2010

Contributions to accelerated destructive degradation test planning

Ying Shi

Iowa State University

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Contributions to accelerated destructive degradation test planning

by

Ying Shi

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Statistics

Program of Study Committee:
William Q. Meeker, Major Professor
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Iowa State University
Ames, Iowa
2010
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ABSTRACT

Many failure mechanisms can be traced to underlying degradation processes. Degradation eventually leads to a weakness that can cause a failure for products. When it is possible to measure degradation, such data often provide more information than traditional failure-time data for purposes of assessing and improving product reliability. For some products, however, degradation rates at use conditions are so low that appreciable degradation will not be observed in a test of practical time length. In such cases, it might be possible to use some accelerating variables (e.g., temperature, voltage, or pressure) to accelerate the degradation processes. In today’s manufacturing industries, accelerated destructive degradation tests (ADDTs) are widely used to obtain timely product reliability information. In designing an experiment, decisions must be made before data collection, and data collection is usually restricted by limited resources. Careful test planning is crucial for efficient use of limited resources: test time, test units, and test facilities. The basic goal in designing an experiment is to improve the statistical inference for the quantities of interest by selecting appropriate test conditions to minimize or control the variability of the estimator of interest. Generally, an ADDT plan specifies a set of testing conditions and the corresponding allocations of test units to each condition. In this dissertation, we study the test planning methods for designing accelerated destructive degradation tests from three aspects, including non-Bayesian and Bayesian methods. First, Chapter 2 presents the non-Bayesian methods for accelerated destructive degradation test planning when there is only one failure cause for the testing products. Second, Chapter 3 describes the non-Bayesian methods for accelerated destructive degradation test planning when more than one failure cause (sometimes known as competing risks) are induced for the produces which are tested at high-stress levels of accelerating variables. Third, Chapter 4 shows the Bayesian methods for accelerated destructive degradation test planning.
CHAPTER 1. GENERAL INTRODUCTION

1.1 Introduction

With the development of new technologies and global competition, today’s manufactures are facing strong pressure to produce high-quality products which are expected to perform their proper functions for years or even decades. This implies the increased need for up-front reliability tests on systems, subsystems, and components (which we generically refer to as “units”), when the products are being designed. With short product development times, reliability tests must be conducted within severe time constraints. Traditional life tests (where time to failure is the response) may result in few or no failures within a long period of time, even when accelerated (e.g., by testing at higher-than-usual stress levels). Thus it is difficult to assess reliability of the products using traditional life tests that record only failure time. For this reason, degradation tests can be useful in manufacturing industries to obtain reliability information more quickly.

For some products, degradation response is the natural response. For example, for a luminescent light, the degradation response is the output of the light. Depending on the application, degradation data may be available continuously or at specific time points where measurements are taken. With degradation data, it is possible to make useful reliability inferences, even with few or no failures. Direct observation of the physical degradation process (e.g., tire wear) or some closely related surrogate may allow direct modeling of the physics-of-failure mechanisms, providing more justification and credibility for reliability estimates and a firmer basis for modeling extrapolation.

Engineers usually increase the levels of stresses (for example, temperature, voltage, humidity, or pressure) to higher than usual levels to accelerate the degradation tests. They expect that at higher levels of stress, the products will degrade more quickly and that they can estimate the lifetime or degradation rates at lower, normal use conditions using extrapolations based on a physically reasonable statistical model. For the degradation models we have considered in this dissertation, only one accelerating variable is to be considered in designing a test plan and that the degradation distributions are log-location-scale distributions. The relationship, after a possible transformation, is linear between the
location parameter of a log-location-scale distribution and the transformed time at a fixed accelerating variable level.

Physical degradation or performance degradation are easy to measure for many applications (e.g., monitoring crack size of a specimen or power output of an electronic device). It may, however, be difficult, costly, or impossible to obtain degradation measures from some components or materials. Often taking degradation measures requires destructive measurements (e.g., destructive strength tests) or disruptive measurements (e.g., disassembly and reassembly of a device) that have the potential to change the degradation processes. In such situations, which is the motivation for the current work, one can obtain only a single measurement on each unit tested. A test with such accelerated degradation data is called “accelerated destructive degradation test” or ADDT.

Careful test planning is important for efficient use of limited resources: test time, test units, and test facilities. In designing a test plan, engineers need to determine a set of test conditions and the corresponding proportional allocations of test units to each condition. ADDT planning requires information that includes planning values for the model parameters, a underlying distribution for the model variability, a specification of the critical degradation level, the range of the accelerating variable available for the experiment. There will also be constraints on the maximum test time and the number of units available for testing. Such information, which is known as “planning information”, is typically obtained based on previous experience with similar products or engineering judgment. The planning values of model parameters are sometimes uncertain, which means that engineers can only provide a range of the planning values that they believe contain the true underlying values.

The role of a statistician in designing an ADDT plan is to help engineers with a specified amount of limited resources to get the most accurate and precise estimates from their experiment. Because maximum likelihood (ML) estimation is commonly used to analyze ADDT data, a useful criterion for planning an ADDT test is to minimize the variance of the ML estimator of some quantiles of interest at use conditions. This is usually done by minimizing the large-sample approximate variance of the estimators obtained from inverting the Fisher information matrix. If there were no uncertainty in the planning values, one could use an optimum test plan which, among all possible test plans, has the smallest large-sample approximate variance of the estimator. In practice, however, there is usually uncertainty in planning information and thus, as an alternative, it has been suggested that one should use a compromise test plan that has good (but not optimum) statistical properties and is also robust to the uncertainties of model specification and planning values.

The main purpose of this research is to study the planning methods for designing accelerated destruc-
tive degradation tests under three different situations. This research is motivated by real applications. The methodology that we developed here, however, is general and can be applied to other practical situations. We know of no previous work that has described the test planning methods in detail for accelerated destructive degradation data.

1.2 Dissertation Organization

This dissertation consists of three main chapters, preceded by the present general introduction and followed by a general conclusion. Each of these main chapters corresponds to a journal article. Chapter 2 presents the methods for accelerated destructive degradation test planning when there is only one failure cause (sometimes referred to as a failure mode) for the testing products. For some applications, however, tests at high-stress levels of accelerating variables like temperature or voltage can induce failure modes that would not be observed at normal operating conditions. If other failure modes (sometimes known as competing risks) are caused only at high-stress levels of the accelerating variables, and primary interest is focused on one particular failure cause that commonly occur at normal conditions, failure from other causes can be viewed as a form of right random censoring in the data analysis (as long as the other failure modes do not completely dominate the failure mode of interest). Chapter 3 studies the methods for accelerated destructive degradation test planning with competing risks. The test planning methods described in Chapters 2 and 3 are non-Bayesian methods which require the exact planning values of model parameters. For many application, however, people only know a general range for model parameters. A prior distribution can be generally used to describe such available information on model parameters. Bayesian methods are useful to formally incorporate prior information into estimation and test planning, providing test plans with better statistical precision (i.e., smaller estimation variance) for applications with model parameters known in a range. Chapter 4 presents the Bayesian methods for accelerated destructive degradation test planning.

1.3 Literature Review


Classical work in the area of Bayesian design of experiments and preposterior analysis is covered, for example, in Raiffa and Schlaifer (1961). Chaloner and Verdinelli (1995) give a broad review of Bayesian design methods. Chaloner and Larntz (1989, 1992) describe Bayesian designs for logistic regression and accelerated testing, respectively. Clyde, Müller, and Parmigiani (1995) describe the Bayesian design methods for heart defibrillators, under a logistic regression model. Polson (1993) provides a general decision theory for accelerated life test (ALT) Bayesian design problem and proposes a preposterior expected information-based utility function. Verdinelli, Polson, and Singpurwalla (1993) describe ALT Bayesian design methods for predictions using utility functions based on Shannon information. Zhang and Meeker (2006) apply Bayesian methods to design an ALT with a log-location-scale distribution and censoring, using preposterior precision on a quantity of interest and large-sample approximation to provide optimum and optimized compromise test plans.

References


Company.


CHAPTER 2. Accelerated Destructive Degradation Test Planning

A paper published in Technometrics

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Abstract

Accelerated Destructive Degradation Tests (ADDTs) provide reliability information quickly. An ADDT plan specifies factor level combinations of an accelerating variable (e.g., temperature) and evaluation time and the allocations of test units to these combinations. This paper describes methods to find good ADDT plans for an important class of destructive degradation models. First, a collection of optimum plans is derived. These plans minimize the large-sample approximate variance of the maximum likelihood (ML) estimator of a specified quantile of the failure-time distribution. The general equivalence theorem (GET) is used to verify the optimality of these plans. Because an optimum plan is not robust to the model specification and the planning information used in deriving the plan, a more robust and useful compromise plan is proposed. Sensitivity analyses show the effects that changes in sample size, time duration of the experiment, levels of the accelerating variable, and misspecification of the planning information have on the precision of the ML estimator of a failure-time quantile. Monte Carlo
simulations are used to evaluate the statistical characteristics of the ADDT plans. The methods are illustrated with an application for an adhesive bond.

**Key Words**: Compromise accelerated destructive degradation test plan; General equivalence theorem; Large-sample approximate variance; Monte Carlo simulation; Optimum accelerated destructive degradation test plan; Reliability.

## 2.1 Introduction

### 2.1.1 Motivation

Manufacturers often conduct up-front reliability tests on materials and components when their products are being designed. Because degradation data provide more information on reliability than traditional failure-time data (where time to failure is the response), especially in applications where few or no failures are expected, degradation tests are used in manufacturing industries to obtain the reliability information of product components and materials. For most applications, however, degradation rates at normal use conditions are so low that appreciable degradation will not be observed in a test of practical time length. For this reason, degradation tests are often accelerated to get reliability information more quickly. Generally, information from tests at high levels of accelerating variables is extrapolated to obtain estimates of lifetime or degradation rates at lower, normal use conditions based on a physically reasonable statistical model.

### 2.1.2 Accelerated Destructive Degradation Test

For some applications, the degradation measurement process destroys or changes the physical/mechanical characteristics of test units so that only one meaningful measurement can be taken on each unit. An accelerated degradation test with such degradation data is called an “accelerated destructive degradation test” or ADDT.

Escobar, Meeker, Kugler, and Kramer (2003) described an application of an accelerated destructive degradation test to evaluate an adhesive bond (Adhesive Bond B). The response was the strength (in Newtons) of the adhesive bond over time. The measurement process was destructive because the strength of a test unit could only be measured once. Additionally, there was special interest in estimating the time at which 1% of the devices would have a strength below 40 Newtons when operating at room temperature of 25 °C (i.e., the 0.01 quantile of the failure-time distribution). To obtain information about the 0.01 quantile of the failure-time distribution, an accelerated destructive degradation test was
used. As a baseline, 8 units with no aging were measured at the start of the experiment. A total of 80 additional units were aged and measured according to the temperature and time schedule presented in Table 2.1.

<table>
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<tr>
<td>Totals</td>
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</tbody>
</table>

### 2.1.3 Related Literature

Nelson (1981) and Nelson (1990, chapter 11) introduced basic models and methods for analyzing ADDT data. Escobar, Meeker, Kugler, and Kramer (2003) provided an application for accelerated destructive degradation data and introduced a more general class of models. There is a large amount of literature on planning accelerated tests. This work has been summarized by Nelson (2005a, 2005b). Some work that is particularly relevant to this paper is included in the following references. Nelson (1990, Chapter 6) described methods for planning accelerated life tests (ALTs) based on a simple model. Meeker and Escobar (1998, chapter 20) provided details and examples on how to plan a single-variable ALT. Escobar and Meeker (1995) described methods for planning ALT’s with two or more variables. There are some important differences between accelerated life tests and accelerated degradation tests. The most important difference is that ALTs almost always result in censored data. Censoring is not as common in ADDTs. Boulanger and Escobar (1994) proposed methods for planning repeated measures accelerated degradation tests. In this paper, we use the application in Escobar, Meeker, Kugler, and Kramer (2003) and describe methods for planning accelerated destructive degradation tests.

### 2.1.4 Overview

The remainder of this paper is organized as follows. Section 2.2 presents a class of models for ADDT data and gives formulas for the degradation distribution. Section 2.3 gives formulas for the failure-time distribution induced by the degradation models. Section 2.4 outlines the framework for accelerated
destructive degradation test planning. Section 2.5 gives optimum ADDT plans and applies the general equivalence theorem (GET) to verify the optimality of test plans. Section 2.6 describes alternative ADDT plans and compares the results of different test plans. Section 2.7 illustrates the effects of changing constraints and investigates sensitivity to misspecification of the planning information. Section 2.8 uses Monte Carlo simulation to evaluate different test plans. Section 2.9 contains some concluding remarks and extensions for future research work. Appendix 2.A provides derivations and technical details about large-sample approximations that are used to evaluate ADDT plans. Appendix 2.B verifies that the ADDT planning problem satisfies the necessary conditions for using the GET.

2.2 Degradation Models

2.2.1 Accelerated Degradation Models

The degradation level for a typical observational unit at time \( t \) and accelerating variable AccVar (e.g., temperature, humidity) is denoted by \( D(\tau, x, \beta) \), where \( \tau = h_t(t) \) and \( x = h_a(\text{AccVar}) \) are known monotone increasing transformations of \( t \) and AccVar respectively, and \( \beta \) is a vector of unknown parameters.

For the class of degradation models used here, transformed degradation \( Y \) for a unit at transformed time \( \tau \) and transformed accelerating variable level \( x \) is

\[
Y = \mu(\tau, x) + \epsilon \\
= \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \epsilon
\]  

(2.1)

where \( \mu(\tau, x) = h_d(D) \), a monotone increasing transformation of \( D \), is a location parameter for the distribution of \( Y \) that depends on the unknown parameters in \( \beta = (\beta_0, \beta_1, \beta_2) \). \( \epsilon \) is a residual deviation that describes unit-to-unit variability with \( (\epsilon/\sigma) \sim \Phi(z) \), where \( \Phi(z) \) is a completely specified cdf. For example, \( \Phi(z) \) can be replaced by \( \Phi_{\text{nor}}(z) \), the standardized normal cdf, or \( \Phi_{\text{sev}}(z) \), the standardized smallest extreme value cdf. The model parameters \( \beta \) and \( \sigma \) are fixed but unknown.

Model (4.1) is linear in the sense that for specified \( x \), the mean transformed degradation path \( \mu(\tau, x) \) is linear in \( \tau \). For purposes of estimation, however, the model in (1) is nonlinear in the parameters. \( \beta_0 \) is the location parameter of the transformed degradation when \( \tau = 0 \). The degradation rate of \( \mu(\tau, x) \) with respect to \( \tau \) at \( x \) is \( \omega(x) = \beta_1 \exp(\beta_2 x) \). The sign of \( \beta_1 \) determines whether the degradation is increasing or decreasing over time. For example if the degradation response is size of a crack or the concentration of a harmful material, \( \beta_1 \) would be expected to be positive. On the other hand, if the degradation response is light output of a LED or the strength of an adhesive bond, \( \beta_1 \) would be negative.
For the Adhesive Bond B application, the strength degradation model used in Escobar, Meeker, Kugler, and Kramer (2003) is as given in (1) with

\[
\begin{align*}
Y &= h_d(\text{Strength}) = \log(\text{Strength in Newtons}) \\
\tau &= h_t(\text{Time}) = \sqrt{\text{Time in Weeks}} \\
x &= h_a(\text{Temperature}) = -\frac{11605}{\text{Temperature in } ^\circ C + 273.15} \\
(\epsilon/\sigma) &\sim \Phi_{\text{nor}}(z).
\end{align*}
\]

The accelerating variable for this application is temperature. The denominator in \(x\) is temperature on the kelvin (K) scale and the numerator is the reciprocal of Boltzmann’s constant in units of electronvolt per kelvin (eV/K). For this parametrization, \(\beta_2\) has the interpretation of an effective activation energy.

### 2.2.2 Degradation CDF

For given time and accelerating variable level, the CDF for the transformed degradation \(Y\) is

\[
F_Y(y; \tau, x) = \Pr(Y \leq y; \tau, x) = \Phi \left[ \frac{y - \mu(\tau, x)}{\sigma} \right]
\]

where \(\mu(\tau, x) = \beta_0 + \beta_1 \exp(\beta_2 x)\tau\).

For the Adhesive Bond B example, the CDF of \(F_Y(y; \tau, x)\) at a fixed factor level combination of time and temperature can be obtained by replacing \(\Phi\) with \(\Phi_{\text{nor}}\). Figure 2.1 shows the degradation distributions at 25 \(^\circ\)C and different values of time for particular values of the parameters \(\beta_0, \beta_1, \beta_2, \sigma\) corresponding to the maximum likelihood (ML) estimates given in Escobar, Meeker, Kugler, and Kramer (2003).

### 2.2.3 Degradation Quantiles

The \(p\) quantile function for the transformed degradation at \((\tau, x)\) is

\[
\begin{align*}
y_p &= \mu(\tau, x) + \sigma \Phi^{-1}(p) \\
&= \beta_0 + \beta_1 \exp(\beta_2 x)\tau + \sigma \Phi^{-1}(p)
\end{align*}
\]

where \(\Phi^{-1}(p)\) is the \(p\) quantile of the standard location-scale distribution.

Substituting \(\Phi_{\text{nor}}^{-1}(p)\) for \(\Phi^{-1}(p)\), one obtains the \(p\) quantile of the transformed degradation (log Newtons) for the Adhesive Bond B example, such as the 0.01 and 0.001 quantiles shown in Figure 2.1.
2.3 Failure-Time Distribution for Degradation Models

2.3.1 Relationship Between Degradation and Failure

For some products, there is a gradual loss of performance with increasing time (e.g., decreasing strength of an adhesive bond). Then failure would be defined at a specified degradation level. This failure-definition is known as a “soft failure” (see Chapter 13 of Meeker and Escobar 1998). We use \( D_t \) to denote the critical level for the degradation distribution at which failure is assumed to occur. The failure-time \( T \) is defined as the time when the observed degradation crosses the critical level \( D_t \).

