ITERATIVE BOUNDARY INTEGRAL SOLUTION
FOR CURVED INTERFACE TRANSMISSION

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INTRODUCTION

Ultrasonic transmission through a moderately curved fluid-solid interface is a problem of obvious practical interest to QNDE. Extreme surface curvatures result in an erratic transmitted beam geometry ill-suited for a measurement probe. However, in many (if not the majority) of cases, a component surface has a moderate machined or cast curvature which distorts the beam in a well-behaved manner, making the prospect of an ultrasonic inspection viable. The quantitative utility of such an inspection will depend on the ability to model this beam distortion.

If the incident beam avoids all critical transmission angles at all points on the moderately curved surface, simple ray theory will likely suffice as an analytical tool. If the incident beam impinges at some point on the surface near the critical angle for compressional or shear waves, simple ray theory will not be adequate for the modeling of surface wave motions which can occur. For this purpose, ray theory must be augmented by the geometrical theory of diffraction (GTD). In practice, application of GTD can be cumbersome, ad-hoc, and in certain situations, incomplete. For this reason, other methods of solution are being assessed.

An alternative analytical tool is the Green function - boundary integral formulation. In this approach, a singular integral equation governs the transmission phenomena. Conventional methods of solving such an equation represent the solution in a discrete basis, thereby transforming the problem into a linear matrix equation. The difficulty with this approach is that, in the problem at hand, the resulting matrix equation can be too large to explicitly invert using standard algorithms. For example, transmission of a 0.5 in. dia., 10 MHz beam with collocation points at 0.5 wavelengths in water results in a 115600x115600 complex-valued scattering matrix. Double precision storage (a must for inversion of a matrix this large) will require 200 gigabytes. A primary goal of this work is to invert the integral equation without retaining a large scattering matrix. An iterated approach is examined, in which an otherwise-diverging Neumann series is modified to assure convergence and reduce non-essential computation.

PROBLEM STATEMENT

The integral equation governing ultrasonic transmission at a fluid-solid interface is
\[ \mathcal{V}_{i}(s) K_{kl}(s | s') ds = \mathcal{V}_{i}(s'), \quad i, k = 0, 1, 2, 3 \quad (1a) \]
\[ v_{0}(s) = p(s) \quad \text{(pressure in fluid)} \quad (1b) \]
\[ v_{i}(s) = u_{i}(s), \quad i = 1, 2, 3 \quad \text{(displacement in solid)} \quad (1c) \]
\[ n_{j}(s) = \text{outward area element vector}, \quad j = 1, 2, 3 \quad (1d) \]
\[ K_{00}(s | s') = 0.5 \delta(s - s') - n_{j}(s) \partial_{j} p^{G}(x(s) | x(s')) \quad (1e) \]
\[ K_{0i}(s | s') = n_{j}(s) \rho \omega^{2} p^{G}(x(s) | x(s')) \quad , \quad i = 1, 2, 3 \quad (1f) \]
\[ K_{k0}(s | s') = n_{j}(s) u_{j}^{G}(x(s) | x(s')) \quad , \quad k = 1, 2, 3 \quad (1g) \]
\[ K_{kl}(s | s') = 0.5 \delta_{kl} \delta(s - s') + n_{j}(s) r_{ij}^{G}(x(s) | x(s')) \quad , \quad i, k = 1, 2, 3 \quad (1h) \]

where the superscript "in" denotes the incident field vector. Index summation convention is used. The integral is defined over the component boundary \( S \), which is parameterized by the mapping \( x(s) \), where \( s \) has dimension one less than \( x \). The outward (to the solid body) normal vector \( n_{i}(x) \) contains the Jacobian of this mapping. The superscript "G" denotes whole-space Green pressure, displacement and stress fields. Material density is denoted \( \rho \), and \( \omega \) denotes time harmonic frequency. The materials are assumed linearly elastic and isotropic. A non-zero incident pressure field is the only case considered here, i.e., \( u_{i}^{in} \) is identically zero.

To motivate the present discussion, let the solution vector be represented as an integral transform

\[ v_{i}(s) = \int_{S} \mathcal{V}_{i}(s'') M_{il}(s'' | s) ds'' \quad , \quad i, l = 0, 1, 2, 3 \quad (2) \]

Substituting eq.(2) into eq.(1) yields

\[ \int_{S} M_{il}(s'' | s) K_{kl}(s | s') ds = \delta_{kl} \delta(s'' - s') \quad (3) \]

Eq.(3) represents the problem of transmission of a point-focused incident field, the solution of which is the inverse kernel of the integral equation eq.(1).

