Optimal control of stochastic flow

Ju Ming
Iowa State University

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Optimal control of stochastic flow

by

Ju Ming

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Applied Mathematics

Program of Study Committee:
Lisheng Hou, Major Professor
Scott Hansen
Hailiang Liu
Gary Lieberman
Paul Sacks

Iowa State University
Ames, Iowa
2009

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CHAPTER 1. Overview

1.1 Introduction

In recent years, as an effective mathematical tool in complex physical modeling and theoretic dynamical problems, there has been an increasing interest in the study of stochastic partial differential equations (SPDEs). In many instances, a physical system may involve various uncertainties such as inexact knowledge of the force system, initial and boundary conditions, and lacking information of the medium, hence the resulting equations may contain some random inputs and are stochastic. Examples include wave propagation [33], [56], turbulence [39], [54], flows in porous media [3], [17]. In addition, in the analysis of climate change, biology, finance, and other fields, SPDEs models are also adopted as an essential component. Analogous to deterministic partial differential equations, there are only very few SPDEs can be solved analytically. Therefore, it is important to investigate the numerical solutions of SPDEs, and by now, various numerical methods and tools have been developed, [2], [9], [11], [15], [32], [34], [59].

Currently the most popular numerical method implemented in solving SPDEs is the Monte Carlo (MC) method [49], [59]. To solve a SPDE numerically, one firstly samples the randomness, then the solution can be obtained by performing the deterministic computation, by repeating the realizations and taking the mean of the solutions, the statistical information of the solution, such as expectation, can be acquired eventually. But according to the law of large number theorem, the rate of the convergence of the MC method is only $1/\sqrt{N}$, where $N$ is the number of the realizations, which is rather slow. Even though a number of techniques have been proposed to accelerate the rate, the MC method is still not very effective, especially for nonlinear SPDEs’ problem.
1.2 Polynomial Chaos Expansion for Stochastic Partial Differential Equations

Besides the MC method, the polynomial chaos expansion (PCE) method provides another direction to solve SPDEs numerically, see e.g. [2], [18], [36], [37], [65], [70], [17], [43], [61], [62], [63]. In this dissertation, we are primely concerned with the Wiener-Itô chaos expansion (WCE) and the polynomial chaos expansion (PCE) based on the Karhunen-Loève expansion (KLE). Particularly for some nonlinear SPDEs, e.g. stochastic Burgers’ equations (SBEs) and stochastic Navier-Stokes equations (SNSEs), with or without the random force, the expansion method may be a more efficient and accurate numerical method than Monte Carlo simulation [38], [44], [45].

These expansion methods are in some sense the Fourier expansion in the corresponding probability spaces. Suppose that \( u(x, t, \omega) \) is the probability solution of a given SPDE, and under some assumptions, for instance, \( u \in L^2(\mu) \), then let

\[
  u(x, t, \omega) = \sum u_\alpha(x, t) H_\alpha(\omega),
\]

each coefficient \( u_\alpha = E[u H_\alpha] \) is deterministic, and \( \{H_\alpha\} \) is an orthonormal basis of the probability space and hence is stochastic. In other words, by expanding \( u \) into the expansion form, we can decompose a stochastic function into the deterministic part and the random part. Therefore, for a solution \( u(x, t, \omega) \) of a nonlinear SPDE, after plugging the expansion of \( u(x, t, \omega) \) into the associated SPDE, we can establish a deterministic partial differential equation (PDE) system for the WCE coefficients \( u_\alpha \), which can be solved efficiently by regular deterministic numerical methods. Furthermore, some techniques can be applied to reduce the size of the resulting deterministic system.

1.3 Wick Product and Gaussian Field

In this dissertation, we will study the SPDEs defined on a special probability space, the so-called Kondratiev space [29]. The reason to define such a space is based on the fact that in general a solution of a SPDE can be viewed as a distribution function, therefore, such solution
should be discussed in the associated Schwartz space. Similar as the deterministic case, a well-defined product of the distribution functions in the Kondratiev space is needed, such a product is called Wick Product, denoted by \( \circ \), and in fact it can be considered as a regularization of the ordinary multiplication; that is, for two distribution functions \( u \) and \( v \), suppose the associated WCE as \( u = \sum \alpha u_{\alpha} H_{\alpha} \) and \( v = \sum \alpha v_{\alpha} H_{\alpha} \), respectively, then \( u \circ v := \sum \alpha v_{\beta} H_{\alpha+\beta} \). The Wick type stochastic Burgers’ equation with additive white noise, for example, is expressed as

\[
 u_t + u \circ u_x = \nu u_{xx} + \dot{W},
\]

where \( W \) is a Brownian motion and \( \nu > 0 \) denotes a viscosity parameter. In [28], the uniqueness and existence of the solutions for such type of SBE is proved by using a Wick version Cole-Hopf transformation. Some examples can be found in [5], [14], [30], etc. The numerical approximation of the Wick type SPDE is studied in [40], [41], [42].

In addition, we will consider the SNSE in a backward facing step channel with stochastic boundary conditions. For this benchmark problem, we assume that there are some random perturbations on the boundary satisfying the standard Gaussian distribution. By applying the KLE, a deterministic system of the PCE can be derived and solved. The original work of the PCE is made by Ghanem and Spanos [18], and a plenty of literatures have been published to study this method, see [52] for a summary.

### 1.4 Optimal Control Problem of Stochastic Flow

On the other hand, the optimal control of flow is an active field of research in fluid dynamics, a vast amount of literatures are devoted to discussing such problems, see e.g. [20], [23], [24], [26], [53]. The aim of this dissertation is to discuss the numerical approximation of SPDEs and the associated optimal control problems by implementing the WCE and the PCE based on KLE. Due to the nonlinearity and randomness of the system, such problems are still very challenging and difficult to handle, and there are only very sparse literatures available.

The first problem under consideration is the optimal control of SBE with additive white noise, which is presented in [8] as a step to develop a method for control of turbulent flows.
In this dissertation, our objective is to keep the distance between the solution of the Burgers equation and the target velocity as small as possible, which is called the velocity tracking problem. The control used in such problem is the forcing function (distributed control) and the boundary conditions (boundary control), separately. By adjusting the control, the distance is expected to be minimized in $L^2$ sense.

Since the Burgers equation can be viewed as a simplified version of the Navier-Stokes equation, we could then develop some useful conclusions applicable to the control problems subject to the SNSE. In other words, this work implies the applicable of Winer-Itô method in the optimal control problems of stochastic flows. Therefore, the velocity-tracking problem of the SNSE with additive white noise will also be investigated.

Aside from above control problems, we will study the vorticity reduction problem of the SNSE in a backward facing step channel. In such problem, we assume that the boundary has some small random perturbations satisfying Gaussian distribution. Very similar to the WCE, the KLE can expand a stochastic function in Gaussian field into a series, and thus the deterministic part can be separated from the function. Therefore, by applying the PCE based on the KLE, the SNSE can be solved numerically by applying deterministic numerical methods, e.g. finite element method. Some discussions can be found in [44] and [45].

The algorithm of the optimal control problems in this dissertation is based on an adjoint-based iterative method [21], that is, firstly we solve the SPDE numerically, then derive a variational formulation for the control problems, repeat this process until we acquire the satisfied solution. Figure 6.4 presents the outline of our optimization algorithm, here $F(u, \phi) = 0$ denotes the state equation, regards to the SBE or SNSE, $\phi$ is the control, $J$ is the objective functional, $\lambda_\alpha$ is the Lagrange multiplier function corresponding to $H_\alpha$.

Figure 1.1 The outline of the optimal control problems
1.5 Summary

This dissertation is organized as follows: in Chapter 2 we review some definitions for the probability space and the WCE method. Chapter 3 is devoted to defining the Wick product and the associated probability space. The distributed and boundary control problems of the SBEs with random forcing are presented, and a variational formulation is derived in Chapter 4, in which some numerical results and experiments are given. In Chapter 5, the distributed control problem of the SNSE with additive white noise is studied, some numerical tests are made to testify our algorithm. In Chapter 6, we will review KLE and study the optimal control problem of the SNSE in backward facing channel.
CHAPTER 2. Brocher-Minlos Theorem and Wiener-Itô Chaos Expansion

2.1 Brocher-Minlos Theorem

For a fixed positive integer \(d\), let \(S(\mathbb{R}^d)\) be the Schwartz space of rapidly decreasing \(C^\infty\) functions on \(\mathbb{R}^d\), which is actually a Fréchet space under the family of semi-norms

\[
||f||_{k,\alpha} := \sup_{x \in \mathbb{R}^d} \{(1 + |x|^k)|\partial^\alpha f(x)|\},
\]

where \(k\) is a nonnegative integer, \(\alpha = (\alpha_1, \cdots, \alpha_d)\) is a \(d\)-dimensional multi-index, and

\[
\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \text{ where } |\alpha| := \alpha_1 + \cdots + \alpha_d.
\]

(2.1)

Equipped with the weak-* topology, the dual \(S'(\mathbb{R}^d)\) of \(S(\mathbb{R}^d)\) is the space of tempered distributions.

Our approach to stochastic partial differential equations is based on the existence of the probability measure \(\mu\) on \(S'(\mathbb{R}^d)\), the following Brocher-Minlos theorem [29] provides the foundation of our numerical algorithm that will be used later.

Theorem 2.1.1 (Brocher-Minlos) For a family of Borel subsets of \(\mathcal{B}(S'(\mathbb{R}))\), there is a unique probability measure \(\mu\) (normalized Gaussian measure) on \(\mathcal{B}(S'(\mathbb{R}))\) with the following property:

\[
E_{\mu}[e^{i\langle \cdot, \phi \rangle}] = \int_{S'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-1/2||\phi||^2}
\]

(2.3)

for all \(\phi \in S\), where \(||\phi||^2 = ||\phi||^2_{L^2(\mathbb{R}^d)}\), \(< \omega, \phi >\) is the action of \(\omega \in S'(\mathbb{R}^d)\) on \(\phi \in S\), and \(E_{\mu}\) denotes the expectation with respect to \(\mu\).

The space \((S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \mu)\) is called the one-dimensional white noise probability space. Furthermore, we have the following lemma.
Lemma 2.1.2 ([29] Lemma 2.2.1) Let $\zeta_1, \cdots, \zeta_n$ be functions in $S(R^d)$ that are orthonormal in $L^2(R^d)$. Let $\lambda_n$ be the normalized Gaussian measure on $R^d$, i.e.

$$d\lambda_n(x) = (2\pi)^{-n/2}e^{-|x|^2/2}dx_1 \cdots dx_n; x = (x_1, \cdots, x_n) \in R^n$$ (2.4)

Then the random variable

$$\omega \mapsto (\omega, \zeta_1), \cdots, (\omega, \zeta_n)$$ (2.5)

has a distribution $\lambda_n$. Equivalently,

$$E_\mu[f(\omega, \zeta_1), \cdots, (\omega, \zeta_n)] = \int_{R^n} f(x)d\lambda_n(x) \text{ for all } f \in L^1(\lambda_n).$$ (2.6)

Remark 2.1.3 If $\zeta_1, \cdots, \zeta_n$ are orthonormal in $L^2(R^d)$, then the random variables $\omega \mapsto \langle \omega, \zeta_1 \rangle, \cdots, \langle \omega, \zeta_n \rangle$ defined on the one-dimensional white noise probability space $(S'(R), B(S'(R)), \mu)$ are independent and normally distributed with zero mean and variance equal to one.

2.2 Wiener-Itô Chaos Expansion

2.2.1 Hermite Polynomial

In statistics, for a probability space $\Omega$, the $L^2$ space with respect to measure $\mu$ is defined as

$$L^2(\mu) = \{ f : \int_{\Omega} f(x)^2 d\mu(x) < \infty \}.$$ (2.7)

In other words,

$$L^2(\mu) = \{ f : E[f^2] < \infty \},$$ (2.8)

where $E$ is the expectation operator. The associated inner product is defined as

$$(f, g)_\mu := E[f g] = \int_{\Omega} f(x)g(x)d\mu.$$ (2.9)

In addition, from the viewpoint of analysis, the $L^2$ space with weight function $\rho(x)$ is defined as

$$L^2_\rho(R) = \{ f : \int_R \rho(x)f(x)^2 dx < \infty \}.$$ (2.10)
with the corresponding inner product

\[(f, g)_\rho := \int_R \rho(x)f(x)g(x)dx. \quad (2.11)\]

Therefore, if \( \Omega = \mathbb{R} \) and the probability measure \( \mu \) is defined as

\[d\mu = \rho(x)dx, \quad (2.12)\]

where \( dx \) is the Lebesgue measure, then the \( L^2(\mu) \) space is exactly the \( L^2_\rho(\mathbb{R}) \), and

\[E[f(\zeta)g(\zeta)] = (f, g)_\rho, \quad (2.13)\]

where \( \zeta \) is a random variable satisfying the standard normal distribution \( N(0, 1) \).

The Hermite polynomials \( h_n(x) \) are defined as

\[h_n(x) = (n!)^{-\frac{1}{2}}(-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}), n = 0, 1, 2, \ldots, \quad (2.14)\]

which constitutes an orthogonal basis of the \( L^2_\rho(\mathbb{R}) \) space with weight function \( e^{-x^2/2} \), i.e. \( \{h_n\} \) is an orthogonal basis of Hilbert space \( L^2(\mathbb{R}, \mu) \), where \( \mu \) is the Gaussian measure.

\[E[h_n(\zeta)h_j(\zeta)] = (h_n, h_j)_\rho = n! \delta_{nj}, \quad (2.15)\]

where \( \rho(x) = e^{-x^2/2} \).

Differentiating (2.14) we have

\[h'_n(x) = xh_n(x) - h_{n+1}(x). \quad (2.16)\]

On the other hand, one can show that the generating function \( g(x, t) \) [10] of \( h_n(x) \) is

\[g(x, t) = e^{-t^2/2+xt} = \sum_{n=0}^\infty h_n(x) \frac{t^n}{n!}. \quad (2.17)\]

Actually, the Taylor expansion of \( g(x, t) \) with respect to \( t \) is

\[g(x, t) = \sum_{n=0}^\infty \frac{\partial^n g(x, t)}{\partial t^n}|_{t=0} t^n. \quad (2.18)\]

Note that

\[g(x, t) = e^{x^2/2}e^{-(t-x)^2/2} \quad (2.19)\]
and
\[ \frac{\partial}{\partial t} e^{-(t-x)^2/2} \bigg|_{t=0} = -\frac{d}{dx} e^{-x^2/2}, \tag{2.20} \]
hence,
\[ g(x, t) = e^{x^2/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} (e^{-x^2/2}) t^n = \sum_{n=0}^{\infty} h_n(x) \frac{t^n}{n!}. \tag{2.21} \]
Now differentiating equation (2.17) with respect to \( x \), we get
\[ \frac{\partial g(x, t)}{\partial x} = tg(x, t) = \sum_{n=0}^{\infty} h'_n(x) \frac{t^n}{n!}. \tag{2.22} \]
Compare the coefficients of \( t^n \) on both sides, apparently
\[ h'_n(x) = nh(x). \tag{2.23} \]
Thus the Hermite polynomials constitute an Appell sequence.

From (2.16) and (2.23), the recursive relation of the Hermite polynomials is given by
\[ h_{n+1}(x) = xh_n(x) - nh_{n-1}(x), \quad n = 0, 1, 2, \cdots \tag{2.24} \]
where \( h_{-1} = 0 \) and \( h_1 = 1 \). The expressions for the Hermite polynomials thus can be obtained.

