INTRODUCTION

The "wedge" method of generating guided waves in isotropic layers was analyzed both theoretically and experimentally by Viktorov et. al., in 1965 [1]. The main parts of the work were later reproduced in Viktorov's now famous book on Rayleigh and Lamb waves [2]. Of several detailed observations made in these investigations, one was that: For optimal generation of a mode of a given wavenumber, \( k \), the angle of the wedge should be "in the neighborhood" of the Snell's law angle, \( \theta = \sin^{-1}(k/k_w) \), where \( k_w \) represents the wavenumber of the wave in the wedge[2]. Such a choice of incident angle was being used by experimentalists utilizing Lamb waves for nondestructive evaluation purposes [3-5] even before Viktorov's analysis. The use of such an angle no doubt arose from the theory of (infinite) plane wave reflection/refraction at planar interfaces. In those cases, which are strictly of academic interest or for approximating real experimental conditions, Snell's law holds exactly as a result of satisfaction of boundary conditions along the entire (infinite) interface.

Another result of Viktorov's analysis, which forms the basis for the current study, was that instead of their being a single angle at which a particular mode could be generated, there was actually a continuous dependence of the excitation amplitude of any mode on the angle of incidence. This phenomenon is entirely due to the fact that the incident beam only insonifies the layer over a finite portion of its surface. It should be noted that beam spreading was completely neglected in Viktorov's (and the present) analysis and hence the continuous dependence of the...
excitation amplitude of a given mode on the angle of incidence is not due to beam spreading within the wedge. Beam spreading gives rise to additional physical phenomenon, as recently discussed by Zeroug et al. [6].

In this paper, the wedge method of generating guided waves is re-analyzed with particular attention being focused on the relationship between the angularly dependent excitation amplitude of a given mode and the physical parameters of the transducer and wedge used to excite the mode. Most of the theoretical analysis relies on a recently published extension of Viktorov's work to encompass arbitrary transducer pressure distributions and generally anisotropic layers [7]. The results of an experimental study to analyze the applicability of the extended theory for predicting actual excitation amplitude variations with incident angle are reported in [8].

ANALYSIS

Two common arrangements of the "wedge" method are shown in Fig. 1. In the immersion approach, the layer is immersed in a fluid bath whereas in the contact method, a (usually non-viscous) couplant is used to couple ultrasonic energy from the wedge into the layer. In either case, a finite sized transducer emits a beam of ultrasonic waves which subsequently impinges on, and generates waves in, the layer. The pressure profile across the face of the beam is denoted by \( p(\alpha) \).

The incident beam will cause quasi-longitudinal and quasi-transverse waves to be excited in the layer, each with its own excitation amplitude. For observation distances sufficiently far from the excitation region, the collection of generated and mode converted waves in the layer can be most efficiently interpreted (both mathematically and conceptually) as a summation of the guided wave modes permissible in the layer with certain amplitudes [9].

An analysis of a two-dimensional model of the wedge method, applicable to arbitrary pressure distributions and generally anisotropic layers has recently been carried out [7]. In this analysis, the transducers were assumed to be infinite in one dimension, thus creating a state of plane strain deformation in the layer. For simplicity, the time variation of the loading was assumed to be harmonic, i.e., \( e^{i\omega t} \). Denoting by \( A_{\nu}(z; \omega, \theta_i) \) the "z" dependent amplitude with which mode "\( \nu \)" of the layer
is excited by the wedge with frequency $\omega$ and incident angle $\theta_i$, it was shown that,

$$A_\nu(z; \omega, \theta_i) = \frac{\tilde{v}_{\nu y}(b/2) e^{-ik_\nu z}}{4P_{\nu \nu}} \int_{-\infty}^{\infty} p(\alpha) e^{i\chi \alpha} d\alpha$$  \hfill (1)$$

where,

$$\chi \triangleq \frac{k_\nu - k_w \sin \theta_i}{\cos \theta_i}.$$  \hfill (2)

In Eq. (1), $\tilde{v}_{\nu y}(b/2)$ denotes the complex conjugate of the “$y$” component of the particle velocity of mode “$\nu$” underneath the wedge and $P_{\nu \nu}$ denotes the time average power flux carried along the layer by the mode $\nu$ per unit waveguide width [10]. Although not shown explicitly, both of these quantities are functions of frequency, $\omega$ or alternatively, of the point on the dispersion curve at which mode $\nu$ is generated. Eq. (1) assumes that the reflected waves in the wedge do not contribute to the traction at the interface between wedge and layer, or at least that their contribution can be considered roughly constant over the angular range of interest [11].

