WAVE SCATTERING BY AN INCLUSION HAVING IMPERFECT INTERFACES WITH THE MATRIX

Jianmin Qu

School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia 30332--0405

INTRODUCTION

Ultrasonic waves have been used in recent years to detect defects and damages in fiber-reinforced composite materials (see [1] for relevant references). A priori knowledge of how sound wave propagates through the composite medium is essential for using ultrasonic techniques in nondestructive evaluation. In the past, the problem of scattering by inclusions, such as fibers, has been studied extensively [2]--[3]. Most of the studies in the literature assume that the inclusions are perfectly bonded to the matrix. However, defects are likely to exist along the fiber--matrix interfaces. As more interfacial defects are initiated, damages are developed within a layer of materials near the interface which will eventually lead to complete fiber debonding. Since interfacial damages may adversely affect the overall strength of the composites, it is important to be able to characterize and monitor the damage accumulation during manufacturing processes and engineering applications of the composites. Thus, developing theories and techniques to characterize interfacial strength, toughness and the amount of damages is warranted. To serve this purpose, the effects of interfacial damages on the behavior of wave propagation in fiber--reinforced composites need to be studied.

To understand wave propagation in damaged composites, the problem of scattering by a fiber imperfectly bonded to the matrix is studied in the present paper. The boundary value problem corresponding to the scattering problem is stated first. Then the interface conditions are discussed and integral equations are derived based on the elastodynamic representation integrals. Finally, the expressions for the far--field displacements are presented. The paper is concluded by a numerical example.

STATEMENT OF THE PROBLEM

Consider an elastic inclusion D embedded in an elastic matrix of infinite extent. The interface between the inclusion and the matrix is denoted by S with the unit outward normal being \( \hat{n}_i \), see Fig. 1. Let the elastic constants and mass density of the matrix be denoted by \( C_{ijkl} \) and \( \rho \), while those of the inclusion by \( C^*_{ijkl} \) and \( \rho^* \), respectively. In this paper, we consider the case of plane wave incidence. In the Cartesian coordinate system shown in Fig. 1, the incident wave may be written as

\[
\mathcal{u}_i^{\text{in}} = u_0 d_1 \exp[i(kz_1 p_j - \omega t)],
\]

(1)
where $d_i$ and $p_i$ are the displacement and propagation vectors, respectively. $k$ is the wave number and $\omega$ is the circular frequency.

It is convenient to express the total displacement field as the sum of the incident and scattered fields:

$$u_i = u_i^{in} + u_i^s,$$

where the scattered field $u_i^s$ satisfies the radiation conditions [4]. The objective is to find the scattered field $u_i^s$ and thus the total displacement field $u_i$.

**INTERFACE CONDITIONS**

The scattering problem stated in previous section has been studied by many investigators under the assumption that the inclusion and the matrix are perfectly bonded together. In this model, the interface is taken to be a surface of zero thickness across which properties are discontinuous but displacements are continuous, see Fig. 2a. If the displacement and stress fields outside the inclusion are represented by $u_i$ and $\sigma_{ij}$, while those inside the inclusion by $u_i^*$ and $\sigma_{ij}^*$, the perfect bonding condition implies

$$u_i = u_i^*, \quad t_i = t_i^*, \quad x_i \in S$$

where $t_i$ and $t_i^*$ are the interfacial tractions given by

$$t_i = \sigma_{ij} n_j, \quad t_i^* = \sigma_{ij}^* n_j.$$

**Fig. 2 Interface conditions; (a) perfect bonding, (b) interphase-layer, (c) spring-layer.**
However, an interface layer surrounding the inclusion is often introduced during the material fabrication process. As demonstrated by recent experimental and atomistic observations [5]–[6], interfaces are regions of distinct atomic structure and, possibly, distinct composition. Therefore, the interface region tends to have properties which are distinct from the bulk properties on either side of the interface. The strength and toughness of this layer may have significant influence on the overall behavior of the composites. The continuum description given by (3) with continuous displacements across the interface cannot adequately describe behaviors that inherently depend on the distinct properties in the interface region.

