1958

Dynamics of a fly-ball motion sensing device

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DYNAMICS OF A FLY-BALL MOTION SENSING DEVICE

by

Edwin Richard Chubbuck

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Theoretical and Applied Mechanics

Approved:

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In Charge of Major Work

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Dean of Graduate College

Iowa State College

1958
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INTRODUCTION

The measurement of motion has been a necessary operation in many fields of endeavor for a long time. Recently, however, guided missiles, modern torpedoes, fast aircraft, and other moving vehicles have made imperative an increased emphasis on non-human methods of motion measurement; in fact, such measurement has been necessary in the development of the indicated vehicles. Sometimes motion of a vehicle is to be measured by an instrument carried on the subject vehicle; sometimes the instrument is on another vehicle or on the ground.

This investigation is directed toward a proposed device for measurement of angular motion of a vehicle carrying the device. The idea for this device was suggested several years ago by the flight stabilization method possessed by ordinary houseflies and some other two-winged insects. These insects have a set of rudimentary wings which vibrate rapidly and somehow stabilize the flight of such insects (5 and 10). These rudimentary wings, located closely behind the regular wings, are called halteres, and the loss of one or both of them completely disorganizes the flight of the insect. Recently, an instrument called a vibragyro has been developed which is described by Chatterton (7). This vibratory gyro is a closer approach to the insect stabilizer than is the device under
consideration here. However, the insects' mode of stabilization was the original source of the idea.

Briefly, the proposed fly-ball motion sensing device could be likened to a fly-ball governor with the spin axis horizontal and spun at a constant speed. The angular motion to be sensed would be that occurring around a vertical axis which passes through the point of attachment of the fly-ball arms. The problem is to find the motion of the balls as the turning of the entire device (around vertical axis) takes place. The device and assumptions are fully described later. It is envisaged that the spin axis (horizontal axis) would be rigidly aligned with the vehicle whose angular motion is to be measured.

The objectives of this investigation are threefold: (a) to ascertain whether or not such a fly-ball device could be used to detect and measure angular motion, (b) to establish a method of solution of the problem, and (c) to solve the problem of determining the position of the balls at any time after commencement of the problem for given starting conditions, problem parameters, and turning function.

One of the features of such a motion-sensing instrument as the one proposed is that the positional information could be "picked off" capacitively, inductively, by reflected light, or other ways, any of which could be made to have almost no influence on the motion of the fly-balls.
Another feature of the proposed device is that the actuating motion is applied directly by the motion of the vehicle to the spin axis of the instrument; no erecting motors are needed.

Inasmuch as the use of a fly-ball motion sensing device would most likely be in some vehicle which maneuvers in its passage from one point to another, the subject of guidance is immediately involved.
REVIEW OF LITERATURE

The development of guidance systems with an emphasis on angular motion measurement will be reviewed briefly since the proposed device would most likely find use in some guidance system. Other uses, unsuspected at this time, may be found.

The subject of guidance is involved with navigation; an historical treatment of the subject of navigation is given in references (2, 4, 5, 6, 10, 11, 12, 21, 22, 27, 29, 31, and 32).

Definition of Guidance System

A guidance system is, in the most fundamental sense, a means of directing the motion of an object from one point to another. In this discussion the guidance system will not include the ultimate steering devices. One element of a guidance system includes the method of determining position in the reference system and another involves acquisition of the knowledge of the necessary orders to the steering system.

Until recently one essential component of any guidance system was a human being; however, in some of the modern systems the response time of a human is far too long. Thus, the latest guidance systems are completely automatic.
Semi-automatic Guidance Systems

Semi-automatic guidance systems require humans to do some operations. Some use has been made of angular motion measuring gyros in aiming devices for gunfire control even though a human operator is necessary also.

Some guided missiles use semi-automatic systems. Parson (25, p. 52) says that guided missiles must have trajectory control and attitude control, the latter to be able to use commands properly. Two common types of semi-automatic systems are (a) beam riders and (b) command guidance. Parson (25, p. 58) describes the beam rider as a missile which follows a beam of energy, usually radar, from firing point to target. Command guidance is the system whereby the human operator "sees" both missile and target on a radar and gives the missile steering orders by radio (25, p. 55). Mullen (24a) says the main advantage of command guidance is that most of the expensive system is on the ground and not in the missile.

Automatic Guidance Systems

Automatic systems of guidance need no human assistance after the decision to fire the missile is made and the initial action is taken.
In any navigation system, two tasks must be performed: (a) the position on earth or with respect to target must be known at all times, and (b) the attitude of the missile must be known continuously (25, p. 52). Position finding may be by preset program (24a and 25), homing (24a and 25), magnetic reference (21 and 25), altitude reference (25, pp. 54-55), distance reference (25, pp. 54-55), celestial reference (21, 24a, 25, and 28), radio reference (25, p. 57), or inertial reference (8, 13, 21, 25, 28, 32, and 33). An accurate knowledge of time is necessary for any of the indicated systems.

Inertial systems are receiving a great deal of attention at the present time. Table 1 is a brief outline of "hardware" used in inertial guidance systems (reproduced from reference 14).

Instruments of Guidance Systems

The instruments used in guidance systems fall into two categories: (a) those having to do with position control, and (b) those involved with attitude control.

For position control, linear accelerometers are used for inertial systems (21, pp. 339-350). Celestial methods are available for determining position by tracking stars (21, 24a, and 25).
Table 1. Components used in inertial guidance systems

<table>
<thead>
<tr>
<th>Gyros</th>
<th>Accelerometers</th>
<th>Integrators</th>
<th>Computers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single degree of freedom, fluid floated</td>
<td>Magnetically restrained, pendulous</td>
<td>Viscous shear</td>
<td>Analog</td>
</tr>
<tr>
<td>Two degrees of freedom, fluid floated</td>
<td>Elastic-restrained, pendulous</td>
<td>Drag cup</td>
<td>Electro-mechanical analog</td>
</tr>
<tr>
<td>Air bearing suspended</td>
<td>Linear displacement</td>
<td>Integrating servo</td>
<td>Digital</td>
</tr>
<tr>
<td>Torsion wire suspended</td>
<td>Integrating</td>
<td>Thermo</td>
<td></td>
</tr>
<tr>
<td>Gas floated</td>
<td>Double integrating</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For attitude control, several instruments are of interest. The two-degrees-of-freedom gyro is the basic unit for establishing a reference plane in space (8, 13, and 21). The vertical gyro is used to establish a plane perpendicular to the effective gravity vector (2 and 21). The rate gyro uses the precessive torque as a rate-of-turn measure (21, pp. 350-353). The single-axis floating integrating gyro is a newly developed instrument which measures rate of turn and total angle of turn (8). Another recently developed instrument is the vibratory rate gyro; this instrument measures rate of turn but not total angle of turn (7 and 30).
The proposed fly-ball motion sensing device would be an attitude control instrument and not a position control instrument.
DESCRIPTION OF THE FLY-BALL DEVICE AND
STATEMENT OF THE PROBLEM

As indicated earlier, the original idea for the device came from the mode of stabilization of certain insects by means of the halteres. Briefly, the proposed fly-ball motion sensing device consists of a horizontal axis of spin to which two arms are attached, each carrying a spherical mass at the end. The arms are mechanically constrained to make equal angles with the spin axis and a restoring spring is attached to bring the arms toward the spin axis. This axis of spin is turned at a constant angular speed ($K$ radians per second) and the entire assembly is turned around a vertical axis passing through the point of attachment of the arms to the spin axis. The two arms and the axis of spin always remain coplanar.

Idealized Formulation

Even though it would not be possible physically, the assumption is made that the arms are massless and the masses are point masses. Figure 1 is a sketch of the fly-ball device. The bearings of the spin axis would be affixed to a missile's framework. The springs are shown as if they were linear coil springs for clarity only; they would be torsion springs. The plane containing the arms and the spin axis
Figure 1. Isometric sketch of the proposed fly-ball motion sensing device

Masses $M_1$ and $M_2$ are equal and are point masses.

Arms are massless and of equal length, $L$.

Torsion springs each have a spring constant $= C/2$ foot pounds per radian.

Springs are indicated as linear coil springs symbolically only; they are actually torsion springs.

$\vec{K}$ = angular velocity of spin and is constant.

$\vec{\omega}(t)$ = angular velocity of turn and is variable.
SPIN AXIS
MASS
ARM
BEARING
TORSION SPRINGS
TURN AXIS
ATTACHED TO VEHICLE

\[ M_1 \]
\[ M_2 \]
spins with a constant angular speed, $K$, hereafter referred to as turn. For the problem the turn axis is in the direction of effective gravity to avoid the complications of gravity. It should be noted that it would be quite difficult to produce an exactly constant spin speed because of the reaction to the driving mechanism.

Object of the Study

The overall objective of this investigation is to determine the suitability of such a device as the proposed fly-ball instrument for use as a means of measuring angular motion. The immediate objectives are as follows:

1. To develop a method for solving the problem of locating the balls completely at any instant.

2. To solve the problem of finding the position ($\theta$) of the arms at any time for a given turning function $[\omega(t)]$ and given starting conditions.

Limitations

Scope of work

The problem is limited to that of determining the angle as a function of time. No attempt will be made to develop a final method of using the results in a navigational system. Several types of turning functions have been used, but by no
means an exhaustive list. Several variations of starting conditions and problem parameters have been considered, but again it is not an exhaustive list.

Assumptions

The following assumptions were made:

1. Bearing friction may be neglected.
2. No spring hysteresis exists.
3. All parts are rigid except springs.
4. The axis of spin remains perpendicular to the effective force of gravity.
5. Fly-balls are point masses and the mass of the arms may be neglected.
6. Only angular motions of the instrument are considered. The axis of turn is vertical and passes through the point of arm attachment.
7. The plane containing the two arms and the axis of spin is vertical at \( t = 0 \).
DERIVATION OF DIFFERENTIAL EQUATION OF MOTION

Because of the advantages offered by the use of Lagrange's equations, the ordinary method of writing equilibrium equations by the free-body method was abandoned. Since the spin speed is constant and the turning function is known as a function of time, the only generalized coordinate that must be known to specify the configuration of the system at any instant is the angle \( \theta \). Akimoff (1) describes this class of phenomena as a single-degree-of-freedom problem with variable constraints. Time is not a generalized coordinate. The development of the differential equation of motion was accomplished by using Lagrange's equations. Thus, writing the equation of motion reduced to two parts: (a) expression of kinetic energy and (b) use of Lagrange's equations to get the equation of motion. Since there is only one degree of freedom, there is need for only one Lagrange equation.

Expression of Kinetic Energy

The basic scheme used to express the kinetic energy was to obtain the absolute velocity of each mass by the use of a rotating set of axes in a fixed set. Figure 2 is a view of the rotating parts of the device with all lengths shown without the foreshortening accompanying Figure 1. The two arms are constrained mechanically to make equal angles with the
Arms are rigid and massless.

$\theta$ is the angle between spin axis and each arm. The arms are mechanically constrained to maintain equal angles with the spin axis.

Torsion springs each have a spring constant of $C/2$. The restoring torque for both arms combined = $-C\theta$.

The two arms and spin axis remain coplanar; this plane turns with angular speed = $K$. 
TORSION SPRINGS
DIRECTION OF EFFECTIVE GRAVITY

AXIS OF SPIN

M₁

L

θ

θ

M₂

AXIS OF TURN

TORSION SPRINGS

DIRECTION OF EFFECTIVE GRAVITY
spin axis. The two arms and the spin axis are always co-planar, but this plane rotates about the spin axis with angular velocity \( \mathbf{K} \). The torsion springs are adjusted so that there would be no torque on the arms if the arms were aligned with the spin axis. Physically, of course, it would be impossible for the arms to take this position because of interference.

Figure 3 shows the rotating set of axes and the fixed (or inertial) set. The spin axis remains permanently oriented along the \( z \) axis of the movable set. The direction of effective gravity (not shown in Figure 3) is in the negative \( X \) direction (also the negative \( x \) direction). Since the rotational \( [\mathbf{\omega}(t)] \) is around one axis only (the \( X \) axis), the indicated orientation of the two reference sets of axes effects considerable simplification. The fixed axes are denoted by capital letters and the movable axes by lower case letters. The indicated orientation maintains the \( YZ \) plane coincident with the \( yz \)-plane, though these two planes rotate with respect to each other about the \( X \)- and \( x \)-axes at the rate \( \mathbf{\omega}(t) \).

Figure 4 shows the movable system for one ball in spherical coordinates. Spherical coordinates are obviously the appropriate ones to use because the radius is constant and the motions are all angular. The unit vectors are also indicated in Figure 4; these are developed as indicated in Figure 5. It should be remembered that these spherical unit vectors have time derivatives since they change directions.
Figure 3. Reference systems chosen in order to express the kinetic energy

X, Y, and Z are fixed axes.

x, y, and z are movable axes.

i', j', and k' are unit vectors corresponding to X, Y, and Z.

i, j, and k are unit vectors corresponding to x, y, and z.

X and x-axes remain coincident; so i' = i.

Axis of spin is the z-axis, and the spin speed = K radians per second. K is constant.
Figure 4. Movable system in spherical coordinates

Spherical unit vectors in terms of rectangular unit vectors, all in the movable system, are as follows (see Figure 5 for development):

\[
\begin{align*}
\vec{e}_\phi &= -i \sin \phi + j \cos \phi \\
\vec{e}_\theta &= i \cos \theta \cos \phi + j \cos \theta \sin \phi - k \sin \theta \\
\vec{e}_r &= i \sin \theta \cos \phi + j \sin \theta \sin \phi + k \cos \theta
\end{align*}
\]
a. View looking in negative z-direction
\[ \vec{e}_\varphi = - \hat{i} \sin \varphi + \hat{j} \cos \varphi \]

b1. View perpendicular to z-R plane showing \( \Theta \) full sized
\[ \vec{e}_\varphi = - k \sin \theta + \vec{e}_R \cos \theta \]

b2. View looking in the negative z-direction
\[ \vec{e}_\varphi = - k \sin \theta + \hat{i} \cos \theta \cos \phi + \hat{j} \cos \theta \sin \phi \]

c1. View perpendicular to z-R plane
\[ \vec{e}_r = k \cos \theta + \vec{e}_R \sin \theta \]

c2. View looking in negative z-direction
\[ \vec{e}_r = k \cos \theta + \hat{i} \sin \theta \cos \phi + \hat{j} \sin \theta \sin \phi \]

Figure 5. Construction showing the development of spherical unit vectors in terms of rectangular unit vectors (all in moving system)
From Figure 4, it is evident that the radius vector, \( \overrightarrow{r} \), which indicates the position of the mass is

\[
\overrightarrow{r} = \left| \overrightarrow{r} \right| \hat{e}_r = L \hat{e}_r .
\]  

(1)

The relative velocity of the ball in the moving system is the first time derivative of the position vector; thus

\[
\overrightarrow{e}_r = \dot{\theta} L \sin \theta \hat{\phi} + L \dot{e}_\theta \hat{\theta}
\]  

(2)

after making use of the value of \( e_r \) from Figure 4. Since \( \overrightarrow{e}_r \) is the relative velocity in the moving system, the velocity of the ball in the fixed system must be evaluated to be able to obtain kinetic energy. If the position vector in the fixed system is designated \( \overrightarrow{r} \) (in spherical coordinates), the velocity becomes \( \dot{\overrightarrow{r}} \). Then

\[
\dot{\overrightarrow{r}} = \dot{\overrightarrow{P}} + \ddot{\overrightarrow{e}_r} + \omega \times \overrightarrow{r}
\]  

(3)

where \( \overrightarrow{P} \) = radius vector in the fixed system of the origin of the moving system and \( \ddot{\overrightarrow{e}_r} \) = radius vector of \( M_1 \) in the fixed system. Since the origins of the two systems are always coincident, \( \overrightarrow{P} = 0 \) and \( \dot{\overrightarrow{P}} = 0 \). Therefore,

\[
\dot{\overrightarrow{r}} = \ddot{\overrightarrow{e}_r} + \omega \times \overrightarrow{r}
\]  

(4)

or

\[
= \dot{\theta} L \sin \theta \hat{\phi} + L \dot{e}_\theta \hat{\theta} + L \omega (\hat{x} \overrightarrow{e}_r)
\]  

(5)
since \( \overrightarrow{e} = L e_r, \overrightarrow{\omega(t)} = i'\omega(t) = i\omega(t) \), and \( \overrightarrow{e} = L e_r \) if \( e_r \) is expressed in terms of fixed system coordinates. Kinetic energy \( M_1 \) is

\[
T_1 = \frac{1}{2} M_1 \mathbf{v}_1^2 = \frac{1}{2} M_1 (r)^2
\]

or

\[
T_1 = \frac{1}{2} M_1 (\dot{r} \cdot \dot{r})
\]

\[
= \frac{1}{2} M_1 \left[ \dot{r} L \sin \phi \dot{\phi} + L \dot{\phi} \dot{\phi} + \omega L (i \dot{e}_r) \right] \cdot \left[ \dot{r} L \sin \phi \dot{\phi} + L \dot{\phi} \dot{\phi} + \omega L (i \dot{e}_r) \right]
\]

\[
= \frac{1}{2} M_1 \left[ \dot{\phi}^2 L^2 \sin^2 \phi - 2 \dot{\phi} \omega L^2 \sin \phi \cos \phi \cos \theta \\
- 2L^2 \dot{\phi}^2 \sin \phi \cos \phi - L^2 \dot{\theta}^2 \right]
\]

In the preceding work and in work to follow, the following relationships were used:

\[
\dot{e}_\phi = - i \sin \phi + j \cos \phi
\]

\[
\dot{e}_\theta = i \cos \theta \cos \phi + j \cos \theta \sin \phi - k \sin \phi
\]

\[
\dot{e}_r = i \sin \theta \cos \phi + j \sin \theta \sin \phi + k \cos \theta
\]

\[
i \dot{e}_r = k \sin \theta \sin \phi - j \cos \theta
\]

\[
\dot{e}_\phi \cdot i \dot{e}_r = \left( - i \sin \phi + j \cos \phi \right) \cdot \left( k \sin \theta \sin \phi - j \cos \theta \right)
\]

\[
= - \cos \phi \cos \theta
\]
\[ \overline{e}_\phi \cdot \overline{ixe}_r = (i\cos\theta\cos\phi + j\cos\theta\sin\phi - k\sin\phi) \cdot (k\sin\theta\sin\phi - j\cos\theta) \]
\[ = - \cos^2\sin\phi - \sin^2\sin\phi \]
\[ = - \sin\phi \quad (14) \]

\[ (ixe_r) \cdot (ixe_r) = (k\sin\theta\sin\phi - j\cos\theta) \cdot (k\sin\theta\sin\phi - j\cos\theta) \]
\[ = \sin^2\theta\sin^2\phi + \cos^2\phi \quad . \quad (15) \]

These relationships were developed as outlined in Figure 4 and were confirmed by comparing some of the results with those given by Page (24b).

