma] v_{1,x_2} \oplus \mathbb{R}[\beta, \gamma] v_{1,x_4} \oplus \mathbb{R}[\alpha, \beta, \gamma] v_{2,x_4} \oplus \mathbb{R}[\beta, \gamma] v_{2,x_4}.$$ \hspace{1cm} (1.6.7)

Where

$$v_{1,1} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_{1,x_2} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \quad v_{1,x_4} = \begin{bmatrix} x_3 \\ x_4 \\ 0 \\ 0 \end{bmatrix},$$

and similarly for the rest.

### 1.7 Outline of this Thesis

The results in this thesis are based mainly on the work by Murdock in [10], [11], and [12], that is, the application of Murdock's methods for computing Stanley decomposition for the nilpotent systems and their unfolding.

This chapter is an introductory chapter, which consists of some background knowledge for understanding the content of the thesis. We briefly discuss the concept of ring of invariants and the module of equivariants.
In chapter 2, which forms the central part of this thesis, we use Groebner basis methods to compute Groebner basis for the ideal of relations among the basic invariants.

In chapter 3, we compute Stanley decomposition of the ring of invariants. We introduce the concept of partially ordered set ring and develop an easy way to write down a Stanley decomposition, when the Groebner basis for the ideal of relations is large.

In chapter 4, we apply Murdock's algorithm to find the Stanley decomposition of the module of equivariants, from the Stanley decomposition of the ring of invariants.

In chapter 5, we introduce the concept of unfolding of a dynamical system giving an example of a first-order unfolding in the $sl(2)$ normal form case. We also give a brief exposition of Cushman and Sanders method in solving the same problem using a different method, showing the similarities and the differences with our method.
CHAPTER 2. Ring of Invariants

2.1 Goal of this Thesis

A single Takens-Bogdanov system has the form

\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \text{quadratic terms + cubic terms + \ldots}
\]

Let \( N_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), then a coupled Takens-Bogdanov system has the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\vdots \\
\dot{x}_n \\
\dot{y}_n
\end{bmatrix} = \begin{bmatrix} N_2 & & & & \\
& N_2 & & & \\
& & \ddots & & \\
& & & N_2 & \\
& & & & N_2
\end{bmatrix} \begin{bmatrix} x_1 \\
y_1 \\
\vdots \\
x_n \\
y_n
\end{bmatrix} + \text{higher order terms (h.o.t.).}
\]

Our goal is to describe the equivariants (normal forms) for the coupled Takens-Bogdanov systems:

\[
\dot{x} = Nx + h.o.t;
\] (2.1.1)
where, \( x \in \mathbb{R}^n \) and \( N = \begin{pmatrix} N_2 \\ N_2 \\ \vdots \\ N_2 \end{pmatrix} \).

As an example of a normal form, we consider the double Takens-Bogdanov system

\[
\dot{x} = N_{22} x + \cdots
\]

The basic invariants are: \( \alpha_1 = x_1, \ \alpha_2 = x_2, \ \beta_{12} = x_1 y_2 - x_2 y_1 \), and the basic equivariants are:

\[
\begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
x_1 \\
y_1 \\
x_2 \\
y_2
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

hence the normal form is:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\dot{x}_2 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x_1 \\
y_1 \\
x_2 \\
y_2
\end{pmatrix} + \begin{pmatrix}
f_1(\alpha_1, \alpha_2, \beta_{12}) \\
f_2(\alpha_1, \alpha_2, \beta_{12}) \\
f_3(\alpha_2, \beta_{12}) \\
f_4(\alpha_1, \alpha_2, \beta_{12}) \\
f_5(\alpha_1, \alpha_2, \beta_{12}) \\
f_6(\alpha_2, \beta_{12})
\end{pmatrix} \begin{pmatrix}
x_1 \\
y_1 \\
x_2 \\
y_2
\end{pmatrix}
\]
2.2 Groebner Basis for the Invariants

In this section, we find the Groebner basis for the basic invariants, associated with the ring of invariants \( \ker X \) for the \( sl(2) \) normal form for coupled Takens-Bogdanov systems;

\[
x = Nx + h.o.t;
\]  \hspace{1cm} (2.2.1)

where, \( x \in \mathbb{R}^{2n} \) and \( N = \begin{pmatrix} N_2 \\ \vdots \\ N_2 \end{pmatrix} \).

According to section 1.5, four steps are required to complete this calculation.

- Compute a finite set of invariants \( I_1, I_2, \ldots, I_s \) for \( \ker X \) called basic invariants which suffices to generate all invariants up to some chosen degree \( j \).
- Compute a Groebner basis for the ideal of relations among the basic invariants.
- From the Groebner basis, determine a Stanley decomposition for the ring \( \mathcal{R} \) of polynomials in the basic invariants.
- From the Stanley decomposition, set up a two-variable generating function called the table function (Hilbert function), and use it to test that \( \mathcal{R} \) is in fact all of \( \ker X \). If it is not, then not all of the basic invariants have been found. In that case, return back to the first step and increase the value of \( j \).

In this section, we shall be concerned with the first two steps. In the first step the invariants can be found by methods outlined in [12] and are given by the following lemma.

Lemma 2.2.1. Let \( \alpha_i = x_i; \) for \( 1 \leq I \leq n \) and let \( \beta_{ij} = x_iy_j - x_jy_i; \) for \( 1 \leq i < j \leq n \). Then \( \{\alpha_i, \beta_{ij}\} \) is a set of invariants of \( \ker X \).
Proof. Recall that \( X = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + \cdots + x_n \frac{\partial}{\partial y_n} \) and since \( \gamma \) is an invariant of \( \ker X \) if \( X(\gamma) = 0 \), it follows immediately that \( \{ \alpha_i, \beta_{ij} \} \) are invariants of \( X \). \( \square \)

In step two, we compute a Groebner basis for the ideal of relations among the invariants. We proceed as follows:

1 Introduce slack variables \( \{ z_i; 1 \leq i \leq n \} \) and \( \{ w_{ij}; 1 \leq i < j \leq n \} \).

2 Let \( \{ x, y \} = \{ x_1, y_1, x_2, y_2, \cdots, x_n, y_n \} \) and \( \{ z_i; 1 \leq i \leq n, w_{ij}; 1 \leq i < j \leq n \} \).

3 Let \( \Phi : \mathbb{R}[z, w] \rightarrow \mathbb{R}[x, y] \) be a map defined by

\[
\begin{align*}
z_i & \mapsto \alpha_i \\
w_{ij} & \mapsto \beta_{ij}
\end{align*}
\]

**Definition 2.2.2.** A relation among the invariants is a function \( f(z, w) \), such that \( f(\alpha, \beta) = 0 \). That is \( f \in \ker \Phi \), the ideal of relations among the polynomial invariants, for a review see section 1.3.

We shall use Groebner basis methods to find the kernel of \( \Phi \), or more precisely the Groebner basis for the kernel of \( \Phi \), see [1], [2] for more details. Let

\[
\begin{align*}
g_i &= \alpha_i - z_i, \text{ for } 1 \leq i \leq n \\
h_{ij} &= \beta_{ij} - w_{ij}, \text{ for } 1 \leq i < j \leq n
\end{align*}
\]

and set \( F = \{ g_i, h_{ij} \} \). By Theorem (1.4.7) we first compute a Groebner basis \( G \) for the polynomials ideal \( K = \langle \alpha_i - z_i, \beta_{ij} - w_{ij} \rangle \) in \( \mathbb{R}[x, y, z, w] \) generated by \( F \). The polynomials in \( G \) without \( x \) and \( y \) variables form a Groebner basis \( H \) for the kernel of \( \Phi \), the ideal of relations. We shall find \( G \) with respect to lexicographical ordering on the \( xy \)-variables with \( x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n \) and degree reverse lexicographical ordering on the \( zw \)-variables with \( w_{12} > \cdots > w_{1n} > w_{23} > \cdots > w_{2n} > \cdots w_{n-1n} > z_1 > z_2 > \cdots > z_n \), which is an elimination order in which the \( xy \)-variables
are larger than the zw-variables. From section 1.3, recall that the strategy for computing a Groebner basis is to reduce the S-polynomials of \( F \) and if the remainder is non-zero add it to \( F \), do this until there are enough polynomials to make all S-polynomials reduce to zero. Equivalently, we say that \( f \in G \) reduces to zero modulo \( G \), written \( f \rightarrow_G 0 \), if \( f \) can be written in the form \( f = a_1 g_1 + \cdots + a_t g_t \). In other words the remainder on division of \( f \) by \( G \) is zero. If this is true for all S-polynomials of \( G \), then \( G \) is a Groebner basis. The following theorem summarizes the detailed calculations presented thereafter, that provide the framework for the results of this thesis.

\[
\begin{align*}
  g_i &= x_i - z_i, \\
  h_{ij} &= x_i y_j - x_j y_i - w_{ij}, \\
  r_{ij} &= y_i z_i - y_j z_i + w_{ij}, \\
  r_{ijk} &= y_i w_{jk} - y_j w_{ik} + y_k w_{ij}, \\
  r'_{ijk} &= z_i w_{jk} - z_j w_{ik} + w_{ij} z_k, \\
  r_{ijkl} &= w_{il} w_{jk} - w_{ik} w_{jl} + w_{ij} w_{kl},
\end{align*}
\]

is a Groebner basis for the polynomial ideal \( K = \langle F \rangle \).

The proof of this Theorem is accomplished by the following 14 Lemmas, in which the calculations involved are suppressed and only the final result is recorded.

**Lemma 2.2.4.** \( S(g_i, g_j) \rightarrow_G 0 \) for \( i \neq j \).

*Proof.* This follows immediately from Buchberger's first criterion (1.4.4). \( \square \)

**Lemma 2.2.5.**

a. \( S(g_i, h_{jk}) \rightarrow_G 0 \) for \( i \neq j \)

b. \( S(g_i, h_{jk}) \rightarrow_G 0 \) for \( i = j \)

*Proof.** a. Follows from Buchberger’s first criterion (1.4.4). b. Let \( i = j \), since \( S(g_j, h_{jk}) = x_k y_j - y_k z_j + w_{jk} = y_i g_k + r_{jk} \), then b follows. \( \square \)

**Lemma 2.2.6.**

a. \( S(h_{ij}, h_{kl}) \rightarrow_G 0 \) for \( i \neq k, j \neq l \)
b. $S(h_{ij}, h_{kl}) \rightarrow_{G} 0$ for $i = k, j \neq l$

c. $S(h_{ij}, h_{kl}) \rightarrow_{G} 0$ for $i \neq k, j = l$

Proof. a. Follows from Buchberger's first criterion.

b. Suppose $k = i$, then $i < j < l$, and $S(h_{ij}, h_{il}) = -x_jy_iy_l + x_ky_iy_j - y_lw_ij + y_jw_{il} = -y_ih_{ij} + r_{ijl}$, hence b follows.

c. Suppose $l = j$, there are two analogous cases to consider, $i < k < j$ and $k < i < j$.