To incorporate distinct interface properties, an interphase-layer model of interfaces, as schematically illustrated in Fig. 2b for a circular inclusion, has been used [7]. In this model, one needs to specify the interface layer thickness in addition to its properties. The interface layer thickness may vary significantly from one type of interfaces to another. It could be only a few angstroms for some grain boundaries, whereas for adhesive type of interfaces, it could be as thick as a few micrometers. From the viewpoint of using ultrasonics for nondestructive testing on fiber-reinforced composites, the pertinent wavelengths should be comparable to the characteristic dimensions of the fibers. It is also conceivable that the thickness of the interfacial region is much smaller than the fiber diameters. In this case, the thickness of the interface layer can be ignored and further simplifications can be made. Recently, a spring-layer model of the interface has been used to study scattering problems [8]–[9]. This model takes the point of view that the interface is a surface of zero thickness across which bulk properties are discontinuous, but displacement discontinuities are also permitted through interface traction–displacement jump relations. Through the former, the need of specifying the thickness of the interface is circumvented, and through the latter, the distinct properties of the interface region are taken into account approximately. A general form of this spring-layer model may be written as

\[ t_i = t_i^* , \]  
\[ t_i = K_{ij}(u_j - u_j^*) , \]  
when \( t_i n_i \geq 0 \) and

\[ (u_j - u_j^*)n_j = 0 , \quad S_{im}t_m = S_{im}K_{nj}(u_j - u_j^*) \]  
when \( t_i n_i < 0 \), where \( S_{im} = (\delta_{im} - n_i n_m) \) is the tangential operator that extracts the tangential components from the traction vector. The matrix \( K_{ij} \) is known as interface stiffness.

Equation (5) simply states the equilibrium. (6) relates the interfacial traction to the displacement jump. (7) means that once the normal stress becomes negative, the spring in the normal direction is removed and the two surfaces stay in contact. As a consequence, continuity of normal displacement is fulfilled although relative sliding is still allowed. This one-sided constraint ensures the compatibility, i.e., no interpenetrations can take place. Obviously, the interfacial conditions given by (5)–(7) are nonlinear. To the author's knowledge, the general conditions for existence and uniqueness of the elasticity solutions that satisfy the non–linear boundary conditions (6) and (7) have not been established. The existence and uniqueness can only be justified on physical ground. Nevertheless, since the governing equations are linear, the nonlinearity in the boundary conditions can be handled with care. In a recent study, Achenbach and Zhu [10] employed an iteration technique in their boundary element calculations for fiber debonding problems. It is conceivable that the schemes used in [10] can also be applied to time dependent problems in elastodynamics, because the iteration procedure can be
implemented at each time step. However, due to the nonlinearity Laplace transforms of (6)-(7) with respect to time are not possible. This implies that solving the scattering problems with boundary conditions (6)-(7) in the frequency domain is meaningless. It is thus fair to say that the nonlinear conditions (6)-(7) can only be taken care of by solving the problem in the time domain.

To solve scattering problems in the frequency domain (i.e. steady state time harmonic), modifications to the above mentioned spring model are needed. To remove the nonlinearity conceptually, one may think that the interface layer is being replaced by a layer of springs with certain thickness as illustrated in Fig. 2c. This is equivalent to pre-stress the composite so that the interfacial springs will be stretched in tension before the incident wave arrives. This is the point of view we are taking in this paper. In fact, this is also exactly the same conceptual argument for the theory of scattering by traction-free cracks [11].

Based on this simplified spring model, the interface conditions are

\[ t_i(x) = t_i^*(x), \quad x \in S \]  
\[ t_i(x) = K_{ij}(x)[u_j(x) - u_j^*(x)], \quad x \in S. \]  

Another advantage of (8) and (9) is that perfect bonding is achieved by taking the limit \( K_{ij} \to \infty \), whereas cavity solution is obtained by taking the limit \( K_{ij} \to 0 \).

INTEGRAL EQUATION FORMULATION

An integral representation for the total displacement field, which satisfies the equilibrium, may be written as [4]

\[ \int_S [T_{ij}(x,y)u_j(y) - U_{ij}(x,y)t_j(y)]dS(y) = \begin{cases} -u_i^*(x), & x \in D \\ u_i^*(x), & x \notin D \end{cases} \]  

(10)

where \( t_i \) is the traction on \( S \) given by (4) and \( U_{ij} \) is the Green's elastodynamics tensor [4] and \( T_{ij} \) is defined as

\[ T_{ij} = \sigma_{im;j}^* n_m = C_{lm;i} U_{kj,l} n_m. \]

When the field point \( x_i \) approaches the surface \( S \) from outside \( D \), it follows from (10) that

\[ \int_S [U_{ij}(x,y)t_j(y) - T_{ij}(x,y)u_j(y)]dS(y) + \frac{1}{2} u_i(z) = u_i^*(z), \quad x_i \in S \]  

(11)

This is an integral equation for the unknown \( u_i \) and \( t_i \).