For example, the first five (unnormalized) Hermite polynomials are:
\[ h_0(x) = 1 \tag{2.25} \]
\[ h_1(x) = x \tag{2.26} \]
\[ h_2(x) = x^2 - 1 \tag{2.27} \]
\[ h_3(x) = x^3 - 3x \tag{2.28} \]
\[ h_4(x) = x^4 - 6x^2 + 3. \tag{2.29} \]

For normalized Hermite polynomials \( \{h_n(x)\} \), the recursive relation is
\[ \sqrt{n+1}h_{n+1}(x) = xh_n(x) - \sqrt{n}h_{n-1}(x), \quad n = 0, 1, 2, \cdots \tag{2.30} \]
and
\[ h'_n(x) = \sqrt{n}h_{n-1}(x). \tag{2.31} \]
Define the Hermite functions \( \{\xi_n\} \) as
\[
\xi_n(x) = \pi^{-\frac{1}{4}} ((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x),
\] (2.32)
which is a well known complete orthonormal basis for \( L^2(\mathbb{R}^d) \).

The family of tensor products
\[
\xi_{\delta} := \xi_{\delta_1} \otimes \cdots \otimes \xi_{\delta_d}
\] (2.33)
forms an basis of \( L^2_\mu(\mathbb{R}^d) \).

For \( x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \), define function \( \phi_j \) on \( \mathbb{R}^d \) as
\[
\phi_j(x) := \xi_{\delta_1}(x_1) \xi_{\delta_2}(x_2) \cdots \xi_{\delta_d}(x_d), \quad j = 1, 2, \cdots,
\] (2.34)
which is in fact \( \xi_{\delta} \) by assigning an order.

Let \( \mathcal{J} \) denote the set of multi-indices where only finitely many components are nonzero, i.e.
\[
\mathcal{J} = \{\alpha = (\alpha_1, \alpha_2, \cdots) : \alpha_i \in \mathbb{N}_0, \ |\alpha| := \sum_{i=1}^{\infty} \alpha_i < \infty\}. \quad (2.35)
\]

We define \( \alpha! = \alpha_1! \alpha_2! \cdots ; \alpha < \beta \) if \( \alpha_i < \beta_i \) for all \( i \in \mathbb{N} \) and \( \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots) \).

**Definition 2.2.1** : For \( \alpha \in \mathcal{J} \), the Wick polynomial \( H_\alpha(\omega) \) with order \( |\alpha| \) is defined as
\[
H_\alpha(\omega) = \prod_{i=1}^{\infty} h_{\alpha_i}(<\omega, \xi_i>); \ \omega \in \mathcal{S}'(\mathbb{R}),
\] (2.36)
where \( <\omega, \xi_i> \) denotes the standard Gaussian random variable.

For measure \( \mu \) defined by (2.1.1) or a general Gaussian measure, the family \( \{H_\alpha, \alpha \in \mathcal{J}\} \) constitutes an orthonormal basis for \( L^2(\mu) \) with properties:

1. \[
E[H_\alpha H_\beta] = \alpha! \delta_{\alpha\beta},\quad (2.37)
\]
2. \[
E[H_0] = 1, \text{ and } E[H_\alpha] = E[H_0 H_\alpha] = 0, \text{ if } \alpha \neq 0. \quad (2.38)
\]
We then have the following Cameron-Martin theorem [6].

**Theorem 2.2.2**: Every \( f \in L^2(\mu) \) has a unique representation (Wiener chaos expansion)

\[
    f(x, t, \omega) = \sum_{\alpha \in \mathcal{J}} f_\alpha(x, t) H_\alpha(\omega), \quad \alpha(x, t) = E[f H_\alpha],
\]

where \( f_\alpha \) denotes the \( \alpha \)th Wiener chaos coefficient which is deterministic.

Furthermore,

\[
    f_0 = E[f], \quad \text{and the variance } \quad \text{Var}[f] = \sum_{\alpha \in \mathcal{J}, \alpha \neq 0} \alpha! |f_\alpha|^2.
\]

2.2.2 Wiener-Itô Chaos Expansion of White Noise

Let \( W(t) \) denote Brownian motion modeled by Wiener process, that is, \( W(t) \) satisfies the following three facts

1. \( W(0) = 0; \)

2. \( W(t) \) is continuous almost surely;

3. \( W(t) \) has independent increments with distribution \( W(t) - W(s) \sim N(0, t - s) \) (for \( 0 \leq s < t \)),

where \( N(\mu, \sigma^2) \) denotes the normal distribution with expectation \( \mu \) and variance \( \sigma^2 \). Then the derivative of \( W \) is called white noise.

Now we suppose \( \{\phi_j(s)\} \) is the orthogonal basis of \( L^2[0, t] \), by above definition of white noise, one can show that

\[
    W(s) = \int_0^t \chi_{[0,s]}(\tau) dW(\tau).
\]

Consider the Fourier expansion of \( \chi_{[0,s]}(\tau) \) with respect to the basis \( \{\phi(\tau)\} \), we have

\[
    \chi_{[0,s]}(\tau) = \sum_{n=0}^{\infty} \int_0^t \chi_{[0,s]}(w) \phi_n(w) d\omega \phi_n(\tau) = \sum_{n=0}^{\infty} \int_0^s \phi_n(w) dw \phi_n(\tau).
\]

Therefore,

\[
    W(s) = \int_0^t \sum_{n=0}^{\infty} \int_0^s \phi_n(w) d\omega \phi_n(\tau) dW(\tau) = \sum_{n=0}^{\infty} \int_0^t \phi_n(\tau) dW(\tau) \int_0^s \phi_n(w) dw.
\]
On the other hand, from the properties of Itô’s integral and the orthogonality of \( \{ \phi_i \} \), we have

\[
E[\int_0^t \phi_n(\tau)dW(\tau)] = 0, \tag{2.44}
\]

\[
E[(\int_0^t \phi_n(\tau)dW(\tau))^2] = \int_0^t (\phi_n(\tau))^2 d\tau = 1. \tag{2.45}
\]

If we denote \( \int_0^t \phi_n(\tau)dW(\tau) \) as \( \zeta_n \), then

\[
E[\zeta_i \zeta_j] = \int_0^t \phi_i(\tau)\phi_j(\tau) = 0, \text{ if } i \neq j. \tag{2.46}
\]

Hence \( \{ \zeta_n \} \) is independent and identically-distributed (i.i.d.) random variables that satisfies standard normal distribution \( N(0, 1) \), furthermore,

\[
W(s) = \sum_{n=0}^{\infty} \zeta_n \int_0^s \phi_n(w)dw. \tag{2.47}
\]

Now we have the following theorem.

**Theorem 2.2.3** The Brownian motion \( \{ W(s) : 0 \leq s \leq t \} \) has the Fourier expansion

\[
W(s) = \sum_{n=0}^{\infty} \zeta_n \int_0^s \phi_n(w)dw, 0 \leq s \leq t \tag{2.48}
\]

and the convergence is in the mean square sense for all \( s \leq t \);

\[
E[W - \sum_{n=0}^{N} \zeta_n \int_0^s \phi_n(w)dw]^2 \leq C \frac{t}{N} \tag{2.49}
\]

where \( C \) is a constant related to the basis \( \phi_i \).

**Remark 2.2.4** In [29], in the expansion of Brownian motion, the basis function \( \{ \phi_i \} \) is the Hermite functions defined by (2.32). Some discussions of the other basis function \( \{ \phi_i \} \) can be found in [12].
CHAPTER 3. Kondratiev Spaces and Wick Product

3.1 Kondratiev Spaces

From previous chapter, to guarantee the convergence of the Wiener chaos expansion, i.e.

\[ f(\omega) = \sum f_\alpha H_\alpha(\omega) \in L^2(\mu), \quad (3.1) \]

we have to assure that

\[ \sum_\alpha \alpha! f_\alpha^2 < \infty. \quad (3.2) \]

In this chapter, we will modify this condition to define a stochastic space of test functions and distributions. As mentioned in Chapter 1, the motive to introduce a such space is from the properties of SPDE: a solution of SPDE can be interpreted as a distribution function, thus we need to define an associated Schwartz space and the multiplication of such distributions.

Definition 3.1.1 The Kondratiev spaces of stochastic test functions and stochastic distributions

1. The stochastic test function spaces:

   Let \( N \) be a natural number. For \( 0 \leq \rho \leq 1 \), space \( (S)^N_\rho \) consist of the functions

   \[ f(\omega) = \sum f_\alpha H_\alpha(\omega) \in L^2(\mu), \text{ with } f_\alpha \in R^N \quad (3.3) \]

   such that

   \[ ||f||_{\rho,k}^2 := \sum_{\alpha \in J} f_\alpha^2 (\alpha!)^{1+\rho} (2N)^{k\alpha} < \infty \text{ for all } k \in N, \quad (3.4) \]

   where

   \[ f_\alpha^2 = |f_\alpha|^2 = \sum_{k=1}^N (f_\alpha^{(k)})^2 \text{ if } f_\alpha = (f_\alpha^{(1)}, \cdots, f_\alpha^{(N)}) \in R^N. \quad (3.5) \]
2. The stochastic distribution spaces :

For $0 \leq p \leq 1$, space $(S)^N_p$ consist of the formal expansions

$$f = \sum_{\alpha \in J} f_\alpha H_\alpha : f_\alpha \in \mathbb{R}^N$$

such that

$$|f|_{-p,-q} := \sum_{\alpha \in J} f_\alpha^2 (\alpha!)^{1-p} (2N)^{-qa} < \infty, \text{ for some } q \in \mathbb{N},$$

where $(2N)^{k\alpha} = \prod_j (2^j)^{k\alpha_j}$. $(S)^N_p$ and $(S)^N_{-p}$ are called the Kondratiev spaces of stochastic test function and stochastic distributions [35], respectively.

**Remark 3.1.2** For general $\rho \in [0,1]$, we have

$$(S)^N_1 \subset (S)^N_\rho \subset (S)^N_0 \subset L^2(\mu) \subset (S)^N_{-0} \subset (S)^N_{-1}. \quad (3.8)$$

### 3.2 Wick Product

The Wick product was first introduced by Wick [67] in quantum field theory. Hida and Ikeda in [25] used the Wick product to study stochastic analysis. Advanced discussion in physical mathematics and probability theory can be found in [13], [50].

**Definition 3.2.1** For given $f(\omega) = \sum_{\alpha \in J} f_\alpha H_\alpha(\omega)$ and $g(\omega) = \sum_{\beta \in J} g_\beta H_\beta(\omega) \in (S)^N_{-1}$, the Wick product is defined as

$$(f \diamond g)(\omega) := \sum_{\alpha, \beta \in J} f_\alpha g_\beta H_{\alpha + \beta}(\omega). \quad (3.9)$$

**Remark 3.2.2** If $\{H_\alpha\}$ in above definition is an orthonormal basis, then

$$(f \diamond g)(\omega)) := \sum_{\alpha, \beta \in J} \sqrt{\frac{(\alpha + \beta)!}{\alpha!\beta!}} f_\alpha g_\beta H_{\alpha + \beta}(\omega). \quad (3.10)$$

**Remark 3.2.3** By replacing conditions (3.4) and (3.7) by $\sup_{\alpha} f_\alpha^2 (2N)^{k\alpha} < \infty$ for all $k < \infty$, and $\sup_{\alpha} f_\alpha^2 (2N)^{-qa} < \infty$ for some $q < \infty$ respectively, we can define two other probability spaces which are called the Hida test functions space $(S)^N$ and the Hida distribution space $(S)^{*N}$ respectively. Also there is a Hida version of Wick product with the same expression as (3.9).
By comparing the definitions of Knodratiev spaces and Hida spaces, we have

\[(S)^N = (S)_0^N \text{ and } (S)^*N = (S)^N_{-0}.\]  \hspace{2cm} (3.11)

Furthermore, we can show the following important properties

**Lemma 3.2.4**

1. if \( f, g \in (S)_{-1}^N \), then \( f \circ g \in (S)_1^1 \). \hspace{2cm} (3.12)

2. if \( f, g \in (S)_1^N \), then \( f \circ g \in (S)_1^1 \). \hspace{2cm} (3.13)

3. if \( f, g \in (S)^N \), then \( f \circ g \in (S)^1 \). \hspace{2cm} (3.14)

4. if \( f, g \in (S)_1^*N \), then \( f \circ g \in (S)_{1,1}^1 \). \hspace{2cm} (3.15)

**Lemma 3.2.5**

1. **(Commutative law)**

   if \( f, g, h \in (S)_{-1}^N \), then \( f \circ g = g \circ f \). \hspace{2cm} (3.16)

2. **(Distributive law)**

   if \( f, g, h \in (S)_{-1}^N \), then \( f \circ (g + h) = f \circ g + f \circ h \). \hspace{2cm} (3.17)

3. **(Associative law)**

   if \( f, g, h \in (S)_{-1}^N \), then \( (f \circ g) \circ h = f \circ (g \circ h) \). \hspace{2cm} (3.18)

Follows from the definition of Wick product, we have \( E[f \circ g] = E[f]E[g] \). Most properties of the Wick product do not hold for general product.
CHAPTER 4. Optimal Control of Stochastic Burgers’ Equations with Additive White Noise

4.1 Wiener chaos solutions of SBEs

The one-dimensional viscous Wick type Burgers equation with a source \( f \) has the form

\[
\begin{aligned}
    u_t + u \circ u_x &= \nu u_{xx} + f \\
    u(x, 0) &= \phi(x)
\end{aligned}
\]  

(4.1)

where \( \nu \) is the viscosity parameter.

In [29], the authors proved that if the equation is regarded as an equation in \((S)^{L_2}_{-1}\), then a Wick version of the Cole-Hopf solution method can be implemented to show that there exists a unique solution of the equation (4.1).

In this chapter, we will study the application of Wiener chaos expansion method in (4.1), particularly, we will add a time-dependent white noise term to the source.

We define one-dimensional, \( d \)-parameter Brownian by the formal expansion

\[
W(t) = \sum_{j=1}^{\infty} \int_0^t \phi_j(s) ds H_{\varepsilon_j},
\]

(4.2)

where \( \varepsilon_j = (0, \ldots, 0, 1, \cdots) \), \( \{\phi_j\} \) is the basis of \( L^2(\mathbb{R}^d) \) defined in (2.34) and \( H_\alpha \) is defined by (2.36).

Now let us consider the Wiener chaos solutions of SBE

\[
\begin{aligned}
    u_t + u \circ u_x &= \nu u_{xx} + f + \dot{W} \\
    u(0, t) &= \varphi_1(t), \; u(1, t) = \varphi_2(t) \quad (t, x) \in Q_T = (0, T) \times [0, 1], \\
    u(x, 0) &= \psi(x)
\end{aligned}
\]

(4.3)

where \( f \) is deterministic, \( \nu \) is the viscosity parameter and \( W(t) \) is the Brownian motion.
Theorem 4.1.1 Let \( u(x, t, \omega) = \sum_{\alpha \in J} u_{\alpha}(x, t)H_\alpha(\omega) \) be a solution of equation (4.3), then the WCE coefficient \( u_\alpha \) satisfies

1. if \( \alpha = 0 \), then
   \[ u_{\alpha,t} + u_\alpha u_{\alpha,x} = \nu u_{\alpha,xx} + f; \] (4.4)

2. if \( |\alpha| = 1 \) and \( \alpha_j = 1 \), then
   \[ u_{\alpha,t} + (u_0 u_\alpha)_x = \nu u_{\alpha,xx} + \phi_j(t), \] (4.5)

where \( \{\phi_j(t)\} \) is an orthonormal basis in \( L^2([0, T]) \);

3. if \( |\alpha| > 1 \), then
   \[ u_{\alpha,t} + \sum_{0 \leq \beta \leq \alpha} \sqrt{\frac{\alpha!}{\beta!(\alpha - \beta)!}} u_\beta u_{\alpha-\beta,x} = \nu u_{\alpha,xx}. \] (4.6)

Proof: The proof can be obtained by plugging the WCE into the Burgers equations, multiplying by \( H_\alpha \) and then taking expectation.