Suppose $i < k < j$, since $S(h_{ij}, h_{kj}) = x_iy_jy_k - x_ky_jy_i + x_iw_{jk} - x_kw_{ij} = x_jh_{ik} - w_{ik}g_j + w_{kj}g_i - w_{ij}g_k + r_{ikj}$. The other case is treated similarly, hence the result follows.

\[ \square \]

Lemma 2.2.7. a. $S(h_{ij}, r_{kl}) \rightarrow_{G} 0$ for $j \neq k$;

b. $S(h_{ij}, r_{kl}) \rightarrow_{G} 0$ for $k = j$

Proof. a. Follows from Buchberger's first criterion.

b. Suppose $k = j$, then $i < j < l$ and $S(h_{ij}, r_{jl}) = x_iy_iy_jz_j - x_jy_iy_i - x_iw_{jl} - z_{i}w_{ij} = z_jh_{il} - x_jr_{il} + x_ir_{ij} - z_ih_{jl} - w_{ij}g_i + w_{il}g_j - w_{ij}g_l - 2r_{ijl}$, hence b follows.

\[ \square \]

Lemma 2.2.8. a) $S(h_{ij}, r_{kls}) \rightarrow_{G} 0$ for $j \neq k$;

b) $S(h_{ij}, r_{kls}) \rightarrow_{G} 0$ for $k = j$

Proof. a) Follows from Buchberger's first criterion.

b) Suppose $k = j$, since $i < j < l < s$ and $S(h_{ij}, r_{jls}) = x_iy_lw_{js} - x_iy_sw_{jl} - x_jy_lw_{ls} - w_{ij}w_{ls} = w_{js}h_{il} - w_{jl}h_{ls} - x_jr_{ils} + x_ir_{jls} - x_sr_{ijl} - w_{il}h_{js} - w_{il}h_{js} - w_{ij}h_{ls} + 2r_{ijls}$, hence b is true.

\[ \square \]
Lemma 2.2.9. a. $S(r_{ij}, r_{st}) \rightarrow G 0$ for $s \neq i$ and $t \neq j$;

b. $S(r_{ij}, r_{st}) \rightarrow G 0$ for $s = i$, and $t \neq j$;

c. $S(r_{ij}, r_{st}) \rightarrow G 0$ for $r \neq i$, and $t = j$.

Proof. a. Follows from Buchberger's first criterion.

b. Suppose $s = i$ and $t \neq j$, either $i < j < t$ or $i < t < j$, but, since $y$ dominates, it is sufficient to consider only one case, say, $i < j < t$. Then $S(r_{ij}, r_{jt}) = -y_j z_i z_t + y_s z_i z_j - z_j w_{it} + z_i w_{ij} = -z_i r_{jt} + r_{ijt}$, and b follows.

c. Suppose $s \neq i$ and $t = j$, there are two cases to consider, $i < k < j$ or $k < i < j$, but, since $y$'s dominates, one case is sufficient, say, $i < s < j$. Then $S(r_{ij}, r_{sj}) = y_i y_j z_s - y_s y_j z_i - y_i w_{sj} + y_s w_{ij} = y_j r_{is} + r_{isj}$, and c follows.

Lemma 2.2.10. a) $S(r_{ij}, r_{kls}) \rightarrow G 0$ for $i \neq k$;

b) $S(r_{ij}, r_{kls}) \rightarrow G 0$ for $k = i$.

Proof. a. Follows from Buchberger's first criterion.

b) Suppose $k = i$, then $S(r_{ij}, r_{ils}) = -y_j z_i w_{ls} + y_l z_j w_{is} - y_s z_j w_{i} + w_{ij} w_{ls}$. Since, $i < l < s$ and $i < j$ there are five cases to consider:

- case i $i < j < l < s$;
- case ii $i < l < j < s$;
- case iii $i < l < s < j$;
- case iv $i < j = l < s$;
- case v $i < l < j = s$. 
We will treat cases (i), and (vi), the rest are treated analogously.

**case i.** $i < j < l < s$, since $S(r_{ij}, r_{ils}) = y_jz_iw_{ls} - y_iz_jw_{is} + y_sz_jw_{it} - w_{ij}w_{ls} = y_jr_{ils} + w_{is}r_{jl} - w_{it}r_{js} - r_{ijls}$, and

**case iv.** $i < j < s$, since $S(r_{ij}, r_{ij}s) = y_jz_iw_{js} - y_iz_jw_{is} + y_sz_jw_{ij} - w_{ij}w_{js} = y_jr_{ij}s - w_{ij}r_{js}$, b follows.

**Lemma 2.2.11.** a) $S(r_{ij}, r_{ij}^t) \rightarrow_G 0$ for $l \neq j$;

b) $S(r_{ij}, r_{kl}s) \rightarrow_G 0$ for $r = j$.

**Proof.** a) Follows from Buchberger’s first criterion.

b) Suppose $l = j$, there is only one case to consider, $i < j < s < t$. Since $S(r_{ij}, r_{ij}^t) = y_iz_jw_{jt} - y_iz_iw_{js} + y_jz_iw_{st} + w_{ij}w_{st} = w_{jt}r_{ls} - w_{js}r_{st} - z_{it}r_{js} + r_{ijst}$, then $S(r_{ij}, r_{kl}s) \rightarrow_G 0$.

**Lemma 2.2.12.** a) $S(r_{ijk}, r_{ilst}) \rightarrow_G 0$ for $l \neq i$ and $s \neq j$ or $k \neq t$;

b) $S(r_{ijk}, r_{ilst}) \rightarrow_G 0$ for $l = i$ and $t \neq j$;

c) $S(r_{ijk}, r_{ilst}) \rightarrow_G 0$ for $l \neq i$, $s = j$ and $t = k$.

**Proof.** a) Follows from Buchberger’s first criterion.

b) Suppose $l = i$ and $s \neq j$ or $t \neq k$, then $S(r_{ijk}, r_{ilst}) = -y_jw_{ik}w_{st} + y_kw_{ij}w_{st} + y_sw_{it}w_{jk} - y_tw_{is}w_{jk}$. Since $i < j < k$ and $i < s < t$, there are six cases to consider,

**case i** $i < j < k < s < t$

**case ii** $i < j < s < k < t$

**case iii** $i < s < j < k < t$
case iv \( i < j < k = s < t \)

case v \( i < j < s < k = t \)

case vi \( i < s = j < k < t \)

We will treat cases i and iv, the rest follows easily from these.

\textbf{case i.} \( i < j < k < s < t \), then \( S(r_{ijk}, r_{ist}) = -y_jw_{ik}w_{st} + y_kw_{ij}w_{st} + y_sw_{it}w_{jk} - y_tw_{is}w_{jk} = -w_{ik}r_{jst} + w_{ij}r_{kst} + y_sr_{ijkt} - y_tr_{ijks} \), and

\textbf{case iv.} \( s = k \), so \( i < j < k < t \), then \( S(r_{ijk}, r_{ikt}) = -y_jw_{ik}w_{kt} + y_kw_{ij}w_{kt} + y_kw_{it}w_{jk} - y_tw_{ik}w_{jk} = y_kr_{ijkt} - w_{ik}r_{jkt} \). \( S(r_{ijk}, r_{lkt}) \rightarrow_0 \).

c Suppose \( s = j \) and \( t = k \) and \( l \neq i \), there are only two cases to consider, \( i < l < j < k \) and \( l < i < j < k \), but, since they are similar, one case is sufficient, say, \( i < l < j < k \). Then, since \( S(r_{ijk}, r_{ijk}) = y_iy_jw_{lk} - y_iy_kw_{lj} - y_iy_kw_{ik} + y_iy_kw_{ij} = y_jr_{ik} - y_kr_{ij} \), we have \( S(r_{ijk}, r_{lkt}) \rightarrow_0 \).

\begin{lemma}
\textbf{Lemma 2.2.13.} a. \( S(r_{ijk}, r_{lkt}) \rightarrow_0 \) if \( s \neq j \) or \( t \neq k \);

b) \( S(r_{ijk}, r_{lkt}) \rightarrow_0 \) if \( s = j \) and \( t = k \).
\end{lemma}

\textit{Proof.} a. Follows from Buchberger's first criterion.

b. Suppose \( s = j \) and \( t = k \), then \( S(r_{ijk}, r_{ijk}) = -y_jz_{ik}w_{lk} + y_kz_{ij}w_{ij} + y_iy_kw_{ik} - y_iy_kw_{ij} \),

there are two cases to consider,

\textbf{case i.} \( i < l < j < k \)

\textbf{case ii.} \( l < i < j < k \)

\textbf{case i.} \( i < l < j < k \), \( S(r_{ijk}, r_{ijk}) \rightarrow_0 \), this follows from Buchberger's second criterion (1.4.5), Lemma (2.2.10) and Lemma (2.2.11).
Lemma 2.2.14. a. \( S(r_{ijk}, r_{lstu}) \to 0 \) for \( j \neq l \) and \( j \neq s \) or \( k \neq u \) and \( k \neq t \);

b. \( S(r_{ijk}, r_{lstu}) \to 0 \) for \( l = j \) and \( u = k \);

c. \( S(r_{ijk}, r_{lstu}) \to 0 \) for \( j = s \) and \( t = k \).

Proof. a. Follows from Buchberger’s first criterion.

b. Suppose \( l = j \) and \( u = k \), since \( i < j < s < t < k \) (the only case), and \( S(r_{ijk}, r_{jst}) = -y_j w_{ik} w_{st} + y_k w_{ij} w_{st} + y_i w_{js} w_{ak} - y_i w_{js} w_{tk} = w_{st} r_{ij} - w_{tk} r_{js} - y_j r_{istk} + w_{ij} r_{st} \), we have \( S(r_{ijk}, r_{lstu}) \to 0 \).

c. Suppose \( s = j \) and \( t = k \), then \( S(r_{ijk}, r_{ijkl}) = -y_j w_{iu} w_{ik} + y_k w_{iu} w_{ij} + y_i w_{ik} w_{ju} - y_i w_{ij} w_{ku} = w_{jk} r_{ijs} - w_{kt} r_{js} - y_j r_{istk} + w_{ij} r_{st} \). Since \( i < j < k \) and \( l < j < k < u \), there are three cases to consider,

**case i** \( i < l < j < k < u \)

**case ii** \( l < i < j < k < u \)

**case iii** \( i = l < j < k < u \)

case i. \( i < l < j < k < u \), since \( S(r_{ijk}, r_{ijkl}) = y_i w_{ik} w_{ju} - y_i w_{ij} w_{ku} - y_j w_{iu} w_{ik} + y_k w_{iu} w_{ij} = w_{ik} r_{iju} - w_{ij} r_{iku} + y_j r_{iku} - w_{il} r_{iku} - y_k r_{ij} + y_u r_{ij} \),

case ii. \( l < i < j < k < u \), since \( S(r_{ijk}, r_{ijkl}) = y_i w_{ik} w_{ju} - y_i w_{ij} w_{ku} - y_j w_{iu} w_{ik} + y_k w_{iu} w_{ij} = w_{ik} r_{iju} - w_{ij} r_{iku} - y_j r_{iku} - w_{il} r_{iku} + y_k r_{ij} - y_u r_{ij} \) and

case iii. \( i = l \), so \( i < j < k < u \), in this case, \( S(r_{ijk}, r_{ijkl}) = y_i w_{ik} w_{ju} - y_i w_{ij} w_{ku} - y_j w_{iu} w_{ik} + y_k w_{iu} w_{ij} = w_{ik} r_{iju} - w_{ij} r_{iku} \). Hence \( S(r_{ijk}, r_{lstu}) \to 0 \).
Lemma 2.2.15. a. $S(r'_{ijk}, r'_{lst}) \rightarrow_G 0$ for $l \neq i$ and $s \neq j$ or $k \neq t$;

b. $S(r'_{ijk}, r'_{lst}) \rightarrow_G 0$ for $l = i$, and $t \neq j$ or $k \neq t$;

c. $S(r'_{ijk}, r'_{lst}) \rightarrow_G 0$ for $l \neq i$, $s = j$ and $t = k$

Proof. a. Follows from Buchberger's first criterion.