The dependence of the modal amplitude, $A_\nu$, on the wedge angle is partially explicit in Eq. (1) in the $\cos(\theta_i)$ term, and partially implicit through the integral. Incidentally, the integral can be recognized as the familiar Fourier transform of the applied traction distribution, $p(\alpha)$, with (real) transform parameter $\chi$.

To make the dependence of the excitation amplitudes on incident angle completely explicit, a specific form must be assumed for the transducer pressure distribution, $p(\alpha)$. In the experimental work reported in [8], the actual pressure distribution of several transducers were measured and used in Eq. (1). In this case, numerical integration was performed to obtain the predicted amplitude (of a specific mode) versus wedge angle. To proceed further analytically however, it is assumed that the transducer produces a roughly parabolic pressure distribution of the form,

$$p(\alpha) = \begin{cases} \sigma_0 \left(1 - \frac{\alpha^2}{(D/2)^2}\right), & \text{if } |\alpha| \leq \frac{D}{2} \\ 0, & \text{if } |\alpha| > \frac{D}{2} \end{cases}$$  \hfill (3)$$

where $\sigma_0$ represents the maximum pressure which occurs at the center of the transducer face, $\alpha = 0$, and the transducer has a width $D$. Substituting this pressure distribution into Eq. (1) yields for the amplitude of generic mode “$\nu$”,

$$A_\nu(z; \omega, \theta_i) = \frac{2\sigma_0 \tilde{v}_{\nu y}(b/2)e^{-ik_\nu z}}{P_{\nu \nu}D \cos(\theta_i)\chi^2} \left[\frac{2\sin(\chi \frac{D}{2})}{D\chi} - \cos \left(\chi \frac{D}{2}\right)\right].$$  \hfill (4)$$

Equation (4) can be used to predict the excitation amplitude of any mode given the transducer and wedge parameters such as frequency ($\omega$), transducer width ($D$) and incident angle ($\theta_i$). For a given frequency, there will be a finite number of real wavenumbers, $k_\nu$, satisfying the dispersion equation of the (generally anisotropic) layer. Using a particular wavenumber in Eq. (4), and calculating $v_{\nu y}(b/2)$ and $P_{\nu \nu}$ for the given frequency, isolates that mode. Of course, several propagating (and non-propagating) modes may be simultaneously excited by the source and therefore, the total response is actually a summation over all of the modes of the layer at the given frequency.
Fig. 2. Normalized excitation amplitudes of the $A_0$ and $S_0$ modes as functions of incident angle and “Snell’s law” phase velocity for various size transducers. (a) $D = 12.7$ mm (1.0 inch); (b) $D = 38.1$ mm (1.5 inch); (c) $D = 254$ mm (10.0 inch).

As an example of the use of Eq. (4) to predict the excitation amplitudes of Lamb waves for given transducer and wedge parameters, consider a 1.5 MHz transducer insonifying a 1.0 mm aluminum layer. For the given frequency thickness product of 1.5 MHz·mm, only the $A_0$ and $S_0$ modes propagate in the layer. Figure 2 shows the predicted normalized excitation amplitudes of the $A_0$ and $S_0$ modes as functions of the “Snell’s law phase velocity” and incident angle, for three different size transducers, viz., 12.7 mm (1/2 inch), 38.1 mm (1.5 inch) and 254 mm (10.0 inch). Recall that the “Snell’s law phase velocity” can be calculated from the incident angle by $v_{ph} = v_w / \sin(\theta_i)$, where $v_w$ represents the longitudinal wave speed in the wedge. The curves have been normalized so that the maximum value attained by either modal amplitude is unity. Note that the angle scale on the bottom decreases towards the right in a non-linear manner.

