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Coloring and extremal problems in combinatorics

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DEDICATION

To M. and D., for making me who I am today.

If people did not sometimes do silly things, nothing intelligent would ever get done.

– Ludwig Wittgenstein, *Culture and Value*

There's no ulterior practical purpose here. I'm just playing. That's what math is – wondering, playing, amusing yourself with your imagination.

– Paul Lockhart, *A Mathematician's Lament*

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CHAPTER 1. OVERVIEW

1.1 Introduction and motivation

Combinatorics is a very large field of mathematics with many different subfields and applications to other fields, including number theory, relation algebra, design theory, and computer science. Reflecting this diversity, this document contains work which has been done in different areas of the field. We will focus on two types of problems.

Coloring problems concern partitions of structures. The classic problem of partitioning the set of integers into a finite number of pieces so that no one piece has an arithmetic progression of a fixed length was solved in 1927. Van der Waerden's Theorem [77] shows that it is impossible to do so. The theorem states that if the set of positive integers is partitioned into finitely many pieces, then at least one of the pieces contains arbitrarily long arithmetic progressions (formal details contained in Chapter 3).

Another canonical example of a coloring problem is solved by Ramsey's Theorem from 1930 in [64] (formal details contained in Chapter 2). The theorem states that for a fixed integer r, if the edge set of a large enough complete graph are partitioned into a finite number of pieces, there is a complete r-vertex subgraph, all of whose edges are entirely within one piece.

Instead of saying "partitioning" we often say "coloring", and refer to the pieces in the partition as colors. With this language, van der Waerden's Theorem solves a coloring problem on the integers, and Ramsey's Theorem solves an edge-coloring problem on complete graphs.

Problems from other areas of mathematics often can be phrased in the language of Ramsey-type problems. We present an example of this in Chapter 2, where we formulate a conjecture from relation algebra as an edge-coloring problem on complete graphs and solve a special case of the conjecture.

Here, we will focus on three different coloring problems. The first coloring problem is on complete graphs. The flavor of this problem is similar to the classical Ramsey problem, although the conditions we impose on the coloring are stricter. In the second problem, we show that if the vertices of the grid are colored in 2 colors, then monochromatic substructures arise. The third concerns subsets of a finite set ordered by inclusion, where we maximize the number of colors we use on those subsets so as to avoid totally multicolored substructures.

Extremal problems focus on finding the largest (or smallest) structures which exhibit a certain property. For instance, we may wish to find a graph with the most number of edges which does not contain a certain fixed subgraph. The famous theorem of Turán [76] from 1941 is the seminal result in the field of extremal graph theory. The theorem gives the most number of edges in an n-vertex graph which does not contain an r-vertex complete subgraph (formal details in Chapter 4). See [11] for a reference on extremal problems on graphs.

The extremal problems we discuss use the Boolean lattice as the ground structure. The seminal result from this area of mathematics is due to Sperner from 1928 in [69]. This theorem provides the size of the largest collection of subsets of a finite set such that no two subsets are comparable (formal details found in Chapter 4).

Here, we reduce an upper bound on the size of a family of subsets containing no four distinct sets A, B, C, and D such that $A \subseteq B, C \subseteq D$ in a special case. We also prove bounds on the sizes of set families which do not have other configurations of sets.

Next, we provide an overview of the terminology which will be used throughout the document, followed by a brief description of each of the problems addressed herein. We will leave the majority of the formal details concerning the problems to the chapters in which they are discussed.

Each individual chapter is organized by providing background and previous results on the problem at hand, followed by statements and proofs of new results, and finally ending with directions for further research.

1.2 Global definitions

For a positive integer n, $[n]$ denotes the set $\{0, 1, 2, \ldots, n-1\}$. For a finite set X and positive integer k, let $\binom{X}{k} = \{A \subseteq X : |A| = k\}$.

A *graph* G is an ordered pair (V, E), where V is a set and $E \subseteq \binom{V}{2}$ (we allow E to be empty). The set V is referred to as the *vertex set* (or *vertices*) of G, and the set E is referred to as the *edge set* (or *edges*) of G. If $x, y \in V$ and $\{x, y\} \in E$, we say that x is *adjacent* to y. If $\{x, y\} \in E$, we often write that xy is an edge in G. If $xy \in E$, we will say that the edge xy is *incident* to vertex x and vertex y.

For a vertex $x \in V$, the *neighborhood* of x, denoted by $N(x)$, is the set $\{y \in V : xy \in E\}$; that is, every vertex to which x is adjacent. Notice that $x \notin N(x)$. For $x \in V$, the *degree* of x, denoted by $\deg(x)$, is $|N(x)|$. A graph is called *regular* if for all $x, y \in V$, $\deg(x) = \deg(y)$. A regular graph $G = (V, E)$ is called *strongly regular* if

- for any two adjacent vertices x and y, there are exactly λ vertices adjacent to both x and y; and
- for any two nonadjacent vertices x and y, there are exactly μ vertices adjacent to both x and y.

An example of a strongly regular graph is given in Chapter 2.

We will often draw diagrams to represent graphs pictorially. The diagrams will consist of dots and line segments between the dots. Each dot will represent a vertex of G and two vertices connected by a line segment are considered adjacent. Since we only wish to encode the information concerning which vertices are adjacent to one another, we may draw the dots and line segments in whichever way is most convenient. For an example of such a drawing, see Figure 2.3.

If $G = (V, E)$ is a graph and $E = \binom{V}{2}$, we call G the *complete graph on* $|V|$ *vertices* and denote this graph by $K_{|V|}$. If $E = \varnothing$, we say G is the *empty graph*.

We say $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If G' is a subgraph of G and E' contains *all* edges $xy \in E$ with $x, y \in V'$, then we say G' is an *induced* *subgraph* of G. If the complete graph on k vertices is an induced subgraph of G, then we say G contains a *clique of size* k. If the empty graph on k vertices is an induced subgraph of G, then we say G contains an *independent set of size* k.

For comprehensive references on graph theory, see [12], [20], and [78].

A *coloring* of a set X is a surjective function with domain X. A *k-coloring* of a set X is a surjective map $f : X \to \{1, 2, \ldots, k\}$. If f is a k-coloring of X, we will refer to the elements of $\{1, 2, \ldots, k\}$ as *colors*. If $X' \subseteq X$ and f is constant on X', we say that X' is *monochromatic* under f. If $X' \subseteq X$ and for all $a, b \in X'$ with $a \neq b$, $f(a) \neq f(b)$, we say X' is *totally multicolored* or *rainbow* under f.

When speaking about the asymptotic behavior of certain functions, we will use the standard "Big Oh notation". Let f and g be two nonnegative functions defined on some set of natural numbers. If there exist real numbers M and N such that for all $n \geq N$, $f(n) \leq Mg(n)$, then we will write $f = O(g)$. This is an abuse of notation, since $O(g)$ is not a single function, but rather a collection of functions. The reason for this is that we often wish to express only the order of magnitude of a function since it may be difficult (or impossible) to express it explicitly. The notations we will use are outlined below. If f and g represent nonnegative functions, then

- by $f = O(g)$, we mean there exist real numbers M and N such that for all $n \geq N$, $f(n) \leq Mg(n)$;
- by $f = \Omega(g)$, we mean there exist real numbers M and N such that for all $n \geq N$, $f(n) \geq Mg(n)$;
- by $f = o(g)$, we mean that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$; and
- by $f \sim g$, we mean that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

For example, the observe that $\frac{1}{n} = o(1)$ since $\lim_{n\to\infty} \frac{1}{n} = 0$.

A *partially ordered set* (or *poset*) $\mathcal{P}$ is a pair $(X, \leq)$ where X is a set and $\leq$ is a relation which is reflexive (for all $x \in X$, $x \leq x$), antisymmetric (if $x \leq y$ and $y \leq x$, then $x = y$), and transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$). If $x \not\leq y$ and $y \not\leq x$, we say that x and y are *incomparable* and write $x||y$.

A poset $\mathcal{R} = (X', \leq')$ is a *subposet* of $\mathcal{P} = (X, \leq)$ if there exists an injective function $g : X' \to X$ such that if $x_1, x_2 \in X'$ and $x_1 \leq' x_2$, then $g(x_1) \leq g(x_2)$. We say $\mathcal{R}$ is an *induced subposet* of $\mathcal{P}$ if there exists a injective function $g : X' \to X$ such that for all $x_1, x_2 \in X'$, $x_1 \leq' x_2$ if and only if $g(x_1) \leq g(x_2)$.

For a set A, let 2^A denote the power set of A. Let Q_n denote the poset $(2^{[n]}, \subseteq)$, also called the Boolean lattice of order n.

For a comprehensive reference on posets, see [75].

1.3 Chromatic graphs

The first problem we will discuss has its roots in finding representations of relation algebras, but we formulate the problem in terms of an edge-coloring problem on complete graphs. We are given a set of colors and a collection of triangles with colors assigned to the edges (which we call *forbidden triangles*). We wish to find a positive integer N so that it is possible to color the edges of K_N in such a way so that each color emanates from each vertex, no forbidden triangle appears in our coloring, and each colored edge in K_N appears in every possible triangle which is not forbidden. For instance, if the triangle with edges colored red, green, and blue is not forbidden, then each edge with color red *must* participate in a triangle with edges colored red, green, and blue.

We show that if our set of colors is $\{r, b_1, \ldots, b_n\}$ and our set of forbidden triangles is all those triangles not containing color r, then it is possible to color the edges of K_N according to our constraints provided N is sufficiently large.

The conditions we impose on our "good" colorings are stricter than those in "good" Ramsey colorings, yet connections exist between the two.

1.4 Monochromatic subsets of the integer grid

The second problem discussed continues work done by Graham and Solymosi in [34], searching for monochromatic substructures in colorings of the rectangular grid. Formally, the rectangular grid is simply the set $\mathbb{Z}^2$. For this problem, we will color the grid in 2 colors, and see how large of a subgrid we must consider until we can guarantee the appearance of monochromatic substructures. We consider 3 and 4 point configurations in general position; that is, no 3 points collinear. This problem has a definite connection to the classical van der Waerden Theorem and makes use of recent work on finding so-called van der Waerden numbers.

1.5 Posets

We combine the results of work on two separate problems into one chapter here, since the ground structure for both is the Boolean lattice.

Given a subposet $(\mathcal{P}, \subseteq)$ of the Boolean lattice of order n, we seek the minimum number of colors r so that in every r-coloring of $2^{[n]}$, there is a totally multicolored induced subposet of Q_n which is isomorphic to $(\mathcal{P}, \subseteq)$. We also investigate the size of the largest collection $\mathcal{F} \subset 2^{[n]}$ such that $(\mathcal{P}, \subseteq)$ is not a subposet of $(\mathcal{F}, \subseteq)$.

CHAPTER 2. CHROMATIC GRAPHS

2.1 Introduction and previous results

For a positive integer n, let $K_n = (V, E)$ be the complete graph on n vertices. Let L be any finite set and $\mathcal{M} \subseteq L^3$. Let $c : E \to L$.

For $x, y, z \in V$, let $c(xyz)$ denote the ordered triple $(c(xy), c(yz), c(xz))$. We say that c is *good with respect to* $\mathcal{M}$ if the following conditions hold:

(i) for all $x, y \in V$ and $j, k \in L$ such that $(c(xy), j, k) \in \mathcal{M}$, there is $z \in V$ such that $c(xyz) = (c(xy), j, k)$;

(ii) for all $x, y, z \in V$, $c(xyz) \in \mathcal{M}$; and

(iii) for all $x \in V$ and $\ell \in L$, there is $y \in V$ such that $c(xy) = \ell$.

If K_n has a coloring c which is good with respect to $\mathcal{M}$, then we say that K_n *realizes* $\mathcal{M}$ (or that $\mathcal{M}$ is *realizable*).

Conditions (i)–(iii) are given in [16] where the author calls a coloring on K_N that realizes some $\mathcal{M}$ a *symmetric color scheme.* Informally, these conditions say

(i) If the edge xy has color r and r is a member of a triple $(r, j, k) \in \mathcal{M}$, then there is a triangle containing edge xy whose edges have colors r, j, and k;

(ii) if there is a triangle with colors j, k, and ℓ under coloring c, then $(j, k, \ell) \in \mathcal{M}$;

(iii) each color in L is incident to each vertex in V.

Definition 2.1. *Let L be a finite set and let $\mathcal{M} \subseteq L^3$ which is closed under permutation. A color $\alpha \in L$ is called* ***flexible*** *if for all $\beta, \gamma \in L$, $(\alpha, \beta, \gamma) \in \mathcal{M}$.*

Notice that a flexible color does not appear in any triple which is *not* in $\mathcal{M}$.

Comer shows in [17] that if $\mathcal{M}$ is a set of triples that is closed under permutation with at least one flexible color, then $\mathcal{M}$ is realized by a coloring on K_ω, the complete graph on countably (infinitely) many vertices.

Conditions (i)–(iii) may seem quite stringent, but in fact these conditions are satisfied in some constructions of lower bounds of Ramsey numbers. In [64], Ramsey proved the seminal theorem (now called Ramsey's Theorem), a special case which is stated below as Theorem 2.1.

Theorem 2.1 (Ramsey, 1930)**.** *For an integer k, there exists $N = N(k)$ so that any graph on $n \geq N$ vertices contains either a clique of size k or an independent set of size k.*

Ramsey's Theorem is most often stated in terms of an edge coloring problem on complete graphs; that is, given k, there is $N = N(k)$ such that if $n \geq N$ and χ is a 2-coloring of K_n, then K_n contains a complete subgraph on k vertices which is monochromatic under χ. In fact, Ramsey's theorem can be extended for any finite number of colors; we state this extended version below as Theorem 2.2 (such a formulation may be found in [35]).

Theorem 2.2 (Ramsey's Theorem, extended version)**.** *Let ℓ be a positive integer. For positive integers $k_1, \ldots, k_\ell$, there exists an integer N such that if $n \geq N$, then for any ℓ-coloring of the edges K_n there is a monochromatic complete subgraph on k_j vertices in color j for some j.*

Let $R(k_1, k_2, \ldots, k_\ell)$ denote the least such integer N guaranteed by Theorem 2.2. This number is known as a *Ramsey number*, and are notoriously difficult to compute. It is well known that $R(3,3) = 6$, which means that it is possible to color the edges of K_5 in 2 colors such that there is no monochromatic copy of K_3 (a triangle). For a positive integer k, let $L_k = \{r_1, r_2, \ldots, r_k\}$ and let

$$\mathcal{Q}_k = L_k^3 \backslash \{(r_i, r_i, r_i) : 1 \leq i \leq k\}.$$

In fact, the coloring of K_5 which shows $R(3,3) \geq 6$ satisfies conditions (i)–(iii) with $L = L_k$ and $\mathcal{M} = \mathcal{Q}_2$; this is easy to check by inspection, so we provide this coloring in Figure 2.1.

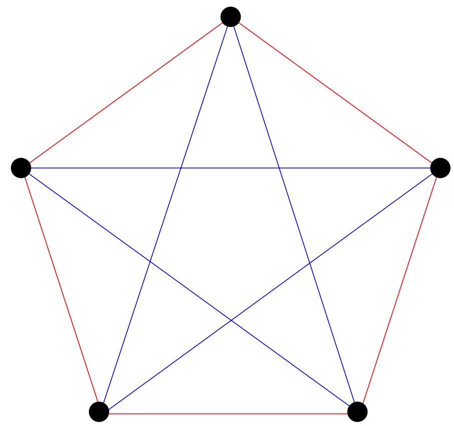

Figure 2.1 The coloring of K_5 which shows that $R(3,3) \geq 6$ and is a realization of $\mathcal{Q}_2$. The color red corresponds to r_1 and the color blue corresponds to r_2. The graph is small enough to check by hand that no red (respectively, blue) edge appears in a monochromatic triangle, and that each red (resp. blue) edge appears in a triangle with two red (resp. blue) and one blue (resp. red) edge as well as a triangle with two blue (resp. red) edges.

In [37], Greenwood and Gleason provide a coloring of K_8 which shows that $R(4,3) \geq 9$; they also prove that $R(4,3) = 9$. Their coloring of K_8 is a realization of $\mathcal{M} = \mathcal{Q}_2 \cup \{(r_1, r_1, r_1)\}$. We depict this coloring in Figure 2.2 below.

Greenwood and Gleason [37] are also responsible for providing colorings of K_{16} which show $R(3,3,3) \geq 17$. These colorings are realizations of $\mathcal{Q}_3$. While it would be too cumbersome to include a figure of the complete coloring here, we note that the edges of each color class (in both the twisted and untwisted colorings) form the so-called Clebsch graph, discovered in [15], and discussed by Godsil in [31]. The graph is constructed in the following way. Each vertex will correspond to a binary string of length 5 with an even number of entries equal to 1, and two vertices are adjacent provided the two strings differ in all but one position. The result is a 16-vertex strongly regular graph of degree 5 with $\lambda = 0$ and $\mu = 2$. We include a figure of the Clebsch graph here for completeness, see Figure 2.3.

The coloring of K_{29} originally given by Kalbfleisch in [44] which shows that $R(4,3,3) \geq 30$ is a realization of $\mathcal{M} = \{r, b, g\}^3 \setminus \{(b,b,b), (g,g,g)\}$. The construction is as follows. Assign

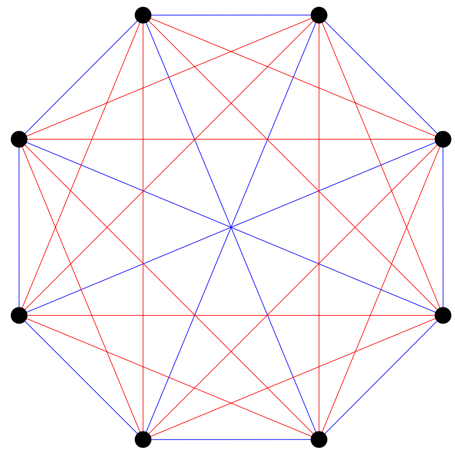

Figure 2.2 The coloring of the edges of K_8 which shows that $R(4,3) \geq 9$ and is a realization of $\mathcal{Q}_2 \cup \{(r_1, r_1, r_1)\}$. Again, red corresponds to r_1 and blue corresponds to r_2.

the vertices in K_{29} the elements of the cyclic group $\mathbb{Z}_{29}$. The edges are colored according to the differences modulo 29 between assignments of the vertices. The first color class is all those differences in the set $\{1, 4, 10, 12\}$ (and the negatives of these modulo 29; that is $\{28, 25, 19, 17\}$), the second color class is all those differences in the set $\{2, 5, 6, 14\}$ (and their negatives modulo 29), and the third color class is all those differences in the set $\{3, 7, 8, 9, 11, 13\}$ (and their negatives modulo 29).

We have cited many of the original sources in our discussion on Ramsey numbers, but for the most up-to-date reference for recent work in the field, we direct the reader to Radziszowski's excellent dynamic survey, found in [63] (most recently updated in August of 2009). The document is available freely online at the website for the Electronic Journal of Combinatorics and contains many tables with all known bounds for Ramsey numbers.

In [16], Comer introduces the number $r(k)$, which is the largest N such that there is a coloring on K_N that realizes $\mathcal{Q}_k$ (if there is no such realization of $\mathcal{Q}_k$, he sets $r(k) = 0$). Since Ramsey's Theorem gives us that any k-coloring χ of K_p where $p = R(\overbrace{3, 3, \ldots, 3}^{k \text{ times}})$ will necessarily contain a triangle which is monochromatic under χ, $r(k) \leq p - 1$. Equality holds for $k = 2$ and $k = 3$. It is not known if $r(k) > 0$ for $k \geq 6$.

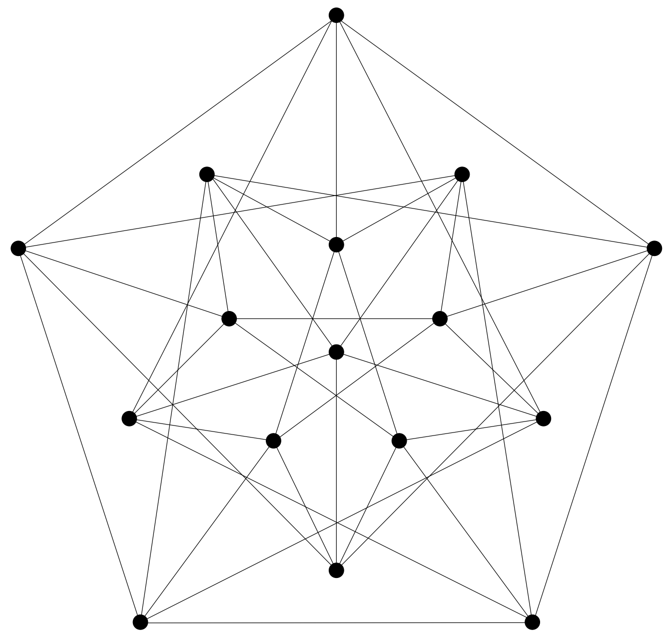

Figure 2.3 A depiction of the Clebsch graph, which is the graph obtained by considering the edges of a color class in the twisted and untwisted colorings of K_{16} which show that $R(3,3,3) \geq 17$. These colorings are also realizations of $\mathcal{Q}_3$.

Realizations of color schemes arise in connection with projective planes as well. For a positive integer $n \geq 2$, a projective plane of order n is a finite set P (called "points") of cardinality n^2+n+1 together with a collection $\mathcal{F} \subseteq 2^P$ (called "lines") of cardinality n^2+n+1 such that the following three things hold:

- given any 2 points $p_1, p_2 \in P$, there is exactly 1 line $X \in \mathcal{F}$ with $p_1, p_2 \in \mathcal{F}$;
- given any 2 lines $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| = 1$; and
- there are 4 points such that no line contains any 3 of them.

For a pictorial representation of the projective plane of order 2, see Figure 2.4. The only examples of projective planes known are of prime power order. The smallest n for which it is not known whether there is a projective plane of that order is $n = 12$, although the work that has been done in [58], [71], [59], [1], and [2] suggests that such a structure does not exist.

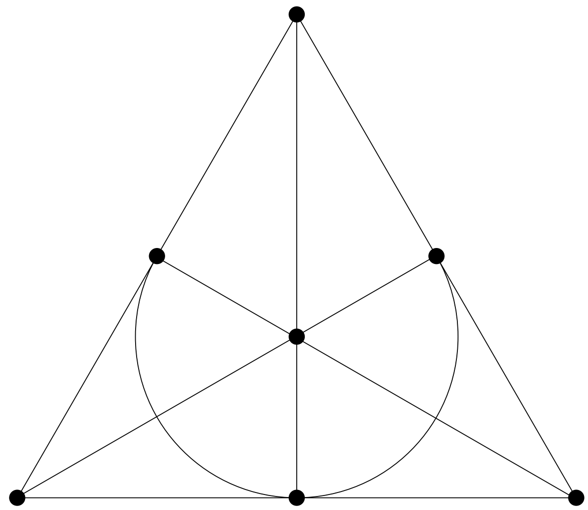

Figure 2.4 An example of a projective plane of order $n = 2$. There are 7 points and 7 lines, each point belongs to 3 lines, and each line is incident with 3 points. The projective plane of order 2 is called the Fano plane.

To see how the theory of projective planes relates to the problem at hand, let $L = \{r_1, \ldots, r_\ell\}$, and let

$$\mathcal{N}_\ell = \{(r_i, r_j, r_k) : |\{i, j, k\}| \in \{1, 3\}\}\,;$$

that is, the only colored triangles allowed are those which are monochromatic or totally multicolored. Lyndon proved in [51] that $\mathcal{N}_\ell$ is realizable in some complete graph if and only if there exists a projective plane of order $\ell - 1$, for $\ell > 2$.

In [43], Jipsen, Maddux, and Tuza show that for $\mathcal{M} = L^3$, K_N realizes $\mathcal{M}$ for arbitrarily large finite N. In the case when $\mathcal{M} = L^3$, every color in L is a flexible color.

2.2 The main result

The principal result of this chapter is that $\mathcal{M}_n$ is realizable in K_N for some $N < \omega$, where $L = \{r, b_1, ..., b_n\}$ and

$$\mathcal{M}_n := \{(r,r,r), (r,r,b_i), (r,b_i,r), (b_i,r,r), (r,b_i,b_j), (b_i,r,b_j), (b_i,b_j,r) : i, j \in \{1, \ldots, k\}\}\,.$$

(Observe that $\mathcal{M}_n = \{r, b_1, \dots, b_n\}^3 \backslash \{b_1, \dots, b_n\}^3$.) This is a special case of a problem that has come to be known as the flexible atom conjecture. The conjecture states that any symmetric color scheme with any nonzero number of flexible atoms is realizable on some finite complete graph. We will prove the special case when there is precisely one flexible color. This problem originates in relation algebra; an explanation of the conjecture in this context can be found in [52]. For a more complete reference on the theory of relation algebras, see [53]. We state our main result below as Proposition 2.1.

Proposition 2.1. *For any positive integer n, there is an integer $\ell = \ell(n)$ such that for every integer k with $k > \ell$, K_N realizes $\mathcal{M}_n$ for $N = \binom{3k-4}{k}$.*

The proof is due to Alm, Maddux, and Manske, and appears in [3]. The proof will proceed as follows. First we will construct realizations of $\mathcal{M}_1$ in K_N for arbitrarily large N. These colorings of K_N will exhibit quite a lot of redundancy; in particular, for any given edge $xy \in E$ and triple $(c(xy), j, k) \in \mathcal{M}_1$, there exist many vertices z such that $c(xyz) = (c(xy), j, k)$, while condition (i) only requires that there be *one* such vertex. The graph K_N, which is colored in colors r and b, can then be recolored by assigning edges colored b to a color from $\{b_1, ..., b_n\}$ uniformly at random. The probability that this recoloring is a realization of $\mathcal{M}_n$ is shown to be nonzero for sufficiently large N.

Note that r is a *flexible color* in $\mathcal{M}_n$. In the case that a flexible color is present, it is not hard to see that condition (iii) is automatically satisfied whenever (i) and (ii) are, and so we make no further mention of it.

2.2.1 Proof of Proposition 2.1

Let $k \in \mathbb{N}$ and let $\binom{[3k-4]}{k}$ denote the collection of k-element subsets of $[3k-4]$. Let G be the complete graph with vertex set $V = \binom{[3k-4]}{k}$.

Lemma 2.1. *If $k \geq 3$, G realizes $\mathcal{M}_1$.*

Proof of Lemma 2.1. Define an edge coloring $c : E(G) \to \{r, b\}$ by

$$c(xy) = \begin{cases} b, & \text{if } 0 \le |x \cap y| \le 1, \\ r, & \text{otherwise.} \end{cases}$$

Let $E_r = \{xy \in E(G) : c(xy) = r\}$ and $E_b = \{xy \in E(G) : c(xy) = b\}$. The following five claims establish that c satisfies condition (i) for $\mathcal{M}_1$.

Let $xy \in E_r$. Since $|x \cap y| \ge 2$, $|x \cup y| \le 2k - 2$.

Claim 1. *$\exists z \in V$ such that $c(xyz) = (r, r, r)$.*

Let $\overline{(x \cup y)}$ denote $[3k-4] \setminus (x \cup y)$ and let ℓ be any subset of $\overline{(x \cup y)}$ with $k - |x \cap y|$ elements. Set $z = \ell \cup (x \cap y)$. We have $|x \cap z| \ge 2$ and $|y \cap z| \ge 2$, so $c(xyz) = (r, r, r)$ and Claim 1 is true.

Claim 2. *$\exists z \in V$ such that $c(xyz) = (r, r, b)$.*

Let $a_1 \in y \setminus x$, $a_2 \in x \cap y$, and ℓ be any $(k-2)$-subset of $\overline{(x \cup y)}$. Set $z = \ell \cup \{a_1, a_2\}$. We have $|x \cap z| = 1$ and $|y \cap z| = 2$, so $c(xyz) = (r, r, b)$ and Claim 2 is true.

Claim 3. *$\exists z \in V$ such that $c(xyz) = (r, b, b)$.*

Let $a_1 \in x \setminus y$, $a_2 \in y \setminus x$. Let ℓ be as in the the proof of Claim 2. Set $z = \ell \cup \{a_1, a_2\}$. We have $|x \cap z| = |y \cap z| = 1$, so $c(xyz) = (r, b, b)$ and Claim 3 is true.

Now let $xy \in E_b$. Since $|x \cap y| \le 1$, $|x \cup y| \ge k - 3$.

Claim 4. *$\exists z \in V$ such that $c(xyz) = (b, r, r)$.*

If $k = 3$, then $|x \cap y| = 1$, so we can pick z to be the 3-subset consisting of $x \cap y$, one point from $x \setminus y$ and one point in $y \setminus x$. For $k \ge 4$, let ℓ_1 be any 2-subset of $x \setminus y$, ℓ_2 be any 2-subset of $y \setminus x$, and ℓ_3 be any $(k-4)$-subset of $\overline{(x \cup y)}$. Set $z = \ell_1 \cup \ell_2 \cup \ell_3$. We have $|x \cap z| = 2$ and $|y \cap z| = 2$, so $c(xyz) = (b, r, r)$ and Claim 4 is true.

Claim 5. *$\exists z \in V$ such that $c(xyz) = (b, b, r)$.*

If $k = 3$, then $|x \cap y| = 1$, so we can pick z to be the 3-subset consisting of $y \setminus x$ together with one point from $x \setminus y$. For $k \geq 4$, let ℓ_1 be any 3-subset of $x \setminus y$ and $a \in y \setminus x$. Let ℓ_3 be as in the proof of claim 4. Set $z = \ell_1 \cup \{a\} \cup \ell_3$. We have $|x \cap z| \geq 2$ and $|y \cap z| = 1$, so $c(xyz) = (b, b, r)$ and Claim 5 is true.

Observe that Claims 1–5 imply that c satisfies condition (i) for $\mathcal{M}_1$. It remains to show that c satisfies condition (ii) for $\mathcal{M}_1$, which we show in Claim 6 below.

Claim 6. *$\forall x, y, z \in V$, $c(xyz) \in \mathcal{M}_1$.*

By way of contradiction, suppose there are $x, y, z \in V$ with $c(xyz) = (b, b, b)$. Since $|x \cup y \cup z| \leq 3k - 4$, the pigeonhole principle implies that one of $|x \cap y|$, $|x \cap z|$, or $|y \cap z|$ is greater than or equal to 2, a contradiction.

Claims 1–6 imply that c is good with respect to $\mathcal{M}_1$, and thus G realizes $\mathcal{M}_1$. □

Let $n \in \mathbb{N}$ and let E_r and E_b be as in the proof of Lemma 2.1. Partition E_b into n parts $E_{b_1}, E_{b_2}, \ldots, E_{b_n}$ probabilistically such that $\forall xy \in E_b$ and $\forall i \in [n]$ the probability that $xy \in E_{b_i}$ is $1/n$.

