REFLECTION AND SCATTERING BY A SLIGHTLY UNDULATING INTERFACE

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INTRODUCTION

Reflection and refraction of plane waves at a perfectly planar interface have been well studied in the literature. Solutions of the coefficients of reflection and refraction can be found in a variety of sources, e.g. [1]. The reflected and refracted waves have been used to develop ultrasonic non-destructive evaluation techniques to detect flaws in fiber-reinforced composite plates [2]–[4].

However, in reality most interfaces are not planar on both micro and macro scales. The purpose of this study is to investigate the interaction between elastic waves and a slightly non-planar interface. The method of asymptotic analysis is used to derive close form solutions for the problem of reflection and scattering of a time-harmonic SH-wave at a slightly, but otherwise arbitrarily, non-planar interface. The case of in-plane waves will be considered elsewhere. Both periodic and non-periodic interface profiles are considered. For simplicity, our discussion in this paper will be restricted to the interface between an elastic half-space and a rigid substrate.

Wave reflection from a traction-free surface of periodic profile has been studied by Roberts et al. [5], and Fokkema and van den Berg [6]. A soft slightly rough surface of an ideal fluid was studied recently by Stickler [7].

PERFECT PLANAR INTERFACE

Let the shear modulus of the elastic solid be \( \mu \), and the mass density be \( \rho \). In this section, we assume that the interface between the elastic half-space and the rigid substrate is a perfect flat surface. A Cartesian coordinate system \((x, y)\) is shown in Fig. 1. For time harmonic SH-waves, the only non-zero displacement component is the anti-plane motion \( u(x, y) \), where the time factor \( \exp(-i\omega t) \) has been omitted. The reduced wave equation for \( u(x, y) \) is

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0 , \quad y > 0
\]

(1)

where \( k \) is the SH-wave number given by \( k = \omega/c \), and \( c = \sqrt{\mu/\rho} \) is the phase velocity of the shear wave.

Now, let a plane wave be incident on the interface from the elastic solid

\[ u_i(x,y) = \exp[ik(x\sin\theta - y\cos\theta_i)] \tag{2} \]

where \( \theta_i \) is the angle of incident shown in Fig. 1.

For perfectly bonded interface, displacements are continuous across the interface \( y = 0 \), i.e.,

\[ u_i(x,0) + u_r(x,0) = 0 \tag{3} \]

where

\[ u_r(x,y) = -\exp[ik(x\sin\theta_i + y\cos\theta_i)] \tag{4} \]

SLIGHTLY NON–PLANAR INTERFACE

Consider an undulating interface as shown in Fig. 2, where the Cartesian coordinate system \((x,y)\) is also indicated. The shape of the interface may
arbitrarily differ from a straight line \( y = 0 \), and is described by the following curve,

\[
y = a h(x) ,
\]

where \( a \) is a constant with dimension of length and \( h(x) \) is a dimensionless continuous function with its norm less than unit, i.e.,

\[
\| h(x) \| \leq 1 .
\]

The interface undulation is considered small if the pertinent wave length \( \Lambda = 2\pi / k \) is much larger than the amplitude of the undulation, i.e.,

\[
\epsilon = a / \Lambda \ll 1 .
\]

Again, consider a plane wave incident. Obviously, due to the interface irregularity, the incident wave will be scattered, as well as reflected at the undulating interface. However, it is also conceivable that the wave field of the slightly undulating interface (where \( \epsilon \ll 1 \)) will be a small perturbation from that of a flat interface. To the first order of \( \epsilon \), the total displacement in the elastic solid may be written as

\[
u(x,y) = u_i(x,y) + u_r(x,y) + \epsilon v(x,y) ,
\]

where \( u_i(x,y) \) is the incident wave given by (2), \( u_r(x,y) \) is the wave reflected by a flat surface, which is given by (4), and \( v \) represents the perturbation of the reflected field due to the non-planar interface.

In (8), the perturbed field \( v \) is the only unknowns. It is to be determined by the wave equation (1) and the interface continuity condition,

\[
u(x,a h(x)) = 0 .
\]

Making use of the Taylor expansion gives

\[
u(x,a h(x)) = u(x,0) + a h(x) \partial u(x,y) / \partial y \bigg|_{y=0} + O(\epsilon^2)
\approx \epsilon v(x,0) + \epsilon \Lambda h(x) \partial [u_i(x,y) + u_r(x,y)] / \partial y \bigg|_{y=0} .
\]

In deriving (10b), the zero–displacement condition (3) has been used. Substitution of (10b) into (9) yields

\[
u(x,0) = 2v_0 h(x) \exp(ik x \sin \theta_i) ,
\]

where

\[
v_0 = -2\pi i \cos \theta_i .
\]