Having found the kinetic energy for one of the balls, one can find the kinetic energy for the other ball by merely substituting the appropriate coordinate changes into the expression for \( T \). Let \( \phi_2, \theta_2, \) and \( L_2 \) be the coordinates of the second ball. From the physical constraints,

\[ \phi_2 = \phi + \pi \quad (16) \]

and

\[ \dot{\phi}_2 = \dot{\phi} \quad . \quad (17) \]

Also

\[ \theta_2 = \theta \quad (18) \]
\[ L_2 = L \quad . \quad (19) \]
Substituting $\phi_2$ for $\phi$, $\theta_2$ for $\theta$, and $L_2$ for $L$ into Equation 8, one obtains $T_2$, the expression for the kinetic energy of $M_2$, as

\[
T_2 = \frac{M_2}{2} \left[ \dot{\phi}_2^2 L_2^2 \sin^2 \theta_2 - 2 \dot{\phi}_2 \omega L_2^2 \cos \phi_2 \cos \theta_2 \sin \theta_2 \\
+ L_2^2 \dot{\theta}_2^2 - 2 L_2^2 \omega \dot{\theta}_2 \sin \phi_2 \\
+ \omega^2 L_2^2 (\sin^2 \theta_2 \sin^2 \phi_2 + \cos^2 \theta_2) \right]
\]  

(20)

or

\[
T_2 = \frac{M}{2} \left[ \dot{\phi}^2 L^2 \sin^2 \Theta - 2 \dot{\phi} \omega L^2 \cos (\phi + \pi) \cos \Theta \sin \Theta \\
+ L^2 \dot{\Theta}^2 - 2 L^2 \omega \dot{\Theta} \sin (\phi + \pi) \\
+ \omega^2 L^2 (\sin^2 \Theta \sin^2 (\phi + \pi) + \cos^2 \Theta) \right]
\]

(21)

after using Equations 16, 17, 18, and 19. Since

\[
\cos (\phi + \pi) = - \cos \phi \\
\sin (\phi + \pi) = - \sin \phi
\]

(22)

(23)

the expression for $T_2$ becomes

\[
T_2 = \frac{M}{2} \left[ \dot{\phi}^2 L^2 \sin^2 \Theta + 2 \omega \dot{\Theta} L^2 \cos \phi \cos \Theta \sin \Theta \\
+ L^2 \dot{\Theta}^2 + 2 L^2 \dot{\Theta} \sin \Theta \\
+ \omega^2 L^2 (\sin^2 \Theta \sin^2 (\phi + \pi) + \cos^2 \Theta) \right]
\]

(24)

after making use of

\[
\phi = K t \\
\dot{\phi} = K \\
M_2 = M_1 = M
\]

(25)

(26)

(27)
The total kinetic energy of $M_1$ and $M_2$ is merely the sum of the kinetic energies $T_1$ and $T_2$ so

$$T = T_1 + T_2 = M \left[ k^2 L^2 \sin^2 \theta + L^2 \dot{\phi}^2 + \omega^2 L^2 (\sin^2 \theta \sin^2 \phi + \cos^2 \theta) \right].$$

(28)

Since considerable algebra was involved in the work up to this point, it should be mentioned that the kinetic energy of $M_1$ was originally carried through entirely in rectangular coordinates. This was long and laborious, but precisely the same final results were obtained for $T_1$ as are shown in Equation 8.

Some question might be raised about the validity of obtaining $T_2$ in the manner indicated. Consequently, a check was made whereby the value of $T_2$ was calculated without using the value of $T_1$ already calculated. This check is as follows:

$$\vec{r}_2 = L_2 \vec{r}_2$$

(29)

$$= L_2 \cos 2\theta \vec{r}_2 - L_2 \sin 2\theta \vec{\phi}_2$$

(30)

$$\dot{\vec{r}}_2 = \dot{\phi}_2 L_2 \sin \theta \vec{e}_\phi + L_2 \dot{\theta}_2 \vec{e}_\theta$$

(31)

$$= \dot{\phi} L \sin \theta (- \vec{e}_\theta) + \dot{\theta} (- \vec{e}_\phi \cos \theta - \vec{r}_\phi \sin \theta)$$

(32)

since

$$\phi_2 = \phi + \pi, \quad \dot{\phi}_2 = \dot{\phi}, \quad L_2 = L, \quad \vec{e}_\phi_2 = - \vec{e}_\phi$$
and

\[ e_{\theta_2} = - e_{\theta}^2 \cos \theta - \overrightarrow{e_r} \sin \theta \, . \]

The development of the last two unnumbered relations is shown in Figure 6. Therefore,

\[ \dot{\overrightarrow{r}}_2 = - \dot{\phi} \overrightarrow{L \sin \theta e_\theta} - \dot{\theta} \overrightarrow{L \cos \theta e_\theta} - \dot{\phi} \overrightarrow{L \sin \theta e_\theta} \, . \]  

(33)

Again use is made of

\[ \overrightarrow{r}_2 = \overrightarrow{r}_2 + \overrightarrow{\omega x} \, . \]  

(3-2)

for the case of \( M_2 \).

\[ \overrightarrow{r}_2 = - \dot{\phi} \overrightarrow{L \sin \theta e_\theta} - \dot{\theta} \overrightarrow{L \cos \theta e_\theta} - \dot{\phi} \overrightarrow{L \sin \theta e_\theta} - k \omega \overrightarrow{L \sin \theta e_\theta} - j \omega \overrightarrow{L \cos \theta} \]  

(34)

\[ (\overrightarrow{r}_2)^2 = (\overrightarrow{r}_2) \cdot (\overrightarrow{r}_2) \]

\[ = \dot{\theta}^2 L^2 \sin^2 \theta + L^2 \omega^2 + \omega^2 L^2 (\sin^2 \theta \sin^2 \theta + \cos^2 \theta) \]

\[ + 2 L \omega \dot{\phi} \sin \theta \cos \theta \cos \phi + 2 L \omega \dot{\phi} \sin \theta \, . \]  

(35)

Considerable algebra has been omitted here in obtaining Equations 34 and 35. After making use of Equations 25 and 26, it is seen that Equation 35 is identical to Equation 24.

Differential Equation from Lagrange's Equations

For a non-conservative system, Akimoff (1) develops the Lagrange equations in the form
Figure 6. Spherical unit vectors of $M_2$ in terms of the spherical unit vectors of $M_1$

$\mathbf{e}_r^2 = \mathbf{e}_r \cos \theta - \mathbf{e}_\phi \sin \theta$

$\mathbf{e}_\phi^2 = -\mathbf{e}_r \sin \theta - \mathbf{e}_\phi \cos \theta$

$\mathbf{e}_\phi^2 = -\mathbf{e}_\phi$
AXIS OF SPIN

\[ \vec{e}_{r_2}, \vec{e}_r, \vec{e}_\theta, \vec{e}_{\theta_2} \]

\[ M_1, M_2 \]

\[ \theta, \theta \]

\[ L, L \]
\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = Q_1
\]  
(36)

where

- \( T \) = kinetic energy of system
- \( q_1 \) = a generalized coordinate (here \( \theta \))
- \( \dot{q}_1 \) = first time derivative of \( q_1 \) (here \( \dot{\theta} \))
- \( Q_1 \) = the generalized force corresponding to \( q_1 \) (here \(-C\theta\))

The fly-ball motion sensing device is certainly a non-conservative system because the spin shaft will be maintained at constant speed externally and because the input turning function, \( \omega(t) \), is present. Both of these inputs will supply energy even though this supply will be oscillatory. In Lagrange's equations, the generalized force need not be a force dimensionally; in the present case it is actually a torque. However, when the generalized force is multiplied by the corresponding generalized coordinate, work must result. Here

\[
Q_1 = -C\theta = \text{a torque}
\]
(37)

\[
Q_1q_1 = (-C\theta)(\theta) = -C\theta^2 = \text{work}
\]
(38)

The negative sign here is necessary because the spring is restorative and work is put into the system when \( \theta \) increases. As mentioned earlier, the system has only one degree of freedom with movable constraints, so only one Lagrange equation is needed.
After substitution of the expression for total kinetic energy from Equation 28 into Lagrange's equation (Equation 36), there results

\[
\frac{\partial^2 T}{\partial q_1^2} = \frac{\partial^2 T}{\partial \dot{\theta}^2} = 2ML^2 \ddot{\theta}
\] (39)

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = 2ML^2 \ddot{\theta}
\] (40)

\[
\frac{\partial T}{\partial q_1} = \frac{\partial T}{\partial \theta} = 2MK^2L^2 \sin \theta \cos \theta + M\omega^2L^2 (2\sin^2 \theta \sin \omega t \cos \theta - 2 \cos \theta \sin \theta) + MK^2L^2 \sin \theta \cos \theta + M\omega^2L^2 \sin^2 \theta - M\omega^2L^2
\] (41)

\[
= 2\sin \theta \cos \theta (MK^2L^2 + M\omega^2L^2 \sin^2 \theta - M\omega^2L^2).
\] (42)

Thus

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_1} \right) - \frac{\partial T}{\partial q_1} = \dot{q}_1
\] (36)

or

\[
2ML^2 \ddot{\theta} - 2\sin \theta \cos \theta (MK^2L^2 + M\omega^2L^2 \sin^2 \theta - M\omega^2L^2) + C\dot{\theta} = 0
\] (43)

or

\[
\frac{d^2 \theta}{dt^2} - \frac{1}{2} \left[ K^2 - \omega(t)^2 \cos^2 \theta \right] \sin \theta \omega^2 \theta + \frac{C}{2ML^2} \theta = 0
\] (44)
C is the torsion spring constant for a single spring connected directly to the total system. Equation 44 is the final form of the differential equation of motion. It is non-linear because of the term Sin2θ.
SOLUTION OF THE DIFFERENTIAL EQUATION OF MOTION

The solution of Equation 44 for $\theta$ as a function of time presents a problem because of non-linearity. Since the angle $\theta$ is not restricted to small values, the common linearization of replacing $\sin 2\theta$ by $2\theta$ was not acceptable. Also, the oscillations of the arms are not necessarily small, so perturbation methods would not be useful.

After considerable study and consultation it was concluded that information about the solution could be obtained from existing methods but that an actual solution is out of the question at the present state of development in the field of non-linear differential equations. In view of this major difficulty, three approaches were made to the problem: (a) an iterative integral solution, (b) an analog computer solution, and (c) a digital computer solution. These three methods are discussed below.

It should be pointed out emphatically that the failure to obtain a solution in function form is a serious loss in value of the solution since the three methods indicated above result in solutions for specific numerical values of the parameters involved. Were an analytic solution available, the results for many problems could be obtained easily by merely substitution. However, for the three methods indicated, each problem must be worked anew. This also means that the
results from the three methods will be in graphic or numerical form and very little will be gained at an attempt to synthesize a general result from the results of a variety of problems. Thus, the demonstration of the method of solution is probably of more importance than any specific numerical result.

Iterative Integral Solution

During consultation with a mathematician, it was pointed out that an iterative solution of the differential equation

\[
\frac{d^2 \theta}{dt^2} = \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \sin 2\theta - \frac{C}{2ML^2} \theta \tag{44}
\]

could be expressed in the form

\[
\theta(t) = A \sin \sqrt{N} t + B \cos \sqrt{N} t
\]

\[
+ \frac{1}{2} \sqrt{N} \int_0^t \sin \sqrt{N}(t-s) \left[ K^2 - \omega^2(s) \cos^2 Ks \right] \sin 2\theta_j(s) \, ds \tag{45}
\]

where

\[
N = \frac{C}{2ML^2} \tag{46}
\]

\[
A = \frac{\dot{\theta}(0)}{\sqrt{N}} \tag{47}
\]

\[
B = \theta(0) \tag{48}
\]
If it is assumed that this process is convergent, it can be shown readily that it is a solution of Equation 44 as follows:

From (45),

\[
\frac{d\theta(t)}{dt} = A \sqrt{N} \cos \sqrt{N} t - B \sqrt{N} \sin \sqrt{N} t
\]

\[
+ \frac{1}{2\sqrt{N}} \left[ f(t,t) t' - f(0,t) 0' \right]
\]

\[
+ \int_0^t \frac{\partial f(s,t)}{\partial t} \, ds
\]

(49)

where

\[
f(s,t) = \sin \sqrt{N} (t-s) \left[ k^2 - \omega^2(s) \cos^2 \theta \right] \sin 2\theta_j(s)
\]

(50)

However,

\[
f(t,t) = 0
\]

(51)

\[
0' = 0
\]

(52)

and

\[
\frac{\partial f(s,t)}{\partial t} = \sqrt{N} \cos \sqrt{N} (t-s) \left[ k^2 - \omega^2(s) \cos^2 \theta \right] \sin 2\theta_j(s)
\]

Therefore,
\[
\frac{d^2 \theta(t)}{dt^2} = A \sqrt{N} \cos \sqrt{N} t - B \sqrt{N} \sin \sqrt{N} t
+ \frac{1}{2} \int_0^t \cos \sqrt{N} (t-s) \left[ K^2 - \omega^2(s) \cos^2 Kt \right] \sin 2\theta(s) \, ds
\] .

(54)

Similarly,
\[
\frac{d^2 \theta(t)}{dt^2} = - AN \sin \sqrt{N} t - BN \cos \sqrt{N} t
+ \frac{1}{2} \left\{ \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \sin 2\theta_j(t)
- \sqrt{N} \int_0^t \sin \sqrt{N} (t-s) \left[ K^2 - \omega^2(s) \cos^2 Ks \right] \sin 2\theta_j(s) \, ds \right\} .
\]

(55)

After substituting Equation 55 into Equation 44, the following result is obtained:
\[
- AN \sin \sqrt{N} t - BN \cos \sqrt{N} t
+ \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \sin 2\theta_j(t)
- \frac{\sqrt{N}}{2} \int_0^t \sin \sqrt{N}(t-s) \left[ K^2 - \omega^2(s) \cos^2 Ks \right] \sin 2\theta_j(s) \, ds
+ AN \sin \sqrt{N} t + BN \cos \sqrt{N} t - \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \sin 2\theta_j(t)
+ \frac{N}{2 \sqrt{N}} \int_0^t \sin \sqrt{N}(t-s) \left[ K^2 - \omega^2(s) \cos^2 Ks \right] \sin 2\theta_j(s) \, ds .
\]

(56)
Thus it is evident that Equation 45 is a solution, in the limit, of Equation 44 if the process is convergent. Actually this scheme was not helpful because the integrand expanded rapidly with each integration for the few attempts which were made. The starting function for \( \theta_0(t) \) could be anything, the simpler the better, such as a constant. Of course, the more nearly the ultimate solution could be guessed, the fewer the number of iterations for a given accuracy. At one stage of the problem it was thought that this procedure might be used profitably as the mode of integration on the IBM 650 where the indicated integrations would be done numerically. Further study indicated that this would not be as efficient a method as the conventional finite differences method of solution, and this avenue of approach was dropped.

**Analog Computer Solutions**

The development of analog computer methods is quite new, having come about mostly since World War II. Analogies have been used for quite some time, but these usually make use of one physical system to solve problems in another type system.
in cases where the governing equations are quite similar. The analog computer can be used to a limited degree in this same way, but it is most useful as an equation solver, especially differential equations. Strictly speaking, even the analog computer uses one physical system to solve problems in another, but many special techniques make it a straightforward procedure without a conscious correlation of physical equivalents on the part of the operator.

**Analog method used to solve the fly-ball problem**

The differential equation was solved for the highest derivative and the equivalent terms fed into an integrating amplifier. The main difficulties encountered stemmed from the fact that one of the inputs to the first integrator was not easily generated. Figure 7 shows a composite program for the solution of Equation 44. Since the various series of runs, based on type of turning function, \( \omega(t) \), differed somewhat in the exact hookup, one single circuit did not suffice. In order to trace out the connections for any series, the switches were closed as indicated by the letters A, B, C, or D. These letters refer to the series of runs and the series were determined by the type of turning function, \( \omega(t) \), used. Table 2 shows the displacement function and the resulting turning function for each series. The turning function, \( \omega(t) \), is obtained from the displacement function, \( \vec{T}(t) \), by
Figure 7. Composite schematic of the analog computer solution of

\[
\frac{d^2 \theta}{dt^2} = \frac{1}{2} \left[ k^2 - \Omega(t) \cos^2 Kt \right] \sin 2\theta - N\theta
\]

In order to obtain the circuit for any given series (A, B, C, or D), connect the optional switches as shown.
Table 2. Displacement functions and corresponding turning functions used for each series of runs

<table>
<thead>
<tr>
<th>Series</th>
<th>Displacement function $\phi(t)$</th>
<th>Turning function $\omega(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$U\sin(pt)$</td>
<td>$U\cos(pt) = U\sin(pt - \pi/2)$</td>
</tr>
<tr>
<td>B</td>
<td>$Ue^{-rt}\sin(pt)$</td>
<td>$Ue^{-rt}(p\cos(pt) - r\sin(pt))$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$= Ue^{-rt} p^2 + r^2 \sin(pt + \delta)$</td>
</tr>
<tr>
<td>C</td>
<td>$U\sin^2(pt)$</td>
<td>$2U\sin(pt)\cos(pt) = U\sin(2pt)$</td>
</tr>
<tr>
<td>D1</td>
<td>$U$</td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td>$U^2$</td>
<td>$2U$</td>
</tr>
<tr>
<td>D3</td>
<td>$U^3$</td>
<td>$3U^2$</td>
</tr>
</tbody>
</table>

$^{a}U,$ $p,$ and $r$ are constants but not the same for all problems.

$b \omega(t) = \frac{d\phi(t)}{dt}.$

$c \delta = \tan^{-1}\left(\frac{p}{-r}\right).$

$d$ One cycle only of the function used.

one differentiation with respect to time. The reason displacement functions were used as a starting point instead of going directly to the turning function was the fact that it is much more difficult to visualize velocity than it is to visualize displacement. The specific choices of displacement functions were quite arbitrary other than that most of them were
considered similar to motions a missile or aircraft might make in flight.