2.3.2 Failure-Time Cumulative Distribution Function

As mentioned in Section 2.2.1, degradation can be decreasing or increasing over time, depending on the sign of \( \beta_1 \). For decreasing degradation (i.e., when \( \beta_1 \) is negative), failure-time \( T \) being less than \( t \) is equivalent to an observed degradation being less than the critical level \( D_t \) at time \( t \) (i.e., the event \( T \leq t \) is equivalent to the event \( Y \leq y_f \), where \( y_f = h_d(D_t) \)), as illustrated in Figure 2.1. Then the
failure-time CDF is

\[ F_T(t; x) = \Pr(T \leq t) = \Pr(Y \leq y_f) = F_Y(y_f; \tau, x) \]
\[ = \Phi \left( \frac{y_f - \mu(\tau, x)}{\sigma} \right) = \Phi \left( \frac{\tau - \nu}{\varsigma} \right), \quad \text{for } t \geq 0 \quad (2.2) \]

where

\[ \nu = \frac{(\beta_0 - y_f) \exp(-\beta_2 x)}{|\beta_1 |} \quad \text{and} \quad \varsigma = \frac{\sigma \exp(-\beta_2 x)}{|\beta_1 |}. \]

With a time transformation \( h_t(t) \) for which \( \tau = 0 \) when \( t = 0 \), the failure-time distribution for decreasing degradation is a mixture with a spike \( \Pr(T = 0) = \Phi \left( (y_f - \beta_0)/\sigma \right) = \Phi (\nu/\varsigma) \) at \( t = 0 \). This spike represents the probability of failure for a new unit that experiences no aging and is sometimes called the dead-on-arrival (or DOA) probability. For \( t > 0 \) the cdf of failure-time in (4.2) is continuous and it agrees with the cdf of a log-location-scale variable with standardized cdf \( \Phi(\cdot) \), location parameter \( \nu \) and scale parameter \( \varsigma \).

For increasing degradation (i.e., \( \beta_1 \) is positive), failure-time \( T \) being less than \( t \) is equivalent to an observed degradation being greater than the critical level \( D_f \) at time \( t \) (i.e., the event \( T \leq t \) is equivalent to the event \( Y \geq y_f \), where \( y_f = h_d(D_f) \)). Then

\[ F_T(t; x) = \Pr(T \leq t) = \Pr(Y \geq y_f) = 1 - F_Y(y_f; \tau, x) \]
\[ = 1 - \Phi \left( \frac{y_f - \mu(\tau, x)}{\sigma} \right) = 1 - \Phi \left( \frac{-\tau - \nu}{\varsigma} \right), \quad \text{for } t \geq 0. \quad (2.3) \]

In this case the spike at \( t = 0 \) is \( \Pr(T = 0) = 1 - \Phi \left( (y_f - \beta_0)/\sigma \right) = 1 - \Phi (\nu/\varsigma) \).

### 2.3.3 Failure-Time Quantiles

From (4.2), the \( p \) quantile of the failure-time for decreasing degradation is

\[ t_p = \begin{cases} 
    h_t^{-1} \left[ \nu + \varsigma \Phi^{-1}(p) \right] & \text{if } p \geq \Phi (\nu/\varsigma) \\
    0 & \text{otherwise.}
\end{cases} \]

From (2.3), the \( p \) quantile of the failure-time for increasing degradation is

\[ t_p = \begin{cases} 
    h_t^{-1} \left\{ - \left[ \nu + \varsigma \Phi^{-1}(1-p) \right] \right\} & \text{if } p \geq 1 - \Phi (\nu/\varsigma) \\
    0 & \text{otherwise.}
\end{cases} \]

In both cases, \( \nu \) and \( \varsigma \) are as defined in Section 2.3.2.
2.4 Accelerated Destructive Degradation Test Planning

2.4.1 Planning Information

ADDT planning requires information that includes planning values for the model parameters, a plausible distribution for the model variability, a specification of the critical degradation level, the range of the accelerating variable available for the experiment. There will also be constraints on the maximum test time and the number of units available for testing.

For the Adhesive Bond B example, the degradation model is as described in Section 2.2.1. The critical degradation level is specified as $D_f = 40$ Newtons. Some constraints for this application are:

- 88 test units.
- $70 \, ^\circ C$ is the maximum temperature that can be used (higher temperatures would cause the model to break down).
- 16 weeks are available for testing.

The goal is to develop a test plan to evaluate a new adhesive bond material similar to the material used in the example described in Escobar, Meeker, Kugler, and Kramer (2003). Test plan properties will depend on the unknown parameters $\theta = (\beta_0, \beta_1, \beta_2, \sigma)'$. The planning values of the parameters are $\beta_0 = 4.471$, $\beta_1 = -86,406,4160$, $\beta_2 = 0.6364$, and $\sigma^2 = 0.1580$. These values derive from the data analysis in Escobar, Meeker, Kugler, and Kramer (2003). The planning information defines the mean transformed degradation paths $\mu(\tau, x)$ at all levels of temperature and the degradation distribution at a given factor level combination of time and temperature, as depicted in Figure 2.2. Note that the strength axis is a logarithmic axis and that the time axis is a square root axis so that the mean transformed degradation paths are linear with respect to the transformed time.

An alternative specification of the planning values is to give the degradation rate (slope of the line), $\omega^2$, of $\mu(\tau, x)$ for a given temperature, say $\omega^2 = -0.1026$ at $50 \, ^\circ C$ instead of $\beta_1^2$. This method has the advantage that each of the model parameters has a clear practical interpretation, making it easier to elicit from experts when needed. With this specification at $50 \, ^\circ C$, one gets

$$\beta_1^2 = \omega^2 \exp(-\beta_2 x) = -0.1026 \exp(0.6364 \times 35.912) = -86,434,2322$$

where $x = -11605/(50 + 273.15) = -35.912$. The difference in the values for $\beta_1^2$ obtained from the two methods is due to rounding in the specifications of $\omega^2$ and $\beta_2^2$. 
2.4.2 ADDT Plan Specification

Denote a factor level combination of transformed time $\tau$ and transformed accelerating variable $x$ as $v = (\tau, x)$. An ADDT plan will specify a set of factor level combinations $v_i$ and the corresponding proportional allocation $\pi_i$ of test units at $v_i$. A test plan with $r$ factor level combinations is denoted as

$$\xi = \begin{bmatrix} v_1, & \pi_1 \\ v_2, & \pi_2 \\ \vdots & \vdots \\ v_r, & \pi_r \end{bmatrix}$$

where $\pi_i > 0$ and $\sum_{i=1}^{r} \pi_i = 1$.

2.4.3 Criterion for Choosing a Plan

The appropriate criterion for planning an ADDT depends on the purpose of the experiment. For accelerated tests, a common objective is to estimate a particular quantile of the failure-time distribution at use conditions, say, $t_p$. For this reason, a commonly used criterion for planning accelerated tests is to minimize $\text{Avar}(\hat{t}_p)$, the large-sample approximate variance of the maximum likelihood (ML) estimator of the specified failure-time quantile. We use this criterion in our work. Because $h_t(t_p)$ is a monotone function of $t_p$, minimizing $\text{Avar}[h_t(\hat{t}_p)]$ gives the same test plan as minimizing $\text{Avar}(\hat{t}_p)$. 

Figure 2.2 Model for degradation evaluated at four different temperatures. The lines for each temperature indicate mean transformed degradation paths $\mu(\tau, x)$ as a function of time.
As explained in Appendix 2.A.3, the optimization criterion is equivalent to finding the test plan $\mathbf{\xi}$ that maximizes the objective function

$$
\Psi(\mathbf{\xi}) = -c'[\mathbf{I}(\mathbf{\xi})]^{-1}c
$$

where $c = \partial t_p(t_p)/\partial \mathbf{\theta}$, $\mathbf{\theta} = (\beta_0, \beta_1, \beta_2, \sigma)'$, and $\mathbf{I}(\mathbf{\xi})$ is the scaled information matrix of the model parameters. This criterion is closely related to $c$ optimality (for example, see Pukelsheim 1993). Details are given in Appendix 2.A.

### 2.5 Optimum Accelerated Destructive Degradation Test Plan

#### 2.5.1 Optimum Plan Structure

The degradation model described in (1) has three regression parameters. This suggests that a non-degenerate optimum ADDT plan for an application with this degradation model should be a 3-point plan (i.e., the test plan should have three factor level combinations). For most practical situations in which accelerated tests are used, an optimum plan will allocate test units on the boundaries of the experimental region. To minimize the large-sample approximate variance for the ML estimator of a specified failure-time quantile, an optimum ADDT plan should spread the three factor level combinations as much as possible, providing better estimates of the regression coefficients than closely-spaced test conditions. Figure 2.3 presents an ADDT optimum plan structure in terms of the transformed experimental variables $\tau$ and $x$. Under the practical constraints of a maximum transformed time $\tau_M$ and a maximum transformed accelerating variable level $x_M$, one particular optimum plan will have some test units allocated at $\mathbf{v}_1^*$ (baseline test condition) with $\tau = 0$, some at the $\mathbf{v}_2^*$ test condition with $\tau_M$ and $x_M$, and some at the $\mathbf{v}_3^*$ test condition with $\tau_M$ and an optimized value $x^*$. The $x^*$ for the $\mathbf{v}_3^*$ test condition and the proportional allocations $\pi_1^*$, $\pi_2^*$ of test units are chosen to optimize the plan. Note that $\pi_3^* = 1 - \pi_1^* - \pi_2^*$ and the degradation model at $\tau = 0$ (and thus test plan properties) does not depend on the level of $x$. Using notation similar to that used in Section 2.4.2, this particular optimum test plan is denoted by

$$
\mathbf{\xi}^* = \begin{bmatrix}
\mathbf{v}_1^*, & \pi_1^* \\
\mathbf{v}_2^*, & \pi_2^* \\
\mathbf{v}_3^*, & \pi_3^*
\end{bmatrix} = \begin{bmatrix}
(0, \bullet), & \pi_1^* \\
(\tau_M, x_M), & \pi_2^* \\
(\tau_M, x^*), & \pi_3^*
\end{bmatrix}
$$

where $\bullet$ indicates that at $\tau = 0$, the level of $x$ is arbitrary. In the next section a variation of Whittle’s (1973) general equivalence theorem (GET) is used to verify the optimality of this test plan.
2.5.2 Checking the Initial Optimum Plan

Here we check the optimality of the test plan $\xi^*$ using the GET. The directional derivative, $\Lambda$, of $\Psi$ at $\xi$ and in the direction of an alternative plan $\eta$ is defined as

$$\Lambda(\xi, \eta) = \lim_{\delta \to 0^+} \frac{\Psi[(1-\delta)\xi + \delta\eta] - \Psi(\xi)}{\delta}$$

As shown in Appendix 2.B.1, $\Lambda(\xi, \eta) = c'[\mathcal{I}(\xi)]^{-1}\mathcal{I}(\eta)[\mathcal{I}(\xi)]^{-1}c - c'[\mathcal{I}(\xi)]^{-1}c$, where $c$, $\mathcal{I}(\xi)$, and $\mathcal{I}(\eta)$ are evaluated at the planning values. Let $\xi_v$ be a singular test plan that puts all units at the $v$ test condition. From the results in Appendix 2.B.1, the plan $\xi^*$ is an optimum plan if it satisfies $\Lambda(\xi^*, \xi_v^1) = \Lambda(\xi^*, \xi_v^2) = \Lambda(\xi^*, \xi_v^3) = 0$ and $\Lambda(\xi^*, \xi_v) \leq 0$ for any singular plan $\xi_v$ in the experimental region.

For the Adhesive Bond B application, a particular optimum plan (obtained numerically) is

$$\xi^* = \begin{bmatrix} (0, \bullet), & \pi_1^\ast \\ (\tau_M, x_M), & \pi_2^\ast \\ (\tau_M, x^*), & \pi_3^\ast \end{bmatrix} = \begin{bmatrix} (0, \bullet), & 0.20374 \\ (4, -33.819), & 0.16160 \\ (4, -35.390), & 0.63466 \end{bmatrix}$$ (2.6)

where $-33.819$ and $-35.390$ are the transformed temperatures corresponding to the maximum $70$ °C and optimized $54.764$ °C, respectively. In terms of the the original variables (Weeks and °C), this plan is shown in Table 2.2.

Figure 2.4 shows the directional derivatives $\Lambda(\xi^*, \xi_v)$ of this optimum plan as a function of temperature and time, where $\xi_v$ is a plan that puts all units at the $v$ test condition. Observe that, as
Table 2.2  Optimum ADDT Plan. The • indicates that at time 0, the level of temperature is arbitrary.

<table>
<thead>
<tr>
<th>Optimum Test Condition</th>
<th>Weeks</th>
<th>Temperature °C</th>
<th>Proportional Allocations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>•</td>
<td>0.20374</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.16160</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>54.764</td>
<td>0.63466</td>
</tr>
</tbody>
</table>

required, the directional derivatives are zero at the three test conditions of the optimum plan. Also, the directional derivatives are zero at all the test conditions with temperature equal to 70 °C. This suggests the existence of alternative optimum plans.

![Figure 2.4](image)

Figure 2.4  Directional derivatives $\Lambda(\xi^*, \xi_v)$ of the optimum plan as a function of temperature and time.

2.5.3 Alternative Optimum Plans

From Figure 2.4 and (2.20) in Appendix 2.5.3, $\Lambda(\xi^*, \xi_v) = 0$ where $\xi_v$ is a test plan putting all units at $v_2 = (\tau, x_M)$, for all $0 \leq \tau \leq \tau_M$. This result suggests that we can move the $v_3^*$ test condition to the left along the horizontal line with $x = x_M$, as shown in Figure 2.3. Using the GET, it can be shown that, for fixed $\tau_0$, $\tau_L \leq \tau_0 \leq \tau_M$, and $\tau_L = \tau_M \pi_2^*/(\pi_1^* + \pi_2^*)$, an alternative optimum plan can be
expressed as

$$\xi^a = \begin{bmatrix} v_1 = (0, \bullet), & \pi_1 = \pi_1^* + \pi_2^* - \frac{\pi_2^{TM}}{\tau_a} \\ v_2 = (\tau_a, x_M), & \pi_2 = \frac{\pi_2^{TM}}{\tau_a} \\ v_3 = (\tau_M, x^*), & \pi_3 = \pi_3^* \end{bmatrix}. \quad (2.7)$$

See Appendix 2.5.3 for the details.

For the Adhesive Bond B application, $\tau_L = 4\pi_2^*/(\pi_1^* + \pi_2^*) = 1.77$. The lower boundary of time for the $v_2$ test condition in Weeks is $t_L = \tau_L^2 = 3.13$. A particular alternative optimum plan can be obtained by substituting the values $\tau_M = 4$, $x_M = -33.819$, $x^* = -35.390$, $\pi_1^* = 0.20374$, $\pi_2^* = 0.16160$, and $\pi_3^* = 0.63466$ from (2.6) into the expression (2.7) and choosing a value of $\tau_a$, $1.77 \leq \tau_a \leq 4$. Figure 2.5 describes how the large-sample approximate standard error of $\hat{t}_{0.01}$ and the proportional allocations of test units change for different optimum plans as the time component of the $v_2$ test condition varies in the experimental time range. Figure 2.5 illustrates the characteristics of the multiple optimum plans, the changing trend of proportional allocations of test units for different optimum plans, and the lower time bound $t_L$. Note that the directional derivatives in Figure 2.4 and the large-sample approximate standard errors of $\hat{t}_{0.01}$ in Figure 2.5 were computed for the continuous test plan with a sample size of 88 (a continuous test plan is one that has non-integer allocations because optimization was done without integer constraints on the number of units allocated to the test conditions).

Figure 2.5 and the results in (2.7) show that the optimized values of $x$ and $\pi_3$ for the $v_3$ test condition are the same for all optimum plans. However, as the value of $\tau_a$ for the $v_2$ test condition increases, the $\pi_1$ increases and $\pi_2$ decreases, as shown in Figure 2.5. These results are not surprising because as the value of transformed time $\tau_a$ for the $v_2$ test condition approaches $\tau_L$, $v_2$ provides information that is similar to the baseline $v_1$ test condition. As the value of $\tau_a$ for $v_2$ increases, the $v_2$ test condition is further away from $v_1$ so that more units are allocated to $v_1$ to get more information about the degradation distribution. Also, when the value of transformed temperature $x$ for the $v_3$ test condition is too small, the information from $v_3$ will be similar to that from $v_1$ (i.e., both $v_1$ and $v_3$ behave like a unit tested at low temperature). When the value of $x$ for $v_3$ is too close to $x_M$, there will not be good information separating the effect that time and the accelerating variable have on the degradation rates. It is interesting that the optimized values of $x$ for the $v_3$ test condition for different optimum plans are the same. Also, as $\tau_a$ approaches $\tau_L$ from above, the limiting plan is degenerate and will not allow estimation of all of the model parameters, even though it will allow estimation of the lifetime distribution at the use conditions. Of course, such “degenerate” test plans have little practical value.
2.6 Other Accelerated Destructive Degradation Test Plans

2.6.1 Traditional Plans

A traditional plan is one that uses equally-spaced levels of the experimental conditions and equal allocations to all factor level combinations. In applications involving extrapolation, like accelerated testing, such traditional plans may not be statistically efficient, which results in less precise estimates.

2.6.2 Motivation for Compromise Plans

An optimum plan provides the smallest large-sample approximate variance of the maximum likelihood estimator of a specified failure-time quantile. Optimum plans, however, have practical deficiencies (e.g., only a small number of factor level combinations) and provide no information to check the adequacy of the model. Generally, optimum plans tend to be highly sensitive to model specification errors and thus are suitable only if the degradation model is correct. Also, planning values that are appreciably in error may result in test plans that are far from optimum. It is highly desirable for a test plan to be robust (i.e., the plan should give useful results even if the assumed model and planning values are inaccurate). A traditional plan with more factor level combinations tends to be robust, but it is less precise than an optimum plan. In general, a compromise plan will be more useful in practical
applications. A compromise plan combines the advantages of optimum and traditional test plans.

2.6.3  Traditional and Optimized Compromise Plans for the Application

A traditional test plan for the Adhesive Bond B application is presented in Table 2.3. This plan has some baseline units at the beginning of the experiment and 12 other combinations using equally spaced levels of time and temperature, each with the same number of test units.

Table 2.3  Traditional ADDT Plan.

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Weeks</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>—</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>60</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>70</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Totals</td>
<td>4</td>
<td>21</td>
</tr>
</tbody>
</table>

To find a compromise between the optimum and the traditional test plans, the number of factor level combinations for a compromise plan should be greater than the optimum plan but less than the traditional plan. As suggested by the traditional plan, a compromise plan for the Adhesive Bond B allocates some test units at the beginning of the experiment and some units at each of nine equally spaced factor level combinations. The nine combinations have three equally spaced time levels and three equally spaced temperature levels. We can not optimize the times for the compromise plan because the optimization would degenerate to a plan with all units (other than the baseline units) allocated to the temperatures at the longest test time. Also, we can not optimize allocations because the optimum allocations would degenerate to a 3-point optimum plan. Therefore, the compromise plan uses three fixed time levels at 12, 14, and 16 weeks respectively and the highest temperature level at 70 °C, as in the traditional plan. The lowest temperature of the compromise plan is chosen to minimize the large-sample approximate variance of the estimated 0.01 failure-time quantile, which is similar to the optimum plan. The middle temperature is the mean of the other two temperature levels. After rounding in allocating the 88 test units, the compromise plan has 9 units at each of the nine equally spaced factor level combinations and 7 test units at the baseline. The optimum lowest temperature for the compromise ADDT plan is 54 °C and the middle temperature is 62 °C, as presented in Table 2.4.
Table 2.4  Compromise ADDT Plan.