b. Suppose $l = i$ and $s \neq j$ or $t \neq k$, then $S(r'_{ijk}, r'_{lst}) = -z_j w_{ik} w_{st} + z_k w_{ij} w_{st} + z_s w_{is} w_{jk}$. Since $i < j < k$ and $i < s < t$, there are four cases to consider,

   case i $i < j < k < s < t$

   case ii $i < j < s < t < k$

   case iii $i < j < k = s < t$

   case iv $i < j < s < k = t$

   case i. $i < j < k < s < t$, then $S(r'_{ijk}, r'_{lst}) = -z_j w_{ik} w_{st} + z_k w_{ij} w_{st} + z_s w_{is} w_{jk} - z_t w_{is} w_{jk} = -w_{ik} r'_{jst} + w_{ij} r'_{kst} + z_s r_{ijkt} - z_t r_{ijks}$.

   case ii. $i < j < s < t < k$, then $S(r'_{ijk}, r'_{lst}) = -z_j w_{ik} w_{st} + z_s w_{is} w_{jk} - z_t w_{is} w_{jk} + z_k w_{ij} w_{st} = -w_{ik} r'_{jst} - z_s r_{ijtk} + w_{ij} r'_{stk} - z_t r_{ijsk}$.

   case iii. $s = k$, so $i < j < k < t$, and $S(r'_{ijk}, r'_{ikt}) = -z_j w_{ik} w_{kt} + z_k w_{ij} w_{kt} + z_k w_{is} w_{jk} - z_t w_{is} w_{jk} = z_k r_{ijkt} - w_{ik} r'_{jkt}$.

   case iv. $t = k$, so $i < j < s < k$, and since $S(r'_{ijk}, r'_{isk}) = -z_j w_{ik} w_{sk} + z_k w_{ij} w_{sk} - z_k w_{is} w_{jk} + z_s w_{is} w_{jk} = z_k r_{ijsk} - w_{ik} r'_{jsk}$. Hence $S(r'_{ijk}, r'_{lst}) \rightarrow_G 0$.

c. Suppose $s = j$ and $t = k$, and $l \neq i$, then $S(r'_{ijk}, r'_{ijk}) = z_i z_j w_{lk} - z_k z_j w_{ij} - z_i z_j w_{ik} + z_l z_k w_{ij}$. Since, $i < j < k$ and $l < j < k$ there are two case to consider,

   case i $i < l < j < k$;

   case ii $l < i < j < k$;
case i. $i < l < j < k$, since \( S(r'_{ijk}, r'_{ijk}) = z_i z_j w_{l k} - z_i z_k w_{l j} - z_i z_j w_{i k} + z_i z_k w_{i j} = z_j r'_{l k} - z_k r_{l i j} \), and

case ii. $l < i < j < k$, since \( S(r'_{ijk}, r'_{ijk}) = z_i z_k w_{i j} - z_i z_j w_{i k} - z_i z_k w_{i j} + z_i z_j w_{i k} = -z_j r'_{l k} + z_k r_{l i j} \). Hence \( S(r'_{ijk}, r'_{l st}) \rightarrow G 0 \).

 Lemma 2.2.16. a. \( S(r'_{ijk}, r'_{lst}) \rightarrow G 0 \) for \( j \neq l \) and \( j \neq s \) or \( k \neq u \) and \( k \neq t \);

b. \( S(r'_{ijk}, r'_{lst}) \rightarrow G 0 \) for \( l = j \) and \( u = k \);

c. \( S(r'_{ijk}, r'_{lst}) \rightarrow G 0 \) for \( j = s \) and \( t = k \)

Proof. a. Follows from Buchberger’s first criterion.

b. Suppose \( l = j \) and \( u = k \), so \( i < j < s < t < k \) (the only case), since \( S(r'_{ijk}, r'_{jst}) = z_i w_{j l} w_{s k} - z_i w_{j s} w_{l k} - z_j w_{i k} w_{s t} + z_k w_{i j} w_{s t} = w_{sk} r'_{ijt} - w_{tk} r'_{ijt} - z_j r'_{i st} + w_{ij} r'_{st} \), then \( S(r'_{ijk}, r'_{lst}) \rightarrow G 0 \).

c. Suppose \( s = j \) and \( t = k \), then \( S(r'_{ijk}, r'_{jku}) = z_i w_{i k} w_{j u} - z_i w_{i j} w_{k u} - z_j w_{i u} w_{i k} + z_k w_{i u} w_{i j} \). Since, \( i < j < k \) and \( l < j < k < u \), there are three cases to be considered,

case i. \( i < l < j < k < u \)

case ii. \( l < i < j < k < u \)

case iii. \( i = l < j < k < u \)

case i. \( i < l < j < k < u \), since \( S(r'_{ijk}, r'_{jku}) = z_i w_{l k} w_{j u} - z_i w_{i j} w_{k u} - z_j w_{i u} w_{i k} + z_k w_{i u} w_{i j} = w_{l k} r'_{iju} - w_{i j} r'_{iku} + z_j r'_{iku} - w_{i j} r'_{jku} - z_k r_{i l j u} + z_u r_{i l j k} \), and

case ii. \( l < i < j < k < u \), since \( S(r'_{ijk}, r'_{jku}) = z_i w_{l k} w_{j u} - z_i w_{i j} w_{k u} - z_j w_{i u} w_{i k} + z_k w_{i u} w_{i j} = w_{l k} r'_{ij u} - w_{i j} r'_{ik u} - z_j r'_{ik u} - w_{i j} r'_{jku} + z_k r_{i l j u} - z_u r_{i l j k} \),
case iii. \( i < j < k < u \); \( l = i \), since, \( S(r_{ijkl}, r_{ijkl}) = z_i w_{ik} w_{ju} - z_j w_{ij} w_{ku} - z_j w_{iu} w_{ik} + z_k w_{iu} w_{ij} = w_{ik} r_{ijy} - w_{ij} r_{iku} \) hence, \( S(r_{ijkl}, r_{ilst}) \to G 0 \).

Lemma 2.2.17. a. \( S(r_{ijkl}, r_{qstu}) \to G 0 \) if \( \gcd(w_{ijkl}, w_{astu}) = 1 \);

b. \( S(r_{ijkl}, r_{qstu}) \to G 0 \), for \( q = i \) and \( u = l \);

c. \( S(r_{ijkl}, r_{qstu}) \to G 0 \), for \( s = j \) and \( t = l \);

d. \( S(r_{ijkl}, r_{qstu}) \to G 0 \), for \( q = j \) and \( u = k \);

e. \( S(r_{ijkl}, r_{qstu}) \to G 0 \), for \( s = i \) and \( t = l \).

Proof. a. Follows from Buchberger’s first criterion 1.4.4.

b. Suppose \( q = i \) and \( u = l \), then \( S(r_{ijkl}, r_{ilst}) = -w_{sl} w_{ik} w_{jl} + w_{st} w_{ij} w_{kl} + w_{jk} w_{il} w_{st} - w_{jk} w_{is} w_{tl} \). Since \( i < j < k < l \) and \( i < s < t < l \), there are five cases to consider,

\[
\text{case i } i < j < k < s < t < l
\]

\[
\text{case ii } i < j < s < k < t < l
\]

\[
\text{case iii } i < s < j < k < t < l
\]

\[
\text{case iv } i < s = j < k < t < l
\]

\[
\text{case v } i < s < j < k = t < l
\]

We prove cases i and iv, cases ii and iii are analogous to case i and case v is analogous to case vi.

\[
\text{case i } i < j < k < s < t < l, S(r_{ijkl}, r_{ilst}) = w_{ij} r_{kssl} + w_{sl} r_{ikkt} - w_{ik} r_{jslt} - w_{it} r_{ijks},
\]

and

\[
\text{case iv } s = j, i < j < k < t < l, \text{ we have } S(r_{ijkl}, r_{ijjt}) = w_{jl} r_{ijkt} - w_{ij} r_{jklt}. \text{ Hence } S(r_{ijkl}, r_{qstu}) \to G 0.
\]
c. Suppose \( s = j \) and \( t = k \), then \( S(r_{ijkl}, r_{ijkl}) = -w_{qu}w_{ik}w_{jl} + w_{qu}w_{ij}w_{kl} + w_{il}w_{qk}w_{ju} - w_{il}w_{qj}w_{ku} \). Since \( i < j < k < l \) and \( i < j < k < u \), there are four cases to consider.

**case i** \( i < q < j < k < l < u \)

**case ii** \( i < q < j < k < u < l \)

**case iii** \( i = q < j < k < l < u \)

**case iv** \( i < q < j < k < l = u \)

**case i**: \( i < q < j < k < l < u \), in this case \( S(r_{ijkl}, r_{ijkl}) = w_{ij}r_{ijkl} - w_{ju}r_{qijkl} + w_{ku}r_{qijl} - w_{iu}r_{qijkl} + w_{qj}r_{ijkl} - w_{jkl}r_{qijkl} - w_{ik}r_{ijkl} - w_{ilk}r_{ijkl} \).

**case ii**: Analogous to case i.

**case iii** \( i = q < j < k < l < u \), \( S(r_{ijkl}, r_{ijkl}) = w_{ij}r_{ik} - w_{ik}r_{ijkl} \).

**case iv**: Analogous to case iii.

d Suppose \( s = i \) and \( t = l \), since \( q < i < l < u \) and \( i < j < k < l \), there is only one case to consider, namely \( q < i < j < k < l < u \). We have, \( S(r_{ijkl}, r_{ijkl}) = -w_{ik}w_{jl}w_{qu} + w_{ij}w_{kl}w_{qu} - w_{jk}w_{qi}w_{iu} - w_{ij}w_{kl}w_{iu} = w_{ij}r_{ijkl} - w_{ik}r_{qijkl} + w_{qj}r_{ijkl} - w_{iu}r_{qijkl} \).

e Suppose \( r = j \) and \( u = k \), since \( i < j < k < l \) and \( j < s < t < k \), there is only one case to consider, \( i < j < s < t < k < l \). Hence, \( S(r_{ijkl}, r_{ijkl}) = -w_{ik}w_{si}w_{jl} + w_{ij}w_{st}w_{kl} + w_{il}w_{js}w_{tk} - w_{il}w_{js}w_{tk} = w_{ij}r_{ijkl} - w_{ik}r_{ijkl} + w_{kl}r_{ijkl} - w_{js}r_{ijkl} \).

Combining a, b, c, d, and e Lemma 14 follows. □

Hence by the 14 lemmas the proof of Theorem (2.2.2) follows. Now by Theorem (1.4.7) the ideal of relations \( I \) is generated by \( H = G \cap \mathbb{R}[w, z] \). Thus \( I = \langle r'_{ijkl}, r_{ijkl} \rangle \), where

\[
\begin{align*}
    r'_{ijk} & = z_iw_{jk} - z_jw_{ik} + w_{ij}z_k, \quad 1 \leq i < j < k \leq n; \\
    r_{ijkl} & = w_{il}w_{jk} - w_{ik}w_{jl} + w_{ij}w_{kl}, \quad 1 \leq i < j < k < \leq n. \quad (2.2.2)
\end{align*}
\]
CHAPTER 3. Stanley Decomposition and Table Function

3.1 Stanley Decomposition

From the Groebner basis for the ideal of relations, found in chapter 2, we write down a Stanley decomposition of the ring of invariants. We know that the Groebner basis for the ideal of relations is $H = \langle r'_{ijk}, r_{ijkl} \rangle$, where

$$
\begin{align*}
    r'_{ijk} &= z_i w_{jk} - z_j w_{ik} + w_{ij} z_k, & 1 \leq i < j < k \leq n; \\
    r_{ijkl} &= w_{il} w_{jk} - w_{lk} w_{ij} + w_{ij} w_{kl}, & 1 \leq i < j < k < l \leq n
\end{align*}
$$

Let $\mathcal{I} = \langle \alpha_i, \beta_{jk}, \beta_{il} \beta_{jk} \rangle$ be the ideal generated by the leading terms of the Groebner basis for $I$, which is a monomial ideal. It is a well known fact that a Stanley decomposition of $\mathbb{R}[\alpha, \beta]/I$ is the same as a Stanley decomposition of $\mathbb{R}[\alpha, \beta]/\mathcal{I}$, see [13], so it suffices to work with $\mathbb{R}[\alpha, \beta]/\mathcal{I}$.