**General description of operation.** The three amplifiers marked 3, 5, and 7 comprise the heart of the system. From Equation 44 it is obvious that the two inputs to Amp. 3 are

\[ \frac{1}{2} \left[ K^2 - \omega^2(t)\cos^2 Kt \right] \sin 2\theta \]  

and

\[ -\frac{C}{2NL^2} \theta = -N\theta \]  

After time scaling by substituting \( \tau/T \) for \( t \) in the differential equation, Term 58 becomes

\[ \frac{0.08 \times 31.25}{5t^2} \left[ K^2 - \omega^2 \frac{\tau}{T} \cos^2 K \frac{\tau}{T} \right] \sin 2\theta \]  

where

- \( t \) = real time in physical system
- \( \tau \) = real time in computer
- \( T \) = time scale.

Also Equation 59 becomes \( -N/T^2 \theta \) in the same manner. If \( T \) is larger than 1, the problem is slowed down on the computer, that is, takes longer on the computer than in physical problem. This was necessary to meet servo-multiplier response time. The three numbers of term 60 were not combined because the 0.08 came from the product of Fn. Mult. No. 2 whose input
is 80\sin 2\theta, the 5 came from the fact that - 5\theta was desired as the output of Amp. 3, and the \(31.25/T^2\) was the other factor necessary after the scaling process. Similarly, the output of Amp. 7 was - 40\theta and - 5N/T^2 \theta was needed; so if - 40\theta is multiplied by \(N/8T^2\), the result is correct. Multiplication by 8 and a second integration was accomplished by Amp. 5; this resulted in an output of 40\theta, a multiple of the desired result. Amp. 7 is merely a sign changer to feed - 40\theta back to Amp. 3. Thus, it is apparent that the equipment indicated in Figure 7 (other than Amps. 3, 5, and 7) was all necessary to generate Term 60. Initial conditions were set for the problem by placing voltages on the feedback capacitors of Amps. 3 and 5. Since all problems were started with \theta = 0, the feedback capacitor of Amp. 3 was shorted until the problems started. The appropriate voltage was initially placed across the feedback capacitor of Amp. 5 (here the voltage was initially = 40\theta_0 \text{ d.c.}).

\sin 2\theta was generated by Fn. Gen. No. 2 and the associated amplifiers (Amps. 13 and 15) by utilizing as the independent variable the output of Amp. 5. The input bias was necessary to allow an input range of \theta between 0 and 3\pi/4 and still let the function generator be varied by an input centered about zero at the input to the function generator proper.

The two multipliers were necessary first to multiply \(\omega^2 (\tau/T)\) by \(\cos^2 K \tau/T\) and then to multiply \([K^2 - \omega^2 (\tau/T)\cos^2 K \tau/T]\) by \sin 2\theta, ignoring constant factors.
easily handled. The factor Q seen in Figure 7 is merely a device to allow changing of the amplitude of a given \( \omega^2(\tau/T) \) without having to set it up again in the function generator.

The group consisting of Amps. 6, 8, 10, 12, and 14 were used to generate \( 20\cos^2K \tau/T \) for all series other than the B-series. This generation was accomplished by solving a differential equation whose solution is \( 40\cos2K \tau/T \) with Amps. 8, 10, and 12 and then adding a constant to this to obtain \( 20\cos^2K \tau/T \). Amp. 6 was used here to obtain a proper amount of "negative damping" to barely prevent the decay of the desired function. For Series B, \( Q\omega^2(\tau/T) \) was generated by using Amp. 6 to generate a non-oscillating decay curve and Amps. 8, 10, and 12 to generate a sine wave (see Appendix B). Then these two were combined in Amp. 14 to produce the damped sine-squared wave which comprised \( \omega^2(\tau/T) \) for this series.

Amp. 9 brings about the subtraction of \( K^2 \) and \( \omega^2(\tau/T)\cos^2K \tau/T \) as necessary in Term 60. Then Amp. 11 merely changes the sign of \( 0.1 \left[ K^2 - \omega^2(\tau/T)\cos^2K \tau/T \right] \) to be able to use Fn. Mult. No. 2 as a four-quadrant multiplier; this is necessary because \( 80\sin2\theta \) is not restricted to positive values for the interesting range of \( \theta \).

Fn. Gen. No. 1 (with Amps. 2 and 4) was used to generate \( \omega^2(\tau/T) \) for all series except B and D1. For Series A and C this was accomplished by the use of a triangle wave generator to provide a time-controlled input to Fn. Gen. No. 1. For
D2 and D3 a ramp function generated by integrating a constant in Amp. 1 was used as input to the function generator. For Series B, the function generator was used to generate $80\cos^2\frac{\gamma}{T}$, again by using the triangular wave as input. For Series D1, the input to Fn. Mult. No. 1 was simply a constant and was applied as shown in Figure 7. When Amp. 1 was used with the triangular wave generator, it served as a signal booster (X10) because the triangular wave generator had a maximum voltage of only about 15 volts and the useful range of the function generator was ±100 volts input.

The group of instruments marked "Synchronized Starting System" were necessary in order to ensure that $\omega^2(\tau/T)$ (or $80\cos^2\frac{\gamma}{T}$ for Series B) started in synchronism with the rest of the problem. A problem with integrators is started by a set of relay switches which release the initial voltages on the feedback capacitors. Since the output of Fn. Gen. No. 1 was produced independently of the rest of the problem, an automatic method of forcing one part to start the other was necessary. The triangular wave generator possessed a synchronization pulse of very short duration which occurred exactly at the upper apex of the triangle wave. This pulse was fed to the electronic switch as the triggering signal. Each time the pulse occurred, the electronic switch alternated in connecting its output with either the A-input or the B-input. From the power supply, 6.3 volts a.c. were connected to input A of the electronic switch. The output of the
electronic switch was amplified further by the a.c. amplifier and transformer in order to obtain sufficient voltage to operate the DPDT relay. Thus, the entire system was dormant until the triangular wave generator pulsed at the top of the triangular wave. This pulse ultimately closed the relay which disconnected the pulse and closed the starting switch on the main console (switch A, Figure 7). At the same instant the pulse occurred, the triangular input to Fn. Gen. No. 1 started producing $Q\omega^2(\tau/T)$. This function generator was set up in such a manner as to produce $Q\omega^2(\tau/T)$ in proper synchronism when the input voltage started at peak value and then continued at the proper frequency. For Series B, this starting sequence was applied to generating the function $80\cos^2K\tau/T$ with the proper synchronization to the rest of the problem. For Series D, no synchronization problems were encountered because Amp. 1 by itself was used to feed Fn. Gen. No. 1 and switch A on the main console started everything in unison.

The manual start switch near the transformer in Figure 7 was used for Series A and B to initiate a problem—from then on the sequence of operations was automatic. For Series C, some other manual operations were necessary to limit $\omega^2(\tau/T)$ to one cycle. Here the sequence of operations was as follows:

1. Turn on manual start switch.
2. Turn on switch A on computer manually after trigger fires the relay.
3. Wait until the next sync pulse releases the relay.
4. Turn off manual starting switch.

This process was necessary to obtain only one cycle of \( \omega^2 (\tau/T) \).

It should be mentioned that the method used here for synchronizing \( \omega^2 (\tau/T) \) with the rest of the problem could have been done much more neatly with a special flip-flop circuit. However, the indicated components were already available at no extra cost so they were used.

**Specific problems solved.** An inspection of Equation 44 and Table 2 shows that a number of problems could have been worked for specific values of the various parameters. Several reasonable values of each parameter were chosen arbitrarily. To avoid making a run for every combination of all parameters, a certain value of each parameter was considered its "central value", and when a given parameter was varied, the other parameters were all held at their central values. Values both larger and smaller than the central value were used for each parameter. Table 3 gives the values of the parameters for each run of each series and the starting values of \( \Theta \), called \( \Theta_0 \). The term \( C/2ML^2 \) has been lumped as one parameter, \( N \), for all problems.

In Table 3, \( \Theta_n \) was calculated by elementary methods for the case of no turning function and \( \Theta \equiv 0 \). This was merely an equilibration between centrifugal moment and the resisting spring moment (see Appendix A).
Table 3. Values of parameters for problems solved by analog computer

<table>
<thead>
<tr>
<th>Setup no.</th>
<th>r</th>
<th>p</th>
<th>U</th>
<th>N</th>
<th>K</th>
<th>(\theta_0 = \theta_n^b)</th>
<th>(\theta \neq \theta_n^b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(sec)</td>
<td>(rad)</td>
<td>(sec(^2))</td>
<td>(rad)</td>
<td>(sec)</td>
<td>Run no.</td>
<td>(\theta_0)</td>
</tr>
<tr>
<td><strong>Series A</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A-1</td>
<td>-</td>
<td>1.0</td>
<td>(\pi/2)</td>
<td>77.19</td>
<td>10(\pi/3)</td>
<td>1</td>
<td>0.70</td>
</tr>
<tr>
<td>A-2</td>
<td>-</td>
<td>2.0</td>
<td>1</td>
<td></td>
<td></td>
<td>3</td>
<td>0.70</td>
</tr>
<tr>
<td>A-3</td>
<td>-</td>
<td>5.0</td>
<td>1</td>
<td></td>
<td></td>
<td>5</td>
<td>0.70</td>
</tr>
<tr>
<td>A-4</td>
<td>-</td>
<td>10.0</td>
<td>1</td>
<td></td>
<td></td>
<td>7</td>
<td>0.70</td>
</tr>
<tr>
<td>A-5</td>
<td>-</td>
<td>2.0</td>
<td>(\pi/20)</td>
<td></td>
<td></td>
<td>9</td>
<td>0.70</td>
</tr>
<tr>
<td>A-6</td>
<td>-</td>
<td>(\pi)</td>
<td></td>
<td></td>
<td></td>
<td>11</td>
<td>0.70</td>
</tr>
<tr>
<td>A-7</td>
<td>-</td>
<td>(\pi/2)</td>
<td>0</td>
<td></td>
<td></td>
<td>13</td>
<td>1.571</td>
</tr>
<tr>
<td>A-8</td>
<td>-</td>
<td>(\pi/2)</td>
<td>20</td>
<td></td>
<td></td>
<td>15</td>
<td>1.325</td>
</tr>
<tr>
<td>A-9</td>
<td>-</td>
<td>(\pi/2)</td>
<td>200</td>
<td></td>
<td></td>
<td>17</td>
<td>0</td>
</tr>
<tr>
<td>A-10</td>
<td>-</td>
<td>77.19</td>
<td>(\pi)</td>
<td></td>
<td></td>
<td>19</td>
<td>0</td>
</tr>
<tr>
<td>A-11</td>
<td>-</td>
<td>5(\pi)</td>
<td></td>
<td></td>
<td></td>
<td>21</td>
<td>1.17</td>
</tr>
<tr>
<td>A-12</td>
<td>-</td>
<td>10(\pi)</td>
<td></td>
<td></td>
<td></td>
<td>23</td>
<td>1.47</td>
</tr>
<tr>
<td><strong>Series B</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B-1</td>
<td>.05</td>
<td>2.0</td>
<td>(\pi/2)</td>
<td>77.19</td>
<td>10(\pi/3)</td>
<td>25</td>
<td>0.70</td>
</tr>
<tr>
<td>B-2</td>
<td>.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>27</td>
<td>0.70</td>
</tr>
<tr>
<td>B-3</td>
<td>.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>29</td>
<td>0.70</td>
</tr>
</tbody>
</table>

\(a\)\(\theta_0 = 0\) for all problems.

\(b\)\(\theta_n\) is the equilibrium angle \(\theta\) for the given parameters with \(\omega(t) = 0\) and \(\ddot{\theta} = 0\).
Table 3 (continued)

<table>
<thead>
<tr>
<th>Setup no.</th>
<th>r (sec)</th>
<th>p (sec)</th>
<th>U (rad)</th>
<th>N (rad)</th>
<th>K (sec)</th>
<th>( \theta_0 = \theta_n )</th>
<th>( \theta_0 = \theta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B-4</td>
<td>0.10</td>
<td>1.0</td>
<td>( \pi/2 )</td>
<td>77.19</td>
<td>10( \pi/3 )</td>
<td>31 0.70</td>
<td>32 0.20</td>
</tr>
<tr>
<td>B-5</td>
<td>5.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>33 0.70</td>
<td>34 0.20</td>
</tr>
<tr>
<td>B-6</td>
<td>10.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>35 0.70</td>
<td>36 0.20</td>
</tr>
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<td>46 0.500</td>
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<tr>
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<td></td>
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<td>( \pi )</td>
<td></td>
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<td>48 0.500</td>
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<td>B-13</td>
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<td>5( \pi )</td>
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Zero Series

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<th>r (sec)</th>
<th>p (sec)</th>
<th>U (rad)</th>
<th>N (rad)</th>
<th>K (sec)</th>
<th>( \theta_0 = \theta_n )</th>
<th>( \theta_0 = \theta_n )</th>
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<tbody>
<tr>
<td>0-1</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>77.19</td>
<td>10( \pi/3 )</td>
<td>53 0.70</td>
<td>54 0.20</td>
</tr>
<tr>
<td>0-2</td>
<td>-</td>
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<td>55 1.571</td>
<td>56 1.071</td>
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<td>58 0.825</td>
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<td>60 0.67</td>
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<tr>
<td>0-5</td>
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<td>( \pi )</td>
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<td>62 0.97</td>
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<td>10( \pi )</td>
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Table 3 (continued)

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<th>U (sec)</th>
<th>N (rad)</th>
<th>K (rad)</th>
<th>$\theta_0 = \theta_n^b$</th>
<th>$\theta \neq \theta_n^b$</th>
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<td>$\pi/20$</td>
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<td>$\pi$</td>
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<td>76 1.071</td>
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<td>84 0.67</td>
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<td>-</td>
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Table 3 (continued)

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<th>sec² (rad)</th>
<th>rad (sec)</th>
<th>Run no.</th>
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<td></td>
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<td>.97</td>
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<td>.20</td>
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<td>.70 134</td>
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<td>139</td>
<td>1.325 140</td>
<td>.825</td>
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Table 3 (continued)

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<tr>
<th>Setup no.</th>
<th>r (rad)</th>
<th>p (sec)</th>
<th>U (rad)</th>
<th>N (sec)</th>
<th>K (rad)</th>
<th>( \theta_0 = \theta_n^b )</th>
<th>( \theta \neq \theta_n^b )</th>
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<td>142 0.50</td>
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<td>-</td>
<td>77.19</td>
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<td>0 144</td>
<td>0.50</td>
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<td>1.17</td>
<td>146 0.67</td>
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<td>10\pi</td>
<td>147</td>
<td>1.47</td>
<td>148 0.97</td>
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</tbody>
</table>

Figure 8 is a photograph of the analog computer setup except that the instruments of the starting synchronization system have been omitted. Figure 9 is a close-up of the control panel alone, with components but without wiring. The operational amplifiers are at the top of the control panel; each amplifier has three tubes and one knob protruding from the top. Fifteen amplifiers were used for all series except D3.

Considerable attention to detail was necessary in operating the computer. There were so many steps that had to be made in the proper order that a checkoff list was used. A sample of this check off list is included in Appendix B. Also included in Appendix B is a sample data sheet for Series B; other data sheets were similar. The results of the...
Figure 8. Photograph of analog computer equipment except the synchronized starting devices
CP — CONTROL PANEL  CS — COMPONENT STOR.
FG — FUNCTION GENERATOR  LS — LEAD STORAGE
RA — RECORDER AMPLIFIER  PS — POWER SUPPLY
RP — RECORDER PENMOTOR  M — MULTIPLIER
MP — MULTIPLIER PATCHBOARD  T — TIMER
Figure 9. Analog computer control panel with components but without wiring
solutions were recorded on a 4-channel recorder tape as indicated in Figure 8.

Results of Analog Computer Solutions

The results of the analog solutions were in the form of graphs on a time-fed paper tape. While only the angle \( \theta \) was actually desired, three other values were recorded also. These other three values were \( \omega^2 (\gamma/T) \), \( 20 \cos^2 k \gamma/T \) and \( 80 \sin 2\theta \). These values were necessary to make certain that key components of the computer were functioning properly. The plotting of \( 20 \cos^2 k \gamma/T \) was especially necessary for all series (except Series A) because the negative feedback had to be adjusted to maintain the amplitude exactly constant. Figure 10 is a photostat taken of the actual tape for Run 21. Some india ink work has obviously been done to label clearly the four tracks and their ranges of values. The time marks at the top of the chart are one second (computer time) apart. The values shown at the left are voltages and must be interpreted properly to convert to physical system variables. For Figure 10, the turning function happened to be \( \omega(t) = \pi \cos 2t \) or \( \omega^2(t) = \pi^2 \cos^2 2t \) and thus \( Q = 8.11 \) in order that the amplitude of the wave for \( Q \omega^2(t) \) be 80 volts. This scaling obviated resetting the function generator with which \( Q \omega^2(t) \) was formed for every change of magnitude of \( \omega(t) \). Then the output, \( Q \omega^2 (\gamma/T) \), was cut down at an input to an
Figure 10. Sample tape used to record the solutions by analog computer.
operational amplifier (No. 9) so as to give the proper amplitude of \( \omega^2(\tau/T) \) in the problem. Also in Figure 10, the trace marked \( 80\sin2\theta \) shows the output of Fn. Gen. No. 2 (actually Amp. 15).