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Weeks</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>54</td>
<td>9</td>
<td>9 9 27</td>
</tr>
<tr>
<td>62</td>
<td>9</td>
<td>9 9 27</td>
</tr>
<tr>
<td>70</td>
<td>9</td>
<td>9 9 27</td>
</tr>
</tbody>
</table>

2.6.4  Comparison of Accelerated Destructive Degradation Test Plans

As explained in Section 2.4.3, the purpose of the test is to estimate \( t_p \), the \( p \) quantile of the failure-time distribution. Denote the ML estimate of \( t_p \) by \( \hat{t}_p \). An approximate 100(1 − \( \alpha \))% confidence interval for \( \log(t_p) \) is

\[
\log(\hat{t}_p) \pm z_{(1-\alpha/2)} \sqrt{\text{Var}[\log(\hat{t}_p)]} = \log(\hat{t}_p) \pm \log(\hat{R}).
\]

Exponentiation yields an approximate confidence interval for \( t_p \)

\[
[\hat{t}_p/\hat{R}, \hat{t}_p\hat{R}]
\]

where

\[
\hat{R} = \exp \left[ z_{(1-\alpha/2)} \sqrt{\text{Avar}[\log(\hat{t}_p)]} \right].
\]  (2.8)

The estimated variance \( \text{Var}[\log(\hat{t}_p)] \) can be obtained from the local information matrix in the usual way (for example, see Appendix B.3 of Meeker and Escobar 1998). We call \( \hat{R} \) the “observed precision factor.” To estimate \( t_p \) precisely, the confidence interval for \( t_p \) should be as narrow as possible.

For test planning purposes, \( \text{Var}[\log(\hat{t}_p)] \) in (4.4) is replaced with \( \text{Avar}[\log(\hat{t}_p)] \), the large-sample approximation of \( \text{Var}[\log(\hat{t}_p)] \), in the evaluations. This gives the precision factor

\[
\hat{R} = \exp \left[ z_{(1-\alpha/2)} \sqrt{\text{Avar}[\log(\hat{t}_p)]} \right]
\]

which can be used in test planning because it is a function of the model and its parameters (planning values) and does not depend on the data. Because \( R \) is an increasing function of \( \text{Avar}[\log(\hat{t}_p)] \), minimizing the \( R \) precision factor is equivalent to minimizing \( \text{Avar}[\log(\hat{t}_p)] \) and approximately equivalent to minimizing \( \text{Var}[\log(\hat{t}_p)] \). \( R \) is easier to interpret as a measure of precision for a positive parameter when compared with \( \text{Avar}[\log(\hat{t}_p)] \), so we will use it for the comparisons among different ADDT
plans. The upper (lower) endpoint of the confidence interval for \( t_p \) is approximately \( 100(R - 1)\% \) larger (smaller) than the ML estimate \( \hat{t}_p \).

For the Adhesive Bond B application, Table 2.5 compares four different ADDT plans: the optimum plan (in Table 2.2), the compromise plan (in Table 2.4), the original plan (in Table 2.1), and the traditional plan (in Table 2.3) in terms of the \( R \) precision factors for estimating the 0.01 failure-time quantile at normal use conditions of 25 °C.

<table>
<thead>
<tr>
<th>ADDT Plan</th>
<th>Number of Factor Levels</th>
<th>Lowest Temperature °C at the Maximum Time</th>
<th>Precision Factor ( R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimum</td>
<td>3</td>
<td>54.764</td>
<td>1.910</td>
</tr>
<tr>
<td>Compromise</td>
<td>10</td>
<td>54</td>
<td>2.208</td>
</tr>
<tr>
<td>Original</td>
<td>16</td>
<td>50</td>
<td>2.465</td>
</tr>
<tr>
<td>Traditional</td>
<td>13</td>
<td>50</td>
<td>2.512</td>
</tr>
</tbody>
</table>

The optimum ADDT plan has the smallest \( R \) precision factor and provides the most precise estimate of \( t_{0.01} \). The original test plan has a smaller \( R \) precision factor than the traditional plan. This is because the original plan has more factor level combinations spaced over the whole experimental region, which can give more information about the failure-time distribution. As expected, the \( R \) precision factor for the compromise plan is smaller than that for the traditional plan but larger than that for the optimum plan. We would recommend the compromise plan in Table 2.5 or a similar compromise plan.

2.7 Effect of Test Plan Changes

The precision of estimating a specified failure-time quantile depends on the constraints of the application, such as sample size, maximum accelerating variable level (temperature in the Adhesive Bond B application), maximum test time, etc. In this section, we evaluate the effects that changes in the sample size and the factor-level (time and temperature) constraints have on the \( R \) precision factors (for estimating the 0.01 failure-time quantile at the 25 °C use conditions) for the Adhesive Bond B experiment. We also conduct a sensitivity analysis to study the effect of misspecification of the planning information. For some of these analyses, we omit details for the original and traditional test plans when they would require extra space.
2.7.1 Effect of Sample Size Changes

As noted in Section 2.6.4,

\[ R = \exp \left[ z_{1-\alpha/2} \sqrt{\text{Avar} \left[ \log(\hat{t}_p) \right]} \right]. \]

The variance factor, defined as \( n \text{Avar} \left[ \log(\hat{t}_p) \right] \), depends on the actual values of the parameters but does not depend on the sample size \( n \). Thus the \( R \) precision factor for any sample size can be predicted from large-sample approximation theory once we know one such factor. Relative to the precision factor for a sample size of 88, the precision factor as a function of \( n \) can be written as

\[ R_n = \exp \left[ \sqrt{\frac{88}{n} \times \log(R_{88})} \right] = R_{88}^{88/n}, \]

where \( R_{88} \) is the precision factor with \( n = 88 \). Figure 2.6 shows the \( R \) precision factors as a function of \( n \) for the four ADDT plans.

![Figure 2.6](image)

Figure 2.6 The \( R \) precision factors for estimating \( t_{0.01} \) as a function of \( n \) for the four ADDT plans.

2.7.2 Effect of Maximum Temperature and Time Changes

Table 2.6 presents the \( R \) precision factors that would be obtained from the optimum and compromise plans if we were to change the maximum temperature and maximum test time. Note that in the actual application, 80 °C is thought to be too high, but we want to show the potential improvement in precision
if temperature could be increased. The results for the initial optimum and compromise plans are marked in bold. The compromise plans have a fixed lower time level at 12 weeks, a changing maximum time, and a third time level which is the halfway between 12 weeks and the maximum time.

Table 2.6 The $R$ precision factors for estimating $t_{0.01}$ for optimum and compromise plans with changing maximum temperature and range of time.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>3.004</td>
<td>2.780</td>
<td>2.610</td>
<td>2.475</td>
<td>4.266</td>
<td>3.957</td>
<td>3.690</td>
<td>3.463</td>
</tr>
<tr>
<td>70</td>
<td>2.003</td>
<td>1.910</td>
<td>1.837</td>
<td>1.778</td>
<td>2.302</td>
<td>2.208</td>
<td>2.127</td>
<td>2.057</td>
</tr>
<tr>
<td>80</td>
<td>1.559</td>
<td>1.519</td>
<td>1.487</td>
<td>1.466</td>
<td>1.642</td>
<td>1.612</td>
<td>1.588</td>
<td>1.568</td>
</tr>
</tbody>
</table>

As the maximum temperature or time increases, the observations are spread out so that the variance for the estimated failure-time quantile becomes smaller, which results in a decrease in the $R$ precision factor for both optimum and compromise plans. And, the optimum plan always has a smaller precision factor than the compromise plan provided the same factor-level constraints.

2.7.3 Sensitivity to Misspecification of the Planning Information

Our method of assessing sensitivity to misspecification of the planning information follows the general approach used in Meeker (1984). From (4.2), the probability that an observational unit at the test condition $(\tau, x)$ will fail is

$$\Phi \left[ \frac{y \tau - \mu(\tau, x)}{\sigma} \right] = \Phi \left[ \frac{h_d(D_\tau) - \beta_0 - \beta_1 \exp(\beta_2 x) \tau}{\sigma} \right]$$

$$= \Phi \left[ \frac{h_d(D_\tau) - \beta_0 - \omega_{50} \exp \left[ \beta_2 (x - x_{50}) \right] \tau}{\sigma} \right]$$

where, $x_{50}$ is the transformed temperature of 50 °C, and $\omega_{50}$ is the degradation slope at 50 °C, as described in Section 2.4.1. Instead of being a function of the four parameters $\beta_0, \beta_1, \beta_2$ and $\sigma$, the probability in (2.9) only depends on the three standardized parameters $(h_d(D_\tau) - \beta_0)/\sigma, \omega_{50}/\sigma, \beta_2$ and the two explanatory variables $\tau$ and $x$. This implies that $n/\sigma^2$ times the large-sample approximate variance of the estimated failure-time quantile can be expressed in terms of these three standardized parameters. Using the standardized parameters makes it possible to do a general evaluations of the robustness of ADDT plans to misspecifications of the planning values by perturbing values in only three dimensions.
The idea of sensitivity analysis is to explore the effect of planning value misspecification across some plausible region of the parameter space. We chose nine sets of standardized parameter values in the parameter space (as shown in Tables 2.7 and 2.8) to evaluate the sensitivity of the test plans to misspecification of the planning information. The first set corresponds to the planning values given in Section 2.4.3 which were used for getting the optimum plan (2.6) for the application of interest. The other eight sets were chosen as the vertices of a cube in the parameter space that would not lead into nonsense degenerate optimized test plans.

For each set of actual parameter values, the optimum plan for estimating the 0.01 failure-time quantile was obtained. For comparison, another eight “optimum” plans were found for misspecification of ± 0.5 in \( h_d(D) - \beta_0 \)/\( \sigma \) and ± 0.2 in both \( \omega_{50}/\sigma \) and \( \beta_2 \). Table 2.7 shows the worst and best ratios of the precision factors \( R \) for the “optimum” plans obtained with the misspecified values to the precision factor for the correct optimum plan. Similar comparisons for compromise plans are given in Table 2.8. Comparisons of the ratios in the two tables show that the compromise plans, when compared to the optimum plans, are much more robust to misspecified planning values.

<table>
<thead>
<tr>
<th>Actual Values</th>
<th>Deviation Yielding Worst Ratio</th>
<th>Deviation Yielding Best Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_d(D) - \beta_0 )</td>
<td>( \omega_{50}/\sigma )</td>
<td>( \beta_2 )</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6</td>
<td>1.20</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>1.31</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6</td>
<td>1.19</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>1.30</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>1.19</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>1.23</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>1.14</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5</td>
<td>1.20</td>
</tr>
</tbody>
</table>

### 2.8 Monte Carlo Simulation to Evaluate Test Plans

Monte Carlo simulation is a powerful tool to provide visualization of the results that might be obtained from a given test plan, and to check the large-sample approximations used to evaluate and optimize ADDT plans.
Table 2.8 Effects of misspecifying \((h_d(D_t) - \beta_0)/\sigma, \omega_{50}/\sigma\) and \(\beta_2\): worst and best precision factor ratios of the compromise plans for estimating the \(p = 0.01\) failure-time quantile.

<table>
<thead>
<tr>
<th>Misspecification of ((h_d(D_t) - \beta_0)/\sigma) by (\pm 0.5)</th>
<th>Misspecification of (\omega_{50}/\sigma) and (\beta_2) by (\pm 0.2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual Values</td>
<td>Deviation Yielding Worst Ratio</td>
</tr>
<tr>
<td>(h_d(D_t) - \beta_0)</td>
<td>(\omega_{50}/\sigma)</td>
</tr>
<tr>
<td>(h_d(D_t) - \beta_0)</td>
<td>(\omega_{50}/\sigma)</td>
</tr>
<tr>
<td>-5</td>
<td>-0.65</td>
</tr>
<tr>
<td>-5</td>
<td>-0.5</td>
</tr>
<tr>
<td>-5</td>
<td>-0.5</td>
</tr>
<tr>
<td>-5</td>
<td>-0.6</td>
</tr>
<tr>
<td>-5</td>
<td>-0.6</td>
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<td>-6</td>
<td>-0.5</td>
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<td>-6</td>
<td>-0.5</td>
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<tr>
<td>-6</td>
<td>-0.6</td>
</tr>
<tr>
<td>-6</td>
<td>-0.6</td>
</tr>
</tbody>
</table>

Simulation and analytical evaluation, based on the large-sample approximations, are complementary tools for test planning. Simulation is best suited for exact evaluation of test plans and for providing useful insight through visualization of sampling variability in the parameter estimates. Simulation also provides a check on the adequacy of the large-sample approximations. The large sample approximations are, however, important for doing computations quickly, as is needed in optimization or comparing a large number of different alternative test plans to assess sample size needs. Generally it takes orders of magnitude more computer time to evaluate a plan with simulation relative to the use of a large sample approximation.

For each test plan in Table 2.5, a simulation trial consists of a set of 88 observations obtained according to the test plan, the given model, and the planning information. The simulated data are used to obtain the maximum likelihood estimates of the parameters, the estimate of the covariance matrix for the ML estimates, and the observed precision factor of an approximate 95% confidence interval for estimating \(t_{0.01}\). The simulation was repeated 1000 times for each test plan.
(a) Optimum Plan
\[ \hat{R} = 1.795, R_{0.90} = 1.923 \]

(b) Compromise Plan
\[ \hat{R} = 2.092, R_{0.90} = 2.361 \]

(c) Original Plan
\[ \hat{R} = 2.353, R_{0.90} = 2.748 \]

(d) Traditional Plan
\[ \hat{R} = 2.414, R_{0.90} = 2.888 \]

Figure 2.7  Simulation of 0.01 failure-time quantile estimates versus temperature for the optimum plan, the compromise plan, the original plan, and the traditional plan.

Figure 2.7 shows estimates of the 0.01 failure-time quantile versus temperature for the first 50 realizations of the simulation for each test plan. The longer lines represent the values computed from
the planning values, which we call the “true” values. The geometric mean and the 0.9 quantile of the 1000 observed precision factors $\hat{R}$, denoted by $\bar{R}$ and $R_{0.90}$ respectively, are given in the figure caption for each plan.

The $\bar{R}$ values tend to be close to the $R$ values obtained from the large-sample approximations given in Table 2.5. The optimum plan has the narrowest group of simulated lines. The spread of the group of simulated lines for the compromise plan is wider than that for the optimum plan but narrower than that for the traditional plan or for the original plan. This is consistent with our previous comments about the estimation precision of different test plans, based on the $R$ precision factors. We have studied the distribution of $\hat{R}$ for all of the test plans in Table 2.5. The distributions are similar although there tends to be less spread in the distribution with the optimum plan and more spread with the traditional plan. This is not surprising given that the variability in $\hat{R}$ is related to the variability of the ML estimators of the model parameters.

To assess the effectiveness of the test plan to estimate the 0.01 failure-time quantile at 25 °C, the 1000 simulated estimates of $t_{0.01}$ for the compromise plan are depicted in Figure 2.8. The geometric mean of these estimates is 775.01 weeks, relative to the “true” value of $t_{0.01} = 755.48$ weeks from the planning values. The distribution of the $t_{0.01}$ values is skewed to the right. Although the probability is small (estimated to be about 0.019 from the simulation results), if the planning values were correct, it would be possible that one could get an estimate of $t_{0.01}$ exceeding 2000 weeks.

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![Histogram of the simulated estimates of 0.01 failure-time quantile for the compromise plan.](image-url)
2.9 Conclusions and Extensions

Accelerated destructive degradation testing is an important tool for making reliability inferences and predictions, especially when test time is limited and few or no failures are expected at lower levels of the accelerating variables. The methodology presented in this paper can be extended in several important directions, suggesting areas for future research. These include the following:

1. For some products, there may be more than one failure mechanism. This can cause observations to be right censored. For such observations, the strength is unknown and is greater than the censoring value.

2. A model with multiple accelerating variables (e.g., temperature and humidity) could be developed.

3. As explained in Section 2.2.1, the relationship between the mean transformed degradation path and the transformed time is linear at a fixed accelerating variable level. The work in this paper can be extended to degradation models that have a nonlinear relationship.

4. While the basic ideas and numerical methods in this paper hold for any log-location-scale distribution, the results in the appendix are for the normal distribution as in our example. It may be possible to derive similar results in general for other log-location-scale distributions, but it will be more difficult when the information matrix is not block diagonal.

5. For some applications, prior knowledge about the failure mechanism might provide information about some model parameters that could be useful to improve the precision of estimating specified quantities. Bayesian methods could be used in such situations.

6. If one is going to use prior information in the estimation of model parameters, then generally, it is important that prior information should also be used in test planning. It would be useful to apply methods like those described in Zhang and Meeker (2006) for ADDT planning.

Acknowledgments

We would like to thank a referee, the associate editor, and the editor who made useful comments that helped us improve our paper.
2.A Large-Sample Approximate Variance Technical Details

2.A.1 Large-Sample Approximate Covariance Matrix

Let \( \hat{\theta} \) be the ML estimate of \( \theta = (\beta_0, \beta_1, \beta_2, \sigma)' \) based on \( n \) observations. Under the usual regularity conditions (for example, see Appendix B.4 of Meeker and Escobar 1998), the following results hold for large samples.

- \( \hat{\theta} \sim \text{MVN}(\theta, \Sigma_{\hat{\theta}}) \), where \( \Sigma_{\hat{\theta}} = I^{-1}_\theta \), and the Fisher information matrix \( I_\theta \) with \( r \) test conditions is
  \[
  I_\theta = E \left[ -\frac{\partial^2 L}{\partial \theta \partial \theta'} \right] = n \sum_{i=1}^{r} \pi_i \frac{\partial^2 L_i}{\partial \theta \partial \theta'} = n \sum_{i=1}^{r} \pi_i I_i. \tag{2.10}
  \]

\( L_i = \log[L_i(\theta)] \) is the contribution of a single observation at test condition \( v_i = (\tau_i, x_i) \) to the log-likelihood, and \( L_i(\theta) = \frac{1}{\sigma} \phi((Y_i - \mu_i)/\sigma) \). \( E(\bullet) \) is the expectation operator and the expectation is with respect to the data to be collected in the ADDT. \( I_i \) is the contribution of one observation at \( v_i \) to \( I_\theta \). Let \( I(\xi) \) denote the large sample scaled Fisher matrix for a particular test plan \( \xi \), then \( I(\xi) = \frac{1}{n} I_\theta \) (see B.3 of Meeker and Escobar 1998).

- For a scalar \( \tilde{g} = g(\hat{\theta}) \sim \text{NOR}(g(\theta), \text{Avar}(\tilde{g})) \), the delta method gives
  \[
  \text{Avar}(\tilde{g}) = \left[ \frac{\partial g(\theta)}{\partial \theta} \right]' \Sigma_{\hat{\theta}} \left[ \frac{\partial g(\theta)}{\partial \theta} \right],
  \]
  allowing us to compute the large-sample approximate variance for a desired function of \( \theta \).