Define a new edge coloring c' of G given by

$$c'(xy) = \begin{cases} b_i & \text{if } xy \in E_{b_i}, \\ r & \text{if } xy \in E_r. \end{cases}$$

We claim that for sufficiently large k, c' is good with respect to $\mathcal{M}_n$, and thus G realizes $\mathcal{M}_n$; for this reason, we assume that $k \geq 4$. Since c satisfies condition (ii) for $\mathcal{M}_1$, it is easy to see c' satisfies condition (ii) for $\mathcal{M}_n$. We show that the probability that c' does not satisfy condition (i) for $\mathcal{M}_n$ is less than 1.

Claim 7. *The probability P_1 that given $xy \in E_r$, there are $i, j \in [n]$ such that for all $z \in V$ $c'(xyz) \neq (r, b_i, b_j)$ is bounded from above by $n^2(1 - 1/n^2)^{(k-2)^2}$.*

Proof of Claim 7. Let $Z := \{z \in V : c(xyz) = (r, b, b)\}$. For fixed $i, j \in [n]$ and $z \in Z$, the probability

$$(xz \in E_{b_i}) \wedge \left(yz \in E_{b_j}\right)$$

is $1/n^2$, so the probability

$$(xz \notin E_{b_i}) \vee \left(yz \notin E_{b_j}\right)$$

is $1 - 1/n^2$. Considering all $z \in Z$, we have that the probability

$$\bigwedge_{z \in Z} \left[(xz \notin E_{b_i}) \vee \left(yz \notin E_{b_j}\right)\right]$$

is $\left(1 - 1/n^2\right)^{|Z|}$. Summing over all n^2 combinations of i and j, we arrive at

$$P_1 = n^2 \left(1 - 1/n^2\right)^{|Z|}. \tag{2.1}$$

For an upper bound on P_1 we compute a lower bound on $|Z|$. Since we seek a lower bound, we may assume $|x \cap y| = 2$. Note that $|\overline{(x \cup y)}| = k - 2$. Let $a_x \in x \setminus y$ and $a_y \in y \setminus x$. If $z = \overline{(x \cup y)} \cup \{a_x, a_y\}$, then $z \in Z$. Since there are $(k-2)^2$ distinct z of this form, $(k-2)^2 \leq |Z|$. This fact together with (2.1) gives $P_1 \leq n^2 \left(1 - 1/n^2\right)^{(k-2)^2}$, as desired. □

Claim 8. *The probability P_2 that given $xy \in E_r$, there is $j \in [n]$ such that for all $z \in V$ $c'(xyz) \neq (r, r, b_j)$ is bounded from above by $n\,(1 - 1/n)^{\binom{k-2}{2}}$.*

Proof of Claim 8. Let $Z := \{z \in V : c(xyz) = (r, r, b)\}$. If $z \in Z$, then $c(yz) = r$, so the probability that $yz \in E_r$ is equal to 1. Hence, for fixed $j \in [n]$ and $z \in Z$, the probability $(xz \in E_{b_j}) \wedge (yz \in E_r)$ is equal to the probability that $xz \in E_{b_j}$. This probability is equal to $1/n$, so the probability

$$(xz \notin E_{b_j})$$

is $1 - 1/n$. Considering all $z \in Z$, we have that the probability

$$\bigwedge_{z \in Z} (xz \notin E_{b_j})$$

is $(1 - 1/n)^{|Z|}$. Summing over all j, we arrive at

$$P_2 = n(1 - 1/n)^{|Z|}. \tag{2.2}$$

For an upper bound on P_2, we compute a lower bound on $|Z|$. As in the proof of Claim 7, we may assume $|x \cap y| = 2$ so $|\overline{(x \cup y)}| = k-2$. Let ℓ be any 2-subset of $\overline{(x \cup y)}$. If $z = (y \backslash x) \cup \ell$, then $z \in Z$. Since there are $\binom{k-2}{2}$ distinct z of this form, $\binom{k-2}{2} \leq |Z|$. This fact together with (2.2) gives $P_2 \leq n\,(1-1/n)^{\binom{k-2}{2}}$, as desired. □

Claim 9. *The probability P_3 that given $i \in [n]$ and $xy \in E_{b_i}$, there is $j \in [n]$ such that for all $z \in V$, $c'(xyz) \neq (b_i, r, b_j)$ is bounded from above by $n\,(1-1/n)^{\binom{k}{4}}$.*

Proof of Claim 9. Fix $i \in [n]$ and $xy \in E_{b_i}$. Let

$$Z := \{z \in V : c(yz) = r \text{ and } c(xz) = b\}.$$

For $j \in [n]$, the probability that $xz \in E_{b_j}$ is $1/n$, so the probability that $xz \notin E_{b_j}$ is $1 - 1/n$. Continuing as in the proof of Claim 8, we have

$$P_3 = n(1-1/n)^{|Z|}. \tag{2.3}$$

Again, we seek a lower bound for $|Z|$, so we may assume $|x \cap y| = 0$. Note that this gives $|\overline{(x \cup y)}| = k - 4$. Let ℓ be any 4-subset of y. If $z = \overline{(x \cup y)} \cup \ell$, then $z \in Z$. Since there are $\binom{k}{4}$ distinct z of this form, $\binom{k}{4} \leq |Z|$. This fact together with (2.3) gives $P_3 \leq n(1-1/n)^{\binom{k}{4}}$.
□

Observe that for all $q \in \{1, 2, 3\}$, $P_1 \geq P_q$. Hence, we can use the upper bound in Claim 7 for P_1 as an upper bound for the probability that c' does not satisfy condition (i) for a given edge $xy \in E$. Since G has less than $\binom{3k-4}{k}^2$ edges, an upper bound for the probability P that c' fails to satisfy condition (i) for $\mathcal{M}_n$ is

$$P \leq \sum_{e \in E} P_1 \leq \binom{3k-4}{k}^2 P_1 \leq \binom{3k-4}{k}^2 n^2 \left(1 - \frac{1}{n^2}\right)^{(k-2)^2}. \tag{2.4}$$

Next, we show that the right hand side of the expression in (2.4) can be made less than 1 by choosing k large enough. Since $1 - x \leq e^{-x}$ for all x, we have

$$\binom{3k-4}{k}^2 n^2 \left(1-\frac{1}{n^2}\right)^{(k-2)^2} \leq \binom{3k-4}{k}^2 n^2 \left(e^{-(k-2)^2/n^2}\right)$$
$$\leq \left(2^{3k-4}\right)^2 n^2 \left(e^{-(k-2)^2/n^2}\right)$$
$$\leq 2^{6k} n^2 \left(e^{-(k-2)^2/n^2}\right). \qquad (2.5)$$

Note that the expression in (2.5) is less than 1 if and only if

$$\log\left[2^{6k} n^2 \left(e^{-(k-2)^2/n^2}\right)\right] < 0,$$

which is equivalent to

$$6k \log 2 + 2\log n - \frac{(k-2)^2}{n^2} < 0. \qquad (2.6)$$

To ensure that the inequality in (2.6) will hold, we first assume that $k = cn^2$ for some $c \in \mathbb{R}$ and realize the above as a quadratic polynomial in c. Since the coefficient of c^2 is negative, the function is concave down. By finding the zeros of this polynomial in terms of n and then maximizing (over n) the greatest of them, we can find the c which will guarantee the inequality in (2.6). For $n \geq 2$, it is sufficient to take $c \geq 5.2$. Note that for the case where $n = 1$, we may use Lemma 2.1, and thus we need not consider this inequality.

For such k, we have that $P < 1$, so there exists an edge coloring $c : E(G) \to \{r, b_1, \ldots, b_n\}$ which is good with respect to $\mathcal{M}_n$. Hence, G realizes $\mathcal{M}_n$ and the proof of Theorem 2.1 is complete.

The following corollary is an interpretation of Proposition 2.1 in the language of relation algebras. As stated earlier, it is a special case of the flexible atom conjecture (the case where exactly one atom is flexible).

Corollary 2.1. *Any finite integral symmetric relation algebra with one flexible atom and with all (mandatory) diversity cycles involving the flexible atom is representable on arbitrarily large finite sets.*

2.3 Further research

We have only considered finding which symmetric color schemes are realizable in complete graphs. Once we find a symmetric color scheme $\mathcal{M}$ which is realizable in some complete graph, a natural question to ask next is for which positive integers n do there exist colorings of K_n which are good with respect to $\mathcal{M}$. For instance, we showed earlier that there is a realization of $\mathcal{M}_0$ on K_5; indeed, K_5 is the *only* complete graph for which there exists a good coloring with respect to $\mathcal{M}_0$. This coloring was provided in Figure 2.1. It is easy to check that no colored subgraph of the one provided in Figure 2.1 will satisfy condition (i) for $\mathcal{Q}_2$ (naturally, as this is the strictest of the 3 conditions).

Given a finite set L and $\mathcal{M} \subseteq L^3$, let $Sp(L, \mathcal{M}) \subseteq \{1, 2, \ldots, \omega\}$ (Sp stands for *spectrum*) such that for each $n \in Sp(L, \mathcal{M})$ there exists a coloring of K_n which is good with respect to $\mathcal{M}$. With this notation and the argument above, we have $Sp(\{r, b\}, \mathcal{Q}_2) = \{5\}$. We provided a coloring of K_8 in Figure 2.2 which showed that $\mathcal{M}' = \mathcal{Q}_2 \cup \{r_1, r_1, r_1\}$ is realizable; interestingly, there is no good coloring for $\mathcal{M}'$ on K_q if $q \leq 7$, but $\mathcal{M}'$ is realizable on K_p for any integer $p \geq 8$.

We also provided a good coloring of K_{16} of the color scheme $\mathcal{Q}_3$. As this coloring was the construction for the lower bound in the proof that $R(3, 3, 3) \geq 17$, we may expect that $Sp(\{r_1, r_2, r_3\}, \mathcal{Q}_3) = \{16\}$, just as $Sp(\{r, b\}, \mathcal{Q}_2) = \{5\}$. However, this is not the case; in fact, $Sp(\{r_1, r_2, r_3\}, \mathcal{Q}_3) = \{13, 16\}$. For its beauty and symmetry, we include Figure 2.5, a 3-coloring of the edges of K_{13} which is good with respect to $\mathcal{Q}_3$. A description of how the coloring is obtained is contained in the caption for Figure 2.5.

Ramsey's Theorem applied to this problem shows us that for every positive integer k, $\omega \notin Sp(\{r_1, \ldots, r_k\}, \mathcal{Q}_k)$. This brings us to a conjecture.

Conjecture 2.1. *If $k \geq 1$ is an integer, then $Sp(\{r_1, \ldots, r_k\}, \mathcal{Q}_k) \neq \varnothing$.*

Conjecture 2.1 is true for $k \in \{2, 3, 4, 5\}$, but not known for $k = 6$.

The flexible atom conjecture can also be formulated using this notation in the following way.

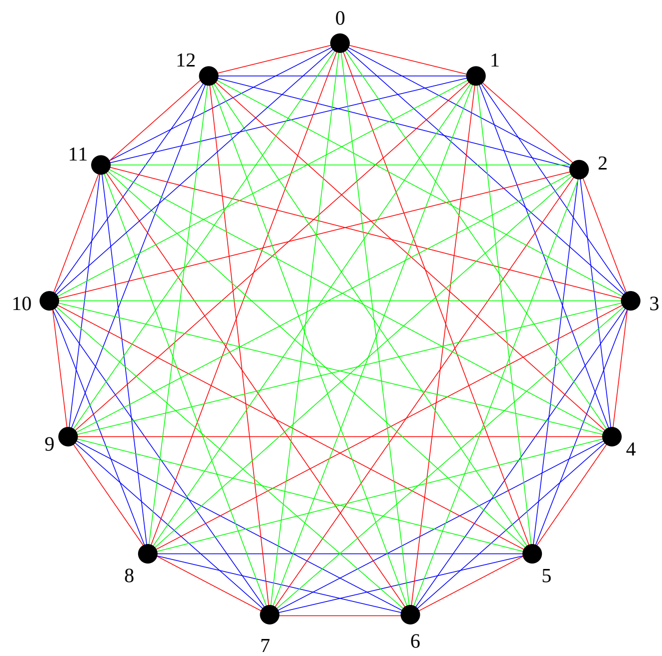

Figure 2.5 A 3-coloring of the edges of K_{13} which is good with respect to $\mathcal{Q}_3$. The coloring is obtained by assigning each vertex in K_{13} a unique element from the group $\mathbb{Z}_{13}$, and the red edges are those differences (modulo 13) in $\{1, 5, 8, 12\}$, the blue are those differences in $\{2, 3, 10, 11\}$, and the green are those differences in $\{4, 6, 7, 9\}$.

Conjecture 2.2. *Let L be a finite set and let $\mathcal{M} \subseteq L^3$. If L has a flexible color, then $Sp(L, \mathcal{M})$ contains a finite number.*

Theorem 2.1 shows that Conjecture 2.2 is true if $\mathcal{M}$ is consists of every colored triangle that the flexible color can appear in.

It can be shown that if $Sp(L, \mathcal{M})$ contains arbitrarily large integers, then $\omega \in Sp(L, \mathcal{M})$. With this in mind, we take the result from Theorem 2.1 and we can partially recover the result of Comer in [17], which is that a symmetric color scheme with at least one flexible color is realizable on K_ω.

For more information and a list of all open problems, see presentations of Maddux [54] on the topic.

Lastly, the problem stated in Section 2.1 remains open, and provides the connection between

Ramsey numbers and finding realizations of the symmetric color scheme $\mathcal{Q}_k$. Recall that $r(k)$ is the largest integer N such that there is a coloring of the edges of K_N which is good for $\mathcal{Q}_k$.

Conjecture 2.3. *For every integer* $k \geq 2$, $r(k) = R(\overbrace{3, 3, \ldots, 3}^{k \text{ times}}) - 1$.

CHAPTER 3. MONOCHROMATIC SUBSETS OF THE INTEGER GRID

3.1 Introduction and previous results

Let n, k, and d be positive integers. For an integer x, an *arithmetic progression* of length k (also referred to as k-AP) is the set $\{x, x+d, x+2d, \ldots x+(k-1)d\}$. The classical theorem of van der Waerden from [77] is one of the first theorems concerning arithmetic progressions in partitions of $[n]$.

Theorem 3.1 (van der Waerden, 1927)**.** *If the positive integers are partitioned into two classes, then at least one of the classes must contain arbitrarily long arithmetic progressions.*

Schur's Theorem, published originally in [66], from 1916 was the motivation for van der Waerden's Theorem. We state it below as Theorem 3.2. We will include a short proof relying on Ramsey numbers, see Chapter 2 or Graham, Rothschild, and Spencer [35] for a more in-depth discussion of Ramsey theory.

Theorem 3.2 (Schur, 1916)**.** *For any positive integer r, there exists an integer $N = N(r)$ such that if $\{1, 2, \ldots, N\}$ is partitioned into r parts, then one of the parts contains integers x, y, and z such that $x + y = z$.*

Proof of Theorem 3.2. Let r be a positive integer, and set $N = R(\overbrace{3, 3, \ldots, 3}^{r\text{ times}})$. Partition the set $\{1, 2, \ldots, N\}$ into r parts, say $C_1, C_2, \ldots, C_r$. Let $K_N = (V, E)$ be the complete graph on N vertices, whose vertices correspond to the elements of $\{1, 2, \ldots, N\}$. Define an r-coloring $f : E \to \{1, 2, \ldots, r\}$ by

$$f(xy) = i \text{ if } |x - y| \in C_i.$$

Since $N = R(\overbrace{3, 3, \ldots, 3}^{r \text{ times}})$ and f is an r-coloring of the edges of K_N, there is a monochromatic triangle under f in K_n, say with vertices $i < j < k$. This means there exists r' with $1 \leq r' \leq r$ such that $(k-j), (k-i), (j-i) \in C_{r'}$. By taking $x = j - i$, $y = k - j$ and $z = k - i$, we have $x, y, z \in C_{r'}$ and $x + y = z$, as desired. □

Schur actually proved something stronger (inspired by Fermat's Last Theorem), which was: *For all integers $m \geq 1$, if p is prime and sufficiently large, the equation $x^m + y^m = z^m$ has a nonzero solution in the integers modulo p.*

Theorem 3.1 can be restated in the following way:

Theorem 3.3 (van der Waerden, alternate form 1)**.** *For all positive integers n and k, there exists an integer $N_0 = N_0(k, n)$ so that if $N \geq N_0$ and the set of integers $[N]$ is partitioned into k classes, then at least one class contains an arithmetic progression of length n.*

Let $VW(k, n)$ be the least such integer guaranteed by van der Waerden's Theorem. Instead of thinking of partitioning the integers into k classes, it is sometimes helpful to think of coloring the integers with k colors. This leads to a third statement of van der Waerden's Theorem, and is the version we will use here.

Theorem 3.4 (van der Waerden, alternate form 2)**.** *For all positive integers n and k, there is an integer $VW(k, n)$ such that if $N \geq VW(k, n)$ and f is a k-coloring of $[N]$, then there exists a monochromatic arithmetic progression of length n under f.*

The number $VW(n) = VW(2, n)$ is usually referred to as the classical van der Waerden number. The best known bounds are

$$(n-1)2^{n-1} \leq VW(n) \leq 2^{2^{2^{2^{2^{n+9}}}}},$$

with the lower bound valid for values of $n - 1$ which are prime. Here, the lower and upper bounds are due to Berlekamp [9] and Gowers [32], respectively; see also the work of Graham in [33] on the growth of functions like $VW(n)$. The only known exact values for VW are $VW(3) = 9$, $VW(4) = 35$, and $VW(5) = 178$, $VW(6) = 1132$; the first two are due to Chvátal

[14], while the third is due to Stevens and Shantaram [70]. Kouril proved in his PhD thesis [47] that $VW(6) \geq 1132$, and conjectured that equality holds; Kouril and Paul later proved this conjecture in [48]. The text from Landman and Robertson [49] also provides a helpful background on van der Waerden numbers and related topics.

There is also a density version of van der Waerden's Theorem. Erdős and Turán conjectured in [26] that for any d with $0 < d < 1$, there is a number $N(d, k)$ such that if $N \geq N(d, k)$, every subset A of $[N]$ of cardinality at least dN contains a k-AP. This statement is known as the Erdős–Turán conjecture. Roth confirmed the conjecture for the case $k = 3$ in [65], and Szemerédi confirmed the conjecture for the case $k = 4$ in [72]. In [73], the conjecture was confirmed for general k by Szemerédi using combinatorial methods. As such, the result is known as Szemerédi's Theorem. The theorem was later proved using ergodic methods by Furstenberg [29]. In [32], Gowers supplied another proof of Szemerédi's Theorem using Fourier analysis. Just recently, Green and Tao [36] posted a preprint of another proof of Szemerédi's Theorem which relies on some of the techniques used by Roth and Gowers.

In search for better bounds and better understanding of van der Waerden numbers, some connections between higher-dimensional problems and the original problem have been established by Graham and Solymosi [34]. In [68], Shkredov proved a 2-dimensional analogue of Szemerédi's Theorem. We continue this effort by studying a problem of independent interest where instead of arithmetic progressions in $[n]$, configurations in $[n]^2$ are considered. The problem has a similar flavor to the Erdős–Szekeres Theorem, even though their result works with the ground set $\mathbb{R}^2$ and our ground set is $\mathbb{Z}^2$. In [25], Erdős and Szekeres proved a theorem about configurations of points in the plane. We state their theorem below as Theorem 3.5. While this is just one of many theorems from this area, we direct the reader to the text from Pach and Agarwal [57], which has a more comprehensive list of related problems and results.

Theorem 3.5 (Erdős and Szekeres, 1935)**.** *For all positive integers m, there exists an integer $N = N(m)$ such that for any set X of at least N points in the Euclidean plane with no three points collinear there exists a subset of X of m points which forms a convex m-gon.*

We will often refer to $\mathbb{Z}^2$ as the *grid*. For a set $V \subseteq \mathbb{Z}^2$, $c \in \mathbb{R} \setminus \{0\}$, and $\mathbf{b} \in \mathbb{Z}^2$, define

$cV + \mathbf{b} = \{c\mathbf{v} + \mathbf{b} : \mathbf{v} \in V\}$. We say that a subset U of the grid is *homothetic* to a set V in the grid if $U = cV + \mathbf{b}$, for some constants $c \in \mathbb{R} \setminus \{0\}$, and $\mathbf{b} \in \mathbb{Z}^2$. In particular, we consider the set of all squares with sides parallel to the axes, i.e., sets homothetic to $S = \{(0,0),(0,1),(1,0),(1,1)\}$. We shall refer to sets which are homothetic to S as simply *squares*. We also consider and the collection of sets homothetic to $L = \{(0,0),(0,1),(1,0)\}$, called *L-sets*. In this section we consider a stronger notion, when the coefficient c above is a natural number. Let

$$Hom(V) = \{cV + \mathbf{b} : c \in \mathbb{N}, \mathbf{b} \in \mathbb{Z}^2\}.$$

Given $k \in \mathbb{N}$, let

$$R_k(V) = \min\{n : \text{any } k\text{-coloring of } [n]^2 \text{ contains a monochromatic set from } Hom(V)\}.$$

Next, we state Gallai's Theorem, where the argument from the proof in [35] can be used to show that $R_k(V)$ is finite. (We direct the reader to Section 3.3 to see this argument.) While Gallai himself never published the proof, its original statement is given by Rado in [61] and again in [62]. Another proof is contained in [4]. We state the concise version given by Graham, Rothschild, and Spencer in [35].

Theorem 3.6 (Gallai)**.** *Let m be a positive integer, and let the vertices of $\mathbb{R}^m$ be finitely colored. For all finite $V \subset \mathbb{R}^m$, there exists a monochromatic set $W \in Hom(V)$.*

Gallai's Theorem together with results of Shelah found in [67] immediately give the upper bound in terms of Hales-Jewett numbers, as

$$R_2(S) \le 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}};$$

where the height of the tower is 25, see Section 3.3 for details concerning the Hales-Jewett Theorem and the derivation of this bound. Here, we improve this bound to

$$R_2(S) \le \min\{VW(8), 5 \cdot 2^{2^{40}}\}.$$

(Currently, the best known lower bound for $VW(8)$ is 11495, and can be found in [42].) One of the results we use is the bound by Graham and Solymosi [34]:

$$R_k(L) \le 2^{2^k}. \tag{3.1}$$

Note that the density results of Shkredov [68] give an upper bound on $R_k(L)$ of $2^{2^{k^{73}}}$.

3.2 New results

A collection of points in the plane in *general position* is a collection of points with the property that no three of them are collinear. An immediate lower bound on $R_k(V)$ for any V in general position with $|V| \geq 3$ is $R_k(V) \geq k$. This can be seen by coloring the i^{th} row of $[k]^2$ with color i. Since each row has its own color and no three points of any $X \in Hom(V)$ can lie on one row, we avoid a monochromatic homothetic copy of V.

The new results concern $R_2(V)$ when V is a 3 or 4-element set in general position. The new results for such V are stated below as Propositions 3.1 and 3.2. Proposition 3.1 is proved using forbidden configuration for squares. Proposition 3.2 provides bounds for arbitrary 3 and 4-element sets in a general position in terms of $R_k(L)$; the proof involves a reduction argument (independent of Proposition 3.1) treating a smaller grid but using more colors (see also presentations of Gasarch [30] on the topic).

Proposition 3.1. $13 \leq R_2(S) \leq VW(8)$.

For a set $A \subseteq [n]^2$, let the *square-size* s_A of A be

$$s_A = \min\{\ell : \ell \in \mathbb{N}, \exists X \subseteq [\ell]^2 \text{ such that } X \in Hom(A)\};$$

i.e., the size of the smallest square containing A.

Proposition 3.2. *Let T and Q be sets of three and four points of $\mathbb{Z}^2$ in general position, respectively. Then*

$$R_k(T) \leq 2s_T R_k(L)$$

and

$$R_2(Q) \leq 40 s_Q^2 R_{40}(L).$$

Note that (3.1) and Proposition 3.2 imply that $R_2(Q) \leq 40(s_Q)^2 2^{2^{40}}$. We can also reduce the bound slightly in the case of the square S to $R_2(S) \leq 5 \cdot 2^{2^{40}}$. We prove these two Propositions in the next sections, leaving the routine case analysis of the proof of Proposition 3.1 for Section 3.5. The results are due to Axenovich and Manske, and appear in [5].

3.2.1 Proof of Proposition 3.1

When we consider 2-colorings of the grid, we assume that mapping is from the set $[n]^2$ to the set $\{\circ, \bullet\}$ for ease in notation. That is, under an arbitrary 2-coloring χ, if $\chi((x,y)) = \circ$ we say that (x,y) is colored white, and if $\chi((x,y)) = \bullet$, we say that (x,y) is colored black.

Upper bound. Let $n \geq VW(8)$. Let $\chi : [n]^2 \to \{\circ, \bullet\}$ be a coloring of $[n]^2$ in two colors. By van der Waerden's Theorem, every row of $[n]^2$ contains a monochromatic 8-AP; in particular, the middle row contains an 8-AP $P = \{X, X+d, \ldots, X+7d\}$. Without loss of generality, we may assume $d = 1$ and $\chi(P) = \circ$. Let $\overline{P} = P + (0,1)$, $\underline{P} = P + (0,-1)$, and $* \in \{\circ, \bullet\}$. We consider cases according to whether either $\overline{P}$ or $\underline{P}$ have four consecutive black vertices, three consecutive black vertices in the center, two consecutive black vertices in the center, or none of the above. We show that there is a monochromatic square in each of these cases.

In the case analysis (details in Section 3.5), we use facts about four configurations in the grid which are outlined in Figure 3.5.

Case 1: $\overline{P}$ or $\underline{P}$ contains 4 consecutive black vertices.

Figure 3.6 deals with the case when there are three vertices to one side of these 4 consecutive vertices. In Figure 3.6, both diamonds marked 1 must have color $\circ$, while both diamonds marked 2 must have color $\bullet$, else we have a monochromatic square. (1) examines the case where the diamond marked 3 has color $\bullet$; here, the diamond marked 4 cannot be colored. (2) examines the case where the diamond marked 3 has color $\circ$; here, the diamond marked 5 cannot be colored.

Figure 3.7 deals with the case when these 4 consecutive vertices are in the center. In Figure 3.7, both diamonds marked 1 must have color $\circ$, otherwise forbidden configuration (1) gives us a square. Both diamonds marked 2 must have color $\bullet$, otherwise forbidden configuration (1) gives us a square. This immediately shows that the diamond marked 3 cannot be colored, concluding Case 1.

Case 2: Case 1 does not hold and there are three consecutive black vertices in $\overline{P}$ or in $\underline{P}$ with at least two vertices on both sides.

Figure 3.8 deals with this case. In Figure 3.8, the diamond marked 1 must have color $\circ$, otherwise forbidden configuration (1) gives us a square. The diamond marked 2 must have color $\bullet$, otherwise forbidden configuration (1) gives us a square. However, the diamonds marked 3 cannot be colored. This concludes Case 2.

Case 3: Cases 1 and 2 do not hold and there are two consecutive black vertices in the center of $\overline{P}$ or in the center of $\underline{P}$.

Figure 3.9 deals with this case. In Figure 3.9, the diamonds marked 1 and 2 cannot both have color $\circ$, otherwise we have a monochromatic square immediately. Without loss of generality (due to symmetry), we color the diamond marked 1 $\circ$. Since the diamonds marked 3 cannot both have color $\circ$, we examine the cases where both have color $\bullet$ and where one has color $\bullet$ and the other has color $\circ$. Similarly, either the diamond marked 4 or the vertex above the upper diamond marked 3 must have color $\bullet$, so by symmetry we say that the diamond marked 4 has color $\bullet$. (1) examines the case where both diamonds marked 3 have color $\bullet$; here, the diamond marked 5 cannot be colored. (2) examines the case where one diamond marked 3 has color $\circ$ and the other has color $\bullet$; here, the diamond marked 6 cannot be colored. This concludes case 3.

Case 4: Cases 1, 2, 3 do not hold.

This case implies that the two central positions above and below P are occupied by white and black vertices. Since it is impossible to have a white vertex x right above P and a white vertex exactly below x and P (see Figure 3.5 (2)), this case (up to reflection) gives the colorings of $\overline{P}$ and $\underline{P}$, respectively: $* * * \bullet \circ \bullet * *$ and $* * \bullet \circ \bullet * * *$.

Figure 3.10 displays two gray diamonds marked 1. Figures 3.10, 3.11, and 3.12 deal with the case that these both have color $\circ$. (Symmetry reduces four cases to three.)

In Figure 3.10, under the hypothesis that the diamonds marked 1, 2, and 3 all have color $\circ$, the diamond marked 4 cannot be colored without there being a monochromatic square.

In Figure 3.11, under the hypothesis that the diamonds marked 1 have color $\circ$ and the diamonds marked 2 and 3 have color $\bullet$, the diamond marked 4 must have color $\circ$ (staggered rows). The diamonds marked 5 cannot be colored.

In Figure 3.12, under the hypothesis that the diamonds marked 1 have color $\circ$, the diamond marked 2 has color $\bullet$, and the diamond marked 3 has color $\circ$, the diamond marked 4 must have color $\circ$ (staggered rows). The diamond marked 5 cannot be colored.

Figures 3.13, 3.14, and 3.15 deal with the case that the gray diamonds in Figure 3.10 both have color $\bullet$. In Figure 3.13, under the hypothesis that the diamonds marked 1, 2, and 3 all have color $\bullet$, the diamonds marked 4 must have color $\circ$ (stacked rows). The diamond marked 5 cannot be colored.

In Figure 3.14, under the hypothesis that both diamonds marked 1 have color $\bullet$, the diamond marked 2 has color $\bullet$, and the diamond marked 3 has color $\circ$, the diamond marked 4 must have color $\circ$ (stacked rows). This shows that the diamond marked 5 cannot be colored. (We need not consider the case where the diamond marked 2 has color $\circ$ and the diamond marked 3 has color $\bullet$; we use symmetry to take care of this.)

In Figure 3.15, under the hypothesis that both diamonds marked 1 have color $\bullet$ and both diamonds marked 2 and 3 have color $\circ$, the diamonds marked 4 must have color $\circ$ (stacked rows). This shows that the diamond marked 5 cannot be colored.

Lastly, Figures 3.16 and 3.17 deal with the case when these gray diamonds in Figure 3.10 have different colors. In Figure 3.16, under the hypothesis that one of the diamonds marked 1 has color $\circ$, the other has color $\bullet$, and that the diamond marked 2 has color $\bullet$, the diamond marked 3 must have color $\circ$ (stacked rows). The diamond marked 4 cannot be colored.

In Figure 3.17, under the hypothesis that one of the diamonds marked 1 has color $\circ$, the other has color $\bullet$, and that the diamond marked 2 has color $\circ$, the diamond marked 3 must have color $\circ$ (stacked rows). The diamond marked 4 cannot be colored. This concludes Case 4, and completes the proof of the upper bound.