Main group of problems

The analog method was recognized from the start as having limited accuracy because of system errors entering the problem. Hence, the rather extensive set of problems indicated in Table 3 was solved on the analog computer primarily as a survey of the solutions for a large number of combinations of parameters. This is the philosophy of using the analog computer as a probe and then making much more accurate solutions of interesting problems with a digital computer. Table 4b summarizes the results of the problems shown in Table 3. Table 4a is the key to two of the stub headings of Table 4b. Since the same information is not applicable to all runs, this device is used here to avoid an extremely large table with several headings inapplicable to given runs. Figures 11 and 12 are to be used with Table 4. Figure 11 is a group of traces of 409 each of which is typical of several runs; each of the entries in Table 4b indicates which type of result the trace for a particular run resembles. The effects of the turning function are indicated in Table 4b qualitatively as (a) pronounced, (b) moderate or (c) slight.
Table 4a. Meaning of column headings A and B of Table 4b

<table>
<thead>
<tr>
<th>Type run no.</th>
<th>Col. hdg.</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1</td>
<td>-</td>
<td>{Types 1 and 2 show traces of constant value throughout the duration of the problem}</td>
</tr>
<tr>
<td>Type 2</td>
<td>A</td>
<td>Beat frequency as a multiple of the frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum magnitude of amplitude of beat (for $\theta$, not $40\theta$), radians</td>
</tr>
<tr>
<td>Type 3</td>
<td>A</td>
<td>Beat frequency as a multiple of the frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum magnitude of amplitude of beat (for $\theta$, not $40\theta$), radians</td>
</tr>
<tr>
<td>Type 4</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Type 5</td>
<td>A</td>
<td>Beat frequency as a multiple of the frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum magnitude of amplitude of beat (for $\theta$, not $40\theta$), radians</td>
</tr>
<tr>
<td>Type 6</td>
<td>A</td>
<td>Maximum deviation of $\theta$ from the trace for $\omega^2(t) = 0$, radians</td>
</tr>
<tr>
<td>Type 7</td>
<td>-</td>
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<tr>
<td>Type 8</td>
<td>A</td>
<td>Amount of decrease of axis of wave at end of run, radians</td>
</tr>
<tr>
<td>Type 9</td>
<td>A</td>
<td>Beat frequency as a multiple of the frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum magnitude of amplitude of beat (for $\theta$, not $40\theta$), radians</td>
</tr>
<tr>
<td>Type 10</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Type 11</td>
<td>A</td>
<td>Magnitude of downward shift of centerline ($\theta$, not $40\theta$), radians</td>
</tr>
<tr>
<td>Type 12</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Type 13</td>
<td>A</td>
<td>Time for $\theta$ to = 0, seconds</td>
</tr>
</tbody>
</table>
Table 4a (continued)

<table>
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<th>Type run or run no.</th>
<th>Col. hdg.</th>
<th>Meaning</th>
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<td>Number of cycles before the valid limit of $\theta$ was reached</td>
</tr>
<tr>
<td>Type 15a A</td>
<td></td>
<td>Number of cycles before valid limit of $\theta$ was reached</td>
</tr>
<tr>
<td>Type 16 A</td>
<td></td>
<td>Frequency of the superimposed ripple on the later portion of the trace as a multiple of the frequency of $\cos^2\theta$</td>
</tr>
<tr>
<td>Type 17 A</td>
<td></td>
<td>Frequency of the ripple on the later portion of the trace as a multiple of the frequency of $\cos^2\theta$</td>
</tr>
<tr>
<td>Runs 3 and 4 A</td>
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<td>Beat frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Beat amplitude of $\theta$, radians</td>
</tr>
<tr>
<td>Run 5 A</td>
<td></td>
<td>Basic frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td>Runs 11 and 12 A</td>
<td></td>
<td>Frequency of ripple on bottom of trace segments as a multiple of frequency of $\cos^2\theta$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Beat frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td>Run 27 A</td>
<td></td>
<td>Beat frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Maximum magnitude of amplitude of beat (for $\theta$, not rOe), radians</td>
</tr>
<tr>
<td>Run 33 A</td>
<td></td>
<td>Basic frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
</tbody>
</table>

$^a$Types 14 and 15 are quite similar other than, for Type 14, $\theta$ becomes $= 0$ whereas for Type 15, $\theta$ exceeds the valid limit.
Table 4a (continued)

<table>
<thead>
<tr>
<th>Type run run no.</th>
<th>Col.</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Run 34</td>
<td>A</td>
<td>Basic frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td>Runs 39 and 40</td>
<td>A</td>
<td>Basic frequency as a multiple of frequency of $\omega^2(t)$</td>
</tr>
<tr>
<td>Runs 91 and 92</td>
<td>A</td>
<td>Frequency of ripple superimposed on wave, especially noticeable in later part of trace</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum deviation from trace for $\omega^2(t) = 0$, radians</td>
</tr>
<tr>
<td>Run 103</td>
<td>A</td>
<td>Time in seconds for $\theta$ to $= 0$, seconds</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>Maximum deviation of $\theta$ from trace for $\omega^2(t) = 0$, radians</td>
</tr>
</tbody>
</table>
Table 4b. Summary of results of problems listed in Table 3

<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp. zero</th>
<th>Dur. of effect</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.930</td>
<td>slight</td>
<td>1a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0.830</td>
<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>3</td>
<td>Fig. 12</td>
<td>0.88</td>
<td>mod.</td>
<td>1a</td>
<td>cont.</td>
<td>1.0</td>
<td>0.075</td>
</tr>
<tr>
<td>4</td>
<td>Fig. 12</td>
<td>0.88</td>
<td>mod.</td>
<td>1b</td>
<td>cont.</td>
<td>1.0</td>
<td>0.10</td>
</tr>
<tr>
<td>5</td>
<td>Fig. 12</td>
<td>1.16</td>
<td>pron.</td>
<td>1a</td>
<td>cont.</td>
<td>1.0</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>14</td>
<td>0.59</td>
<td>pron.</td>
<td>1b</td>
<td>cont.</td>
<td>2.0</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>-</td>
<td>pron.</td>
<td>1a</td>
<td>cont.</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>13</td>
<td>-</td>
<td>pron.</td>
<td>1b</td>
<td>cont.</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0.94</td>
<td>slight</td>
<td>1a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>0.85</td>
<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
</tbody>
</table>

\(^{a}\)See Figure 11 for type number if a number alone appears here; see Figure 12 if indicated and find the run indicated.

\(^{b}\)Basic frequency is the one most apparent and is usually given as a multiple of the natural frequency, \(\sqrt{N}\).

\(^{c}\)Qualitative magnitude given as slight, moderate or pronounced.

\(^{d}\)Comparative zero - indicates the corresponding run for which \(\omega(t) = 0\); see Figure 13.

\(^{e}\)Duration of effect of turning function; n.m. = not measurable, cont. = continuous.

\(^{f}\)See Table 4a for specific meaning for the given run.
<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp. zero</th>
<th>Dur. of effect</th>
<th>$A^f$</th>
<th>$B^f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>Fig. 12</td>
<td>0.88 pron.</td>
<td>1a</td>
<td>cont.</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>Fig. 12</td>
<td>0.88 pron.</td>
<td>1b</td>
<td>cont.</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>10.2$^g$ mod.</td>
<td>2a</td>
<td>last half</td>
<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>7.9$^g$ slight</td>
<td>2b</td>
<td>near end</td>
<td>2.0</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>9</td>
<td>2.43 slight</td>
<td>3a</td>
<td>last half</td>
<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>14</td>
<td>2.16 slight</td>
<td>3b</td>
<td>last part</td>
<td>3.0</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>slight</td>
<td>4a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>13</td>
<td>slight</td>
<td>4b</td>
<td>n.m.</td>
<td>0.16</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>slight</td>
<td>5a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>13</td>
<td>slight</td>
<td>5b</td>
<td>n.m.</td>
<td>0.2</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10</td>
<td>slight</td>
<td>6a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>12</td>
<td>slight</td>
<td>6b</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>10</td>
<td>slight</td>
<td>7a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>12</td>
<td>slight</td>
<td>7b</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>3</td>
<td>0.86 mod.</td>
<td>1a</td>
<td>first 25 sec.</td>
<td>1.0 0.075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>5</td>
<td>0.81 mod.</td>
<td>1b</td>
<td>first 17 sec.</td>
<td>1.0 0.075</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>Fig. 12</td>
<td>0.86 mod.</td>
<td>1a</td>
<td>first 22 sec.</td>
<td>1.0 0.056</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^g$This is the actual frequency in radians per second since the natural frequency ($\sqrt{N}$) is = 0 here for $\omega(t) = 0$. 
<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp zero</th>
<th>Dur. of effect</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>5</td>
<td>0.81</td>
<td>mod.</td>
<td>lb</td>
<td>first 18 sec.</td>
<td>0.63</td>
<td>0.075</td>
</tr>
<tr>
<td>29</td>
<td>3</td>
<td>0.95</td>
<td>mod.</td>
<td>la</td>
<td>first 8 sec.</td>
<td>1.0</td>
<td>0.050</td>
</tr>
<tr>
<td>30</td>
<td>5</td>
<td>0.82</td>
<td>mod.</td>
<td>lb</td>
<td>first 10 sec.</td>
<td>0.80</td>
<td>0.063</td>
</tr>
<tr>
<td>31</td>
<td>3</td>
<td>0.88</td>
<td>mod.</td>
<td>la</td>
<td>first 10 sec.</td>
<td>1.0</td>
<td>0.063</td>
</tr>
<tr>
<td>32</td>
<td>5</td>
<td>0.86</td>
<td>mod.</td>
<td>lb</td>
<td>first 10 sec.</td>
<td>1.0</td>
<td>0.075</td>
</tr>
<tr>
<td>33</td>
<td>Fig. 12</td>
<td>1.01</td>
<td>pron.</td>
<td>la</td>
<td>first 15 sec.</td>
<td>0.92</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>Fig. 12</td>
<td>0.89</td>
<td>pron.</td>
<td>lb</td>
<td>first 15 sec.</td>
<td>0.79</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>13</td>
<td>-</td>
<td>pron.</td>
<td>la</td>
<td>cont.</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>13</td>
<td>-</td>
<td>pron.</td>
<td>lb</td>
<td>cont.</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>1</td>
<td>-</td>
<td>slight</td>
<td>la</td>
<td>n.m.</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>5</td>
<td>0.84</td>
<td>slight</td>
<td>lb</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>39</td>
<td>Fig. 12</td>
<td>0.87</td>
<td>pron.</td>
<td>la</td>
<td>first 15 sec.</td>
<td>1.94</td>
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</tr>
<tr>
<td>40</td>
<td>Fig. 12</td>
<td>0.86</td>
<td>pron.</td>
<td>lb</td>
<td>first 14 sec.</td>
<td>1.94</td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>10</td>
<td>10.5</td>
<td>slight</td>
<td>2a</td>
<td>n.m.</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>15</td>
<td>9.5</td>
<td>slight</td>
<td>2b</td>
<td>n.m.</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>10</td>
<td>2.4</td>
<td>slight</td>
<td>3a</td>
<td>n.m.</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>
Table 4b (continued)

<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp. zero</th>
<th>Dur. of effect</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>44</td>
<td>14</td>
<td>2.07</td>
<td>slight</td>
<td>3b</td>
<td>n.m.</td>
<td>2.5</td>
<td>-</td>
</tr>
<tr>
<td>45</td>
<td>2</td>
<td>-</td>
<td>slight</td>
<td>4a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>46</td>
<td>13</td>
<td>-</td>
<td>slight</td>
<td>4b</td>
<td>n.m.</td>
<td>0.14</td>
<td>-</td>
</tr>
<tr>
<td>47</td>
<td>2</td>
<td>-</td>
<td>slight</td>
<td>5a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>48</td>
<td>13</td>
<td>-</td>
<td>slight</td>
<td>5b</td>
<td>n.m.</td>
<td>0.17</td>
<td>-</td>
</tr>
<tr>
<td>49</td>
<td>10</td>
<td>1.79</td>
<td>slight</td>
<td>6a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>1.57</td>
<td>slight</td>
<td>6b</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>51</td>
<td>10</td>
<td>3.65</td>
<td>slight</td>
<td>7a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>52</td>
<td>12</td>
<td>3.44</td>
<td>slight</td>
<td>7b</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>63</td>
<td>1</td>
<td>-</td>
<td>slight</td>
<td>1a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>64</td>
<td>5</td>
<td>0.77</td>
<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>65</td>
<td>1</td>
<td>-</td>
<td>slight</td>
<td>1a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>66</td>
<td>5</td>
<td>0.81</td>
<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>67</td>
<td>6</td>
<td>0.88</td>
<td>pron.</td>
<td>1a</td>
<td>first</td>
<td>0.1</td>
<td>25 sec.</td>
</tr>
<tr>
<td>68</td>
<td>5</td>
<td>0.85</td>
<td>pron.</td>
<td>1b</td>
<td>first</td>
<td>no</td>
<td>no beat</td>
</tr>
<tr>
<td>69</td>
<td>6</td>
<td>0.87</td>
<td>pron.</td>
<td>1a</td>
<td>first</td>
<td>0.41</td>
<td>-</td>
</tr>
<tr>
<td>70</td>
<td>5</td>
<td>0.88</td>
<td>pron.</td>
<td>1b</td>
<td>first</td>
<td>no</td>
<td>no beat</td>
</tr>
<tr>
<td>71</td>
<td>1</td>
<td>-</td>
<td>slight</td>
<td>1a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 4b (continued)

<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp. zero</th>
<th>Dur. of effect</th>
<th>A^f</th>
<th>B^f</th>
</tr>
</thead>
<tbody>
<tr>
<td>72</td>
<td>5</td>
<td>0.82</td>
<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>73</td>
<td>Fig. 12</td>
<td>0.91</td>
<td>mod.</td>
<td>1a</td>
<td>first 2.5 sec.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>74</td>
<td>5</td>
<td>0.82</td>
<td>pron.</td>
<td>1b</td>
<td>first 9 sec.</td>
<td>1.0</td>
<td>0.175</td>
</tr>
<tr>
<td>75</td>
<td>10</td>
<td>10.25^n</td>
<td>slight</td>
<td>2a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>76</td>
<td>15</td>
<td>9.50^n</td>
<td>slight</td>
<td>2b</td>
<td>n.m.</td>
<td>1.0</td>
<td>-</td>
</tr>
<tr>
<td>77</td>
<td>10</td>
<td>2.48</td>
<td>slight</td>
<td>3a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>78</td>
<td>14</td>
<td>2.20</td>
<td>slight</td>
<td>3b</td>
<td>n.m.</td>
<td>1.5</td>
<td>-</td>
</tr>
<tr>
<td>79</td>
<td>2</td>
<td>-</td>
<td>slight</td>
<td>4a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>80</td>
<td>13</td>
<td>-</td>
<td>slight</td>
<td>4b</td>
<td>n.m.</td>
<td>0.16</td>
<td>-</td>
</tr>
<tr>
<td>81</td>
<td>2</td>
<td>-</td>
<td>slight</td>
<td>5a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>82</td>
<td>13</td>
<td>-</td>
<td>slight</td>
<td>5b</td>
<td>n.m.</td>
<td>0.2</td>
<td>-</td>
</tr>
<tr>
<td>83</td>
<td>10</td>
<td>1.82</td>
<td>slight</td>
<td>6a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>84</td>
<td>12</td>
<td>1.52</td>
<td>slight</td>
<td>6b</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>85</td>
<td>10</td>
<td>3.59</td>
<td>slight</td>
<td>7a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>86</td>
<td>14</td>
<td>-</td>
<td>slight</td>
<td>7b</td>
<td>n.m.</td>
<td>0.5</td>
<td>-</td>
</tr>
<tr>
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<td>-</td>
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<td>n.m.</td>
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<td>-</td>
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<tr>
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<td>slight</td>
<td>1b</td>
<td>n.m.</td>
<td>no beat</td>
<td>no beat</td>
</tr>
<tr>
<td>91</td>
<td>Fig. 12</td>
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<td>92</td>
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<td>1b</td>
<td>cont.</td>
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<td>pron.</td>
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</table>
Table 4b (continued)

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<th>Basic frequency&lt;sup&gt;b&lt;/sup&gt;</th>
<th>Qual. mag.&lt;sup&gt;c&lt;/sup&gt;</th>
<th>Comp. zero&lt;sup&gt;d&lt;/sup&gt;</th>
<th>Dur. of effect&lt;sup&gt;e&lt;/sup&gt;</th>
<th>A&lt;sup&gt;f&lt;/sup&gt;</th>
<th>B&lt;sup&gt;f&lt;/sup&gt;</th>
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<td>pron.</td>
<td>1b</td>
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<td>-</td>
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<td>mod.</td>
<td>2a</td>
<td>last 2/3</td>
<td>1.48&lt;sup&gt;h&lt;/sup&gt;</td>
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<tr>
<td>98</td>
<td>15</td>
<td>9.6&lt;sup&gt;g&lt;/sup&gt;</td>
<td>slight</td>
<td>2b</td>
<td>n.m.</td>
<td>1.0</td>
<td>-</td>
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<td>99</td>
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<td>pron.</td>
<td>3a</td>
<td>cont.</td>
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<td>1.5&lt;sup&gt;h&lt;/sup&gt;</td>
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<td>n.m.</td>
<td>0.15</td>
<td>-</td>
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<td>Fig. 12</td>
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<td>cont.</td>
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<td>6b</td>
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<td>-</td>
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<tr>
<td>107</td>
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<td>3.6</td>
<td>slight</td>
<td>7a</td>
<td>n.m.</td>
<td>-</td>
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<tr>
<td>108</td>
<td>14</td>
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<td>pron.</td>
<td>7b</td>
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<td>110</td>
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<td>1b</td>
<td>n.m.</td>
<td>-</td>
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<td>111</td>
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<td>mod.</td>
<td>1a</td>
<td>last 2.3</td>
<td>1.0</td>
<td>-</td>
</tr>
</tbody>
</table>

<sup>a</sup>Here this is the actual frequency in radians per second since $\omega^2(t)$ has no frequency for this case.
<table>
<thead>
<tr>
<th>Run no.</th>
<th>Type of trace&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Basic freq&lt;sup&gt;b&lt;/sup&gt;</th>
<th>Qual. mag.&lt;sup&gt;c&lt;/sup&gt;</th>
<th>Comp. zero&lt;sup&gt;d&lt;/sup&gt;</th>
<th>Dur. of effect&lt;sup&gt;e&lt;/sup&gt;</th>
<th>A&lt;sup&gt;f&lt;/sup&gt;</th>
<th>B&lt;sup&gt;f&lt;/sup&gt;</th>
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<td>8</td>
<td>0.75</td>
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<td>1b</td>
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<td>0.85</td>
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<td>1a</td>
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<td>3.0</td>
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<td>2a</td>
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<td>6b</td>
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<td>-</td>
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<td>pron.</td>
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<td>0.5</td>
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<td>1.0</td>
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</table>
Table 4b (continued)

<table>
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<th>Run no.</th>
<th>Type of trace</th>
<th>Basic frequency</th>
<th>Qual. mag.</th>
<th>Comp. zero</th>
<th>Dur. of effect</th>
<th>A^f</th>
<th>B^f</th>
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<td>1b</td>
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<td>slight</td>
<td>2a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
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<tr>
<td>138</td>
<td>15</td>
<td>9.0^g</td>
<td>slight</td>
<td>2b</td>
<td>n.m.</td>
<td>1.0</td>
<td>-</td>
</tr>
<tr>
<td>139</td>
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<td>2.51</td>
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<td>last 2/3</td>
<td>0.05</td>
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<td>last half</td>
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<td>2</td>
<td>-</td>
<td>slight</td>
<td>4a</td>
<td>n.m.</td>
<td>-</td>
<td>-</td>
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<tr>
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<td>-</td>
<td>slight</td>
<td>4b</td>
<td>n.m.</td>
<td>0.16</td>
<td>-</td>
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<td>n.m.</td>
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<td>-</td>
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<td>148</td>
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<td>pron.</td>
<td>7b</td>
<td>cont.</td>
<td>0.5</td>
<td>-</td>
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</table>
Figure 11. Characteristic traces of $\theta$ versus time for most of the runs as referred to in Table 4b
Figure 12. Traces of $\Theta$ versus time for several runs which did not fit the types shown in Figure 11.
Obviously, if a turning function is present there will be some effect but this may be too small to detect visually from the analog results on the tape. Figure 12 shows the results for individual runs which were unique and did not fit very well any of the types of Figure 11. The effects of the turning function were determined by comparing the trace of 400 for a given problem with the trace for an identical setup but with $\omega^2(t) = 0$. In order to reduce the effects of amplifier drift to a minimum, the problems were scaled to run in as short a time as was compatible with the response time of the servo multiplier as recommended by the manufacturer of the gear and motor unit. It turned out that the trace of $\theta$ was very sensitive to even the slight phase shift in the multiplier product, but this was not learned until the runs were completed. This will be discussed further later, but the apparent damping out or instability of some runs as shown in Figures 10, 11, and 12 is probably not truly representative of the action of the physical motion sensing device. The runs were not repeated at a slower rate inasmuch as the information was not to be used in a quantitative manner, but the comparisons of each run to its zero run (corresponding setup but with $\omega^2(t) = 0$) made use of the zero runs for which the multiplier was used in the circuit. These zero runs are shown in Figure 13; they were run at the same time-scale as the corresponding runs where $\omega^2(t) \neq 0$. 
Figure 13. Traces of $\theta(t)$ vs. time for the zero series of Table 3 for which $\omega(t) \equiv 0$
Phase shift from servo-multiplier

As mentioned earlier, some trouble was encountered as a result of the phase shift in output of the multiplier from the input to the amplifier-motor. It was impossible, of course, to solve any of the problems without using the multiplier other than those where $\omega^2(t) \neq 0$. In order to check the results, the zero runs were repeated with the multiplier (for multiplying $80\sin2\theta$ by $1/2 K^2$) bypassed. This was possible when $\omega^2(t) \neq 0$ since a variable can be multiplied by a constant with an operational amplifier.