From (4.2) and (4.5), the randomness of the WCE solution is generated from the part of \( |\alpha| = 1 \), which is called Gaussian part.

As for the numerical Wiener chaos solution, one of the essential problems is the number of WCE coefficients in our truncation, because we are expecting to achieve satisfactory results without having to solve a large system. In fact, for a \( N \)th order truncation with \( K \) Gaussian random variables, the number of WCE coefficients would be \( \sum_{n=0}^{N} \frac{(n+K)!}{n!K!} \), thus the number would grow rapidly as \( N \) and \( K \) increase. One basic technique that can reduce the number is to choose some 'important' Wick polynomials that can capture the main information of the solution.

In particular, let
\[ r = (\alpha_1, \alpha_2, \ldots, \alpha_K), \text{ where } N = r_1 \geq r_2 \geq \cdots \geq r_K, \] (4.7)

and define the index truncation
\[ J_{K,N} = \{ (\alpha_1, \cdots, \alpha_K) : |\alpha| \leq N, \alpha_i \leq r_i \}. \] (4.8)
then our truncation is defined as
\[
u_{K,N}(x, t, \omega) = \sum_{\alpha \in J_{K,N}} u_{\alpha}(x, t) H_\alpha(\omega).
\] (4.9)

Take the fifth order truncation with seven Gaussian random variables as an example (\(N = 5, K = 7\)). For a simple truncation, the number of WCE coefficients would be 792, by implementing above technique, let
\[
\begin{align*}
  r &= (5 5 4 3 2 1 1), \text{ if } |\alpha| = 1 \\
  r &= (5 5 4 3), \text{ if } |\alpha| = 2 \\
  r &= (5 5 4), \text{ if } |\alpha| = 3 \\
  r &= (5 5), \text{ if } |\alpha| = 4 \text{ and } |\alpha| = 5,
\end{align*}
\]
the number can be reduced to only 39.

![Figure 4.1](image)

(a) Mean, \(t=0.5\)

(b) Variance, \(t=0.5\)

**Figure 4.1** The Mean and variance of the WCE solution

Figure(4.1) presents a numerical example of the WCE solution for equation (4.3)on \([0, 1] \times [0, 1]\), where \(f = 0\), the viscosity parameter \(\nu = 0.1\), the boundary conditions \(\varphi_1(t) = \varphi_2(t) = 0\),
Figure 4.2  The $L^2([0,1] \times [0,1])$ norm of the fifth order WCE coefficients

and the initial condition $\psi(x) = 0.1 \sin(4\pi x)$. The solution is tested by the second order ($N=2$, $K=3$), fourth order ($N=3$, $K=5$), fifth order ($N=4$, $K=6$) and seventh order ($N=7$, $K=9$) WCE approximation with above sparse truncation technique, with 7, 32, 39, and 87 WCE coefficients respectively. From this figure, for the variance, the fourth order WCE approximation has almost the same accuracy as seventh order. The $L^2$ norm of the 39 WCE coefficients with $|\alpha| = 5$ on $[0,1] \times [0,1]$ is presented on Figure(4.2). Obviously the coefficients decay rapidly, the first coefficient is the zeroth order of Wick polynomial, which is the solution of (4.4); and the next 7 coefficients are the first order of Wick polynomials, which are the solutions of (4.5). The remaining coefficients correspond to the solution of (4.6). From the result of this test, the first several coefficients, including the zeroth Wick polynomial and the Gaussian part, dominate in magnitude. Further analysis and numerical experiments can be found in [38] and [65]. Hence for the control problem, this implies that we may acquire the numerical approximation by only considering several of the leading terms.
4.2 Distributed problems

4.2.1 Formulation of distributed control problem

Due to the randomness of our WCE solutions of the Wick type SBEs, the control problem is necessary to be considered in the stochastic sense. Now define the objective functional as

$$J(u, f) := E \left[ \frac{1}{2} \int_0^T \int_0^1 |u - v|^2 dx dt + \frac{1}{2} \int_0^T \int_0^1 |f|^2 dx dt \right]. \quad (4.10)$$

We wish to minimize $J$ subject to

$$\begin{cases}
  u_t + u \circ u_x = \nu u_{xx} + f + \dot{W} \\
  u(0, t) = u(1, t) = 0 \\
  u(x, 0) = \psi(x)
\end{cases} \quad (t, x) \in Q_T = (0, T) \times [0, 1], \quad (4.11)$$

where we assume the control $f$ is deterministic and the initial condition $\psi \in L^2(Q_T)$, i.e. we want to match the velocity $u$ to a target velocity $v$, where $v$ can be random variable as well.

Suppose $v(x, t, \omega) = \sum_{\alpha \in J} v_\alpha(x, t) H_\alpha(\omega)$, by Theorem 2.2.2, we have

$$J(u, f) = \frac{1}{2} \sum_{\alpha \in J} \int_0^T \int_0^1 |u_\alpha - v_\alpha|^2 dx dt + \frac{1}{2} \int_0^T \int_0^1 |f|^2 dx dt. \quad (4.12)$$

Remark 4.2.1: If the target velocity $v$ is deterministic, then

$$J = \frac{1}{2} \int_0^T \int_0^1 |u_0 - v|^2 dx dt + \frac{1}{2} \sum_{|\alpha| \geq 1, \alpha \in J} \int_0^T \int_0^1 |u_\alpha|^2 dx dt + \frac{1}{2} \int_0^T \int_0^1 |f|^2 dx dt$$

$$= \frac{1}{2} ||u_0 - v||^2_{L^2(Q_T)} + \frac{1}{2} ||\text{Var}[u]||^2_{L^2(Q_T)} + \frac{1}{2} ||f||^2_{L^2(Q_T)}.$$

By introducing the adjoint functions $\lambda_\alpha$ for $\alpha \in J$, and letting

$$\begin{cases}
  \lambda_\alpha(x, T) = 0 \\
  \lambda_\alpha(0, t) = \lambda_\alpha(1, t) = 0
\end{cases},$$

the associated Lagrangian is expressed as
\[ L(u, f; \lambda) := J(u, f) + \int_0^T \int_0^1 \lambda_0(u_{0,t} + u_0u_{0,x} - \nu u_{0,xx} - f) \, dx \, dt \]
\[ + \sum_{|\alpha|=1, \alpha_j=1} \int_0^T \int_0^1 \lambda_0(u_{\alpha,t} + (u_0u_\alpha)_x - \nu u_{\alpha,xx} - \phi_j) \, dx \, dt \]
\[ + \sum_{|\alpha|>1} \int_0^T \int_0^1 \lambda_0(u_{\alpha,t} + \sum_{0 \leq \beta \leq \alpha} \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}} u_\beta u_{\alpha,x} - \nu u_{\alpha,xx} \lambda_0) \, dx \, dt. \]

Note that we assume \( \psi(x) \) is deterministic, so we have the initial conditions for the WCE coefficients,

\[ u_0(x, 0) = \psi(x) \text{ and if } |\alpha| \geq 1, u_\alpha(x, 0) = 0 \]

Therefore, by using integration by parts, we have

\[ L(u, f; \lambda) = J(u, f) - \int_0^1 u_0(x, 0) \lambda_0(x, 0) \, dx \]
\[ - \int_0^T \int_0^1 \lambda_0 f \, dx \, dt - \sum_{|\alpha|=1, \alpha_j=1} \int_0^T \int_0^1 \phi_j \lambda_0 \, dx \, dt \]
\[ - \sum_{\alpha \in J} \int_0^T \int_0^1 (u_\alpha - v_\alpha - \lambda_{\alpha,t} - \sum_{\gamma \geq \alpha, \gamma \in J} \sqrt{\frac{\gamma!}{\alpha!(\gamma-\alpha)!}} u_{\gamma-\alpha} \lambda_{\gamma,x} - \nu \lambda_{\alpha,xx}) \, dx \, dt \]
\[ + \int_0^T \int_0^1 (f - \lambda_0) \delta f \, dx \, dt. \]

Now taking variation, note that since the initial condition is fixed, the admissible variation \( \delta u_0 \) has \( \delta u_0(x, 0) = 0, \)

\[ \delta L = \sum_{\alpha \in J} \frac{\partial L}{\partial u_\alpha} \delta u_\alpha + \frac{\partial L}{\partial f} \delta f \]
\[ = \sum_{\alpha \in J} \int_0^T \int_0^1 (u_\alpha - v_\alpha - \lambda_{\alpha,t} - \sum_{\gamma \geq \alpha, \gamma \in J} \sqrt{\frac{\gamma!}{\alpha!(\gamma-\alpha)!}} u_{\gamma-\alpha} \lambda_{\gamma,x} - \nu \lambda_{\alpha,xx}) \delta u_\alpha \, dx \, dt \]
\[ + \int_0^T \int_0^1 (f - \lambda_0) \delta f \, dx \, dt. \]

Thus if we let

\[ \lambda_{\alpha,t} + \sum_{\gamma \geq \alpha, \gamma \in J} \sqrt{\frac{\gamma!}{\alpha!(\gamma-\alpha)!}} u_{\gamma-\alpha} \lambda_{\gamma,x} + \nu \lambda_{\alpha,xx} = u_\alpha - v_\alpha, \quad (4.13) \]

the variation of \( L \) is simplified as

\[ \delta L = \int_0^T \int_0^1 (f - \lambda_0) \delta f \, dx \, dt, \quad (4.14) \]
which gives us a descent direction to update the control $f$,

$$\delta f = -\eta (f - \lambda_0)$$  (4.15)

where $\eta$ is the step size.

4.2.2 The discrete control problem

In practice, we solve the discrete version of above distributed control problem, and thus some ‘good’ discretization needs to be chosen. Firstly, partition the time interval $[0, T]$ (respectively, the space interval $[0, 1]$) into $M$ (respectively, into $I$) subintervals of length $\Delta t = T/M$ (respectively $\Delta x = 1/I$). For the set of index truncation $J_{K,N}$, the discrete objective functional is given by

$$J_{K,N,I}^{M} = \frac{\Delta x \Delta t}{2} \sum_{\alpha \in J_{K,N}} \sum_{i=1}^{I} \sum_{j=1}^{M} |u_{\alpha,i}^j - v_{\alpha,i}^j|^2 + \frac{\Delta x \Delta t}{2} \sum_{i=1}^{I} \sum_{j=1}^{M} |f_{ij}^j|^2,$$  (4.16)

where $u_{\alpha,i}^j$ and $f_{ij}^j$ is the approximation of $u_{\alpha}(i\Delta t, j\Delta x)$ and $f(i\Delta t, j\Delta x)$ respectively, and we truncate the WCE solution by the technique in Section 4.1. Denote the first order and the second order central difference in space as

$$\delta u_{ij} := u_{i+1}^j - u_{i-1}^j; \quad \delta^2 u_{ij} := u_{i+1}^j - 2u_i^j + u_{i-1}^j.$$

Use Crank-Nicolson in time and second order central difference in space to discretize equations (4.4), (4.5) and (4.6), we have

1. If $|\alpha| = 0$, then
   (i) initial condition: $u_{\alpha,i}^0 = \psi(i\Delta x), 0 \leq i \leq I$;
   (ii) boundary condition: $u_{\alpha,0}^j = u_{\alpha,I}^j = 0, 1 \leq j \leq M$;
   (iii) scheme: for $j = 1, 2, \cdots, M - 1$ and $1 \leq i \leq I - 1$,

   $$\frac{1}{\Delta t}(u_{\alpha,i}^{j+1} - u_{\alpha,i}^j) + \frac{1}{4\Delta x}(u_{\alpha,i}^j \delta u_{\alpha,i}^j + u_{\alpha,i}^{j+1} \delta u_{\alpha,i}^{j+1}) - \frac{\nu}{2\Delta x^2}(\delta^2 u_{\alpha,i}^j + \delta^2 u_{\alpha,i}^{j+1}) - \frac{1}{2}(f_{ij}^j + f_{ij}^{j+1}) = 0,$$
2. If $|\alpha| = 1$ and $\alpha_l = 1$, then

(i). initial condition: $u_{\alpha,i}^0 = 0$, $0 \leq i \leq I$;
(ii) boundary condition: $u_{\alpha,0}^j = u_{\alpha,i}^j = 0$, $1 \leq j \leq M$;
(iii) for $j = 1, 2, \ldots, M - 1$ and $1 \leq i \leq I - 1$,

$$
\frac{1}{\Delta t} (u_{\alpha,i}^{j+1} - u_{\alpha,i}^j) + \frac{1}{4\Delta x} (u_{0,i}^j \delta u_{\alpha,i}^j + u_{0,i}^{j+1} \delta u_{\alpha,i}^{j+1}) + \frac{1}{4\Delta x} (u_{0,i}^j \delta u_{\alpha,i}^j + u_{0,i}^{j+1} \delta u_{\alpha,i}^{j+1})
$$

$$
- \frac{\nu}{2\Delta x^2} (\delta^2 u_{\alpha,i}^j + \delta^2 u_{\alpha,i}^{j+1}) - \frac{1}{2} (\phi_{i}^{j+1} + \phi_{i}^{j}) = 0.
$$

3. If $|\alpha| \geq 2$, then

(i). initial condition: $u_{\alpha,i}^0 = 0$, $0 \leq i \leq I$;
(ii). boundary condition: $u_{\alpha,0}^j = u_{\alpha,i}^j = 0$, $1 \leq j \leq M$;
(iii). for $j = 1, 2, \ldots, M - 1$ and $1 \leq i \leq I - 1$,

$$
\frac{1}{\Delta t} (u_{\alpha,i}^{j+1} - u_{\alpha,i}^j) + \frac{1}{4\Delta x} \sum_{0 \leq \beta \leq \alpha} \sqrt{\frac{\alpha!}{\beta!(\alpha - \beta)!}} (u_{\beta,i}^j \delta u_{\alpha-\beta,i}^j + u_{\beta,i}^{j+1} \delta u_{\alpha-\beta,i}^{j+1})
$$

$$
- \frac{\nu}{2\Delta x^2} (\delta^2 u_{\alpha,i}^j + \delta^2 u_{\alpha,i}^{j+1}) = 0.
$$

The discretization of adjoint functions can be established as follows

(i). final conditions: $\lambda_{\alpha,i}^M = 0$, $0 \leq i \leq I$;
(ii). boundary conditions: $\lambda_{\alpha,0}^j = \lambda_{\alpha,i}^j = 0$, $1 \leq j \leq M$;
(iii). for $j = M, M - 1, \ldots, 1$,

$$
\frac{1}{\Delta t} (\lambda_{\alpha,i}^j - \lambda_{\alpha,i}^{j-1}) + \frac{1}{4\Delta x} \sum_{\gamma \geq \alpha, \gamma \in J_{K,N}} \sqrt{\frac{\gamma!}{\alpha!(\gamma - \alpha)!}} (u_{\gamma-\alpha,i}^j \delta \lambda_{\alpha,i}^{j-1} + u_{\gamma-\alpha,i}^j \delta \lambda_{\alpha,i}^j)
$$

$$
+ \frac{1}{2\Delta x^2} (\delta^2 \lambda_{\alpha,i}^j + \delta^2 \lambda_{\alpha,i}^{j-1}) = \frac{1}{2} (u_{\alpha,i}^{j+1} - v_{\alpha,i}^{j+1} + u_{\alpha,i}^j - v_{\alpha,i}^j).
$$

From this discretization, we have

$$
\delta J_{K,N,I}^r = f_i^j - \lambda_{0,i}^j.
$$

(4.17)

Our optimization algorithm is summarized as follows:

Step 1. Determine the index set $J_{K,M}$ and choose a tolerance $\varepsilon$,
Step 2. Set initial value to $f_i^M$ and let $\eta$ be sufficient small,
Step 3. Solve for \((u_{M,I}^{M}, \lambda_{M,I}^{M})\) from the corresponding discrete equations,

Step 4. Evaluate \(J_{K,N,I}^{r,M}(0) = J_{K,N,I}^{r,M}(u_{I}^{M}, f_{I}^{M})\),

Step 5. Set \(f_{I}^{M} = f_{I}^{M} - \eta(f_{I}^{M} - \lambda_{0,I}^{M})\),

Step 6. Solve for \((u_{M,I}^{M}, \lambda_{M,I}^{M})\) from the corresponding discrete equations, and set \(k = k + 1\),

Step 7. Evaluate \(J_{K,N,I}^{r,M}(k) = J_{K,N,I}^{r,M}(u_{I}^{M}, f_{I}^{M})\).