Eq.(3) is discretized by representing the solution vector in terms of an appropriate discretely orthogonal basis set \( b_{\alpha}(s) \)

\[ M_{il}(s_{\alpha} | s) = \sum_{\beta} M_{il\beta} b_{\beta}(s) \quad (4a) \]
\[ b_{\beta}(s_{\alpha}) = \delta_{\alpha\beta} \quad (4b) \]

where \( s_{\alpha} \) denote discrete positions in the s-plane. Greek indices are used to indicate dis-
crete s-plane positions, and the index summation convention is not applied to Greek indices.

In this work, basis functions are one- or two-dimensional Gaussian-multiplied sinc functions defined over an equi-spaced grid in the s-plane, with zero spacing at \( 0.4 \lambda_w \), the wavelength in water. The discrete problem is written

\[
\sum_{\alpha \in D} M_{i\alpha}^k \cdot \delta_{kl} \delta_{\chi^\beta} = \delta_{kl} \delta_{\chi^\beta} 
\]

(5a)

\[
K_{k\beta} = \int_{S} b_{\alpha}(s) \cdot K_{k\beta}(s^\prime) \, ds
\]

(5b)

The summation is performed over a sub-region of \( S \), referred to as the solution domain \( D \). Ideally, the solution domain is chosen to be the non-zero support of the exact solution (which, of course, is generally unknown a priori), so as to avoid solution truncation errors. A difficulty encountered in the 3-dimensional application of the above formulation is that the matrix required to cover the non-zero support of the solution is far too large to hold, much less numerically invert.

In this work, it was noted under certain conditions that, given a point-focus beam incident at position \( \chi \) in eq.(5), the solution at \( \chi \), \( M_{i\alpha}^\chi \), can be accurately obtained using a sum domain centered about the point \( \chi \) which is smaller than the non-zero support of the exact solution, and that the convergence of \( M_{i\alpha}^\chi \) is controlled by the radius of this domain. In other words, under certain conditions, given an error \( \epsilon \), there is a corresponding radius \( R_{\epsilon} \) in the s-plane, centered at \( \chi \), beyond which the summation can be truncated, such that the solution error at \( \chi \) will be less than \( \epsilon \). This domain of radius \( R_{\epsilon} \) centered at \( \chi \) will be referred to as the "inner" summation (integration) domain \( I_\chi \). The approximate solution at \( \chi \) is thereby obtained as the solution to

\[
\sum_{\alpha \in I_\chi} M_{i\alpha}^\chi \cdot K_{k\beta} = \delta_{kl} \delta_{\chi^\beta}
\]

(6)

The complement to the domain \( I_\chi \) within \( D \) will be referred to as the outer domain \( O_\chi \). For future notational convenience, eq.(5) is re-written

\[
\sum_{\alpha \in I_\chi} M_{i\alpha}^\chi \cdot K_{k\beta} = \delta_{kl} \delta_{\chi^\beta} - \sum_{\alpha \in O_\chi} M_{i\alpha}^\chi \cdot K_{k\beta}
\]

(7)

where the summation on the right hand side of eq.(7) is treated as a modification of the input data vector. It is seen that eq.(7) is equivalent to eq.(6) when the \( M_{i\alpha}^\chi \) are identically zero over the outer domain \( O_\chi \).

Consider next \( M_{i\alpha}^{\chi+1} \) at a nearest neighboring point, denoted \( \chi+1 \). To get a correspondingly accurate solution at this point, the summation domain will have to be increased slightly. An effective approach is to define the new sum domain as the union of the domains \( I_\chi \) and \( I_{\chi+1} \), where \( I_{\chi+1} \) is a domain of radius \( R_{\epsilon} \), centered at point \( \chi+1 \). However, consider the solution outside of \( I_{\chi+1} \) yet inside \( I_\chi \). The solution in this sub-domain will be relatively unaffected by the increase in the sum domain. Therefore, rather than calculating new solution values over this sub-domain, the current values will sum in the right hand side of eq.(7).

\[
\sum_{\alpha \in I_{\chi+1}} M_{i\alpha}^{\chi+1} \cdot K_{k\beta} = \delta_{kl} \delta_{\chi^\beta} - \sum_{\alpha \in O_{\chi+1}} M_{i\alpha}^{\chi+1} \cdot K_{k\beta}
\]

(8)

The process continues by considering the next-nearest neighboring point to \( \chi \), denoted \( \chi+2 \). Again, the matrix of elements within \( I_{\chi+2} \) is inverted explicitly, while those elements having non-zero coefficients within \( O_{\chi+2} \) are summed into the right hand side input vector. The sequence continues, denoted by the general expression
where point $X+n$ denotes the $n^{th}$ nearest neighbor to point $X$.