Similarly, one can show [4] that for the field inside the inclusion \( D \), we have

\[ \int_S [T_{ij}^*(x,y)u_j^*(y) - U_{ij}^*(x,y)t_j^*(y)]dS(y) = \begin{cases} u_i^*(x), & x \in D \\ 0, & x \notin D \end{cases} \]  

(12)

where the quantities with an asterisk are associated with the fields inside the inclusion, e.g.

\[ T_{ij}^* = C_{lm;i} U_{kj,l} n_m. \]
Another integral equation is obtained from (12) if we let the field point $z_i$ approach the surface $S$ from the inside of the inclusion, namely,

$$
\int_S \left[ U_{ij}(z,y)l_i^*(y) - T_{ij}(z,y)u_j^*(y) \right] dS(y) - \frac{1}{2}u_i^*(z) = 0 , \quad z_i \in S .
$$

Equations (11) and (13) are coupled through (8)-(9). A set of integral equations for $u_i$ and $u_i^*$ are obtained by using (8)-(9) in (12) and (13)

$$
\int_S \left\{ U_{im}(z,y)K_{mj}(y) - T_{ij}(z,y)u_j(y) - U_{im}(z,y)K_{mj}(y)u_j^*(y) \right\} dS(y) + \frac{1}{2}u_i(z) = u_i^*(z) .
$$

$$
\int_S \left\{ U_{ij}(z,y)K_{mj}(y)u_j(y) - [U_{ij}(z,y)K_{mj}(y) + T_{ij}(z,y)u_j^*(y)] \right\} dS(y) + \frac{1}{2}u_i^*(z) = 0 .
$$

Once the integral equations (14) and (15) have been solved for $u_i$ and $u_i^*$, then $t_i$ and $t_i^*$ can be computed from (5) and (6). With $u_i, t_i, u_i^*$ and $t_i^*$ known, the scattered displacement field outside the inclusion follows from (10)

$$
u_i^*(z) = \int_S [T_{ij}(z,y)u_j(y) - U_{ij}(z,y)l_i^*(y)] dS(y) .
$$

The displacement field inside the inclusion is given by (12)

$$
u_i^*(z) = \int_S [T_{ij}(z,y)u_j^*(y) - U_{ij}(z,y)l_i^*(y)] dS(y) .
$$

THE FAR-FIELD DISPLACEMENT

For a field point $z$ far from the source point $y$, the following approximation can be made

$$
|x - y| \approx r - (\hat{x} \cdot y)
$$

where $\hat{x}$ is a unit vector in the direction of $x$, $\hat{x}_i = x_i/r$, and

$$
r = |z| = (x_1^2 + x_2^2)^{1/2} .
$$

Making use of (18) in the expression of the Green's elastodynamics tensor $U_{ij}$ in conjunction with (16) yields the scattered displacement field far from the inclusion

$$
\begin{align*}
\nu_i^*(z) & \approx \sum_{\alpha} V_{i}^{\alpha}(z) \frac{\exp[i(\pi/4 + k_{\alpha}^* \hat{x} \cdot y)]}{\sqrt{8\pi k_{\alpha}^* r}} , \quad \alpha = L, T
\end{align*}
$$

1285
where

\[ V^\alpha_i(\hat{z}) = \int_S [ik\alpha B^\alpha_{i m;j}(\hat{z})n_m u_j^s - A^\alpha_{i j}(\hat{z})\delta^i_j \exp(-ik\alpha \cdot y)dS(y)]. \]  (20)