Step 8. If \(J_{K,N,I}^{r,M}(k) > J_{K,N,I}^{r,M}(k - 1)\), set \(\eta = \eta/2\) and go to Step 5; otherwise, continue,

Step 9. If \(||J_{K,N,I}^{r,M}(k) - J_{K,N,I}^{r,M}(k - 1)||/||J_{K,N,I}^{r,M}(k)|| > \varepsilon\), set \(\eta = 1.2\eta\) and go to Step 5; otherwise, stop.

4.2.3 Numerical Test

Two numerical examples are tested for the above algorithm with following settings:

\(\nu = 0.1, \ T = 1, \ I = 50, \ M = 50, \ f = 0, \)

\(\psi(x) = \sin(2\pi x), \)

and we choose the sine basis

\[ \phi_1(t) = \frac{1}{\sqrt{T}}, \quad \phi_i(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{(i-1)\pi t}{T}\right), \quad i = 2, 3, \ldots . \]

for \(L^2[0,T]\), where \(0 \leq t \leq T\).

4.2.3.1 Numerical Test 1

First let us consider that the target velocity is deterministic. We choose

\(v(x, t) = e^{-t}x(1-x)(2x^3 - 5x^2 + 7), \)

whose graph is presented in Figure (4.3).
From the remark of Section 4.1, if the desired velocity is deterministic, then we are expecting the mean of our optimal solution to get as close as possible to the target velocity in the $L^2$ sense. In other words, the associated variance should be as small as possible. With a similar idea to the proper orthogonal decomposition (POD) technique in the deterministic control problems, we don’t have to control every WCE coefficient here, but only consider the most ‘important’ coefficients in the second order moment sense. Another technique to reduce our
The mean of the solution under control, which is actually the first WCE coefficient $u_0$, is quite close to the desired velocity after a short time.

Computation cost is instead of taking control to the higher order WCE solutions directly, we can 'predict' control $f$ first by using lower order WCE approximation, then take it back to our higher-order WCE system as the initial value and then compute the final results.

In this numerical experiment, we use the fifth order WCE approximation with 8 Gaussian random variables ($N=5, K=8$) and apply the technique of truncation in Section 4.1 to choose 117 WCE coefficients. Figure(4.4) lays out the mean of the numerical WCE solution without control.

Let the tolerance $\varepsilon = 0.01$ and the initial step size $\eta = 0.001$. The CPU time cost (Dual Core 2.4GHs and 2 GB of RAM) without prediction is 1555.39 seconds. Next we apply our optimization algorithm after predicting $f$ first by a second order approximation ($N=2, K=3$) with 7 coefficients, the elapsed time is only 176.19 seconds in total, including 51.16 seconds for prediction.

Figure(4.6) shows the optimal results for the two cases: $J$ decreases from 0.5573 to 0.1707 by 13 steps of iteration without prediction and from 0.5573 to 0.1728 by two steps of iteration after prediction. The results are very close.

Figure(4.5) presents the mean of our WCE solution under control, and some snapshots are taken to compare our results with the target velocity $v(t)$ in detail, see Figure (4.7).
Figure 4.6 The objective functional $J$. The result without predicting the control $f$ by using lower order WCE may be not as good as the one using prediction, but it reduces the elapsed CPU time from 1555.39 seconds to 176.19 seconds, therefore, taking a prediction of control $f$ is still an efficient technique to handle this type of control problem.

In Figure(4.8), the $L^2$ norm of WCE coefficients on $Q_T$ is presented. The first coefficient is actually the mean $u_0$ of our WCE solution, the next eight coefficients correspond to the first order Wick polynomials, and the rest are all the coefficients with order $2 \leq |\alpha| \leq 5$. 
Figure 4.7 The mean of controlled, desired and uncontrolled velocity at time $t=0, 0.26, 0.5, 0.76, 1$. At the beginning $t=0$, the mean of velocity under control is determined by the initial condition, but after a short time, the result matches the desired velocity quite well.

4.2.3.2 Numerical Test 2

In this test, we consider the case that our desired velocity is a random variable. This assumption is plausible under stochastic circumstances. For most cases, we do not expect our desired velocity is dominant by the random part, but may only be with a small perturbation.

Let

$$v(x, t) = 2e^t \sin(3\pi x) + \frac{1}{2} \cos(2\pi x) \dot{W}(t),$$

where $\dot{W}(t)$ is a Brownian motion.

From the properties of white noise, we know that $E[v] = v_0 = 2e^t \sin(3\pi x)$, $Var[v] = \cos^2(2\pi x)/4$. Figure (4.9) shows the simulation of $v$, where the white noise $\dot{W}$ is generated by the ‘randn’ command of Matlab.

From (4.2), the WCE of $v(x, t)$ would be

$$v(x, t) = \sum_{\alpha \in \mathcal{J}, |\alpha| \leq 1} v_\alpha(x, t) H_\alpha,$$  \hspace{1cm} (4.18)

where $v_0 = e^t \sin(3\pi x)$, for $|\alpha| = 1$. Suppose the $j^{th}$ component of $\alpha$ is 1, i.e. $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in \mathcal{J}$, $\alpha_i = \delta_{ij}$, then $v_\alpha = \cos(2\pi x)\phi_j(x)/2$, and $v_\alpha(x, t) = 0$, if $|\alpha| > 1$.

For this example, we use the technique of prediction with the same WCE truncation as the
The $L^2$ norm of the WCE coefficients. Note that the first coefficient corresponds to the mean of our optimal solution, which is quite close to the desired function, and in the meantime the magnitude of the remaining WCE coefficients decay rapidly. This implies the optimization of the variance.

first numerical test, and let

$$I = 100, \ M = 50,$$

the tolerance $\varepsilon = 10^{-6}$ for our prediction and $\varepsilon = 0.01$ for the fifth order WCE approximation. As for the desired velocity $v$, due to the smallness of the random part, we also take 8 Gaussian random variables in our WCE approximation. Figure (4.10) and Figure (4.11) show the means and some snapshots respectively. The following Table presents our optimal result.

<table>
<thead>
<tr>
<th>CPU time (seconds)</th>
<th>iterations (prediction)</th>
<th>iterations (after prediction)</th>
<th>initial $J$</th>
<th>final $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3322</td>
<td>359</td>
<td>2</td>
<td>3.9559</td>
<td>1.6738</td>
</tr>
</tbody>
</table>

4.3 **Boundary control problems**

4.3.1 **Formulation of boundary control problems**

The boundary control problem is presented as follows: consider the objective functional

$$J(u, f) := E\left[\frac{1}{2} \int_0^T \int_0^1 |u - v|^2 dx dt + \frac{1}{2} \int_0^T (|\varphi_1|^2 + |\varphi_2|^2) dt \right]. \quad (4.19)$$
We wish to minimize $J$ subject to

$$
\begin{align*}
&u_t + u \odot u_x = \nu u_{xx} + \dot{W} \\
&u(0, t) = \varphi_1(t) \\
&u(1, t) = \varphi_2(t) \\
&u(x, 0) = \psi(x)
\end{align*}
$$

where the functions $\varphi_1$ and $\varphi_2$ are our controls which are assumed to be deterministic.

Firstly, the objective functional can be rewritten as

$$
J(u, f) = \frac{1}{2} \sum_{\alpha \in J} \int_0^T \int_0^1 |u_\alpha - v_\alpha|^2 dx dt + \frac{1}{2} \int_0^T (|\varphi_1|^2 + |\varphi_2|^2) dt.
$$

With similar arguments as in the distributed control problems, define the adjoint functions as $\lambda_\alpha$ for $\alpha \in J$, and let

$$
\begin{align*}
&\lambda_\alpha(x, T) = 0 \\
&\lambda_\alpha(0, t) = \lambda_\alpha(1, t) = 0
\end{align*}
$$

Figure 4.9  Simulation of the desired function.
Figure 4.10 Means of desired velocity (left) and WCE solution under control (right).

The associated Lagrangian is expressed as

\[
L(u, f; \lambda) = J(u, f) + \int_0^T \int_0^1 \lambda_0(u_{0,t} + u_0u_{0,x} - \nu u_{0,xx})dxdt \\
+ \sum_{|\alpha|=1, \alpha_j=1} \int_0^T \int_0^1 \lambda_\alpha(u_{\alpha,t} + (u_0u_\alpha)_x - \nu u_{\alpha,xx} - \phi_j)dxdt \\
+ \sum_{|\alpha|>1} \int_0^T \int_0^1 \lambda_\alpha(u_{\alpha,t} + \sum_{0 \leq \beta \leq \alpha} \sqrt{\frac{\alpha!}{\beta!(\alpha - \beta)!}} u_\beta u_{\alpha-\beta,x} - \nu u_{\alpha,xx})dxdt \\
= J(u, f) + \int_0^1 u_0(x, t)\lambda_0(x, t)|_{t=0}dx - \sum_{|\alpha|=1, \alpha_j=1} \int_0^T \int_0^1 \phi_j \lambda_\alpha dxdt \\
+ \nu \int_0^T (\varphi_2(t)\lambda_0 x(1, t) - \varphi_1(t)\lambda_0 x(0, t))dt \\
- \sum_{\alpha \in \mathcal{J}} \int_0^T \int_0^1 (u_\alpha \lambda_{0,t} + \frac{1}{2} \sum_{0 \leq \beta \leq \alpha} \sqrt{\frac{\alpha!}{\beta!(\alpha - \beta)!}} u_\beta u_{\alpha-\beta}\lambda_{\alpha,x} + \nu u_{\alpha,xx})dxdt.
\]

(4.22)

If we let

\[
\lambda_{\alpha,t} + \sum_{\gamma \geq \alpha, \gamma \in \mathcal{J}} \sqrt{\frac{\gamma!}{\alpha!(\gamma - \alpha)!}} u_{\gamma-\alpha}\lambda_{\gamma,x} + \nu \lambda_{\alpha,xx} = u_\alpha - v_\alpha,
\]

(4.23)
then we have

\[
\delta L = \sum_{\alpha \in J} \frac{\partial L}{\partial u_{\alpha}} \delta u_{\alpha} + \frac{\partial L}{\partial \phi_1} \delta \phi_1 + \frac{\partial L}{\partial \phi_2} \delta \phi_2 \tag{4.24}
\]

\[
= \sum_{\alpha \in J} \int_0^T \int_0^1 (u_{\alpha} - v_{\alpha} - \lambda_{\alpha,t} - \sum_{\gamma \geq \alpha, \gamma \in J} \frac{\gamma!}{\alpha!(\gamma - \alpha)!!} u_{\gamma-a} \lambda_{\gamma,x} - \nu \lambda_{\alpha,x}) \delta u_{\alpha} dx dt
\]

\[
+ \int_0^T (\phi_2(t) + \nu \lambda_{0,x}(1,t)) \delta \phi_2(t) dt + \int_0^T (\phi_1(t) - \nu \lambda_{0,x}(0,t)) \delta \phi_1(t) dt
\]

\[
= \int_0^T (\phi_2(t) + \nu \lambda_{0,x}(1,t)) \delta \phi_2(t) dt + \int_0^T (\phi_1(t) - \nu \lambda_{0,x}(0,t)) \delta \phi_1(t) dt. \tag{4.25}
\]

Thus the descent direction to update the control \( \phi_1 \) and \( \phi_2 \) can be assigned as

\[
\begin{align*}
\delta \phi_1 &= \eta (\phi_1(t) - \nu \lambda_{0,x}(0,t)) \\
\delta \phi_2 &= \eta (\phi_2(t) + \nu \lambda_{0,x}(1,t)),
\end{align*}
\tag{4.26}
\]

where \( \eta \) is the step size.

For the optimization algorithm, we still apply the Crank-Nicolson scheme here. Thus for
the discrete boundary control problems, we have the gradient of $L$ as
\[
\begin{align*}
\frac{\partial J_{r,M}^{K,N,I}}{\partial \varphi_1} &= \eta (\varphi_1(i\Delta t) - \nu \lambda_{0,x}(0,i\Delta t)) \frac{\Delta x}{\partial x} \\
\frac{\partial J_{r,M}^{I,K,N}}{\partial \varphi_2} &= \eta (\varphi_2(i\Delta t) + \nu \lambda_{1,x}(0,i\Delta t)) \frac{\Delta x}{\partial x}
\end{align*}
\]
for $i = 0, 1, \cdots, m$. (4.27)

4.3.2 Numerical Test

Similarly we test our algorithm by two numerical examples. Let
\[
\nu = 1, \ T = 1, \\
\varphi_1 = 2 - t, \varphi_2 = t - 2, \psi(x) = 2\cos(\pi x).
\]

4.3.2.1 Numerical Test 1

For the first example, the target velocity $v(x,t)$ is
\[
v(x,t) = e^{-t}\sin(2\pi x) + \cos(3\pi t),
\]
which is deterministic and plotted in Figure (4.12).

![Figure 4.12 The target velocity.](image)

In this example, we still choose fifth order WCE approximation with 117 coefficients and use second order WCE with 7 coefficients for our prediction. The control results including the
CPU elapsed time, iterations for prediction and after prediction, initial objective functional and final objective functional are given in the following table.

Table 4.2 Test 1 of the boundary control

<table>
<thead>
<tr>
<th>CPU time (seconds)</th>
<th>iterations (prediction)</th>
<th>iterations (after prediction)</th>
<th>initial $J$</th>
<th>final $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>352</td>
<td>11</td>
<td>2</td>
<td>2.8985</td>
<td>1.1049</td>
</tr>
</tbody>
</table>
Figure 4.14  The mean of WCE solution under control,

Figure 4.15  Snapshots of the target velocity, the mean of WCE solutions under control and without control at x=0, 0.26, 0.5, 0.76, 1.
Figure 4.16 $L^2(Q_T)$ of the WCE coefficients. Note that the first coefficient is the mean of our optimal solution, the next 8 coefficients are the first order Wick polynomials, and the remaining coefficients are the Wick polynomials with order $2 \leq |\alpha| \leq 5$, which are almost identically zero ($O(10^{-4})$).

The effect of the optimization algorithm is presented in Figure (4.14) and Figure (4.15). Similarly to Example 4.3.1, as $t = 0$, the solution is determined by the initial function $\psi(x) = 2 \cos(\pi x)$, and after a short time the mean of the optimal solution starts to approximate the target velocity. Also see the snapshots taken at $x = 0, 0.26, 0.5, 0.76, 1$ (Figure (4.15)). Since the controls here are functions of time $t$, the results would be obvious if we "cut the slice" based on position. Note that the snapshots at $x = 0$ and $x = 1$ are actually our controls $\varphi_1$ and $\varphi_2$.