2.A.2 ADDT Model Fisher Information Matrix

The contribution \( I_i \) to the Fisher information matrix (2.10) in terms of parameters \( \theta \) is

\[
I_i = \frac{1}{\sigma^2} \begin{bmatrix}
  f_{11}(\zeta)u_i u_i' & f_{12}(\zeta)u_i \\
  f_{12}(\zeta)u_i' & f_{22}(\zeta)
\end{bmatrix}
\]

where \( f_{11}(\zeta), f_{12}(\zeta), f_{22}(\zeta) \) elements can be computed using the LSINF algorithm (see Escobar and Meeker 1994). In a situation where there is censoring, \( \zeta \) depends on the test conditions. For the normal distribution and no censoring, it can be shown that \( f_{11} = 1, f_{12} = 0, \) and \( f_{22} = 2 \). \( u_i \) is the vector of partial derivatives of the degradation with respect to the \( \beta \) parameters. That is

\[
u_i = \begin{bmatrix}
  \frac{\partial \mu_i}{\partial \beta_0} \\
  \frac{\partial \mu_i}{\partial \beta_1} \\
  \frac{\partial \mu_i}{\partial \beta_2}
\end{bmatrix} = \begin{bmatrix}
  1 \\
  \exp(\beta_2 x_i) \tau_i \\
  \beta_1 x_i \exp(\beta_2 x_i) \tau_i
\end{bmatrix}
\]
where \( \mu_i = \beta_0 + \beta_1 \exp(\beta_2 x_i) \tau_i, \tau_i = h_i(t_i) \), and \( x_i = h_a(\text{AccVar}_i) \). For ADDT planning, the Fisher information matrix is evaluated at planning values \( \theta^\| \).

2.A.3 Large-Sample Approximate Variance of \( h_i(\hat{t}_p) \) and \( \hat{t}_p \)

We can write \( \text{Avar}[h_i(\hat{t}_p)] \) as a function of \( \Sigma \theta^\| \). Define \( c = \partial h_i(t_p)/\partial \theta \). Then direct computations yield

\[
\text{Avar}[h_i(\hat{t}_p)] = c' \Sigma \theta^\| c = c' I_\theta^{-1} c = \frac{1}{n} c' [\mathcal{I}(\xi)]^{-1} c. \tag{2.11}
\]

For decreasing degradation, \( h_i(t_p) = \nu + \varsigma \Phi^{-1}(p) \) for \( p \geq \Phi (-\nu/\varsigma) \). The elements of \( c \) are:

\[
\begin{align*}
\frac{\partial h_i(t_p)}{\partial \beta_0} &= -\frac{1}{\beta_1 \exp(\beta_2 x)} \\
\frac{\partial h_i(t_p)}{\partial \beta_1} &= -h_i(t_p) \\
\frac{\partial h_i(t_p)}{\partial \beta_2} &= -x h_i(t_p) \\
\frac{\partial h_i(t_p)}{\partial \sigma} &= -\Phi^{-1}(p)
\end{align*}
\]

For increasing degradation, \( h_i(t_p) = -[\nu + \varsigma \Phi^{-1}(1-p)] \) for \( p \geq 1 - \Phi (-\nu/\varsigma) \). The elements of \( c \) are the same as those for decreasing degradation except for

\[
\frac{\partial h_i(t_p)}{\partial \sigma} = -\Phi^{-1}(1-p) \frac{1}{\beta_1 \exp(\beta_2 x)}.
\]

Using the delta method,

\[
\text{Avar}(\hat{t}_p) = \left( \frac{\partial h_i^{-1}(z)}{\partial z} \bigg|_{h_i(t_p)} \right)^2 \text{Avar}[h_i(\hat{t}_p)].
\]

2.B Optimum Plan Technical Details

2.B.1 General Equivalence Theorem

The following results apply to the ADDT planning problem:

1. The objective function \( \Psi(\xi) \) defined in (3.3) is strictly concave. That is, if \( 0 < \alpha < 1 \) and \( \xi, \eta \) are two test plans \( (\xi \neq \eta) \) then

\[
\Psi [\alpha \xi + (1-\alpha)\eta] > \alpha \Psi(\xi) + (1-\alpha)\Psi(\eta). \tag{2.12}
\]

2. The directional derivative, \( \Lambda \), of \( \Psi \) at \( \xi \) and in the direction of an alternative plan \( \eta \) is

\[
\Lambda(\xi, \eta) = c'[\mathcal{I}(\xi)]^{-1} \mathcal{I}(\eta)[\mathcal{I}(\xi)]^{-1} c - c'[\mathcal{I}(\xi)]^{-1} c \tag{2.13}
\]

where \( c, \mathcal{I}(\xi), \) and \( \mathcal{I}(\eta) \) are evaluated at the planning values.
3. For fixed \( \xi \), the directional derivative \( \Lambda(\xi, \eta) \) is linear in \( \eta \) in the following sense (Whittle 1973, equation 7). For \( a_i \geq 0, \sum_i a_i = 1 \)

\[
\Lambda \left( \xi, \sum_i a_i \eta_i \right) = \sum_i a_i \Lambda(\xi, \eta_i)
\]

where the \( \eta_i \)'s are alternative test plans. That is, for an alternative test plan specified by the convex combination \( \sum_i a_i \eta_i \), the directional derivative is the corresponding convex combination of the \( \Lambda(\xi, \eta_i) \).

4. Consider the directional derivative \( \Lambda(\xi, \xi_v) \) in the direction of the plan that puts all units at \( v \).

Then the plan \( \xi^* \) is optimal if and only if \( \sup_v \Lambda(\xi^*, \xi_v) \leq 0 \).

5. The test conditions \( v^*_i \) in the optimal plan are a subset of the conditions \( v \) satisfying \( \Lambda(\xi^*, \xi_v) = 0 \).

**Proof of Results**

To prove (2.12), the concavity of the objective function, use (3.3) to get

\[
\Psi [a\xi + (1 - a)\eta] - a\Psi(\xi) - (1 - a)\Psi(\eta)
\]

\[
= c' \left\{ a[I(\xi)]^{-1} + (1 - a)[I(\eta)]^{-1} - [aI(\xi) + (1 - a)I(\eta)]^{-1} \right\} c > 0
\]

where the inequality on the right hand side of the last equation follows from the fact that the matrix \( a[I(\xi)]^{-1} + (1 - a)[I(\eta)]^{-1} - [aI(\xi) + (1 - a)I(\eta)]^{-1} \) is positive definite; see Moore (1973).

To prove (2.13), start with the definition

\[
\Lambda(\xi, \eta) = \lim_{\delta \to 0^+} \frac{\Psi[(1 - \delta)\xi + \delta\eta] - \Psi(\xi)}{\delta}
\]

(2.14)

From (2.14), using l'Hôpital's rule for limits, the chain rule for derivatives, and

\[
\frac{\partial \Psi(\xi)}{\partial \xi} = [I(\xi)]^{-1} c c' [I(\xi)]^{-1}
\]

one gets

\[
\Lambda(\xi, \eta) = -\text{tr} \left[I(\xi) \frac{\partial \Psi(\xi)}{\partial \xi} \right] + \text{tr} \left[I(\eta) \frac{\partial \Psi(\xi)}{\partial \xi} \right] = c' [I(\xi)]^{-1} I(\eta) [I(\xi)]^{-1} c - c' [I(\xi)]^{-1} c.
\]

(2.15)

To show the linearity of \( \Lambda(\xi, \eta) \) with respect to \( \eta \), use (2.15) to write

\[
\Lambda \left( \xi, \sum_i a_i \eta_i \right) = c' [I(\xi)]^{-1} I \left( \sum_i a_i \eta_i \right) [I(\xi)]^{-1} c - c' [I(\xi)]^{-1} c.
\]

(2.16)

Using the fact that \( I \left( \sum_i a_i \eta_i \right) = \sum_i a_i I(\eta_i) \), expanding the first term on the right hand side of (2.16), and after some simplications, one obtains \( \Lambda \left( \xi, \sum_i a_i \eta_i \right) = \sum_i a_i \Lambda(\xi, \eta_i) \).
Because the objective function $\Psi(\xi)$ is concave and its directional derivative $\Lambda(\xi, \eta)$ is linear in $\eta$. Results 4 and 5 follow immediately from Theorem 1 parts (i), (ii), (iii), and (c) in Whittle (1973, page 125).

### 2.B.2 Alternative Optimum Plans

This section shows an alternative plan $\xi^a$ in (2.7) is optimum. The information matrix for the optimal plan $\xi^*$ given in equation (2.5) is

$$
\mathcal{I}(\xi^*) = \pi_1^2 \mathcal{I}(v_1^*) + \pi_2^2 \mathcal{I}(v_2^*) + \pi_3^2 \mathcal{I}(v_3^*)
$$

(2.17)

where $e_3^* = [\exp(\beta_2 x_2^*), \beta_1 x_2^* \exp(\beta_2 x_2^*)]'$ and $e_4^* = [\exp(\beta_2 x_3^*), \beta_1 x_3^* \exp(\beta_2 x_3^*)]'$.

For the alternative plan $\xi^a$, $\pi_3 = \pi_3^a$, $\pi_2 = \pi_2^a \tau_M / \tau_a$, and $\pi_1 = \pi_1^a + \pi_2^a - \pi_2^a \tau_M / \tau_a$, then after some simplifications, one obtains $\mathcal{I}(\xi^a) = \mathcal{I}(\xi^*) - m_0 u_2^a u_2^a'$, where $m_0 = \pi_2^a (\tau_2^a - \tau_M \tau_a)$ and $u_2^a = (0, e_2^a, 0)$. Consequently, using a result to compute the inverse of a sum of matrices (see Problem 2.8 on page 33 of Rao, 1973), one gets

$$
c' [\mathcal{I}(\xi^a)]^{-1} c = c' [\mathcal{I}(\xi^*) - m_0 u_2^a u_2^a']^{-1} c
\quad = c' [\mathcal{I}(\xi^*)^{-1} m_0 c' [\mathcal{I}(\xi^*)]^{-1} u_2^a u_2^a' [\mathcal{I}(\xi^*)]^{-1} c]
\quad = c' [\mathcal{I}(\xi^*)]^{-1} c.
$$

(2.18)

The last step in obtaining (2.18) above follows from the fact that $c' [\mathcal{I}(\xi^*)]^{-1} u_2^a = 0$ (see details below). Equation (2.18) shows that the alternative plan $\xi^a$ is optimum.

Now we prove that $c' [\mathcal{I}(\xi^*)]^{-1} u_2^a = 0$. First we derive a simple general expression for the directional derivatives. Consider the test plan $\xi_v$ with $v = (\tau, x)$. Define $e = [\exp(\beta_2 x), \beta_1 x \exp(\beta_2 x)]'$. Then using the fact that $\Lambda(\xi^*, \xi_v) = 0$ (recall that $v_1^*$ is a test condition in the optimum plan), after simple manipulations, one gets

$$
\Lambda(\xi^*, \xi_v) = \tau c' [\mathcal{I}(\xi^*)]^{-1} [0 \quad e'] e\tau e' [\mathcal{I}(\xi^*)]^{-1} c = \tau r_2 e (2r_1 + \tau r_3 e)
$$

(2.19)

where $r_1$ is a scalar and $r_2$ is a vector with two components defined by $(r_1, r_2, r_3) = c' [\mathcal{I}(\xi^*)]^{-1}$. 


Because $\xi^*$ is optimum, $\Lambda(\xi^*, \xi_{v1}) = \Lambda(\xi^*, \xi_{v2}) = \Lambda(\xi^*, \xi_{v3}) = 0$. Then in view of (2.19)

$$
r'_2 e^*_2 (2r_1 + \tau_M r'_2 e^*_3) = 0
$$

(2.20)

$$
r'_2 e^*_3 (2r_1 + \tau_M r'_2 e^*_3) = 0.
$$

(2.21)

Because $x^*_3$ is optimum, the directional derivative function must have a relative maximum at that point. Then

$$
\frac{\partial \Lambda(\xi^*, \xi_{w})}{\partial x} = 2\tau_M (r_1 + \tau_M r'_2 e^*_3) \left( r'_2 \frac{\partial e}{\partial x} \right)_{x^*_3} = 0.
$$

(2.22)

Equations (2.20), (2.21), and (2.22) imply $r'_2 e^*_2 = 0$, $r'_2 e^*_3 = -r_1/2$, and $r'_2 \partial e/\partial x|_{x^*_3} = 0$. Consequently, $e'[I(\xi^*)]^{-1} u^*_2 = 0$ as required.

References


CHAPTER 3. Planning Accelerated Destructive Degradation Test with Competing Risks

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Abstract

Accelerated destructive degradation tests (ADDTs) are widely used in manufacturing industries to obtain timely product reliability information, especially in applications where few or no failures are expected under use conditions in tests of practical length. An ADDT plan specifies the test conditions of accelerating variables, running time, and the corresponding allocation of test units to each condition. Usually, variables such as temperature, voltage, or pressure can be used as accelerating variables to accelerate degradation of a product. For some applications, however, tests at high-stress test conditions would result in more than one type of failure for test units, called competing risk problems. Careful test planning is important for efficient use of limited resources: test time, test units, and test facilities. This paper describes methods to find unconstrained and constrained optimum test plans for competing risk applications under a given test optimization criterion, such as minimizing the large-sample approximate variance of a failure-time distribution quantile at use conditions. A modified general equivalence theorem (GET) is used to verify the optimality of a given ADDT plan. Generally, an optimum test plan provides insight for constructing a good compromise test plan which tends to be more robust and practical. Monte Carlo simulations are used to provide visualization of the results that might be obtained from given test plans. The methods are illustrated with an application for an adhesive bond.

KEY WORDS: compromise plan; competing risk; general equivalence theorem; Monte Carlo simulation; optimum plan; reliability.
3.1 Introduction

3.1.1 Background

With the development of new technologies and strong global competition, manufacturers are facing pressure to produce high reliability products and to do so quickly. This has motivated a strong interest in conducting up-front accelerated reliability tests on materials and components while products are being designed. Degradation tests are becoming more and more popular nowadays because degradation data can provide considerably more reliability information, compared to the traditional failure-time data (where time to failure is the response), especially in applications where few or no failures are expected within a long period of time. For most applications, to get reliability information quickly, degradation tests are accelerated by testing at higher than usual levels of accelerating variables such as temperature, voltage or humidity. Information obtained from tests at high levels of the accelerating variables is then extrapolated to the use conditions using a reasonable statistical model, often based on physics of failure knowledge. For some applications, degradation tests are destructive because the measurement process destroys or changes the physical/mechanical characteristics of test units. An accelerated degradation test with such degradation data is called an accelerated destructive degradation test (ADDT).

3.1.2 Motivation: Adhesive Bond C

The application motivating this work was an adhesive bond (adhesive bond C) which was to be used to attach two components of a product together. The experimental response was strength (in Newtons) of the adhesive bond and the strength deteriorated over time. In particular, the product engineers wanted to estimate the time at which 1% of the product would have an adhesive strength below 40 Newtons when operating at a room temperature of 25 °C (i.e., they wanted to estimate the 0.01 quantile of the failure-time distribution at use conditions). To obtain the degradation information quickly, temperature was to be used as an accelerating variable for the test. To measure the strength of the test units, engineers applied an increasing force until the two components attached by the adhesive bond broke apart. Thus the measurement process for this application was destructive. For the adhesive bond C, two types of failure were observed for the units tested at high-stress test conditions (i.e., high temperature for long running time). In particular, some units failed from adhesive failure and some failed from cohesive failure. Adhesive failures occurred when one of the components detached from the adhesive bond. Cohesive failures occurred when the adhesive bonds to the material remained intact, but the components came apart due to a loss of cohesion within the layer of the adhesive material.
Because of the destructiveness of measurement process, only one type of failure or the other could be observed on each test unit. Because of differences in the two effective activation energy values, the engineers were confident that only adhesive failure would occur on units operating at the normal operating temperatures, even after a long running time. Hence, the primary interest was to estimate the 0.01 quantile of the failure-time distribution for the adhesive failures.

Table 3.1 shows the accelerated destructive degradation test (ADDT) plan originally proposed for the adhesive bond C application in order to estimate the 0.01 quantile of the failure-time distribution at use conditions. The 15 units tested at time 0 would have no aging and were to serve as baseline units. A total of 59 additional test units were aged and measured after running according to the test conditions as shown in Table 3.1. Note that no test units were assigned at the highest stress test condition (16 Weeks, 70 °C), in an attempt to avoid cohesive failures.

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Weeks 0</th>
<th>Weeks 2</th>
<th>Weeks 4</th>
<th>Weeks 6</th>
<th>Weeks 12</th>
<th>Weeks 16</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>5</td>
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<td>5</td>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>6</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>5</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>—</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td>15</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>Totals</td>
<td>15</td>
<td>15</td>
<td>4</td>
<td>15</td>
<td>15</td>
<td>10</td>
<td>74</td>
</tr>
</tbody>
</table>

### 3.1.3 Related Literature

There is a large amount of literature on planning accelerated tests, especially for accelerated life tests (ALTs) where failure time is the response. Nelson (2005a, 2005b) summarizes such test planning work. Some work that is particularly relevant to this paper is included in the following references. Nelson (1990, Chapter 6) describes methods for planning ALTs based on a simple model. Meeker and Escobar (1998, chapter 20) provide details and examples on how to plan a single-variable ALT. Escobar and Meeker (1995) describe methods for planning ALTs with two or more explanatory variables. There are some important differences between ALTs and accelerated degradation tests because of their different responses. Boulanger and Escobar (1994) give methods for planning repeated measures accelerated degradation tests. Shi, Escobar, and Meeker (2009) show how to construct accelerated destructive degradation tests for applications with only one type of failure. In this paper, we describe methods for
planning ADDTs for applications with more than one type of failure.

3.1.4 Overview

The remainder of this paper is organized as follows. Section 3.2 describes a class of degradation models for ADDT data with more than one type of failure mode and gives expressions for the degradation distribution and quantiles. Section 3.3 illustrates the failure-time distribution and quantiles induced by the degradation models. Section 3.4 talks about the criterion of accelerated destructive degradation test planning for applications with more than one type of failure. Section 3.5 gives a constrained optimum ADDT plan and applies a modification of the general equivalence theorem (GET) to verify the optimality of this test plan. Section 3.5 also presents a compromise ADDT plan that meets practical constraints. Section 3.6 uses Monte Carlo simulation to provide visualization of the results that might be obtained from given test plans. Section 3.7 contains some concluding remarks and extensions for future research work. Appendix A provides derivations and technical details about the large-sample approximations used in this work.

3.2 Degradation Models with Competing Risks

For applications resulting in data with more than one type of failure, the statistical competing risk model (e.g., David and Moeschberger 1978, and Crowder 2001) can be used. In this paper, we mainly focus on accelerated test planning in the applications with two types of failure where the marginal distribution of time to failure for one of the failure types is of primary interest. The generic responses corresponding to two types of failure are called the primary response and the competing response, respectively. In an ADDT with competing risks, for each experimental unit, we observe only the minimum of the primary response and the competing response.