Lower bound. Let $n = \lceil (VW(k,4)-1)/3 \rceil$. We construct a k-coloring χ' of $[n]^2$ which contains no monochromatic square. Let $\chi : \{0,1,\ldots,VW(k,4)-2\} \to \{1,2,\ldots,k\}$ be a coloring which admits no 4-AP. Define a k-coloring χ' on $[n]^2$ by $\chi'(x,y) = \chi(x+2y)$. By way of contradiction, assume there is a monochromatic square under χ'; that is, there exist $(x, y) \in [n]^2$ and $d \in \mathbb{N}$ such that $\chi'((x,y)) = \chi'((x+d,y)) = \chi'((x,y+d)) = \chi'((x+d,y+d))$. By definition of χ', this means that

$$\chi(x+2y) = \chi(x+2y+d) = \chi(x+2y+2d) = \chi(x+2y+3d),$$

which is a monochromatic 4-AP in $[VW(k,4)-1]$ under χ. This is a contradiction, so $R_k(S) \geq \lceil (VW(k,4)-1)/3 \rceil$, as desired. Using a 2-coloring of [34] with no 4-AP due to Chvátal [14], we can construct a specific 2-coloring of $[12]^2$ which contains no monochromatic square, and hence $R_2(S) \geq 13$; see Figure 3.1.

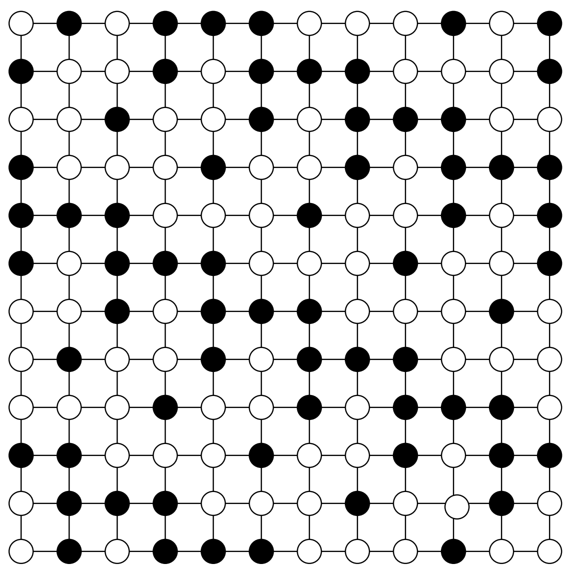

Figure 3.1 A 2-coloring of $[12]^2$ with no monochromatic square.

Using the best known lower bounds for $W(k,4)$ due to Rabung [60] and Herwig, et al. [42], we have the following 4 corollaries of Proposition 3.1. Each follows immediately from the proof of the lower bound in Proposition 3.1.

Corollary 3.1. $R_3(S) > 97$.

Corollary 3.2. $R_4(S) > 349$.

Corollary 3.3. $R_5(S) > 751$.

Corollary 3.4. *$R_6(S) > 3259$.*

3.2.2 Proof of Proposition 3.2

Define the *diagonal* D_n of $[n]^2$ to be $D_n := \{(x, y) : x + y = n - 1\}$, and the *lower triangle* $T_n = \{(x, y) : (x, y) \in [n]^2,\ x + y \leq n - 1\}$. Throughout this section we shall be using a map which allows us to deal with arbitrary three point configurations as L-sets. We say that a subset $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of three distinct elements in the grid forms a 3-AP, if, up to reordering, there is a vector $\mathbf{u}$ such that $\mathbf{u}_3 = \mathbf{u}_2 + \mathbf{u}$, $\mathbf{u}_2 = \mathbf{u}_1 + \mathbf{u}$. Given $X \subseteq \mathbb{Z}^2$ and $m, k \in \mathbb{N}$, we say that a collection of subsets $\mathcal{X} \subseteq [m]^2 \cap Hom(X)$ is a *forcing set* (with respect to parameters X, m, and k) if in any k-coloring of $[m]^2$ there is a monochromatic set from $\mathcal{X}$. Let $forc(X, m, k)$ denote the cardinality of the smallest such collection $\mathcal{X}$. In the next two Lemmas we find bounds for $R_k(T)$, where T is a three point configuration and we prove that that for any such T and $k = 2$, there is a forcing set with 20 sets in it.

Lemma 3.1. *$R_2(L) = 5$. Furthermore, $forc(L, 5, 2) \leq 20$.*

Proof of Lemma 3.1. **Lower bound.** To see that $R_2(L) \geq 5$, consider the coloring of $[4]^2$ with no monochromatic L-set shown in Figure 3.2.

Upper bound. Consider a 2-coloring of $[5]^2$. At least 3 elements on the diagonal, D_5, are of the same color, say black. If D_5 has a 3-AP, then we immediately have a monochromatic L-set contained in the lower triangle. If D_5 has at least 4 black vertices, then either there is a 3-AP in it, or, there are exactly four black vertices on this diagonal and the central vertex is white. Then one of

$$\{(0,0), (0,4), (4,0)\},$$

$$\{(0,4), (0,3), (1,3)\},$$

$$\{(3,1), (3,0), (4,0)\}, \text{ or}$$

$$\{(0,3), (0,0), (3,0)\}$$

will be a monochromatic L-set. Therefore, there are exactly three black vertices on the diagonal, and they do not form a 3-AP. The possible colorings (up to symmetries) of the diagonal in this case are shown in Figure 3.3. In each of these cases, it is easy to conclude that there is a monochromatic L-set in the lower triangle. Hence, $R_2(L) \leq 5$.

This gives that $R_2(L) = 5$. Since the number of L-sets in T_5 is 20, $\mathit{forc}(L, 5, 2) \leq 20$. □

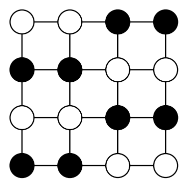

Figure 3.2 A 2-coloring of $[4]^2$ with no monochromatic L-set.

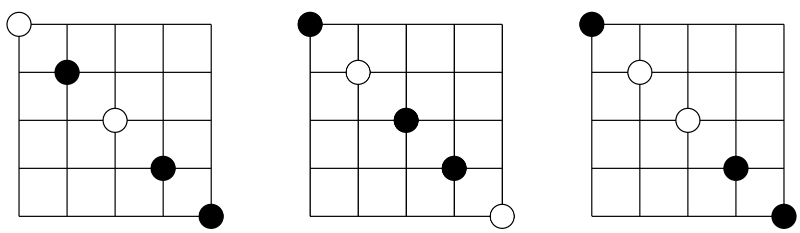

Figure 3.3 Colorings of D_5 with three black points not forming 3-AP.

For a given three point subset T of $\mathbb{Z}^2$ in general position, define the *parallelogram size* p_T to be the minimum square size of a parallelogram defined by T. Recall that the *square size* of a set X is the size of the smallest square containing X. For example, when $T = L$, $p_T = 1$; when $T = \{(0,0), (1,2), (-1,3)\}$, $p_T = 4$. Note that $p_T \leq 2s_T$. By choosing an appropriate linear transformation, we will produce a bound on $R_k(T)$ in terms of $R_k(L)$.

Lemma 3.2. *If $T \subseteq \mathbb{Z}^2$ is in general position with $|T| = 3$ then $R_k(T) \leq p_T R_k(L) \leq 2s_T R_k(L)$. Furthermore, $R_2(T) \leq 5p_T \leq 10s_T$ and $\mathit{forc}(T, 5p_T, 2) \leq 20$.*

Proof. Let $T = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\} \subset \mathbb{Z}^2$ be a set in general position. Suppose that a parallelogram defined by T with smallest square size has two sides corresponding to vectors $\mathbf{u} = \mathbf{t}_2 - \mathbf{t}_1$

and $\mathbf{v} = \mathbf{t}_3 - \mathbf{t}_1$. Let $k \geq 2$ be an integer. Set $q = R_k(L)$. Let $n = p_T q$ and let Q be the parallelogram with sides given by the vectures $\mathbf{u}$ and $\mathbf{v}$. Then qQ is contained in an $n \times n$ square grid. Formally, let $\mathbf{x} \in \mathbb{Z}^2$ such that $qQ + \mathbf{x} \subseteq [n]^2$.

Let $X = [n]^2 \cap \{k\mathbf{u} + l\mathbf{v} + \mathbf{x} : k, l \in \mathbb{N} \cup \{0\}\}$. Define $\phi : X \to [n/p_T]^2$ by

$$\phi(k\mathbf{u} + l\mathbf{v} + \mathbf{x}) = (k, l).$$

Let χ be a k-coloring of $[n]^2$. This induces a k-coloring χ' of $[n/p_T]^2$ by taking

$$\chi'(k, l) = \chi(k\mathbf{u} + l\mathbf{v} + \mathbf{x}).$$

As $q = R_k(L)$, there is a monochromatic L-set under χ', say $\{(l, l'), (l + d, l'), (l, l' + d)\}$. By definition of ϕ, this corresponds to a monochromatic set

$$\{l\mathbf{u} + l'\mathbf{v} + \mathbf{x}, (l + d)\mathbf{u} + l'\mathbf{v} + \mathbf{x}, l\mathbf{u} + (l' + d)\mathbf{v} + \mathbf{x}\}$$

which is a triangle with sides $d\mathbf{u}, d\mathbf{v}$, a homothetic image of T. Since there exists a forcing set X with parameters L, 5, 2 and $|X| \leq 20$, we may take $\phi^{-1}(X)$ to be a forcing set for T in $[p_T R_2(L)]^2 = [5p_T]^2$ to see that there exists a forcing set with respect to parameters T, $5p_T$, and 2 of cardinality at most 20. □

Note that for any four point subset Q of $\mathbb{Z}^2$, there is a three point subset $T \subseteq Q$ such that $s_T = s_Q$. This is easily seen by taking T to be two points of Q with maximum Euclidean distance together with any third point of Q. This leads us to our next lemma. First, for n an even positive integer and d any positive integer less than n, we define the *middle square of width* d of $[n]^2$ to be the $d \times d$ subgrid $\{\frac{n}{2} - \lfloor \frac{d}{2} \rfloor, \frac{n}{2} - \lfloor \frac{d}{2} \rfloor + 1, \ldots, \frac{n}{2} - \lfloor \frac{d}{2} \rfloor + d - 1\}^2$.

Lemma 3.3. *Let Q be a set of four points of $\mathbb{Z}^2$ in general position and let $T \subseteq Q$, $|T| = 3$ such that $s_T = s_Q$. Then $R_2(Q) \leq 40 s_Q R_{40}(T)$, and $R_2(S) \leq 5R_{40}(L)$.*

Proof. Let $q = 10s_T = 10s_Q$, $n = 4qR_{40}(T)$, and $\chi : [n]^2 \to \{\bullet, \circ\}$. We shall construct another coloring $\chi' : [n/q]^2 \to \{1, 2, \ldots, 40\}$ generated by χ. We shall first show that χ' has a monochromatic homothetic image T' of T in $[n/q]^2$. Using this T', we shall find a monochromatic homothetic image of Q in the original coloring.

By Lemma 3.2, we have that $R_2(T) \leq q$ and $forc(T, q, 2) \leq 20$. Let $\{X_1, \ldots, X_{20}\}$ be a forcing set with respect to parameters T, q, and 2, and let $(Y_1, \ldots, Y_{40}) = ((X_1, \circ), (X_2, \circ), \ldots, (X_{20}, \circ), (X_1, \bullet), (X_2, \bullet), \ldots, (X_{20}, \bullet))$. Any 2-coloring of the $q \times q$ grid has some set X_i colored in $\circ$ or $\bullet$ which corresponds to either Y_i or Y_{20+i}, respectively, $1 \leq i \leq 20$.

Split $[n]^2$ into $q \times q$ grids

$$A_{(x,y)} = \{(a, b) : qx \leq a < q(x + 1), qy \leq b < q(y + 1), 0 \leq x, y \leq n/q - 1\}.$$

Let

$$\chi'((x, y)) = \min\{i : A_{(x,y)} \text{ has a colored set } Y_i \text{ under } \chi\}.$$

Note that χ' is a coloring of $[n/q]^2$ in at most 40 colors.

To allow for us to later choose additional points which belong to the grid, we consider the middle square M, of $[n/q]^2$ of width $\dfrac{n}{4q} = R_{40}(T)$. Then M contains, under χ', a monochromatic set $T' = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, $T' \in Hom(T)$. Let $\mathbf{x}_4$ be the point such that the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is in $Hom(Q)$.

Since $\chi'(\mathbf{x}_1) = \chi'(\mathbf{x}_2) = \chi'(\mathbf{x}_3)$, the corresponding subgrids $A_{\mathbf{x}_1}, A_{\mathbf{x}_2}$, and $A_{\mathbf{x}_3}$ have a three element set from $Hom(T)$ in the same position and of the same color. More formally, we can say $T'' = \{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3\} \in Hom(T)$, $T'' \subseteq [q]^2$. Next, define

$$\begin{aligned} T_1 &= T'' + q\mathbf{x}_1, \\ T_2 &= T'' + q\mathbf{x}_2, \text{ and} \\ T_3 &= T'' + q\mathbf{x}_3, \end{aligned}$$

and notice that $T_i \in A_{\mathbf{x}_i}$ for $i \in \{1, 2, 3\}$. We have that T_1, T_2, and T_3 are all monochromatic; without loss of generality, say they are colored black (see Figure 3.4 (1)). Let $\mathbf{t}_4$ be the grid vertex such that $\{\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{t}_4\} \in Hom(Q)$ and let $T_4 = T'' + q\mathbf{x}_4$. Since T_1, T_2, T_3 are monochromatic, we may assume T_4 is monochromatic (white); otherwise if one of its points, say $\mathbf{t}_1 + q\mathbf{x}_4$ is black under χ, then $\{\mathbf{t}_1 + q\mathbf{x}_1, \mathbf{t}_1 + q\mathbf{x}_2, \mathbf{t}_1 + q\mathbf{x}_3, \mathbf{t}_1 + q\mathbf{x}_4\} \in Hom(Q)$, and is a monochromatic set. Similarly, we may assume $\chi(\mathbf{t}_4 + q\mathbf{x}_1) = \chi(\mathbf{t}_4 + q\mathbf{x}_2) = \chi(\mathbf{t}_4 + q\mathbf{x}_3) = \circ$, and $\chi(\mathbf{t}_4 + q\mathbf{x}_4) = \bullet$. Let $Q' = \{q\mathbf{x}_1 + \mathbf{t}_1, q\mathbf{x}_2 + \mathbf{t}_2, q\mathbf{x}_3 + \mathbf{t}_3, q\mathbf{x}_4 + \mathbf{t}_4\}$. (See Figure 3.4 for a pictorial representation.)

Claim. *We have that $Q' \in Hom(Q)$ and Q' is monochromatic under χ.*

Let $Q = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$. For $i \in \{1, 2, 3, 4\}$, for some $a, a' \in \mathbb{N}$, and for some $\mathbf{b}, \mathbf{b}' \in \mathbb{Z}^2$, we may write

$$\mathbf{x}_i = a\mathbf{q}_i + \mathbf{b}$$

and

$$\mathbf{t}_i = a'\mathbf{q}_i + \mathbf{b}'.$$

So, we have that $q\mathbf{x}_i + \mathbf{t}_i = q(a\mathbf{q}_i + \mathbf{b}) + (a'\mathbf{q}_i + \mathbf{b}') = (qa + a')\mathbf{q}_i + (q\mathbf{b} + \mathbf{b}')$. This concludes the proof of the Claim.

What remains for us to check is that indeed all the selected points $\mathbf{t}_j + q\mathbf{x}_i$, $i, j \in \{1, 2, 3, 4\}$ belong to the grid $[n]^2$. Note that $q\mathbf{x}_1, q\mathbf{x}_2, q\mathbf{x}_3$ are in the middle grid M'' of $[n]^2$ of width $n/4$. Since $s_T = s_Q$, all four points $q\mathbf{x}_i$ for $i \in \{1, 2, 3, 4\}$ are contained in a square of size at most $n/4$, so $q\mathbf{x}_4$ is in the middle square of $[n]^2$ of width $3n/4$. Since $\mathbf{t}_j \in [q]^2$ for $j \in \{1, 2, 3\}$ and $s_T = s_Q$, we have that $\mathbf{t}_j$ is in a $3q \times 3q$ grid for $j \in \{1, 2, 3, 4\}$. Hence, $\mathbf{t}_j + q\mathbf{x}_i$ are in the middle square of $[n]^2$ of width $3n/4 + 6q$ for $i, j \in \{1, 2, 3, 4\}$. Since $n = 4qR_{40}(T) \geq 4q \cdot 40 \geq 4q \cdot 6$, we have $6q \leq n/4$ and hence $\mathbf{t}_j + q\mathbf{x}_i$ belong to $[n]^2$ for $i, j \in \{1, 2, 3, 4\}$. □

Notice that when $Q = S$, we can take $n = qR_{40}(L)$, instead of $4qR_{40}(T)$ in the proof of Lemma 3.3 because the point $\mathbf{x}_4$ will be in the square determined by $\mathbf{x}_i$, $i \in \{1, 2, 3\}$. Similarly each of the points $q\mathbf{x}_i + \mathbf{t}_4$ for $i \in \{1, 2, 3, 4\}$ will be in the squares determined by corresponding $q\mathbf{x}_i + \mathbf{t}_j$ for $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, 3\}$.

We now have the machinery in place to prove the second of our new results, Proposition 3.2.

Proof of Proposition 3.2. Let $Q \subseteq \mathbb{Z}^2$ be a set in general position with $|Q| = 4$. By Lemmas 3.2 and 3.3 together with inequality (3.1), we have $R_2(Q) \leq 20s_Q \cdot 2s_Q R_{40}(L) \leq 40s_Q^2 2^{2^{40}}$. □

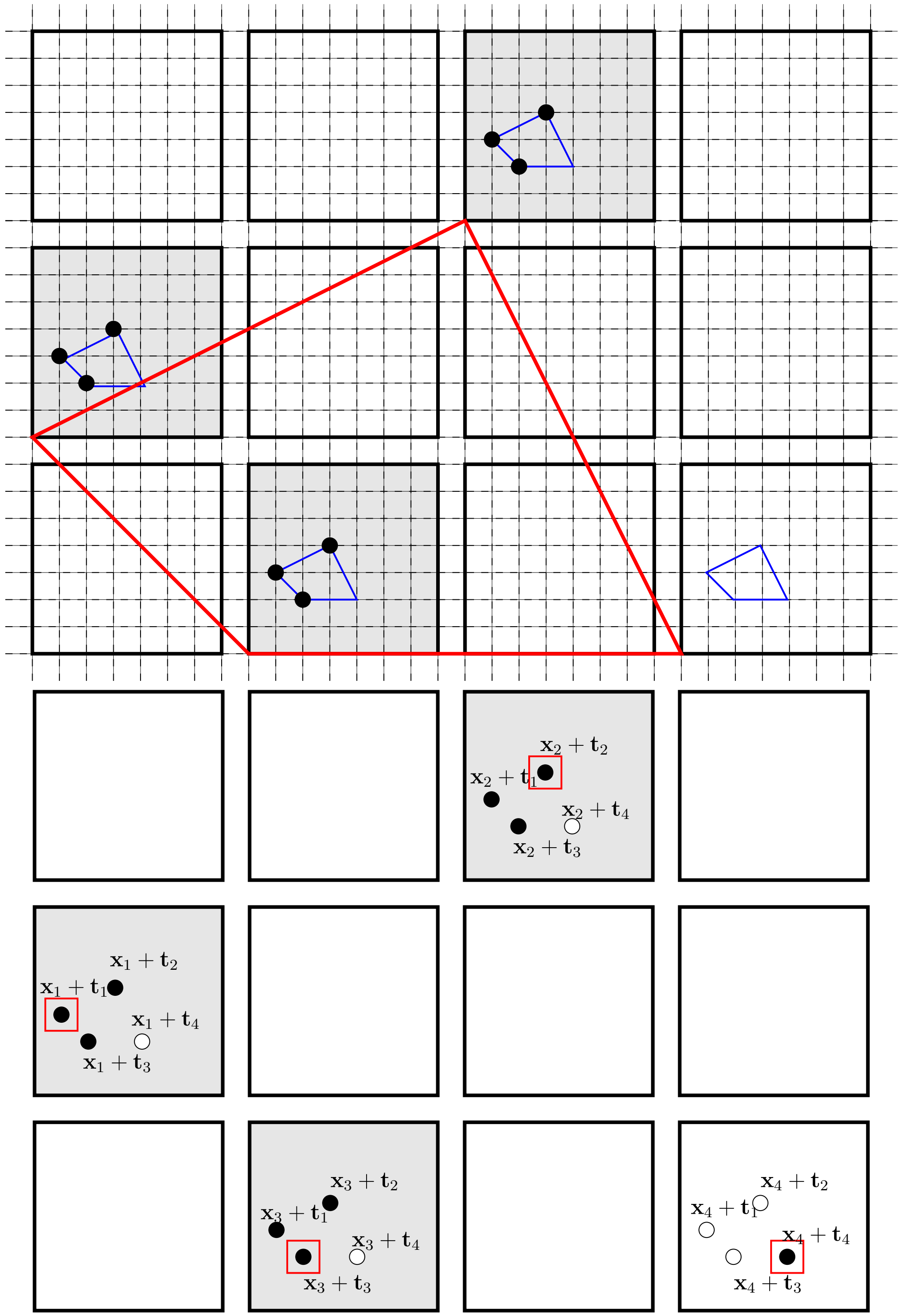

Figure 3.4 An example of the configuration Lemma 3.3 is describing. In this example, the points $\mathbf{t}_j + q\mathbf{x}_i$ are elements of shaded subgrids $A_{\mathbf{x}_i}$.

3.3 The Hales-Jewett number

In this section, we will provide background on the Hales-Jewett Theorem and the Hales-Jewett numbers, which were referenced in Section 3.1. First, we define C_t^n, the n-cube over t elements to be

$$C_t^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1, \ldots, t-1\}\}.$$

We define a *line* (or *combinatorial line*) in C_t^n to be a collection of t distinct points $\mathbf{x}_0, \ldots, \mathbf{x}_{t-1}$ (where $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ for $0 \leq i \leq t-1$) such that for each j with $1 \leq j \leq n$, either

(i) $x_{0j} = x_{1j} = \cdots = x_{t-1\ j}$, or

(ii) $x_{sj} = s$ for $0 \leq s \leq t-1$.

Essentially, what this definition means is that all the points $\mathbf{x}_i$ have the same value in each coordinate or it is possible to order the $\mathbf{x}_i$ in a suitable fashion so that in coordinate j, the values of the coordinates increase from 0 to $t-1$ In this light, we see that in order for a collection of t distinct points to be a line in C_t^n, condition (ii) above must be satisfied for at least one j, $0 \leq j \leq t-1$.

As an example of a combinatorial line in C_t^n, take $n = 4$ and $t = 3$. We may take the 3 points $(0,1,0,2), (1,1,1,2)$, and $(2,1,2,2)$ as a line in C_3^4. As stated above, we sometimes say *combinatorial line* instead of *line* to make it apparent that our definition differs from that of the ordinary *geometric line*. For example, in C_3^2, the 3 points $(0,2), (1,1)$, and $(2,0)$ form a geometric line, but not a combinatorial one. The reason for this is to make C_t^n independent of the underlying set $[t]$. Instead of using $[t]$ as the underlying set, any t element set $A = \{a_0, \ldots, a_{t-1}\}$ can be used as the coordinates in a typical n-tuple from C_t^n.

Hales and Jewett proved in [41] a result about coloring the points of C_t^n. We state their result below as Theorem 3.7. For a proof, we direct the reader to [35].

Theorem 3.7 (Hales and Jewett, 1963)**.** *For all r and t, there exists N so that if $N' \geq N$ and the elements of $C_t^{N'}$ are colored in r colors, there is a monochromatic combinatorial line.*

Let $HJ(r,t)$ be the least such number guaranteed by the Hales-Jewett Theorem. Let $N = HJ(2,4)$. Since C_4^N is independent of the underlying set, we may take any 4-element set; we choose the square $S = \{(0,0),(0,1),(1,0),(1,1)\}$. More precisely, we will write C_t^N as

$$\{(x_1, x_2, \ldots, x_N) : x_i \in \{(0,0),(0,1),(1,0),(1,1)\}\},$$

with the elements of S ordered as listed above. We may consider N to be the smallest positive integer such that in any 2-coloring of C_t^N there is a monochromatic combinatorial line. The mapping $f : S^N \to [2^N]^2$ defined by $f(x_0, x_1, \ldots, x_{N-1}) = \sum_{j=0}^{N-1} 2^j x_j$ is injective, hence a 2-coloring of $[2^N]^2$ gives a 2-coloring of S^N which has a monochromatic combinatorial line, say $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Recall that x_{ij} is the entry in the j-th coordinate of N-tuple $\mathbf{x}_i$. From the definition of a combinatorial line, there exists a coordinate j so that

$$\begin{aligned} x_{0j} &= (0,0) \\ x_{1j} &= (0,1) \\ x_{2j} &= (1,0) \\ x_{3j} &= (1,1), \end{aligned}$$

so this line in turn gives a monochromatic homothetic copy of S in $[2^N]^2$, and hence we have the bound $R_2(S) \leq 2^N$. The recursive bound on $N = HJ(2,4)$ gives $HJ(2,4) \leq 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$, where the tower has height 24 (this last inequality is found in [8]).

3.4 Further research

In a preprint, Bacher and Eliahou have proven in [7] that $R_2(S) = 15$ by an exhaustive computer search. Our proof here still represents the best analytic result without the use of a computer. Bacher and Eliahou also provide 232228 2-colorings on $[14]^2$ which avoid monochromatic squares.

Their original motivation for working on the problem is the due to an open problem stated by Erickson in [28], which is: *Find the minimum n such that if the n^2 lattice points of $[n] \times [n]$*

are 2-colored, there exist four points of one color lying on the vertices of a square with sides parallel to the coordinate axes.

Bacher and Eliahou show that it is possible to 2-color the $13 \times \infty$ grid and the 14×14 grid without any monochromatic sets homothetic to the square S. They also show that every 2-coloring of the 14×15 grid has a monochromatic square.

3.5 Case analysis for Proposition 3.1

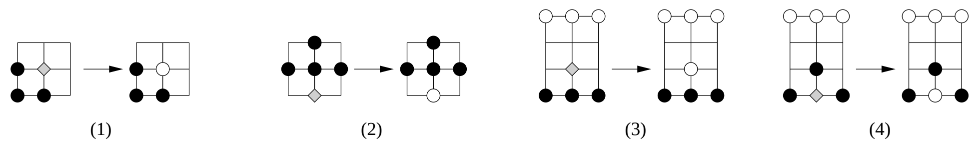

Figure 3.5 The configurations used in the case analysis. Trivially, the diamond in (1) must have color ∘. We refer to the Figure above labeled (2) as the *cross*; note that if the diamond in (2) has color •, we can no longer avoid a monochromatic square. We refer to (3) as *stacked rows* and (4) as *staggered rows.* In each, the diamond must have color ∘.

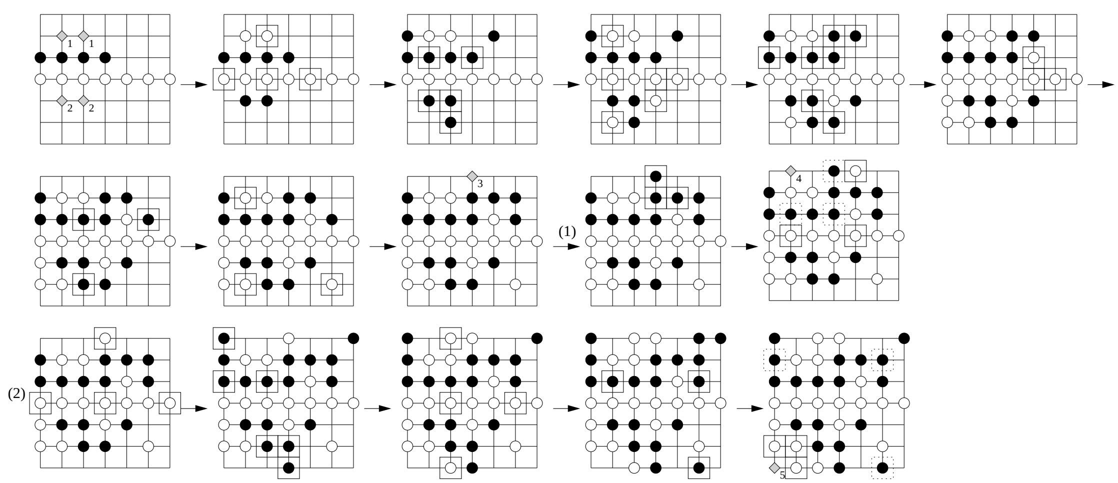

Figure 3.6 Both diamonds marked 1 must have color ∘, while both diamonds marked 2 must have color •, else we have a monochromatic square. (1) examines the case where the diamond marked 3 has color •; here, the diamond marked 4 cannot be colored. (2) examines the case where the diamond marked 3 has color ∘; here, the diamond marked 5 cannot be colored.

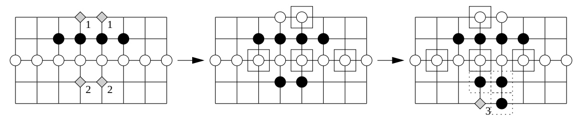

Figure 3.7 Both diamonds marked 1 must have color ∘, otherwise forbidden configuration (1) gives us a square. Both diamonds marked 2 must have color •, otherwise forbidden configuration (1) gives us a square. This immediately shows that the diamond marked 3 cannot be colored, concluding Case 1.

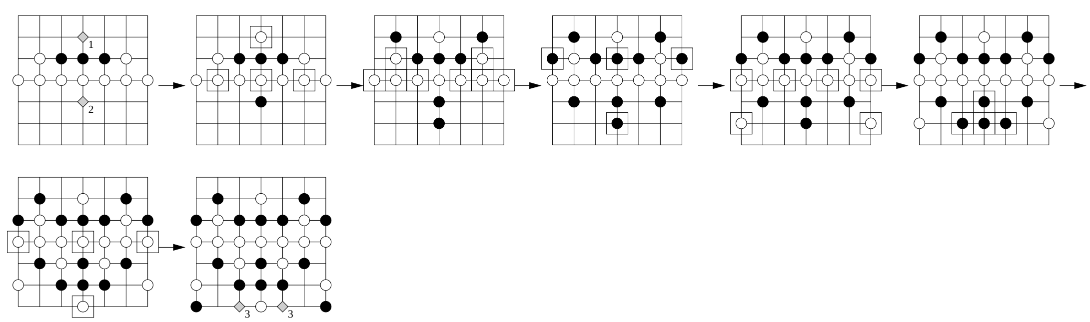

Figure 3.8 The diamond marked 1 must have color ∘, otherwise forbidden configuration (1) gives us a square. The diamond marked 2 must have color •, otherwise forbidden configuration (1) gives us a square. However, the diamonds marked 3 cannot be colored. This concludes Case 2.