In Figure (4.16), the $L^2$ norm of WCE coefficients decay rapidly, and for the coefficients of order $|\alpha| \geq 2$, the norms are almost identically zero ($O(10^{-4})$), which implies our optimal solution can be viewed as a approximation of the target velocity ($|\alpha| \leq 1$).

### 4.3.3 Numerical Test 2

Similarly to Example 4.3.2, we will consider that our target velocity has a perturbation. Let

$$v(x, t) = e^{-x} \cos(2\pi t) + \frac{1}{2} \sin(\pi x)\dot{W}(t).$$

And the preset is the same as before except for letting

$$I = 100, \ M = 50.$$
This test also shows that a good approximation to the target velocity is given by our algorithm. See the following table.

<table>
<thead>
<tr>
<th>CPU time (seconds)</th>
<th>iterations (prediction)</th>
<th>iterations (after prediction)</th>
<th>initial $J$</th>
<th>final $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>725.30</td>
<td>12</td>
<td>2</td>
<td>3.3285</td>
<td>0.7971</td>
</tr>
</tbody>
</table>

In Figures (4.17) and (4.18), we compare the mean of the target velocity and our solution under control. One interesting result is that our optimization result seems still to be a Gaussian approximation to the target velocity. See Figure (4.19).

Figure 4.17  The mean of the target velocity (left) and the WCE solution under control (right) .
Figure 4.18  Snapshots of the mean of the target velocity, the WCE solution under control and without control at \( x=0, 0.25, 0.5, 0.75, 1 \).

Figure 4.19  \( L^2(Q_T) \) of the WCE coefficients. The results are similar to Figure (4.16) and shows that our optimization solution is a Gaussian approximation to the target velocity.
4.4 Conclusions

In this chapter, by applying the Wiener Chaos Expansion (WCE) and an adjoint algorithm for control problem, we propose an algorithm for the control problems subject to the stochastic Burgers’ equations. In our numerical test, we use fifth order WCE with 8 Gaussian random variables as our approximation. For a $Nth$ order with $K$ Gaussian random variables approximation, the number of WCE coefficients would be $\binom{K + N}{K}$, thus the number would increase dramatically as $N$ and $K$ increase, which will leads to an unacceptable computational cost, even for the fifth order approximation. To avoid this, we firstly use the technique of sparse truncation, see e.g. [38] and [65], which can reduce the number of coefficients efficiently without losing much accuracy. On the other hand, we employ our algorithm to a lower order WCE approximation to predict our controls, and next use the results as the initial control guess for the higher-order WCE approximation. By comparing the results achieved by third order WCE approximation with only 7 coefficients to the fifth order truncation with 117 coefficients, we found that the difference between the controls is surprisingly small, which actually shows the stability of our optimal solution. This aspect can be explained roughly as that our optimal solutions are acquired by controlling the most important coefficients. It is highly unlikely that our control outcome are the best possible results, however, due to the randomness, one can not expect to find a perfect control.

Furthermore, the algorithm can also be implemented to control problems subject to general stochastic Burgers’ equations with a random source, like the problems in [8]. One only needs to change the product of WCE from Wick type to general one.
CHAPTER 5. Optimal Control Problem of Stochastic Navier-Stokes Equations with Additive White Noise

5.1 Wick-type Navier-Stokes equations

In this chapter, we will extend our previous results to the stochastic Navier-Stokes equations.

Let $\mathcal{D}$ be a bounded open set, the Wick type two-dimensional stochastic Navier-Stokes equation with additive random forcing is given as follows: Given $\tilde{f} \in (S)_{-1}^k(H(\mathcal{D}))$, find $(\vec{u}, p) \in (S)_{-1}^k(H_0^1(\mathcal{D})) \times (S)_{-1}^k(L^2(\mathcal{D}))$ satisfying

$$
\vec{u}_t - \nu \Delta \vec{u} + (\vec{u} \diamond \nabla) \vec{u} + \nabla p = \tilde{f} + \sigma \dot{\vec{W}}(t),
$$

$$
\nabla \cdot \vec{u} = 0,
$$

with initial velocity

$$
\vec{u}|_{t=0} = \vec{u}_0,
$$

and Dirichlet boundary conditions

$$
\vec{u}|_{\partial \mathcal{D} \times [0,T]} = \vec{0},
$$

where $\diamond$ is the Wick product, $\nu > 0$ denotes a viscosity parameter, $\vec{u} = [u, v]^T, \vec{u}_0 = [\varphi_1, \varphi_2]^T$, $\sigma = \text{diag}[\sigma_1(x, y), \sigma_2(x, y)]$, and $\dot{\vec{W}}$ is a Brownian motion vector. Hence the term $\sigma \dot{\vec{W}}$ represents the additive random term, and $\tilde{f} = [f_1, f_2]^T$ is the external forces.

Similar to the previous chapter, we have the following theorem of the Weiner-Itô expansion solution:

**Theorem 5.1.1** Let $u = \sum_{\alpha \in \mathcal{J}} u_\alpha H_{\alpha}$, $v = \sum_{\alpha \in \mathcal{J}} v_\alpha H_{\alpha}$, and $p = \sum_{\alpha \in \mathcal{J}} p_\alpha H_{\alpha}$ be a WCE solution
of (5.1), \( \varphi_1 = \sum_{\alpha \in J} \varphi_1^\alpha H_\alpha, \varphi_2 = \sum_{\alpha \in J} \varphi_2^\alpha H_\alpha, \) and suppose both \( \vec{f} \) and the boundary condition is deterministic. Then the WCE coefficients \((u_\alpha, v_\alpha, p_\alpha)\) satisfy

1. if \(|\alpha| = 0, \)

\[ u_{\alpha,t} - \nu \Delta u_\alpha + u_\alpha u_{\alpha,x} + v_\alpha u_{\alpha,y} + p_{\alpha,x} = f_1, \]

\[ v_{\alpha,t} - \nu \Delta v_\alpha + u_\alpha v_{\alpha,x} + v_\alpha v_{\alpha,y} + p_{\alpha,y} = f_2, \]

\[ u_{\alpha,x} + v_{\alpha,y} = 0, \]

\[ u_\alpha|_{t=0} = \varphi_1^\alpha, v_\alpha|_{t=0} = \varphi_2^\alpha, \]

\[ u_\alpha|_{\partial D \times [0,T]} = 0, v_\alpha|_{\partial D \times [0,T]} = 0; \]

(5.2)

2. if \(|\alpha| = 1, \) where \( \alpha = (0, \cdots, 0, 1, 0, \cdots) \),

\[ u_{\alpha,t} - \nu \Delta u_\alpha + u_\alpha u_{0,x} + v_\alpha u_{0,y} + u_0 u_{\alpha,x} + v_0 u_{\alpha,y} + p_{\alpha,x} = \sigma_1(x, y)\psi_j(t), \]

\[ v_{\alpha,t} - \nu \Delta v_\alpha + u_\alpha v_{0,x} + v_\alpha v_{0,y} + u_0 v_{\alpha,x} + v_0 v_{\alpha,y} + p_{\alpha,y} = \sigma_2(x, y)\psi_j(t), \]

\[ u_{\alpha,x} + v_{\alpha,y} = 0, \]

\[ u_\alpha|_{t=0} = \varphi_1^\alpha, v_\alpha|_{t=0} = \varphi_2^\alpha, \]

\[ u_\alpha|_{\partial D \times [0,T]} = 0, v_\alpha|_{\partial D \times [0,T]} = 0, \]

(5.3)

where \( \{\psi_j(t)\} \) is an orthonormal basis in \( L^2([0,T]) \);

3. if \(|\alpha| > 1, \)

\[ u_{\alpha,t} - \nu \Delta u_\alpha + u_\alpha u_{\alpha,x} + v_\alpha u_{\alpha,y} + u_0 u_{\alpha,x} + v_0 u_{\alpha,y} + p_{\alpha,x} \]

\[ = - \sum_{0<\beta<\alpha} (u_\beta v_{\alpha-\beta,x} + v_\beta v_{\alpha-\beta,y}), \]

\[ v_{\alpha,t} - \nu \Delta v_\alpha + u_\alpha v_{\alpha,x} + v_\alpha v_{\alpha,y} + u_0 v_{\alpha,x} + v_0 v_{\alpha,y} + p_{\alpha,y} \]

\[ = - \sum_{0<\beta<\alpha} (u_\beta v_{\alpha-\beta,x} + v_\beta v_{\alpha-\beta,y}), \]

\[ u_{\alpha,x} + v_{\alpha,y} = 0, \]

\[ u_\alpha|_{t=0} = \varphi_1^\alpha, v_\alpha|_{t=0} = \varphi_2^\alpha, \]

\[ u_\alpha|_{\partial D \times [0,T]} = 0, v_\alpha|_{\partial D \times [0,T]} = 0. \]

(5.4)
**Proof**: The proof can be obtained directly by plugging the Wiener-Itô chaos expansion into the SNEs (5.1). Or see [41].

### 5.1.1 Weak Formulation of the Wick type Navier-Stokes problems

Similar to the weak form of the deterministic Navier-Stokes problems, we set

\[
a(\vec{u}, \vec{v}) = \nu \sum_{i,j=1}^{n} \left( \frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j} \right), \quad \forall \vec{u}, \vec{v} \in H^1(D),
\]

\[
b(\vec{v}, q) = -(q, \nabla \cdot \vec{v}), \quad \forall \vec{v} \in H^1(D),
\]

\[
c(\vec{u}; \vec{v}, \vec{w}) = 2 \sum_{i,j=1}^{2} \int_{\Omega} w_j \frac{\partial u_i}{\partial x_j} v_i \, dx, \quad \forall \vec{u}, \vec{v}, \vec{w} \in H^1(D).
\]

From above equations (5.2)-(5.4), a weak form of the system (5.2)-(5.4) can be written as

1. if \(|\alpha| = 0\),

\[
(\vec{u}_{\alpha,t}, \vec{w}) + a(\vec{u}_{\alpha}, \vec{w}) + c(\vec{u}_{\alpha}; \vec{u}_{\alpha}, \vec{w}) + b(\vec{w}, p_{\alpha}) = (\vec{f}, \vec{w}),
\]

2. if \(|\alpha| = 1\),

\[
(\vec{u}_{\alpha,t}, \vec{w}) + a(\vec{u}_{\alpha}, \vec{w}) + c(\vec{u}_{\alpha}; \vec{u}_{0}, \vec{w}) + c(\vec{u}_{0}; \vec{u}_{\alpha}, \vec{w}) + b(\vec{w}, p_{\alpha})
= (\vec{σ}_\psi, \vec{w}),
\]

3. if \(|\alpha| > 1\),

\[
(\vec{u}_{\alpha,t}, \vec{w}) + a(\vec{u}_{\alpha}, \vec{w}) + c(\vec{u}_{\alpha}; \vec{u}_{0}, \vec{w}) + c(\vec{u}_{0}; \vec{u}_{\alpha}, \vec{w}) + b(\vec{w}, p_{\alpha})
= - \sum_{0 < \beta < \alpha} c(\vec{u}_{\beta}; \vec{u}_{\alpha-\beta}, \vec{w}),
\]

where \(\forall \vec{w} \in H(D)\).

As for the numerical approximation of Wiener Chaos solution, we let

\[
r = (\alpha_1, \alpha_2, \cdots, \alpha_K), \text{ where } N = r_1 \geq r_2 \geq \cdots \geq r_K,
\]

and define the index truncation

\[
J_{K,N}^r = \{(\alpha_1, \cdots, \alpha_K) : |\alpha| \leq N, \alpha_i \leq r_i\},
\]
then our truncation of the Wiener Chaos solution is defined as

$$\bar{u}_{K,N}(x,t,\omega) = \sum_{\alpha \in \mathcal{J}_{K,N}} \bar{u}_{\alpha}(x,t) H_{\alpha}(\omega).$$  \hspace{1cm} (5.11)

Figure 5.1  Column 1 and 3 are the deterministic Navier-Stokes flows, Column 2 and 4 are the stochastic Navier-Stokes flows, which are generated by the Wiener-Itô expansion solution (5.11).

A numerical example defined on $D \times [0,T] = [0,1]^2 \times [0,1]$ is presented by figure 5.1.1. Here we set $\nu = 0.01$, $\Delta t = 0.05$, $\Delta x = \Delta y = 0.1$, the initial conditions are given by

$$\varphi_1(x,y) = -\phi(x)\phi'(y), \quad \varphi_2(x,y) = \phi'(x)\phi(y),$$

where $\phi(z) = 10z^2(1-z)^2$, and let $\sigma$ in equation (5.3) be defined as

$$\sigma_1(x,y) = \cos(x)\sin(y), \quad \sigma_2(x,y) = \sin(x)\sin(y).$$
The solution is tested by $N = 4$ and $K = 6$, and taking 50 WCE coefficients by applying the sparse truncation technique mentioned in previous chapter. Here we use the finite element method and the details of the algorithm will be given in the next section.

5.2 Distributed Control Problems

5.2.1 Formulation of the Optimality System

In this section, we will discuss the application of the Wiener Chaos method in distributed optimal control problems subject to the stochastic Navier-Stokes equations (5.1). Also an adjoint system will be derived by the gradient method.

The objective functional considered is defined as

$$J = E\left[ \ell_1 \| \vec{u} - \vec{U} \|^2 + \ell_2 \| \vec{f} \|^2 \right],$$  \hspace{1cm} (5.12)

where $\vec{u}$ is the solution of equation (5.1) and $\vec{U}$ is the target velocity, and $\vec{f}$ is our control. The goal of the minimization of the objective functional is to keep the solution $\vec{u}$ close to target velocity $\vec{U}$.