Whereas convergence (such as $1b$ in Figure 13) or divergence (such as $2a$ in Figure 13) had been obtained previously, no such traces were obtained when the multiplier was bypassed. After considerable experimenting the trouble was found to be caused by the lag in output of the multiplier; divergence or convergence was determined by the ratio of $N$ to $K$. For a large $N$ and small $K$, convergence prevailed; for a large $K$ and small $N$, divergence was obtained (see Table 3 and Figure 13).

In order to improve the situation, several runs (54, 55, and 56) were repeated with the time-scaling such that the problems required longer to run on the computer. Figure 14 shows the traces for runs 54, 55, and 56 for the following conditions: (a) original setup, multiplier used, $T = 5$, corresponds to zero $1b$; (b) slower setup, multiplier used,
Figure 14. Traces of $409 \times t$ vs. time for $\omega^2(t) \equiv 0$ showing effect of time-scaling and bypassing of multiplier.
T = 10; and (c) multiplier bypassed, T = 5. It is quite noticeable that slowing down the problem (multiplier in) is beneficial. However, with the multipliers being used, the divergence or convergence was still present at a reduced rate no matter how slowly the problem was run off. Thus the stability is not determined by the analog computer. It should be mentioned that the slight convergence in runs 54-c and 56-c is caused by the finite resistance in the capacitors--were the capacitors perfect, no convergence would exist.

**Resonance with turning function**

One question which came up concerns the matter of resonance. As shown in Appendix A, the value of K must be greater than \( \sqrt{N} \) for \( \theta_n \) to be greater than 0. It can be shown that the condition for resonance (lowest mode in each case) is given by

\[
K = p = \frac{\sqrt{N}}{2}.
\]

Since \( K \) is not greater than \( \sqrt{N} \), the device will not exhibit resonance between spin speed and natural frequency.

Therefore, two resonance tests were made with the analog computer: (a) runs 149-a,b,c, and 150 in which \( K = p \), and (b) runs 151 and 152 in which \( p = \sqrt{N}/2 \). Table 5 gives the details of the setup and Figure 16 shows the results of
a. $U = \pi/20$

b. $U = \pi/10$

c. $U = \pi/3.71$, the largest value possible with $p = 10\pi/3$ without $\theta$ going to 0 immediately

Runs 149-a,b,c: Turning function frequency same as spin frequency and $\theta_o = \theta_n = 0.70$ rad.

Run 150: Same as 149-c except $\theta_o = \theta_n - 0.5 = 0.20$ rad.

Run 151: Turning function frequency equal one half natural frequency and $\theta_o = 0.70$ rad.

Run 152: Same as run 151 except $\theta_o = 0.20$ rad.

Figure 15. Plot of $40\theta$ vs. time for the check on resonance of oscillation of $\theta$ with critical frequency inputs of turning function and spin speed (Table 5 for parameters)
Table 5. Values of parameters of runs 149-a,b,c; 150; 151; and 152 for determination of resonance

<table>
<thead>
<tr>
<th>Run no.</th>
<th>$p$ (rad./sec.)</th>
<th>$U$ (rad.)</th>
<th>$N$ (sec.$^{-2}$)</th>
<th>$K$ (rad./sec.)</th>
<th>$\theta_0$ (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>149-a</td>
<td>$10\pi/3$</td>
<td>$\pi/20$</td>
<td>77.19</td>
<td>$10\pi/3$</td>
<td>0.70</td>
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<tr>
<td>149-b</td>
<td>$\pi/10$</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>149-c</td>
<td>$\pi/3.71^c$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>150</td>
<td>$\pi/3.71^c$</td>
<td></td>
<td></td>
<td>$0.20$</td>
<td></td>
</tr>
<tr>
<td>151</td>
<td>$1.4\pi^d$</td>
<td>$\pi/2$</td>
<td></td>
<td></td>
<td>0.70</td>
</tr>
<tr>
<td>152</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.20</td>
</tr>
</tbody>
</table>

These are similar to Series B runs because $\omega(t) = Up\cos pt$.

Use this table with Figure 15.

This value of $U$ was the largest for which $\theta$ did not run to zero immediately when $p = 10\pi/3$.

This value of $p = \sqrt{N}/2$ for $N = 77.19$.

these runs in the usual graphical form of $40\theta$ plotted versus time by the recording penmotor.

IBM 650 Digital Computer Solutions

In order to obtain more accuracy than that possible from the analog computer, a digital computer (the IBM 650) was used to solve a small number of problems. The high speed
digital computers are fairly new in the engineering picture and render solvable many problems formerly left unsolved because of excessive calculation effort.

The analog computers suffer from the fact that voltages are used to represent quantities and these voltages are accurate only to approximately one percent. The digital computer, on the other hand, performs each of the limited number of basic operations exactly accurately up to the number of significant places it can handle. Fundamentally, the digital computer can perform arithmetical operations, remember, and look up tabular material; however, it can do these operations at a rate much faster than humans (for example multiplication may take only 0.065 seconds).

In order to obtain a comparison between analog and digital computer results, twelve problems were solved by the digital computer. Four problems from Series A, four from Series B, and four zero runs were chosen to be solved by the IBM 650. Table 6 shows the problem breakdown and indicates the identity of the problems in Table 3 by giving the run number. A logical reason for the grouping indicated in Table 6 is that two problems were solved simultaneously with the digital computer; set Aa, consisting of runs 3 and 4, were done together; also set Ab, runs 21 and 22; and so on. The reason for doing pairs of problems together was the reduction in calculations of the common term \(1/2 \left[ K^2 - \left(1 + t_0^2 \right) \cos^2 Kt \right] \) which was rather time consuming for the computer.
Table 6. Values of parameters for problems solved by digital computer

<table>
<thead>
<tr>
<th>Prob. ident.</th>
<th>Run no. (Table 3)</th>
<th>$r$ (sec.)</th>
<th>$p$ (rad.)</th>
<th>$U$ (rad./sec.)</th>
<th>$N$ (sec. sec.)</th>
<th>$K$ (sec.)</th>
<th>$t$ (sec.)</th>
<th>$\theta_0$ (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IAa</td>
<td>3</td>
<td>0</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>0.70</td>
</tr>
<tr>
<td>IIAa</td>
<td>4</td>
<td>0</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>0.20</td>
</tr>
<tr>
<td>IAb</td>
<td>21</td>
<td>0</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>1.16</td>
</tr>
<tr>
<td>IIAb</td>
<td>22</td>
<td>0</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>-$\pi/2$</td>
</tr>
<tr>
<td>IBa</td>
<td>27</td>
<td>0.1</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>0.70</td>
</tr>
<tr>
<td>IIBa</td>
<td>28</td>
<td>0.1</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>0.20</td>
</tr>
<tr>
<td>IBBa</td>
<td>49</td>
<td>0.1</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>1.16</td>
</tr>
<tr>
<td>IIBBb</td>
<td>50</td>
<td>0.1</td>
<td>2.0</td>
<td>$\pi/2$</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>-$\pi/2$</td>
</tr>
<tr>
<td>ICa</td>
<td>53</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>-</td>
</tr>
<tr>
<td>IICa</td>
<td>54</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>77.19</td>
<td>10$\pi/3$</td>
<td>0.0075</td>
<td>-</td>
</tr>
<tr>
<td>ICb</td>
<td>59</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>1.16</td>
</tr>
<tr>
<td>IICb</td>
<td>60</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>77.19</td>
<td>5$\pi$</td>
<td>0.005</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Turning function: $\mathbf{\omega}(t) = -U p \sin(p t + \delta)$ from $\mathbf{\Phi}(t) = U a \sin p t$.

Turning function: $\mathbf{\omega}(t) = -U \sqrt{p^2 + r^2} e^{-\gamma t} \sin(p t + \delta)$ from $\mathbf{\Phi}(t) = U e^{-\gamma t} \sin p t$.

Turning function: $\mathbf{\omega}(t) \equiv 0$, corresponds to zero 1-b.

Turning function: $\mathbf{\omega}(t) \equiv 0$, corresponds to zero 6-b.
Solution of differential equation by finite differences

The method of solution of ordinary differential equations by difference methods is well known and will not be developed here other than to present the particular scheme used for the solution of Equation 44. Essentially, the difference methods hinge on replacing derivatives with differences, thus turning the problem of integration into one of arithmetic. Thus, necessary in the method is an appropriate quadrature formula whereby the integration is accomplished.

Table 7 is the difference array used in solving Equation 44; the use of this array is explained in detail following the table. The entries in Table 7 are indexed at any general time in the problem, and it is assumed that $\theta_n$, $\theta_{n-\frac{1}{2}}$, $\theta_{n-1}$, and all corresponding values above them in the table are known for the particular step being calculated. The solution of Equation 44 by difference methods is fortunately an initial value problem instead of a boundary value problem. Following are the detailed steps of the difference method of solution of Equation 44 using Table 7:

$$\ddot{\theta} = \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \sin 2\theta - N\theta \quad (44)$$

where

$$\omega(t) = -U \sqrt{p^2 + r^2} \, e^{-rt} \sin(pt + \delta)$$

and

$$\delta = \tan^{-1}(p/r) - \pi.$$
Table 7. Array of differences used to solve Equation 4.4 by the method of finite differences

<table>
<thead>
<tr>
<th>Index</th>
<th>θ</th>
<th>δθⁿᵃ</th>
<th>δ²θⁿᵇ</th>
<th>θⁿᶜ</th>
<th>δ⁴θⁿᵈ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>θ</td>
<td>δθⁿ⁻²</td>
<td>δ²θⁿ⁻²</td>
<td>θⁿ⁻²</td>
<td>δ⁴θⁿ⁻²</td>
</tr>
<tr>
<td>tₙ₋₂</td>
<td>θ₋₂</td>
<td>δθ₋₂</td>
<td>δ²θ₋₂</td>
<td>θ₋₂</td>
<td>δ⁴θ₋₂</td>
</tr>
<tr>
<td></td>
<td>δθ₋₁</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tₙ₋₁</td>
<td>θ₋₁</td>
<td>δθ₋₁</td>
<td>δ²θ₋₁</td>
<td>θ₋₁</td>
<td>δ⁴θ₋₁</td>
</tr>
<tr>
<td></td>
<td>δθ₋₁</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tₙ</td>
<td>θ</td>
<td>δθⁿ</td>
<td>δ²θⁿ</td>
<td>θⁿ</td>
<td>δ⁴θⁿ</td>
</tr>
<tr>
<td></td>
<td>δθ₊½</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tₙ₊₁</td>
<td>θ₊₁</td>
<td>δθ₊₁</td>
<td>δ²θ₊₁</td>
<td>θ₊₁</td>
<td>δ⁴θ₊₁</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tₙ₊₂</td>
<td>θ₊₂</td>
<td>δθ₊₂</td>
<td>δ²θ₊₂</td>
<td>θ₊₂</td>
<td>δ⁴θ₊₂</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ a \delta θₙ ± ½ = θₙ ± θₙ₋₁ \text{ where } i = 1, 3. \]

\[ b \delta²θₙ = δθ₊½ - δθ₋½ = θ₊₁ - 2θₙ + θ₋₁. \]

\[ cθ'' = \frac{d²θ}{dt²} = \frac{1}{2} \left[ K² - \omega²(t)cos²Kt \right] \sinθ - Nθ. \]

\[ d\delta⁴θ'' = θ''₊₂ - 4θ''₊₁ + 6θ''ₙ + 4θ''₋₁ - θ''₋₂. \]
0. Compute \( \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 Kt \right] \) for \( t_{n+1} \). \( \theta_n, \theta_n', \) and \( \theta_n'' \) have been computed and confirmed.

1. Extrapolate \( \theta_n'' \) into a trial value for \( \theta_n'' \). Call this \( T_1 \theta_n'' \), and \( T_1 \theta_n'' = \theta_n'' \) (the \( T_1 \) refers to the first trial value).

2. Compute \( \theta_{n+1} \) from \( T_1 \theta_n'' \) by addition.

\[
T_1 \theta_{n+1/2} = \theta_{n-1/2} + T_1 \theta_n''
\]

\[
T_1 \theta_{n+1} = \theta_n + T_1 \theta_n''/2.
\]

3. Compute \( T_1 \theta_{n+1}'' \) from differential equation.

\[
T_1 \theta_{n+1}'' = \frac{1}{2} \left[ K^2 - \omega^2(t_{n+1}) \cos^2 Kt_{n+1} \right] \sin 2T_1 \theta_{n+1}
\]

\[
\text{(this part already available from step 0)}
\]

\[
- N T_1 \theta_{n+1}.
\]

4. Compute \( T_2 \theta_n'' \) from \( \theta_n'' \), \( \theta_n'', \) and \( T_1 \theta_{n+1}'' \) from quadrature formula. This formula comes from the first two terms of Equation 39.3 of Milne (23, p. 84),

\[
T_2 \theta_n'' = h^2/12 \left[ \theta_n'' + \theta_n'' + T_1 \theta_{n+1}'' \right]
\]

where \( h = \) time increment.

5. Compare \( T_2 \theta_n'' \) with \( T_1 \theta_n'' \). If they do not agree to the desired accuracy, go to step 6. If they do agree go to step 7.
6. Replace $T_1 \delta^2 \theta_n$ by $T_2 \delta^2 \theta_n$ and store in location for $\delta^2 \theta_n$. Loop by entering step 2 again and continue the loop until agreement is reached between $T_{i+1} \delta^2 \theta_n$ and $T_i \delta^2 \theta_n$. When agreement is reached, go to step 7.

7. Store $\theta_{n+1}$ and $\theta''_{n+1}$ in punch storage.

8. Compute $\delta^4 \theta^n_{n-1}$ from the formula. $\delta^4 \theta^n_{n-1} = \theta^n_{n+2} - 4\theta^n_{n+1} + 6\theta^n_n - 4\theta^n_{n-1} + \theta^n_{n-2}$. If this is the first problem of the set of two, exit to step 1 for the second problem. If this is problem two of the set, exit to step 9.

9. Punch $t_{n+1}$, $\theta_{n+1}$, $\theta''_{n+1}$, and $\delta^4 \theta^n_{n-1}$ for both problems.

10. Step up time by one increment and move the values of the array of Table 7 back one address.

11. Compare $t$ with $t = 10.00$ seconds. If $t \neq 10.00$, go to step 0 for the next line ($t_{n+2}$) of this set of problems. If $t = 10.00$, stop and reload for next set of two problems.

Starting values. Following through the array of Table 7 and the steps of procedure following, one observes that values of $\theta$ for three consecutive times are necessary to get the problem started. Once underway, the procedure given will suffice. Taylor's series expanded about time $= 0$ was used to calculate $\theta(o - 2\Delta t)$, $\theta(o - \Delta t)$, $\theta(o)$, and $\theta(o + \Delta t)$. Since the time increment was quite small (.0075 seconds for $K = 10\pi/3$ and .005 seconds for $K = 5\pi$), only the first three terms of Taylor's series were used. The time increment was arrived at by dividing into twenty parts the duration of
one-half of a cycle of the highest frequency input. For the problems solved by IBM 650, the frequency of the term $\cos^2 Kt$ was the governing value.

In order to compute the terms of Taylor's series, several derivatives evaluated at $t = 0$ were computed and used in the following form:

$$\theta(t) = \theta(o) + t\dot{\theta}(o) + \frac{t^2\ddot{\theta}(o)}{2!} + \frac{t^3\dddot{\theta}(o)}{3!}.$$  \hspace{1cm} (62)

Since $\theta(o)$ and $\dot{\theta}(o)$ were given by the initial conditions, only $\ddot{\theta}(o)$ and $\dddot{\theta}(o)$ remained to be calculated. The differential equation is fairly involved algebraically to differentiate because of the various products contained therein. This was done, however, and evaluated at $t = 0$ and will not be shown here because of the lengthy algebra. There were no fundamental difficulties involved—only a lot of algebra. The starting values of $\theta''(o - 2\Delta t)$, $\theta''(o - \Delta t)$, $\theta''(o)$, and $\theta''(o + \Delta t)$ could have been computed by the digital computer; however, in order to simplify the program, these values were computed by hand on a desk calculator. Once the problem was underway, of course, the program had the computer obtain $\theta''$ as a part of its operation. Table 8 gives the starting values of $\theta$ for each of the problems.