Before generalizing the result of writing down the Stanley decomposition, we give an example as a motivation. Let us consider a normal form with linear part $N = N_{222}$. Then

$$
\mathcal{X} = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_3}
$$

The invariants are:

$$
\begin{align*}
    \alpha_1 &= x_1, & \alpha_2 &= x_2, & \alpha_3 &= x_3, \\
    \beta_{12} &= x_1 y_2 - x_2 y_1, & \beta_{13} &= x_1 y_3 - x_3 y_1, \\
    \beta_{23} &= x_2 y_3 - x_3 y_2. & \text{There is only one generator (which therefore forms a Groebner basis) for the ideal of relations among the basic invariants, } \alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \alpha_3 \beta_{12} = 0
\end{align*}
$$
with \( \bar{T} = \langle \alpha_1, \beta_{23} \rangle \). This relation can be used to eliminate the combination \( \alpha_1 \beta_{23} \) (the leading term of the relation) from any polynomial in the basic invariants. A Stanley decomposition is given by

\[
R = \frac{\mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \alpha_1, \alpha_2, \alpha_3]}{(\alpha_1 \beta_{23})} = \mathbb{R}[\beta_{12}, \beta_{13}, \alpha_1, \alpha_2, \alpha_3] \oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \alpha_2, \alpha_3].
\]

(3.1.1)

Hence every element of \( \mathcal{R} \) can then be written uniquely as:

\[
f_1(\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}) + f_2(\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}) \beta_{23}.
\]

Writing down the Stanley decompositions can be complicated, especially when there are more than one relation among the basic invariants. We describe an easier way of writing down the Stanley decomposition when this is the case.

**Definition 3.1.1. (The partially ordered set (poset) ring).** Let \( K \) be any field and let \( P = \{x_1, x_2, \ldots, x_n\} \) be a poset. Let \( R = K[x_1, x_2, \ldots, x_n] \) be a polynomial ring, where the elements of \( P \) are regarded as the independent indeterminates. Let \( I_P \) be the ideal of \( R \) generated by all products \( x_i x_j \), such that \( x_i \) and \( x_j \) are incomparable as elements of \( P \), that is, neither \( x_i \leq x_j \) nor \( x_j \leq x_i \). Setting \( R_P = R/I_P \), we call \( R_P \) the *poset ring* corresponding to the poset \( P \).

Now, it is not hard to see that our ring \( \mathcal{R} = \mathbb{R}[\alpha_i, \beta_{ij}] / \bar{T} \), for \( 1 \leq i \leq n \) and \( 1 \leq i < j \leq n \), is a poset ring corresponding to the poset \( P_n = \{\alpha_i, \beta_{ij}\} \) with the ordering defined by the relations,

\[
\begin{align*}
\beta_{ij} &\leq \beta_{kl}, \quad \text{if } i \leq k \text{ and } j \leq l; \\
\alpha_i &\leq \alpha_j, \quad \text{if } i \leq j; \\
\beta_{ij} &\leq \alpha_i, \quad \text{for all } i,
\end{align*}
\]

since the generators of \( \bar{T} \) are products of incomparable elements of \( P_n \). For example for \( n = 4 \), the poset \( P_n \) diagrammatically (written like this for convenience) is:
where $s \to t$ means $s \leq t$. We define paths from $\beta_{12}$ to $\alpha_4$ to be any moves in the direction of the arrows, that is to be made up of moves left or moves up. Such paths are called maximal monotone paths.

Every chain takes for example the form

\[ \begin{array}{c}
\uparrow \\
\leftarrow \ast \\
\uparrow \\
\leftarrow \ast \\
\uparrow \\
\end{array} \]

Each of the points marked * will be called a corner of a maximal monotone path. A Stanley decomposition of the poset ring $R$ is then given in general by:

\[ R = \bigoplus_i R[\text{variables on the } i^{th} \text{ path }](\text{product of corners on the } i^{th} \text{ path}). \]

The fact that this in indeed a Stanley decomposition follows from [13].

Let us revisit the example with linear part $N_{222}$. Recall that for $n = 3$, $\tilde{I} = \langle \alpha_1 \beta_{23} \rangle$, and the poset $P_3$ of the poset ring $\mathbb{R}[\alpha, \beta]/\tilde{I}$ is therefore:
with two maximal monotone paths \((\beta_{12} \to \beta_{13} \to \alpha_1 \to \alpha_2 \to \alpha_3)\) with no corner and 
\((\beta_{12} \to \beta_{13} \to \beta_{23} \to \alpha_2 \to \alpha_3)\) with \(\beta_{23}\) as corner. From the definition a Stanley decomposition is easily written down as:

\[
\mathcal{R} = \mathbb{R}[\beta_{12}, \beta_{13}, \alpha_1, \alpha_2, \alpha_3] \oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \alpha_2, \alpha_3] \beta_{23}
\]

as before. This gives us an elegant way of writing down a Stanley decomposition for any of our rings \(\text{ker } \mathcal{X}\).

Next, we illustrate this method by writing down a Stanley decomposition for \(N_{2222}\).

We have

\[
\mathcal{X} = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_3} + x_4 \frac{\partial}{\partial y_4}
\]

The basic invariants are:

\[
\alpha_i = x_i \; ; \; 1 \leq i \leq 4; \quad \beta_{ij} = x_i y_j - y_i x_j \; ; \; 1 \leq i < j \leq 4.
\]

A Groebner basis (generators) for the ideal of relations among the basic invariants is:

\[
\begin{align*}
\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 &= 0 \\
\alpha_1 \beta_{24} - \alpha_2 \beta_{14} + \beta_{12} \alpha_4 &= 0 \\
\alpha_1 \beta_{34} - \alpha_2 \beta_{14} + \beta_{13} \alpha_4 &= 0 \\
\alpha_2 \beta_{34} - \alpha_3 \beta_{24} + \beta_{23} \alpha_4 &= 0 \\
\beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} &= 0
\end{align*}
\]
with the monomial ideal $\mathcal{I} = \langle \alpha_1 \beta_{23}, \alpha_1 \beta_{24}, \alpha_1 \beta_{34}, \alpha_2 \beta_{34}, \beta_{14} \beta_{23} \rangle$ of the leading terms of $I$. These relations can be used to eliminate any of the leading terms from any polynomial in the basic invariants, leading to a Stanley decomposition for the ring $R = \mathbb{R}[\alpha, \beta]/\mathcal{I}$.

The Stanley decomposition is easily obtained from the poset $P_4$ given by:

\[
\begin{align*}
\alpha_4 & \\
\uparrow & \\
\alpha_3 & \leftarrow \beta_{34} \\
\uparrow & \uparrow \\
\alpha_2 & \leftarrow \beta_{24} \leftarrow \beta_{23} \\
\uparrow & \uparrow \uparrow \\
\alpha_1 & \leftarrow \beta_{14} \leftarrow \beta_{13} \leftarrow \beta_{12}
\end{align*}
\]

with the following maximal monotone paths:

$(\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4)$, with no corners.

$(\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4)$, with $\beta_{23}$ as corner.

$(\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4)$, with $\beta_{24}$ as corner.

$(\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \alpha_3 \rightarrow \alpha_4)$, with $\beta_{34}$ as corner.

$(\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \alpha_3 \rightarrow \alpha_4)$, with $\beta_{23}$ and $\beta_{34}$ as corners.

Hence we obtain the following Stanley decomposition for $\mathcal{R}$.

\[
\mathcal{R} = \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{14}, \alpha_1, \alpha_2, \alpha_3, \alpha_4] \oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}, \alpha_2, \alpha_3, \alpha_4] \beta_{23} \quad (3.1.2)
\]

\[
\oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}, \alpha_2, \alpha_3, \alpha_4] \beta_{24} \oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}, \beta_{34}, \alpha_3, \alpha_4] \beta_{34}
\]

\[
\oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}, \beta_{34}, \alpha_3, \alpha_4] \beta_{23} \beta_{34}
\]
3.2 Table Function

Given a Stanley decomposition for the ring $\mathcal{R} = \mathbb{R}[\alpha, \beta]/\mathcal{I}$, we develop a table function denoted by $T_{2n}$, where $2n$ is the dimension of the nilpotent matrix $N$. According to section 1.4 Lemma (1.5.1), to prove that $\mathcal{R} = \text{ker } \mathcal{X}$, that is to show that we have found all the basic invariants, it suffices to verify that $\left. \frac{\partial}{\partial w}(wT_{2n}) \right|_{w=1} = \frac{1}{(1-d)^{2n}}$. From example $N_{222}$ above, the Stanley decomposition is

$$\mathcal{R} = \mathbb{R}[\beta_{12}, \beta_{13}, \alpha_1, \alpha_2, \alpha_3] \oplus \mathbb{R}[\beta_{12}, \beta_{13}, \beta_{23}, \alpha_2, \alpha_3]$$

and the table function is given by

$$T_6 = \frac{1}{(1-dw)^3(1-d^2)} + \frac{d^2}{(1-dw)^2(1-d^2)^3}.$$

Now

$$wT_6 = \frac{w}{(1-dw)^3(1-d^2)^2} + \frac{wd^2}{(1-dw)^2(1-d^2)^3},$$

and it is easily shown that

$$\frac{d}{dw}(wT_6)_{w=1} = \frac{1}{(1-d)^6}$$

as required. Hence all the basic invariants have been found for all normal forms with $N=N_{222}$ as linear part, thus $\mathcal{R} = \text{ker } \mathcal{X}$.

For another example, we consider a normal form with linear part $N_{222}$, whose Stanley decomposition is given by equation (3.1.2). We obtain the following table function:

$$T_8 = \frac{1}{(1-dw)^4(1-d^2)^3} + \frac{2d^2}{(1-dw)^3(1-d^2)^4} + \frac{d^2}{(1-dw)^2(1-d^2)^5} + \frac{d^4}{(1-dw)^2(1-d^2)^5}.$$

Again it is not hard to show that

$$\frac{d}{dw}(wT_8)_{w=1} = \frac{1}{(1-d)^8}.$$
as required. This shows that all the basic invariants for $N_{2222}$ have been found, hence $\mathcal{R} = \ker \mathcal{X}$.

We are now faced with the following question: can we generalize these results to any given $N_{2222}$, that is, can we claim that we have found all the basic invariants for any given linear part $N_{2222}$? To answer this question we proceed as follows.

It is clear from the definition of a table function and from the above examples that, for each term of a table function the denominator corresponds to a maximal monotone path and the numerator to the corners associated to that path. This brings us to our next question: Consider a poset ring $\mathcal{R}$ corresponding to the poset $P_n = \{\alpha_i, \beta_{ij}\}$ with

$$\begin{cases}
\beta_{ij} \leq \beta_{kl}, & \text{if } i \leq k \text{ and } j \leq l; \\
\alpha_i \leq \alpha_j, & \text{if } i \leq j; \\
\beta_{ij} \leq \alpha_i, & \text{for all } i, 1, 2, \ldots, n.
\end{cases}$$

The poset $P_n$ looks like:

\[
\begin{array}{c}
\alpha_n \\
\uparrow \\
\vdots \leftarrow \cdots \\
\uparrow \\
\alpha_i \\
\uparrow \\
\vdots \leftarrow \cdots \leftarrow \beta_{ij} \\
\uparrow \\
\vdots \\
\uparrow \\
\alpha_1 \\
\uparrow \\
\vdots \leftarrow \cdots \leftarrow \beta_{1i} \\
\leftarrow \cdots \leftarrow \beta_{12}
\end{array}
\]

**Remark.** In every case there is only one maximal monotone path with no corner, namely, the outermost path containing all the $\alpha$'s.