Particularly, if $\vec{f}$ is deterministic, $\vec{u} = \sum_{\alpha \in \mathcal{J}} \vec{u}_{\alpha} H_{\alpha}$ and $\vec{U} = \sum_{\alpha \in \mathcal{J}} \vec{U}_{\alpha} H_{\alpha}$, then

$$J = \ell_1 \sum_{\alpha \in \mathcal{J}} \alpha! \| \vec{u}_{\alpha} - \vec{U}_{\alpha} \|^2_{L^2(\Omega \times [0,T])} + \ell_2 \| \vec{f} \|^2_{L^2(\Omega \times [0,T])}.$$  \hspace{1cm} (5.13)

To develop the adjoint system, we introduce the family of the adjoint functions $\{\vec{\lambda}_{\alpha} : \vec{\lambda}_{\alpha} \in H_0(\Omega)\}_{\alpha \in \mathcal{J}}$, with $\nabla \cdot \vec{\lambda}_{\alpha} = 0$, and the associated Lagrangian is expressed as

$$L(\vec{u}, \vec{f}) = \frac{\ell_1}{2} \sum_{\alpha \in \mathcal{J}} \alpha! (\vec{u}_{\alpha} - \vec{U}_{\alpha}, \vec{u}_{\alpha} - \vec{U}_{\alpha}) + \frac{\ell_2}{2} (\vec{f}, \vec{f})$$

$$- \sum_{\alpha \in \mathcal{J}} [(\vec{u}_{\alpha, t}, \vec{\lambda}_{\alpha}) + a(\vec{u}_{\alpha}, \vec{\lambda}_{\alpha}) + \sum_{0 \leq \beta \leq \alpha} c(\vec{u}_{\beta}; \vec{u}_{\alpha-\beta}, \vec{\lambda}_{\alpha}) + b(\vec{\lambda}_{\alpha}, p_{\alpha})]$$

$$+ (\vec{f}, \vec{\lambda}_{\alpha}) + \sum_{j=1, |\alpha| = 1, \alpha_j = 1} (\vec{\sigma}_j \psi_j, \vec{\lambda}_{\alpha}).$$  \hspace{1cm} (5.14)

Note that if $\nabla \cdot \vec{u} = 0$, then the variation

$$\delta((\vec{u} \cdot \nabla)\vec{v}, \vec{w}) = ((\nabla \vec{v})^T \vec{w}, \delta \vec{u}) - ((\vec{u} \cdot \nabla)\vec{w}, \delta \vec{v}),$$
hence,
\[
\delta L(\bar{u}, \bar{f}) = \ell_1 \sum_{\alpha \in J} \alpha! (\bar{u}_\alpha - \bar{U}_\alpha, \delta \bar{u}_\alpha) + \ell_2 (\bar{f}, \delta \bar{f}) \\
- \sum_{\alpha \in J} \left[ - (\lambda_{\alpha,t}, \delta \bar{u}_\alpha) + a(\bar{\lambda}_{\alpha}, \delta \bar{u}_\alpha) + \sum_{0 \leq \beta \leq \alpha} (((\nabla \bar{u}_\alpha - \beta)^T \bar{\lambda}_{\alpha}, \delta \bar{u}_{\beta}) \\
- ((\bar{u}_\beta \cdot \nabla) \bar{\lambda}_{\alpha}, \delta \bar{u}_{\alpha - \beta})) + b(\bar{\lambda}_{\alpha}, \delta p_\alpha) \right] + (\bar{\lambda}_0, \delta \bar{f}).
\] (5.15)

Thus if we let
\[
- (\bar{\lambda}_{\alpha,t}, \delta \bar{u}_\alpha) + a(\bar{\lambda}_{\alpha}, \delta \bar{u}_\alpha) \\
+ \sum_{\alpha \leq \gamma} (((\nabla \bar{u}_{\gamma - \alpha})^T \bar{\lambda}_{\gamma}, \delta \bar{u}_\alpha) - ((\bar{u}_{\gamma - \alpha} \cdot \nabla) \bar{\lambda}_{\gamma}, \delta \bar{u}_{\alpha})) + b(\bar{\lambda}_{\alpha}, \delta p_\alpha) \\
= \ell_1 \alpha! (\bar{u}_\alpha - \bar{U}_\alpha, \delta \bar{u}_\alpha),
\] (5.16)

then the variation of \( L \) is simplified as
\[
\delta L = (\ell_2 \bar{f} + \bar{\lambda}_0, \delta \bar{f}),
\] (5.17)
where \( \bar{\lambda}_0 \) represents \( \bar{\lambda}_\alpha \) as \( |\alpha| = 0 \), which gives us a descent direction to update the control \( \bar{f} \),
\[
\delta \bar{f} = \bar{f} - \varepsilon (\ell_2 \bar{f} + \bar{\lambda}_0),
\] (5.18)
where \( \varepsilon \) is the step size in the descent direction.

### 5.2.2 Finite Element Discretization

Now let us consider the discrete version of the distributed control problem (5.12). Due to the determinism of the WCE coefficient system, we can use the finite element method to solve the system.

For simplicity, we consider the domain \( D \) to be a square, that is \( D = [0, L]^2 \). Firstly, partition the time interval \([0, T]\) (respectively, the space interval \([0, L]\)) into \( M \) (respectively, into \( I \)) subintervals of length \( \Delta t = T/M \) (respectively \( \Delta x = \Delta y = L/I \)). In current context, we solve the system (5.2)-(5.4) by semi-discrete approximation in time and finite element methods in space, see e.g. [19], [22]. Assume that \( M_h \) and \( N_h \) are finite element subspaces of \( H_0(D) \)
and $L^2_0(D)$ respectively. We define the finite dimension vector space

$$\mathcal{P}_h := \{ \bar{u}_h = \sum_{\alpha \in \mathcal{J}^K_h} \bar{u}_{\alpha,h} H_{\alpha,h} \in (S(H_0(D)))^k_{-1} : \bar{u}_{\alpha,h} \in (M^2_h) \}.$$  \hspace{1cm} (5.19)

$$\mathcal{Q}_h := \{ p_h = \sum_{\alpha \in \mathcal{J}^K_h} p_{\alpha,h} H_{\alpha} \in (S(L^2_0(D)))^k_{-1} : p_{\alpha,h} \in (N^2_h) \}.$$  \hspace{1cm} (5.20)

Apply the single step $\theta$ scheme approximation in time: for $0 \leq \theta \leq 1$, let $\bar{u}_{\alpha,h}^0 = \theta \bar{u}_{\alpha,h}^n + (1 - \theta)\bar{u}_{\alpha,h}^{n+1}$, $\bar{p}_{\alpha,h}^0 = \theta \bar{p}_{\alpha,h}^n + (1 - \theta)\bar{p}_{\alpha,h}^{n+1}$, and $\bar{f}^0 = \theta \bar{f}^n + (1 - \theta)\bar{f}^{n+1}$, $n = 1, 2, \cdots, M$ and denote $\bar{u}_{\alpha,h}$ as $\bar{u}_{0,h}$ as $|\alpha| = 0$, then the Galerkin semi-discrete approximation of the coefficient system is formed as follows: for each $t \in [0, T]$, find $\bar{u}_h \in \mathcal{P}_h$ and $\bar{p}_h \in \mathcal{Q}_h$ such that

1. if $|\alpha| = 0$, 

$$\frac{1}{\theta \Delta t} (\bar{u}_{0,h}^0, \bar{\varphi}_h) + a_h(\bar{u}_{0,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{0,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{0,h}^0, \bar{\varphi}_h) + b_h(\bar{\varphi}_h, \bar{p}_{\alpha,h}^0) = \frac{1}{\theta \Delta t} (\bar{u}_{0,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{0,h}^0, \bar{\varphi}_h),$$

2. if $|\alpha| = 1$

$$\frac{1}{\theta \Delta t} (\bar{u}_{1,h}^0, \bar{\varphi}_h) + a_h(\bar{u}_{1,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{1,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{1,h}^0, \bar{\varphi}_h) + b_h(\bar{\varphi}_h, \bar{p}_{\alpha,h}^0) = \frac{1}{\theta \Delta t} (\bar{u}_{1,h}^0, \bar{\varphi}_h) + (\bar{\sigma}_j^0, \varphi_h),$$

3. if $|\alpha| > 1,$

$$\frac{1}{\theta \Delta t} (\bar{u}_{\alpha,h}^0, \bar{\varphi}_h) + a_h(\bar{u}_{\alpha,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{\alpha,h}^0, \bar{\varphi}_h) + c_h(\bar{u}_{\alpha,h}^0, \bar{\varphi}_h) + b_h(\bar{\varphi}_h, \bar{p}_{\alpha,h}^0) = \frac{1}{\theta \Delta t} (\bar{u}_{\alpha,h}^0, \bar{\varphi}_h) - \sum_{0 < \beta < \alpha} c_h(\bar{u}_{\beta,h}^0, \bar{\varphi}_h),$$

where $\forall \bar{\varphi}_h \in (M^2_h)$,

and

$$b_h(\bar{u}_{\alpha,h}^0, q_h) = \begin{cases} (1 - \theta)b(\bar{u}_{\alpha,h}^0, q_h), & \text{if } n = 1, \\ 0, & \text{if } n > 1 \end{cases},$$

(5.24)

for $\forall q_h \in (M^2_h)$.
Similarly, the discretization of the adjoint system (5.16) can be formed as follows: find \( \vec{\lambda}_{\alpha,h} \in (M_h)^2 \), such that
\[
\frac{1}{\theta \Delta t} (\vec{\lambda}_{\alpha,h}^\theta, \varphi_h) + a_h(\vec{\lambda}_{\alpha,h}^\theta, \varphi_h) + \left((\nabla u_0^0)^T \vec{\lambda}_{\alpha,h}^\theta, \varphi_h\right) - \left((u_0^0 \cdot \nabla) \vec{\lambda}_{\alpha,h}^\theta, \varphi_h\right) \\
+ b(\varphi_h, \vec{\lambda}_{\alpha,h}^\theta) = \sum_{\gamma > \alpha} \left( -\left((\nabla u_{\gamma-\alpha,h}^0)^T \vec{\lambda}_{\gamma,h}^\theta, \varphi_h\right) + \left((u_{\gamma-\alpha,h}^0 \cdot \nabla) \vec{\lambda}_{\gamma,h}^\theta, \varphi_h\right) \\
+ \frac{1}{\theta \Delta t} (\vec{\lambda}_{\alpha,h}^{\alpha+1}, \varphi_h) \right) + \ell_1 \alpha! (\vec{u}_{\alpha,h}^\theta - \vec{U}_{\alpha,h}^\theta, \varphi_h),
\]
(5.25)
where \( \vec{\lambda}_{\alpha,h}^\theta = \theta \vec{\lambda}_{n,h}^\alpha + (1 - \theta) \vec{\lambda}_{n+1,h}^\alpha \).

Now we denote the discrete version of our objective functional as
\[
J_{r,M} = \frac{\ell_1 \Delta t}{2} \sum_{a \in J_{K,N}} \sum_{j=1}^{M} || \vec{u}_{\alpha,j}^a - \vec{U}_{\alpha,j}^a ||^2 + \frac{\ell_2 \Delta t}{2} \sum_{j=1}^{M} || \vec{j}^2 ||^2,
\]
(5.26)
where \( \vec{u}_{\alpha,j}^a = \vec{u}_{a,t=(j-1)\Delta t}, \vec{U}_{\alpha,j}^a = U_{\alpha,t=(j-1)\Delta t} \) and \( \vec{j}^2 = \vec{j}_{t=(j-1)\Delta t} \).

The algorithm used here is as same as gradient-based iterative algorithm in Chapter 4.

### 5.3 Numerical Test

To check the validation of our methodology, a numerical example is tested with the following settings:
\[
\nu = 0.01, \ T = 1, \ L = 1, \ I = 8, \ M = 10, \ \ell_1 = 1, \ \ell_2 = 0.001,
\]
\[
\sigma_1(x,y) = \cos(x) \sin(y), \ \sigma_2(x,y) = \sin(x) \sin(y),
\]
and we choose the sine basis
\[
\varphi_i(t) = \frac{1}{\sqrt{T}}, \ \varphi_i(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{(i-1)\pi t}{T}\right), \quad i = 2, 3, \ldots
\]
for \( L^2[0,T] \), where \( 0 \leq t \leq T \).

In this example, we assume that the stochastic target flow with velocity \((U, V)\) is given as follows:
\[
U = -10 \phi(x) \phi'(y) + \frac{1}{20} \cos(\pi x) \sin(\pi y) \dot{W}(t),
\]
(5.27)
\[
V = -10 \phi'(x) \phi(y) + \frac{1}{20} \sin(\pi x) \cos(\pi y) \dot{W}(t),
\]
(5.28)
Figure 5.2 Column 1 and 3 are the mean and simulation of the target Navier-Stokes flow respectively, Column 2 and 4 are the mean and simulation of the controlled stochastic Navier-Stokes flow, which is generated by Wiener-Itô chaos expansion solution (5.11). Row 1 - Row 5 are the instants as t=0.1, 0.3, 0.5, 0.7, 0.9 respectively.
where we still let $\phi = 10z^2(1 - z)^2$.

In addition, the tolerance $\varepsilon = 5 \times 10^{-3}$. We use the fourth order Wiener chaos approximation with 5 Gaussian random variables ($N=4, K=5$) and apply the technique of sparse truncation to choose 33 Wiener Chaos coefficients. From our numerical test, 33 coefficients are enough to capture the main information of the stochastic solution. In fact, we compared it with the solution of 66 coefficients as $N = 6, K = 5$, and found that the quotient of their second moments $E[\tilde{u}^2]_{N=5,K=4}/E[\tilde{u}^2]_{N=6,K=5}$ is almost identically 1 (> 98%).

The effect of the above optimization algorithm is presented in figure(5.3). At the beginning, the flow is determined by the initial condition, and at about the instant between 0.2 and 0.3, the flow shape, not only the mean of our stochastic flow, starts to change to match the target flow. However, due to the randomness, we cannot expect that our controlled flow could match the target flow exactly.

The error between the controlled (or uncontrolled) flow $\tilde{u}$ (or $\tilde{w}$) and target flow $\tilde{U}$ is presented in figure(5.3). Our result shows that the magnitude of the objective functional (5.26) decreases from 7.0348 to 0.9496. In fact, a better optimization result can be obtained if we set a smaller value to $\varepsilon$. 

Figure 5.3  Errors
5.4 Conclusions

In this chapter, a distributed control problem subject to the stochastic Navier-Stokes equations is studied, where the product in the governing equations is defined in the Wick sense. To solve this optimal control problem, we apply Wiener chaos expansion to convert the original SNSEs to a deterministic system, then we use the the stochastic Galerkin finite element method to find the solutions. Furthermore, based on these coefficients, we developed an adjoint system which leads to a descent direction to minimize our objective functional. A numerical test is used to verify our algorithm.
CHAPTER 6. Optimal Control of Stochastic Fluid Flow in A Backward Facing Step Channel

Recently, polynomial chaos expansion (PCE) has been widely used in computational fluid dynamics (CFD), such as flow in porous media [16], [17]; incompressible flow [44], [71], [47], thermofluid flow [45], [46], reaction flow [58], [63], compressible flow [7], [57], [48].

In this chapter, we will investigate the application of PCE based on Karhunen-Loeve expansion (KLE) in stochastic Navier-Stokes equations and the associated control problem, in particular, the control in a backward facing step channel.

Using the stochastic boundary problem as an example, we can outline the basic principles of the PCE method follows: firstly apply the KLE to expand the boundary conditions as a power series of Gaussian random variables. Then project the stochastic Navier-Stokes equations onto the probability space, the resulting system can be solved numerically by the stochastic finite element method. In addition, for the optimal control problem, a deterministic adjoint system can be derived and hence a decent direction can be found.

6.1 Karhunen-Loeve Expansion

6.1.1 Mercer’s Theorem

Define the kernel function $K$ as

$$ K : [a, b] \times [a, b] \to R, \quad (6.1) $$

which is a symmetric continuous function, that is ,

$$ K(x, y) = K(y, x). \quad (6.2) $$
Then $K$ is said to be non-negative definite (or positive semidefinite) if and only if
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} K(x_i, x_j)c_i c_j \geq 0, \tag{6.3}
\]
for all finite sequences of points $\{x_k\}_{k=1}^{n}$ of $[a, b]$ and all choices of real numbers $\{c_k\}_{k=1}^{n}$.

Define the associated integral operator $T_K$ as
\[
(T_K \phi)(x) = \int_{a}^{b} K(x, y)\phi(y)dy. \tag{6.4}
\]
For technical considerations we assume $\phi$ can range through the space $L^2[a, b]$. Due to the linearity of $T_k$, we have the following Mercer’s Theorem for the eigenvalues and eigenfunctions of $T_K$.

**Theorem 6.1.1 (Mercer’s Theorem):** Suppose $K$ is a continuous symmetric non-negative definite kernel. Then there is an orthonormal basis $\{e_i\}_i$ of $L^2[a, b]$ consisting of eigenfunctions of $T_K$ such that the corresponding sequence of eigenvalues $\{\lambda_i\}_i$ is nonnegative, i.e.
\[
(T_K e_i)(x) = \lambda_i e_i(x), \quad \text{and} \quad \lambda_i \geq 0. \tag{6.5}
\]
The eigenfunctions corresponding to non-zero eigenvalues are continuous on $[a, b]$ and $K$ has the representation
\[
K(x, y) = \sum_{i=1}^{\infty} \lambda_i e_i(x)e_i(y), \tag{6.6}
\]
where the convergence is absolute and uniform.