**Program for digital computer**

It was decided to program the problems in so-called basic
Table 8. Starting values of $\theta$ for use with finite difference method used with the digital computer

<table>
<thead>
<tr>
<th>Prob. ident.</th>
<th>$\Delta t$ (sec.)</th>
<th>$\theta(0-2\Delta t)$ (rad.)</th>
<th>$\theta(0-\Delta t)$ (rad.)</th>
<th>$\theta(0)$ (rad.)</th>
<th>$\theta(0+\Delta t)$ (rad.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>IAa 3</td>
<td>0.0075</td>
<td>0.69945291</td>
<td>0.69986323</td>
<td>0.700000</td>
<td>0.69986323</td>
</tr>
<tr>
<td>IIAa 4</td>
<td>0.0075</td>
<td>0.20044915</td>
<td>0.20011229</td>
<td>0.200000</td>
<td>0.20011229</td>
</tr>
<tr>
<td>IAb 21</td>
<td>0.005</td>
<td>1.16305231</td>
<td>1.16318699</td>
<td>1.16323189</td>
<td>1.16318699</td>
</tr>
<tr>
<td>IIAb 22</td>
<td>0.005</td>
<td>0.66641801</td>
<td>0.66402842</td>
<td>0.66323189</td>
<td>0.66402842</td>
</tr>
<tr>
<td>IBa 27</td>
<td>0.0075</td>
<td>0.69945182</td>
<td>0.69986309</td>
<td>0.700000</td>
<td>0.69986337</td>
</tr>
<tr>
<td>IIBa 28</td>
<td>0.0075</td>
<td>0.20044872</td>
<td>0.20011223</td>
<td>0.200000</td>
<td>0.20011234</td>
</tr>
<tr>
<td>IIBb 49</td>
<td>0.005</td>
<td>1.16305207</td>
<td>1.16318696</td>
<td>1.16323189</td>
<td>1.16318702</td>
</tr>
<tr>
<td>IIBb 50</td>
<td>0.005</td>
<td>0.66641801</td>
<td>0.66402842</td>
<td>0.66323189</td>
<td>0.66402842</td>
</tr>
<tr>
<td>ICa 53</td>
<td>0.0075</td>
<td>0.70000000</td>
<td>0.70000000</td>
<td>0.70000000</td>
<td>0.70000000</td>
</tr>
<tr>
<td>IICa 54</td>
<td>0.0075</td>
<td>0.20066564</td>
<td>0.20016652</td>
<td>0.20000000</td>
<td>0.20016652</td>
</tr>
<tr>
<td>ICb 59</td>
<td>0.005</td>
<td>1.16323190</td>
<td>1.16323190</td>
<td>1.16323190</td>
<td>1.16323190</td>
</tr>
<tr>
<td>IICb 60</td>
<td>0.005</td>
<td>0.66665922</td>
<td>0.66408824</td>
<td>0.66323189</td>
<td>0.66408824</td>
</tr>
</tbody>
</table>

\textsuperscript{a}The identification symbols have the following meaning:
- $A - \omega(t) = -U \sin(pt + \delta)$
- $B - \omega(t) = U \sqrt{p^2 + r^2} e^{-rt} \sin(pt + \delta)$
- $C - \omega(t) \equiv 0$
- $a - K = 10\pi/3$ and $\Delta t = 0.0075$ sec.
- $b - K = 5\pi$ and $\Delta t = 0.005$ sec.
- 27 - run number 27 (see Tables 3 and 6).
language instead of the Bell Laboratories' interpretive system (15). The reason behind this decision was to reduce machine running time and thus reduce cost; the Bell system can be expected to require four to five times as much machine time as does the basic language method. The scheme shown above in Table 7 and the subsequent steps are by no means the form useful to the computer.

The first step toward programming was to construct a flow sheet which follows diagrammatically the logic used in the procedure following Table 7; this flow diagram is shown in Figure 16. The flow diagram was then used as the basis to write the detailed, step-by-step program useful to the machine.

A sample page of the program for the IBM 650 digital computer is included in Appendix C. The method using the basic language is described in the IBM 650 operation manual (16). Several other types of programming are available including the Bell Laboratories' interpretive system already mentioned. A system abbreviated S. O. A. P. accomplishes optimum location of items on the memory drum so as to minimize drum turning time.

Results of Digital Computer Solutions

In order to compare the digital computer results with the analog computer results, a composite plot was made of the digital results for each problem directly on one of the
Figure 16. Flow diagram for use in programming the solution of
\[ \theta = \frac{1}{2} \left[ K^2 - \omega^2(t) \cos^2 kt \right] \sin \theta - N \theta \]
by the digital computer (see Table 7 and the procedure following).
\( \frac{1}{2} [K^2 - \omega^2(t) \cos^2 Kt] \)

FOR \( t_{n+1} \) AND STORE

EXTRAPOLATE \( \delta^2 \theta_{n-1} \) INTO TRIAL VALUE OF \( \delta^2 \theta_n \) AND STORE

COMPUTE \( \theta_{n+1} \) FROM \( t_i \delta^2 \theta_n \) BY ADDITION AND STORE

REPLACE \( t_i \delta^2 \theta_n \) BY \( t_2 \delta^2 \theta_n \)

IS \( t_2 \delta^2 \theta_n = t_i \delta^2 \theta_n \) ?

COMPUTE \( t_2 \delta^2 \theta_n \) FROM \( \theta^{''}_{n-1}, \theta''_n \), \( \delta^0 \theta''_{n+1} \) AND STORE

COMPUTE \( \theta^{'''}_{n+1} \) FROM DIFF. EQUA. AND STORE

STORE \( \theta_{n+1} \) & \( \theta^{'''}_{n+1} \) IN PUNCH STOR

COMPUTE \( \delta^0 \theta_{n-1} \), INTO PUNCH STOR

STORE \( \theta_{n+1} \) & \( \theta^{'''}_{n+1} \) IN PUNCH STOR

PUNCH OUT \( \theta_{n+1}, \theta''_{n+1}, \delta^0 \theta_{n-1} \) FOR BOTH PROBS. OF THE SET FOR \( t_{n+1} \)

STOP MACHINE AND LOAD FOR NEXT SET OF TWO PROBLEMS

IS \( t = t_f \) ?

STEP UP \( t \) BY ONE AND MOVE ALL VALUES IN TABLE 7 BACK ONE ADDRESS

12

11

10
recorder tapes from the analog computer. This is the main mode of presentation of the digital results. To have attempted to plot the analog results on rectangular paper would have been rather inaccurate since the analog results were in the trace form only. Since the time increment was very small, a value of $\theta$ was not plotted for every value of time available in the digital results.

Figure 17 shows the results of the digital computer solution (dots) plotted on the analog computer tape for the case where $K = 10\pi/3$. Problems IAa (run 3), IBa (run 27), and ICa (run 53) all started with $\theta_0 = .700$ radians. The other problems of Figure 18 all started with $\theta_0 = .200$ radians. It should be noticed that problems IAa and IBa are plotted at an expanded scale so as to show the trace more clearly. The time axis shown is in real time, not the scaled-down analog computer time.

Figure 18 corresponds exactly to Figure 17 except that this set of runs is for a spin speed of $K = 5\pi$. Again problems IAb (run 21) and IBb (run 49) are plotted at an expanded scale in $\theta$. At the higher spin speed, the equilibrium angle, $\theta_n$, was greater than before; here $\theta_n = 1.1632319$ radians. Problems IIAb (run 22), IIBb (run 50), and IICb (run 60) were started at $\theta_0 = .6632319$ radians; this value of $\theta_0$ is .5 radian less than $\theta_n$. Again, real time is the basis of the time axis. Problem IBb was traced at a lower paper.
IAa (run 3). \( \omega(t) = -\pi \sin(2t - \pi/2) \)

IIAa (run 4). \( \omega(t) = -\pi \sin(2t - \pi/2) \)

IBa (run 27). \( \omega(t) = -\frac{\pi}{2} \sqrt{4.01} e^{-1t} \sin(2t - 1.5208+) \)

IIBa (run 28). \( \omega(t) = -\frac{\pi}{2} \sqrt{4.01} e^{-1t} \sin(2t - 1.5208+) \)

ICA (run 53). \( \omega(t) \neq 0 \)

IICA (run 54). \( \omega(t) \neq 0 \)

Figure 17. Plot of \( \Theta \) vs. time for digital computer and for analog computer at a spin speed of \( K = 10\pi/3 \) (Tables 6 and 8 for parameters)
IAb (run 21). $\omega(t) = -\pi \sin(2t - \pi/2)$

IIAb (run 22). $\omega(t) = -\pi \sin(2t - \pi/2)$

IBb (run 49). $\omega(t) = -\pi/2 \sqrt{4.01} e^{-1t} \sin(2t - 1.5208^+)$

IIBb (run 50). $\omega(t) = -\pi/2 \sqrt{4.01} e^{-1t} \sin(2t - 1.5208^+)$

ICb (run 59). $\omega(t) \equiv 0$

IICb (run 60). $\omega(t) \equiv 0$

Figure 18. Plot of $\theta$ vs. time for digital computer and for analog computer at a spin speed of $K = 5\pi$ (Tables 6 and 8 for parameters)
speed than the others, and is thus shown for the full 10 seconds. An attempt always was made when the analog computer was used to plot at a scale and paper speed to give appreciable amplitude and about 45 degree slopes.

In order to visualize the effect of the turning function, Figure 19 was plotted, again on the recorder paper. Here the results of the digital computer only are shown. The results of problems IIAa (run 4), IIBa (run 28), IIAb (run 22), and IIBb (run 50) are plotted in dots compared to the results for the corresponding problem but with $\omega(t) \equiv 0$. For problems IIAa and IIBa, the corresponding "zero" is problem IICa (run 54). For problems IIAb and IIBb, the corresponding "zero" is problem IICb (run 60). This sort of comparison plotting was not necessary for the problems where $\theta_0 = \theta_n$ because $\theta(t) \equiv \theta_n$ if $\omega(t) \equiv 0$ as indicated by the results for problems ICa (Figure 17) and ICb (Figure 18).

In order to get a detailed look at some of the digital results, a portion of problem IAa (run 3) was plotted from $t = 0.1200$ seconds to $t = 0.3825$ seconds as shown in Figure 20. In this plot, a value of $\theta$ is plotted for every time increment ($\Delta t = .0075$) in the range indicated.

No tabular values of the digital results are included because of the large number of entries; each problem of the group for $K = 10\pi/3$ contains 1334 entries and each problem for $K = 5\pi$ has 2000 entries.
IIAa (run 4). \( \omega(t) = -\pi \sin(2t - \pi/2) \)

\[ K = 10\pi/3 \]

IIBa (run 28). \( \omega(t) = -\pi \sin(2t - \pi/2) \)

\[ K = 10\pi/3 \]

IIAb (run 22). \( \omega(t) = -\pi/2 \sqrt{4.01} e^{-1t} \sin(2t - 1.5208^+) \)

\[ K = 5\pi \]

IIBb (run 50). \( \omega(t) = -\pi/2 \sqrt{4.01} e^{-1t} \sin(2t - 1.5208^+) \)

\[ K = 5\pi \]

Figure 19. Plot of \( \theta \) vs. time from the digital computer results showing the effect of the turning function \( \omega(t) \) for \( \theta_0 \neq \theta_n \).
IIAa

IIBa

IIAb

IIBb
Figure 20. Detailed plot of θ vs. time from digital computer results for a short interval of time of problem IAa (run 3)
Comparison of Analog and Digital Computers

Controversy exists about the relative merits of analog versus digital computers. Neither kind of computer is better for every problem. Since a computer of each kind was available for the fly-ball problem, each was used; and for more than merely a check on the other computer.

Following is a listing of the strong and weak points of the two types of computers:

**Analog computer**

**Advantages:**

1. The operational amplifier can integrate directly any function fed to it. The analog computer is excellent for solving ordinary differential equations of any order.

2. The analog computer is an excellent probing tool; a parameter may be changed while a problem is progressing in order to observe the effect of variation of that parameter.

3. The analog computer is convenient to use; new problems are quickly set up.

4. Interpretation of results is straight-forward.

5. No special language is necessary for programming a problem.

6. Low cost computers are available.

7. Trouble shooting is rather simple as far as hookup mistakes are concerned.

8. The system may be added to as new problems demand more extensive equipment.
9. Auxiliary equipment of different manufacturer than the basic unit may be used as long as power specifications are met.

Disadvantages:

1. Analog computers are not entirely accurate; component values are not known exactly and vary with temperature and humidity. Recording equipment is normally accurate to only about one percent.

2. Capacitors necessary for integration exhibit "soakage", the retention of some charge after being shorted and the short removed.

3. The operational amplifiers drift; that is, exhibit an output when there is no input voltage.

4. Auxiliary equipment such as function generators and multipliers have unavoidable errors.

5. Servo multipliers exhibit slow response and hunting characteristics. Electronic multipliers are expensive and are less accurate than are servo multipliers.

6. Variation of reference voltages may be troublesome.

7. The computer is not well suited to the solution of partial differential equations.

Digital computer

Advantages:

1. The digital nature of the computer makes it well suited to any operations which can be broken down into arithmetic steps. Simultaneous equations and difference methods are two examples for which the digital computer is well adapted.

2. The computer can be used in solving partial differential equations by using difference methods.

3. The computer has a memory.

4. The digital computer performs every operation accurately up to the number of significant figures which can be used.

5. Operations are performed rapidly.
Disadvantages:

1. Programming of problems into machine language is tedious and subject to errors. Location of program errors may be time consuming.

2. A useful computer is expensive, and a steady supply of problems is necessary to justify the rental price. This limits the availability.

3. The computer is not a good probing device; the effect of a change of a parameter cannot be seen easily during a problem.

4. The digital computer cannot integrate directly.

5. Demand for the services of a digital computer may result in considerable delay in solving problems.
DISCUSSION

The solution of the differential equation of motion (Equation 44) was performed successfully by an analog computer and a digital computer procedure. As pointed out previously, the solutions are for specific problems and do not lend themselves well to the generalization that could be done were an analytic solution available. Thus, if a new combination of parameters were desired, it would be unsafe to predict \( \theta(t) \) on the basis of the solution of other specific problems. Instead, it would be necessary to solve again by one of the two methods the equation of motion for the specific set of inputs. The method is extremely useful, however, since specific solutions are often desired in many engineering problems.

Analog Computer Results

The problems solved by the analog method were indicated in Table 3; the results were given in Table 4b with the meaning of two columns indicated in Table 4a. Even though the servo-multiplier gave some trouble from phase shift as indicated, useful results were still obtained.

Frequency of turning function

For a turning function which exhibited a cyclic nature
(Series A, B, and C), an increase of frequency (p) of the turning function resulted in increased effect as shown in Table 4b.

Magnitude of turning function

For all problems, an increase in the constant coefficient (U) of the turning function resulted in an increased effect of the turning function. This also is apparent from an inspection of Tables 3 and 4b.

Damping factor for Series B

Series B was the only series for which the turning function was damped. Here an increase of the damping factor (r) resulted in very little change in the early magnitude of the effect of the turning function, but the duration of the effect was reduced.

Design parameter N

An increase of the combined parameter N from 0 to 200 resulted in an increase in effect from the turning function up to the central value of 77.19 and then a decrease. This is reasonable because, at very low N, the angle θ will be nearly equal to π/2; also for large N, the angle θ will tend to be near zero as indicated in Appendix A. Thus, it is apparent that a design of a useful motion-sensing instrument
must incorporate an appropriate value of $N$ for a given spin speed in order to be sensitive.

**Spin speed $K$**

A pronounced reduction in the effect of the turning function was observed as the spin speed, $K$, was increased. This sort of action was expected by an inspection of the differential equation of motion (Equation 44) since the term $\omega^2(t) \cos^2 Kt$ was subtracted from $K^2$. Because of this reduction in effect of turning function with increased spin speed, the spin speed must be designed on the basis of expected angular velocities of the vehicle whose motion is to be sensed.

**Basic frequency**

There was little consistency of the observed basic frequency as a multiple of the natural frequency, $\sqrt{N}$. This simply means that the dynamic forces from centrifugal action overbalance in many cases the forces tending to oscillate the system at the circular frequency of $\sqrt{N}$. Even when $N$ was zero, there was a definite basic frequency; this indicates that the centrifugal forces acted in the capacity of restoring springs.

**Duration of effect**

In some problems such as the ones of Series B the action
of the turning function soon becomes quite small but the oscillation of $\theta$ may continue indefinitely, because of this early displacement. The notation in Table 4b was based on a comparison of a given trace to its corresponding zero.

Stability of operation

In an instrument such as the proposed motion sensing device question of stability naturally arises. There are two types of stability to consider: (a) inherent stability and (b) operative stability. The inherent stability question may be likened to the problem of placing a simple pendulum so that the mass is above the pivot as compared to the usual position below the pivot. The upper position would exhibit unstable equilibrium; the lower position would be stable. Operative stability, on the other hand, could still be in doubt for the pendulum whose mass is below the pivot if a driving force of proper frequency resulted in resonance. The analogous problem arises with the motion sensing device. No evidence was obtained for any combination of parameters tried that inherent instability exists. Figure 15 shows, however, that if the turning function frequency is one-half the natural frequency \[ \omega(t) = -U \sin(pt + \delta) \text{,} \] resonance does occur and the angle $\theta$ becomes zero; that is, the masses $M_1$ and $M_2$ impinge upon the spin axis. If the turning function frequency is the same as the spin frequency, no
resonance occurs. As discussed before, it is not possible to have spin frequency and natural frequency resonant because of the requirement that $K/\sqrt{N}$ be greater than 1.0 as developed in Appendix A. Thus, the instrument must be designed to have a natural frequency, $\sqrt{N}$, well away from twice any possible turning frequency of the vehicle whose motion is to be sensed.

Digital Computer Results

The digital computer results are summarized in Figures 17, 18, 19, and 20. The digital computer is considerably more precise than the analog computer in the individual operations. The two types of errors inherent in the difference method of solution are (a) generated and (b) round-off. Since the time increment was short, small generated errors were expected. The round-off errors are difficult to analyze. The results indicate that errors were small because of the constancy of amplitude and frequency of the plots in Figures 17, 18, and 19.

Comparison to analog results

In Figures 17 and 18 the digital computer results (dots) are compared to the analog computer results (solid line). The most noticeable discrepancy is in frequency. However, the analog results are believed to be in error because
a similar order of magnitude of frequency change was observed between two apparently identical runs made about three days apart. The values of the resistors and capacitors were determined at room temperature and at ambient humidity. When used, the components are at a higher temperature; also the humidity may change from day to day. Thus, the frequency discrepancy is probably not serious.

Table 9 is a summary of the values of the discrepancies between the analog and digital results for frequency and amplitude.

Wave shape can best be compared visually. The values given in Table 9 are presented with no allowance for the divergence or convergence of the analog trace caused by the servo-multiplier. From Table 9, it is apparent that amplitudes at \( t = 1.0 \) seconds agreed within 14 percent with one exception and that frequencies agreed within 11 percent for all runs. Runs IA.b and IB.b exhibit large discrepancies in amplitude in terms of percent, but the entire amplitude was so small that reading errors would be a large fraction of the base amplitude.