The question is: How many paths starting from $\beta_{12}$, exiting at $\alpha_i$ and ending at $\alpha_n$, for $i = 1, 2, 3, \ldots, n-1$, have 1 corner, 2 corners, 3 corners, $\ldots$, $k$ corners?
This is a problem in combinatorics on the number of lattice paths that never go below the line \( y = kx \) for a positive \( k \). The result which we repeat here for convenience is known and is given as Theorem 3.4.3 in [14].

**Theorem 3.2.1.** Let \( \mu \) be a positive integer and \( n \geq \mu m \geq 0 \). The number of all lattice paths from \((0,0)\) to \((m,n-1)\) with \( k \) up-right corners, that never go below the line \( y = \mu x \) is

\[
\binom{m-1}{k-1} \binom{n}{k} - \mu \binom{m}{k} \binom{n-1}{k-1}.
\]

Hence it turns out (with \( \mu =1 \)) that the number of maximal monotone paths with \( k \) corners starting at \( \beta_{12} \) and exiting at \( \alpha_i \), denoted by \( C_{n,k} \) is

\[
C_{n,k} = \binom{m-1}{k-1} \binom{n}{k} - \binom{m}{k} \binom{n-1}{k-1} = \binom{n-i}{k-1} \binom{n-2}{k-1} \binom{i-2}{k-1}.
\]

For a further illustration, let us consider now the example \( N_{22222} \).

The invariants for \( N_{22222} \) are:

\[
\alpha_i = x_i, \text{for } i \leq 5; \\
\beta_{ij} = x_i y_j - y_i x_j, \text{for } 1 \leq i < j \leq 5.
\]

These are related by the following family of relations:

\[
\begin{align*}
\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 &= 0, \quad \alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 = 0 \\
\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 &= 0, \quad \alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 = 0 \\
\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 &= 0, \quad \alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 = 0 \\
\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 &= 0, \quad \alpha_1 \beta_{23} - \alpha_2 \beta_{14} + \beta_{12} \alpha_4 = 0 \\
\alpha_1 \beta_{34} - \alpha_2 \beta_{14} + \beta_{13} \alpha_4 &= 0, \quad \alpha_2 \beta_{34} - \alpha_3 \beta_{24} + \beta_{23} \alpha_4 = 0 \\
\beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} &= 0, \quad \beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} = 0 \\
\beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} &= 0, \quad \beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} = 0 \\
\beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} &= 0,
\end{align*}
\]
whose monomial ideal is \( \mathcal{I} = (\alpha_1 \beta_{23}, \alpha_1 \beta_{24}, \alpha_1 \beta_{25}, \alpha_1 \beta_{34}, \alpha_1 \beta_{35}, \alpha_2 \beta_{34}, \alpha_2 \beta_{35}, \alpha_2 \beta_{45}, \alpha_3 \beta_{45}, \beta_{14} \beta_{23}, \beta_{15} \beta_{23}, \beta_{15} \beta_{24}, \beta_{15} \beta_{34}, \beta_{25} \beta_{34}) \). \( R = \mathbb{R}[w, z]/\mathcal{I} \) is the poset ring of the poset \( P_5 \) given by

\[
\begin{align*}
\alpha_5 \\
\uparrow \\
\alpha_4 &\leftarrow \beta_{45} \\
\uparrow &
\uparrow \\
\alpha_3 &\leftarrow \beta_{35} \leftarrow \beta_{34} \\
\uparrow &
\uparrow &
\uparrow \\
\alpha_2 &\leftarrow \beta_{25} \leftarrow \beta_{24} \leftarrow \beta_{23} \\
\uparrow &
\uparrow &
\uparrow &
\uparrow \\
\alpha_1 &\leftarrow \beta_{15} \leftarrow \beta_{14} \leftarrow \beta_{13} \leftarrow \beta_{12},
\end{align*}
\]

with the following chains:

\[
\begin{align*}
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{15} \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with no corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{23} \text{ as the corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{24} \text{ as the corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{15} \rightarrow \beta_{25} \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{25} \text{ as the corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \beta_{35} \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{34} \text{ as the corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{35} \text{ as the corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \beta_{35} \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{23} \text{ and } \beta_{34} \text{ as corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{23} \text{ and } \beta_{35} \text{ as corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{24} \text{ and } \beta_{35} \text{ as corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \beta_{45} \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_5 \text{ as corner.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \beta_{45} \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{23} \text{ and } \beta_{45} \text{ as corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{14} \rightarrow \beta_{24} \rightarrow \beta_{25} \rightarrow \beta_{35} \rightarrow \beta_{45} \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{24} \text{ and } \beta_{45} \text{ as corners.} \\
(\beta_{12} &\rightarrow \beta_{13} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \beta_{35} \rightarrow \beta_{45} \rightarrow \alpha_4 \rightarrow \alpha_5), \text{ with } \beta_{34} \text{ and } \beta_{45} \text{ as corners.}
\end{align*}
\]
\((\beta_{12} \rightarrow \beta_{13} \rightarrow \beta_{23} \rightarrow \beta_{24} \rightarrow \beta_{34} \rightarrow \beta_{35} \rightarrow \beta_{45} \rightarrow \alpha_{4} \rightarrow \alpha_{5})\), with \(\beta_{23}, \beta_{34}\) and \(\beta_{45}\) as corners. The corresponding table function is

\[
T_5 = \frac{1}{(1 - dw)^5(1 - d^2)^4} + \frac{3d^2}{(1 - dw)^4(1 - d^2)^5} + \frac{2d^2}{(1 - dw)^3(1 - d^2)^6} \\
+ \frac{3d^4}{(1 - dw)^2(1 - d^2)^6} + \frac{d^2}{(1 - dw)^2(1 - d^2)^7} + \frac{d^6}{(1 - dw)^2(1 - d^2)^7}
\]  

(3.2.2)

It is not hard to show that

\[
\frac{\partial}{\partial w}(wT_5) \bigg|_{w=1} = \frac{1}{(1 - d)^{10}},
\]

confirming that all the invariants for \(N_{22222}\) have been found, and \(\ker X = \mathbb{R}[\alpha, \beta]/\widetilde{I}\).

We observe that the coefficient on each term corresponds to the number of paths with a fixed number of \(\alpha\)'s and a fixed number of corners (= (the power of \(d))/2). For instance, the term \(\frac{3d^4}{(1 - dw)^2(1 - d^2)^7}\) on the table function above has 3 paths with two corners and two \(\alpha\)'s (i.e., starting at \(\beta_{12}\) and exiting at \(\alpha_{3}\)). Observe that \(C_{332} = 3\).

Now for a given \(n\) we summarize these results in the following theorem.

**Theorem 3.2.2.** (Ring of invariants) Let \(\mathcal{R} = \mathbb{R}[\alpha_i, \beta_{ij}]/\widetilde{I}\) where \(\widetilde{I} = \langle \alpha_i \beta_{jk}, \beta_{il} \beta_{jk} \rangle\) for \(1 \leq i \leq n, 1 \leq i < j \leq n\) and for \(i < j < k, i < j < k < l\). Then \(\mathcal{R} = \ker X\), the ring of invariants.

From the above description the table function for \(\mathcal{R}\) is

\[
T_{2n} = \frac{1}{(1 - d^2)^n(1 - dw)^n} + \sum_{k=1}^{n-1} \sum_{i=k+1}^{n-1} \frac{C_{nik}d^{2k}}{(1 - d^2)^{n+i-2}(1 - dw)^{n-i+1}}
\]  

(3.2.3)

where \(C_{nik} = \frac{n - i}{k} \binom{n - 2}{k - 1} \binom{i - 2}{k - 1}\).

To prove this theorem we just need to verify that

\[
\frac{\partial}{\partial w}(wT_{2n}) \bigg|_{w=1} = \frac{1}{(1 - d)^{2n}}.
\]  

(3.2.4)
This verification requires some knowledge from combinatorics and symbolic summation. Before the proof, we give a brief account of Zeilberger's algorithm for symbolic summation. For a formal description of Zeilberger's algorithm and the proof of its correctness, see [15], [16]. For tutorial description of this algorithm, see [9], [17], [18].

The mission of Zeilberger's algorithm, also known as the method of creative telescoping, is to produce a recurrence relation given a summand function. That is, Zeilberger's algorithm takes as an input a terminating hypergeometric sum and computes a linear recurrence with polynomial coefficients that is satisfied by this sum. Additionally it delivers a rational function, the so called certificate, which contains all information necessary to validate the result independently. More precisely, let $f_{n,k}$ be a double-indexed sequence over a field $K$ of characteristic 0, where $n$ ranges over the nonnegative integers and $k$ over all integers. We call $f_{n,k}$ hypergeometric in both parameters if both quotients

$$\frac{f_{n+1,k}}{f_{n,k}} \quad \text{and} \quad \frac{f_{n,k+1}}{f_{n,k}}$$

are rational functions in $n$ and $k$ over $K$. For Example the sequence $f_{n,k} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ is hypergeometric in $n$ and $k$.

**Theorem 3.2.3.** (Existence of a k-free recurrence) Let $f_{n,k}$ be a hypergeometric sequence, then $f_{n,k}$ satisfies a nontrivial recurrence of the form

$$\sum_{i=0}^{m} a_i(n) f_{n+i,k} = g_{n,k+1} - g_{n,k},$$

where the $a_i$'s are polynomials in $n$ over $K$ and $g_{n,k}$ is a rational function multiple of $f_{n,k}$ and therefore a hypergeometric sequence too.

Now suppose that $f_{n,k}$ has finite support (i.e., for each nonnegative integer $n$ there exists a finite interval $I_n$ such that $f_{n,k}$ vanishes for $k \notin I_n$. Then $S_n := \sum_k f_{n,k}$, where $k$ runs through all integers, denotes a finite sum, for which a recurrence can be easily deduced from equation (3.2.5), namely

$$\sum_{i=0}^{m} a_i(n) S_{n+i} = 0.$$

\[ f_{n,k} \cdot \sum_{i=0}^{m} a_i(n) \frac{f_{n+i,k}}{f_{n,k}}, \quad (3.2.7) \]

a rational function multiple of the original summand \( f_{n,k} \) with undetermined \( a_i \), can be
used to compute both the polynomials \( a_i \) and the sequence \( g_{n,k} \) from equation (3.2.5).
Several forms of Zeilberger's algorithms have been developed; the most relevant ones to
us are due to Paule and Schorn[17] and Zeilberger[18]. We demonstrate with an example
how Zeilberger's algorithm can be applied. Suppose that we have a sum that involves
factorials, binomial coefficients and so on, such as

\[ f(n) = \sum_{k=0}^{n} \binom{x-k+1}{k} \binom{x-2k}{n-k}. \quad (3.2.8) \]

The following is then done:

1. Let \( F(n,k) \) be the summand, i.e., the function that is being summed. The first
task is to find a recurrence that \( F \) satisfies.

2. If you are using Mathematica (which we are), get the program \texttt{Zb}, and for example
\((3.2.8)\) type

\[ \texttt{Zb[Binomial[x-k+1, K]Binomial[x-2k, n-k], k, n, 2].} \]

The result is the recurrence

\[ (n+2)(n-x)F(n,k) - (n+2)(n-x)F(n+2,k) = G(n,k+1) - G(n,k). \quad (3.2.9) \]

3. The next step is to sum this recurrence (3.2.9) over all the desired values of \( k \). It
turns out that the right hand side telescopes to zero (see [16]), and you end up
with the recurrence that your unknown sum \( f(n) \) satisfies, in the form
\[ f(n) - f(n + 2) = 0. \]

4. Lastly find and check (when there is an identity to prove) the initial conditions.

In example (3.2.8) the initial conditions are \( f(0) \) and \( f(1) \).