3.2.1 Accelerated Degradation Model for the Primary Response

An important class of the degradation models for the primary response at time $t$ and the accelerating variable AccVar is

$$Y = \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \epsilon$$

(3.1)

where $Y = h_d$(primary response) is transformed degradation, $\tau = h_t(t)$ and $x = h_a$(AccVar) are known monotone increasing transformations of time and the accelerating variable, respectively. These functions may be suggested by physics of failure or determined empirically. $\epsilon$ is residual deviation that describes
unit-to-unit variability. Here $\epsilon/\sigma \sim \Phi(z)$, where $\Phi(z)$ is a completely specified cumulative distribution function (cdf), such as, the standardized normal cdf, $\Phi_{nor}(z)$, or the standardized smallest extreme value cdf, $\Phi_{sev}(z)$. The model parameters $\theta = (\beta_0, \beta_1, \beta_2, \sigma)$ are fixed but unknown, and will be estimated from ADDT data.

Model (4.1) is linear in the sense that for a specified $x$, $\mu(\tau, x) = \beta_0 + \beta_1 \exp(\beta_2 x)\tau$ is linear in $\tau$. The full model in (4.1) is, however, nonlinear in the parameters. The intercept $\beta_0$ is the location parameter of the transformed degradation distribution when $\tau = 0$. The degradation rate of $\mu(\tau, x)$ with respect to $\tau$ at $x$ is $\omega(x) = \beta_1 \exp(\beta_2 x)$. The sign of $\beta_1$ determines whether the degradation is increasing or decreasing over time. For example if the degradation response is the size of a crack, $\beta_1$ would be expected to be positive. On the other hand, if the degradation response is the strength of an adhesive bond, $\beta_1$ would be expected to be negative.

### 3.2.2 Accelerated Degradation Model for the Competing Response

The degradation model for the competing response is parallel to the one for the primary response, and is given as

$$Y^{(c)} = \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x)\tau + \epsilon^{(c)}$$

where $Y^{(c)} = h_d(\text{competing response})$ is the transformed competing degradation, $\tau$ and $x$ are the same as in Section 3.2.1, and again $\epsilon^{(c)}/\sigma^{(c)} \sim \Phi(z)$. The model parameters for this competing model are $\theta^{(c)} = (\beta_0^{(c)}, \beta_1^{(c)}, \beta_2^{(c)}, \sigma^{(c)})$ and they are also fixed but unknown.

### 3.2.3 Degradation Models for Adhesive Bond C

The degradation model for adhesive bond C is an extension of the model derived by Escobar et al. (2003) for a similar application without competing risks, adhesive bond B. Product engineers believe that these two types of bonds have the same characteristics for the adhesive strength, but for adhesive bond C, cohesive failures can, after a long running time, occur for units tested at high temperatures.

For the adhesive bond C application, the degradation models for the adhesive strength and cohesive strength are, respectively,

$$Y = \beta_0 + \beta_1 \exp(\beta_2 x)\tau + \epsilon$$ and $$Y^{(c)} = \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x)\tau + \epsilon^{(c)}$$
where

\[
\begin{align*}
Y &= \log(\text{Adhesive Strength in Newtons}) \\
Y^{(c)} &= \log(\text{Cohesive Strength in Newtons}) \\
x &= -\frac{11605}{\text{Temperature in } ^\circ\text{C} + 273.15} \\
\tau &= \sqrt{\text{Time in Weeks}} \\
\left(\frac{\epsilon}{\sigma}\right) &\sim \Phi_{\text{nor}}(z) \quad \text{and} \quad \left(\frac{\epsilon^{(c)}}{\sigma^{(c)}}\right) \sim \Phi_{\text{nor}}(z).
\end{align*}
\]

For this application, the accelerating variable is temperature and \( x \) is the Arrhenius transformed temperature. The denominator in \( x \) is temperature on the kelvin (K) scale and the numerator is the reciprocal of Boltzmann’s constant in units of electronvolt per kelvin (eV/K). Under this parametrization, \( \beta_2 \) and \( \beta_2^{(c)} \) have the interpretations of effective activation energy corresponding to the degradation mechanism for adhesive and cohesive strengths, respectively.

Figure 3.1 is an illustration of degradation distributions for the adhesive strength at 25 \(^\circ\text{C}\) and different values of time for some specific values of the parameters \( \beta_0, \beta_1, \beta_2, \sigma \) corresponding to the maximum likelihood (ML) estimates given in Escobar et al. (2003). This figure gives a visual impression of the adhesive strength degradation distribution over time. For the cohesive strength, the degradation distribution should have a similar decreasing trend over time, but at a different degradation rate from the adhesive strength because of differences in the parameters.

Due to the competing risk between two types of failure, we can only observe \( \min(Y, Y^{(c)}) \). If an adhesive failure occurs, then \( Y < Y^{(c)} \); if a cohesive failure occurs, then \( Y^{(c)} < Y \). Because the adhesive strength is of primary interest, we consider an observation to be \textit{exact} for an adhesive failure and \textit{right-censored} for a cohesive failure. For right-censored observations, the adhesive strength is unobserved but known to be greater than the censored value. Such right-censored observations contain relatively little information about the adhesive strength. Thus, using the competing risk model for test planning should result in a test plan that limits the probability of having such censored observations in the accelerated test.

### 3.2.4 Degradation Distribution and Quantiles

For a given time and accelerating variable level, the cdf of the primary transformed degradation \( Y \) is

\[
F_Y(y; \tau, x) = \Pr(Y \leq y; \tau, x) = \Phi\left(\frac{y - \mu(\tau, x)}{\sigma}\right).
\]
Figure 3.1 Adhesive bond C adhesive strength degradation distributions as a function of time at 25 °C. The strength axis is a logarithmic axis and the time axis is a square root axis. The horizontal line at $D_t = 40$ Newtons is the threshold for the failure-definition degradation level. At each time, the shaded area below the horizontal line is the failure probability at that specific time.

For the adhesive bond C example, the cdf for $Y$, the logarithm of the adhesive strength at a fixed test condition of time and temperature, can be obtained by replacing $\Phi$ with $\Phi_{\text{nor}}$.

The $p$ quantile function for the primary transformed degradation distribution at $(\tau, x)$ is

$$y_p = \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \sigma \Phi^{-1}(p)$$

where $\Phi^{-1}(p)$ is the $p$ quantile of a standard location-scale distribution. Substituting $\Phi_{\text{nor}}^{-1}(p)$ for $\Phi^{-1}(p)$, one obtains the $p$ quantile of the transformed degradation (log Newtons) for the adhesive strength in the adhesive bond C application, as shown in Figure 3.1 for $p = 0.01$ and $p = 0.001$.

3.3 Failure-Time Distribution with Competing Risks

3.3.1 Relationship Between Degradation and Failure

For some products, there is a gradual loss of performance as usage time increases (e.g., decreasing adhesive strength of an adhesive bond). A “soft failure” (see Section 13.4 of Meeker and Escobar 1998) occurs when performance reaches a critical level $D_t$ after a certain usage time. Because we are mainly interested in evaluating the failure-time distribution for the primary response, the failure-time $T$, for
the competing risk applications, is defined as the time when the observed primary degradation crosses
the critical level $D_f$.

### 3.3.2 Failure-Time Distribution and Quantiles

For decreasing degradation, when $\beta_1$ is negative, the event of failure-time $T$ being less than $t$ is
equivalent to an observed primary degradation response being less than the critical level $D_f$ at time $t$
(i.e., $Y \leq y_t$, where $y_t = h_d(D_f)$). Then the failure-time cdf is

$$F_T(t; x) = \Pr(T \leq t) = \Pr(Y \leq y_t) = F_Y(y_t; \tau, x)$$

$$= \Phi \left( \frac{y_t - \mu(\tau, x)}{\sigma} \right) = \Phi \left( \frac{\tau - \nu}{\zeta} \right), \quad \text{for } t \geq 0 \quad (3.2)$$

where

$$\nu = \frac{(\beta_0 - y_t) \exp(-\beta_2 x)}{|\beta_1|} \quad \text{and} \quad \zeta = \frac{\sigma \exp(-\beta_2 x)}{|\beta_1|}.$$  

Note that in this model, there is a positive $\Pr(T = 0) = \Phi(-\nu/\zeta)$ at $t = 0$. This spike is sometimes
called the dead-on-arrival (or DOA) probability.

From (4.2), the $p$ quantile of the failure-time for decreasing degradation is

$$t_p = \begin{cases} 
    h_t^{-1} [\nu + \zeta \Phi^{-1}(p)] & \text{if } p \geq \Phi(-\nu/\zeta) \\
    0 & \text{otherwise.}
\end{cases}$$

For increasing degradation, when $\beta_1$ is positive, the derivations of the failure-time cdf and quantiles are
similar.

### 3.4 Test Planning with Competing Risks

#### 3.4.1 ADDT Planning Information

For ADDT planning, there are usually practical constraints. For example, for the adhesive bond C
application, there are three constraints for the ADDT planning, which are:

- 74 test units is the maximum sample size.
- 70 $^\circ$C is the maximum temperature that can be used (The engineers involved in the testing felt
that higher temperatures would cause the degradation models to break down).
- 16 weeks is the maximum time available for testing.
To do test planning, planning information for the parameters of the given degradation models is required. This is because the test plan evaluation criteria depend on the model parameters. Such planning information could come from previous experiments, past data, engineers' experience, etc. For the adhesive bond C application, planning values for parameters in the primary response degradation model are obtained from the data analysis in Escobar et al. (2003) because it is believed that the adhesive degradation properties of bond C are similar to those of bond B, as mentioned in Section 3.2.3. That is, $\beta_0 = 4.471$, $\beta_2 = 0.6364$, $\sigma = 0.158$, and the degradation rate at 50 °C, $\omega_{50} = -0.1026$. These parameters have clear practical interpretations, which makes them easier to elicit from experts when needed. With the given planning values, one gets $\beta_1 = \omega_{50} \exp(-\beta_2 x_{50}) = -0.1026 \exp(0.6364 \times 35.912) = -8.643 \times 10^8$, where $x_{50} = -11605/(50 + 273.15) = -35.912$ is the transformed temperature of 50 °C.

Planning values for parameters in the competing response degradation model are $\beta_0^{(c)} = 5.7$, $\beta_2^{(c)} = 0.6$, $\omega_{50}^{(c)} = -0.23$, and $\sigma^{(c)} = 0.2$. This information was obtained from a combination of limited previous experience with cohesive failures in accelerated tests and engineering judgement. In some preliminary experiments, engineers did not find any cohesive failures at test conditions with low temperature and they would not expect to see any cohesive failures at use conditions, even after a long period of running time (i.e., with probability approaching one, adhesive failures would occur first at use conditions). Cohesive failures only happened at some high-stress test conditions. Under the specified planning values for the model parameters, Figure 3.2 shows a contour plot of the probability that the cohesive strength is less than adhesive strength (i.e., the right-censored probability) as a function of test conditions within the experimental region, and Figure 3.3 compares the mean degradation responses of adhesive and cohesive strength at various test conditions.

### 3.4.2 Criterion for ADDT Planning with Competing Risks

For ADDT planning, the appropriate test planning criterion depends on the purpose of the experiment. For our application, the objective is to estimate $t_p$, a particular quantile of the failure-time distribution at use conditions. A commonly used criterion for test planning is to minimize $\text{Avar}(\hat{t}_p)$, the large-sample approximate variance of the maximum likelihood (ML) estimator of the specified failure-time quantile.

We use $v = (t, \text{AccVar})$ to denote a test condition specifying the time $t$ and the accelerating variable level AccVar. An ADDT plan, denoted by $\xi$, will specify a set of test conditions $v_i$, and the corresponding proportional allocation $\pi_i$ of test units at each $v_i$. If a test plan has $r$ test conditions, then
Figure 3.2 The probability of cohesive strength being less than adhesive strength (i.e., probability of a right-censored observation) under specified planning values for the model parameters.

the proportional allocations satisfy $\pi_i > 0$, for each $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} \pi_i = 1$.

Because $h_i(t_p)$ is a monotone increasing function of $t_p$, minimizing $\text{Avar}[h_i(\hat{t}_p)]$ is equivalent to minimizing $\text{Avar}(\hat{t}_p)$. Thus, from Appendix 3.A.2, the optimization criterion is equivalent to finding a test plan $\xi$ that maximizes the objective function

$$\Psi(\xi) = -c'[I_{\theta,\theta}(\xi)]^{-1}c,$$

where $c = \partial h_i(t_p)/\partial \theta$ and $I_{\theta,\theta}(\xi)$ is the Fisher information matrix for test plan $\xi$. Details are defined in Appendix 3.A. This criterion is known as $c$ optimality.

3.5 ADDT Plan with Competing Risks

3.5.1 Initial Optimized ADDT Plan with Competing Risks

For applications with no competing risks and a single degradation model like that in (4.1), Shi, Escobar, and Meeker (2009) present an ADDT optimum plan structure, illustrated in Figure 3.4. Under the practical constraints of a maximum time $t_M$ and a maximum accelerating variable level $x_M$, the optimum plan with no competing risks is not unique. That is, there are many plans that provide the same smallest large-sample approximate variance for the ML estimator of a specified failure-time
Adhesive/Cohesive Strength at 10 weeks

Adhesive/Cohesive Strength at 16 weeks

Adhesive/Cohesive Strength at 65 Degrees C

Adhesive/Cohesive Strength at 70 Degrees C

Figure 3.3  The mean degradation responses of adhesive and cohesive strength at various test conditions under specified planning values for the model parameters. The strength axis is a logarithmic axis, the time axis is a square root axis, and the temperature axis is an Arrhenius axis.

As shown in Figure 3.4, such optimum plans have some test units allocated at $v_1$ (the baseline test condition) with $t = 0$ (note from the model that the level of AccVar is not a factor at time $t = 0$), some at the $v_2$ test condition with $x_M$ and a time level larger than a lower boundary $t_L$, and some at the $v_3$ test condition with $t_M$ and an optimized AccVar value. The AccVar value for the $v_3$ test condition and the proportional allocations $\pi_1$, $\pi_2$ of test units are chosen to minimize the large-sample approximate variance for each value of time for $v_2$ between $t_L$ and $t_M$. For the no-competing-risk model, each such plan has the same optimum value for the objective function.

Generally, in a regression model, if all of the parameters are linear with respect to the explanatory variables, the number of test conditions for the optimum plan will not be any larger than the number of parameters needed to define the response surface. Although no such definite conclusion exists for
models that are nonlinear in their parameters, such as our degradation model, numerical experiments for the ADDT model have not found a counter example. Because there are three regression parameters in the degradation model described in (4.1), we expect that a non-degenerate optimum ADDT plan for an application with the competing risk degradation model could also be a three-point plan (i.e., the optimum test plan should have three test conditions). We will check this using the GET.

For the adhesive bond C application which has competing risks, our first attempt to find an optimum plan was to investigate test plans that have a structure similar to that for the no-competing-risk model, as shown in Figure 3.4. For most practical situations in which accelerated tests are used, all the test conditions for an optimum plan are on the boundaries of the experimental region and they spread out as much as possible, providing better estimates of the regression coefficients than closely-spaced test conditions. For the competing risk model, the probability of getting a cohesive failure (a right-censored observation, providing relatively little information about adhesive failures) is highest when \( v_2 \) is in the NE corner so we might expect the optimum \( v_2 \) to be at an interior point. Hence, we assumed similar \( v_1 \) and \( v_3 \) test conditions but explored the properties of test plans with the \( v_2 \) point moving within a subset of the experimental region. Then at each fixed test condition for \( v_2 \), temperature for the \( v_3 \) and the proportional allocations \( \pi_1, \pi_2 \) of test units were chosen to minimize \( \text{Avar}(\hat{t}_{0.01}) \). Figure 3.5 shows the contour plot of the large-sample approximate standard error of \( \hat{t}_{0.01} \), \( \text{Ase}(\hat{t}_{0.01}) \), for each optimized
plan as the test condition $v_2$ varies within the experimental region, under the usual constraint that proportional allocations for any test condition should be nonnegative and that they sum to 1.

Figure 3.5 indicates that having $v_2$ on the 70 °C boundary with time around 3 weeks results in a best test plan with minimum $\text{Ase}(\hat{t}_{0.01})$. This optimization result for the competing risk model is different from the one with no-competing risk where many optimum plans are derived by moving the time for the $v_2$ point between a lower boundary $t_L$ and $t_M = 16$ weeks at 70 °C. The reason for different optimization conclusions between the two models is because that testing at temperatures approaching 70 °C will tend to cause cohesive failures.

![Figure 3.5](image1)

**Figure 3.5** Contour plot of the large-sample approximate standard error of $\hat{t}_{0.01}$ for optimized plans when the test condition $v_2$ moves within the experimental region.

As mentioned before, the boundary of the experimental region is of particular interest. Figure 3.6 shows $\text{Ase}(\hat{t}_{0.01})$ and the proportional allocations of test conditions obtained for different optimized test plans, when the point $v_2$ moves along the maximum 70 °C temperature line. Because all of the test plans are optimized under the condition that the proportional allocation for each test condition must be nonnegative and sum to one, $\pi_1$ is close to zero for optimized plans when the time for $v_2$ is less than around 3 weeks. At the minimum $\text{Ase}(\hat{t}_{0.01})$ value 272.06, the corresponding proportional allocation $\pi_1$ is close to zero, which is nearly a degenerate optimum test plan with little practical appeal.
3.5.2 Constrained Optimum ADDT Plan with Competing Risks

In practical applications, engineers prefer to allocate a certain number of test units at the baseline condition at time 0. For the adhesive bond C application, instead of the initial optimized “degenerate” plan derived in Section 3.5.1, it is more meaningful to assign a fixed proportion of test units at the baseline condition and then find a constrained optimum ADDT plan. For purposes of illustration, we assign 10% of test units to the baseline test condition $v_1$. This amount is typical of what engineers have used in previous ADDT experiments that we have seen. The time of condition $v_2$, the temperature of condition $v_3$ and the proportional allocation $\pi_2$ are then chosen to minimize $A\text{var} (\hat{t}_{0.01})$. Table 3.2 shows the constrained optimum ADDT plan. The optimum $\text{Ase}(\hat{t}_{0.01})$ for this test plan is 274.62, only slightly larger than that for the initial optimum degenerate plan.