The analog traces of Figures 17 and 18 were obtained at a time scale about double that originally used to reduce the servo-multiplier difficulties discussed earlier; the divergence or convergence is still present, however.
Table 9. Comparison of analog and digital results in terms of amplitude and frequency of $\Theta(t)$

<table>
<thead>
<tr>
<th>Ident.</th>
<th>Run no.</th>
<th>Amplitude discrepancy$^a$</th>
<th>Frequency discrepancy$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>at $t = 1.0$ sec. (percent)</td>
<td>at $t = 5.0$ sec. (percent)</td>
<td></td>
</tr>
<tr>
<td>IAa</td>
<td>3</td>
<td>$+3.6^b$</td>
<td>$-8.9^b$</td>
</tr>
<tr>
<td>IIAa</td>
<td>4</td>
<td>$+4.9$</td>
<td>$+6.8$</td>
</tr>
<tr>
<td>IBa</td>
<td>27</td>
<td>$-2.0$</td>
<td>$-5.0$</td>
</tr>
<tr>
<td>IIBa</td>
<td>28</td>
<td>$+3.8$</td>
<td>0</td>
</tr>
<tr>
<td>ICa</td>
<td>53</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>IICa</td>
<td>54</td>
<td>$+1.5$</td>
<td>$-3.2$</td>
</tr>
<tr>
<td>IAb</td>
<td>21</td>
<td>$-10.0$</td>
<td>$-20.0$</td>
</tr>
<tr>
<td>IIAb</td>
<td>22</td>
<td>$+13.9$</td>
<td>$+28.6$</td>
</tr>
<tr>
<td>IIBb</td>
<td>49</td>
<td>$-25.0$</td>
<td>$+5.0$</td>
</tr>
<tr>
<td>ICb</td>
<td>59</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>IICb</td>
<td>60</td>
<td>$+11.1$</td>
<td>$+22.2$</td>
</tr>
</tbody>
</table>

$^a$Discrepancies are given as the percent difference of the value of the digital result. Amplitudes are the peak-to-peak magnitudes of oscillation.

$^b$If the value of analog result is greater, the positive sign is used; if smaller, the negative sign.
The agreement of analog and digital results does not mean agreement of either method with the performance of an actual instrument; it merely means that the two ways of solving the equation of motion are probably correct since the two methods followed very different paths.

Effects of various parameters

Since the digital and analog methods agreed reasonably well (Table 9) for the twelve problems solved by both methods, similar agreement could be expected for those problems which were solved by the analog but not by the digital computer. Thus, the observations made concerning effect of variation of parameters in the analog results can be expected to hold true for the digital results. So few problems were solved by the digital computer that effects of parameter change were observed for spin speed only.

A comparison of Figures 17 and 18 shows that the effect of an increase of spin speed is to greatly reduce the response to the turning function. This is also shown very plainly in Figure 19.

An interesting thing shown in Figure 19 is the frequency change in $\Theta$ caused by the turning function for problems IIAa and IIBb where $K = 10\pi/3$. This frequency change may possibly be useful as a method of measurement. The plots of Figure 19 are entirely digital, so this frequency change is not of the same origin as that referred to above.
Further Work Desirable

Prior to serious consideration of actually using a fly-ball device as a motion sensing instrument, several other aspects should be considered on about the same basis as the present work. Following is a list of the most obvious areas to investigate:

1. Introduce friction into the problem. Bearing friction cannot be eliminated; air friction can be greatly reduced by operating in a high partial vacuum.

2. Release the restriction that the plane containing the arms and spin axis must also contain the vector of effective gravity at the start of the problem.

3. Allow the spin axis to take an attitude other than perpendicular to the vector of effective gravity.

4. Extend the lower limit of spin speed, $K$, to approach a static system similar to ordinary accelerometers.

5. Consider the effects of acceleration in the direction parallel to the spin axis.

While it is believed by the writer that all of the above difficulties, except number 1, can be overcome by utilizing certain techniques with the present design, certainly such information would be desirable and would probably result in a more useful ultimate instrument.

Friction can be readily introduced into the problem for both the analog and the digital computer. The analog
computer would require a relay or flip-flop circuit sensitive to sign of \( \cdot \). The digital program could be modified easily to branch on the sign of \( \cdot \) and introduce the proper value of friction.

All of the other items above will require changes in the basic development of the equation of motion, except number 4 which can be varied as done already.

Method of Use of Motion-sensing Device

The usefulness of the motion-sensing device is dependent upon the determination of the turning experienced by a missile, aircraft, or other vehicle; this is the inverse of the problem solved in this thesis. Thus, in a practical use, the function \( \theta(t) \) will be available by measurement and the angular motion of the missile is to be determined from this information. It was necessary, of course, to work first from \( \omega(t) \) to \( \Theta(t) \) to ascertain whether or not the function \( \theta(t) \) was sufficiently responsive to possible turning functions. The results of this study indicate that the response is adequate to provide useful information provided that the instrument is designed and operated properly.

There are two functions, \( \omega(t) \) and \( \dot{\Theta}(t) \), available from a continuous record of \( \theta(t) \). The function \( \omega(t) \) is the turning function referred to throughout this thesis; only now it would be unknown and is to be found from \( \theta(t) \). This
is easily accomplished by solving Equation 44 for \( \omega(t) \) as follows:
\[
\omega(t) = \frac{\sqrt{K^2 - \frac{2(\dot{\Theta} + \ddot{\Theta})}{\sin 2\Theta}}}{\cos Kt}.
\] (63)

The proposed instrument is not sensitive to direction of turn, only magnitude. However, a simple rate gyro could be used with the fly-ball motion sensing device to provide the sign (+ or -) for \( \omega(t) \) in Equation 63.

To provide angular displacement, \( \Phi(t) \), it is merely necessary to integrate Equation 63 once with respect to time,
\[
\Phi(t) = \int_0^t \omega(t) dt
= \int_0^t \frac{\sqrt{K^2 - \frac{2(\dot{\Theta} + \ddot{\Theta})}{\sin 2\Theta}}}{\cos Kt} dt + \Phi(0)
\] (64)

Again the proper sign of the integrand would have to be provided by some external means such as the small rate gyro already mentioned.

The angle \( \Theta \) can be measured by several methods, so this poses no real problems. However, the measurement of \( \ddot{\Theta} \) does involve difficulties. The analog type computers are not good for measurement of derivatives because of the confusion of electrical "noise" with the desired signal; this results in
erratic values for derivatives. The value of $\theta$ probably could be best determined by the method of approximation by differences,

$$
\ddot{\theta}_i = \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta t)^2}.
$$

(65)

The smaller the time increments (for example, 0.01 second) the better will be the approximation of $\theta$ by Equation 65.

Airborne digital computers are available and will probably be miniaturized further. Thus, a numerical integration as indicated in Equation 64 will give the present heading of the vehicle about the axis of turn. The digital computer, of course, can also provide $\ddot{\theta}$ as necessary in Equation 64 from measurements of $\theta$ as indicated in Equation 65.
CONCLUSIONS AND SUMMARY

Conclusions

In terms of the three objectives stated in the introduction; the following conclusions may be drawn:

1. The proposed fly-ball device is capable of detecting and measuring angular motion though not the sense of such motion.

2. The equation of motion was developed by using Lagrange's equations; then, since this differential equation was non-linear, two computer methods of solution were developed and used.

3. A number of specific problems were solved by both analog and digital computer methods for various parameters and starting conditions. For $K = 10\pi/3$, the percent error in amplitude was less than 5.0 at $t = 1.0$ second and less than 9.0 percent at $t = 5.0$ seconds; also percent error in frequency was less than 11.0. For $K = 5\pi$, amplitude errors were less than 30 percent and frequency errors less than 3 percent. For some runs, Table 9 shows that the errors are considerably smaller than the limits indicated here. Correspondence of wave shape was good.

In addition, several other conclusions are pertinent within the range of variables studied:
4. An increase in spin speed greatly reduces the sensitivity of the instrument. Digital methods may overcome much of this objection.

5. The instrument is inherently stable except for resonance conditions which can be induced by a turning function of the proper frequency. Resonance can be avoided by proper design of the instrument.

6. The theoretical method of using instrument output information was developed and appears quite straightforward (see "Method of Use of Motion Sensing Device").

7. Several simplifying assumptions were made in the formulation of the problem; further work has been recommended to eliminate these restrictions.

8. Since an analytic solution to the equation of motion (Equation 44) was not found, the results were for specific problems and generalization was not possible. Thus, each new problem must be solved individually on one of the computers.

Summary

The possibility of using a fly-ball device as an angular motion sensing instrument was investigated. This device is similar to a fly-ball governor operated at constant spin speed and turned about an axis perpendicular to the spin axis (Figure 1). Use of such an instrument would be in missiles, aircraft, torpedoes, or other vehicles whose angular motion
is to be controlled or recorded.

Frictionless operation and alignment of axis of turn with the effective gravity vector were assumed.

The development of the differential equation of motion was accomplished by writing the expression for total kinetic energy of the system and then using Lagrange's equations. The differential equation of motion is as follows:

\[
\frac{d^2 \theta}{dt^2} = \frac{1}{2} \left[ K^2 - \omega(t)^2 \cos^2 Kt \right] \sin 2\theta - N \theta .
\]

No progress was made in solving Equation 44 analytically, so computer methods were used.

One hundred and fifty problems were explored with an analog computer and twelve were solved with an IBM 650 digital computer. Excellent agreement of the results was obtained.

The problems solved in this thesis were actually the inverses of those that would be interesting in practice. Reasonable turning functions were assumed in the development in this work; then the action of the device was determined in terms of the angle (\( \theta \)) between spin axis and either arm as a function of time. In a practical use of the instrument, \( \theta(t) \) would be available by measurement and the angular motion of the vehicle carrying the instrument would be calculated from \( \theta(t) \). This process was outlined.
The conclusions are that a fly-ball device such as the one proposed could well be used to detect and measure angular motion provided the instrument is designed and operated properly with relation to the magnitude of the turning velocity.
REFERENCES


ACKNOWLEDGMENTS

As in most developments, any individual relies heavily upon the knowledge and help of others. The following people and the one organization have been especially helpful:

Dr. Glenn Murphy as major professor and department head for his superb job of guidance and ready store of ideas;

Dr. A. H. Hausrath who made the analog computer approach possible;

Dr. H. J. Weiss for a number of excellent suggestions including the method of generating $\omega^2(t)$ for Series B;

Dr. George Seifert for investigation the possibilities of an analytic solutions to the equation of motion and suggesting the iterative integral solution;

The National Science Foundation for their generous faculty fellowship enabling full-time pursuit of the work;

Mrs. E. R. Chubbuck for typing the first draft of the manuscript; and

Mrs. Marjorie Downs for typing the final draft.
APPENDIX A

Centrifugal Equilibrium $\theta$ for Given $N$ and $K$

Since the most important initial value of $\theta$ is $\theta_n^1$, the centrifugal equilibrium value for $\omega(t) = 0$, perhaps the determination of $\theta_n$ should be included and a curve drawn to show the results. The equilibrium value of $\theta$ is determined, of course, by the spin speed ($K$) and the lumped parameter $N$ which is made up of the spring constant ($C$), length of arm ($L$), and the point mass ($M$). Figure 21 is a sketch of one-half of the fly-ball system showing pertinent forces. Here, the spring is half as stiff as it would be for two masses and is designated $C'$. The development here is merely an equation of the torques from centrifugal force on $M$ with the resisting torque of the spring. The spring has no torque at $\theta = 0$.

Using Figure 21:

\begin{align*}
F &= MrK^2 = MK^2LS\sin\theta_n \quad (66) \\
P &= F\cos\theta_n = MK^2LS\sin\theta_n\cos\theta_n \quad (67)
\end{align*}

or

\begin{equation}
P = \frac{MLK^2}{2} \sin 2\theta_n. \quad (68)
\end{equation}

$^1\theta_n$ is the most important value of $\theta_0$ because this is the most likely value of $\theta_0$ to be used in an instrument used as a motion-sensing device.
Figure 21. Sketch of one mass of the motion-sensing device showing forces and torques for determining $\theta_n$. 
SPIN SPEED = $|\mathbf{\omega}|$ RAD. PER SEC.

$r = L \sin \theta$

$P = F \cos \theta$

TORQUE = $PL$

TORQUE = $-C'\theta$
Torque around point 0 from centrifugal force = PL

\[ \frac{ML^2K^2}{2} \sin 2\theta_n \]  \hspace{1cm} (69)

Torque from spring = \(-C'\theta_n\)  \hspace{1cm} (70)

Therefore,

\[ \frac{ML^2K^2\sin 2\theta_n}{2} = C'\theta_n \]  \hspace{1cm} (71)

or

\[ \frac{\sin 2\theta_n}{2\theta_n} = \frac{C'}{MK^2L^2} \]  \hspace{1cm} (72)

By definition

\[ N = \frac{C}{2ML^2} \]

and

\[ C' = C/2 \]

so

\[ \frac{\sin 2\theta_n}{2\theta_n} = \frac{N}{K^2} \]  \hspace{1cm} (73)

or

\[ \frac{K}{\sqrt{N}} = \sqrt{\frac{2\theta_n}{\sin 2\theta_n}} \]  \hspace{1cm} (74)

The same result can be obtained more easily but with less feeling for the meaning by making use of the fact that \(\dot{\theta} \equiv 0\) for
the case of equilibrium and thus $\ddot{\theta} = 0$. This leads to the relation

$$0 = \frac{1}{2} \left[ K^2 - 0 \right] \sin 2\theta_n - N\theta_n$$  \hspace{1cm} (75)

since $\omega(t) \equiv 0$ for the equilibrium condition. Again,

$$\frac{K^2}{2} \sin 2\theta_n = N\theta_n$$  \hspace{1cm} (76)

and

$$\frac{\sin 2\theta_n}{2\theta_n} = \frac{N}{K^2}$$  \hspace{1cm} (77)

or

$$\frac{K}{\sqrt{N}} = \sqrt{\frac{2\theta_n}{\sin 2\theta_n}}$$  \hspace{1cm} (78)

Figure 22 is a plot of $\theta_n$ vs. $K/\sqrt{N}$ for values of $K/\sqrt{N}$ up to 4.4799. The curve is asymptotic to the line $\theta_n = \pi/2$ and to the line $K/\sqrt{N} = 1.0$. The $\pi/2$ asymptote is certainly expected, but the other asymptote points out the necessity of having the spin speed ($K$ radians per second) greater than $\sqrt{N}$ in order that the angle $\theta$ not become 0 quickly.
Figure 22. Plot of angle $\theta$ (radians) vs. $K/\sqrt{N}$ for the centrifugal equilibrium condition.
\[ \sqrt{\frac{2 \theta_n}{\sin 2 \theta_n}} = \frac{K}{\sqrt{N}} \]

\( K \) = SPIN SPEED

\( N = \frac{C}{2ML^2} \)

\( \theta_n \) = CENTRIFUGAL EQUIL. ANGLE
APPENDIX B

Auxiliary Analog Computer Topics

Method of generating $\omega^2(\zeta/T)$ for Series B

The generation of the function

$$\omega^2(\zeta/T) = U^2(p^2 + r^2)e^{-2rT} \sin^2(p\zeta/T + \delta)$$  \hspace{1cm} (79)

posed a problem because no standard method of doing this was available. Had another multiplier been available, this function could have been formed by generating a damped sine wave of proper shape and squaring it.

By starting with Equation 79 in its original equivalent form

$$\omega^2(\zeta/T) = U^2e^{-2rT} \left[ p^2\cos^2p\zeta/T + r^2\sin^2p\zeta/T ight. \\
- \left. 2pr\cos p\zeta/TS\sin p\zeta/T \right]$$  \hspace{1cm} (80)

it is possible to write it in the form of the sum of a damped sine wave and a decaying exponential. Both of these can be generated and then added to produce the desired result, thus

$$\omega^2(\zeta/T) = U^2e^{-2rT} \left[ \frac{p^2}{2}(1 + \cos 2p\zeta/T) \\
+ \frac{r^2}{2}(1 - \cos 2p\zeta/T) - pr\sin 2p\zeta/T \right]$$  \hspace{1cm} (81)

\[1\]The phase angle $\delta = \tan^{-1}(p/-r)$. 
\[ U^2 e^{-2r \tau / T} \left[ \frac{p^2 + r^2}{2} + \left( \frac{p^2 - r^2}{2} \right) \cos 2p \tau / T \right. \\
\left. - pr \sin 2p \tau / T \right] \]  

or

\[ \omega^2(\tau / T) = U^2 \frac{(p^2 + r^2)}{2} e^{-2r \tau / T} \]

\[ + e^{-2r \tau / T} \left[ U^2 \frac{(p^2 - r^2)}{2} \cos 2p \tau / T \right. \\
\left. - U^2 pr \sin 2p \tau / T \right] . \]  

Let

\[ X = e^{-2r \tau / T} \left[ U^2 \frac{(p^2 - r^2)}{2} \cos 2p \tau / T \right. \\
\left. - U^2 pr \sin 2p \tau / T \right] \]  

and

\[ Y = U^2 \frac{(p^2 + r^2)}{2} e^{-2r \tau / T} . \]  

Therefore,

\[ \omega^2(\tau / T) = X + Y . \]  

Both X and Y can be generated, and the initial conditions will now be determined:

\[ \frac{dY}{d\tau} = \frac{-2r}{T} U^2 \frac{(p^2 + r^2)}{2} e^{-2r \tau / T} = \frac{-2r}{T} Y \]  

\[ Y_0 = U^2 \frac{(p^2 + r^2)}{2} \]
\[
\frac{dX}{d\tau} = e^{-2r/\tau} \left[ \frac{rU^2}{T} (r^2 - 3p^2) \cos 2p \tau/T + \frac{U^2T}{T} (3r^2 - p^2) \sin 2p \tau/T \right].
\]

The result of Equation 89 is a simplified form.

\[
X_0 = \frac{u^2(p^2 - r^2)}{2}
\]

\[
\dot{X}_0 = \frac{rU^2}{T} (r^2 - 3p^2).
\]

Figure 23 shows how \( \omega^2(\tau/T) \) was generated; \( X \) was generated with Amps. 8, 10, and 12, \( Y \) by Amp. 6, and they were added together in Amp. 14.

The generation of \( Y \) was fairly straightforward by merely making

\[
M = \frac{2r}{T}
\]

and

\[
Y_0 = \frac{u^2(p^2 + r^2)}{2}.
\]

In order to get proper magnitude scaling of the output, the factor \( Q \) was introduced in the initial voltages across the feedback capacitors for Amps. 6, 8, and 10.