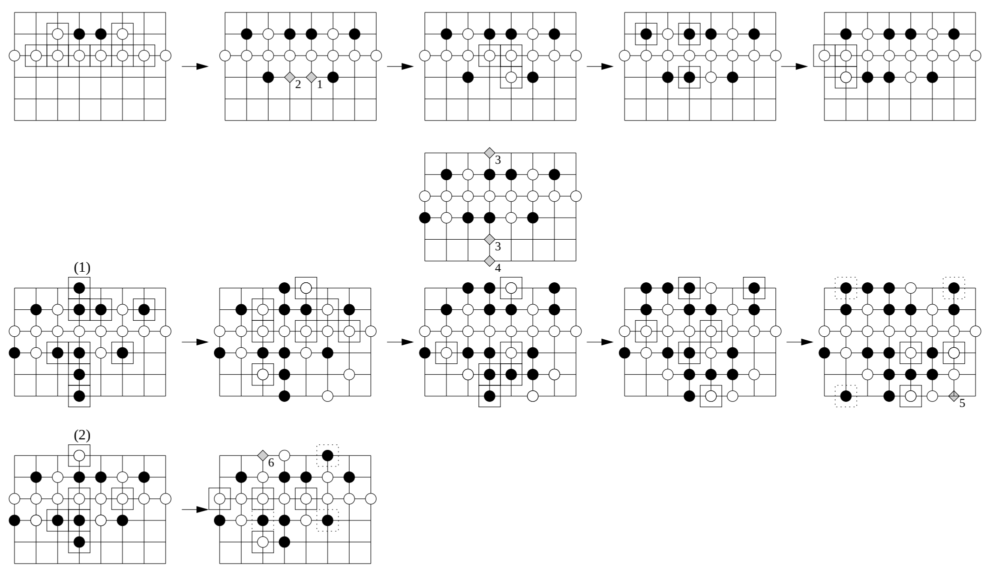

Figure 3.9 The diamonds marked 1 and 2 cannot both have color ∘, otherwise we have a monochromatic square immediately. Without loss of generality (due to symmetry), we color the diamond marked 1 ∘. Since the diamonds marked 3 cannot both have color ∘, we examine the cases where both have color • and where one has color • and the other has color ∘. Similarly, either the diamond marked 4 or the vertex above the upper diamond marked 3 must have color •, so by symmetry we say that the diamond marked 4 has color •. (1) examines the case where both diamonds marked 3 have color •; here, the diamond marked 5 cannot be colored. (2) examines the case where one diamond marked 3 has color ∘ and the other has color •; here, the diamond marked 6 cannot be colored. This concludes Case 3.

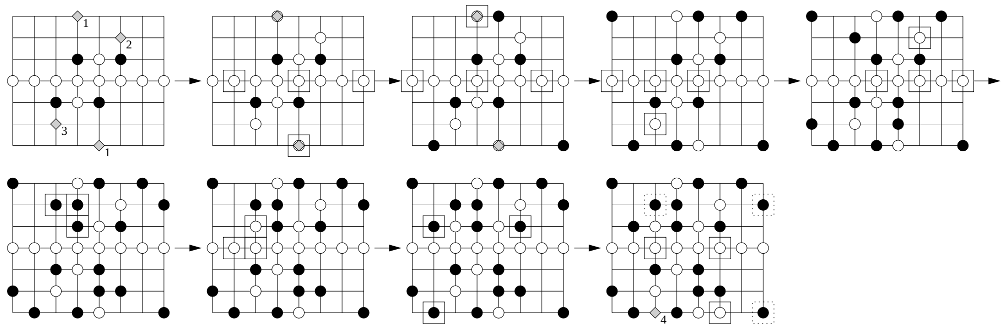

Figure 3.10 Under the hypothesis that the diamonds marked 1, 2, and 3 all have color ∘, the diamond marked 4 cannot be colored without there being a monochromatic square.

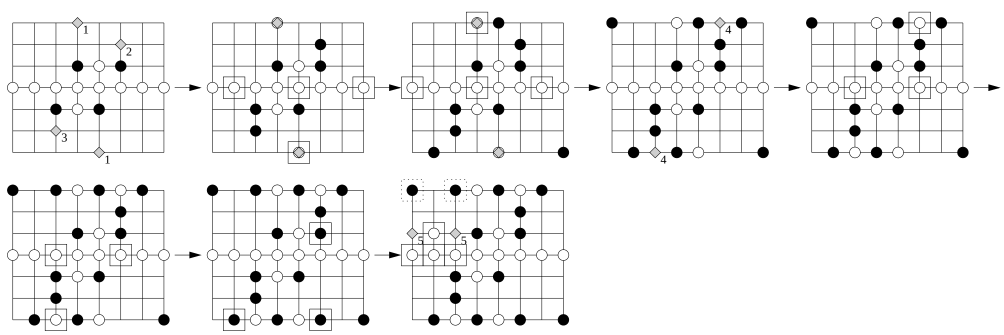

Figure 3.11 Under the hypothesis that the diamonds marked 1 have color ∘, and the diamonds marked 2 and 3 have color •, the diamond marked 4 must have color ∘ (staggered rows). The diamonds marked 5 cannot be colored.

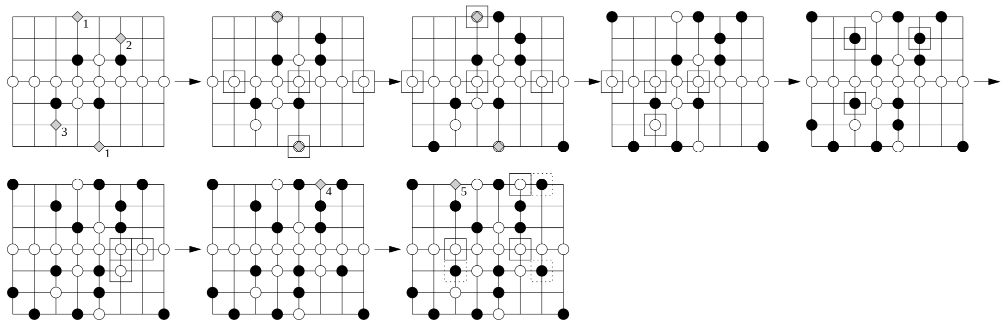

Figure 3.12 Under the hypothesis that the diamonds marked 1 have color ○, the diamond marked 2 has color ●, and the diamond marked 3 has color ○, the diamond marked 4 must have color ○ (staggered rows). The diamond marked 5 cannot be colored.

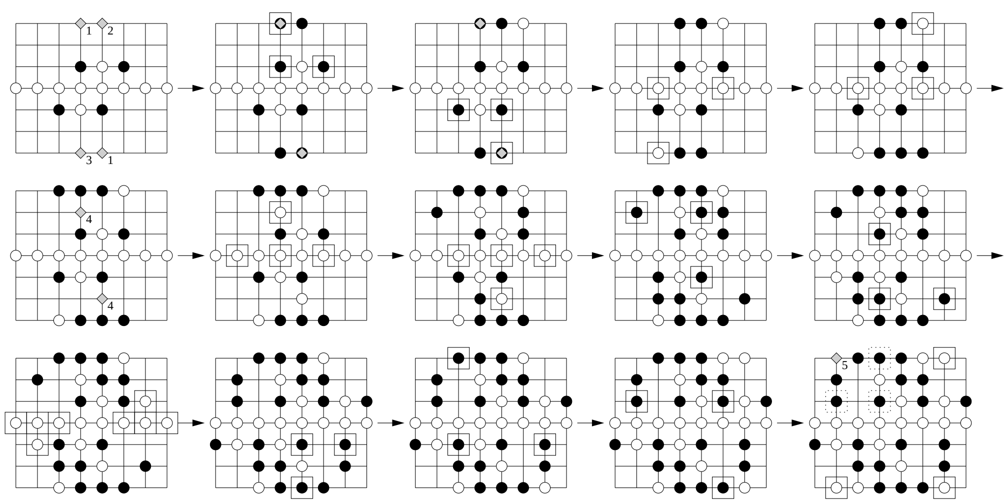

Figure 3.13 Under the hypothesis that the diamonds marked 1, 2, and 3 all have color ●, the diamonds marked 4 must have color ○ (stacked rows). The diamond marked 5 cannot be colored.

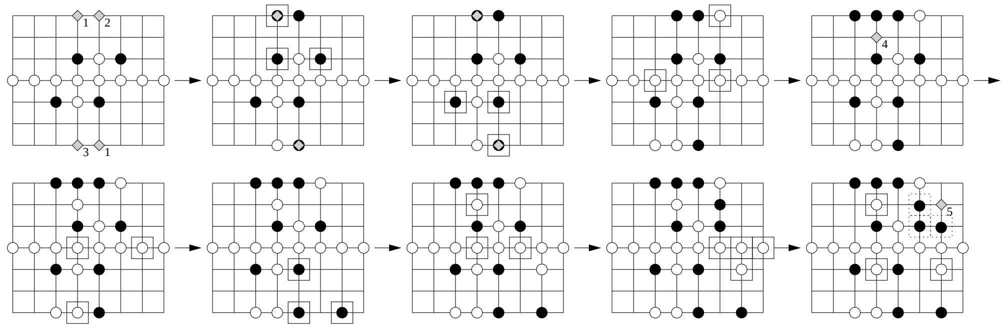

Figure 3.14 Under the hypothesis that both diamonds marked 1 have color •, the diamond marked 2 has color •, and the diamond marked 3 has color ∘, the diamond marked 4 must have color ∘ (stacked rows). This shows that the diamond marked 5 cannot be colored. (We need not consider the case where the diamond marked 2 has color ∘ and the diamond marked 3 has color •; we use symmetry to take care of this.)

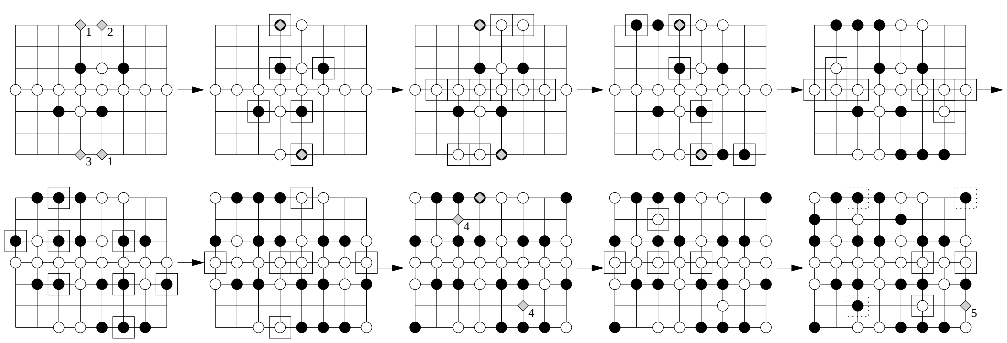

Figure 3.15 Under the hypothesis that both diamonds marked 1 have color • and both diamonds marked 2 and 3 have color ∘, the diamonds marked 4 must have color ∘ (stacked rows). This shows that the diamond marked 5 cannot be colored.

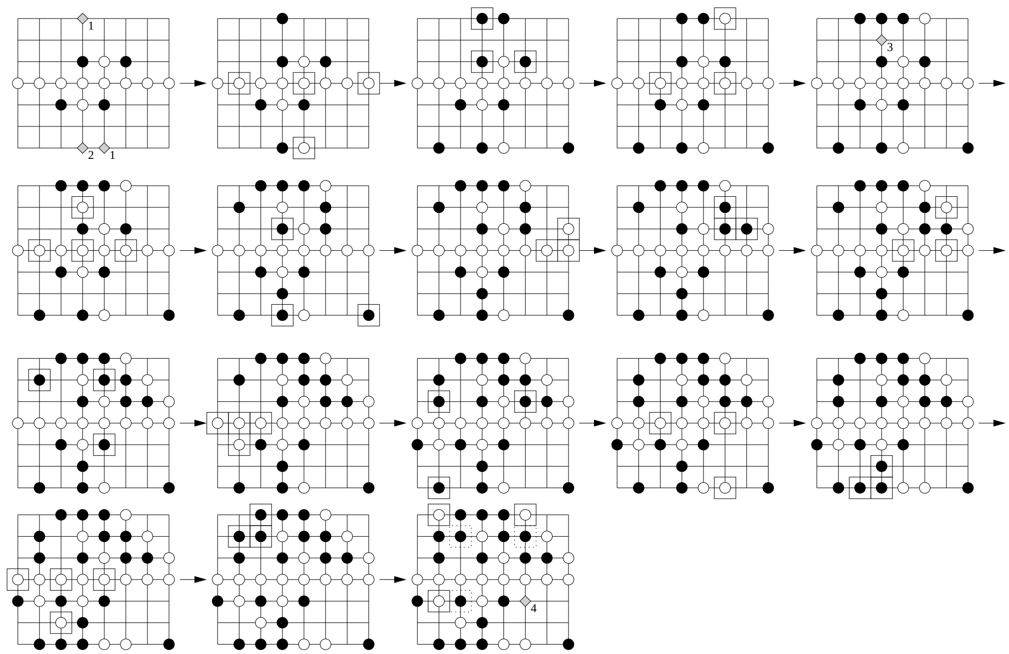

Figure 3.16 Under the hypothesis that one of the diamonds marked 1 has color ◦ and the other has color • and that the diamond marked 2 has color •, the diamond marked 3 must have color ◦ (stacked rows). The diamond marked 4 cannot be colored.

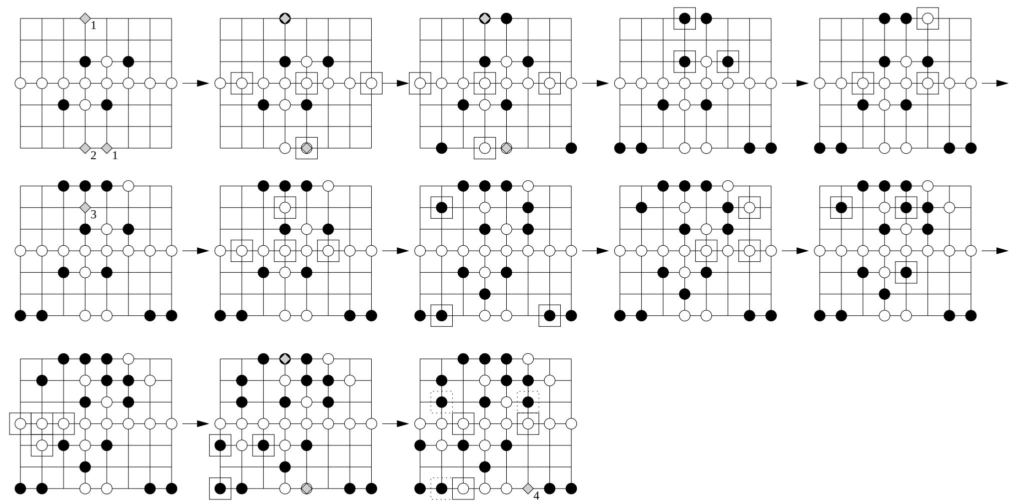

Figure 3.17 Under the hypothesis that one of the diamonds marked 1 has color ○ and the other has color ● and that the diamond marked 2 has color ○, the diamond marked 3 must have color ○ (stacked rows). The diamond marked 4 cannot be colored. This concludes Case 4.

CHAPTER 4. POSETS

4.1 Introduction

For a positive integer n, recall $[n]$ denotes the integer set $\{0, 1, 2, \ldots, n-1\}$. For a set A, let 2^A denote the power set of A. Let Q_n denote the poset $(2^{[n]}, \subseteq)$, also called the Boolean lattice of order n. The kth *layer* of Q_n for $0 \leq k \leq n$, denoted by $\mathcal{L}_k$ is the collection of all k-element subsets of $[n]$. Let $N = \binom{n}{\lfloor n/2 \rfloor}$.

We can represent finite posets pictorially with what are referred to as *Hasse diagrams*. The Hasse diagram of a finite poset $\mathcal{P} = (P, \leq)$ is a picture with each node corresponding to an element of P, and a line segment or curve going upward from x to y if $x \leq y$ and there is no element z such that $x \leq z \leq y$. In this case, we say y covers x. Since $\leq$ is a transitive relation, if there is a line segment upward from x to y and one from y to z, we deduce that $x \leq z$. For an example, see Figure 4.1. Since any finite family of sets $\mathcal{F}$ together with the inclusion relation is a poset, we also can draw Hasse diagrams for set families.

Related to the Hasse diagram of a poset $\mathcal{P} = (P, \leq)$ is the *cover relation graph*. The cover relation graph of a poset $\mathcal{P}$ is a graph with vertex set P where two vertices p_1 and p_2 are adjacent when either p_1 covers p_2 or vice versa.

For any subset X of $[n]$, we define the *downset* of X, denoted by $\mathcal{D}_X$, to be the collection $\{Y \subseteq [n] : Y \subseteq X\}$. Similarly, the *upset* of X is the collection $\{Y \subseteq [n] : X \subseteq Y\}$, denoted by $\mathcal{U}_X$. Notice that $(\mathcal{D}_X, \subseteq)$ is isomorphic to $Q_{|X|}$ and that $(\mathcal{U}_X, \subseteq)$ is isomorphic to $Q_{n-|X|}$.

For a poset $\mathcal{P} = (X, \leq)$ and a set $X' \subseteq X$, the poset *spanned* by X' in $\mathcal{P}$ is the poset $(X', \leq')$ where $\leq'$ and $\leq$ agree for all pairs taken from X', so we write $(X', \leq') = (X', \leq)$. The first problem we discuss is the following. Fix a family of sets $\mathcal{R}$ of $[n]$ and find the cardinality of the largest collection $\mathcal{F}$ of subsets of $[n]$ such that $(\mathcal{R}, \subseteq)$ is not a subposet of $(\mathcal{F}, \subseteq)$. Note that this type of extremal problem does not require that $(\mathcal{R}, \subseteq)$ be an induced subposet of $(\mathcal{F}, \subseteq)$ (and hence not of Q_n). If $(\mathcal{R}, \subseteq)$ is not a subposet of $(\mathcal{F}, \subseteq)$, we say $\mathcal{F}$ is $\mathcal{R}$-free. The second problem we discuss is finding the largest collection $\mathcal{F}$ of subsets of $[n]$ such that $(\mathcal{R}, \subseteq)$ is not an *induced* subposet of $(\mathcal{F}, \subseteq)$.

Definition 4.1. *The cardinality of the largest collection $\mathcal{F}$ of subsets of $[n]$ which is $\mathcal{R}$-free is denoted by $ex(n, \mathcal{R})$.*

Definition 4.2. *The cardinality of the largest collection $\mathcal{F}$ of subsets of $[n]$ such that $(\mathcal{R}, \subseteq)$ is not an induced subposet of $(\mathcal{F}, \subseteq)$ is denoted by $ex_{\text{ind}}(n, \mathcal{R})$.*

Immediately, we see that $ex(n, \mathcal{R}) \leq ex_{\text{ind}}(n, \mathcal{R})$, since if $(\mathcal{R}, \subseteq)$ is an induced subposet of $(\mathcal{F}, \subseteq)$, then it is certainly a subposet.

The third problem we discuss is of a different nature, but closely related to the problem of finding $ex_{\text{ind}}(n, \mathcal{R})$. Recall that a surjective function $\chi : 2^{[n]} \to \{1, 2, \ldots, k\}$ is called a k-coloring of $2^{[n]}$. Let $\mathcal{H}$ be a collection of subsets of $[n]$. If χ is a k-coloring of $2^{[n]}$ and there exists a collection of subsets $\mathcal{H}'$ of $[n]$ such that $(\mathcal{H}', \subseteq)$ is isomorphic to $(\mathcal{H}, \subseteq)$ and that for all $G, H \in \mathcal{H}'$, $\chi(G) \neq \chi(H)$, we say that χ *admits* a rainbow (or totally multicolored) copy of $\mathcal{H}$ and write $\chi \to \mathcal{H}$. We seek to find the smallest k such that every k-coloring χ of $2^{[n]}$ admits a rainbow copy of $\mathcal{H}$.

Definition 4.3. *If $\mathcal{H}$ is a collection of subsets of $[n]$, let*

$$f(n, \mathcal{H}) = \min\{q : \textit{for every } q\textit{-coloring } \chi \textit{ of } 2^{[n]},\ \chi \to \mathcal{H}\}.$$

We see immediately that for any collection $\mathcal{H}$ of subsets of $[n]$, $f(n, \mathcal{H}) \leq 2^n$, so $f(n, \mathcal{H})$ is well-defined. We also have the bound of $f(n, \mathcal{H}) \leq ex_{\text{ind}}(n, \mathcal{H}) + 1$, since any $(ex_{\text{ind}}(n, \mathcal{H}) + 1)$-coloring of $2^{[n]}$ admits a totally multicolored collection of sets $\mathcal{F}$ of size $ex_{\text{ind}}(n, \mathcal{H}) + 1$, and by definition of ex_{ind}, any collection $\mathcal{F}$ with $|\mathcal{F}| \geq ex_{\text{ind}}(n, \mathcal{H}) + 1$ has the property that $(\mathcal{H}, \subseteq)$ is an induced subposet of $(\mathcal{F}, \subseteq)$.

Lastly, a *chain of size k* or a *k-chain* (denoted P_k) is a set family with k distinct elements $F_1, F_2, \ldots, F_k$ such that $F_1 \subset F_2 \subset \cdots \subset F_k$. The *height* of a poset $\mathcal{P}$ is the size of the largest chain which is a subposet of $\mathcal{P}$.

4.2 Previous results

We begin by discussing previous results on $ex(n, \mathcal{P})$ and $ex_{\text{ind}}(n, \mathcal{P})$. The first result in extremal poset theory is credited to Sperner in [69], which was that $ex(n, P_2) = N$. Since $(P_k, \subseteq)$ is a subposet of $(\mathcal{F}, \subseteq)$ if and only if $(P_k, \subseteq)$ is an induced subposet of $(\mathcal{F}, \subseteq)$, we also have $ex_{\text{ind}}(n, P_2) = N$. We state the version of his result found in [22] below as Theorem 4.1, for which we need one more definition. An *antichain* is a set family $\mathcal{P}$ so that no two elements are comparable with one another. An *antichain of size* k or k*-antichain* (denoted A_k) is an antichain with k distinct elements.

Theorem 4.1 (Sperner, 1928)**.** *Let $\mathcal{F}$ be a collection of subsets of $[n]$ such that $(\mathcal{F}, \subseteq)$ is an antichain. Then $|\mathcal{F}| \leq N$ with equality if and only if*

$$\mathcal{F} = \begin{cases} \{X \subseteq [n] : |X| = n/2\} & \text{if } n \text{ is even,} \\ \{X \subseteq [n] : |X| = \lfloor n/2 \rfloor\} \text{ or } \{X \subset [n] : |X| = \lceil n/2 \rceil\} & \text{if } n \text{ is odd.} \end{cases}$$

Erdős generalized Sperner's theorem in [23]:

Theorem 4.2 (Erdős, 1945)**.** *For n and k integers with $n \geq k+1$, $ex(n, P_{k+1})$ is equal to the sum of the k largest binomial coefficients of order n.*

An important lemma of Erdős, also found in [23], gives us information about the number of disjoint chains between certain layers in Q_{2n}, we will state it and its short proof (to differentiate it from the proof of Lemma 4.2) below.

Lemma 4.1 (Erdős, 1945)**.** *For integers n, k with $n + k \leq 2n$, let $m = \binom{2n}{n+k}$. There exist m chains $\mathcal{C}_1, \ldots, \mathcal{C}_m$ in Q_{2n} such that $|\mathcal{C}_i \cap \mathcal{C}_j| = \varnothing$ for $1 \leq i < j \leq m$, all whose minimal element is in layer $\mathcal{L}_{n-k}$ and all whose maximal element is in layer $\mathcal{L}_{n+k}$.*

Proof of Lemma 4.1. The proof is a consequence of Menger's Theorem, originally from [55], which is: *Let $G = (V, E)$ be a graph and let $A, B \subseteq V$. The minimum number of vertices separating A from B is equal to the maximum number of disjoint A–B paths in G.*

Fix integers n and k with $n + k \leq 2n$, and let $m = \binom{2n}{n+k}$. Let G be the cover relation graph of Q_{2n}. Let A be the subset of vertices of G corresponding to the subsets of $[2n]$ of size

$n-k$, and let B be the subset of vertices of G corresponding to the subsets of $[2n]$ of size $n+k$. If we can show that A and B cannot be separated by less than m vertices, we may apply Menger's Theorem to finish the proof. Since there are $(n+k)(n+k-1)\cdots(n-k+1)$ chains connecting a fixed subset in layer $\mathcal{L}_{n+k}$ to layer $\mathcal{L}_{n-k}$, the number of A–B paths is

$$m(n+k)(n+k-1)\cdots(n-k+1). \tag{4.1}$$

However, these paths are not disjoint. For any i with $-k \le i \le k$, let z be a vertex in G corresponding to a subset of $[2n]$ of size $n+i$. The number of A–B paths that pass through vertex z is the same as the number of chains connecting z to layer $\mathcal{L}_{n+k}$ times the number of chains connecting z to layer $\mathcal{L}_{n-k}$. This number is then

$$(n+i)(n+i-1)\cdots(n-k+1)(n-i)(n-i-1)\cdots(n-k+1) \le (n+k)(n+k-1)\cdots(n-k+1). \tag{4.2}$$

By dividing the expression in (4.1) by that in (4.2), we get that the number of vertices separating A from B is at least m. We then apply Menger's Theorem to obtain m disjoint A–B paths. Since each A–B path corresponds to a chain in Q_{2n} with minimal element in layer $\mathcal{L}_{n-k}$ and maximal element in layer $\mathcal{L}_{n+k}$, the proof is complete. □

In [46], Kleitman confirmed a conjecture of Erdős and Katona, proving a stability result concerning Sperner's theorem.

Theorem 4.3 (Kleitman, 1968)**.** *If $\mathcal{F}$ is a family of subsets of $[n]$ having $N+x$ members, then there are at least Nx distinct pairs (A,B) of members of $\mathcal{F}$ satisfying $A \subset B$, $A \neq B$.*

Another result concerning antichains is the celebrated LYM inequality (also called YBLM inequality), proven independently by Yamamoto [79], Bollobás [10], Lubell [50], and Meshalkin [56].

Theorem 4.4 (LYM inequality)**.** *If $\mathcal{F}$ is a family of subsets of $[n]$ such that $(\mathcal{F},\subseteq)$ is an antichain, then $\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \le 1$.*

It is easy to see how one can prove Sperner's theorem using the LYM inequality, and we will include the details here for completeness. Let $\mathcal{F}$ be family of subsets of $[n]$ with $(\mathcal{F},\subseteq)$

an antichain, and let f_k denote the number of sets in $\mathcal{F}$ of size k. Since for any $0 \leq k \leq n$, $N \geq \binom{n}{k}$, the LYM inequality gives

$$\sum_{k=0}^{n} f_k N^{-1} \leq \sum_{k=0}^{n} f_k \binom{n}{k}^{-1} \leq 1.$$

Multiplying both sides by N gives

$$|\mathcal{F}| = \sum_{k=0}^{n} f_k \leq N,$$

as desired.

A *chain partition* $\mathcal{C}$ of a poset $\mathcal{P} = (P, \leq)$ is a partition of the elements of P into sets $X_1, \ldots, X_m$ such that $(X_i, \leq)$ is a chain for every i. Dilworth's theorem from [21] is also well-known, and links the notions of chain partitions and antichains:

Theorem 4.5 (Dilworth, 1950)**.** *For any finite poset $\mathcal{P}$, the maximum size of an antichain in $\mathcal{P}$ is equal to the minimum number of chains in a chain partition of $\mathcal{P}$.*

The r-fork V_r is a set family with $r+1$ elements $F, G_1, \ldots, G_r$ with inclusions $F \subset G_1, F \subset G_2, \ldots, F \subset G_r$ (see Figure 4.1 for its Hasse diagram). Katona and Tarján found an asymptotic lower bound and an upper bound on $ex(n, V_2)$ in [45].

Theorem 4.6 (Katona and Tarján, 1983)**.** *For a positive integer n,*

$$N\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq ex(n, V_2) \leq N\left(1 + \frac{2}{n}\right).$$

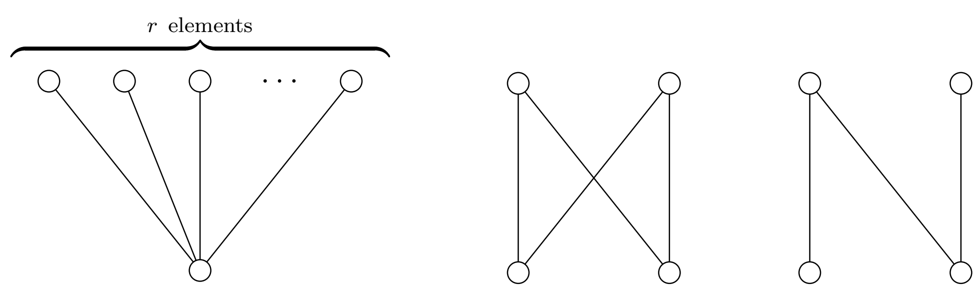

Figure 4.1 The Hasse diagrams of the set families V_r, $\bowtie$, and N, respectively

This result was then generalized by De Bonis and Katona in [18] as well as independently by Thanh in [74], stated below in Theorem 4.7.

Theorem 4.7 (De Bonis and Katona 2007; Thanh 1998). *For positive integers n and r with $r \geq 1$,*

$$N\left(1+\frac{r}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \leq ex(n, V_{r+1}) \leq N\left(1+\frac{2}{n}r+O\left(\frac{1}{n^2}\right)\right).$$

A set family with four distinct elements A, B, C, and D satisfying $A \subset B$, $C \subset B$, $A \subset D$, $C \subset D$ is called a *butterfly* and denoted by $\bowtie$. See Figure 4.1 for its Hasse diagram. De Bonis, Katona, and Swanepoel found an exact result for $ex(n, \bowtie)$ in [19], stated below in Theorem 4.8.[1] They also prove an LYM-type inequality, stated below in Theorem 4.9.

Theorem 4.8 (De Bonis, Katona, and Swanepoel, 2005). *If $n \geq 3$ is a positive integer, then* $ex(n, \bowtie) = \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.

Theorem 4.9 (De Bonis, Katona, and Swanepoel, 2005). *Suppose $n \geq 3$. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $\varnothing, [n] \notin \mathcal{F}$. If $\mathcal{F}$ is butterfly-free, then* $\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} \leq 2$.

A set family with four distinct elements A, B, C, and D such that $A \subset B$, $C \subset B$, $C \subset D$ is denoted by N. See Figure 4.1 for its Hasse diagram. In [39], Griggs and Katona find asymptotic lower and upper bounds on $ex(n, \mathsf{N})$, stated below in Theorem 4.10.