Obviously, when \( h(x) = 0 \), the interface becomes flat. In this case, \( v = 0 \) will satisfy both the governing equation (1) and the interface continuity condition (11). The solution to the flat interface is thus recovered. In what follows, the perturbation \( v \) due to non–zero \( h(x) \) will be discussed for different types of \( h(x) \).
Periodic Interface Profile

Assume that \( h(x) \) is a periodic function of \( x \). Without loss of generality, let

\[
h(x) = \cos(\gamma x)
\]

where \( \gamma \) is the wave number of the interface undulation. For convenience, we write

\[
h(x) = \frac{1}{2} [\exp(i\gamma x) + \exp(-i\gamma x)]
\]

Substitution of (14) into (11) yields

\[
v(x,0) = v_0 [\exp(izk_1) + \exp(izk_2)]
\]

where

\[
k_{1,2} = k \sin \theta_i \pm \gamma
\]

It can be shown by direct substitution that the solution to \( v \) that satisfies the governing equation (1) and the interface condition (15) is

\[
v(x,y) = v_0 \sum_{n=1}^{2} \exp[izk_n - y(\sqrt{k_n^2 - k^2})^{1/2}]
\]

The branch cut of the radical in (17) is defined as follows:

\[
(\sqrt{k_n^2 - k^2})^{1/2} = \begin{cases} \\
\sqrt{k_n^2 - k^2} & \text{for } |k_n| > k \\
-i\sqrt{k_n^2 - k^2} & \text{for } |k_n| < k 
\end{cases} 
\]

It thus follows from (17) that the directions of propagation for the reflected waves can be defined by the angles between the vectors of propagation and the \( y \)-axis. They are, respectively, given by

\[
\theta_n = \tan^{-1}[k_n/(k^2 - k_n^2)^{1/2}]
\]

where the angles are positive when measured from the \( y \)-axis counterclockwise, and negative when measured from the \( y \)-axis clockwise. The wave becomes evanescent when its angle of propagation is \( \pm 90^\circ \), or imaginary.

It is seen from (17) that there are two reflected waves. However, depending upon the values of \( \gamma/k \) and the angle of incident, one, or both of the waves may be evanescent. For example, when \( \gamma/k < 1 \), the second wave \( k_2 \) is always a propagating wave for \( 0 \leq \theta_i < 90^\circ \), while the first wave \( k_1 \) has a critical angle at \( \theta_c = \sin^{-1}(1 - \gamma/k) \). For \( \theta_i < \theta_c \) it is a propagating wave and for \( \theta_i > \theta_c \) it becomes evanescent.

Non–periodic Interface Profile

It is assumed that the non–periodic interface profile \( h(x) \) studied in this section satisfies the following requirement, namely, the integral,
exists. It can then be shown that the following integral also exists
\[ \eta(\xi) = \int_{-\infty}^{\infty} h(x) \exp(iz \sin \theta_i) \exp(i \xi x) \, dx. \] (21)

Under this assumption, the Fourier transform of \( \nu(v) \) can be taken to yield
\[ \nu^*(\xi, 0) = 2v_0 \eta(\xi), \] (22)
where \( \nu^*(\xi, y) \) is the Fourier transform of \( \nu \). By direct substitution, one can easily show that \( \nu^*(\xi, y) \) given below satisfy (22) and the transformed governing equation of motion (1),
\[ \nu^*(\xi, y) = 2v_0 \eta(\xi) \exp[-y(\xi^2 - k^2)^{1/2}]. \] (23)

In (23), the branch cut is defined as
\[ (\xi^2 - k^2)^{1/2} = \begin{cases} \sqrt{\xi^2 - k^2} & \text{when } \xi > k \\ -i\sqrt{k^2 - \xi^2} & \text{when } \xi < k \end{cases} \] (24)

By taking the inverse Fourier transform on (23), we obtain
\[ \nu(x, y) = (v_0 / \pi) \int_{-\infty}^{\infty} \eta(\xi) \exp[-ix - y(\xi^2 - k^2)^{1/2}] \, d\xi. \] (25)

Using the steepest-descent method, the integral in (25) can be carried out asymptotically for \( kr \gg 1 \),
\[ \nu(r, \theta) = -iv_0(2\pi kr)^{-1/2} \eta(k \sin \theta) \exp[i(kr + \pi/4)], \] (26)
where \( x = -r \sin \theta, \ y = r \cos \theta, \ -\pi/2 \leq \theta \leq \pi/2 \) as shown in Fig. 2.