The only remaining difficulty was to generate \( X \) in the proper form. The \( X \)-part is a damped sine wave of proper phase, and the phase can be determined by initial values on the capacitors.
Figure 23. Analog computer program for generating $\omega^2(\tau/T)$ of Series A

$$\omega^2(\tau/T) = U^2(p^2 + r^2)e^{-2ar\tau/T} \sin^2(pt + \delta)$$

$$M = \frac{2r}{T}$$

$$L = \frac{4(p^2 + r^2)}{T^2}$$

$$R = \frac{4r}{T}$$

$T = $ time scale

$\tau = $ computer time
\[ Q(X+Y) = Q \omega^2 \left( \frac{t}{T} \right) \]
The method of solving a differential equation of second order was given earlier; here the same methods were used to solve the equation

$$\dddot{x} + rx + lx = 0$$  \hspace{1cm} (93)

which has as a solution the desired damped sine wave. Phase, frequency, initial magnitude, and degree of damping are all determined by the constants R and L and the initial conditions of \(\ddot{x}\) and \(\dot{x}\). The Equation 93 is a second order linear one with constant coefficients and the solution can be readily expressed as

$$x = e^{-R/2 \tau} \left[ C_1 \cos \sqrt{L^2 - R^2/4} \tau ight. + C_2 \sin \sqrt{L^2 - R^2/4} \tau \right]$$  \hspace{1cm} (94)

if \(L^2 - R^2/4 \geq 0\). Also, the first derivative is necessary in order to determine initial conditions, thus

$$\frac{dx}{d\tau} = e^{-R/2 \tau} \left[ \left( C_2 \sqrt{L^2 - R^2/4} - R/2 C_1 \right) \cos \sqrt{L^2 - R^2/4} \tau ight. \\
- \left. \left( C_1 \sqrt{L^2 - R^2/4} + R/2 C_2 \right) \sin \sqrt{L^2 - R^2/4} \tau \right]$$  \hspace{1cm} (95)

Therefore,

$$x_0 = x(\tau = 0) = C_1$$  \hspace{1cm} (96)

and

$$\ddot{x}_0 = C_2 \sqrt{L^2 - R^2/4} - R/2 C_1$$  \hspace{1cm} (97)
Therefore,

\[ C_2 = \frac{\frac{\ddot{x}_0 + R/2}{\sqrt{L^2 - R^2/4}} x_0}{\sqrt{L^2 - R^2/4}} \]  \quad (98)

Finally,

\[ \bar{x} = e^{-R/2} \left[ \frac{L_2}{\sqrt{L^2 - R^2/4}} \right] + \frac{\dot{x}_0 + R/2}{\sqrt{L^2 - R^2/4}} x_0 \sin \sqrt{L^2 - R^2/4} \tau \]  \quad (99)

Comparing Equation 99 with Equation 84, the following results are obtained:

(a) \[ \frac{R}{2} = \frac{2r}{T} \text{ or } R = \frac{4r}{T} \]  \quad (100)

(b) \[ \bar{x}_0 = \frac{u^2(p^2 - r^2)}{2} \]  \quad (101)

(c) \[ \sqrt{L^2 - R^2/4} = \frac{-2p}{T} \]  \quad (102)

or

(d) \[ L = \frac{4(p^2 + r^2)}{T^2} \]  \quad (103)

after using Equation 100

(e) \[ \frac{\ddot{x}_0 + R/2}{\sqrt{L^2 - R^2/4}} x_0 = -u^2 p r \]  \quad (104)
from which

\[ \dot{X}_0 = \frac{rU^2}{T} (r^2 - 3p^2) \]  

(105)

and this checks Equation 91.

The values of M, L, and R are determined by the values of r, T, and p; thus the settings of the input potentiometers of Amps. 6 and 8 are found. In order to phase the problem properly the values indicated for \( \dot{X}_0 \) and \( \dot{X}_0 \) need be used.

Figure 23 shows the method \( X \) was generated with Amps. 8, 10, and 12. Then \( X \) and \( Y \) were added together with Amp. 14, the output of which is the desired total function of the form of Equation 80.

**Sample data sheet and sample check list**

The sample data sheet shown in Figure 24 is fairly typical of those used and is filled out for runs 49 and 50. The double numbers indicated at Amp. 3 and Amp. 5 are the values for runs 49 and 50 respectively.

Figure 25 shows a sample check list for Series D. The many operations necessary prior to making a run necessitated some organized way of insuring that no step would be omitted.
Figure 24. Photostat of a sample data sheet used with the analog computer
### Data Sheet for Analog Computer Solution of $\frac{d^2\theta}{dt^2} + \frac{k}{\tau} \theta + \frac{1}{\tau^2} \theta = 0$

$K = 5 \pi^2$; $\tau = 77.79$; $T = 10 \tau$ ($= 77.79$); $f = K/\tau = 5.692$; $Q = 80/\pi^2 = 2.511$

$t = \int e^{-\frac{u}{\tau}} \left[ C \cos pt + A \sin pt \right] du$

\[ t = \frac{1}{u}; \quad \frac{d}{dt} = \frac{d}{du} \cdot \frac{du}{dt} = \frac{d}{du} \cdot \frac{1}{u} = -\frac{1}{u^2} \quad \text{p} = 2 \]

**Initial Conditions:**

- $\theta(0) = 0^\circ / 0$; $\theta(0) = 46.8/24.1$; $\theta(0) = -40.1$
- $\theta(0) = 80r/T \times (1 - r^2/p^2)$

<table>
<thead>
<tr>
<th>Amp.</th>
<th>Component</th>
<th>Nom. val.</th>
<th>Actual value</th>
<th>Gain</th>
<th>Pot. No.</th>
<th>Pot Setting for Indicated Run</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$31.25T^2$</td>
<td>$R_1$ C13</td>
<td>R13 0.25</td>
<td>0.985</td>
<td>3A</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$R_2$ B91</td>
<td>B91 1.00</td>
<td>0.2501</td>
<td>3B</td>
<td>0.965</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$R_3$ B37</td>
<td>B37 1.00</td>
<td>0.975</td>
<td>3B</td>
<td>0.975</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>$R_1$ C31</td>
<td>C31 1.00</td>
<td>0.987</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$R_1$ R71</td>
<td>R71 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>$R_1$ R73</td>
<td>R73 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>$R_1$ R10</td>
<td>R10 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>$R_2$ R15</td>
<td>R15 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>$R_1$ R20</td>
<td>R20 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>$R_1$ R25</td>
<td>R25 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>$R_1$ R30</td>
<td>R30 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>$R_1$ R35</td>
<td>R35 1.00</td>
<td>0.965</td>
<td>5B</td>
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</tr>
<tr>
<td>14</td>
<td>1</td>
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<tr>
<td>15</td>
<td>1</td>
<td>$R_1$ R45</td>
<td>R45 1.00</td>
<td>0.965</td>
<td>5B</td>
<td>0.800</td>
</tr>
</tbody>
</table>

**Gain Values:**

- $31.25T^2$
- $R/2T^2$
- $0.0125q^2$
- $0.125q^2$

**Potentiometer Values:**

- $R_1 = 31.25T^2$
- $R_2 = R/2T^2$
- $R_3 = 0.0125q^2$
- $R_4 = 0.125q^2$

**Gain Values:**

- $31.25T^2$
- $R/2T^2$
- $0.0125q^2$
- $0.125q^2$

**Initial Conditions:**

- $\theta(0) = 0^\circ / 0$
- $\theta(0) = 46.8/24.1$
- $\theta(0) = -40.1$
- $\theta(0) = 80r/T \times (1 - r^2/p^2)$
Figure 25. Photostat of one of the checklists used with the analog computer
1. Warm up computer and recorders one hour or more.

2. Set -100 volts on center tap of auxiliary Pot #2 (by connecting to HP power supply) and connect to Pots hA, hB, hA, and lSA. Null against Amp #1.

3. Set -100 volts on I.C. #3; this goes to patchboard.

4. Set +100 volts on positive side of I.C. #6; this goes to multiplier patchboard.

5. Calibrate recorders with +100 volts from Amp #1.

6. Set up function generators and plug them in:

A. Sin25 (Fn. Gen. #2)
   1. Set Pot I5B = 1.000.
   2. Set +60.0 volts on output of Amp #1 and hook to X-input of Fn. Gen. through I.C. #5 (+ side of I.C. #5 to Amp #1).
   4. Set +73 volts on output of Amp #15 with Pot 15A.
   5. Set output of Amp #1 to zero and read zero volts on output of Amp #15.
   6. Check further and set up Fn. Gen. again if 1 volt agreement is not reached.
   7. Reconnect + side of I.C. #5 to output of Amp #5.

B. Sin(t) (Fn. Gen. #1)
   1. Set Pot la = 1.000 and set Pot lb = 0.
   2. Set zero volts on output of Amp #1 and hook to X-input of Fn. Gen.
   3. Read zero volts on output of Amp #1.
   4. Bias output of Amp #1 to +60 volts with Pot 1B.
   5. Set +70 volts on output of Amp #1.
   6. Read zero volts on output of Amp #1.
   7. Check further and set up Fn. Gen. if one volt agreement is not reached.

7. Set -10 volts on IA with reference supply and Pot IB.

8. Set e0 = 2π/8 volts on Amp #5 with I.C. #1.

9. Set initial condition for Amp #1 = -100 volts with I.C. #2.

10. Set initial condition for Amp #10 = +10 volts with I.C. #1.

11. Check for proper components.

12. Set all pots to correct values except Pots hA, hB, hA, and lSA.

13. Adjust Pot 6A so as to get no change of Cos2Kt with time.

14. Turn on multipliers and pen recorder.

15. Operate.
APPENDIX C

Sample of the IBM 650 Program

The entire program was fairly lengthy and is not included, but Figure 26 is a photostat of one sheet of it. The program was in the so-called basic language and is obviously quite unintelligible to anyone not already familiar with the work.
Figure 26. Sample page of the program for the IBM 650 digital computer
PROBLEM: I of the pair

<table>
<thead>
<tr>
<th>INSTR. NO.</th>
<th>LOCATION OF INSTR.</th>
<th>CODE</th>
<th>DATA</th>
<th>INSTRUCTION</th>
<th>OPERATION</th>
<th>ABBRV.</th>
<th>REMARKS</th>
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<td></td>
<td>69</td>
<td>0328</td>
<td>0288</td>
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<td>Begin   ( \theta_{n-1} ) routine</td>
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<tr>
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<td></td>
<td>24</td>
<td>0245</td>
<td>0298</td>
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<td>0253</td>
<td>0307</td>
<td>RAU</td>
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<td></td>
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<tr>
<td>0307</td>
<td></td>
<td>19</td>
<td>0310</td>
<td>0255</td>
<td>MUTT</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0295</td>
<td></td>
<td>31</td>
<td>0501</td>
<td>0301</td>
<td>SRD</td>
<td>6 places</td>
<td></td>
</tr>
<tr>
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<td>0308</td>
<td>STL</td>
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<td>0349</td>
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<td>0001</td>
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<td>SRD</td>
<td>6 places</td>
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<td>SRD</td>
<td>6 places</td>
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</tr>
<tr>
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<td>AL</td>
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<tr>
<td>0359</td>
<td></td>
<td>20</td>
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<td>0250</td>
<td>STL       ( \theta_{n-1} ) is now in punch store 0329</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0387</td>
<td></td>
<td>60</td>
<td>0328</td>
<td>0381</td>
<td>RAU       ( \theta_{n-1} ) in upper</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0381</td>
<td></td>
<td>45</td>
<td>0534</td>
<td>0385</td>
<td>BRMNT</td>
<td>Is ( \theta_{n-1} ) negative?</td>
<td></td>
</tr>
<tr>
<td>0534</td>
<td></td>
<td>11</td>
<td>0527</td>
<td>0511</td>
<td>SU</td>
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<td></td>
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<td></td>
<td>65</td>
<td>0544</td>
<td>0506</td>
<td>BRMNT</td>
<td>Is ( \theta_{n-1} ) greater than tolerance?</td>
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<tr>
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<td>01</td>
<td>0527</td>
<td>0598</td>
<td>STOP</td>
<td>If Yes. stop</td>
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</tr>
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<td></td>
<td>69</td>
<td>0501</td>
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<td>Is ( \theta_{n-1} ) greater than tolerance?</td>
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<td>0288</td>
<td>ID</td>
<td>If no. begin ( \theta_{n-1} ) routine</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX D

List of Symbols Used

A, E, C, D  series designations of the several sets of runs
A, B, C   coefficient constants used in an example analog computer problem (Equation 57)
A, E   constants of integration in the iterative integral solution of the equation of motion;
\[ A = \frac{\dot{\theta}(c)}{\sqrt{N}}, \quad B = \theta(c) \] (Equations 47 and 48)
C   torsion spring constant for the spring of both fly-balls combined, ft. lbs./radian
C'   torsion spring constant for one fly-ball \((C' = C/2),\) ft. lbs./radian

C_1, C_2  constants of integration for solution of the intermediate equation used in generating one part of \(\omega^2(\tau/T)\) for Series B (Appendix B)

\(e\)  base of natural logarithms
\(\vec{e}_r\)  radial unit vector of \(M_1\) in movable system
\(\vec{e}_\phi\)  \(\phi\) unit vector of \(M_1\) in movable system
\(\vec{e}_\theta\)  \(\theta\) unit vector of \(M_1\) in movable system
\(\vec{e}_{r_2}\)  radial unit vector of \(M_2\) in movable system
\(\vec{e}_{\phi_2}\)  \(\phi\) unit vector of \(M_2\) in movable system
\(\vec{e}_{\theta_2}\)  \(\theta\) unit vector of \(M_2\) in movable system

\(F\)  centrifugal force on one fly-ball (Appendix A)

\(h\)  time increment used in quadrature formula (step 4 after Table 7), seconds
a Cartesian unit vector in fixed system

a Cartesian unit vector in fixed system

a Cartesian unit vector in fixed system

a Cartesian unit vector in movable system

a Cartesian unit vector in movable system

a Cartesian unit vector in movable system

the spin velocity vector

the magnitude of K, radians per second

length of each fly-ball arm, feet

the constant

$$\frac{4(p^2 + r^2)}{T^2}$$

used in generating $\omega^2(\tau/T)$ for Series B on the analog computer (Appendix B)

either of the two masses used as fly-balls,

$$\frac{\text{lb. sec.}^2}{\text{ft.}}$$

individual fly-ball masses,

$$\frac{\text{lb. sec.}^2}{\text{ft.}}$$

the constant $2r/T$ used in generating $\omega^2(\tau/T)$ for Series B (Appendix B)

the combined parameter

$$\frac{C}{2ML^2} \left( \frac{1}{\text{sec.}^2} \right)$$

compact of force F perpendicular to fly-ball arm in development of equilibrium angle, $\theta_n$ (Appendix A)
$\vec{F}$
position vector of origin of movable system in fixed system; here $\vec{F}$ = 0

$p$
circular frequency (radians per second) of $\omega(t)$ for Series A, B, and C

$Q$
coefficient of $\omega^2(T/T)$ used in analog computer setup to reduce the number of function generator adjustments

$q_i$
any generalized coordinate as used in Lagrange's equations; here only one $q_i$ was necessary and $q_i = \theta$ (radians)

$q_i$
first time derivative of $q_i$

$Q_i$
a generalized force corresponding to $q_i$; here $Q_i = -C\theta$ (lb. ft.)

$R$
the constant $4r/T$ used in generating $\omega^2(T/T)$ for Series B (Appendix B)

$R$
a radial coordinate of $M_1$ in the xy-plane of the movable system (Figure 5)

$r$
position vector of $M_1$ in the fixed coordinate system

$r_2$
position vector of $M_2$ in the fixed coordinate system

$r$
damping factor for $\omega(t)$ of Series B, l/sec.

$r$
radial distance from spin axis to $M_1$ in Figure 22 (Appendix A)

$T$
time scale for analog computer where $t = \tau/T$

$T_1$, $T_2$
kinetic energy of fly-balls $M_1$ and $M_2$ respectively, ft. lbs.

$T$
total kinetic energy of $M_1$ and $M_2$ or $T = T_1 + T_2$

$T_1\delta^2\phi_{n}$
first trial value of $\delta^2\phi_{n}$

$T_1\theta_{n+1}$
first trial value of $\theta_{n+1}$

$T_1\phi_{n+1}$
first trial value of $\phi_{n+1}$
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t real time in dynamic system, sec.

U coefficient constant in \( \omega(t) \); values for the various problems in Table 3

X, Y, Z Cartesian coordinates in the fixed system

X, Y the two parts of \( \omega^2(T/T) \) generated with the analog computer for Series B (Appendix B)

X dependent variable of Equation 93 which was involved in generating \( \omega^2(T/T) \) for Series B (Appendix B)

x, y, z Cartesian coordinates in the movable system

\( \Delta t \) time increment in difference methods, sec.

\( \phi \) phase angle (radians) of \( \omega(t) \) for Series A, B, and C; \( \phi = \tan^{-1}(p/r) \)

\( \delta \) operator for first central difference

\( \delta^2 \) operator for second central difference

\( \delta^4 \) operator for fourth central difference

\( \theta \) angle between spin axis and either fly-ball arm, radians

\( \dot{\theta}, \ddot{\theta}, \dddot{\theta} \) first, second, and third time derivatives of \( \theta \)

\( \theta_n \) centrifugal equilibrium angle, radians (Appendix A)

\( \theta_{n+i} \) the angle \( \theta \) for a given time index as used in the difference method of solving the equation of motion (i is an integer and may be zero)

\( \theta_j(t) \) the j-th iteration in the solution for \( \theta(t) \) by the iterative integral method (Equations 45, 49, 53, 54, and 55)

\( \vec{r} \) radius vector of \( M_1 \) in the fixed system

\( \vec{r}_2 \) radius vector of \( M_2 \) in the fixed system
\( \vec{r}_1 \) \hspace{1cm} \text{radius vector of } M_1 \text{ in the movable system}

\( \vec{r}_2 \) \hspace{1cm} \text{radius vector of } M_2 \text{ in the movable system}

\( \tau \) \hspace{1cm} \text{computer time for analog computer, sec.}

\( \varphi \) \hspace{1cm} \text{total rotation from time zero around the}
\hspace{1cm} \text{z-axis from spin vector } \vec{K}, \text{ radians } (\varphi = Kt)

\( \vec{\phi}(t) \) \hspace{1cm} \text{angular displacement vector of the motion-}
\hspace{1cm} \text{sensing device around the axis of turn }

\( \vec{\phi}(t) \) \hspace{1cm} \text{magnitude of } \vec{\phi}(t), \text{ radians}

\( \vec{\omega}(t) \) \hspace{1cm} \text{turn velocity vector}

\( \omega(t) \) \hspace{1cm} \text{magnitude of } \vec{\omega}(t), \text{ radians per sec.}