Theorem 4.10 (Griggs and Katona, 2008). *If n is a positive integer,*

$$N\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right) \leq ex(n, \mathsf{N}) \leq N\left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right).$$

The set family $P_k(s,t)$ (referred to as a *baton*) consists of distinct elements $F_1, F_2, \ldots, F_s$, $G_2, \ldots, G_{k-1}$, $H_1, \ldots, H_t$ with $F_i \subset G_2$ for $1 \leq i \leq s$, $G_{k-1} \subset H_j$ for $1 \leq j \leq t$, and $G_2 \subset G_3 \subset \cdots \subset G_{k-1}$. See Figure 4.2 for its Hasse diagram. In [40], Griggs and Lu prove an asymptotic upper bound on $ex(n, P_k(s,t))$, stated below in Theorem 4.11.

Theorem 4.11 (Griggs and Lu, 2009). *For any $s, t \geq 1$ and $k \geq 3$,*

$$ex(n, P_k(s,t)) \leq \sum_{i=\lfloor \frac{n-(k-2)}{2} \rfloor}^{\lfloor \frac{n+(k-2)}{2} \rfloor} \binom{n}{i} + \binom{n}{\lfloor \frac{n+k}{2} \rfloor}\left(\frac{2k(s+t-2)}{n} + O(n^{-3/2}\sqrt{\ln n})\right).$$

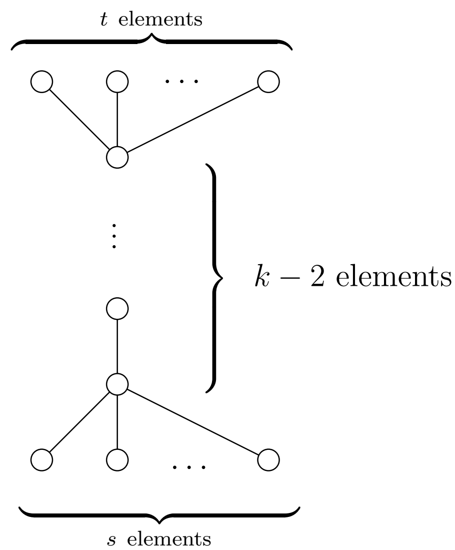

Figure 4.2 The Hasse diagram of the set family $P_k(s,t)$

The idea of the set family $\bowtie$ is generalized to that of $\mathcal{O}_{2k}$ (sometimes called the *crown*) in the following way. The family $\mathcal{O}_{2k}$ is a set family so that $(\mathcal{O}_{2k}, \subseteq)$ is a poset of height 2 whose cover relation graph is a cycle of length $2k$. We depict the set family $\mathcal{O}_8$ in Figure 4.3 to aid the reader. Notice that $\mathcal{O}_4$ is simply $\bowtie$. Griggs and Lu prove asymptotic bounds on $ex(n, \mathcal{O}_{4k})$ and $ex(n, \mathcal{O}_{4k-2})$ in [40], stated below in Theorem 4.12.

Theorem 4.12 (Griggs and Lu, 2009)**.** *For positive integers n and k with $k \geq 2$,*

$$ex(n, \mathcal{O}_{4k}) = (1 + o(1))N, \text{ and}$$
$$ex(n, \mathcal{O}_{4k-2}) \leq \left(1 + \frac{\sqrt{2}}{2} + o(1)\right) N.$$

Hence, the smallest crown $\mathcal{O}_i$ for which the problem of determining $ex(n, \mathcal{O}_i)$ asymptotically remains open is $\mathcal{O}_6$ (which, incidentally, is isomorphic to the union of the two middle layers of Q_3.

[1]The authors in [19] note that Griggs wryly suggested that a family of subsets of $[n]$ which is $\bowtie$-free be referred to as a *butterfly-free meadow*.

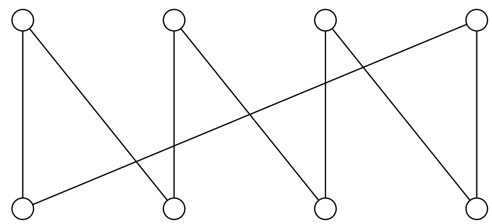

Figure 4.3 The Hasse diagram of the set family $\mathcal{O}_8$

Many of these results suggest that the size of the largest $\mathcal{R}$-free family is equal to the size of the maximum number of the largest layers of the Boolean lattice whose union is $\mathcal{R}$-free. This is in fact conjectured by Bukh in [13], where he proves this to be true in the special case of a set family whose cover relation graph is a tree. The full conjecture, if true, may be thought as an analogue of the Erdős-Stone Theorem from extremal graph theory, found in [24] (which we will address shortly). First, we will state the conjecture formally.

Let $\mathrm{Mon}(\mathbb{Z})$ denote the collection of all functions $f : \mathbb{Z} \to \{0, 1\}$ such that $f(n) = 1$ and $f(-n) = 0$ for all sufficiently large n. The functions in $\mathrm{Mon}(\mathbb{Z})$ can be thought of as eventually monotone functions. For $f, g \in \mathrm{Mon}(\mathbb{Z})$, we say $f \leq g$ if $f(n) \leq g(n)$ for all n. With this relation, $(\mathrm{Mon}(\mathbb{Z}), \leq)$ is a poset. Notice that if $f, g \in \mathrm{Mon}(\mathbb{Z})$, then $\sum_n f(n) - g(n)$ is finite.

A level in $\mathrm{Mon}(\mathbb{Z})$ is a maximal family $\mathcal{L} \subseteq \mathrm{Mon}(\mathbb{Z})$ such that for all $f, g \in \mathcal{L}$, $\sum_n f(n) - g(n) = 0$. We will show quickly that a level in $\mathrm{Mon}(\mathbb{Z})$ is an antichain. Let $\mathcal{L}$ be a level in $\mathrm{Mon}(\mathbb{Z})$, and suppose $f, g \in \mathcal{L}$ with $f \neq g$. If $f \leq g$, then $g(n) = 1$ whenever $f(n) = 1$, but there are values of n for which $f(n) = 0$ and $g(n) = 1$. Hence, $\sum_n f(n) - g(n) \neq 0$, a contradiction. As such, $f || g$, and $\mathcal{L}$ is an antichain.

We can now state the conjecture made by Bukh in [13]

Conjecture 4.1. *Let $\mathcal{P}$ be a finite set family and let $\ell(\mathcal{P})$ be the maximum number of levels in* $\mathrm{Mon}(\mathbb{Z})$ *so that their union does not contain $(\mathcal{P}, \subseteq)$ as a subposet. We have*

$$ex(n, \mathcal{P}) = \ell(\mathcal{P}) N \left(1 + O\left(\frac{1}{n}\right)\right).$$

To see how Conjecture 4.1 relates to the Erdős-Stone Theorem, we first discuss the function $f(n, \mathcal{P})$ and how it relates to extremal graph theory.

The problem of finding $f(n, \mathcal{P})$ has not been widely studied, but it finds its roots in anti-Ramsey theory. (Recall the discussion of Ramsey theory in Chapter 2, although the problem originally discussed in [27] is closer to that of Turán type problems than to that of Ramsey theory). We begin by stating the classic Turán Theorem, which concerns the most number of edges in a graph which does not contain a specified complete subgraph. More precisely, for integers n and r with $n \geq r-1$, the *Turán graph* $T_{r-1}(n)$ is the complete $(r-1)$-partite graph on n vertices whose partition sets differ in size by at most 1 (see Figure 4.4 for an example of a Turán graph). Clearly such a graph does not contain a complete r-vertex subgraph. Turán's theorem, stated in Theorem 4.13, is that $T_{r-1}(n)$ is the graph with the most number of edges which does not contain an r-vertex complete subgraph. The original proof is found in [76].

Theorem 4.13 (Turán, 1941)**.** *Let n and r be integers with $n \geq r-1$. If G is an n-vertex graph with more than $\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}$ edges, then G contains K_r as a subgraph. Moreover, the unique graph (up to isomorphism) achieving this bound is $T_{r-1}(n)$.*

The next theorem, due to Erdős and Stone, is surprising in the sense that while Turan's Theorem tells us that any n-vertex graph with more than $\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}$ edges will contain K_r as a subgraph (and hence any graph H with chromatic number r), that having an ϵ proportion number of edges more will give us a multitude of vertex-disjoint copies of H. Notice how Conjecture 4.1 can be thought of as an analogue of Theorem 4.14.

Theorem 4.14 (Erdős and Stone, 1946)**.** *Let H be a graph with chromatic number $r \geq 2$. For any integer $s \geq 1$ and real number $\epsilon > 0$, there exists an integer $N = N(r, s, \epsilon)$ such that any graph G on $n \geq N$ vertices such with at least $\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2} + \epsilon n^2$ edges contains s vertex-disjoint copies of H.*

The stability result due to Simonovits essentially states that if G does not have $T_{d+1}(n)$ as a subgraph, but the number of edges of G is close to that of the Turán graph $T_d(n)$, then its structure is similar to that of the Turán graph $T_d(n)$. We state the result formally below as Theorem 4.15.

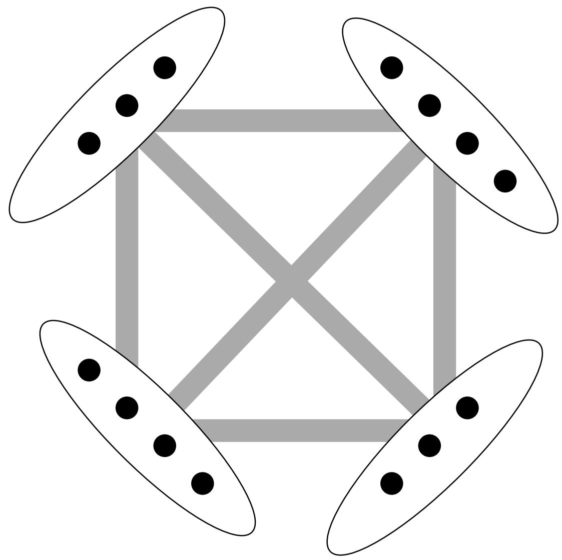

Figure 4.4 Example of the Turán graph with $r = 5$ and $n = 14$. A gray line from one part to another indicates that each vertex in the first part is adjacent to each vertex in the second part. Notice that the graph is 4-partite and the partition sets differ in size by at most one. There is no copy of K_5 as a subgraph, since any collection of 5 vertices necessarily contains 2 from the same partite set, and 2 vertices in the same partite set are not adjacent.

Theorem 4.15 (Simonovits, 1968). *If $r \geq 2$ and $d \geq 2$ are given positive integers and $\epsilon > 0$, then there exists $\delta > 0$ and an integer n_0 such that if $n > n_0$ and G is a graph on n vertices which does not contain $T_{d+1}(r(d+1))$ as a subgraph and $|E(G)| \geq |E(T_d(n))| - \delta n^2$, then we may omit $\lfloor \epsilon n^2 \rfloor$ of the edges of G so that the resulting graph has chromatic number d.*

Unfortunately, there can be no similar stability result in the vein of Conjecture 4.1. We see this by examining $ex(n, 2^{[2]})$. If true, Conjecture 4.1 gives $ex(n, 2^{[2]}) = 2N\left(1 + O\left(\frac{1}{n}\right)\right)$. We have 2 extremal families of this size. The first is obtained by taking the union of the two largest layers of the Boolean lattice. The second is obtained in the following way. Assume $n = 2k$. Let

$$\begin{aligned} F_1 &= \{X \in \mathcal{L}_{k-1} : 0 \in X\} \text{ and} \\ F_2 &= \{X \in \mathcal{L}_{k+1} : 0 \notin X\}. \end{aligned}$$

Let $\mathcal{F} = F_1 \cup F_2 \cup \mathcal{L}_k$. It is clear that $\mathcal{F}$ is $2^{[2]}$-free, as $\mathcal{F}$ does not contain a chain of size 3. The cardinality of $\mathcal{F}$ is

$$\begin{aligned}
|F_1| + |F_2| + |\mathcal{L}_k| &= \binom{2k-1}{k-1} + \binom{2k-1}{k+1} + N \\
&= \frac{(2k-1)!}{(k-1)!k!} + \frac{(2k-1)!}{(k+1)!(k-2)!} + N \\
&= \frac{(2k-1)!(k)(k+1) + (2k-1)!(k)(k-1)}{(k+1)!k!} + N \\
&= \frac{2k^2(2k-1)!}{(k+1)!k!} + N \\
&= \frac{(2k)!}{(k+1)!(k-1)!} + N \\
&= \binom{2k}{k+1} + N.
\end{aligned}$$

This is exactly the same cardinality of the family consisting of the union of the two largest layers of the Boolean lattice, but the structure of the family is quite different.

For a more comprehensive reference work in extremal graph theory, see [11].

The seminal 1973 paper by Erdős, Simonovits, and Sós introduces anti-Ramsey theory. As with Ramsey theory, anti-Ramsey theory is a graph coloring problem, although as mentioned, is more closely related to Turán-type problems than Ramsey-type problems. For an integer n and graph H, let $ext(n, H)$ denote the most number of edges in an n-vertex graph which does not contain H as a subgraph. Turán-type problems are concerned with finding $ext(n, H)$. Let $AR(n, H)$ denote the maximum number of colors the edges of K_n can be colored with if it does not contain a totally multicolored copy of H. Anti-Ramsey problems are concerned with finding $AR(n, H)$. To solidify the link between anti-Ramsey problems and Turán-type problems, consider the main theorem from [27], stated below as Theorem 4.16.

Theorem 4.16 (Erdős, Simonovits, and Sós, 1973)**.** *Let p be an integer with $p \geq 4$. There exists an integer $N = N(p)$ such that if $n \geq N$,*

$$AR(n, K_p) = ext(n, K_{p-1}) + 1.$$

The authors in [27] also describe an extremal coloring. Notice that K_n contains $T = T_{p-2}(n)$ as a subgraph. By coloring the edges of T all differently and using a final color on all the edges in $\overline{T}$, we will have a coloring of the edges of K_n which does not contain a totally multicolored copy of K_p and uses $ext(n, K_{p-1}) + 1$ colors by Turán's Theorem. Moreover, they prove that the coloring that uses $ext(n, K_{p-1}) + 1$ colors is unique (up to permutation of the colors).

4.3 New results

In Section 4.2, we mentioned work that has been done on finding $ex(n, \mathcal{P})$ for certain set families $\mathcal{P}$. Very little has been done toward finding $ex_{\text{ind}}(n, \mathcal{P})$, and almost no work has been done toward finding $f(n, \mathcal{P})$. This section is devoted to listing new results, most of which concern the function $f(n, \mathcal{P})$. The proofs of these results will follow in the subsections.

Before we state the results, we define two more set families. The family Λ_r consists of $r+1$ distinct sets $B_1, B_2, \ldots, B_r$, and C such that $C \subset B_i$ for all $1 \leq i \leq r$ (compare Λ_r and V_r). The family $\dot{P}_2$ consists of 3 distinct sets A, B, and C such that $A \subset B$.

To aid the reader, Table 4.1 contains all the previous and new results for the functions f, ex and ex_{ind}.

Now we will state the new results here, with their proofs to follow in the later subsections.

Proposition 4.1. *For integers $k \geq 2$ and $n \geq k$, $f(n, P_k) = ex(n, P_{k-1}) + 2$.*

Lemma 4.2. *For a positive integer n, there exist n maximal chains $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in Q_n such that $\mathcal{C}_i \cap \mathcal{C}_j = \{\varnothing, [n]\}$ for $i \neq j$.*

Proposition 4.2. *For positive integers $k \geq 2$ and n,*

$$\min\{k-1, n\} \cdot (n-1) + 2 \leq ex_{\text{ind}}(n, A_k) \leq (k-1)(n-1) + 2.$$

Proposition 4.3. *For an integer $n \geq 2$, $f(n, A_2) = 4$.*

Proposition 4.4. *For an integer $n \geq 4$, $f(n, A_3) = n + 3$, and $f(3, A_3) = 7$.*

Proposition 4.5. *For positive integers k and n with $k \geq 4$ and $n \geq 4k - 8$,*

$$f(n, A_k) = (k-2)(n-1) + 4.$$

$\mathcal{F}$	$f(n,\mathcal{F})$	$ex(n,\mathcal{F})$	$ex_{\text{ind}}(n,\mathcal{F})$
P_k	$ex(n,P_{k-1})+1^{(*)}$	$\sum_{i=\lfloor\frac{n-k+1}{2}\rfloor}^{i=\lfloor\frac{n+k-1}{2}\rfloor}\binom{n}{i}$	$ex(n,P_k)$
$\bowtie$	$\geq N+\binom{n}{\lfloor n/2\rfloor+1}+4^{(*)}$	$N+\binom{n}{\lfloor n/2\rfloor+1}$	
N		$N\left(1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)$ $N\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right)$	
$A_2^{(*)}$	4	2	$n+2$
$A_3^{(*)}$	$n+3$	3	$2n+1$
$A_k; k\geq 4^{(*)}$	$(k-2)(n-1)+4$; $n\geq 4k-8$	k	$(k-1)(n-1)+2$ $\min\{k-1,n\}(n-1)+2$
V_2	$5^{(*)}$	$N\left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)$ $N\left(1+\frac{1}{n}+\Omega\left(\frac{1}{n^2}\right)\right)$	
V_k		$N\left(1+2\frac{k-1}{n}+O\left(\frac{1}{n^2}\right)\right)$ $N\left(1+\frac{k-1}{n}+\Omega\left(\frac{1}{n^2}\right)\right)$	
$\mathcal{O}_{4k}$		$(1+o(1))N$	
$\mathcal{O}_{4k-2}$		$\leq\left(1+\sqrt{2}/2+o(1)\right)N$	
$2^{[2](*)}$	$ex(n,V_4)+2$ $N+2$	$2.6N$ $\binom{n}{\lfloor n/2\rfloor}+\binom{n}{\lfloor n/2\rfloor+1}$	

Table 4.1 Upper and lower bounds on $f(n,\mathcal{F})$, $ex(n,\mathcal{F})$ and $ex_{\text{ind}}(n,\mathcal{F})$. When both upper and lower bounds are known, the top value in a cell denotes an upper bound and the bottom value in a cell denotes a lower bound. A single value in a cell denotes equality when no other qualifiers are present. If a cell is blank, then no nontrivial bounds are known. Those marked with an asterisk are those which are new results whose proofs are contained in this dissertation. Those without asterisks are mentioned in Section 4.2, where credit is attributed appropriately.

Lemma 4.3. *Let n be a positive integer. If $\mathcal{R}$ and $\mathcal{P}$ are families of subsets of $[n]$ such that there is a bijection $g:\mathcal{R}\to\mathcal{P}$ with the property that for all $X,Y\in\mathcal{R}$, $X\subset Y$ if and only if $g(Y)\subset g(X)$, then $f(n,\mathcal{R})=f(n,\mathcal{P})$.*

Proposition 4.6. *If $n \geq 3$ is an integer, $f(n, V_2) = f(n, \Lambda_2) = 5$.*

Proposition 4.7. *If $n \geq 4$ is an integer, $f(n, \dot{P}_2) = 5$. Furthermore $f(3, \dot{P}_2) = 6$.*

Proposition 4.8. *If $n \geq 4$, then $f(n, \bowtie) \geq N + \binom{n}{\lfloor n/2 \rfloor + 1} + 4$.*

Proposition 4.9. *If n is a positive integer, $N + 2 \leq f(n, 2^{[2]}) \leq ex(n, V_4) + 2$.*

Proposition 4.10. *If n is an integer, then $\binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1} \leq ex(n, 2^{[2]}) \leq 2.6N$.*

Proposition 4.11. *Let $\mathcal{F}$ be a collection of subsets of $[n]$ which is Q_2-free. Suppose $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$, where $\mathcal{S}$ is the collection of minimal elements of $\mathcal{F}$, $\mathcal{U}$ is the collection of maximal elements of $\mathcal{F} \setminus \mathcal{S}$ and $\mathcal{T} = \mathcal{F} \setminus (\mathcal{S} \cup \mathcal{U})$ such that for any $T \in \mathcal{T}$, $S \in \mathcal{S}$, $U \in \mathcal{U}$, $|T| = k$, $|U| > k$, $|S| < k$. We have*

$$|\mathcal{F}| \leq \left(\frac{3+\sqrt{2}}{2}\right) N + o(N) \leq 2.20711N + o(N).$$

In particular, if $\mathcal{F}$ is a Q_2-free family of subsets of $[n]$ contained in only three layers of Q_n, then $|\mathcal{F}| \leq 2.20711N + o(N)$.

Before we move to the proofs of the above propositions, we make a few observations.

- Proposition 4.1 has the same flavor as that of the Erdős-Simonovits-Sós Theorem (stated as Theorem 4.16).

- Lemma 4.2 is used to construct colorings which do not admit certain rainbow substructures (and is a corollary of a lemma of Erdős from [23], stated as Lemma 4.1, but the proof presented here is constructive).

- Note that if $k-1 \leq n$, then Proposition 4.2 implies that $ex_{\text{ind}}(n, A_k) = (k-1)(n-1)+3$.

- Note that $f(n, A_3)$ agrees with the result for $f(n, A_k)$ in the table, but the proof of Proposition 4.5 cannot be applied for the case $k = 3$. This is why we state Proposition 4.4 separately with its own proof.

- Recall that for any family $\mathcal{H}$, $f(n, \mathcal{H}) \leq ex_{\text{ind}}(n, \mathcal{H}) + 1$. This fact shows us that Propositions 4.1 and 4.5 are nontrivial improvements. This also gives us that the upper bound on $f(n, A_k)$ for $n < 4k - 8$ is $(k-1)(n-1) + 3$.

- Lemma 4.3 is intuitively obvious, but requires a rigorous proof.

- Recall from Theorem 4.7 that $ex(n, V_4) \leq N\left(1 + \frac{6}{n} + O\left(\frac{1}{n^2}\right)\right)$, so the bounds in Proposition 4.9 have the correct leading term in the asymptotic.

- The upper bound in Proposition 4.10 has actually been beaten by Griggs and Lu, but remains unpublished. See presentations by Griggs or Lu for the proof [38].

- Propositions 4.8, 4.9, 4.10, and Theorem 4.8 imply that $f(n, \bowtie) \geq ex(n, \bowtie)$ as well as $f(n, 2^{[2]}) < ex(n, 2^{[2]})$. As such, there is no inequality possible between $f(n, \mathcal{H})$ and $ex(n, \mathcal{H})$ for all n and $\mathcal{H}$.

4.3.1 Proof of Proposition 4.1

We begin with a lemma.

Lemma 4.4. *Let $\mathcal{F}$ be a family of subsets of $[n]$ with $|\mathcal{F}| \geq ex(n, P_{k-1}) + 2$. If either $[n] \in \mathcal{F}$ or $\varnothing \in \mathcal{F}$, then P_k is a subposet of $(\mathcal{F}, \subseteq)$.*

Proof of Lemma 4.4. Without loss of generality, assume $[n] \in \mathcal{F}$ (since the argument will be symmetric for the case where $\varnothing \in \mathcal{F}$). Let $\mathcal{F}' = \mathcal{F} \setminus \{[n]\}$. Then $|\mathcal{F}'| \geq ex(n, P_{k-1}) + 1$, so $P_{k-1} \subseteq \mathcal{F}'$. Since $[n] \notin \mathcal{F}'$, $P_{k-1} \cup \{[n]\} = P_k \subseteq \mathcal{F}$, as desired. □

Proof of Proposition 4.1. Fix integers n and k with $n \geq k \geq 2$.

Lower bound. By the definition of $ex(n, P_{k-1})$, there exists a family $\mathcal{F}$ of subsets of $[n]$ of size $ex(n, P_{k-1})$ such that $(P_{k-1}, \subseteq)$ is not a subposet of $(\mathcal{F}, \subseteq)$. Write $\mathcal{F} = \{F_1, F_2, \ldots, F_{ex(n, P_{k-1})}\}$. We define a $(ex(n, P_{k-1}) + 1)$-coloring χ of $2^{[n]}$ by

$$\chi(A) = \begin{cases} i, & \text{if } A = F_i \text{ for some } i, \\ ex(n, P_{k-1}) + 1 & \text{if } A \neq F_j \text{ for all } 1 \leq j \leq ex(n, P_{k-1}). \end{cases}$$

Since the height of $(\mathcal{F}, \subseteq)$ is at most $(k-2)$ and because any totally multicolored k-chain under χ can contain at most one set in $2^{[n]} \setminus \mathcal{F}$, χ does not admit a rainbow k-chain.

Upper bound. For ease in notation, let $m = ex(n, P_{k-1}) + 2$. Let χ be a m-coloring of $2^{[n]}$. Without loss of generality, assume that $\chi([n]) = m$. There exists a family $\mathcal{F}$ of subsets of $[n]$ such that

(i) $[n] \notin \mathcal{F}$, and

(ii) for each i with $1 \leq i \leq m-1$, there is a set $F \in \mathcal{F}$ such that $\chi(F) = i$.

Choose $\mathcal{F}$ so that $|\mathcal{F}$ is minimal among such families of sets; i.e., $|\mathcal{F}| = m-1$ and for all $F \in \mathcal{F}$, $\chi(F) \neq m$. Let $\mathcal{F}' = \mathcal{F} \cup \{[n]\}$. Since $|\mathcal{F}'| = ex(n, P_{k-1}) + 2$ and $[n] \in \mathcal{F}'$, by Lemma 4.4, $(P_k, \subseteq)$ is a subposet of $(\mathcal{F}', \subseteq)$. Since $\mathcal{F}'$ is totally multicolored, $\chi \to P_k$. Since χ was an arbitrary m-coloring, $f(n, P_k) \leq m = ex(n, P_{k-1}) + 2$. □

4.3.2 Proof of Lemma 4.2

Proof. Let n be a positive integer. We will construct the n chains we need. Since each of $\mathcal{C}_1, \ldots, \mathcal{C}_n$ will be a chain, each $\mathcal{C}_j$ for $1 \leq j \leq k$ contains only one set per layer of Q_n. Hence, we may define each $\mathcal{C}_j$ by which element it contains from layer $\mathcal{L}_i$ for $1 \leq i \leq n-1$. We refer the reader to Table 4.2 to see how to construct these chains. The set in chain $\mathcal{C}_j$ in layer $\mathcal{L}_i$ is the set consisting of the elements of $[n]$ in row j in columns 1 through i (arithmetic is done modulo n). From the table, it is clear that all the chains are different. To see that if $j_1 < j_2$ that $\mathcal{C}_{j_1} \cap \mathcal{C}_{j_2} = \{\varnothing, [n]\}$, it suffices to show that $\mathcal{C}_{j_1}$ and $\mathcal{C}_{j_2}$ have different sets from layer $\mathcal{L}_i$; i.e., that the two sets

$$\{j_1 - 1, j_1, j_1 + 1, \ldots, j_1 - 2 + i\} \text{ and } \{j_2 - 1, j_2, j_2 + 1, \ldots, j_2 - 2 + i\}$$

are different. Since $j_1 \neq j_2$, these two sets above can intersect in a set of size at most $(i-1)$, and hence are not the same set in layer $\mathcal{L}_i$. □

Notice that the proof of Lemma 4.2 differs from that of Lemma 4.1 in the sense that the proof here is constructive. The proof of Lemma 4.1 (stated in Section 4.2) uses Menger's Theorem, which is nonconstructive.

	$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\cdots$	$\mathcal{L}_i$	$\cdots$	$\mathcal{L}_{n-1}$
$\mathcal{C}_1$	0	1	2	$\cdots$	$i-1$	$\cdots$	$n-2$
$\mathcal{C}_2$	1	2	3	$\cdots$	i	$\cdots$	$n-1$
$\mathcal{C}_3$	2	3	4	$\cdots$	$i+1$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\mathcal{C}_j$	$j-1$	j	$j+1$	$\cdots$	$j-2+i$	$\cdots$	$j+n-3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$\mathcal{C}_k$	$k-1$	k	$k+1$	$\cdots$	$k-2+i$	$\cdots$	$k+n-3$

Table 4.2 A table showing how to construct the chains in Lemma 4.2.

4.3.3 Proof of Proposition 4.2

Proof. Let n and k be positive integers with $k \geq 2$.

Lower bound. Let $m = \min\{k-1, n\}$. Since $m \leq n$, by Lemma 4.2, there exist m maximal chains $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that for every $i \neq j$, $\mathcal{C}_i \cap \mathcal{C}_j = \{\varnothing, [n]\}$. Let $\mathcal{F} = \bigcup_{i=1}^{m} \mathcal{C}_i$. Since $m < k$, any subcollection of sets from $\mathcal{F}$ with cardinality k must contain at least 2 elements from some $\mathcal{C}_i$. Hence, there can be no antichain of size k in $(\mathcal{F}, \subseteq)$. Since $|\mathcal{F}| = m(n-1)+2$, $m(n-1)+2 \leq ex_{\text{ind}}(n, A_k)$.

Upper bound. Let $\mathcal{F}$ be a family of subsets of $[n]$ with $|\mathcal{F}| = (k-1)(n-1)+3$. Since there are $n-1$ layers in Q_n with cardinality greater than 1, there exists an integer i with $1 \leq i \leq n-1$ such that $|\mathcal{L}_i \cap \mathcal{F}| \geq k$. Since any layer in Q_n is an antichain, we have found an antichain of size k in $(\mathcal{F}, \subseteq)$. Since $\mathcal{F}$ was an arbitrary family of subsets of $[n]$, $ex_{\text{ind}}(n, A_k) \leq (k-1)(n-1)+2$.

□

4.3.4 Proof of Proposition 4.3

Proof. Let $n \geq 2$.

Lower bound. Let χ be the 3-coloring of $2^{[n]}$ such that $\chi(\varnothing) = 1$, $\chi([n]) = 2$, and $\chi(X) = 3$ for every subset X of $[n]$ which is not $\varnothing$ or $[n]$. Since $\varnothing$ and $[n]$ never participate in any antichain of size greater than 1, there is no totally multicolored A_2 under χ.

Upper bound. Let χ be a 4-coloring of $2^{[n]}$. We may assume that χ is constant on the layer $\mathcal{L}_1$. Say $\chi(X) = 1$ for each $X \in \mathcal{L}_1$. Since χ is a 4-coloring, there exists some subset Y of $[n]$ which is not $\varnothing$ or $[n]$ such that $\chi(Y) \neq 1$. Hence, there is $a \in [n] \setminus Y$, and $\{\{a\}, Y\}$ is a rainbow 2-antichain, so $\chi \to A_2$. Since χ was an arbitrary 4-coloring of $2^{[n]}$, $f(n, A_2) \leq 4$. □

4.3.5 Proof of Proposition 4.4

Proof. Let $n \geq 4$.

Lower bound. We will find an $(n+2)$-coloring of $2^{[n]}$ which does not admit a rainbow A_3. Let $\mathcal{P}$ be a maximal chain. Define $\chi : 2^{[n]} \to \{1, 2, \ldots, n+2\}$ by

$$\chi(X) = \begin{cases} |X|, & \text{if } X \in \mathcal{P}; \\ n+2, & \text{if } X \notin \mathcal{P}. \end{cases}$$

Since any totally multicolored collection of subsets $\mathcal{F}$ with $|\mathcal{F}| = 3$ must contain at least 2 sets from $\mathcal{P}$, we cannot have a totally multicolored induced copy of A_3, so $f(n, A_3) \geq n + 3$.