3.5.3 General Equivalence Theorem

Shi, Escobar, and Meeker (2009) uses a variation of Whittle’s (1973) general equivalence theorem (GET) to check the optimality of a given ADDT plan without competing risks. For an ADDT with competing risks, the objective function shown in (3.3) has a form that is similar to the one in Shi,
Table 3.2 Constrained optimum ADDT plan with competing risks. The — indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Test Condition</th>
<th>Time in weeks</th>
<th>Temperature in °C</th>
<th>Proportional Allocation</th>
</tr>
</thead>
<tbody>
<tr>
<td>v₁</td>
<td>0</td>
<td>—</td>
<td>0.1</td>
</tr>
<tr>
<td>v₂</td>
<td>4.04</td>
<td>70</td>
<td>0.279</td>
</tr>
<tr>
<td>v₃</td>
<td>16</td>
<td>53.519</td>
<td>0.621</td>
</tr>
</tbody>
</table>

Escobar, and Meeker (2009), except that the expression of \( I_{θ,θ^{(c)}}(ξ) \) is different. The derivation of \( I_{θ,θ^{(c)}}(ξ) \) for the competing risk model is given in Appendix 3.A. To use GET appropriately for checking the optimality of an ADDT plan with competing risks, some conditions need to be satisfied. The proof for the satisfaction of GET conditions can be done in a manner that is similar to that presented in Appendix B.1 of Shi, Escobar, and Meeker (2009). To save space, however, the details of proof are not included here but just show the idea of GET for optimality check. For the GET, the directional derivative, \( Λ(ξ,η) \), of \( Ψ(ξ) \) in the direction of an alternative plan \( η \) is defined as

\[
Λ(ξ,η) = \lim_{δ→0^+} \frac{Ψ(ξ) + δη - Ψ(ξ)}{δ} = c'[I_{θ,θ^{(c)}}(ξ)]^{-1}I_{θ,θ^{(c)}}(η)[I_{θ,θ^{(c)}}(ξ)]^{-1}c - c'[I_{θ,θ^{(c)}}(ξ)]^{-1}c
\]

where \( c, I_{θ,θ^{(c)}}(ξ) \), and \( I_{θ,θ^{(c)}}(η) \) are evaluated using the planning values of the model parameters \( θ^{(c)} \) and \( θ^{(c)} \). Let \( ξ_v \) be a singular test plan that puts all units at the test condition \( v \). Suppose a given test plan \( ξ \) has \( r \) test conditions, \( v_1, v_2, \ldots, v_r \). Then the plan \( ξ \) is optimal if it satisfies \( Λ(ξ,ξ_{v_1}) = Λ(ξ,ξ_{v_2}) = \ldots = Λ(ξ,ξ_{v_r}) = 0 \) and \( Λ(ξ,ξ_v) \leq 0 \) for any other singular plan \( ξ_v \) in the experimental region.

For a constrained optimum ADDT plan with a fixed proportion of test units at the baseline condition, a modification to the GET is needed. Instead of allocating all units at a single test condition, the alternative test plan \( η \) must now put the same proportion of test units at the baseline condition \( v_1 \) as in the constrained optimum plan, and the remaining proportion of units at a single test condition \( v \). Let \( ξ_{v_1,v} \) denote a plan with two test conditions, one baseline condition \( v_1 \) with a fixed proportion of test units, and the other condition \( v \) with all the remaining proportion of test units. For a given test plan \( ξ \) with a fixed test condition \( v_1 \) and \( r-1 \) additional test conditions, \( v_2, v_3, \ldots, v_r \), the plan \( ξ \) is a constrained optimum plan if and only if it satisfies \( Λ(ξ,ξ_{v_1,v_2}) = Λ(ξ,ξ_{v_1,v_3}) = \ldots = Λ(ξ,ξ_{v_1,v_r}) = 0 \) and \( Λ(ξ,ξ_{v_1,v}) \leq 0 \) for any other single condition \( v \) in the experimental region.
For the adhesive bond C application, this modified GET is used to check the optimality of the constrained test plan illustrated in Table 3.2. Figure 3.7 shows the directional derivatives $\Lambda(\xi, \xi_{v_1,v})$ of this constrained optimum plan as a function of temperature and time for the test condition $v$, where $\xi_{v_1,v}$ is a plan that puts 10% of the test units at the baseline condition $v_1$ and the remaining 90% of the test units at the test condition $v$. Observe that, as expected, the directional derivatives are zero at those two unconstrained test conditions $v_2, v_3$ of the constrained optimum plan and less than zero anywhere else. This behavior of the directional derivative curves also implies that the constrained optimum plan is unique.

![Figure 3.7](image)

**Figure 3.7** Directional derivatives $\Lambda(\xi, \xi_{v_1,v})$ of the constrained optimum plan as a function of temperature and time for the test condition $v$ in the adhesive bond C application. $\xi_{v_1,v}$ is a plan that puts 10% test units at the baseline condition $v_1$ and the remaining 90% test units at the test condition $v$.

### 3.5.4 Compromise ADDT Plan with Competing Risks

An optimum plan results in the smallest large-sample approximate variance of the estimated failure-time quantile. An optimum plan, however, provides no information to check the adequacy of the model and tends to be highly sensitive to planning information specification errors (e.g., Meeker 1984). An
optimum plan does, however, provide insight for obtaining good compromise test plans which tend to be more robust and useful in practical applications. A compromise test plan is generally suggested by the optimum plan but uses more test conditions to achieve robustness.

Shi, Escobar, and Meeker (2009) proposed an optimized compromise plan, whose structure is shown in Figure 3.8, for an adhesive bond application with no competing risks. This compromise plan has ten test conditions, numbered 1 to 10, including the baseline condition and nine additional test conditions consisting of equally spaced levels of time and temperature within the experimental region. For practical reasons (a limited number of temperature-controlled test chambers and the test groups of units at one time) the rectangular pattern is preferred by engineers. For our adhesive bond C application with competing risks, we expect to obtain little useful information about adhesive strength by allocating test units at high-stress test conditions because of the high right-censored probabilities at those conditions, as shown in Figure 3.2. For this reason, some of the high-stress test conditions in the compromise plan structure of Figure 3.8 should be omitted to reduce the risk of cohesive failures and to get a better estimate of $\hat{t}_{0.01}$ for competing risk applications. Table 3.3 compares Ase($\hat{t}_{0.01}$) for various compromise plans obtained by omitting different combinations of the high-stress test conditions. The proportional allocations for all the remaining test conditions in the plan are equal. For each compromise plan, beyond a baseline test condition, there are three equally spaced time levels and three equally spaced temperature levels. The highest time level is fixed at 16 weeks and the highest temperature level is fixed at 70 °C. The lowest time level and the lowest temperature level are chosen to minimize Ase($\hat{t}_{0.01}$). The middle time level is the mean of the other two time levels and the middle temperature level is the mean of the other two temperature levels. As can be seen from Table 3.3, the plan after omitting three high-stress test conditions, numbered 7, 9, 10, appears to be the best optimized compromise plan with competing risks in the sense that it provides the smallest Ase($\hat{t}_{0.01}$). Note that there is a trade off between the estimation precision and robustness of the test plan. We can obtain a test plan with a smaller Ase($\hat{t}_{0.01}$) by omitting more test conditions, but we have to pay for this by some loss of robustness. We would recommend the last compromise plan in Table 3.3 or a similar compromise plan.

### 3.6 Monte Carlo Simulation to Evaluate Test Plans

Monte Carlo simulation is a useful tool to provide visualization of the results obtained from a test plan. Simulation also provides a check on the adequacy of the large-sample approximations. We do simulations to evaluate the three ADDT plans illustrated before: the original plan in Table 3.1, the constrained optimum plan in Table 3.2, and the last compromise plan in Table 3.3. For the adhesive
bond C application, the large-sample approximate standard error of the estimated 0.01 failure-time quantile, $Ase(\hat{t}_{0.01})$, for these test plans are 396.68, 274.62, and 364.44 respectively. For each of these test plans, a simulation trial consists of a set of 74 observations obtained according to the test plan, the given degradation models, and the planning information.

Figure 3.9 shows estimates of the 0.01 failure-time quantile versus temperature for 50 realizations of simulations for each test plan. The longer lines represent the values computed from the planning values, which we call the “true” values. The constrained optimum plan has the narrowest group of simulated lines. The spread of simulated lines for the compromise plan is wider than that for the constrained optimum plan but narrower than that for the original plan. These results are exactly consistent with the estimation precisions of different test plans, based on the large-sample approximate $Ase(\hat{t}_{0.01})$.

3.7 Conclusions and Extensions

Accelerated destructive degradation testing is an important tool for making reliability inferences and predictions, especially when few or no failures are expected at use conditions within severe testing time
Table 3.3 Comparison of different compromise ADDT plans with competing risks.

<table>
<thead>
<tr>
<th>Omitted Conditions</th>
<th>Lowest Time in Weeks</th>
<th>Lowest Temperature in °C</th>
<th>Proportional Allocation</th>
<th>Asy(\hat{t}_{0.01})</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>8.19</td>
<td>50.437</td>
<td>1/10</td>
<td>389.83</td>
</tr>
<tr>
<td>10</td>
<td>7.56</td>
<td>50.182</td>
<td>1/9</td>
<td>372.47</td>
</tr>
<tr>
<td>9, 10</td>
<td>10.36</td>
<td>52.087</td>
<td>1/8</td>
<td>373.88</td>
</tr>
<tr>
<td>7, 10</td>
<td>7.89</td>
<td>48.673</td>
<td>1/8</td>
<td>368.26</td>
</tr>
<tr>
<td>7, 9, 10</td>
<td>9.39</td>
<td>51.552</td>
<td>1/7</td>
<td>364.44</td>
</tr>
</tbody>
</table>

Figure 3.9 Simulation of 0.01 failure-time quantile estimates versus temperature for (a) the original plan, (b) the compromise plan, and (c) the constrained optimum plan.

constraints. Good ADDT plans can yield significant benefits to industry. However, accelerated tests might result in more than one type of failure for units tested at high-stress test conditions. Because of the competing risks problem, designing an appropriate ADDT plan could give more useful information about the primary degradation responses within the limited test time. The methodology presented in this paper can be extended in several important directions, suggesting areas for future research. These include the following:

1. Test plans for an accelerated degradation model with multiple accelerating variables (e.g., temperature and humidity) could be developed.

2. As explained in Section 3.2.1, the relationship between the location parameter and the transformed
time is linear at a fixed accelerating variable level. The work in this paper could be extended to
degradation models that have a nonlinear relationship (such a relationship might, for example, be
suggested by a physics of failure model).

3. For some applications, prior knowledge about the failure mechanism might provide information
about some of the model parameters. It is important that such prior information be used in test
planning as well as analysis. It would be useful to apply Bayesian methods like those described
in Zhang and Meeker (2006) for ADDT planning.

3. A  Technical Details

3. A. 1  The Fisher Information Matrix for ADDT with Competing Risks

Because of the competing risks, each obtained observation could be an exact or right random cen-
soring value (for example, see Chapter 5 of Nelson 1982 or Chapter 9 of Lawless 2003). Escobar and
Meeker (1998) show in detail how to compute the Fisher information matrix and large-sample ap-
proximate covariance matrix of maximum likelihood estimators for a wide class of parametric models,
including models with right random censoring data. The specific approach used here follows the general

Let \( \hat{\theta} \) be the ML estimator of the parameters \( \theta = (\beta_0, \beta_1, \beta_2, \sigma)' \) in the primary response model
based on \( n \) observations. Under the usual regularity conditions (for example, see Appendix B.4 of
Meeker and Escobar 1998), \( \hat{\theta} \sim \text{MVN}(\theta, \Sigma) \), and the Fisher information matrix
\( I_{\theta,\theta^{(c)}} \) with \( r \) test conditions is
\[
I_{\theta,\theta^{(c)}} = n \sum_{i=1}^{r} \pi_i I_i.
\]
Here \( I_i \) is the contribution of one observation at a specific test condition \( v_i = (\tau_i, x_i) \) to \( I_{\theta,\theta^{(c)}} \). For an ADDT with competing risks, following the
approach given in Escobar and Meeker (1998), \( I_i \) can be expressed in terms of parameters \( \theta \) and \( \theta^{(c)} \) as
\[
I_i = 1 \sigma^2 \begin{bmatrix}
\int_{-\infty}^{\infty} f_{11}(\cdot, y - \mu_i^{(c)}) h_i(y) dy & u_i' \\
\int_{-\infty}^{\infty} f_{12}(\cdot, y - \mu_i^{(c)}) h_i(y) dy & u_i'
\end{bmatrix}
\]
where \( \mu_i \) is the location parameter for the primary response, defined in Section 3.2.1 as \( \mu_i = \beta_0 + \beta_1 \exp(\beta_2 x_i) \tau_i \) with \( \tau_i = h_i(t_i) \) and \( x_i = h_a(\text{AccVar}_i) \). \( f_{jk}(\cdot, y - \mu_i), jk = 11, 12, 22 \) are the elements
of the Fisher information matrix, multiplied by \( \sigma^2 \), for right-censored observations (see Escobar and
Meeker 1998 for details). These elements can be computed using the LSINF algorithm (see Escobar
and Meeker 1994). Here \( h_i(y) = \frac{1}{\sigma \phi} \phi \left( \frac{y - \mu_i^{(c)}}{\sigma} \right) \) is the pdf of right random censoring point \( y \) due
to competing risks, where \( \mu_i^{(c)} = \beta_0^{(c)} + \beta_1^{(c)} \exp(\beta_2^{(c)} x_i) \tau_i \), and \( u_i \) is the vector of partial derivatives of \( \mu_i \).
with respect to the parameters \((\beta_0, \beta_1, \beta_2)\). That is

\[
u_i = \begin{bmatrix}
\frac{\partial \mu_i}{\partial \beta_0} \\
\frac{\partial \mu_i}{\partial \beta_1} \\
\frac{\partial \mu_i}{\partial \beta_2}
\end{bmatrix}
= \begin{bmatrix}
1 \\
\exp(\beta_2 x_i) \tau_i \\
\beta_1 x_i \exp(\beta_2 x_i) \tau_i
\end{bmatrix}
\]

The Fisher information matrix is evaluated based on planning information for model parameters \(\theta^\Omega\) and \(\theta^{(c)\Omega}\).

### 3.A.2 Large-Sample Approximate Variance of \(h_t(\widehat{t}_p)\) and \(\widehat{t}_p\)

When \(\widehat{\theta} \sim \text{MVN}(\theta, \Sigma)\), then for a scalar function \(\widehat{g} = g(\widehat{\theta}) \sim \text{NOR}[g(\theta), \text{Avar}(\widehat{g})]\), and the delta method gives

\[
\text{Avar}(\widehat{g}) = \left[ \frac{\partial g(\theta)}{\partial \theta} \right]' \Sigma \left[ \frac{\partial g(\theta)}{\partial \theta} \right],
\]

providing the large-sample approximate variance for the ML estimator of a desired function of \(\theta\).

\[
\text{Avar}[h_t(\widehat{t}_p)] = c' \Sigma c = c' \mathbf{I}^{-1}_{\theta, \theta(c)} c,
\]

where \(c = \partial h_t(t_p)/\partial \theta\). For decreasing degradation, \(h_t(t_p) = \nu + \varsigma \Phi^{-1}(p)\) for \(p \geq \Phi(-\nu/\varsigma)\). In this case the elements of \(c\) are:

\[
\begin{align*}
\frac{\partial h_t(t_p)}{\partial \beta_0} &= -\frac{1}{\beta_1 \exp(\beta_2 x)} \\
\frac{\partial h_t(t_p)}{\partial \beta_1} &= -\frac{h_t(t_p)}{\beta_1} \\
\frac{\partial h_t(t_p)}{\partial \beta_2} &= -x h_t(t_p) \\
\frac{\partial h_t(t_p)}{\partial \sigma} &= -\frac{\Phi^{-1}(p)}{\beta_1 \exp(\beta_2 x)}.
\end{align*}
\]

Using the delta method again,

\[
\text{Avar}(\widehat{t}_p) = \left( \frac{\partial h_t^{-1}(z)}{\partial z} \bigg|_{h_t(t_p)} \right)^2 \text{Avar}[h_t(\widehat{t}_p)].
\]

### References


CHAPTER 4. Bayesian Methods for Accelerated Destructive Degradation

Test Planning

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Abstract

Accelerated Destructive Degradation Tests (ADDTs) provide timely product reliability information in practical applications. This paper describes Bayesian methods for ADDT planning under a class of nonlinear degradation models with one accelerating variable. We use a Bayesian criterion based on the estimation precision of a specified failure-time distribution quantile at use conditions to find optimum test plans. A large-sample approximation for the posterior distribution provides a useful simplification to the planning criterion. The general equivalence theorem (GET) is used to verify the global optimality of the numerically optimized test plans. Optimum plans usually provide insight for constructing compromise plans which tend to be more robust and practically useful. We present a numerical example with a log-location-scale distribution to illustrate the Bayesian test planning methods and to investigate the effects of the prior distribution and sample size on test planning results.

Key Words: Compromise plan; General equivalence theorem; Large-sample approximation; Log-location-scale distribution; Optimum plan; Reliability.
4.1 Introduction

4.1.1 Background and Motivation

Many modern high-quality products are expected to perform their functions properly for years or even decades. Traditional life tests result in few or no failures, providing little or no information about reliability. \textit{Accelerated destructive degradation tests} (ADDTs) are sometimes used in manufacturing industries to obtain reliability information more quickly. Usually, variables like temperature, voltage, or pressure can be used as accelerating variables to accelerate the degradation of a material or product. In designing an experiment, decisions must be made before data collection, and data collection is usually restricted by limited resources. Careful test planning is crucial for efficient use of limited resources: test time, test units, and test facilities. The basic idea in designing an experiment is that statistical inference for the quantities of interest can be improved by selecting appropriate test conditions to minimize or otherwise control the variability of the estimator of interest. Generally, ADDT plans specify the test conditions of the accelerating variables, the amount of running time, and the corresponding allocation of test units to each condition. One can find an optimum test plan for a given criterion, such as the estimation precision of a failure-time distribution quantile at use conditions. Optimum test plans provide insight for constructing good practical test plans.

For some applications, specific information about the underlying models or parameters is usually available from past studies or empirical knowledge of the failure mechanism from previous experimentation. When using Bayesian design methods, a prior distribution is used to describe the available information on model parameters. The primary motivation for this paper is to address the need to use such prior information in accelerated destructive degradation test planning. Bayesian methods can be used to formally incorporate prior information into estimation and test planning, providing test plans with better statistical precision (i.e., smaller estimation variance).

4.1.2 Related Literature

Shi, Escobar, and Meeker (2009) describe methods for ADDT planning using a non-Bayesian approach and outline much of the related literature in this area. Chaloner and Verdinelli (1995) give a nice review of Bayesian design methods. Hamada, Martz, Reese, and Wilson (2001) show methods to find near-optimal Bayesian experimental designs for regression models using genetic algorithms. Clyde, Müller, and Parmigiani (1995) describe Bayesian design methods for a logistic regression model. There is an extensive literature on Bayesian accelerated life tests (ALTs) planning. Work of particular rele-
vance to this paper includes the following. Polson (1993) provides a general decision theory for ALT Bayesian design problems. Zhang and Meeker (2006) present Bayesian methods for planning accelerated life tests (ALTs) to estimate a specific quantile of interest. In this paper, we follow the general Bayesian design framework proposed by Zhang and Meeker (2006), but apply it to ADDT planning.