We are now ready to verify Theorem 3.2.2. Recall the table function for \( R \),

\[ T_{2n} = \frac{1}{(1 - d^2)^{n-1}T_0^n} + \sum_{k=1}^{n-1} \sum_{i=k+1}^{n-1} \frac{C_{n,k}d^{2k}}{(1 - d^2)^{n+i-2}(1 - d^n)^{n-i+1}}, \quad (3.2.10) \]

where \( C_{n,k} = \frac{n-i}{k} \left( \frac{n-2}{k-1} \right) \left( \frac{i-2}{k-1} \right) \), and \( n \geq 2 \). We want to show that

\[ \frac{\partial}{\partial w} (u T_{2n}) \bigg|_{w=1} = \frac{1}{(1 - d)^{2n}}. \quad (3.2.11) \]

Working out the \( LHS \) of (3.2.11) we get

\[ \frac{1}{(1 + d)^{2n-3}(1 - d)^{2n}} [(1 + (n - 1)d)(1 + d)^{n-2} + \sum_{k=1}^{n-1} \sum_{i=k+1}^{n-1} \frac{C_{n,k}(1 + (n - i)d)(1 + d)^{n-i-2}d^{2k}}{(1 - d)^{2n}} = \frac{1}{(1 - d)^{2n}}. \quad (3.2.12) \]

After simplifying, it is enough to prove the identity,

\[ (1 + (n - 1)d)(1 + d)^{n-2} + \sum_{k=1}^{n-1} \sum_{i=k+1}^{n-1} \frac{C_{n,k}(1 + (n - i)d)(1 + d)^{n-i-1}d^{2k}}{(1 - d)^{2n-3}} = (1 + d)^{2n-3}, \quad (3.2.13) \]

or equivalently (after rearranging) the identity,

\[ \sum_{k=1}^{n-1} \sum_{i=k+1}^{n-1} \frac{C_{n,k}(1 + (n - i)d)(1 + d)^{n-i-1}d^{2k}}{(1 - d)^{2n-3}} = (1 + d)^{2n-3} \cdot (1 + (n - 1)d)(1 + d)^{n-2}. \quad (3.2.14) \]

It turns out that the \( LHS \) of (3.2.14) is a hypergeometric sum (thanks to Zeilberger's algorithm). We now use Zeilberger's algorithm to find a recurrence relation for the sum on the \( LHS \) of (3.2.14).

Define \( F(n,k) = \sum_{r=k+1}^{n-1} C_{n,k}(1 + (n - i)d)(1 + d)^{n-i-1}d^{2k} \) and let \( f(n) = \sum_{k=1}^{n-1} F(n,k) \).

By Zeilberger's algorithm following steps 1 through 3 above we get,

\[ \{-d(1 + d)^2 f(n) + df(n + 1) = d^3(1 + d)^{n-1}(n - 1) \}. \quad (3.2.15) \]
Rearranging we find that the \textit{lhs} of (3.2.14) satisfies the recurrence,

\[ f(n + 1) - (1 + d)^2 f(n) = (n - 1)d^2(1 + d)^{n-1}. \]  

(3.2.16)

To verify our identity (3.2.14), we need to show that the \textit{rhs} satisfies this recurrence and check that both sides satisfy the same initial condition, i.e., \( f(2) \), since we have that \( n \geq 2 \).

Since \( \text{rhs} = (1 + d)^{2n-3} - (1 + (n - 1)d)(1 + d)^{n-2} \), then

\[
(1 + d)^2 f(n) = (1 + d)^{2n-1} - (1 + (n - 1)d)(1 + d)^n
\]

\[
f(n + 1) = (1 + d)^{2n-1} - (1 + d)(1 + d)^{n-1}
\]

adding and simplifying we get

\[
f(n + 1) - (1 + d)^2 f(n) = (n - 1)d^2(1 + d)^{n-1} = \text{lhs}
\]

Hence we have shown that both sides satisfy the same recurrence. Lastly we check the initial condition. Let \( L(n) = \text{lhs} \) and \( R(n) = \text{rhs} \) of (3.2.14), clearly \( L(2) = 0 \) and \( R(2) = 0 \). This proves the identity, and hence Theorem 3.2.2. We have shown that all the invariants for the coupled Takens-Bogdanov systems have been found.
CHAPTER 4. Normal Form Module of Equivariants

In this chapter we shall apply an algorithm developed by Murdock in [12] to obtain the Stanley decomposition of the module $\ker \mathcal{X}^r$, given the Stanley decomposition of $\ker \mathcal{X}$. Then we shall use Theorem (1.6.1) to obtain a Stanley decomposition of the normal form module of equivariants $\ker \mathcal{X}$ from the Stanley decomposition of $\ker \mathcal{X}^r$.

4.1 Stanley Decomposition of the Normal Form Module $\ker \mathcal{X}$

According to Lemma (1.6.4) the chain tops of $\mathcal{P}(\mathbb{R}^n, \mathbb{R})$ under the triad $\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$ can be taken to be the standard monomials in the basic invariants, say $I_1, I_2, \cdots, I_s$ with respect to the Stanley decomposition of $\ker \mathcal{X}$. The chains under these chain tops are the Jordan chains of $\mathcal{Y}$ and can be obtained by repeated application of $\mathcal{Y}$ (regarded as mapping downwards). These chains are also modified Jordan chains for $\mathcal{X}$ (regarded as mapping upwards). Clearly a vector space basis for $\ker \mathcal{X}^r$ is obtained by computing the $\mathcal{Y}$ iterates to depth $r$.

Remark. By modified Jordan chains for $\mathcal{X}$ we mean that $\mathcal{X}$ operating on a given vector in the chain gives a nonzero constant times the vector above it, or zero if the given vector is at the top of a chain.

We now describe how to obtain the Stanley decomposition of $\ker \mathcal{X}^r$ for any Stanley decomposition of $\ker \mathcal{X}$.

We recall here the definition of prefix ring from section 1.6. Let $m_1, \cdots, m_p$ be the leading monomials of the Groebner basis $\gamma_1, \cdots, \gamma_p$ of the ideal of relations. Let $g$ be a
stripped suffix of suffix $S$ and let $I_{i(g)}$ be the first basic invariant (from the left) appearing in $g$. If $f$ is a prefix monomial associated with suffix $S$ then $f$ is a standard monomial with respect to the ideal $J_S = \langle m'_1, \ldots, m'_p, I_{i(g)}+1, \ldots, I_s \rangle$, where $m'_i = m_i/gcd(m_i, g)$. Therefore the prefix monomials $f$ associated with the given stripped suffix $g$ are the standard monomials with respect to the (new) ideal $J_g = \langle m'_1, \ldots, m'_p, I_{i(g)}, \ldots, I_s \rangle$ (ordering the basic invariants by $I_i < I_j$, if $j < i$). Now let $f$ be a prefix monomial associated with a given suffix $S$. Then the collection of polynomials which are linear combination of such prefix monomials for a given suffix $S$ is a ring, called the prefix ring for $S$, which has a Stanley decomposition (defined by its standard prefix monomials).

**Remark.** By length of a basic invariant $I$ we mean the first power $r$ of $y$ such that $y^r(I) = 0$, this is the length of the Jordan chain for $y$ headed by $I$.

**Algorithm to find** $\ker X^r$

The following four steps apply which will become clear after the examples that follow.

1. Order the basic invariants putting the longest (with largest $r$, see remark above) basic invariant last (to the far right).

2. Determine the finite set of suffixes that can occur to depth $r$. All standard monomials can be classified by their endings. These endings determine the suffixes that can occur when the $y's$ are applied repeatedly from the last as described in Lemma (1.6.6).

3. Determine the prefix ring (coefficient ring) for each suffix. The prefix ring for a given suffix is the set of admissible prefixes (standard monomials not containing the endings) that can appear with that suffix.

4. Write down the Stanley decomposition of $\ker X^r$ as given by Theorem (1.6.7), that is

$$\ker X^r = SD(\ker X) \oplus \bigoplus_{S} P(S)S,$$
or equivalently by the above discussion

$$
\ker X = SD(\ker X) \oplus \bigoplus_S (\ker X/J_s)S,
$$

where $SD(\ker X)$ is the Stanley decomposition of $\ker X$ and $P(S)$ is the Stanley decomposition of the prefix ring corresponding to suffix $S$.

Next, we consider some examples to illustrate the above steps.

**Example** $N_{222}$. The basic invariants in this case are:

$$
\alpha_i = x_i; 1 \leq i \leq 3;
$$

$$
\beta_{ij} = x_iy_j - y_ix_j; 1 \leq i < j \leq 3.
$$

These are related by the single relation $\alpha_1\beta_{23} - \alpha_2\beta_{13} + \alpha_3\beta_{12} = 0$, with $\alpha_1\beta_{23}$ as the leading monomial. To compute $\ker X^2$ it is necessary to apply $Y$ to depth 2. Order the basic invariants by $\beta_{23} < \beta_{13} < \beta_{12} < \alpha_3 < \alpha_2 < \alpha_1$, with length 1, 1, 1, 2, 2, and 2 respectively. The standard monomials cannot contain both $\alpha_1$ and $\beta_{23}$ and can be classified into those ending in either $\alpha_1, \alpha_2$ or $\alpha_3$ and those ending in one of the trivial basic invariants $\beta_{12}, \beta_{13}$ or $\beta_{23}$. For those ending in $\alpha_1$ the suffix is $Y\alpha_1$ and the prefix ring is $R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]$, for those ending in $\alpha_2$ the suffix is $Y\alpha_2$ and the prefix ring is $R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]\beta_{23}$, for those ending in $\alpha_3$ the suffix is $Y\alpha_3$ and the prefix ring is $R[\alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]\beta_{23}$. Since the standard monomials ending in any of the trivial basic invariants have no suffixes, the suffix set is $\{Y\alpha_1, Y\alpha_2, Y\alpha_3\}$. The Stanley decomposition for the module $\ker X^2$ as given by Theorem (1.6.7) is

$$
\ker X^2 = R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]\beta_{23}
$$

$$
\oplus R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]Y\alpha_1
$$

$$
\oplus (R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]\beta_{23})Y\alpha_2
$$

$$
\oplus (R[\alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]\beta_{23})Y\alpha_3.
$$
Having obtained the Stanley decomposition for ker $X^2$, the $sl(2)$ normal form ker $X$, that is, the module of equivariants is easily obtained by Theorem (1.6.1). Recall that ker $X \cong ker X^2 \oplus ker X^3 \oplus \cdots \oplus ker X^k$. Hence For $N_{222}$ we have ker $X \cong ker X^2 \oplus ker X^2 \oplus ker X^2$; explicitly by Lemma (1.6.2) the Stanley decomposition for the $sl(2)$ normal form is

$$
ker X = (R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(1,1)})
$$

$$
\oplus R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]v_{(1,\alpha_1)}
$$

$$
\oplus (R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(1,\alpha_2)})
$$

$$
\oplus (R[\alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(1,\alpha_3)})
$$

$$
\oplus (R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(2,2)})
$$

$$
\oplus R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]v_{(2,\alpha_1)}
$$

$$
\oplus (R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(2,\alpha_2)})
$$

$$
\oplus (R[\alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(2,\alpha_3)})
$$

$$
\oplus (R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(3,3)})
$$

$$
\oplus R[\alpha_1, \alpha_2, \alpha_3, \beta_{12}, \beta_{13}]v_{(3,\alpha_1)}
$$

$$
\oplus (R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_2, \alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(3,\alpha_2)})
$$

$$
\oplus (R[\alpha_3, \beta_{12}, \beta_{13}] \oplus R[\alpha_3, \beta_{12}, \beta_{13}, \beta_{23}]v_{(3,\alpha_3)})
$$