Upper bound. Let χ be a $(n+3)$-coloring of $2^{[n]}$. Since $\varnothing$ and $[n]$ cannot be members of any antichain of size 3, we may assume $\chi(\varnothing) \neq \chi([n])$. Say $\chi(\varnothing) = n+2$ and $\chi([n]) = n+3$. Define a bipartite graph G with partite sets $A = \{a_1, \ldots, a_{n-1}\}$ and $B = \{b_1, \ldots, b_{n+1}\}$ where vertices a_i and b_j are adjacent if and only if there is a set $X \in \mathcal{L}_i$ with $\chi(X) = j$. (Notice that $N(a_i)$ corresponds to the set of colors used on layer $\mathcal{L}_i$.) Immediately, we see if there is $a \in A$ with $|N(a)| \geq 3$, then χ admits a rainbow A_3 since each layer in Q_n is an antichain. Hence, we may assume $|N(a)| \leq 2$ for all $a \in A$.

Claim. *There exist integers ℓ and m with $1 \leq \ell < m \leq n-1$ with $|N(a_\ell)| = |N(a_m)| = 2$ and $N(a_\ell) \cap N(a_m) = \varnothing$.*

Proof of Claim. Let $A' = \{a \in A : \deg(a) = 2\}$. Since χ uses $n+1$ colors on $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$, we know that for each $b \in B$, $\deg(b) \geq 1$. By the pigeonhole principle, $|A'| \geq 2$.

Let $A'' = A \setminus A'$, so $|A''| = n - 1 - |A'|$. Suppose by way of contradiction that there do not exist two vertices a_ℓ and a_m in A' such that $N(a_\ell) \cap N(a_m) = \varnothing$. This gives $|N(A')| \leq |A'| + 1$.

By definition of A'', we have $|N(A'')| \leq n-1-|A'|$. Since $B = N(A') \cup N(A'')$, we have

$$\begin{aligned} |B| &= |N(A') \cup N(A'')| \\ &\leq |N(A')| + |N(A'')| \\ &\leq |A'| + 1 + n - 1 - |A'| \\ &\leq n, \end{aligned}$$

a contradiction, so the Claim is true.

Let ℓ and m be as in the Claim. We have two cases to consider.

Case 1. $\ell \neq 1$ or $m \neq n-1$.

Assume $m \neq n-1$. Choose sets $X_1, X_2 \in \mathcal{L}_\ell$ with $\chi(X_1) \neq \chi(X_2)$, and choose $x_1 \in X_1$ and $x_2 \in X_2$ (not necessarily distinct). Choose $Y \in \mathcal{L}_m$ such that $Y \subset [n] \setminus \{x_1, x_2\}$ (such Y exists since $m \leq n-2$). This gives us that $\{X_1, X_2, Y\}$ is an antichain, and is totally multicolored by the Claim, so $\chi \to A_3$. The case where $\ell = 1$ is symmetric, concluding Case 1.

Case 2. $\ell = 1$ and $m = n-1$.

Suppose $N(a_\ell) = \{b_1, b_2\}$ and $N(a_m) = \{b_3, b_4\}$. Since $n \geq 4$, there must exist an integer j with $2 \leq j \leq n-2$ such that $b_5 \in N(a_j)$; that is, color 5 is used on layer $\mathcal{L}_j$. If $b_1, b_2 \notin N(a_j)$, then (as in Case 1) choose $X_1, X_2 \in \mathcal{L}_\ell$ with $\chi(X_1) \neq \chi(X_2)$. Choose $x_1 \in X_1$ and $x_2 \in X_2$ and $J \in \mathcal{L}_j$ with $J \subset [n] \setminus \{x_1, x_2\}$ (such J exists as $j \leq n-2$). This gives us that $\{X_1, X_2, J\}$ is an antichain, and is totally multicolored, so $\chi \to A_3$. On the other hand, if one of b_1 or b_2 is in $N(a_j)$, then since $|N(a_j)| \leq 2$, $N(a_j) \cap N(a_m) = \varnothing$, and $j \geq 2$. This puts us back in Case 1, so $\chi \to A_3$.

Case 1 and Case 2 together give us that $f(n, A_3) \leq n+3$.

To finish the proof, we must show that $f(3, A_3) = 7$.

Lower bound. Define $\chi : 2^{[3]} \to \{1,2,3,4,5,6\}$ by

$$\chi(X) = \begin{cases} 1, & \text{if } X = \varnothing; \\ 2, & \text{if } X = \{0\}; \\ 3, & \text{if } X = \{0,1\}; \\ 4, & \text{if } X = [3]; \\ 5, & \text{if } X \in \mathcal{L}_1 \setminus \{\{0\}\}; \\ 6, & \text{if } X \in \mathcal{L}_2 \setminus \{\{0,1\}\}. \end{cases}$$

See Figure 4.5 for a pictorial representation. It is easy to check by inspection that χ does not admit a rainbow A_3. We have $f(3, A_3) \geq 7$.

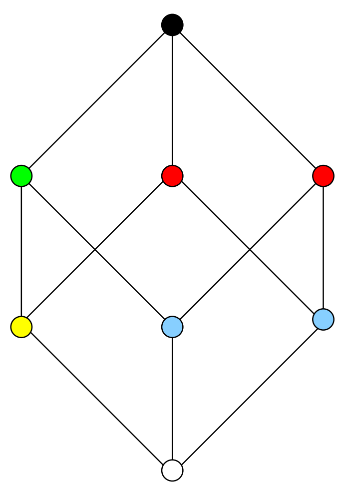

Figure 4.5 A 6-coloring χ of $2^{[3]}$ which does not admit a rainbow A_3.

Upper bound. Let χ be a 7-coloring of $2^{[3]}$. This means χ uses at least 5 colors on layers $\mathcal{L}_1$ and $\mathcal{L}_2$. By the pigeonhole principle, χ must use at least 3 colors on one of these layers. Since any layer in Q_n is an antichain, $\chi \to A_3$, so $f(3, A_3) \leq 7$. □

4.3.6 Proof of Proposition 4.5

Recall that for a graph $G = (V, E)$, for a vertex $x \in V$, the *neighborhood* of x is denoted by $N(x)$. If $W \subseteq V$, then $N(W) = \bigcup_{w \in W} N(w)$.

Proof. Let k and n be integers with $k \geq 4$ and $n \geq 4k-8$.

Lower bound. We will construct a $((k-2)(n-1)+3)$-coloring χ of $2^{[n]}$ which does not admit a rainbow A_k. Since $n \geq 4k-8$ and $k \geq 4$, $k-2 \leq n$, so by Lemma 4.2 there exist $(k-2)$ maximal chains $\mathcal{C}_1, \ldots, \mathcal{C}_{k-2}$ in Q_n such that for all i and j with $1 \leq i < j \leq k-2$, $\mathcal{C}_i \cap \mathcal{C}_j = \{\varnothing, [n]\}$. Define $\chi : 2^{[n]} \to \{1, 2, \ldots, (k-2)(n-1)+3\}$ by

$$\chi(X) = \begin{cases} |X| + (j-1)(n-1), & \text{if } X \in \mathcal{C}_j,\ X \neq \varnothing, [n]; \\ (k-2)(n-1)+1, & \text{if } X = \varnothing; \\ (k-2)(n-1)+2, & \text{if } X = [n]; \\ (k-2)(n-1)+3, & \text{otherwise.} \end{cases}$$

Since any antichain contains at most one set from each of $\mathcal{C}_1, \ldots, \mathcal{C}_{k-2}$, χ does not admit a rainbow A_k.

Upper bound. Let χ be an arbitrary $((k-2)(n-1)+4)$-coloring of $2^{[n]}$. We will show that $\chi \to A_k$. Since $\varnothing$ and $[n]$ cannot be members of any antichain of size k for $k \geq 2$, we may assume that $\chi(\varnothing) \neq \chi([n])$. Say $\chi(\varnothing) = (k-2)(n-1)+3$ and $\chi([n]) = (k-2)(n-1)+4$.

Define a bipartite graph G with partite sets $A = \{a_1, \ldots, a_{n-1}\}$ and $B = \{b_1, \ldots, b_{(k-2)(n-1)+2}\}$ where vertices a_i and b_j are adjacent if and only if there is a set $X \in \mathcal{L}_i$ with $\chi(X) = j$. Note that $N(a_i)$ corresponds to the set of colors used on $\mathcal{L}_i$. If there exists a vertex a_i such that $|N(a_i)| \geq k$, we are finished, so we may assume $|N(a_i)| \leq k-1$ for $1 \leq i \leq n-1$.

Claim 1. *There exist ℓ and m with $1 \leq \ell < m \leq n-1$ such that $|N(a_\ell)| = |N(a_m)| = k-1$ and $N(a_\ell) \cap N(a_m) = \varnothing$.*

Proof of Claim 1. Let $A' = \{a \in A : \deg(a) = k-1\}$. Since χ uses colors $1, \ldots, (k-2)(n-1)+2$ on layers $\mathcal{L}_1, \ldots, \mathcal{L}_{n-1}$, we know that for each $b \in B$, $\deg(b) \geq 1$. By the pigeonhole principle, $|A'| \geq 2$.

Let $A'' = A \setminus A'$, so $|A''| = n-1-|A'|$. Suppose by way of contradiction that there do not exist two vertices a_ℓ and a_m in A' such that $N(a_\ell) \cap N(a_m) = \varnothing$. This gives the inequality $|N(A')| \leq |A'|(k-2)+1$. By definition of A'', we have $|N(A'')| \leq (n-1-|A'|)(k-2)$. Since

$B = N(A') \cup N(A'')$, we have

$$\begin{aligned} |B| &= |N(A') \cup N(A'')| \\ &\leq |N(A')| + |N(A'')| \\ &\leq |A'|(k-2) + 1 + (n-1-|A'|)(k-2) \\ &= (n-1)(k-2)+1, \end{aligned}$$

a contradiction, so the Claim 1 is true.

Let ℓ and m be as in Claim 1. We have three cases to consider.

Case 1. $1 \leq \ell < m \leq n-(k-1)$.

Choose $X_1, X_2, \ldots, X_{k-1} \in \mathcal{L}_\ell$ such that $\chi(X_i) \neq \chi(X_j)$ for $1 \leq i, j \leq k-1$ whenever $i \neq j$. Choose $x_i \in X_i$ (not necessarily distinct). Since $m \leq n-(k-1)$, there is a set $Y \subset [n] \setminus \{x_1, \ldots, x_{k-1}\}$ with $Y \in \mathcal{L}_m$. Since $Y \not\supset X_i$ for $1 \leq i \leq k-1$, $\{X_1, \ldots, X_{k-1}, Y\}$ is an antichain. Since $N(a_\ell) \cap N(a_m) = \varnothing$, $\chi \to A_k$.

Case 2. $k-1 \leq \ell < m \leq n-1$.

Similar to Case 1, choose $Y_1, Y_2, \ldots, Y_{k-1} \in \mathcal{L}_m$ such that $\chi(Y_i) \neq \chi(Y_j)$ for $1 \leq i, j \leq k-1$ whenever $i \neq j$. Choose $y_i \in [n] \setminus Y_i$ (not necessarily distinct). Since $\ell \geq k-1$, there is a set $X \supset \{y_1, \ldots, y_{k-1}\}$ with $X \in \mathcal{L}_\ell$. Since $X \not\subset Y_i$ for $1 \leq i \leq k-1$, $\{Y_1, \ldots, Y_{k-1}, X\}$ is an antichain. Since $N(a_\ell) \cap N(a_m) = \varnothing$, $\chi \to A_k$.

Case 3. $\ell \leq k-2$ and $m \geq n-(k-2)$.

We begin with a claim.

Claim 2. *There is j with $k \leq j \leq n-k$ such that $|N(a_j) \setminus (N(a_\ell) \cup N(a_m))| \geq k-2$.*

Proof of Claim 2. By way of contradiction, assume that for all j with $k \leq j \leq n-k$, $|N(a_j) \setminus (N(a_\ell) \cup N(a_m))| \leq k-3$. We will find an upper bound on $|B|$ by considering the neighborhoods of the vertices in A.

$$\begin{aligned} |B| &\leq \sum_{0<\ell'\leq\ell} |N(a_{\ell'})| + \sum_{\ell<j<m} |N(a_j)| + \sum_{m\leq m'<n} |N(a_{m'})| \\ &\leq (k-1)(k-2) + (k-3)(n-1-(2k-4)) + (k-1)(k-2) \\ &= 2(2k-4) + (k-3)(n-1). \end{aligned}$$

Since $n \geq 4k-8$, $2(2k-4)+(k-3)(n-1) < (k-2)(n-1)+2$, a contradiction, so Claim 2 is true.

Let j be as in Claim 2. Choose $X_1, \ldots, X_{k-1} \in \mathcal{L}_\ell$ with $\chi(X_p) \neq \chi(X_q)$ for $1 \leq p < q \leq k-1$. Again, choose $x_1, x_2, \ldots, x_{k-1}$ with $x_i \in X_i$ (not necessarily distinct), and let $J_1 \in \mathcal{L}_j$ with $J_1 \subset [n] \setminus \{x_1, \ldots, x_{k-1}\}$. The collection $\{X_1, \ldots, X_{k-1}, J_1\}$ is an antichain, so we may assume $\chi(J_1) = \chi(X_i)$ for some i with $1 \leq i \leq k-1$ (else $\chi \to A_k$).

Now choose $Y_1, \ldots, Y_{k-1} \in \mathcal{L}_m$ with $\chi(Y_p) \neq \chi(Y_q)$ for $1 \leq p < q \leq k-1$. Choose $y_1, y_2, \ldots, y_{k-1}$ with $y_i \in [n] \setminus Y_i$ (not necessarily distinct), and let $J_2 \in \mathcal{L}_j$ with $\{y_1, \ldots, y_{k-1}\} \subseteq J_2$. Notice that if $J_2 = J_1$, Claim 1 implies that the collection $\{Y_1, \ldots, Y_{k-1}, J_2\}$ is a totally multicolored antichain of size k, so we may assume $J_2 \neq J_1$ and that $\chi(J_2) = \chi(Y_p)$ for some p (else $\chi \to A_k$).

Since $|N(a_j) \setminus (N(a_\ell) \cup N(a_m))\,| \geq k-2$, this gives $|N(a_j)| \geq k$. Since any layer in Q_n is an antichain, $\chi \to A_k$ and $f(n, A_k) \leq (k-2)(n-1)+4$. □

4.3.7 Proof of Lemma 4.3

Lemma 4.3 is a technical lemma. Lemma 4.3 says that if $\mathcal{R}$ and $\mathcal{P}$ are families of sets such that the Hasse diagram of $(\mathcal{R}, \subseteq)$ is simply that of $(\mathcal{P}, \subseteq)$ turned "upside-down", then $f(n, \mathcal{R}) = f(n, \mathcal{P})$. See Figure 4.6 for a pictorial representation of two set families satisfying the hypotheses of Lemma 4.3.

Proof. Fix a positive integer n. Let $\mathcal{R}$ and $\mathcal{P}$ be families of subsets of $[n]$ such that there is a bijection $g : \mathcal{R} \to \mathcal{P}$ with the property that for all $X, Y \in \mathcal{R}$, $X \subset Y$ if and only if $g(Y) \subset g(X)$. Let χ be any coloring of $2^{[n]}$, and let $\chi'(X) = \chi([n] \setminus X)$ for $X \in 2^{[n]}$. Notice that this definition also gives us $\chi(X) = \chi'([n] \setminus X)$. It suffices to show that $\chi' \to \mathcal{R}$ if and only if $\chi \to \mathcal{P}$.

Let $\gamma : 2^{[n]} \to 2^{[n]}$ be given by $\gamma(X) = [n] \setminus X$ for $X \in 2^{[n]}$, and notice that $X \subseteq Y$ if and only if $\gamma(Y) \subseteq \gamma(X)$ and that $\gamma = \gamma^{-1}$.

Suppose $\chi' \to \mathcal{R}$. This means there is a family of sets $\mathcal{R}'$ which is totally multicolored and $(\mathcal{R}', \subseteq)$ is isomorphic to $(\mathcal{R}, \subseteq)$. Let $h : \mathcal{R}' \to \mathcal{R}$ be a poset isomorphism (a bijection which

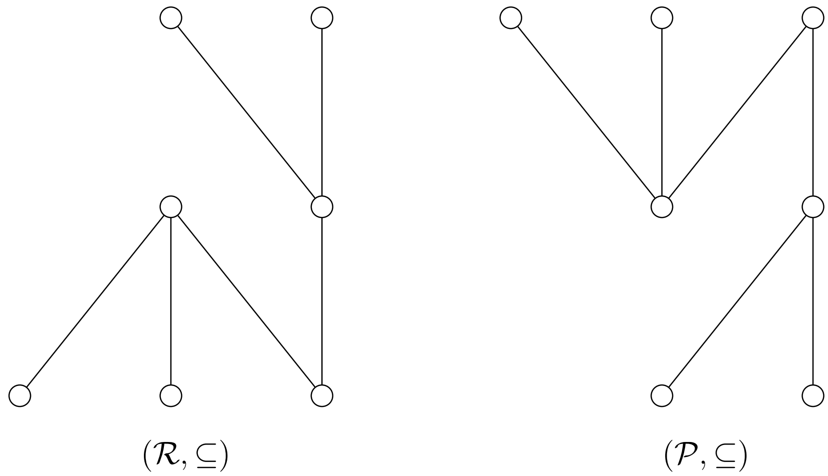

Figure 4.6 The Hasse diagrams for two set families $\mathcal{R}$ and $\mathcal{P}$ which satisfy the hypotheses of Lemma 4.3. It is natural to assume that $f(n, \mathcal{R}) = f(n, \mathcal{P})$, and that is exactly what Lemma 4.3 states.

preserves order). Consider the family $\gamma(\mathcal{R}') = \{\gamma(R) : R \in \mathcal{R}'\}$.

Claim 1. $(\mathcal{P}, \subseteq)$ *is isomorphic to* $(\gamma(\mathcal{R}'), \subseteq)$.

Proof of Claim 1. Let $\alpha : \mathcal{P} \to \gamma(\mathcal{R}')$ be given by $\alpha(X) = \gamma(h^{-1}(g^{-1}(X)))$. Let $X, Y \in \mathcal{P}$. By assumption, $g^{-1}(Y) \subseteq g^{-1}(X)$ if and only if $X \subseteq Y$. Since h is an isomorphism, $h^{-1}(g^{-1}(Y)) \subseteq h^{-1}(g^{-1}(X))$ if and only if $g^{-1}(Y) \subseteq g^{-1}(X)$. By definition of γ, $\gamma(h^{-1}(g^{-1}(X))) \subseteq \gamma(h^{-1}(g^{-1}(Y)))$ if and only if $h^{-1}(g^{-1}(Y)) \subseteq h^{-1}(g^{-1}(X))$, so $\alpha(X) \subseteq \alpha(Y)$ if and only if $X \subseteq Y$. Since g, h, and γ are all bijections, α is a bijection, so Claim 1 is true.

(To elucidate which maps are between which families of sets, see Figure 4.7.)

Now all we need to show is that $\gamma(\mathcal{R}')$ is totally multicolored under χ. Let $R \in \mathcal{R}'$. Notice that $\chi'(R) = \chi([n] \setminus R) = \chi(\gamma(R))$. Since for all $R_1, R_2 \in \mathcal{R}'$ with $R_1 \neq R_2$, $\chi'(R_1) \neq \chi'(R_2)$, $\chi(\gamma(R_1)) \neq \chi(\gamma(R_2))$. This gives us that $\gamma(\mathcal{R}')$ is totally multicolored under χ. This fact, together with Claim 1, gives $\chi \to \mathcal{P}$.

Now suppose $\chi \to \mathcal{P}$. We essentially mirror the proof that $\chi' \to \mathcal{R}$ implies $\chi \to \mathcal{P}$. Since $\chi \to \mathcal{P}$, there is a family of sets $\mathcal{P}'$ which is totally multicolored and $(\mathcal{P}', \subseteq)$ is isomorphic to

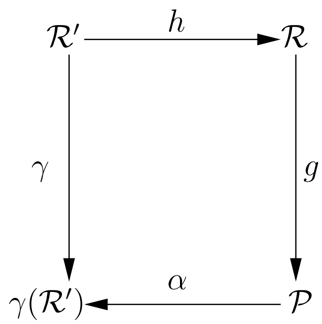

Figure 4.7 The maps used in Lemma 4.3. Recall that each of the maps is a bijection, and that $\alpha = \gamma \circ h^{-1} \circ g^{-1}$.

$(\mathcal{P}, \subseteq)$. Let $q : (\mathcal{P}', \subseteq) \to (\mathcal{P}, \subseteq)$ be an isomorphism. Let $\gamma(\mathcal{P}') = \{\gamma(P) : P \in \mathcal{P}'\}$.

Claim 2. *$(\mathcal{R}, \subseteq)$ is isomorphic to $(\gamma(\mathcal{P}'), \subseteq)$.*

Proof of Claim 2. Let $\beta : \mathcal{R} \to \gamma(\mathcal{P}')$ be given by $\beta(X) = \gamma(q^{-1}(g(X)))$. Let $X, Y \in \mathcal{R}$. By assumption, $g(Y) \subseteq g(X)$ if and only if $X \subseteq Y$. Since q is an isomorphism, $q^{-1}(g(Y)) \subseteq q^{-1}(g(X))$ if and only if $g(Y) \subseteq g(X)$. By definition of γ, $\gamma(q^{-1}(g(X))) \subseteq \gamma(q^{-1}(g(Y)))$ if and only if $q^{-1}(g(Y)) \subseteq q^{-1}(g(X))$, so $\beta(X) \subseteq \beta(Y)$ if and only if $X \subseteq Y$. Since g, q, and γ are all bijections, β is a bijection, so Claim 1 is true.

Now all we need to show is that $\gamma(\mathcal{P}')$ is totally multicolored under χ'. Let $P \in \mathcal{P}'$. Notice that $\chi(P) = \chi'([n] \setminus P) = \chi'(\gamma(R))$. Since for all $P_1, P_2 \in \mathcal{P}'$ with $P_1 \neq P_2$, $\chi(P_1) \neq \chi(P_2)$, $\chi'(\gamma(P_1)) \neq \chi'(\gamma(P_2))$. This gives us that $\gamma(\mathcal{P}')$ is totally multicolored under χ'. This fact together with Claim 2 give $\chi' \to \mathcal{R}$. □

4.3.8 Proof of Proposition 4.6

Proof. Let n be an integer with $n \geq 3$. Lemma 4.3 gives that $f(n, V_2) = f(n, \Lambda_2)$, so we only need to show that $f(n, V_2) = 5$.

Lower bound. Let $Z \in \mathcal{L}_{n-2}$. There exist exactly 2 supersets Z_1 and Z_2 of Z in $\mathcal{L}_{n-1}$. We define a 4-coloring χ of $2^{[n]}$ by

$$\chi(X) = \begin{cases} 1, & \text{if } X = Z; \\ 2, & \text{if } X = Z_1 \text{ or } X = Z_2; \\ 3, & \text{if } X = [n]; \\ 4, & \text{otherwise.} \end{cases}$$

See Figure 4.8 for a pictorial representation of this coloring. We check that χ does not admit a rainbow V_2. Notice that the set Z cannot be the minimal element in any rainbow copy of V_2 since $\chi(Z_1) = \chi(Z_2)$. Since $|Z_1| = |Z_2| = n-1$, neither Z_1 nor Z_2 can be the minimal element in any copy (rainbow or otherwise) of V_2. As such, the minimal element in any rainbow copy of V_2 must have color 4. Since any rainbow antichain of size 2 will necessarily consist of at least one set with color 4, χ does not admit a rainbow copy of V_2.

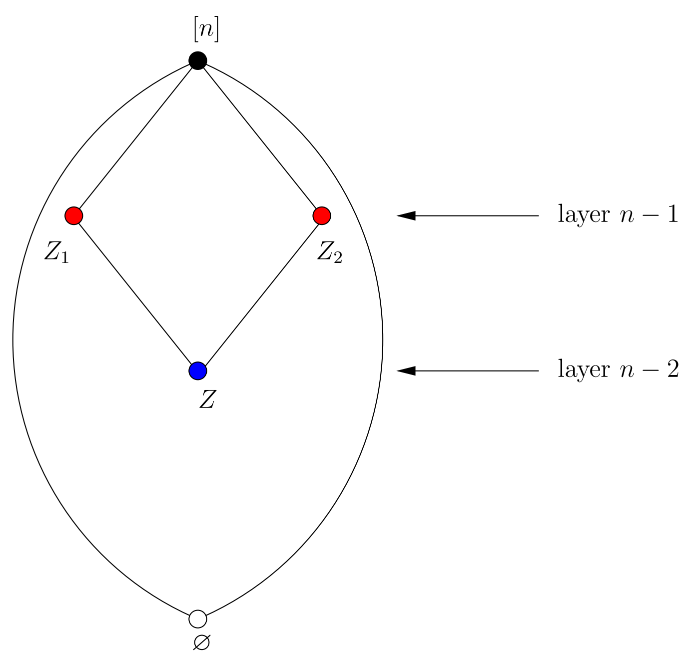

Figure 4.8 A 4-coloring of Q_n with no rainbow V_2. Any subset of $[n]$ not depicted above is colored the same as $\varnothing$.

Upper bound. Let χ be an arbitrary 5-coloring of $2^{[n]}$, and assume $\chi(\varnothing) = 1$. Choose $X \subset [n]$ such that $|X|$ is minimal among those subsets of $[n]$ with $\chi(X) \neq 1$. Without loss of generality, say $\chi(X) = 2$. Let $Y \subset [n]$ with $\chi(Y) \in \{3, 4, 5\}$. If we assume that χ does not admit a rainbow copy of V_2, then we must have $X \subset Y$ by the minimality of $|X|$. Since Y was an arbitrary subset of $[n]$ with $\chi(Y) \in \{3, 4, 5\}$, we have that $X \subset Y$ for every Y with $\chi(Y) \in \{3, 4, 5\}$. Hence, if $\chi(Y) \in \{3, 4, 5\}$, then $Y \in \mathcal{U}_X$.

Recall that $(\mathcal{U}_X, \subseteq)$ is isomorphic to $Q_{n-|X|}$, so if we restrict χ to $\mathcal{U}_X$, we have a 4-coloring of $\mathcal{U}_X$ and by Proposition 4.3, χ admits a rainbow A_2 in $\mathcal{U}_X$.

Claim. *There are two sets* $W, Z \in \mathcal{U}_X$ *with* $W||Z$, $\chi(W) \neq \chi(Z)$, *and either* $\chi(W) \in \{3, 4, 5\}$ *or* $\chi(Z) \in \{3, 4, 5\}$.

Proof of Claim. Since χ is a 5-coloring, there is a set $W \neq [n]$ with $\chi(W) \neq \chi(X)$. Without loss of generality, assume $\chi(W) = 3$. Assume the claim is false, so for all $Z \in \mathcal{U}_X$ with $Z||W$, $\chi(Z) = 3$. This implies that every set in $\mathcal{L}_{|W|} \cap \mathcal{U}_X$ must have color 3. However, if the claim is false, then *every* set in $\mathcal{U}_X$ which is incomparable with *some* set in $\mathcal{L}_{|W|} \cap \mathcal{U}_X$ must have color 3. Notice that for any set $Y \in \mathcal{U}_X \setminus \{X, [n]\}$, there exists a set $Y' \in \mathcal{L}_{|W|} \cap \mathcal{U}_X$ with $Y||Y'$. This gives that every set in $\mathcal{U}_X$ (save X and $[n]$) must have color 3. Hence, χ uses at most 3 colors on $\mathcal{U}_X$, a contradiction, so the Claim is true.

Let W and Z be as in the Claim, and assume $\chi(W) = 3$. We have two cases to consider.

Case 1. $\chi(Z) = 1$ or $\chi(Z) \notin \{1, 2\}$.

Notice that $(\{X, W, Z\}, \subseteq)$ is isomorphic to V_2 and $\{X, W, Z\}$ is totally multicolored, so $\chi \to V_2$.

Case 2. $\chi(Z) = 2$.

Notice that $(\{\varnothing, W, Z\}, \subseteq)$ is isomorphic to V_2 and $\{\varnothing, W, Z\}$ and is totally multicolored, so $\chi \to V_2$.

This gives $f(n, V_2) \leq 5$. □

4.3.9 Proof of Proposition 4.7

Proof. Let n be an integer with $n \geq 4$.

Lower bound. Let χ be the 4-coloring of $2^{[n]}$ defined by

$$\chi(X) = \begin{cases} 1, & \text{if } X = \varnothing; \\ 2, & \text{if } X = \{0\}; \\ 3, & \text{if } X = [n]; \\ 4, & \text{otherwise.} \end{cases}$$

Since $\varnothing$ and $[n]$ are comparable with every other set in $2^{[n]}$, they are not members of any induced copy of $\dot{P}_2$. Since only 2 colors are used on the rest of the sets in $2^{[n]}$, χ does not admit a rainbow $\dot{P}_2$, so $f(n, \dot{P}_2) \geq 5$.

Upper bound. Let χ be a 5-coloring of $2^{[n]}$. We consider three cases.

Case 1. Every layer in Q_n is monochromatic under χ.

Since χ is a 5-coloring of $2^{[n]}$, by the pigeonhole principle there must be 3 integers k, ℓ, and m with $1 \leq k < \ell < m \leq n-1$ such that the layers $\mathcal{L}_k$, $\mathcal{L}_\ell$, and $\mathcal{L}_m$ use different colors. Let $K \in \mathcal{L}_k$ with $K = \{x_1, x_2, \ldots, x_k\}$. Let $M \in \mathcal{L}_m$ with $M = \{x_1, x_2, \ldots, x_k, y_1, \ldots, y_{m-k}\}$. Since $m \neq n$, there is $z \in [n] \setminus M$, so we may take $L \in \mathcal{L}_\ell$ to be $\{z, x_2, \ldots, x_k, y_1, \ldots, y_{\ell-k}\}$. (If $k = 1$, take $L = \{z, y_1, \ldots, y_{\ell-1}\}$.) Since $\{K, L, M\}$ is a multicolored set by assumption and $(\{K, L, M\}, \subseteq)$ is isomorphic to $\dot{P}_2$ by construction, χ admits a rainbow $\dot{P}_2$.

Case 2. There is a layer in Q_n which uses 3 colors under χ.