4.1.3 Overview

The remainder of this paper is organized as follows. Section 4.2 presents the ADDT degradation model and provides the degradation distribution and failure-time distribution induced by the model. Section 4.3 describes the Bayesian planning criterion, prior distribution specification, and general equivalence theorem (GET) used to verify the optimality of test plans. Section 4.4, based on an application, illustrates the methods of finding Bayesian optimum plans and optimized compromise plans under different situations for the specification of prior information. Section 4.4 also investigates the effects that changing the amount of prior information and sample size will have on Bayesian test plans. Section 4.5 gives some conclusions and areas for future research.

4.2 Degradation Models

4.2.1 The Model

Shi, Escobar, and Meeker (2009) describe non-Bayesian methods of finding ADDT plans for an important class of destructive degradation models. This paper illustrates the Bayesian ADDT planning methods based on the same degradation model. The ADDT regression model is

\[ Y = \beta_0 + \beta_1 \exp(\beta_2 x) \tau + \epsilon \]  

where \( Y \) is the transformed degradation response, \( \tau \) and \( x \) are known monotone increasing transformations of time \( t \) and accelerating variable \( \text{AccVar} \), respectively. \( \beta_0 \) is the location parameter of the transformed degradation when \( \tau = 0 \). The degradation rate with respect to \( \tau \) at the accelerating variable level \( x \) is \( \beta_1 \exp(\beta_2 x) \). The sign of \( \beta_1 \) determines whether the degradation is increasing or decreasing over time. A positive value of \( \beta_1 \) corresponds to increasing degradation and a negative value of \( \beta_1 \) corresponds to decreasing degradation. \( \epsilon \) is a residual deviation that describes unit-to-unit variability with \( (\epsilon/\sigma) \sim \Phi(z) \), where \( \Phi(z) \) is a completely specified cumulative distribution function (cdf). For example, one could use \( \Phi_{\text{nor}}(z) \) for the standardized normal cdf, or \( \Phi_{\text{sev}}(z) \) for the standardized smallest extreme value cdf. The model in (4.1) is nonlinear with respect to the parameters \( \theta = (\beta_0, \beta_1, \beta_2, \sigma) \), and the elements of \( \theta \) are fixed but unknown.
4.2.2 Degradation Distribution and Failure-Time Distribution

At a given test condition of running time and accelerating variable level, the cdf for the transformed degradation $Y$ is

$$F_Y(y; \tau, x) = \Pr(Y \leq y; \tau, x) = \Phi \left[ \frac{y - \mu(\tau, x)}{\sigma} \right],$$

where $\mu(\tau, x) = \beta_0 + \beta_1 \exp(\beta_2 x) \tau$.

For applications where performance degrades gradually as usage time increases, failure time, $T$ is defined as the time when the degradation level reaches a specified critical level. This is known as a “soft failure” (see Section 13.4 of Meeker and Escobar 1998). Let $y_f$ denote the transformed critical level for the degradation distribution at which failure is assumed to occur.

For a decreasing degradation (i.e., when $\beta_1$ is negative), the event that the failure time $T$ is less than $t$ ($T \leq t$), is equivalent to the event that the observed transformed degradation $Y$ is less than the transformed critical level $y_f$ ($Y \leq y_f$) at time $t$. Then the failure time cdf is

$$F_T(t; x) = \Pr(T \leq t) = \Pr(Y \leq y_f) = F_Y(y_f; \tau, x) = \Phi \left[ \frac{y_f - \mu(\tau, x)}{\sigma} \right], \quad \text{for } t \geq 0 \quad (4.2)$$

where

$$\nu = \frac{(\beta_0 - y_f) \exp(-\beta_2 x)}{|\beta_1|} \quad \text{and} \quad \varsigma = \frac{\sigma \exp(-\beta_2 x)}{|\beta_1|}.$$  

For increasing degradation (i.e., when $\beta_1$ is positive), the derivation of the failure-time cdf is similar.

Let $h_t()$ denote the monotone increasing transformation function for time. That is, $\tau = h_t(t)$. From (4.2), the $p$ quantile of the failure-time distribution for decreasing degradation is

$$t_p = \begin{cases} 
  h_t^{-1} [\nu + \varsigma \Phi^{-1}(p)] & \text{if } p \geq \Phi(-\nu/\varsigma) \\
  0 & \text{otherwise}.
\end{cases}$$

4.2.3 Reparametrization of the Model for Prior Distribution Specification

In our numerical computation for either estimation or test planning, we use an alternative set of “stable” parameters (as defined by Ross 1990). This reparameterization breaks the otherwise strong correlations between some pairs of estimators. It also speeds convergence of the estimation algorithms (both ML and MCMC). Another important advantage of the new parameters is that they have meaningful interpretations. This makes it easier to elicit marginal prior distributions from the engineers working in the area.
Let \( \bar{x} \) denote the mean of the accelerating variable and let \( \bar{\tau} \) denote the average transformed time. Then the model (4.1) can be reparameterized as

\[
Y = \gamma_0 + \gamma_1 \left\{ \exp \left[ (x - \bar{x}) \gamma_2 \right] \tau - \bar{\tau} \right\} + \epsilon,
\]

where \( \gamma_0 \) is the intercept of the average accelerating line (i.e., degradation line for \( \bar{x} \)) at \( \bar{\tau} \); \( \gamma_1 \) is the slope of the average accelerating line; and \( \gamma_2 \) is the regression coefficient corresponding to the \( x \) variable. The vector \( \varphi = (\gamma_0, \gamma_1, \gamma_2, \sigma) \) denotes the stable parameters.

4.3 Bayesian ADDT Planning

4.3.1 Fisher Information Matrix

We denote a test condition in terms of transformed time \( \tau \) and transformed accelerating variable \( x \) by \( v = (\tau, x) \). An ADDT plan, denoted by \( \xi \), will specify a set of test conditions, \( v_i \), and the corresponding proportional allocation \( \pi_i \) of test units at each condition. A test plan \( \xi \) with \( r \) test conditions is denoted by

\[
\xi = \begin{bmatrix}
v_1, & \pi_1 \\
v_2, & \pi_2 \\
\vdots & \vdots \\
v_r, & \pi_r
\end{bmatrix},
\]

where \( \pi_i > 0 \) and \( \sum_{i=1}^{r} \pi_i = 1 \).

Let \( L_i(\varphi) \) denote the likelihood of a single observation at test condition \( v_i = (\tau_i, x_i) \). Then

\[
L_i(\varphi) = \frac{1}{\sigma} \phi\left( \frac{Y_i - \mu_i}{\sigma} \right), \quad \mu_i = \gamma_0 + \gamma_1 \left\{ \exp \left[ (x_i - \bar{x}) \gamma_2 \right] \tau_i - \bar{\tau} \right\}
\]

and \( \phi(z) \) is a standardized pdf. It can be shown that, under the stable parametrization \( \varphi \), the Fisher information for test plan \( \xi \) is

\[
I_{\varphi}(\xi) = n \sum_{i=1}^{r} \pi_i E \left[ -\frac{\partial^2 L_i}{\partial \varphi \partial \varphi'} \right] = \frac{n}{\sigma^2} \sum_{i=1}^{r} \pi_i F_i,
\]

where \( L_i = \log[L_i(\varphi)] \), \( n \) is the total sample size, and

\[
F_i = \begin{bmatrix}
f_{11} u_i u'_i & f_{12} u_i \\
f_{12} u'_i & f_{22}
\end{bmatrix}
\]
is the scaled Fisher information for a single unit at \( v_i \). The vector \( u_i \) contains the partial derivatives of the degradation \( \mu_i \) with respect to the parameters \( \gamma = (\gamma_0, \gamma_1, \gamma_2) \) and can be expressed as

\[
\begin{bmatrix}
\frac{\partial \mu_i}{\partial \gamma_0} \\
\frac{\partial \mu_i}{\partial \gamma_1} \\
\frac{\partial \mu_i}{\partial \gamma_2}
\end{bmatrix}
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2
\end{bmatrix}
\begin{bmatrix}
\exp[\gamma_2(x_i - \bar{x})] \tau_i - \bar{\tau} \\
\gamma_1 (x_i - \bar{x}) \exp[\gamma_2(x_i - \bar{x})] \tau_i
\end{bmatrix}
\]

The basic elements \( f_{11}, f_{12}, f_{22} \) can be computed using the LSINF algorithm (see Escobar and Meeker 1994). For the special case of a normal distribution and no censoring, the basic elements are \( f_{11} = 1, f_{12} = 0, f_{22} = 2 \).

### 4.3.2 Bayesian Planning Criterion

The appropriate criterion for test planning depends on the goal of the experiment. To plan an efficient experiment, one should specify a utility function reflecting the purpose of the experiment and then select a test plan that maximizes the expected utility. In most reliability applications, the objective is to estimate a particular quantile, such as the \( p \) quantile, of the failure-time distribution at use conditions, say, \( t_p \). Because \( h_t(t_p) \) is a monotone increasing function of \( t_p \), we can use a Bayesian ADDT criterion based on the estimation precision of \( h_t(t_p) = c^t \phi \), where \( c = \partial h_t(t_p) / \partial \phi \). For decreasing degradation, the elements of \( c \) are

\[
\begin{align*}
\frac{\partial h_t(t_p)}{\partial \gamma_0} &= -\frac{1}{\gamma_1 \exp[\gamma_2(x - \bar{x})]} \\
\frac{\partial h_t(t_p)}{\partial \gamma_1} &= \frac{1}{\gamma_1} \left[ \frac{\tau}{\exp[\gamma_2(x - \bar{x})]} - h_t(t_p) \right] \\
\frac{\partial h_t(t_p)}{\partial \gamma_2} &= -(x - \bar{x}) h_t(t_p) \\
\frac{\partial h_t(t_p)}{\partial \sigma} &= -\frac{\Phi^{-1}(p)}{\gamma_1 \exp[\gamma_2(x - \bar{x})]} 
\end{align*}
\]

where \((\bar{x}, \bar{\tau})\) are as defined in Section 4.2.3. For increasing degradation, the elements of \( c \) are the same as those for decreasing degradation except that

\[
\frac{\partial h_t(t_p)}{\partial \sigma} = -\frac{\Phi^{-1}(1-p)}{\gamma_1 \exp[\gamma_2(x - \bar{x})]}.
\]

We define the utility function as minus the posterior variance of \( h_t(t_p) \). An optimum ADDT plan maximizes this utility function. Since the posterior variance depends on the unobserved data, a marginal expectation of the posterior variance over all possible data can be used as an objective function under a Bayesian test planning criterion. Generally, approximations must be used for the posterior variance because the exact expected utility is, in general, a complicated integral that has no closed form and that is computationally intractable. When sample sizes are reasonably large, the posterior variance can be expressed as a simple function of information from the prior distribution and the data. Let \( S \) denote
the variance-covariance matrix of the prior distribution for $\varphi$. Then $S^{-1}$ is the prior precision matrix for $\varphi$. Let $p(\varphi)$ denote the joint prior distribution for $\varphi$. Following the general approach in Zhang and Meeker (2006), for large-sample approximations, the Bayesian test planning criterion is to find a test plan $\xi$ that maximizes the objective function

$$
\Psi(\xi) = -\int c^t [S^{-1} + I_\varphi(\xi)]^{-1} c p(\varphi) \, d\varphi.
$$

(4.3)

A similar approximation was used in Clyde, Müller, and Parmigiani (1995).

To estimate $t_p$ precisely, the confidence interval for $t_p$ should be as narrow as possible. Denote the ML estimate of $t_p$ by $\hat{t}_p$. An approximate 100(1 - $\alpha$)% confidence interval for $\log(t_p)$ is

$$
\log(\hat{t}_p) \pm z_{(1-\alpha/2)} \sqrt{\text{Var} \left[ \log(\hat{t}_p) \right]} = \log(\hat{t}_p) \pm \log(\hat{R}).
$$

Exponentiation yields an approximate confidence interval for $t_p$,

$$
[\hat{t}_p/\hat{R}, \hat{t}_p \hat{R}]
$$

where

$$
\hat{R} = \exp \left[ z_{(1-\alpha/2)} \sqrt{\text{Var} \left[ \log(\hat{t}_p) \right]} \right].
$$

(4.4)

For Bayesian test planning, the estimated variance $\text{Var} \left[ \log(\hat{t}_p) \right]$ in (4.4) is replaced by the large-sample approximation of the expected posterior variance of $\log(t_p)$. Similar to deriving the objective function in (4.3), the large-sample approximation of the expected posterior variance of $\log(t_p)$ can be expressed as

$$
\int \frac{1}{\hat{t}_p} \left( \frac{\partial t_p}{\partial h(t_p)} \right)^2 c^t [S^{-1} + I_\varphi(\xi)]^{-1} c p(\varphi) \, d\varphi.
$$

This gives

$$
R = \exp \left[ z_{(1-\alpha/2)} \sqrt{\int \frac{1}{\hat{t}_p} \left( \frac{\partial t_p}{\partial h(t_p)} \right)^2 c^t [S^{-1} + I_\varphi(\xi)]^{-1} c p(\varphi) \, d\varphi} \right].
$$

We call $R$ the “precision factor.” The upper (lower) endpoint of the confidence interval for $t_p$ is approximately 100($R - 1$)% larger (smaller) than the ML estimate $\hat{t}_p$. Minimizing the $R$ precision factor is equivalent to maximizing the objective function in (4.3). Because $R$ is easier to interpret as a measure of precision for a positive parameter $t_p$, we can use it for the comparisons among different Bayesian ADDT plans.
4.3.3 The Prior Distribution

Prior distributions for the parameters can be obtained from engineering judgement, previous experiments and past data. In Bayesian experimental design, it is often necessary to specify two different prior distributions:

- The prior distribution to be used to design the experiment,
- The prior distribution to be used in the inference.

Some papers, for example, Tsutakawa (1972) and Etzione and Kadane (1993), have discussed the need to use different prior distributions for the design and for the inference. One motivation for this need is that the risk of those conducting the experiment is different from that those who are concerned with the accuracy of the inference from the experiment. This idea can be seen from the objective function (4.3), in which the precision matrix $S^{-1}$ quantifies the prior information for the inference, and $p(\varphi)$ represents the prior distribution for the design. Generally, the prior distribution to be used to design the experiment must be informative for all parameters. An experimenter may, however, prefer to use a non-informative prior for the inference by having $S^{-1}$ be identically zero, as was done in Chaloner and Larntz (1989).

We will explore several different combinations of prior distributions in doing Bayesian ADDT planning. In particular, we will use a point-mass prior $p_0(\varphi)$, an informative prior for all parameters $p_1(\varphi)$, an informative prior for partial parameters $p_2(\varphi)$, and a non-informative prior $p_3(\varphi)$. Table 4.1 summarizes different cases that we will use for test planning in terms of the specification of the prior distribution for the design $p(\varphi)$ and for the inference $S^{-1}$ separately. As mentioned above, the non-informative prior $p_3(\varphi)$ for the inference is implemented by setting $S^{-1} = 0$. Note that informative prior distributions are used for the design in all cases. This is because that test planning criteria are highly sensitive to the particular form of any diffuse prior for the design. Some information about the model parameters is required in order to obtain sensible test planning results.

<table>
<thead>
<tr>
<th>Case</th>
<th>Design $p(\varphi)$</th>
<th>Inference $S^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>point-mass $p_0(\varphi)$</td>
<td>non-informative $p_3(\varphi)$</td>
</tr>
<tr>
<td>B</td>
<td>informative for all parameters $p_1(\varphi)$</td>
<td>non-informative $p_3(\varphi)$</td>
</tr>
<tr>
<td>C</td>
<td>informative for all parameters $p_1(\varphi)$</td>
<td>informative for partial parameters $p_2(\varphi)$</td>
</tr>
<tr>
<td>D</td>
<td>informative for all parameters $p_1(\varphi)$</td>
<td>informative for all parameters $p_1(\varphi)$</td>
</tr>
</tbody>
</table>
4.3.4 General Equivalence Theorem

Whittle’s (1973) general equivalence theorem (GET) can be used to check the optimality of test plans. The outputs for an application of the GET can also suggest that an optimum plan is unique or not or suggest why a given plan is not optimum. We will use the GET to check the optimality of the test plans that we find.

The directional derivative, $\Lambda$, of $\Psi$ at $\xi$ and in the direction of an alternative plan $\eta$ is defined as

$$
\Lambda(\xi, \eta) = \lim_{\delta \to 0^+} \frac{\Psi[(1 - \delta)\xi + \delta \eta] - \Psi(\xi)}{\delta}.
$$

In Bayesian ADDT planning, the derivative function of (4.3) at $\xi$ can be derived as

$$
\Lambda(\xi, \eta) = \int c' V(\varphi, \xi)V(\varphi, \eta)^{-1} V(\varphi, \xi)c p(\varphi) d\varphi + \Psi(\xi),
$$

(4.5)

where $V(\varphi, \xi) = [S^{-1} + I_{\varphi}(\xi)]^{-1}$. Let $\xi_v$ be a singular test plan that puts all units at a single test condition $v$. Suppose that a given ADDT plan $\xi$ has $r$ test conditions, $v_1, v_2, \ldots, v_r$. Then this plan $\xi$ is an optimum plan for the Bayesian criterion if it satisfies $\Lambda(\xi, \xi_v) = \Lambda(\xi, \xi_{v_1}) = \ldots = \Lambda(\xi, \xi_{v_r}) = 0$ and $\Lambda(\xi, \xi_v) \leq 0$ for any other singular plan $\xi_v$ in the experimental region.

4.4 Numerical Example

In this section, we use the adhesive bond B application in Shi, Escobar, and Meeker (2009) to illustrate the Bayesian ADDT planning methods.

4.4.1 The Application

Adhesive bond B was to be evaluated for use in manufacturing an inkjet cartridge. The adhesive is used to bond a protective coating to protect the printhead of the inkjet cartridge. When the adhesive becomes weak, there can be delamination and ink can attack the electronics in the printhead, causing failure. The degradation response is the strength (in Newtons) of the adhesive bond over time. There was special interest in estimating the time at which 1% of the product would have a strength below 40 Newtons when operating at the use condition of 25 °C (i.e., the 0.01 quantile of the failure-time distribution). For this application, the accelerating variable is temperature, and the degradation strength
model is as given in (4.1) with

\[ Y = \log(\text{Strength in Newtons}) \]
\[ \tau = \sqrt{\text{Time in Weeks}} \]
\[ x = \frac{-11605}{\text{Temperature in } ^\circ\text{C} + 273.15} \]
\[ (\epsilon/\sigma) \sim \Phi_{\text{nor}}(z). \]

Figure 4.1 provides a visualization of the degradation distributions as a function of time at 25 °C for specific values of the parameters  \( \beta_0, \beta_1, \beta_2, \) and \( \sigma. \) The strength axis is a logarithmic axis and the time axis is a square root axis, corresponding to model assumptions that imply linear degradation in these scales. The horizontal line at 40 Newtons is the failure-definition degradation level for this application. At each point in time with a vertical line, the shaded area below the horizontal line is the failure probability at that time.