For isotropic materials, \( A^\alpha_{i j} \) and \( B^\alpha_{i m;j} \) in (20) are given by

\[ A^L_{i j}(\hat{z}) = \hat{z}_i \hat{z}_j/\left(\lambda + 2\mu\right) , \quad A^T_{i j}(\hat{z}) = \left(\delta_{ij} - \hat{z}_i \hat{z}_j\right)/\mu , \]

\[ B^L_{i j k}(\hat{z}) = [2\kappa^2\hat{z}_i \hat{z}_j + (1 - 2\kappa^2)\delta_{ij}]\hat{z}_k , \quad B^T_{i j k}(\hat{z}) = \delta_{ik}\hat{z}_j + \delta_{jk}\hat{z}_i - 2\hat{z}_i \hat{z}_j \hat{z}_k , \]

\[ \kappa = [(\lambda + 2\mu)/\mu]^{1/2} . \]

**NUMERICAL EXAMPLE**

For simplicity, we consider a rigid circular inclusion of radius \( a \) as shown in Fig. 3. The matrix material is taken to be aluminium with the phase velocities and mass density given by

\[ c_L = 6300m/s , \quad c_T = 3100m/s , \quad \rho = 2700kg/m^3 . \]

The incident wave is a longitudinal wave propagating along the \( x_1 \)-axis

\[ u_1^{in} = u_0 \exp[ik_L(x_1 - c_L t)] , \quad u_2^{in} = u_3^{in} = 0 . \]

The calculations are performed for frequency \( k_T a = 1.5 \) with the interfacial stiffness

\[ K_{11} = K_{22} = \rho c_T^2 k_0/a , \quad K_{12} = K_{21} = 0 , \]

where \( k_0 \) is a non-dimensional parameter ranging from 0 to infinity. \( k_0 = 0 \) corresponds to a cavity and \( k_0 = a \) is for the case of perfect bonding. Normalized radial displacement components, \( \hat{u}_r \) and \( \hat{u}_r^s \), are plotted in Figs. 4 - 7. The normalizations are defined as follows:

\[ \hat{u}_r = |u_r/u_0| , \quad \hat{u}_r^s = \sqrt{3\pi} \left| u_r^s/u_0 \right| . \]

Fig. 3 A longitudinal wave incident upon a rigid circular inclusion.
Fig. 4a shows the radial displacement on the inclusion surface versus polar angle $\theta$ for various values of $k_0$. The solution for $k_0 = 0$ corresponds to a cavity. The results we obtained for this case are in good agreement with the analytical solutions in [2]. The very thin line on the bottom of the figure is the solution for $k_0 = 100$, which is almost zero. This is expected since the inclusion is rigid and $k_0 \to \infty$ means perfect bonding. Fig. 4b presents the far-field scattering pattern. Once again, the thin line at the bottom is for $k_0 = 100$. It is seen from Figs. 4a and 4b that the solutions for various values of $k_0$ fall more or less in between the two limit cases, namely, $k_0 = 0$ and $k_0 = \infty$. It is also seen from Fig. 4b that the far-field displacement amplitude does not change very significantly for different $k_0$ values. This implies that any method using the far-field displacement amplitude to assess the interface stiffness $k_0$ will not be very accurate.

To simulate interfacial defects, calculations are carried out for the case when $k_0$ is not a constant. For example, a circumferential crack of length $L = \phi a / \pi / 90$ along the interface with its center at $\theta = \theta_0$ may be simulated by

$$k_0(\theta) = \begin{cases} 0 , & \text{when } \theta_0 - \phi < \theta < \theta_0 + \phi \\
\infty , & \text{elsewhere} \end{cases}$$

where the angles are in degrees. In Figs. 5 and 7, results for an interface crack of length $L = 2\pi a / 3$ at three different locations, $\theta_0 = 0^\circ, 90^\circ, 180^\circ$, are presented. Instead of assigning $k_0$ to be infinite, we use $k_0 = 10$ outside the cracked interface region in the numerical calculations. The crack opening displacements are clearly seen from Figs. 5a, 6a and 7a. The far-field displacement amplitudes given in Figs. 5b, 6b and 7b show significant increase due to the existence of the interfacial crack indicating that an inclusion with damaged interface is a much more effective scatterer. This observation makes it possible to use ultrasonics for assessing interfacial damages.
Fig. 6a Interface displacement for crack at $\theta = 90^\circ$.

Fig. 6b Far-field displacement for crack at $\theta = 90^\circ$.

Fig. 7a Interface displacement for crack at $\theta = 180^\circ$.

Fig. 7b Far-field displacement for crack at $\theta = 180^\circ$.

REFERENCES