### 6.1.2 Karhunen-Loeve Expansion

Now we let $Y(x, \omega), x \in \Omega$ be a random variable with finite second order moment
\[
E[Y^2(x, \omega)]dx < \infty. \tag{6.7}
\]
We denote the associated covariance function as $R(X, Y)$. For simplicity, we assume that $E[Y] = 0$, then
\[
R(X, Y) = Cov(X, Y) = E[(X - EX)(Y - EY)] = E[XY] - EX \cdot EY = E[XY]. \tag{6.8}
\]
It is easy to show that \( R(X, Y) \) is bounded, symmetric and positive definite by the definition of the covariance. Thus by Mercer’s Theorem, \( R \) has the specific decomposition

\[
R(X, Y) = \sum_{i=1}^{\infty} \lambda_i \phi_i(X) \phi_j(Y),
\]

(6.9)

where \( \lambda_i \) and \( \phi_i(X) \) are the eigenvalues and the eigenfunctions of the covariance kernel, respectively. In other words, they satisfy

\[
\int_{\Omega} R(X, Y) \phi_i(Y) dY = \lambda_i \phi_i(X), \quad i = 1, 2, \ldots,
\]

(6.10)

where \( \Omega \) is the probability space.

Clearly, \( \{ \phi_i \} \) constitutes an orthogonal basis of \( L^2(\Omega) \), for simplicity, we still use \( \{ \phi_i \} \) to denote the associated normalized basis functions.

Now for a random variable \( Y(x, \omega) \in L^2(\Omega) \), it can thus be expanded as a Fourier series

\[
Y(x, \omega) = \sum_{i=1}^{\infty} Y_i(\omega) \phi_i(x),
\]

(6.11)

where

\[
Y_i(\omega) = \int_{\Omega} Y(x, \omega) \phi_i(x) dx.
\]

Furthermore,

\[
E[Y_i Y_j] = \int_{\Omega} \int_{\Omega} E[Y(x, \omega) Y(y, \omega)] \phi_i(x) \phi_j(y) dxdy = \int_{\Omega} \phi_j(y) \int_{\Omega} R(x, y) \phi_i(x) dxdy = 0.
\]

(6.12)

Therefore, \( Y_i \) is uncorrelated, i.e. \( E[Y_i Y_j] = 0 \), if \( i \neq j \), and

\[
E[Y_i Y_i] = \int_{\Omega} \phi_i(y) \int_{\Omega} R(x, y) \phi_i(x) dxdy = \int_{\Omega} \lambda_i(\phi_i(y))^2 dy = \lambda_i > 0.
\]

Define \( \theta_i = Y_i / \sqrt{\lambda_i} \), \( Y(x, \omega) \) then can be expressed as the following expansion:

\[
Y(x, \omega) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \theta_i(\omega) \phi_i(x),
\]

(6.13)

where \( \theta_i \) satisfy \( E(\theta_i) = 0 \) and \( E(\theta_i \theta_j) = \delta_{ij} \).
Definition 6.1.2 The expansion (6.13) is called the Karhunen-Loeve expansion (KLE) of the stochastic process $Y(x, \omega)$.

Remark 6.1.3 For a Gaussian process $Y(x, \omega)$, the coefficients $Y_i(\omega)$ are independent and identically distributed (i.i.d.) Gaussian random variables.

From the definition of $\{\phi_i\}_i$, $\{\lambda_i\}_i$ and Mercer’s Theorem, the KLE is similar to the eigenvalue decomposition of a symmetric semi-positive definite matrix. Also the KLE is able to expand a random process as a series of uncorrelated random variables, the eigenfunctions $\phi_i(x)$ constitutes an orthogonal basis of $L^2(\Omega)$.

Suppose that the KLE of a given stochastic process $Y(x, \omega)$ is truncated as

$$Y_{N_{KL}}(x, \omega) = \sum_{i=1}^{N_{KL}} \sqrt{\lambda_i} \theta_i(x) \phi_i(\omega).$$  \hspace{1cm} (6.14)

Then $Y_{N_{KL}}(x, \omega)$ converges to $Y(x, \omega)$ in the mean square sense, that is

$$\lim_{N \to \infty} E[(Y - Y_{N_{KL}})^2] = 0.$$ \hspace{1cm} (6.15)

Furthermore, the essential factor to determine the convergence rate of the KLE is the smoothness of the associated covariance function, not the structure of the stochastic process being expanded. Using this fact, when making truncation to approximate the random variables whose covariance is very smooth, we only need to keep the first several leading terms to capture the main information or energy of the stochastic variable.

For the truncation (6.13), the energy ratio is defined as

$$e(N_{KL}) := \frac{\int_{\Omega} E[Y_{N_{KL}}^2] dx}{\int_{\Omega} E[Y^2] dx} = \frac{\sum_{i=1}^{N_{KL}} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i}.$$ \hspace{1cm} (6.16)

Due to rapid decay of the eigenvalues $\lambda_i, i = 1, 2, \cdots$, the truncation $Y_{N_{KL}}$ can provide a good approximations of the stochastic process.

Now we compare the KLEs of $R(x, y) = e^{-|x-y|}$ and $R(x, y) = e^{-|x-y|^2}$. 

For the covariance function $R(x, y) = e^{-|x-y|}$, we can acquire the explicit expressions for the associated eigenvalues [18], that is,

$$\lambda_n = \frac{2}{\omega_n^2 + 1},$$  \hspace{1cm} (6.17)

and

$$\lambda_n^* = \frac{2}{\omega_n^{*2} + 1},$$  \hspace{1cm} (6.18)

where $\omega_n$ and $\omega_n^*$ are defined by

$$\begin{cases} 
1 - \omega \tan(\omega) = 0 \\
\omega^* + \tan(\omega^*) = 0.
\end{cases} \hspace{1cm} (6.19)$$

![Figure 6.1 Eigenvalues, covariance function $R(x, y) = e^{-|x-y|}$](image)

The energy ratio $E(N_d)$ for $R(x, y) = e^{-|x-y|}$ and $R(x, y) = e^{-|x-y|^2}$ is approximately 92.3% and 99.8%, respectively. From Figure (6.1) and Figure (6.3), apparently the eigenvalues for covariance function $R(x, y) = e^{-|x-y|^2}$ decay faster than $R(x, y) = e^{-|x-y|}$, since the former covariance function is smoother.
Remark 6.1.4 The ideas of the KLE are also widely applied in deterministic optimal control problems, which is called proper orthogonal decomposition (POD) method, see e.g. [60], [20].

6.1.3 Galerkin Approximation

Consider the truncation

$$Y(x, \omega) = \sum_{i=1}^{N_{KL}} Y_i(\omega) \phi_i(x).$$

(6.20)

Let $h_j(x)$ be a complete set of functions in the Hilbert space $H$. Then each eigenfunction of the kernel $R(x, y)$ can be represented as

$$\phi_i(x) \approx \sum_{j=1}^{N} d^{(i)}_j h_j(x), i = 1, 2, \cdots, N_{KL}.$$  

(6.21)

From (6.10), we have the following expression for the error

$$\varepsilon_N = \sum_{j=1}^{N} d^{(i)}_j \left( \int_{\Omega} R(x, y) h_j(y) dy - \lambda_i h_j(x) \right).$$

(6.22)

Now consider the Galerkin approximation, we let

$$(\varepsilon_N, h_l(x)) = 0, \ l = 1, 2, \cdots, N.$$  

(6.23)

In other words,

$$\sum_{j=1}^{N} d^{(i)}_j \left[ \int_{\Omega} \int_{\Omega} R(x, y) h_j(y) dy h_l(x) dx - \lambda_i \int_{\Omega} h_j(x) h_l(x) dx \right] = 0.$$  

(6.24)

If we let $M = N$, then above identity can be expressed as

$$KD = BDA,$$  

(6.25)

where $K$, $D$, and $B$ are N-dimensional matrices whose elements of $K$, $D$, and $B$ are given as

$$K_{jl} = \int_{\Omega} \int_{\Omega} R(x, y) h_j(x) h_l(y) dx dy$$  

(6.26)

$$D_{jl} = d^{(i)}_j,$$  

(6.27)

$$B_{jl} = \int_{\Omega} h_j(x) h_l(y) dx dy,$$  

(6.28)
Λ is a N-dimensional diagonal matrix whose elements are

\[ \Lambda_{jl} = \delta_{jl}\lambda_j. \] (6.29)

Equation (6.25) represents a generalize eigenvalue problem which may be solved for the matrix \( D \) and the eigenvalues \( \lambda_j \).

See figure (6.2) and (6.3) for the numerical test, we let the covariance function be

\[ R(x, y) = e^{-|x-y|^2}; \Omega = [-1, 1]^2; N_{KL} = N = 10. \] (6.30)

Remark 6.1.5: The Rayleigh-Ritz method guarantees the convergence of the approximation eigenfunctions and eigenvalues to the real ones, see [4].

6.1.4 Polynomial Chaos Expansion

Suppose that \( \{\zeta_i(\omega)\} \) is a sequence of i.i.d Gaussian random variables. Denote the space of all polynomials in \( \{\zeta_i(\omega)\} \) of degrees not exceeding \( p \) as \( \hat{\Gamma}_p \), then let \( \Gamma_p \) present the space of all the polynomials in \( \hat{\Gamma}_p \) but are orthogonal to \( \hat{\Gamma}_{p-1} \). Then any function with finite variance
Figure 6.3  The first 11 eigenvalues (left); The first 4 eigenfunctions (right)

can be represented as

\[
u(\omega) = u_0 + \sum_{i_1=1}^{\infty} u_{i_1} \Gamma_1(\zeta_{i_1}(\omega)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} u_{i_1 i_2} \Gamma_2(\zeta_{i_1}(\omega), \zeta_{i_2}(\omega)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{i_3=1}^{\infty} u_{i_1 i_2 i_3} \Gamma_3(\zeta_{i_1}(\omega), \zeta_{i_2}(\omega), \zeta_{i_3}(\omega)) + \cdots,
\]

(6.31)

Where \( \Gamma_n(\zeta_{i_1}, \zeta_{i_2}, \ldots, \zeta_{i_n}) \) represents the Hermite-chaos polynomial of order \( n \), that is, for a finite index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \), define \( \Gamma_n \) as

\[
\Gamma_n(\zeta_{i_1}, \zeta_{i_2}, \ldots, \zeta_{i_n}) = \prod_{k=1}^{n} h_{\alpha_k}(\zeta_{i_k}).
\]

(6.33)

They are products of one-dimensional Hermite polynomials \( h_{\alpha} \), defined by (2.14).

The homogenous chaos expansion was first introduced by Wiener [68]. The theorem of Cameron- Martin introduced in Chapter 2 states that it can represent any functional in \( L^2 \) and converges in an \( L^2 \) sense. Thus a general stochastic process with finite variance can be expressed in the (Hermite) polynomial chaos (PC) expansion.

We can rewrite the expansion (6.31) as

\[
u(\omega) = \sum_{i=0}^{\infty} u_i \Phi_i(\zeta),
\]

(6.34)

where \( \zeta = (\zeta_1(\omega), \zeta_2(\omega), \ldots) \) and there is a one-to-one correspondence between the coefficients and the polynomials in (6.31) and (6.34).
If we denote $E[f \cdot g]$ as $\langle f, g \rangle$, then by the orthogonality of $\{\Phi_i\}$,

$$u_i = \frac{\langle u, \Phi_i \rangle}{\langle \Phi_i, \Phi_i \rangle}, i = 0, 1, 2, \ldots .$$

(6.35)

Furthermore,

$$E[u] = u_0, \ E[(u - u_0)^2] = \sum_{i=1}^{\infty} u_i^2 < \Phi_i, \Phi_i > .$$

(6.36)

Similar to Wiener-Itô chaos expansion, in practical computation, we need to truncate the PCE in both the order of the polynomials $\Phi_i$ and the number of the random variables, therefore, the finite PCE can be represented as

$$u_{p,n}(\omega) = \sum_{i=0}^{p} u_i \Phi_i(\zeta_1(\omega), \zeta_2(\omega), \cdots, \zeta_n).$$

(6.37)

6.2 Vorticity reduction by stochastic boundary action in backward facing step channel

6.2.1 Stochastic Navier-Stokes Equation

Consider the incompressible fluid through a channel with a backward facing step (Figure (6.4)), which can be described by the following incompressible Navier-Stokes equation.

$$\ddot{\vec{u}} - \nu \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \vec{f} \tag{6.38}$$

$$\nabla \cdot \vec{u} = 0, \tag{6.39}$$
where \( \vec{u} = (u, v) \). The boundary conditions are given by

\[
\begin{align*}
\vec{u}\mid_{\Gamma_{\text{in}}} &= (u_{\text{in}}, 0), \\
(P\vec{u} - \nu \frac{\partial \vec{u}}{\partial n})\mid_{\Gamma_{\text{out}}} &= (0, 0), \\
\vec{u}\mid_{\Gamma_{\text{t}} \cup \Gamma_{\text{b}} \cup \Gamma_{\text{d}}} &= (0, 0), \\
\vec{u}\mid_{\Gamma_{\text{c}}} &= \phi(t)\vec{h}(y),
\end{align*}
\]

(6.40) (6.41) (6.42) (6.43)

where \( \mu \) is viscosity, \( \vec{u} \) is the velocity, \( p \) is the pressure.

Now suppose there is a small perturbation on boundary \( \Gamma_{\text{c}} \), which is expressed as

\[
\vec{h}(y) = \vec{h} + \sigma \zeta(\omega),
\]

(6.44)

where \( \zeta(\omega) \) is a Gaussian random process with symmetry, nonnegative covariance function

\[
K(x, y) = \sigma^2 e^{-\frac{|x-y|^2}{L}},
\]

(6.45)

\( \sigma^2 \) is the variance vector and \( L \) is the normalized correlation length.

Denote the Karhunen-Loève expansion of \( \zeta(\omega) \) as

\[
\zeta(\omega) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} h_i(t) \Phi_i(\omega)
\]

(6.46)

where \( \{h_i\} \) and \( \{\lambda_i\} \) are the eigenfunctions and eigenvalues of the covariance of \( \zeta \), respectively. Then we truncate the random variable \( \omega = (\omega_1, \cdots, \omega_{NK\text{L}}) \), hence the corresponding truncation of random process \( \zeta \) can be expressed as

\[
\zeta_{NK\text{L}}(\omega) = \sum_{i=1}^{NK\text{L}} \sqrt{\lambda_i} h_i(t) \Phi_i(\omega).
\]

(6.47)

therefore,

\[
\vec{h}(y, \omega) = \vec{h}(y) + \sum_{i=1}^{NK\text{L}} \alpha_i \Phi_i(\omega) = \sum_{i=0}^{NK\text{L}} \alpha_i \Phi_i(\omega),
\]

(6.48)

where \( \alpha_0 = \vec{h}, \alpha_i = \sigma \sqrt{\lambda_i} h_i(y) \) for \( i = 1, 2, \cdots, NK\text{L} \), and by the one-to-one correspondence between \( \Phi_i \) and \( \Gamma_i \), and the order of polynomial \( \Phi_i \) is 1, hence

\[
\Phi_i(\omega) = \omega_i, \text{ for } i = 1, 2, \cdots, NK\text{L}.
\]

(6.49)
Using (6.37) and denoting the truncation of the PCE solution of (6.38) as

$$\vec{u}_M = \sum_{i=0}^{M} \vec{u}_i \Phi_i(\omega), P_M = \sum_{i=0}^{M} P_i \Phi_i(\omega),$$

(6.50)

we have the truncated version of Navier-Stokes equations:

$$(\vec{u}_M)_t - \nu \triangle \vec{u}_M + (\vec{u}_M \cdot \nabla)\vec{u}_M + \nabla P_M = f,$$

(6.51)

$$\nabla \cdot \vec{u}_M = 0.$$  

(6.52)

By multiplying $\Phi_i$ for $i = 0, 1, 2, \cdots, M$ and taking expectation, also note the orthogonality of $\Phi_i$, we obtain that

$$\partial_t \vec{u}_i < \Phi_i, \Phi_i > - \nu \triangle \vec{u}_i < \Phi_i, \Phi_i > + \sum_{l,j=1}^{M} (\vec{u}_l \cdot \nabla)\vec{u}_j < \Phi_l, \Phi_j, \Phi_i > + \nabla P_i(\Phi_i, \Phi_i)$$

$$= < f, \Phi_i >.  \quad (6.53)$$

therefore,

$$\partial_t \vec{u}_i - \nu \triangle \vec{u}_i + \sum_{l,j=0}^{M} C(i,j,l)(\vec{u}_l \cdot \nabla)\vec{u}_j + \nabla P_i = f_i,  \quad (6.54)$$

where $C(i,j,l) = < \Phi_l \Phi_j, \Phi_i > / < \Phi_i, \Phi_i >$ and $f_i = < f, \Phi_i >$.