![Figure 4.1](image)

Figure 4.1 Adhesive Bond B degradation distributions as a function of time at 25 °C.

The original ADDT plan for this application used 88 test units. As a baseline, 8 units with no aging were measured at the start of the experiment. A total of 80 additional units were aged and measured according to the temperature and time schedule in Table 4.2.

For the reparameterization to stable parameters, the centroid of the accelerating variable \( \bar{x} \) and the average transformed time \( \bar{\tau} \) are obtained based on this original test plan. In particular, we use \( \bar{x} = \sum \pi_i x_i \) and \( \bar{\tau} = \sum \pi_i \tau_i \), where \( \pi_i \) is the proportional allocation at test condition \((\tau_i, x_i)\). The numerical values for this centroid are \( \bar{x} = -34.833 \) and \( \bar{\tau} = 2.455 \).
Table 4.2 Original ADDT plan. The — indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Weeks</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>—</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>60</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Totals</td>
<td>8</td>
<td>20</td>
</tr>
</tbody>
</table>

4.4.2 Specification of the Prior Distribution

For this application, parameter $\gamma_2$ can be interpreted as an effective activation energy. Given previous experience with a failure mechanism, engineers often have useful prior information for this parameter. Prior distributions for the other three parameters $\gamma_0$, $\gamma_1$, and $\sigma$ could be obtained from previous experiments and past data (e.g., Escobar et al. 2003). This application has decreasing degradation so the degradation slope $\gamma_1$ is negative. The effective activation energy $\gamma_2$ and standard deviation of the residuals $\sigma$ are positive parameters. Hence, the uncertainty for the four parameters ($\exp(\gamma_0), -\gamma_1, \gamma_2, \sigma$) can be described by an independent multivariate lognormal distribution with specified 0.01 and 0.99 quantiles (any particular quantiles could be used, but we find these particular values to be useful in eliciting prior information from engineers). Due to the prior specification issues mentioned in Section 4.3.3, we will illustrate Bayesian ADDT planning by specifying four possible prior distributions [i.e., a point-mass $p_0(\varphi)$, informative for all parameters $p_1(\varphi)$, informative for parameter $\gamma_2$ only $p_2(\varphi)$, and a non-informative prior $p_3(\varphi)$ (implemented by setting $S^{-1} = 0$)].

**Point-mass Prior Distribution** $p_0(\varphi)$: Shi, Escobar, and Meeker (2009) describe non-Bayesian methods of finding ADDT plans for the same adhesive bond B application. The locally optimum test plans developed in that paper require the specification of planning values for the model parameters. The planning values of the parameters used there will be used to specify a point-mass prior distribution for the parameters. This will allow us to compare, directly, the non-Bayesian and Bayesian test plans. A point-mass prior distribution is assumed to be highly informative centered around the planning values. Thus the point-mass prior distribution can be approximately specified by normal distributions with the mean at the planning values and a small standard deviation. For this application, the approximate point-mass prior distribution is $\gamma_0 \sim N(3.97, 0.002), \log(-\gamma_1) \sim N(1.59, 0.002), \log(\gamma_2) \sim N(-0.45, 0.002)$,
and $\log(\sigma) \sim N(-1.84, 0.002)$.

**Informative Prior Distribution** $p_1(\varphi)$: Table 4.3 summarizes the independent multivariate log-normal distribution for parameters ($\exp(\gamma_0), -\gamma_1, \gamma_2, \sigma$) and the corresponding log-location scale hyper-parameters for the informative prior information.

<table>
<thead>
<tr>
<th>Prior specification</th>
<th>Hyperparameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>0.01 quantile</td>
</tr>
<tr>
<td>$\exp(\gamma_0)$</td>
<td>51</td>
</tr>
<tr>
<td>$-\gamma_1$</td>
<td>0.15</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.55</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**Partial Informative Prior Distribution** $p_2(\varphi)$: For this application, engineers often have access to highly informative prior information for the effective activation energy (i.e., parameter $\gamma_2$ in our model). Often there is strong information about this parameter, based on previous experience and knowledge of the physics or chemistry of the failure mechanism (indeed, in some applications of accelerated testing, the effective activation energy is assumed to be known). For other parameters, often, the prior information is limited. Table 4.4 summarizes the marginal lognormal distribution for the parameters ($\exp(\gamma_0), -\gamma_1, \gamma_2, \sigma$) and their corresponding log-location scale hyperparameters for the partial informative prior information.

<table>
<thead>
<tr>
<th>Prior specification</th>
<th>Hyperparameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>0.01 quantile</td>
</tr>
<tr>
<td>$\exp(\gamma_0)$</td>
<td>40</td>
</tr>
<tr>
<td>$-\gamma_1$</td>
<td>0.05</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>0.55</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.05</td>
</tr>
</tbody>
</table>
4.4.3 Bayesian Optimum Test Plans

There are usually practical constraints for test planning. For the adhesive bond B application, the constraints are:

- 88 test units available for the sample size,
- 70 °C is the maximum temperature,
- 16 weeks is maximum available time for testing.

After the specification of the prior distributions, we can explore Bayesian test planning for the different cases listed in Table 4.1. Optimum test plans obtained under each situation in the following part are all continuous test plans (a continuous test plan is one that has non-integer allocations because optimization was done without integer constraints on the number of units allocated to the test conditions).

**Case A:** In nonlinear models, the non-Bayesian optimum plans are expected to be special cases of Bayesian optimum plans which correspond to point-mass prior distributions for the design and non-informative prior distributions for the inference. Shi, Escobar, and Meeker (2009) present a non-Bayesian optimum ADDT plan structure in terms of transformed accelerating variable $x$ and transformed time $\tau$, as shown in Figure 4.2.

![Figure 4.2 Optimum plan structure.](image-url)
For Bayesian test planning with a point-mass prior distribution, we explore optimum plans following the same structure as Figure 4.2. $x_M$ and $\tau_M$ are the maximum transformed accelerating variable and transformed time, respectively. Because (4.1) has three parameters, there are expected to be three test conditions for an optimum plan. The three test conditions include a baseline condition $v_1^*$, a highest stress test condition $v_2^*$ at $x_M$ and $\tau_M$, and a right boundary test condition $v_3^*$ at $\tau_M$. The variable $x^*$ for the condition $v_3^*$ and two proportional allocations $\pi_1^*$, $\pi_2^*$ are optimized to maximize the objective function in (4.3). For this case, the resulting Bayesian optimum plan $\xi^*$ is shown in Table 4.5. As expected, this optimum plan from Bayesian test planning with a point-mass prior for the design and a non-informative prior for the inference is the same as the one obtained from non-Bayesian test planning methods in Shi, Escobar and Meeker (2009). As from (4.3), when the prior information for the inference $S^{-1}$ is 0, the absolute value of the objective function is inversely proportional to the sample size $n$ which is reflected in $I_{\phi}(\xi)$. Under the sample size of $n = 88$, the objective function of this optimum plan, $\Psi(\xi^*)$, is $-20.43$, and the $R$ precision factor is 1.907. The optimality of this Bayesian test plan can be verified using the GET, as described in Section 4.3.4. The plot of the directional derivatives $\Lambda(\xi^*, \xi_0)$ is the same as the one obtained from the non-Bayesian method in Shi, Escobar and Meeker (2009), and is omitted here to avoid redundancy.

<table>
<thead>
<tr>
<th>Optimum Test Condition</th>
<th>Weeks</th>
<th>Temperature $^\circ$C</th>
<th>Proportional Allocations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>—</td>
<td>0.203</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.162</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>54.765</td>
<td>0.635</td>
</tr>
</tbody>
</table>

**Case B:** Following the same steps used in case A, a Bayesian optimum plan $\xi^*$ for case B is shown in Table 4.6. This optimum test plan is close to the one obtained in case A. It verifies the conclusion from Chaloner and Larntz (1989), that for prior distributions with support over a small region, Bayesian optimum plans are similar to non-Bayesian optimum plans. Again for this case, the absolute value of the objective function is inversely proportional to the sample size. Under the sample size of 88, the objective function of this optimum plan, $\Psi(\xi^*)$, is $-24.07$, and the $R$ precision factor is 1.881.
Table 4.6: A Bayesian optimum ADDT plan $\xi^*$ with an informative prior for the design and a non-informative prior for the inference. The $-$ indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Optimum Test Condition</th>
<th>Weeks</th>
<th>Temperature °C</th>
<th>Proportional Allocations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>$-$</td>
<td>0.213</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.162</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>55.331</td>
<td>0.625</td>
</tr>
</tbody>
</table>

**Cases C and D:** For these two cases, we incorporate some prior information for inference by specifying an informative $S^{-1}$. From the objective function (4.3), we can see that the sample size $n$ reflected in $I_\phi(\xi)$, relative to the amount of prior information, plays a role in the posterior distribution and test planning. When the sample size is large, the posterior distribution will tend to be driven by the data and will not be sensitive to the prior distribution for the inference. In contrast, when the sample size is small, the prior distribution will have more effect on both the posterior distribution and the design. Hence, we investigate Bayesian optimum plans under two different sample sizes: a small sample size $n = 88$ and a large one $n = 300$. As before, we explore optimum plans following the structure in Figure 4.2. Tables 4.7 and 4.8 show test conditions, the values of the objective function, and the $R$ precision factors for the optimum plans of two cases under sample sizes 88 and 300, respectively. Note that the first two columns in both tables are common for the two cases. The optimality of these test plans can be verified using the GET in the same way as was done above. Figure 4.3 shows a plot of the directional derivatives $\Lambda(\xi^*, \xi_v)$ for case D with $n = 88$. The shapes of the directional derivatives plots for the other cases are similar except for magnitude changes in the values.

From Tables 4.7 and 4.8, we can see that, with a large sample size, the prior distribution for the inference will not strongly influence Bayesian test planning, as compared to test plans using a small sample size. As the sample size becomes larger and larger, the prior information becomes less influential in the inference, and the resulting optimum plans should approach a plan for which the available prior information is to be used in test planning but not for the inference (i.e., the test plan obtained in Case B).

For all of the cases mentioned above, alternative optimum plans exist. As seen from Figure 4.3, the directional derivatives $\Lambda(\xi^*, \xi_v)$ are all zero when the alternative singular plan $\xi_v$ puts all the test units at a test condition with 70 °C as the temperature level. This indicates the existence of alternative
Table 4.7 Bayesian Optimum ADDT plans $\xi^*$ for cases C and D under sample size of $n = 88$. The — indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Weeks</th>
<th>$\degree C$</th>
<th>$\pi^*$</th>
<th>$\Psi(\xi^*)$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>—</td>
<td>0.159</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.143</td>
<td>-15.66</td>
<td>1.664</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>55.061</td>
<td>0.698</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

All cases

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Weeks</th>
<th>$\degree C$</th>
<th>$\pi^*$</th>
<th>$\Psi(\xi^*)$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>—</td>
<td>0.199</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.156</td>
<td>-6.00</td>
<td>1.371</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>55.009</td>
<td>0.645</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 4.8 Bayesian Optimum ADDT plans $\xi^*$ for cases C and D under sample size of $n = 300$. The — indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Weeks</th>
<th>$\degree C$</th>
<th>$\pi^*$</th>
<th>$\Psi(\xi^*)$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1^*$</td>
<td>0</td>
<td>—</td>
<td>0.199</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>$v_2^*$</td>
<td>16</td>
<td>70</td>
<td>0.174</td>
<td>-5.79</td>
<td>1.362</td>
</tr>
<tr>
<td>$v_3^*$</td>
<td>16</td>
<td>55.315</td>
<td>0.622</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

optimum test plans that can be obtained by moving the test condition $v_2^*$ along the upper temperature boundary in Figure 4.2. Using the plan specification notation in Section 4.3.1, an alternative optimum plan $\xi^a$ can be expressed in terms of $\pi_1^a, \pi_2^a, x^*$ of the initial optimum plan $\xi^*$, which is given as

$$
\xi^a = \begin{bmatrix}
\psi_1 = (0, -), & \pi_1 = \pi_1^* + \pi_2^* - \frac{\tau_M}{\tau_a} \\
\psi_2 = (r_a, x_M), & \pi_2 = \frac{\tau_M}{\tau_a} \\
\psi_3 = (\tau_M, x^*), & \pi_3 = \pi_3^*
\end{bmatrix},
$$

(4.6)

where $\tau_L \leq \tau_a \leq \tau_M$, and $\tau_L = \tau_M \pi_2^* / (\pi_1^* + \pi_2^*)$. The optimality of alternative plans can be proved in a way that is similar to the proof given in Appendix B.2 of Shi, Esboar and Meeker (2009), and is omitted here.
4.4.4 Bayesian Optimized Compromise Test Plans

Optimum plans have some disadvantages. For example, they tend to be highly sensitive to model specification errors and such plans provide little or no information about departures from the acceleration model. For this reason, it has been suggested (e.g., Chapter 6 of Nelson 1990, and Chapter 20 of Meeker and Escobar 1998) to construct compromise test plans that tend to be more robust and practical. An optimum plan can usually provide useful insight for obtaining good compromise test plans.

For the adhesive bond B application, Shi, Escobar, and Meeker (2009) propose an optimized compromise plan for non-Bayesian test planning. The idea there can also be used to find Bayesian optimized compromise plans. For the compromise plan, we allocate some test units at the baseline conditions and an equal proportion of units at each of nine additional equally-spaced test conditions. The nine equally-spaced test conditions use three fixed time levels at 12, 14, and 16 weeks and a fixed highest temperature level at 70 °C. The lowest temperature level is chosen to maximize the objective function \( \Psi(\xi) \) in (4.3). The middle temperature level is the mean of the other two temperature levels. For case
D, the informative prior distribution \( p_1(\varphi) \) is used for both the design and for inference. With sample size of \( n = 88 \), after rounding in the allocations, the compromise plan has 7 units at the baseline and 9 units at each of the other nine test conditions. The optimized lowest temperature level is 53.2 °C and the middle temperature level is 61.6 °C. This Bayesian optimized compromise plan is presented in Table 4.9. The objective function \( \Psi(\xi) \) for this compromise plan is \(-16.46\). And the \( R \) precision factor for this plan is 1.676, compared with 1.617 for the corresponding optimum plan under case D with sample size \( n = 88 \), suggesting there is little loss in the estimation precision. For other cases listed in Table 4.1, similar Bayesian optimized compromise plans can be found in the same way but the details are not given here.

**Table 4.9** Bayesian optimized compromise ADDT plan for case D under sample size of \( n = 88 \). The — indicates that at time 0, the level of temperature has no effect on the model.

<table>
<thead>
<tr>
<th>Temperature</th>
<th>Weeks</th>
<th>Totals</th>
</tr>
</thead>
<tbody>
<tr>
<td>°C</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>—</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>53.2</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>61.6</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>70</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>Totals</td>
<td>7</td>
<td>27</td>
</tr>
</tbody>
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### 4.5 Conclusions and Areas for Future Research

Planning ADDTs with prior information is useful for making reliability inferences in practical applications. In this paper, we present Bayesian test planning methods for ADDT problems under an important class of nonlinear regression models when prior information is available on the model parameters. We use a Bayesian criterion based on the estimation precision of a failure-time distribution quantile at use conditions. A large-sample approximation provides a useful simplification for the posterior distribution. The GET is an important tool to verify that the numerically optimized plans are globally or near-globally optimum. We also examine the effects of changing the amount of prior information and sample size on doing Bayesian test planning.

The Bayesian methods illustrated in this paper can be extended to the ADDT planning problems with more complicated degradation models, such as models with multiple accelerating variables (e.g.,
temperature, humidity) or nonlinear relationships between degradation and time. For some products, there may be more than one failure mechanism. This can cause degradation observations to be right-censored, as described in Escobar et al. (2003). The statistical competing risk model (see David and Moeschberger 1978) can be used as the degradation model for such applications. Bayesian test planning methods could be used for ADDT problems with such competing risk models. In addition, Monte Carlo simulation methods could complement the results obtained from the large-sample approximation approach used in this paper. Such simulations are particularly useful for providing visualization of sampling variability resulting from different test plans, as illustrated in Chapter 20 of Meeker and Escobar (1998).

References


CHAPTER 5. GENERAL CONCLUSIONS

5.1 General Discussion

Accelerated destructive degradation testing is an important tool for making reliability inferences and predictions, especially when test time is limited and few or no failures are expected at normal use conditions. Good ADDT plans can yield significant benefits to industry.

In Chapter 2, we described methods to find good ADDT plans for an important class of destructive degradation models when there is only one type of failure cause for applications. In this chapter, a collection of optimum plans was first derived, which can minimize the large-sample approximate variance of the maximum likelihood (ML) estimator of a specified quantile of the failure-time distribution. The general equivalence theorem (GET) was used to verify the optimality of these plans. Then we proposed a compromise plan which tended to be more robust to the model specification and the planning information. We also did sensitivity analyses for both optimum plans and compromise plans to study the effects that changes in sample size, time duration of the experiment, levels of the accelerating variable, and misspecification of the planning information had on the precision of the ML estimator of a specified quantile. At last, we used Monte Carlo simulations to evaluate the statistical characteristics of different ADDT plans.

In Chapter 3, we described methods to find unconstrained and constrained optimum test plans for competing risk applications under a given test optimization criterion, such as minimizing the large-sample approximate variance of a failure-time distribution quantile at use conditions. A modified general equivalence theorem (GET) was used to verify the optimality of a given ADDT plan. And, we constructed a good compromise test plan which was more robust and practical. Finally, Monte Carlo simulations were used to provide visualization of the results that might be obtained from given test plans.

In Chapter 4, we described Bayesian methods for ADDT planning under a class of nonlinear degradation models with one accelerating variable. We used a Bayesian criterion based on the estimation precision of a specified failure-time distribution quantile at use conditions to find optimum test plans.
A large-sample approximation for the posterior distribution provided a useful simplification to the test planning criterion. The general equivalence theorem (GET) was used to verify the global optimality of the numerically optimized Bayesian test plans. Again, a more robust and practically useful compromise plans was constructed. We also investigated the effects of the prior distribution and sample size on Bayesian test planning results.

### 5.2 Recommendation for Future Research

The methodology that we presented in this research study is general and can be extended in several important directions, suggesting areas for future research. These include the following:

1. Test plans for an accelerated destructive degradation model with multiple accelerating variables (e.g., temperature and humidity) could be developed.

2. In this study, for the degradation model, the relationship between the location parameter and the transformed time is linear at a fixed accelerating variable level. The work here could be extended to degradation models that have a nonlinear relationship (such a relationship might, for example, be suggested by a physics of failure model).

3. Bayesian test planning methods for applications with competing risks could also be studied.