Let $\mathcal{L}_k$ be such a layer. Without loss of generality, $k \leq \lfloor n/2 \rfloor$. Let $A, B, C \in \mathcal{L}_k$ such that $\chi(A)$, $\chi(B)$, and $\chi(C)$ are all distinct, say $\chi(A) = 1$, $\chi(B) = 2$, and $\chi(C) = 3$. Since $n \geq 4$, there are sets $A', B', C' \in \mathcal{L}_{k+1}$ such that

$$\begin{aligned} A \subset A', &\qquad B || A', \text{ and } C || A'; \\ B \subset B', &\qquad A || B', \text{ and } C || B'; \\ C \subset C', &\qquad A || C', \text{ and } B || C'. \end{aligned}$$

If χ does not admit a rainbow $\dot{P}_2$, we must have $\chi(A') = 1$, $\chi(B') = 2$, and $\chi(C') = 3$.

Let G be the bipartite graph with partite sets $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$. Two vertices are adjacent if and only if the sets to which they correspond are comparable. Since this graph is connected, there is a path $\mathcal{P}$ from A' to B'.

Claim. *If χ does not admit a rainbow $\dot{P}_2$, then every vertex in $\mathcal{P}$ must have color* 1.

Proof of Claim. We first enumerate the elements of $\mathcal{P}$ in order of the path, as $A' = X_1, X_2, \ldots, X_m = B'$. Let ℓ be the least integer $1 \leq \ell \leq m$ such that $\chi(X_\ell) \neq 1$. If $\chi(X_\ell) = 2$, then both $\{X_{\ell-1}, X_\ell, C\}$ and $\{X_{\ell-1}, X_\ell, C'\}$ are totally multicolored. We show that one of $(\{X_{\ell-1}, X_\ell, C\}, \subseteq)$ and $(\{X_{\ell-1}, X_\ell, C'\}, \subseteq)$ must be isomorphic to $\dot{P}_2$. Suppose otherwise, so then C is comparable one of with $X_{\ell-1}$ or X_ℓ and C' is comparable with the other of $X_{\ell-1}$ and X_ℓ. Hence, $(\{X_{\ell-1}, X_\ell, C, C'\}, \subseteq)$ is isomorphic to $\bowtie$. This is a contradiction, as $\{X_{\ell-1}, X_\ell, C, C'\} \subseteq \mathcal{L}_k \cup \mathcal{L}_{k+1}$, and $\bowtie$ is not a subposet of the union of two consecutive layers. Hence, if $\chi(X_\ell) = 2$, then χ admits a rainbow $\dot{P}_2$.

The proof is similar if $\chi(X_\ell) \in \{3, 4, 5\}$; consider the two set families $\{X_{\ell-1}, X_\ell, B\}$ and $\{X_{\ell-1}, X_\ell, B'\}$ and proceed as above. The proof of the Claim is complete.

By the Claim, since $B' \in V(\mathcal{P})$ and $\chi(B') = 2 \neq 1$, χ admits a rainbow $\dot{P}_2$; this concludes Case 2.

Case 3. There is a layer in Q_n which uses 2 colors under χ.
Let $\mathcal{L}_k$ be such a layer, and let $A, B \in \mathcal{L}_k$ with $\chi(A) = 1$ and $\chi(B) = 2$. Since χ is a 5-coloring, one of the colors 3, 4, or 5 must be used on a set $X \in 2^{[n]} \setminus \{\varnothing, [n]\}$. Without loss of generality, assume $\chi(X) = 3$. We have that $X \in \mathcal{L}_m$, with $1 \leq m \leq n-1$ and $m \neq k$.

If χ does not admit a rainbow $\dot{P}_2$, then there are two cases to consider.

Case 3.1. X is comparable with both A and B.
Since $X \notin \{\varnothing, [n]\}$, there exists $Y \in \mathcal{L}_k$ such that $X||Y$. Note that $\chi(Y) \in \{1, 2\}$. If $\chi(Y) = 1$, then $\{X, Y, B\}$ is totally multicolored and $(\{X, Y, B\}, \subseteq)$ is isomorphic to $\dot{P}_2$. If $\chi(Y) = 2$, then $\{X, Y, A\}$ is totally multicolored and $(\{X, Y, A\}, \subseteq)$ is isomorphic to $\dot{P}_2$. Hence, χ admits a rainbow $\dot{P}_2$.

Case 3.2. X is not comparable with A and X is not comparable with B.
Let $Y \in \mathcal{L}_k$ such that Y is comparable with X. Note that $\chi(Y) \in \{1, 2\}$. If $\chi(Y) = 1$, then $\{X, Y, B\}$ is totally multicolored and $(\{X, Y, B\}, \subseteq)$ is isomorphic to $\dot{P}_2$. If $\chi(Y) = 2$, then $\{X, Y, A\}$ is totally multicolored and $(\{X, Y, A\}, \subseteq)$ is isomorphic to $\dot{P}_2$, so χ admits a rainbow $\dot{P}_2$.

Case 3.3. X is comparable with A and not B, or X is comparable with B and not A. This case is easy to handle; since $(\{X, A, B\}, \subseteq)$ is isomorphic to $\dot{P}_2$ and $\{X, A, B\}$ is totally multicolored, χ admits a rainbow $\dot{P}_2$.

Hence, $f(n, \dot{P}_2) \leq 5$ for $n \geq 4$.

To complete the proof of Proposition 4.7, we need to show that $f(3, \dot{P}_2) = 6$.

Lower bound. Define the 5-coloring χ of $2^{[3]}$ by

$$\chi(X) = \begin{cases} 1, & \text{if } X \in \{\{0\}, \{1, 2\}\}; \\ 2, & \text{if } X \in \{\{1\}, \{0, 2\}\}; \\ 3, & \text{if } X \in \{\{2\}, \{0, 1\}\}; \\ 4, & \text{if } X = [3]; \\ 5, & \text{if } X = \varnothing. \end{cases}$$

See Figure 4.9 for a pictorial representation of this coloring, and to see that χ does not admit a rainbow $\dot{P}_2$.

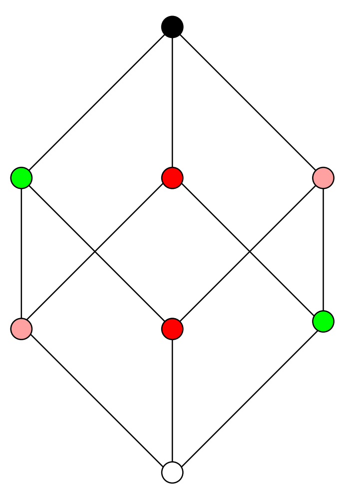

Figure 4.9 A 5-coloring of [3] which shows that $f(3, \dot{P}_2) \geq 6$.

Upper bound. We begin with a Claim.

Claim. *If $\mathcal{F} \subseteq 2^{[3]}$ such that $\varnothing, [3] \notin \mathcal{F}$ and $|\mathcal{F}| \geq 4$, then $\dot{P}_2$ is an induced subposet of $(\mathcal{F}, \subseteq)$.*

Let $\mathcal{F} \subseteq 2^{[3]}$ such that $\varnothing, [3] \notin \mathcal{F}$ and $|\mathcal{F}| \geq 4$. The size of the largest antichain in $\mathcal{F}$ is either 2 or 3.

If it is 3, then $\mathcal{F}$ consists of a layer of Q_3 and one other set F. Since $F \notin \{\varnothing, [n]\}$, there is a set $F' \in \mathcal{F}$ such that $F'||F$. Since $\mathcal{F}$ contains an entire layer of $2^{[3]}$, there is a set F'' which is comparable with F; and $(\{F, F', F''\}, \subseteq)$ is isomorphic to $\dot{P}_2$.

If the size of the largest antichain in $\mathcal{F}$ is 2, then $\mathcal{F}$ consists of two subsets of cardinality 2 (say F and F') and two subsets of cardinality 1 (say G and G'). Without loss of generality, $F \supset G$ and $F' \supset G'$. Since $F \neq F'$, at least one of F and F' is not equal to $G \cup G'$; say F. This gives that $(\{F, G, G'\}, \subseteq)$ is isomorphic to $\dot{P}_2$, and hence the Claim is true.

If χ is a 6-coloring of $2^{[3]}$, at least 4 colors must be used the sets in $2^{[3]} \setminus \{\varnothing, [3]\}$. By the Claim, χ admits a rainbow $\dot{P}_2$, so $f(3, \dot{P}_2) \leq 6$. □

4.3.10 Proof of Proposition 4.8

Proof. Let n be an integer with $n \geq 4$. Let $m = N + \binom{n}{\lfloor n/2 \rfloor + 1}$. We will construct an $(m+3)$-coloring χ of $2^{[n]}$ which does not admit a rainbow $\bowtie$. We begin by enumerating the elements of $\mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1}$; that is, write

$$\mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1} = \{F_1, F_2, \ldots, F_m\}.$$

Define $\chi : 2^{[n]} \to \{1, 2, \ldots, m+3\}$ by

$$\chi(X) = \begin{cases} i, & \text{if } X = F_i \text{ for some } i \in \{1, 2, \ldots, m\}; \\ m+1, & \text{if } X = \varnothing; \\ m+2, & \text{if } X = [n]; \\ m+3, & \text{if } X \notin \mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1} \cup \{\varnothing, [n]\}. \end{cases}$$

Assume by way of contradiction that $\chi \to \bowtie$. This means there exists a totally multicolored collection of 4 sets $\mathcal{F}$ with $(\mathcal{F}, \subseteq)$ isomorphic to $(\bowtie, \subseteq)$. Since both $\varnothing$ and $[n]$ are comparable with every subset of $[n]$, we have $\varnothing, [n] \notin \mathcal{F}$. Hence, $\mathcal{F}$ contains at least 3 sets from $\mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1}$. Call these sets F_1, F_2, and F_3. Furthermore, we must have $(\{F_1, F_2, F_3\}, \subseteq)$

isomorphic to either $(V_2, \subseteq)$ or $(\Lambda_2, \subseteq)$. This gives us that two of the sets from $\{F_1, F_2, F_3\}$ are in the same layer; without loss of generality, say $F_1, F_2 \in \mathcal{L}_{\lfloor n/2 \rfloor + 1}$, and hence $F_3 = F_1 \cap F_2$.

Since $(\bowtie, \subseteq)$ is not a subposet (induced or otherwise) of $\left(\mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1}, \subseteq\right)$, there is a set $F \in \mathcal{F} \setminus \mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1}$. Furthermore, F must be a subset of both F_1 and F_2, but not of F_3. However, since $F_3 = F_1 \cap F_2$, this is not possible. Hence, χ does not admit a rainbow $\bowtie$, so $f(n, \bowtie) \geq m + 4$, as desired. □

4.3.11 Proof of Proposition 4.9

To get an idea as to how large $ex(n, V_4)$ is, we direct the reader to [18], where we find that the best known asymptotic upper bound for $ex(n, V_4)$ is $\binom{n}{\lfloor n/2 \rfloor}\left(1 + \frac{6}{n} + O\left(\frac{1}{n^2}\right)\right)$.

What Proposition 4.9 shows us is that we have the correct leading term in the asymptotic for $f(n, Q_2)$, and any future work would be toward reducing the error term. To prove Proposition 4.9, we first need a Lemma 4.5.

Lemma 4.5. *If χ is a 6-coloring of $2^{[n]}$ with $\chi(\varnothing) \neq \chi([n])$, then χ admits a rainbow 2^2.*

Proof of Lemma 4.5. Let $\chi : 2^{[n]} \to [6]$ be a 6-coloring, with $\chi(\varnothing) \neq \chi([n])$. Say $\chi(\varnothing) = 1$ and $\chi([n]) = 2$. As χ is a 6-coloring, there exist $W, X, Y, Z \in Q_n$ such that $\chi(W) = 3, \chi(X) = 4, \chi(Y) = 5$, and $\chi(Z) = 6$. If any pair of W, X, Y, and Z are *not* comparable (say $X||Y$), then $\{\varnothing, X, Y, [n]\}$ is totally multicolored and $(\{\varnothing, X, Y, [n]\}, \subseteq)$ is isomorphic to Q_2, so we may assume that $W \subset X \subset Y \subset Z$. Let $z \in Z \setminus Y$. Consider the set $W' = W \cup \{z\}$. Since $\{W', X\}$ and $\{W', Y\}$ are antichains, $\chi(W') \in \{1, 2, 4, 5\}$. Since $(\{W, X, W', Z\}, \subseteq)$ and $(\{W, Y, W', Z\}, \subseteq)$ are isomorphic to Q_2, $\chi(W') \notin \{1, 2\}$, so $\chi(W') \in \{4, 5\}$. If $\chi(W') = 4$, then $\{W, Y, W', Z\}$ is totally multicolored and $(\{W, Y, W', Z\}, \subseteq)$ is isomorphic to Q_2, and if $\chi(W') = 5$, then $\{W, X, W', Z\}$ is totally multicolored and $(\{W, X, W', Z\}, \subseteq)$ is isomorphic to Q_2.

We direct the reader to Figure 4.10 for a pictorial representation of this situation. □

Note that 6 colors is the best we can do in Lemma 4.5 above. We construct a 5-coloring χ of $2^{[n]}$ with $\chi(\varnothing) \neq \chi([n])$ which does not admit a rainbow Q_2 by coloring the downset of

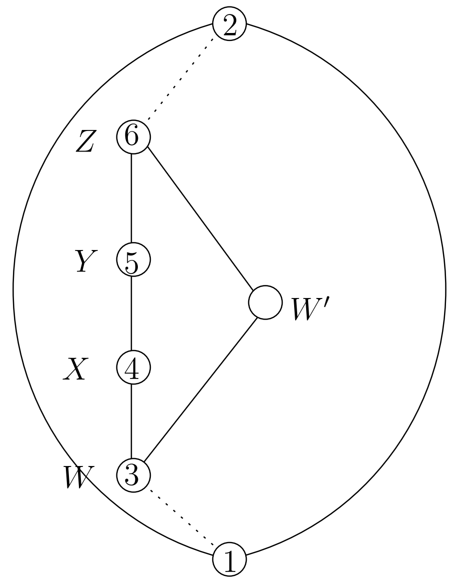

Figure 4.10 The situation described in Lemma 4.5

$\{0, 1, 2\}$ in 4 colors, and giving every other set the fifth color. Specifically, for $X \in 2^{[n]}$, let

$$\chi(X) = \begin{cases} 1 & \text{if } X \in \{\varnothing, \{1\}, \{2\}, \{1,2\}\}; \\ 2 & \text{if } X = \{0\}; \\ 3 & \text{if } X \in \{\{0,1\}, \{0,2\}\}; \\ 4 & \text{if } X = \{0,1,2\}; \\ 5 & \text{if } X \notin \mathcal{D}_{\{0,1,2\}}. \end{cases}$$

See Figure 4.11 for a pictorial representation of the coloring χ restricted $\mathcal{D}_{\{0,1,2\}}$. To see that this coloring does not admit a rainbow Q_2, notice first that $\mathcal{D}_{\{0,1,2\}}$ does not contain an induced subposet isomorphic to Q_2 which is totally multicolored. Hence, any induced copy of Q_2 which is totally multicolored under χ must contain a subset of $[n]$ which is *not* a subset of $\{0, 1, 2\}$. However, since any such induced copy of Q_2 would contain a superset of $\{0, 1, 2\}$, such a Q_2 necessarily has two elements A and B with $\chi(A) = \chi(B) = 5$. Hence, χ does not admit a rainbow Q_2 and the bound in Lemma 4.5 is sharp.

We are now ready to prove Proposition 4.9.

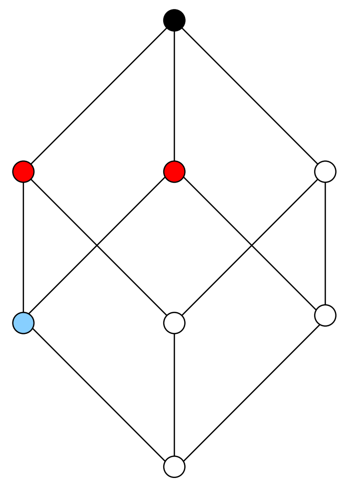

Figure 4.11 The coloring of the downset of $\{0, 1, 2\}$ in Q_n. Every other element in $2^{[n]}$ gets the fifth color.

Proof of Proposition 4.9. **Lower bound.** We define a $(N+1)$-coloring χ by first enumerating the sets of $\mathcal{L}_{\lfloor n/2 \rfloor}$; that is, write $\mathcal{L}_{\lfloor n/2 \rfloor} = \{F_1, \ldots, F_N\}$. Define $\chi : 2^{[n]} \to \{1, \ldots, N+1\}$ by

$$\chi(X) = \begin{cases} i, & \text{if } X = F_i \text{ for some } i \in \{1, \ldots, N\}; \\ N+1, & \text{otherwise.} \end{cases}$$

There can be no rainbow copy of $2^{[2]}$ (induced or otherwise) under this coloring, since every 3-chain will contain at least 2 sets colored the same.

Upper bound. Suppose χ is a $(ex(n, V_4) + 2)$-coloring of $2^{[n]}$. Without loss of generality, assume $\chi([n]) = 1$. This gives us a totally multicolored family $\mathcal{F}$ of size $ex(n, V_4) + 1$ such that for all $X \in \mathcal{F}$, $\chi(X) \neq 1$ and which does not contain $[n]$. If $\mathcal{F}$ contains an induced copy of V_2, then this copy of V_2 together with $[n]$ forms a induced rainbow copy of $2^{[2]}$. If $\mathcal{F}$ does not contain an induced copy of $2^{[2]}$, then as $|\mathcal{F}| = ex(n, V_4) + 1$ must contain a 5-chain. Suppose X is the bottom element of this 5-chain. Consider the upset $\mathcal{U}_X$ of X and the coloring χ' of $\mathcal{U}_X$ inherited from the coloring χ. This coloring uses at least 6 colors and satisfies the hypotheses of Lemma 4.5. Hence, χ' admits a rainbow copy of $2^{[2]}$ in $\mathcal{U}_X$ and thus χ admits a rainbow copy of $2^{[2]}$ in Q_n. $\square$

4.3.12 Proof of Proposition 4.10

We have one more definition for a particular kind of subset of $\{0, 1, \dots, n-1\}$ which we will use to prove Proposition 4.10. A *circular interval* is a subset of $[n]$ of the form $\{k, k+1, \dots, \ell\}$ where $0 \le k, \ell \le n-1$ and addition is taken modulo n. We will call k the *starting point* of the interval. A *cyclic permutation* of the set $[n]$ is a permutation consisting of exactly one cycle.

Before providing the proofs of these Propositions, we state and prove a lemma which counts the size of a family $\mathcal{F} \subseteq 2^{[n]}$ by considering the number of sets in $\mathcal{F}$ which are intervals under a given cyclic permutation of $[n]$.

Lemma 4.6. *Let n be a positive integer and let $\mathcal{F} \subseteq 2^{[n]}$. If there is a real number c such that for each cyclic permutation σ of $\{0, 1, \dots, n-1\}$ the number of sets in $\mathcal{F}$ which are intervals under σ is bounded above by cn, then $|\mathcal{F}| \le cN$.*

Proof of Lemma 4.6. Let

$$\mathcal{M} = \{(\sigma, X) : \ \sigma \text{ is a cyclic permutation of } [n],\ X \in \mathcal{F} \text{ is an interval under } \sigma\}.$$

The number of cyclic permutations of $[n]$ is $(n-1)!$. By assumption, the number of sets in $\mathcal{F}$ which are intervals under any cyclic permutation is at most cn. This gives that

$$|\mathcal{M}| \le cn \cdot (n-1)! = c(n!).$$

Since a set $X \in \mathcal{F}$ is an interval in $|X|!(n-|X|)! \ge \left\lfloor \frac{n}{2} \right\rfloor! \left\lceil \frac{n}{2} \right\rceil!$ cyclic permutations, we have

$$|\mathcal{F}| \le \frac{|\mathcal{M}|}{\lfloor n/2 \rfloor! \lceil n/2 \rceil!} \le \frac{cn!}{\lfloor n/2 \rfloor! \lceil n/2 \rceil!} = cN.$$

□

We are now ready to present the proof of Proposition 4.10.

Proof of Propositon 4.10. Let n be a positive integer.

Lower bound. Let $\mathcal{F} = \mathcal{L}_{\lfloor n/2 \rfloor} \cup \mathcal{L}_{\lfloor n/2 \rfloor + 1}$. Notice that Q_2 is not a subposet of $(\mathcal{F}, \subseteq)$, since the height of Q_2 is 3, while the height of $(\mathcal{F}, \subseteq)$ is 2.

Upper bound. Let $\mathcal{F}$ be a family of intervals of $[n]$ which is Q_2-free.

If $i \in [n]$ and $\mathcal{F}$ contains three intervals with starting point i, then we will say $\mathcal{F}$ has a *triple at* i. Let

$$T = T(\mathcal{F}) = \{i : \mathcal{F} \text{ has a triple at } i\}$$

and

$$T' = \{(i, j) : i, j \in T;\ \nexists k \in T \text{ with } |k - i| \leq |j - i|\};$$

that is, T' consists of ordered pairs whose elements are consecutive (modulo n) elements from T.

Since every set in $\mathcal{F}$ is an interval and there are n starting points for these intervals together with the fact that $\mathcal{F}$ is Q_2-free and hence does not have 4 intervals with the same starting point, we have $|\mathcal{F}| \leq 3|T| + 2(n - |T|) = |T| + 2n$.

Assume $|T| \geq 2$, else there is nothing to prove.

For $k \in T$, We will denote the cardinalities of the intervals starting at k by s_k, m_k and ℓ_k so that $s_k < m_k < \ell_k$. That is, s_k is the size of the minimal interval with starting point k and ℓ_k is the size of the maximal interval with starting point k. We will call m_k the size of the *middle interval* with starting point k.

Consider the set $[n]$ arranged on a circle. (See Figure 4.12 for an example using $n = 12$.) Let $|j - i|$ denote the *clockwise distance* from i to j.

Let $i, j \in T$. We must have $\ell_j \leq \ell_i - (|j - i| - 1)$ and $s_j \leq s_i - (|j - i| - 1)$, otherwise $\mathcal{F}$ contains a P_4 and is hence not Q_2-free. To see this, notice that should either of these inequalities fail to hold, $\mathcal{F}$ is not Q_2-free by taking the three intervals with starting point i together with the maximal interval with starting point j in the first case, and the three intervals with starting point i and the minimal interval with starting point j in the second case.

Claim. *If* $(i, j) \in T'$ *then*

$$(s_j - s_i) + (m_j - m_i) + (\ell_j - \ell_i) \geq 2 - 3(|j - i| - 1).$$

Proof of Claim 1. We consider two cases.

Case 1. $s_j \geq m_i - (|j - i| - 1)$.

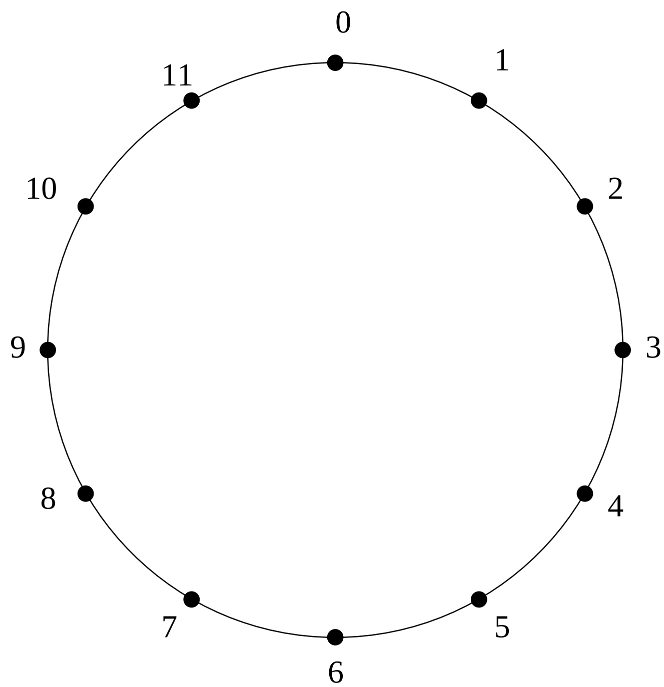

Figure 4.12 The elements of [12] arranged on a circle. The clockwise distance from 10 to 11 is $|11-10| = 1$, whereas the clockwise distance from 11 to 10 is $|10-11| = 11$. The clockwise distance from i to j is simply the equivalence class of the difference $j-i$ modulo n.

In this case, the above inequality implies that the minimal interval starting at j is not contained in the middle interval starting at i. This gives

$$s_i < m_i \leq s_j + (|j-i|-1) < m_j + (|j-i|-1),$$

which gives

$$\begin{aligned} s_j - s_i + (|j-i|-1) &\geq 1 \\ m_j - m_i + (|j-i|-1) &\geq 1. \end{aligned}$$

This gives

$$(s_j - s_i) + (m_j - m_i) + (\ell_j - \ell_i) \geq 2 - 3(|j-i|-1),$$

as desired.

Case 2. $s_j < m_i - (|j-i|-1)$.

Since $\mathcal{F}$ is Q_2-free, we have $m_j \geq \ell_i - (|j-i|-1)$. This implies

$$m_i < \ell_i \leq m_j + (|j-i|-1) < \ell_j.$$

This gives

$$\begin{aligned} m_j - m_i + (|j-i|-1) &\geq 1 \\ \ell_j - \ell_i + (|j-i|-1) &\geq 1. \end{aligned}$$

This gives

$$(s_j - s_i) + (m_j - m_i) + (\ell_j - \ell_i) \geq 2 - 3(|j-i|-1),$$

completing the proof of the claim. □

Now we can show that $|\mathcal{F}| \leq 2.6n$. Notice that since the sum is telescoping,

$$Q = \sum_{(i,j)\in T'} (s_j - s_i) + (m_j - m_i) + (\ell_j - \ell_i) = 0.$$

This fact together with the claim gives

$$\begin{aligned} 0 = Q &\geq \left[\sum_{(i,j)\in T'} 2 - 3(|j-i|-1)\right] \\ &= \sum_{(i,j)\in T'} 2 + 3 - \sum_{(i,j)\in T'} 3(|j-i|) \\ &= 2|T'| + 3|T'| - 3n \\ &= 5|T| - 3n, \end{aligned}$$

where $\sum_{(i,j)\in T'} |j-i|$ is taken modulo n (and hence the sum is 0).

This gives that $|T| \leq \frac{3}{5}n$. We have

$$|\mathcal{F}| \leq 3|T| + 2(n - |T|) = 3 \cdot \frac{3}{5}n + 2 \cdot \frac{2}{5}n = 2.6n.$$

To finish the proof, let $\mathcal{G} \subseteq 2^{[n]}$ which is Q_2-free. Let σ be an arbitrary cyclic permutation of $[n]$. The argument above shows that $\mathcal{G}$ can have at most at most $2.6n$ intervals under σ. By Lemma 4.6, $|\mathcal{G}| \leq 2.6N$. Since $\mathcal{G}$ is an arbitrary family of subsets of $[n]$ which is Q_2-free, $ex(n, Q_2) \leq 2.6N$, as desired.

□

4.3.13 Proof of Proposition 4.11

The proof is due to Axenovich, Manske, and Martin, and appears in [6].

Let n be a positive integer.

We begin with a technical lemma, which shows that we may assume that the majority of the family $\mathcal{F}$ is "near" the middle layer of the Boolean lattice.

Lemma 4.7. *If n is a positive integer,*

$$\sum_{|k-\lfloor n/2\rfloor|\geq n^{2/3}} \binom{n}{k} \leq 2^{n-\Omega(n^{1/3})} = 2^{-\Omega(n^{1/3})}N.$$

Proof of Lemma 4.7. Let n be a positive integer and set

$$x = 2^{-n} \sum_{|k-n/2|\geq n^{2/3}} \binom{n}{k}.$$

Notice that x is the probability that a $B(n, 1/2)$ binomial random variable, X, takes on values outside of the interval $\left(n/2 - n^{2/3}, n/2 + n^{2/3}\right)$. Using a standard Chernoff bound,

$$\begin{aligned} \Pr\left(|X - \lfloor n/2\rfloor| \geq \delta\lfloor n/2\rfloor\right) &\leq 2\exp\left\{-\lfloor n/2\rfloor\delta^2/2\right\} \\ \Pr\left(|X - \lfloor n/2\rfloor| \geq n^{2/3}\right) &\leq 2\exp\left\{-n^{1/3}\right\} \\ \sum_{|k-\lfloor n/2\rfloor|\geq n^{2/3}} \binom{n}{k} &\leq 2^{n+1}e^{-n^{1/3}}. \end{aligned}$$

Since $N = \Omega(n^{-1/2})2^n$, we may conclude that $\sum_{|k-\lfloor n/2\rfloor|\geq n^{2/3}} \binom{n}{k} \leq 2^{-\Omega(n^{1/3})}N.$ □

First we will show that Proposition 4.11 is true for a family $\mathcal{F}$ which is contained in three consecutive layers of Q_n, then show it is true for any Q_2-free family satisfying the hypotheses of Proposition 4.11. Suppose $\mathcal{F}$ is a Q_2-free family from 3 consecutive layers, $\mathcal{L}_{k-1}, \mathcal{L}_k$, and $\mathcal{L}_{k+1}$. Let

$$\begin{aligned} \mathcal{S} &= \mathcal{F} \cap \mathcal{L}_{k-1}, \\ \mathcal{T} &= \mathcal{F} \cap \mathcal{L}_k, \text{ and} \\ \mathcal{U} &= \mathcal{F} \cap \mathcal{L}_{k+1}. \end{aligned}$$

Let Υ be the set of 3-element chains contained in $\mathcal{L}_{k-1} \cup \mathcal{L}_k \cup \mathcal{L}_{k+1}$, and for $i \in \{0, 1, 2, 3\}$, define $\Upsilon_i = \{\sigma \in \Upsilon : |\sigma \cap \mathcal{F}| = i\}$.

Let

$$\mathcal{M} = \{(F, \sigma) : \quad F \in \mathcal{F}, \quad \sigma \in \Upsilon, \quad F \in \sigma\}.$$

Then

$$|\mathcal{M}| = 3|\Upsilon_3| + 2|\Upsilon_2| + |\Upsilon_1| = 2|\Upsilon| + |\Upsilon_3| - |\Upsilon_1| - 2|\Upsilon_0| \leq 2|\Upsilon| + |\Upsilon_3| - |\Upsilon_1|.$$

On the other hand,

$$|\mathcal{M}| = (k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}|.$$

Putting together these expressions for $|\mathcal{M}|$ and using the fact that $|\Upsilon| = \binom{n}{k} k(n-k)$, we have

$$(k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}| \leq 2\binom{n}{k} k(n-k) + |\Upsilon_3| - |\Upsilon_1|. \tag{4.3}$$

Later, we shall use Lemma 4.7 to show that the left hand side of the inequality in (4.3) is approximately $\frac{n^2}{4}(|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}|) = \frac{n^2}{4}|\mathcal{F}|$, which will allow us to obtain an upper bound on the size of the family $\mathcal{F}$.