The corresponding boundary conditions are:

on $\Gamma_{in}$, $\vec{u}_i = \vec{u}_{in}$; if $i = 0$, $\vec{u}_i = 0$, if $i \geq 1$  

(6.55)

on $\Gamma_{out}$, $\nu \frac{\partial \vec{u}_i}{\partial n} - P_i \vec{u}_i = 0$, if $i = 0$, $\vec{u}_i = 0$, if $i \geq 1$  

(6.56)

on $\Gamma_c$, $\vec{u}_i = \phi(t)\alpha_i$, if $0 \leq i \leq N_{KL}$; $\vec{u}_i = 0$, if $N < i \leq M$  

(6.57)

on $\Gamma_l \cup \Gamma_b \cup \Gamma_d$, $\vec{u}_i = (0,0)$, for $i = 0, 1, 2, \cdots, M$.  

(6.58)

6.2.2 Solution Method

To solve the deterministic system of the coefficients $u_i$, we still adapt the semi-implicit scheme and notations $a(\cdot, \cdot), c(\cdot, \cdot, \cdot), b(\cdot, \cdot)$ introduced in Chapter 5, and use the stochastic finite element method.

Define the finite variance space as $L^2_\mu(D) = \{f : E[f^2] < \infty\}$, where $\mu$ is the Gaussian measure, and $H = \{\vec{u} = \sum_{i=1}^{\infty} \vec{u}_i \Phi_i(\zeta(\omega)) \in L^2_\mu(D) : \nabla \cdot \vec{u} = 0, u = 0\}$ on $\partial D \setminus (\Gamma_c \cup \Gamma_{in} \cup \Gamma_{out})$.  

For any $\vec{w} \in H^1_0(D)$, where $D$ is the region of the backward facing channel, the weak formulation of our equation is to seek $\vec{u} = \sum_{i=0}^{\infty} \vec{u}_i \Phi_i(\zeta(\omega)) \in L^2(0, T; H)$, and $u_i \in L^2(0, T; H^1_0(D) \cap L^\infty(0, T; H))$, $u_i|_{\Gamma_c} = \phi(t)\vec{h}(y)$, $u_i|_{\Gamma_m} = \vec{u}_m$ and $p = \sum_{i=0}^{\infty} \vec{P}_i \Phi_i(\zeta(\omega)) \in L^2(0, T; H)$, and $P_i \in L^2(0, T; H^1_0(D))$ satisfy

$$\left( \partial_t \vec{u}_i, \vec{w} \right) + a(\vec{u}_i, \vec{w}) + \sum_{l,j=0}^{M} C(i, j, l) c(\vec{u}_j; \vec{u}_l, \vec{w}) + b(\vec{w}, P_i) = (f, \vec{w});$$

$$b(\vec{u}_i, q) = 0; \text{ for } q \in L^2(D),$$

and the associated semi-discrete system of the coefficients $\vec{u}_i$ is

$$\frac{1}{\theta \Delta t}(\vec{w}_i, \vec{w}) + a(\vec{w}_i, \vec{w}) + \sum_{l,j=0}^{M} \frac{1}{\theta \Delta t} C(i, j, l) c(\vec{w}_j; \vec{w}_l, \vec{w}) + b(\vec{w}, P_i)$$

$$= (f^n_i, \vec{w}) + \frac{1}{\theta \Delta t}(\vec{w}_i^n, \vec{w}) + \sum_{l,j=0}^{M} C(i, j, l) c(\vec{w}_j^n; \vec{w}_l^n, \vec{w}) + (f^n_i, \vec{w});$$

$$b(\vec{w}_i^n, q) = \begin{cases} (1 - \theta)b(\vec{w}_i^0, q), & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases},$$

where $u_i^\theta = \theta u_i + (1 - \theta) u_i^{n+1}$, $\vec{P}_i^\theta = \theta \vec{P}_i + (1 - \theta) \vec{P}_i^{n+1}$ and $f_i^\theta = \theta f_i + (1 - \theta) f_i^{n+1}$.

6.2.3 Vorticity Reducing Problem

In mechanisms, particularly the transition process, it is known that a transition phenomenon is the formulation of vortices and bursts of turbulence in places of high local vorticity. Thus, to reduce the vorticity in the domain can lead to delay the transition, see [64].

Define the objective functional $J$ is defined as

$$J(\vec{u}, \phi) = E[\frac{1}{2} \int_0^T \int_D |\nabla \times \vec{u}|^2 d\Omega dt + \frac{1}{2} \int_0^T |\phi(t)|^2 dt],$$

and then by the definition of PCE of $\vec{u}$, we obtain

$$J(\vec{u}, \phi) = \frac{1}{2} \sum_{i=0}^{M} \left( \Phi_i, \Phi_i \right) \int_0^T \int_D |\nabla \times \vec{u}_i|^2 d\Omega dt + \frac{1}{2} \int_0^T |\phi(t)|^2 dt.$$ (6.65)

Our control problem will be

$$\min_{(\vec{u}, \phi) \in U} J(\vec{u}, \phi),$$ (6.66)
where \( \Omega = \{(\bar{u}, \phi(t)) : (\bar{u}, \phi(t)) \in \bar{u} = \sum_{i=0}^{\infty} \bar{u}_i \Phi_i(\zeta(t)) \in L^2(0, T; D), \Phi_i \in L^2(0, T; H^1_0(D)) \cap L^\infty(0, T; H^1(\Omega)), \bar{u}|_{\Gamma_e} = \phi(t)\bar{h}(y), \bar{u}|_{\Gamma_m} = \bar{u}_{in}\}. 

By using the truncation (6.50) and introducing the adjoint functions \( \bar{\lambda}_i = (\lambda^1_i, \lambda^2_i) \in L^2(0, T; H^1_0(\Omega)), 0 \leq i \leq M, \) let \( \bar{\lambda}_{iT} = (0, 0), \nabla \cdot \bar{\lambda} = 0 \) and \( \lambda|_{\partial \Omega \cap \Gamma_{out}} = 0, \) we define the associated Lagrangian \( L(\bar{u}, \lambda, \phi) \) as

\[
L(\bar{u}, \phi) = \frac{1}{2} \sum_{i=1}^{M} (\Phi_i, \Phi_i) \int_0^T \int_D |\nabla \times \bar{u}_i|^2 d\Omega dt + \frac{\alpha}{2} \int_0^T |\phi(t)|^2 dt \tag{6.67}
\]

\[
- \sum_{i=0}^{M} (\Phi_i, \Phi_i) \int_0^T \int_D \bar{\lambda}_i \cdot (\bar{u}_i - \nu \Delta \bar{u}_i) + \sum_{i,j=0}^{M} C(i, j, l)(\bar{u}_i \cdot \nabla)\bar{u}_j + \nabla P_i d\Omega dt \tag{6.68}
\]

\[
= \frac{1}{2} \sum_{i=0}^{M} (\Phi_i, \Phi_i)(\nabla \times \bar{u}_i, \nabla \times \bar{u}_i) + \frac{\alpha}{2} (\phi, \phi)_{[0,T]} \tag{6.69}
\]

\[
- \sum_{i=0}^{M} (\Phi_i, \Phi_i)[(\partial_k \bar{\lambda}_i, \bar{\lambda}_i) - \nu (\Delta \bar{u}_i, \bar{\lambda}_i)] + \sum_{i,j=1}^{M} C(i, j, l)(\bar{u}_j \cdot \nabla)\bar{u}_i, \bar{\lambda}_j) \tag{6.70}
\]

\[
+ (\bar{\lambda}_i, \nabla P_i) - (\bar{\lambda}_i, f_i]. \tag{6.71}
\]

Moreover, let \((\Phi_i, \Phi_i) = 1, \) we have

\[
\sum_{i=1}^{M} \left( \frac{\delta L}{\delta u_i}, \delta \bar{u}_i \right) + \left( \frac{\delta L}{\delta \phi}, \delta \phi \right)_{[0,T]} \tag{6.72}
\]

\[
= \sum_{i=1}^{M} (\nabla \times (\nabla \times \bar{u}_i), \delta \bar{u}_i) + \alpha (\phi, \delta \phi)_{[0,T]} \tag{6.73}
\]

\[
+ \sum_{i=1}^{M} [(\partial_k \bar{\lambda}_i, \delta \bar{u}_i) + (\Delta \lambda_i, \delta u_i) \tag{6.74}
\]

\[
+ \sum_{j,l=1}^{M} C(i, j, l) - ((\nabla \bar{u}_i)^T \bar{\lambda}_j, \delta \bar{u}_j) + ((\bar{u}_j \cdot \nabla)\bar{\lambda}_i, \delta \bar{u}_i)] \tag{6.75}
\]

\[
+ \sum_{i=1}^{M} [- \sum_{j,l=1}^{M} C(i, j, l)((\bar{u}_i \cdot \bar{n})\bar{\lambda}_j, \delta \bar{u}_j)_{[0,T]} + ((\nabla \bar{u}_i - (\nabla \bar{u}_i)^T) \cdot \bar{n} \tag{6.76}
\]

\[
+ \nu \frac{\partial \bar{\lambda}_i}{\partial \bar{n}}, \delta \bar{u}_i)_{[0,T]}]. \tag{6.77}
\]

Therefore, for \( i = 1, 2, \ldots, M, \) the system of the equations for the family of the adjoint
functions \{\lambda_i\} is
\begin{align*}
-\partial_t \lambda_i - \Delta \lambda_i + \sum_{j,l=1}^{M} C(j, i, l)(\nabla \bar{u}_l)^T \lambda_j - \sum_{j,l=1}^{M} C(l, j, i)(\bar{u}_l \cdot \nabla) \lambda_j + \nabla Q_i &= 0, \quad (6.78) \\
- \sum_{j,l=1}^{M} C(j, i, l)(\bar{u}_l \cdot \bar{n}) \lambda_j + (\nabla \bar{u}_i - (\nabla \bar{u}_i)^T) \cdot \bar{n} - \nu \frac{\partial \lambda_i}{\partial \bar{n}} + Q_i \bar{n} &= 0, \quad \text{on } \Gamma_{out}, \quad (6.79) \\
\nabla \cdot \lambda_i &= 0, \quad (6.80) \\
\lambda_i\big|_{t=T} &= (0,0), \quad (6.81) \\
\lambda_i &= (0,0), \quad \text{on } \partial \Omega \setminus \Gamma_{out}. \quad (6.82)
\end{align*}

Note that \(\bar{u}_i = 0\) on \(\Gamma_t \cup \Gamma_b \cup \Gamma_d\), and if \(i > N_{KL}\), \(u_i = 0\) on \(\Gamma_c\), the variation of Lagrangian is
\begin{align*}
\delta L &= \int_0^T (\alpha \phi(t) + \sum_{i=1}^{N_{KL}} \int_{\Gamma_c} \bar{h} \cdot ((\nabla \bar{u}_i - (\nabla \bar{u}_i)^T) \cdot \bar{n} - \nu \frac{\partial \lambda_i}{\partial \bar{n}} + Q_i \bar{n})d\Gamma) \delta \phi dt, \quad (6.83)
\end{align*}
which gives a direction to update our control \(\phi(t)\).

### 6.2.4 Numerical Test

In this numerical test, we let the initial condition as
\begin{align*}
u_0(x, y) &= \begin{cases} 
10(1-y)(y-0.5), \quad \text{on } \Gamma_m, \\
0, \quad \text{otherwise}
\end{cases}, \quad (6.84)
\end{align*}
also let \(T = 5\), \(\Delta t = 0.05\), viscosity \(\mu = 0.002\), on boundary \(\Gamma_c\), we let \(h(y) = 0.2(0.5 - y)\), the coefficient for the perturbation \(\sigma = 0.5\), and \(\phi(t) = 1\) for computing the flow without control, 320 elements were taken on \(D\).

In addition, we let \(N_{KL} = 3\), the energy ratio \(E_3\) is about 97\%, and the order of polynomial is 2, and the corresponding number of polynomials in our truncation \(M = 10\). For simplicity, we assume \(v = 0\) on \(\Gamma_c\).

Figure (6.5) and Figure (6.6) present the numerical simulation of the mean of the velocity field in the channel, it is clear that around the step, a recirculation bubble is generated. Figure (6.7) presents the snapshot of the coefficients at instant \(t = 4\). Figure (6.8) shows the random part of the PCE solution, that is, \(\bar{u}_M - \bar{u}_0\).
Figure 6.5 The simulation of the stochastic flow (Column 1) and the mean of the stochastic flow (Column 2) at time $t = 2$ (Row 1) and $t = 4$ (Row 2)
Figure 6.6  The simulation of the stochastic flow (Column 1) and the mean of the stochastic flow (Column 2) at time $t = 2$ (Row 1) and $t = 4$ (Row 2)
Figure 6.7 Polynomial chaos coefficients, $u_0$ (Column 1, Row 1), $u_1$ (Column 2, Row 1), $u_2$ (Column 1, Row 2), $u_3$ (Column 2, Row 2) at time $t = 4$
Figure 6.8 $u_M - u_0$ at time $t = 2$ (Column 1, Row 1), $t = 4$ (Column 2, Row 1), $t = 6$ (Column 1, Row 2), $t = 8$ (Column 2, Row 2)
After our control algorithm was applied, the recirculation bubble disappeared, and the objective functional was reduced from $6.5 \times 10^5$ to $2.1 \times 10^3$ after 7 iterating 7 times, see Figure(6.9).

Figure 6.9  The mean of the stochastic flow under control.
The control function $\phi$ is presented by Figure(6.10)

![Control Function Graph](image)

Figure 6.10   The control function $\phi$.

### 6.3 Conclusion

In this chapter we developed a method using polynomial chaos expansion (PCE) based on Karhunen-Loeve expansion for the optimal control of fluid flows with stochastic boundary conditions. The Karhunen-Loeve expansion can capture the information of the randomness of a Gaussian stochastic process efficiently. In our example, the first 3 associated eigenvalues can capture 97% energy. Then, a system of the coefficients in the PCE can be derived, and the SNSE is converted to a deterministic system. An adjoint system hence can be derived, which gives an decent direction of our objective functional. Numerical test was used to verify our
method.
BIBLIOGRAPHY