There are several features of the plots in Fig. 2 which deserve mentioning. First of all, for a frequency thickness product of 1.5 MHz·mm, the phase velocities of the $S_0$ and $A_0$ modes are 5.14 and 2.54 mm/μsec respectively. Using Snell’s law, this gives angles of 17 and 36 deg (using $v_w = 1.5$ mm/μsec for the wedge velocity) for maximum excitation. As can be seen from Fig. 2, the excitation spectra do indeed have maxima very close to these values [12]. A second thing to note is that the maximum excitation amplitude of the $A_0$ mode is, at this frequency thickness product, larger than the maximum excitation amplitude of the $S_0$ mode. This has to do with the fact that the $A_0$ mode (again at the given $fd$) has a predominantly out of plane particle displacement and is, therefore, efficiently generated by
the normal traction which the incident wave applies to the layer. The $S_0$ mode on the other hand, with its predominantly in plane particle motion (at the given $fd$), does not efficiently couple to the normally applied pressure field. Thirdly, note that even for the same size transducer, the width of the excitation spectrum of the $A_0$ mode is narrower than that of the $S_0$ mode. This is essentially due to the fact that at the given frequency thickness product, the wavelength of the $A_0$ mode ($\lambda_{A_0} = 1.7$ mm) is roughly one half that of the $S_0$ mode ($\lambda_{S_0} = 3.4$ mm).

One can also note that the width of each excitation spectrum decreases as the size of the transducer, $D$, increases. It can be shown [7] that the width of the excitation spectrum (considered a function of the Snell’s law phase velocity) normalized by the Snell’s law phase velocity itself can be written in the form,

$$\frac{\Delta V}{V_{ph}^0} = \frac{K}{\pi} \left(\frac{\lambda^0/D}{1 - (\frac{K}{\pi})^2 \left(\frac{\lambda^0/D}{2}\right)^2}\right),$$

where $\Delta V$ represents the width of the excitation spectrum at its -9 dB point (i.e., where the amplitude drops by $1/e$ from its maximum), and $K$ is a constant depending upon the pressure profile of the transducer. For a parabolic source, $K \approx 5.852$ whereas for a piston source, $K \approx 4.398$. Also, the other quantities appearing in Eq. (5) are,

$$\bar{D} \triangleq D/\cos(\theta_i); \quad V_{ph}^0 \triangleq \frac{v_w}{\sin(\theta_i)}; \quad \lambda^0 \triangleq \frac{V_{ph}^0}{f},$$

with $f$ representing the frequency of the transducer. Note that the width of the excitation spectrum (as a function of Snell’s law phase velocity) is only a function of the dimensionless parameter $\lambda^0/\bar{D}$. Recognizing $\bar{D}$ as the length of the loading region, it can be concluded that the width of the phase velocity spectrum of a given mode is only dependent upon the ratio of the loading length to the wavelength of the mode at the Snell’s law phase velocity.

The width of the phase velocity spectrum is directly related to the potential phase velocity measurement error which could occur when using techniques such as the leaky Lamb wave technique to measure phase velocities of guided waves. It is customary when using such techniques to infer the phase velocity of generated modes from the incident angle of the insonifying transducer using Snell’s law. As shown by Fig. 2 however, modes may be excited with appreciable amplitudes even if the incident angle does not satisfy Snell’s law. It is therefore important to consider finite source effects in addition to angular positioning effects when estimating potential phase velocity errors using the wedge technique. Figure 3 is a plot of $\Delta V/V_{ph}^0$ as a function of $\lambda^0/\bar{D}$. As can be seen, in order to keep the width of the phase velocity spectrum within 10% of the predicted Snell’s law phase velocity, the insonification region must be at least 20 times larger than the wavelength of the mode being generated. This number will, in actual experimental arrangements, be reduced somewhat due to the spreading of the incident beam in the wedge. Beam spreading will, among other things, increase the length of the insonification region.