Here

$$
v_{(1,1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_{(1,\alpha_1)} = \begin{bmatrix} \chi_1 \alpha_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_{(1,\alpha_2)} = \begin{bmatrix} \chi_2 \alpha_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_{(1,\alpha_3)} = \begin{bmatrix} \chi_3 \alpha_3 \\ 0 \end{bmatrix}.
$$
Example $N_{2222}$. In this example the basic invariants are:

$$\alpha_i = x_i; \quad 1 \leq i \leq 4;$$

$$\beta_{ij} = x_i y_j - y_i x_j; \quad 1 \leq i < j \leq 4.$$

A Groebner basis for the ideal of relations is:

$$\alpha_1 \beta_{23} - \alpha_2 \beta_{13} + \beta_{12} \alpha_3 = 0$$

$$\alpha_1 \beta_{24} - \alpha_2 \beta_{14} + \beta_{12} \alpha_4 = 0$$

$$\alpha_1 \beta_{34} - \alpha_2 \beta_{14} + \beta_{13} \alpha_4 = 0$$

$$\alpha_2 \beta_{34} - \alpha_3 \beta_{24} + \beta_{23} \alpha_4 = 0$$

$$\beta_{14} \beta_{23} - \beta_{13} \beta_{24} + \beta_{12} \beta_{34} = 0$$

To compute ker $\mathfrak{K}$ it is necessary to apply $\mathfrak{Y}$ to depth 2. Order the basic invariants by $\beta_{jk} < \alpha_i$ for all $i$ and $\alpha_j < \alpha_i$ for $i < j$. The ordering on the $\beta_{ij}$ is not necessary in our case, since each $\beta_{ij}$ is of length 1 and contribute no suffixes. The standard monomials
can be classified into those ending in one of the $\alpha_i$'s and those ending in one of the trivial basic invariants. For those ending in $\alpha_i$ the suffix is $\gamma \alpha_i$ (since each $\alpha_i$ is of length 2) and for those ending in one of the trivial basic invariants have no suffixes. So the suffix set is $\{\gamma \alpha_i, \text{for } 1 \leq i \leq 4\}$. The Stanley decomposition for the module $\ker X^2$ as given by Theorem (1.6.7) is

$$\ker X^2 = R[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}] \oplus R[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{23}$$

$$\oplus R[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{23} \beta_{34}$$

$$\oplus R[\alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}] \beta_{23} \beta_{34}$$

$$\oplus R[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}] \gamma \alpha_1 \oplus (R[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}] \gamma \alpha_1 \oplus (R[\alpha_2, \alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{23} \beta_{34} \oplus R[\alpha_3, \alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{23} \beta_{34} \gamma \alpha_3$$

$$\oplus (R[\alpha_4, \beta_{12}, \beta_{13}, \beta_{14}] \oplus R[\alpha_4, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}] \beta_{23} \oplus R[\alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{24}$$

$$\oplus R[\alpha_4, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{24}] \beta_{23} \beta_{34} \gamma \alpha_4$$

Having obtained the Stanley decomposition for $\ker X^2$, the Stanley decomposition for the $sl(2)$ normal form $\ker X$, the module of equivariants follows easily from Theorem (1.6.1). For $N_{2222}$ we have $\ker X \cong \ker X^2 \oplus \ker X^2 \oplus \ker X^2 \oplus \ker X^2$. The Stanley decomposition for the $sl(2)$ normal form is obtained by placing equation (4.1.3) into the second, fourth, sixth and eighth positions in the vector fields and then applying the reconstruction map:

$$(0, \gamma \alpha_1, 0, \gamma \alpha_2, 0, \gamma \alpha_3, 0, \gamma \alpha_4) \rightarrow (X(\gamma \alpha_1), \gamma \alpha_1, X(\gamma \alpha_2), \gamma \alpha_2, X(\gamma \alpha_3), \gamma \alpha_3, X(\gamma \alpha_4), \gamma \alpha_4)).$$

It is clear from the above examples that we can now find the Stanley decomposition for the $sl(2)$ module of equivariants for any coupled Takens-Bogdanov system. We summarize the above results to any $N_{2222}$ with an arbitrary number of $2 \times 2$ blocks.

**Example** $N_{2222}$. In this case the basic invariants are
\[ \alpha_i = x_i \text{ for } 1 \leq i \leq n \text{ and } \beta_{ij} = x_i y_j - x_j y_i \text{ for } 1 \leq i < j \leq n. \]

The Groebner basis for the ideal of relation is

\[ H = \langle r'_{ijk}, r_{ijkl} \rangle, \]

where

\[ r'_{ijk} = \alpha_i \beta_{jk} - \beta_j w_{ik} + \beta_{ij} z_k, \quad 1 \leq i < j < k \leq n; \]
\[ r_{ijkl} = \beta_{ul} w_{jk} - \beta_{lk} w_{jl} + \beta_{ij} w_{kl}, \quad 1 \leq i < j < k < \leq n, \]

with the monomial ideal \( \overline{I} = \langle \alpha_i \beta_{jk}, \beta_{ul} \beta_{jk} \rangle \). To compute \( \ker X^2 \) once again, it is necessary to apply \( \mathcal{Y} \) to depth 2. Order the basic invariants by \( \beta_{jk} < \alpha_i \) for all \( i \) and \( \alpha_j < \alpha_i \) for \( i < j \). Each \( \beta_{ij} \) is of length 1 and each \( \alpha_i \) is of length 2. The standard monomials can be classified into those ending in one of the \( \alpha_i \)'s and those ending in one of the trivial basic invariants. For those ending in \( \alpha_i \) the suffix is \( \mathcal{Y} \alpha_i \) and those ending in one of the trivial basic invariants have no suffixes. So the suffix set is \( \{ \mathcal{Y} \alpha_i \text{ for } 1 \leq i \leq n \} \). Each of the \( \alpha_i \)'s is a stripped suffix and so \( i(g) = i(\alpha_i) = i \). Therefore the Stanley decomposition for the module \( \ker X^2 \) as given by Theorem (1.6.7) and the discussion before the algorithm above is

\[ \ker X^2 = \ker X \oplus \bigoplus_{i=1}^{n} (\ker X/J_{i}) \mathcal{Y} \alpha_i. \tag{4.1.4} \]

Now the Stanley decomposition for the module of equivariants for \( N_{222\ldots2} \) is obtained by placing equation (4.1.4) into the second, fourth, \( \cdots \), and \( (2n) \)th positions in the vector fields and then applying the reconstruction map.
CHAPTER 5. Unfoldings and Conclusion

In this chapter we will briefly discuss the unfolding of the single Takens-Bogdanov system in $sl(2)$ normal form with the hope that, this will supply more information than the simplified normal form discussed in [10] and can be extended to higher dimensions in future work. We will also briefly mention in conclusion the work of Richard Cushman, Jan Sanders, and Neil White [4] on coupled Takens-Bogdanov systems using a different method.

5.1 Unfolding in the Presence of Generic Quadratic Terms

There are two reasons for doing what is called unfoldings to a dynamical system, imperfection in modelling and bifurcation theory as illustrated by Murdock in [12].

Roughly Speaking, to unfold a system of vector fields is to add parameters to the system, with the intention of studying the behavior of all possible systems close to the original one. On normalizing the perturbed system up to a given degree, the arbitrary parameters that remain are called the unfolding parameters, and the number of such parameters is called the codimension of the unfolding. There exists a natural notion of unfolding known as asymptotic unfolding (for vector fields having a rest point at the origin), under which all systems have unfoldings of finite codimension. Such unfoldings have been computed for many years in applied contexts, and are treated, for example, in [5], [12] and [10]. Loosely speaking an asymptotic unfolding exhibit all behavior which can be detected in perturbation of the original system up to a given degree, such
as existence and stability of certain bifurcations.

The First-Order Unfolding. The goal here is to find a simple form representing all systems close to a given system \( \dot{x} = Ax + Q(x) + \cdots \), where \( A \) is a given matrix and \( Q \) is a given quadratic part. All perturbation of this system can be obtained by adding \( \varepsilon \{ p + Bx + \cdots \} + \cdots \), where \( p \in \mathbb{R}^n \) is an arbitrary constant vector, \( B \) is an arbitrary \( n \times n \) matrix, \( \varepsilon \) is a small parameter, and the second set of dots represent terms of higher order in \( \varepsilon \). Defining the equivalence relation \( \cong \) to mean congruence modulo cubic in \( x \), quadratic terms in \( \varepsilon \) and \( \varepsilon \) times quadratic terms in \( x \), the system to be normalized appears as

\[
\dot{x} \cong Ax + Q(x) + \varepsilon \{ p + Bx \}.
\] (5.1.1)

\( A \) is assumed to be in Jordan form and \( Q \) is in normal form with respect to \( A \) (in our case \( Q \) will be in \( sl(2) \) normal form). Our aim is to simplify \( p \) and \( B \) in (5.1.1) as much as possible with the intention of reducing these \( n + n^2 \) quantities to a much smaller number of unfolding parameters \( \mu_1, \cdots, \mu_k \). The result is called the first-order unfolding of \( \dot{x} = Ax + Q(x) + \cdots \). According to Murdock in [10] and [12], if \( Q \) were in simplified normal form the normalization is achieved in three stages:

1. A coordinate shift \( x = y + \varepsilon k \) called a primary shift, that simplifies \( p \).

2. A linear coordinate transformation \( y = z + \varepsilon Bz \) to simplify \( B \).

3. A coordinate shift \( z = w + \varepsilon h \), called a secondary shift, that has no effect on \( p \) but simplifies \( B \) further.

However, this is not possible when \( Q \) is in \( sl(2) \) normal form, the difficulty being that the secondary shift does not preserve (as shown in [10]) the normal form achieved for the linear part \( B \). We found out that the problem can be resolved by carrying out the normalization of the perturbed linear part and the secondary shift simultaneously (mixed transformation), rather than successively. This idea was proposed by Murdock
in [12] on page 382, and we illustrate it in the following example.

The Normal Form and Unfolding for $N_2$. We find the normal form and unfolding for the single Takens-Bogdanov system $\dot{x} = N_2 x + \cdots$.

**Definition 5.1.1.** A simple striped matrix is a square matrix $C = (c_{ij})$ such that $c_{ij} = 0$ for $j > i$ and $c_{ij} = c_{kl}$ whenever $i - j = k - l$. That is, the entries above the main diagonal are zero, and the entries within any diagonal, on or below the main diagonal are equal.

**Lemma 5.1.2.** If $A_o = N$ is a nilpotent $n \times n$ matrix in upper Jordan form with one Jordan block, then a matrix series with $A_o$ as leading term is in inner product normal form (up to a given order) if and only if the succeeding terms are simple striped matrices. See Murdock [12] for the proof.

To find the $sl(2)$ normal form (in this case the $sl(2)$ normal form coincides with the inner product normal form), we first find the ring of invariants $\ker \mathcal{X}$, where $\mathcal{X} = x_1 \frac{\partial}{\partial x_2} = x \frac{\partial}{\partial y}$, using $(x, y) = (z_1, z_2)$ (see equation (1.3.3).) By inspection $\alpha = x$ is one invariant, and we claim that this generates the entire ring; that is,

$$\ker \mathcal{X} = \mathbb{R}[x]. \quad (5.1.2)$$

To check this, we note that the weight of $x$ (that is, its eigenvalue under $\mathcal{Z} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$) is one, so that the table function of $\mathbb{R}[x]$ is

$$T = \frac{1}{1 - dw}.$$ 

Since

$$\left. \frac{\partial}{\partial w} \right|_{w=1} = \frac{1}{(1 - d)^2},$$

this implies (5.1.2) by Lemma (1.5.1).