It is interesting to notice that the far-field angular variation depend on \( \theta \) only through the transform function \( \eta(\xi) \). Different interface profile \( h(z) \) will give different \( \eta(\xi) \). The three types of interface profiles shown in Fig. 3 are investigated and their corresponding far-field angular variations are computed and plotted in Fig. 4.

Rectangular Indentation:
\[ h(z) = H(\beta - |z|) \] (27)
\[ \eta(\xi) = \sin[\beta(\xi + k \sin \theta_i)] / [\beta(\xi + k \sin \theta_i)] \] (28)

Wedge Indentation:
\[ h(z) = (1 - |z| / \beta) H(\beta - |z|) \] (29)
\[ \eta(\xi) = 2\{1 - \cos[\beta(\xi + k \sin \theta_i)] / [\beta(\xi + k \sin \theta_i)]\}^{1/2} \] (30)
Fig. 3. Simple interface profiles

\[ h(x) = H(\beta - |x|) \]

\[ h(x) = (1 - |z|/\beta)H(\beta - |z|) \]

\[ h(x) = (1 - z^2/\beta^2)^{1/2}H(\beta - |z|) \]

Fig. 4. Angular variations of the scattered far-field for different surface defects under oblique incidence, \( \theta_i = 60^\circ \). The solid line is for rectangular indentation, the dash line is for wedge indentation and the dash-dot line is for elliptical indentation.
Elliptical Indentation:

\[ h(x) = (1 - x^2/\beta^2)^{1/2}H(\beta - |x|) \]  
\[ \eta(\xi) = 2J_1[\beta(\xi + k \sin \theta_1)]/[\beta(\xi + k \sin \theta_1)] \]

(31) \quad (32)

In the above equations, \( \beta \) is a positive number with length dimension, \( H(x) \) is the Heaviside step function, and \( J_1(x) \) is the first order Bessel function of the first kind.

The angular variations of \( \eta(k \sin \theta) = \eta(k \sin \theta)/A \), where \( A \) is given by (20), are plotted in Fig. 4 for \( \theta_1 = 60^\circ \). The solid line is for the rectangular indentation, the dashed line is for the wedge indentation and the dash-dot line is for the elliptical indentation. In making the plot, \( a \) is taken to be \( ak = 0.3 \). This corresponds to \( a/\beta \approx 2 \). It is seen from Fig. 4 that the scattered far-field depends on \( \theta \) very weakly. However, strong \( \theta \) dependence is observed when \( a/\beta \) is less that one.

APPLICATIONS TO NON-DESTRUCTIVE EVALUATION

As mentioned in the introduction, the results obtained in this paper provide some necessary information for understanding wave propagation in laminated materials. This will help us in developing more realistic models of composite materials for non-destructive evaluation of laminated composites. In particular, (21) may be used to develop ultrasonic techniques for the detection of surface defects.

In principle, if the far-field angular variation of the scattered field \( v \) can be measured, i.e., \( \eta(\xi) \) is known, then, the inverse problem of finding the surface profile \( h(x) \) can be solved by taking the inverse Fourier transform of (21),

\[ h(x) = (1/2\pi)\exp(-ik \sin \theta_1) \int_{-\infty}^{\infty} \eta(\xi)\exp(-i\xi x) d\xi \]  

(33)

If only the volume (area in the 2-D case considered here) of the defects are of interest, the amplitude of the displacement \( v \) can be used, since it is proportional to the defect volume \( A \).

The feasibility and applicability of using these methods for NDE of surface defects are discussed in [8].

SUMMARY

Interaction of time harmonic SH-waves with a slightly non-planar interface is studied in this paper. Both periodic and non-periodic interface profiles are considered. The asymptotic solutions obtained in this paper are accurate to the first order of \( \epsilon \), which is the amplitude ratio of the interface undulation and incident wave length. For the periodic interface profile, the reflected wave fields are obtained and the critical angles at which propagating waves become evanescent waves are identified. For non-periodic interface profiles, emphasis is given to interface irregularities of integrable type, i.e., the interface is flat everywhere except in a finite interval \( |x| \leq \beta \). In this case, the far-field contains a plane wave reflected from a planar surface and a cylindrical wave scattered by the surface irregularity. For very small \( \epsilon \), the scattered far-field is second order to the spectral reflection. Angular variations of the scattered far-field are obtained by the steepest-descent method. Applications of the results to non-destructive evaluation of surface defects are discussed. Three profiles representing some typical surface defects are considered in detail.
ACKNOWLEDGEMENTS

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REFERENCES

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