For $X \in \mathcal{L}_{k-1}$, $Y \in \mathcal{L}_k$, and $Z \in \mathcal{L}_{k+1}$, define

$$g(Z) = |\{T \in \mathcal{T} : Z \supset T\}|; \qquad h(X) = |\{T \in \mathcal{T} : X \subset T\}|;$$

$$\breve{g}(Y) = |\{U \in \mathcal{U} : U \supset Y\}|; \qquad \breve{h}(Y) = |\{S \in \mathcal{S} : S \subset Y\}|.$$

Note that we have

$$\sum_{X \in \mathcal{S}} h(X) = \sum_{Y \in \mathcal{T}} \breve{h}(Y)$$

and

$$\sum_{Z \in \mathcal{U}} g(Z) = \sum_{Y \in \mathcal{T}} \breve{g}(Y).$$

Next, we find a bound on $|\Upsilon_3| - |\Upsilon_1|$ by counting the chains that contain an element of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{U}$, then counting the chains containing an element of $\mathcal{T}$, $\mathcal{L}_{k-1} \setminus \mathcal{S}$, $\mathcal{L}_{k+1} \setminus \mathcal{U}$.

$$\begin{aligned}
|\Upsilon_3| - |\Upsilon_1| &\leq \sum_{Y \in \mathcal{T}} \left[\breve{h}(Y)\breve{g}(Y) - \left(k - \breve{h}(Y)\right)(n - k - \breve{g}(Y)) \right] \\
&= \sum_{Y \in \mathcal{T}} \left[(n-k)\breve{h}(Y) + k\breve{g}(Y) - k(n-k) \right] \\
&= (n-k) \sum_{X \in \mathcal{S}} h(X) + k \sum_{Z \in \mathcal{U}} g(Z) - |\mathcal{T}|k(n-k). \qquad (4.4)
\end{aligned}$$

The next Lemma provides a bound on $\sum_{X \in \mathcal{S}} h(X)$ and $\sum_{Z \in \mathcal{U}} g(Z)$ in terms of $|\mathcal{S}|$ and $|\mathcal{U}|$.

Lemma 4.8.

$$\begin{aligned}
\sum_{X \in \mathcal{S}} h(X) &\leq \sqrt{|\mathcal{S}|(N - |\mathcal{U}| + 1)}(k+1), \\
\sum_{Z \in \mathcal{U}} g(Z) &\leq \sqrt{|\mathcal{U}|(N - |\mathcal{S}| + 1)}(n-k+1).
\end{aligned}$$

Proof of Lemma 4.8. For each $X \in \mathcal{L}_{k-1}$, $|\{U \in \mathcal{U} : X \subset U\}| \leq \binom{n-k+1}{2}$. Since $\mathcal{F}$ is Q_2-free, for each $X \in \mathcal{S}$,

$$|\{U \in \mathcal{U} : X \subset U\}| \leq \binom{n-k+1}{2} - \binom{h(X)}{2}.$$

Also, for each $U \in \mathcal{L}_{k+1}$, $|\{X \in \mathcal{S} : X \subset U\}| \leq \binom{k+1}{2}$, and for each $U \in \mathcal{U}$,

$$|\{X \in \mathcal{S} : X \subset U\}| \leq \binom{k+1}{2} - \binom{g(X)}{2}.$$

Thus,

$$
\begin{aligned}
|\mathcal{U}| &\leq \binom{k+1}{2}^{-1} \sum_{X \in \mathcal{L}_{k-1}} |\{U \in \mathcal{U} : X \subset U\}| \\
&= \binom{k+1}{2}^{-1} \left(\sum_{X \in \mathcal{S}} |\{U \in \mathcal{U} : X \subset U\}| + \sum_{X \in \mathcal{L}_{k-1} - \mathcal{S}} |\{U \in \mathcal{U} : X \subset U\}| \right) \\
&\leq \binom{k+1}{2}^{-1} \left(\sum_{X \in \mathcal{S}} \left(\binom{k+1}{2} - \binom{h(X)}{2} \right) + \sum_{X \in \mathcal{L}_{k-1} \setminus \mathcal{S}} \binom{k+1}{2} \right) \\
&= \binom{k+1}{2}^{-1} \left(|\mathcal{L}_{k-1}| \binom{k+1}{2} - \sum_{X \in \mathcal{S}} \binom{h(X)}{2} \right) \\
&\leq |\mathcal{L}_{k-1}| - \binom{k+1}{2}^{-1} \sum_{X \in \mathcal{S}} (h(X) - 1)^2/2 \\
&\leq N - \binom{k+1}{2}^{-1} \frac{1}{2|\mathcal{S}|} \left(\sum_{X \in \mathcal{S}} (h(X) - 1) \right)^2 \qquad (4.5) \\
&\leq N - \binom{k+1}{2}^{-1} \frac{1}{2|\mathcal{S}|} \left(\sum_{X \in \mathcal{S}} h(X) \right)^2 + \frac{1}{2} \binom{k+1}{2}^{-1}.
\end{aligned}
$$

Notice that (4.5) follows from Jensen's Inequality. This gives

$$
\left(\sum_{X \in \mathcal{S}} h(X) \right)^2 \leq |\mathcal{S}|(N - |\mathcal{U}| + 1) \binom{k+1}{2} 2, \qquad (4.6)
$$

and, by a symmetric argument,

$$
\left(\sum_{Z \in \mathcal{U}} g(Z) \right)^2 \leq |\mathcal{U}|(N - |\mathcal{S}| + 1) \binom{n-k+1}{2} 2. \qquad (4.7)
$$

Taking square roots of both sides of the inequalities in (4.6) and (4.7) completes the proof of Lemma 4.8. □

Returning to (4.3) and using the bound from (4.4) and Lemma 4.8, we have:

$$
\begin{aligned}
&(k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}| \\
&\quad \leq 2\binom{n}{k} k(n-k) + |\Upsilon_3| - |\Upsilon_1| \\
&\quad \leq 2Nk(n-k) + (n-k) \sum_{X \in \mathcal{S}} h(X) + k \sum_{Z \in \mathcal{U}} g(Z) - |\mathcal{T}|k(n-k) \\
&\quad \leq 2Nk(n-k) + (n-k)\sqrt{|\mathcal{S}|(N - |\mathcal{U}| + 1)}(k+1) + k\sqrt{|\mathcal{U}|(N - |\mathcal{S}| + 1)}(n-k+1) - |\mathcal{T}|k(n-k).
\end{aligned}
$$

If we set $Q = (k+1)k|\mathcal{U}| + k(n-k)|\mathcal{T}| + (n-k+1)(n-k)|\mathcal{S}|$, we may write

$$Q \le (2N-|\mathcal{T}|)k(n-k)+(n-k)\sqrt{|\mathcal{S}|(N-|\mathcal{U}|+1)}(k+1)+k\sqrt{|\mathcal{U}|(N-|\mathcal{S}|+1)}(n-k+1). \quad (4.8)$$

By Lemma 4.7, we may assume that $\lfloor n/2 \rfloor - n^{2/3} \le k \le \lfloor n/2 \rfloor - n^{2/3}$, so we have that

$$\begin{aligned} \frac{(k+1)k}{n^2/4} &= 1 + o(1), \\ \frac{k(n-k)}{n^2/4} &= 1 + o(1), \text{ and} \\ \frac{(n-k+1)(n-k)}{n^2/4} &= 1 + o(1). \end{aligned}$$

Dividing all terms in inequality (4.8) by $n^2/4$, we have

$$|\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| \le 2N - |\mathcal{T}| + \sqrt{|\mathcal{S}|(N - |\mathcal{U}|)} + \sqrt{|\mathcal{U}|(N - |\mathcal{S}|)} + o(N). \quad (4.9)$$

Thus, by bringing $|\mathcal{T}|$ to the left side of the inequality in (4.9) and adding $|\mathcal{U}| + |\mathcal{S}|$ to both sides, we see

$$\begin{aligned} 2|\mathcal{U}| + 2|\mathcal{T}| + 2|\mathcal{S}| &\le 2N + \left(\sqrt{|\mathcal{S}|(N - |\mathcal{U}|)} + \sqrt{|\mathcal{U}|(N - |\mathcal{S}|)}\right) + |\mathcal{U}| + |\mathcal{S}| + o(N) \\ &\le (3 + \sqrt{2})N + o(N). \quad (4.10) \end{aligned}$$

The inequality in (4.10) is obtained by maximizing the function

$$f(u, s) = 2 + \sqrt{s(1-u)} + \sqrt{u(1-s)} + u + s, \text{ where } 0 \le u, s \le 1.$$

The maximum occurs when $s = u = (2 + \sqrt{2})/4$. Thus

$$|\mathcal{F}| = |\mathcal{U}| + |\mathcal{T}| + |\mathcal{S}| \le \frac{3+\sqrt{2}}{2} N + o(N) \le 2.20711N + o(N),$$

and hence Proposition 4.11 is true if $\mathcal{F}$ is contained in 3 consecutive layers of Q_n.

Next, we move to the more general setting. Let $\mathcal{F} \subseteq 2^{[n]}$ be a Q_2-free family as in the hypotheses of Proposition 4.11; that is, $\mathcal{F}$ has the property that $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$, where $\mathcal{S}$ is the collection of minimal elements in $\mathcal{F}$, $\mathcal{U}$ is the collection of maximal elements in $\mathcal{F} \setminus \mathcal{S}$, and $\mathcal{T}$ is the set of the remaining elements such that $\mathcal{T} \subseteq \mathcal{L}_k$, and that for any $S \in \mathcal{S}$, $|S| < k$ and for any $U \in \mathcal{U}$, $|U| > k$.

We see that $\mathcal{S}, \mathcal{T}$, and $\mathcal{U}$ are antichains. By Lemma 4.7, we may assume that for any element $F \in \mathcal{F}$, $n/2 - n^{2/3} \leq |F| \leq n/2 + n^{2/3}$. Let $k' = \lceil n/2 - n^{2/3} \rceil$, let k'' be the integer greater than $n/2$ such that $\binom{n}{k'} = \binom{n}{k''}$. The argument of Erdős using Menger's theorem, see either [23] or the proof of Lemma 4.1 in Section 4.1, states that there is a collection of disjoint chains $\mathcal{P}_1, \ldots, \mathcal{P}_q$, joining all elements of $\mathcal{L}_{k'}$ to elements in $\mathcal{L}_{k''}$.

We create a new family $\mathcal{F}'$ from $\mathcal{F}$ such that $\mathcal{F}' \subseteq \mathcal{L}_{k-1} \cup \mathcal{L}_k \cup \mathcal{L}_{k+1}$, $|\mathcal{F}| \leq |\mathcal{F}'| + o(N)$, and which is Q_2-free if and only if the original family $\mathcal{F}$ is Q_2-free. Informally, we keep $\mathcal{T}$ unchanged, and shift $\mathcal{S}$ and $\mathcal{U}$ to the layers directly below $\mathcal{T}$ and above $\mathcal{T}$, respectively, along each chain P_i. Formally, let

$$\mathcal{S}' = \{\mathcal{P}_i \cap \mathcal{L}_{k-1} : \text{ there is } S \in \mathcal{S} \cap \mathcal{P}_i, i = 1, \ldots, q\},$$

$$\mathcal{U}' = \{\mathcal{P}_i \cap \mathcal{L}_{k+1} : \text{ there is } U \in \mathcal{U} \cap \mathcal{P}_i, i = 1, \ldots, q\},$$

and set

$$\mathcal{F}' = \mathcal{S}' \cup \mathcal{T} \cup \mathcal{U}'.$$

For a pictorial representation of this construction, see Figure 4.13.

We need to show that $|\mathcal{F}| \leq |\mathcal{F}'| + o(N)$, and that $\mathcal{F}'$ is Q_2-free. Observe that any antichain contains at most $N - \binom{n}{\lceil n/2 + n^{2/3} \rceil} = o(N)$ elements not in $\bigcup_{i=1}^{q} P_i$. Thus, only $o(N)$ elements of $\mathcal{S}$ and $\mathcal{U}$ were not used to create elements in $\mathcal{S}'$ and $\mathcal{U}'$. This gives us $|\mathcal{F}| \leq |\mathcal{F}'| + o(N)$.

Assume by way of contradiction that $\mathcal{F}'$ contains a copy of Q_2. It must have 4 sets T, T', S', U', where $T, T' \in \mathcal{T}$, $U' \in \mathcal{U}'$, $S' \in \mathcal{S}'$. Then we have that there is $S \in \mathcal{S}$, $S \subseteq S'$ and $U \in \mathcal{U}$, $U \supseteq U'$. Thus $(\{S, T, T', U\}, \subseteq)$ is isomorphic to Q_2 and $\{S, T, T', U\} \subset \mathcal{F}$, a contradiction. Since $\mathcal{F}'$ is a Q_2-free family in three consecutive layers, we have that

$$|\mathcal{F}| \leq |\mathcal{F}'| + o(N) \leq 2.20711N + o(N),$$

completing the proof of Proposition 4.11. □

4.4 Further research

Naturally, the first open questions are to fill in any empty spots in Table 4.1. However, there are other questions as well. For $\mathcal{H} \subseteq 2^{[n]}$, define $\pi(\mathcal{H}) = \lim_{n \to \infty} \frac{ex(n, \mathcal{H})}{N}$. It is not known

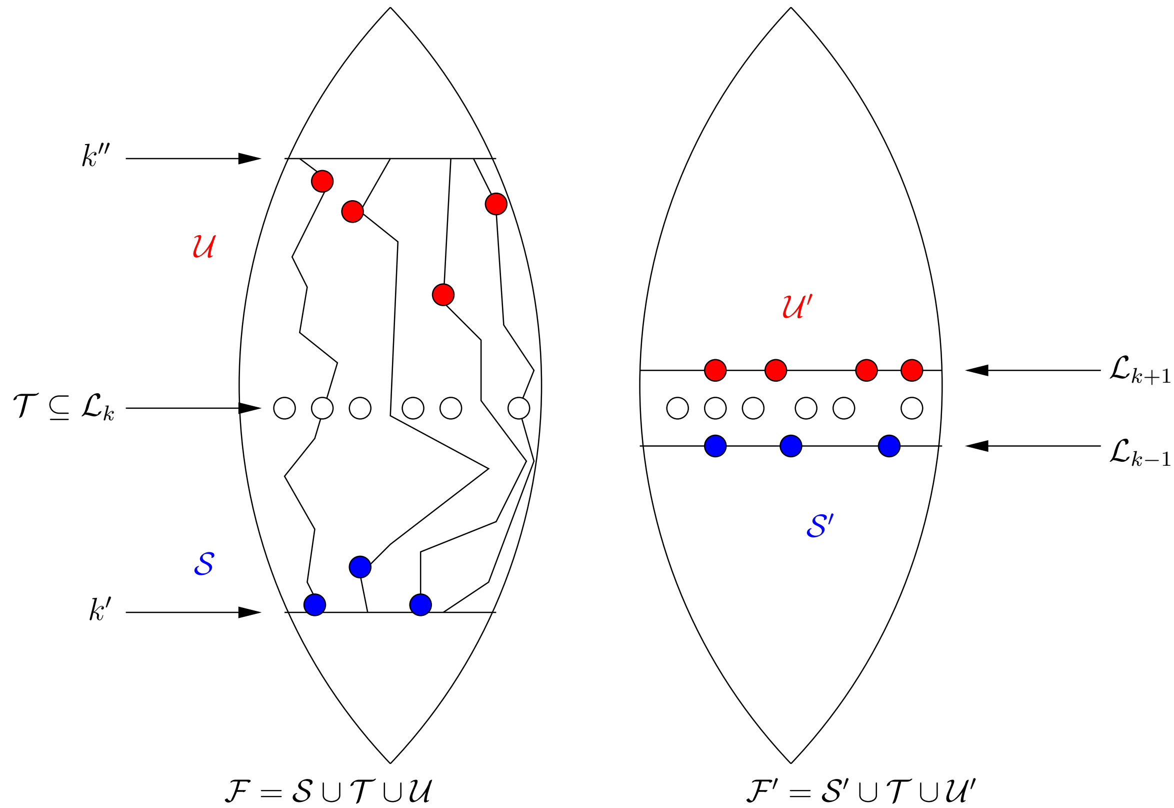

Figure 4.13 The construction of the family $\mathcal{F}'$ from the family $\mathcal{F}$ in Proposition 4.11.

whether $\pi(\mathcal{H})$ exists for every collection of subsets $\mathcal{H}$. It is also not known that *if* $\pi(\mathcal{H})$ exists for every $\mathcal{H} \subseteq 2^{[n]}$, then $\pi(\mathcal{H}) \in \mathbb{Z}$. For every $\mathcal{H}$ for which $\pi(\mathcal{H})$ is known, $\pi(\mathcal{H})$ is integral; in fact, it is equal to the largest number of middle layers of the Boolean lattice such that their union does not contain $(\mathcal{H}, \subseteq)$ as a subposet.

The smallest $\mathcal{H}$ for which $\pi(\mathcal{H})$ is not known is the case that $(\mathcal{H}, \subseteq)$ is isomorphic to Q_2. Proposition 4.10 shows that if $\pi(2^{[2]})$ exists, then it is in the interval $[2, 2.6]$. A recent preprint by Axenovich, Martin, and Manske shows that if $\pi(2^{[2]})$ exists, then it must be in the interval $[2, 2.283261]$. Griggs and Lu suggest in [40] that they believe one can show that if $\pi(2^{[2]})$ exists, then it is in the interval $[2, 2.25]$ (but they do not provide an argument themselves). To see how to show that if $\pi(2^{[2]})$ exists, then it is in the interval $[2, 2.3]$, see presentations by Griggs [38] on the subject.

There is also great interest in determining $\pi(2^{[n]})$, but naturally this must start with determining $\pi(2^{[2]})$. For determining $\pi(2^{[3]})$, it would be helpful to know $\pi(\mathcal{O}_6)$, since $(\mathcal{O}_6, \subseteq)$ is isomorphic to the union of the two middle layers of Q_3.

When wondering if $\pi(\mathcal{H})$ depends only on the height of $(\mathcal{H}, \subseteq)$, we direct the reader again to [40]. Suppose $\mathcal{H} \subseteq 2^{[n]}$ such that $(\mathcal{H}, \subseteq)$ is a poset of height 2. If $\pi(\mathcal{H})$ exists, it is known that $\pi(\mathcal{H}) \in [1, 2]$. However, if $(\mathcal{H}, \subseteq)$ is a poset of height 3, there is no such general upper bound on $\pi(\mathcal{H})$. This can be seen by the following. Let m be a fixed positive integer, and choose n so that $N \geq 2^m$. Take $\mathcal{H}$ to be the set family consisting of an antichain of size $2^{m-1} - 1$, and two more sets A and B such that for each $X \in \mathcal{H}$, $X \subset A$ and $B \subset X$. If n is large enough so that $N \geq 2^{m-1} - 1$, then we may take the family $\mathcal{F}$ to consist of the m largest layers in Q_n. Notice that for any two sets $F, F' \in \mathcal{F}$, there are at most $2^{m-1} - 2$ sets X with $F \subset X \subset F'$, and hence $(\mathcal{H}, \subseteq)$ is not a subposet of $(\mathcal{F}, \subseteq)$. Since $|\mathcal{F}| \sim mN$, we can make $\pi(\mathcal{H}) \geq m$. So, we cannot have an upper bound for $\pi(\mathcal{H})$ when $\mathcal{H}$ is a poset of height 3. By the construction of $\mathcal{H}$, it seems to suggest that $\pi(\mathcal{H})$ is a function of not only the height of $(\mathcal{H}, \subseteq)$ but also the width; i.e., the size of the largest antichain contained in $\mathcal{H}$.

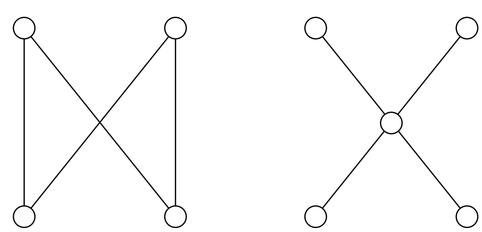

Figure 4.14 The poset $(\bowtie, \subseteq)$ is a subposet of $(\mathsf{X}, \subseteq)$. Notice that $\ell(\bowtie) = \ell(\mathsf{X})$.

There is also more progress to be done toward proving (or disproving) Conjecture 4.1. As stated in Section 4.2, Bukh proves his conjecture true in [13] in the case that the cover relation graph of the poset is a tree. His proof also allows for the one to find $ex(n, \mathcal{P})$ in the case where $(\mathcal{P}, \subseteq)$ is a subposet of a different poset whose cover relation graph is a tree. For example, the set family X consists of five distinct sets A, B, C, D, and E such that $A, B \subset E \subset C, D$. See Figure 4.14 for the Hasse diagram of X. The cover relation graph of X is a tree, and $(\bowtie, \subseteq)$ is a subposet of $(\mathsf{X}, \subseteq)$. Furthermore, since $\ell(\bowtie) = \ell(\mathsf{X}) = 3$,

$$ex(n, \bowtie) = ex(n, \mathsf{X}) = 2N\left(1 + O\left(\frac{1}{n}\right)\right).$$

This is not a new result (recall Theorem 4.8 in Section 4.2 due to De Bonis, Katona, and Swanepoel), but it affirms the conjecture. The smallest set family $\mathcal{F}$ which is not a subposet of any tree with the same value of $\ell(\mathcal{F})$ is $\mathcal{F} = 2^{[2]}$, so we may not apply Bukh's result to find $ex(n, 2^{[2]})$.

BIBLIOGRAPHY

[1] K. Akiyama and C. Suetake. The nonexistence of projective planes of order 12 with a collineation group of order 8. *J. Combin. Des.*, 16(5):411–430, 2008.

[2] K. Akiyama and C. Suetake. On projective planes of order 12 with a collineation group of order 9. *Australas. J. Combin.*, 43:133–162, 2009.

[3] J. F. Alm, R. D. Maddux, and J. Manske. Chromatic graphs, Ramsey numbers and the flexible atom conjecture. *Electron. J. Combin.*, 15(1):Research paper 49, 8, 2008.

[4] P. G. Anderson. A generalization of Baudet's conjecture (van der Waerden's theorem). *Amer. Math. Monthly*, 83(5):359–361, 1976.

[5] M. Axenovich and J. Manske. On monochromatic subsets of a rectangular grid. *Integers*, 8:A21, 14, 2008.

[6] M. Axenovich, J. Manske, and R. Martin. Q_2-free families of the Boolean lattice. *submitted*, 2009.

[7] R. Bacher and S. Eliahou. Extremal binary matrices without constant 2-squares. *submitted*, August 2009.

[8] J. Beck, W. Pegden, and S. Vijay. The Hales-Jewett number is exponential: game-theoretic consequences. In *Analytic number theory*, pages 22–37. Cambridge Univ. Press, Cambridge, 2009.

[9] E. R. Berlekamp. A construction for partitions which avoid long arithmetic progressions. *Canad. Math. Bull.*, 11:409–414, 1968.

[10] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar*, 16:447–452, 1965.

[11] B. Bollobás. *Extremal graph theory*, volume 11 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[12] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[13] B. Bukh. Set families with a forbidden subposet. *Electron. J. Combin.*, 16(1):Research paper 142, 11, 2009.

[14] V. Chvátal. Some unknown van der Waerden numbers. In *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*, pages 31–33. Gordon and Breach, New York, 1970.

[15] A. Clebsch. Ueber die Flächen vierter Ordnung, welche eine Doppelcurve zweiten grades besitzen. *J. für Math.*, 69:142–184, 1868.

[16] S. D. Comer. Color schemes forbidding monochrome triangles. In *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983)*, volume 39, pages 231–236, 1983.

[17] S. D. Comer. Combinatorial aspects of relations. *Algebra Universalis*, 18(1):77–94, 1984.

[18] A. De Bonis and G. O. H. Katona. Largest families without an r-fork. *Order*, 24(3):181–191, 2007.

[19] A. De Bonis, G. O. H. Katona, and K. J. Swanepoel. Largest family without $A \cup B \subseteq C \cap D$. *J. Combin. Theory Ser. A*, 111(2):331–336, 2005.

[20] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[21] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Ann. of Math. (2)*, 51:161–166, 1950.

[22] K. Engel. *Sperner theory*, volume 65 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1997.

[23] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[24] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[25] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935.

[26] P. Erdős and P. Turán. On some sequences of integers. *J. Lond. Math. Soc.*, 11:261–264, 1936.

[27] P. Erdős, M. Simonovits, and V. T. Sós. Anti-Ramsey theorems. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*, pages 633–643. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[28] M. J. Erickson. *Introduction to combinatorics*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1996. A Wiley-Interscience Publication.

[29] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Analyse Math.*, 31:204–256, 1977.

[30] B. Gasarch. Private communication, 2007.

[31] C. D. Godsil. Problems in algebraic combinatorics. *Electron. J. Combin.*, 2:Feature 1, approx. 20 pp. (electronic), 1995.

[32] W. T. Gowers. A new proof of Szemerédi's theorem. *Geom. Funct. Anal.*, 11(3):465–588, 2001.

[33] R. Graham. On the growth of a van der Waerden-like function. *Integers*, 6:A29, 5 pp. (electronic), 2006.

[34] R. Graham and J. Solymosi. Monochromatic equilateral right triangles on the integer grid. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 129–132. Springer, Berlin, 2006.

[35] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, second edition, 1990. A Wiley-Interscience Publication.

[36] B. Green and T. Tao. Yet another proof of Szemerédi's theorem. *arXiv:1002.2254v1 [math.NT]*, 2010.

[37] R. E. Greenwood and A. M. Gleason. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, 7:1–7, 1955.

[38] J. R. Griggs. Private communication, 2009.

[39] J. R. Griggs and G. O. H. Katona. No four subsets forming an N. *J. Combin. Theory Ser. A*, 115(4):677–685, 2008.

[40] J. R. Griggs and L. Lu. On families of subsets with a forbidden subposet. *Combinatorics, Probability and Computing*, 18:731–748, 2009.

[41] A. W. Hales and R. I. Jewett. Regularity and positional games. *Trans. Amer. Math. Soc.*, 106:222–229, 1963.

[42] P. R. Herwig, M. J. H. Heule, P. M. van Lambalgen, and H. van Maaren. A new method to construct lower bounds for van der Waerden numbers. *Electron. J. Combin.*, 14(1):Research Paper 6, 18 pp. (electronic), 2007.

[43] P. Jipsen, R. D. Maddux, and Z. Tuza. Small representations of the relation algebra $\mathcal{E}_{n+1}(1,2,3)$. *Algebra Universalis*, 33(1):136–139, 1995.

[44] J.G. Kalbfleisch. *Chromatic Graphs and Ramsey's Theorem*. Ph.D. dissertation, University of Waterloo, 1966.

[45] G. O. H. Katona and T. G. Tarján. Extremal problems with excluded subgraphs in the n-cube. In *Graph theory (Łagów, 1981)*, volume 1018 of *Lecture Notes in Math.*, pages 84–93. Springer, Berlin, 1983.

[46] D. Kleitman. A conjecture of Erdős-Katona on commensurable pairs among subsets of an n-set. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 215–218. Academic Press, New York, 1968.

[47] M. Kouril. *A Backtracking Framework for Beowulf Clusters with an Extension to Multi-Cluster Computation and Sat Benchmark Problem Implementation*. PhD thesis, University of Cincinnati, 2006.

[48] M. Kouril and J. L. Paul. The van der Waerden number $W(2,6)$ is 1132. *Experiment. Math.*, 17(1):53–61, 2008.

[49] B. M. Landman and A. Robertson. *Ramsey theory on the integers*, volume 24 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2004.

[50] D. Lubell. A short proof of Sperner's lemma. *J. Combinatorial Theory*, 1:299, 1966.

[51] R. C. Lyndon. Relation algebras and projective geometries. *Michigan Math. J.*, 8(1):21–28, 1961.

[52] R. D. Maddux. A perspective on the theory of relation algebras. *Algebra Universalis*, 31(3):456–465, 1994.

[53] R. D. Maddux. *Relation algebras*, volume 150 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2006.

[54] R. D. Maddux. Private communication, 2009.

[55] K. Menger. Über reguläre Baumkurven. *Math. Ann.*, 96(1):572–582, 1927.

[56] L. D. Mešalkin. A generalization of Sperner's theorem on the number of subsets of a finite set. *Teor. Verojatnost. i Primenen*, 8:219–220, 1963.

[57] J. Pach and P. K. Agarwal. *Combinatorial geometry.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[58] A. R. Prince. Projective planes of order 12 and PG$(3, 3)$. *Discrete Math.*, 208/209:477–483, 1999. Combinatorics (Assisi, 1996).

[59] A. R. Prince. Ovals in finite projective planes via the representation theory of the symmetric group. In *Finite groups 2003*, pages 283–290. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[60] J. R. Rabung. Some progression-free partitions constructed using Folkman's method. *Canad. Math. Bull.*, 22(1):87–91, 1979.

[61] R. Rado. Verallgemeinerung Eines Satzes von van der Waerden mit Anwendungen auf ein Problem der Zahlentheorie. *Sonderausg. Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Klasse*, 17:1–10, 1933.

[62] R. Rado. Note on combinatorial analysis. *Proc. London Math. Soc. (2)*, 48:122–160, 1943.

[63] S. P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 42 pp. (electronic), 1994; Revision #12: 4 August 2009.

[64] F. P. Ramsey. On a problem in formal logic. *Proc. London Math. Soc.*, 30:264–286, 1930.

[65] K. F. Roth. On certain sets of integers, I. *J. Lond. Math. Soc.*, 28:104–109, 1956.

[66] I. Schur. Über die Kongruenz $x^m + y^m = z^m \pmod{p}$. *Jber. Deutsch. Math.-Verein.*, 25:114–117, 1916.

[67] S. Shelah. Primitive recursive bounds for van der Waerden numbers. *J. Amer. Math. Soc.*, 1(3):683–697, 1988.

[68] I. D. Shkredov. On a two-dimensional analogue of Szemerédi's theorem in abelian groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 73(5):181–224, 2009.

[69] E. Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27(1):544–548, 1928.

[70] R. S. Stevens and R. Shantaram. Computer-generated van der Waerden partitions. *Math. Comp.*, 32(142):635–636, 1978.

[71] C. Suetake. The nonexistence of projective planes of order 12 with a collineation group of order 16. *J. Combin. Theory Ser. A*, 107(1):21–48, 2004.

[72] E. Szemerédi. On sets of integers containing no four elements in arithmetic progression. *Acta Math. Acad. Sci. Hungar.*, 20:89–104, 1969.

[73] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Acta Arith.*, 27:199–245, 1975. Collection of articles in memory of Juriĭ Vladimirovič Linnik.

[74] H. T. Thanh. An extremal problem with excluded subposet in the Boolean lattice. *Order*, 15(1):51–57, 1998.

[75] W. T. Trotter. *Combinatorics and partially ordered sets*. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.

[76] P. Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[77] B.L. van der Waerden. Beweis einer Baudetchen Vermutung. *Nieuw Arch. Wiskunde*, 15:212–216, 1927.

[78] D. B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.

[79] K. Yamamoto. Logarithmic order of free distributive lattice. *J. Math. Soc. Japan*, 6:343–353, 1954.