The use of Snell’s law relating the incident angle of the wedge to the phase velocity of generated modes would lead to an estimate of the measured phase velocity.
Fig. 3. Plot of $\Delta V/V^0_{\text{ph}}$ as a function of $\lambda^o/\bar{D}$.

error on the order of,

$$\left(\frac{\Delta V}{V^0_{\text{ph}}}\right)_{\Delta \theta_i} \approx -\Delta \theta_i \cot(\theta_i).$$  \hspace{1cm} (7a)

From Eq. (5) and the definitions in Eq. (6), it can be shown that the potential phase velocity measurement error due to finite source effects satisfies,

$$\left(\frac{\Delta V}{V^0_{\text{ph}}}\right)_{\text{Finite} \ D} \geq \frac{2Kv_w}{D} \cot(\theta_i),$$  \hspace{1cm} (7b)

where $K$ is the previously defined constant depending only on the pressure distribution across the face of the transducer. Both sources of error are seen to depend upon the cotangent of the incident angle, but comparing the coefficients and recognizing that $2Kv_w/D \gg \Delta \theta_i$ for all practical situations, it can be concluded that the potential phase velocity measurement errors due to finite source effects are at least an order of magnitude larger than those due to angular positioning errors.

For a given layer thickness, the wavelength of any given mode varies from point to point on the mode’s dispersion curve. A given size transducer will therefore be more or less selective to a given mode depending on where on its dispersion curve it is generated. There are, therefore, regions of the dispersion curves where, due to small wavelengths, the finite size of the transducer is not of great importance. On the other hand, some regions of the dispersion curves represent very large wavelengths and therefore the selectivity of even large (one inch, say) transducers to a particular phase velocity is very low. As an example, Figure 4 shows the -9 dB width of the phase velocity spectra at various points on the dispersion curves of different modes. The layer properties were that of aluminum ($V_L = 6.3 \text{ mm/\mu sec}, V_T = 3.1 \text{ mm/\mu sec}$) and its thickness was 1.0 mm. The size of the transducer used was $D = 12.7 \text{ mm}$, and the longitudinal wave speed of the wedge was 1.5 mm/\mu sec.

As can be seen from Fig. 4, for a given fixed frequency, the width of the excitation spectra decreases as the incident angle increases (or phase velocity decreases). This is due to the fact that as the incident angle increases, the insonification region increases and simultaneously, the phase velocity decreases, both giving rise to an
Fig. 4. Dispersion curves of an aluminum layer and the widths of the excitation spectra at various points on the curves of various modes.

increase in the ratio of $\frac{D}{\lambda^o}$. The width of the excitation spectra also decreases as the frequency increases for a given incident angle (or phase velocity). This is due to the decreasing wavelength with increasing frequency. Overall, it can be concluded that the measurement of phase velocity using single angle amplitude measurements is very difficult at regions of the dispersion curves where the modes have wavelengths with sizes comparable to the insonification region.

DISCUSSION AND CONCLUSIONS

In this paper (and [7]) the wedge technique of generating guided waves has been further analyzed in detail. Specific attention has been placed on the dependence of the excitation amplitude of a given (arbitrary) mode on the angle of incidence of the wedge. It has been shown that the width of the so-called “phase velocity spectrum”, (i.e., the amplitude of a given mode as a function of the Snell’s law phase velocity which is itself calculated from the incident angle), is dependent only upon the ratio of loading length to wavelength of the mode being generated. A given size transducer may therefore be very selective or not at all selective to a particular mode depending on where in the phase velocity — frequency plane the excited region of the mode lies. A general rule of thumb is that to keep the width of the phase velocity spectrum less than 10% of the calculated Snell’s law phase velocity, the ratio of loading length to wavelength should be at least 20. It should be noted, that if the excitation amplitude of a given mode is plotted directly as a function of the incident angle, (for lack of a better term, we call resulting amplitude verses incident angle curve the “angular spectrum”), the width is roughly independent of the wavelength of the mode being excited and only dependent upon the size of the transducer.

Of course, any theory is only good if it predicts results in agreement with carefully taken experimental results; when such results are obtainable. In the present theory, the assumption of plane strain as well as the replacement of the wedge—transducer assembly by an “equivalent” system of applied tractions are indeed questionable. That these assumptions lead to satisfactory results for a broad range of
modes and frequencies has in fact been verified in recent experiments which are re­
ported in these proceedings [8].

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11. The factor $|R(\theta_i)|$ in [7] has thus been set equal to unity.
12. It can be shown by differentiating Eq. (2) with respect to $\theta_i$ and taking the
limit as $\theta_i \rightarrow \sin^{-1}(k_\nu/k_w)$, that $A_\nu(z;\omega,\theta_i)$ is not a maximum at the Snell’s
law angle. The actual maximizing angle is, however, generally very close to this
angle.