The next step is to compute $\ker \mathcal{X}^2$ as a module over $\ker \mathcal{X}$. Since $N_2$ contains one Jordan
block of size 2, this entails going to depth 2 in the Jordan chain for \( y = \frac{\partial}{\partial x} \) with chain top \( x \). We compute

\[
y x = y
\]

and conclude that the required Stanley decomposition is

\[
\ker X^2 = \mathbb{R}[x] \oplus \mathbb{R}[x]y.
\] (5.1.3)

The Stanley basis (equivariants) of the \( sl(2) \) normal form is then \( v_{(1,1)} = (0,1)^T \) and \( v_{(2,y)} = (x,y)^T \), and the Stanley decomposition of the module of equivariants is

\[
\ker X = \mathbb{R}[x] \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus \mathbb{R}[x] \begin{bmatrix} x \\ y \end{bmatrix}.
\]

That is, the differential equations in \( sl(2) \) normal form are

\[
\begin{align*}
\dot{x} &= y + g(x)x = y + (\beta_1 x + \beta_2 x^2)y + \cdots, \\
\dot{y} &= f(x) + g(x)y = (\alpha_1 x + \beta_1 y)x + (\alpha_2 x + \beta_2 y)x^2 + \cdots.
\end{align*}
\] (5.1.4)

Defining the equivalence relation \( \equiv \) to mean congruence modulo cubic in \( x \) and \( y \), quadratic terms in \( \varepsilon \) and \( \varepsilon \) times quadratic terms in \( x \) and \( y \), to unfold the system

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \end{bmatrix}
\]

we start with the following arbitrary perturbation of (5.1.1):

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha x^2 \\ \beta x^2 + \alpha xy \end{bmatrix} + \varepsilon \begin{bmatrix} p \\ q \end{bmatrix} + \varepsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Our first goal is to reduce the number of arbitrary parameters \( p, q, a, b, c, d \) from six to 2 (in the generic cases) or 3. On applying the following transformation
\[
\begin{bmatrix}
    x \\
    y
\end{bmatrix} = \begin{bmatrix}
    x \\
    y
\end{bmatrix} + \begin{bmatrix}
    \alpha x^2 \\
    \beta x^2 + \alpha xy
\end{bmatrix} + \varepsilon \begin{bmatrix}
    h \\
    k
\end{bmatrix} + \varepsilon \begin{bmatrix}
    r \\
    s \\
    t \\
    u
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix},
\]

and computing \( \dot{x} \) and \( \dot{y} \), we get

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{y}
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix} + \begin{bmatrix}
    \alpha \bar{x}^2 \\
    \beta \bar{x}^2 + \alpha \bar{x} \bar{y}
\end{bmatrix} + \varepsilon \begin{bmatrix}
    p + k \\
    q
\end{bmatrix} + \begin{bmatrix}
    a + t + 2\alpha h & b + u - r \\
    c + \alpha k + 2\beta h & \alpha h + d - t
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix}.
\]

Now we need to choose \( h, k, r, s, t, u \) so that by the above Lemma 5.1.2, the system has the form

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{y}
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix} + \begin{bmatrix}
    \alpha \bar{x}^2 \\
    \beta \bar{x}^2 + \alpha \bar{x} \bar{y}
\end{bmatrix} + \varepsilon \begin{bmatrix}
    0 \\
    \bar{q}
\end{bmatrix} + \begin{bmatrix}
    \bar{a} \\
    \bar{c} \\
    \bar{d}
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix}.
\]

That is, we want
\[
\begin{align*}
0 &= p + k, \\
\bar{q} &= q, \\
\bar{a} &= 2\alpha h + t + a, \\
0 &= b + u - r, \\
\bar{c} &= \alpha k + 2\beta h + c, \\
\bar{a} &= \alpha h + d - t,
\end{align*}
\]

so we choose \( k = -p \), choose \( u - r = b \) and choose \( t = \frac{d - a - \alpha h}{2} \) to make the diagonal elements equal.

If \( \beta \neq 0 \) (a generic condition), we can eliminate \( \bar{c} \), by choosing \( h = \frac{-(\alpha k + c)}{2\beta} = \frac{\alpha p - c}{2\beta} \), modifying the \( \bar{a} \), and the resulting system can be rearranged as
Setting $u_1 = \varepsilon q$, and $u_2 = \varepsilon a$ we get
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
\varepsilon q & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \begin{bmatrix}
\alpha x^2 \\
\beta x^2 + \alpha y
\end{bmatrix}
\]

if $\alpha \neq 0$, we can eliminate $\bar{a}$, by choosing $t = \frac{2d - a}{3}$ and $h = \frac{-(a + d)}{3\alpha}$, modify $\bar{c}$, and the resulting system is instead
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
u_1 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \begin{bmatrix}
\alpha x^2 \\
\beta x^2 + \alpha y
\end{bmatrix}
\]

if both $\alpha \neq 0$ and $\beta \neq 0$ we can get $\bar{a} = 0$ or $\bar{c} = 0$ but not both at once.

If we wish to permit arbitrary $\alpha$ and $\beta$ with no condition then we must expect the codimension three unfolding given by
\[
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
u_1 & u_2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \begin{bmatrix}
\alpha x^2 \\
\beta x^2 + \alpha y
\end{bmatrix}
\]

Further investigation is required to provide prove that the above unfoldings exhibit all behavior possible for systems close to equation 5.1.4. The next step would then be to study the behavior of the unfolded system by scaling methods. We leave these as a basis for future research including the following open problems:

- Unfolding using $sl(2)$ for the general case.
- Scaling for $N_{222...2}$ problem.
- Possible reduction of the unfolded problem to a lower-dimensional nonautonomous system using invariants as variables.
• The role of transvectants in the \( N_{22\ldots2} \) problem. In particular:
  
  - Deriving the relation among the basic invariants from the transvectant expressions.
  
  - Is there any way to do the Groebner basis work using the transvectants?
  
  - Relations between the transvectant structure and the bracket algebra that Cushman and Sanders use.

• Generalization to \( N_{33\ldots3} \) and to mixed problems like \( N_{223} \).

• What is the normal form for Hamiltonian systems with \( N_{22\ldots2} \) for their linear part? This should be obtained by taking invariants as Hamiltonians. That is \( f \rightarrow J \nabla f \) should map \( \text{ker } X \rightarrow \text{ker } X \) and should carry the Stanley decomposition for \( \text{ker } X \) to a part of the Stanley decomposition for \( \text{ker } X \). Is there any nice complement to this part? Is there a generalization of the Hamiltonian and Eulerar splitting in the case of \( N_2 \) obtained by Baider and Sanders?

• The \( N_{22\ldots2} \) linear part should be generic for systems in \((x_1, y_1, \ldots, x_n, y_n)\) variables symmetric under interchanges \((x_i, y_i) \leftrightarrow (x_j, y_j)\), when there is a double zero eigenvalue. This symmetric will further strick the normal form. What is the Stanley decomposition in this case?

• What about the combined Hamiltonian and symmetric case?

5.2 Conclusion

In conclusion, the problem of finding Stanley decompositions for the equivariants of \( N_{22\ldots2} \) was first solved by Richard Cushman, Jan Sanders, and Neil White [4] using a method called “covariants of special equivariants.” Whereas our method begins by studying a scalar problem (the problem of invariants) that is smaller and simpler than
the vector problem (of equivariants), their method begins by creating a scalar problem that is larger than the vector problem. Their procedures derive from classical invariant theory. When the scalar problem has been solved, our approach makes it unnecessary to repeat the calculations of classical invariant theory at the level of equavariants. Instead, an algorithm is given (through the construction of suffixes and their prefix rings) that converts a Stanley decomposition of the invariant ring into Stanley decomposition of the module \( \ker \mathcal{X} \). This leads immediately to the structure of the normal form modules \( \ker \mathcal{X} \).

We briefly describe this large scalar problem for the case of a nilpotent matrix

\[
N = \begin{bmatrix} R \\ S \end{bmatrix}
\]

with two Jordan blocks \( R \) and \( S \) of sizes \( r \) and \( s \) with \( r + s = n \). An equivariant vector field (with respect to this matrix) of the form \( v(x) = (v_1(x), \ldots, v_r(x), 0, \ldots, 0) \) is called a special equivariant for the block \( R \). (It is not simply an equivariant of \( R \) with added zeros, because it depends on all of the variables \( (x_1, \ldots, x_n) \).) The first step is to “scalarize” the special equivariants by introducing new variables \( \xi, \eta \) and mapping each vector field \( v(x) = (v_1(x), \ldots, v_r(x), 0, \ldots, 0) \) to \( p(x) = v_1(x)\xi^{r-1} + v_2(x)\xi^{r-2}\eta + \cdots + v_r(x)\eta^{r-1} \). The action of \( N \) (and of the associated copy of \( \mathfrak{sl}(2) \)) is transferred from the vector equivariants to these scalar functions in such a way that when \( v(x) \) is an equivariant, \( p(x) \) is an invariant. What we call an invariant for \( N \) is classically a semi-invariant of \( \mathfrak{sl}(2) \), and the next step is to invoke the Hilbert-Cayley correspondence between semi-invariants and covariants, to produce (for each special equivariant) a covariant \( p(x)X^w + \cdots \) with leading term \( p(x) \). (Here \( w \) is the weight of the semi-invariant, \( X \) and \( Y \) are another pair of new variables, and the terms indicated by \( \cdots \) are homogeneous of degree \( w \) in \( X \) and \( Y \).) The algebra of all such expressions is the algebra of covariants of special equivariants (associated with the block \( R \)).
Now it turns out that when this method is applied to $N_{22...2}$, the algebra obtained in this way is isomorphic to something called a bracket algebra. (These brackets are not Lie brackets, but the brackets of umbral calculus, and are connected with certain Grassmann manifolds.) This bracket algebra has been studied extensively in classical invariant theory, and the results of this study are transferred back to the original problem of equivariants to yield the desired Stanley decomposition.

The method presented here is considerably simpler. It also relies on the application of methods from classical invariant theory to a scalar problem, but the scalar problem is simpler, and the entire scalar algebra that we use is relevant to the problem of equivariants, because it is just the ring over which the equivariants form a module. (The algebra studied in [4] is so large that it contains many polynomials having nothing to do with the equivariants.) Our algebra does not seem to be isomorphic to a classical bracket algebra, although it is very close, and we have studied it "from scratch" by Groebner basis methods rather than borrowing classical results. It is hoped that these methods will extend to other examples, such as $N_{33...3}$ or $N_{44...4}$. Although the method of covariants of special equivariants could be applied to these problems, it does not lead to an algebra that is recognizable in classical terms, so one cannot immediately carry out the program of [4] for these problems. It would be necessary to study the algebra "from scratch," which would be more difficult than studying the algebra of invariants needed for our method.
BIBLIOGRAPHY


dation for Computer Science, Addison Wesley.

izing Beyond the Normal Form, Journal of Differential Equations 143, 151-190.


tems. Springer-Verlag, New York.

rings. Combinatorica 11, 275-293.

Number of Turns, in "Advances in Combinatorial Methods and Applications to Prob-

Identities. Discrete Mathematics 80, 207-211.

Computation, 11, 195-204.

for Proving Binomial Coefficient Identities. Journal of Symbolic Computation, 11,
195-204.

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