The validity of this procedure depends on the validity of the original assumption leading to eq.(6). Numerical experimentation has demonstrated that the sum domain $I_x$ can be truncated if the field at the boundary of the truncated domain is outward propagating, i.e., no reflections are taking place outside $I_x$ which will reflect exiting energy back into $I_x$. In the case of surface wave propagation, this implies that the surface is sufficiently smooth such that surface wave reflections are negligible. This is an intuitively compelling observation, and suggests a conceptual similarity between the method described above and a ray tracing approach.

In the problems examined, a sum domain radius of greater than 3 wavelengths in water produced adequate results, and essentially exact results are obtained if a sum radius of 6 to 8 wavelengths is used. These numbers depend, of course, on the geometry and wave speeds of the component, but they do indicate the approximate size of the required matrices. The required computations are easily handled in a workstation environment.

Several variations of the basic procedure described above have been examined. The more important of these variants will be discussed here. Firstly, the computational requirements for matrix inversion increase as $(RE)^2$ for a two dimensional problem, and $(RE)^3$ for a three dimensional problem. Therefore, the algorithm proceeds much faster if a smaller matrix is used. It was noted that the above procedure will converge when carried out iteratively using a smaller sum radius, say 1 to 2 wavelengths in water, provided certain precautions are taken at the edge of the total domain of solution $D$. It was observed that overall computation time can be significantly less by performing two or three such iterations with a smaller sum radius. The iterative procedure simply repeats the above process once the index $n$ in eq.(9) covers all points in the total solution domain $D$. However, when repeating the process, the $M_{i|\alpha\chi}$ in the right hand side sum of eq.(9) are possibly non-zero over the entire solution domain $D$, and must be considered. Convergence difficulties are encountered when the total solution domain $D$ does not completely cover the non-zero support of the solution, i.e., when the solution is truncated. The iteration is made to converge by generally requiring that the boundary of the "outer" sum domain $O_{X+n}$ be truncated a specified distance from the boundary of the total solution domain $D$, say 6 wavelengths in water. This is apparently due to the fact that, in the zeroth iteration, the solution at the "farthest" edge of the inner domain $I_{X+n}$ (i.e., farthest from the initial point $X$) displays a significant truncation error, containing "inward" propagating spectral components. For inner domains $I_{X+n}$ well away from the boundary of $D$, these errors will be corrected as the process marches outward. However, the truncation errors at the solution boundary $D$ will never be corrected. If the outer domain summations in eq.(9) sum these erroneous solution values, these truncation errors will propagate inward and contaminate the solution in the inner regions of the solution domain $D$. These inward propagating truncation errors will grow without bound as the number of iterations increases, and the solution will diverge. Note that when $I_{X+n}$ contains one point, iteration of eq.(9) produces a Neumann series.

A second important variation of the above procedure is to reduce the overlap in the successive inner domains $I_{X+n}$. Experimentation has shown that the iterative process will converge if there is no overlap at all in successive inner domains (this experiment, for a 3-dimensional transmission problem, utilized rectangular inner domains which marched out "quasi-spirally" according to increasing distance between the current and initial domains). Again, the trade-off in this approach is the number of iterations required verses the time required per iteration.
A third variation to the above process is perhaps most important. The development of
the procedure was based on the canonical problem of an infinitely focused beam, so as to
clearly illustrate the convergence properties of $M_{i \alpha \chi}$ with increasing inner domain size.
The above discussion implies a procedure involving i) the calculation of the scattering ma-
trix $M_{i \alpha \chi}$, followed by ii) contraction with the incident field via eq.(2). When the iterative
procedure is employed, this process can be simplified if the solution domain $D$ completely
covers the non-zero support of the incident field on $S$ (more rigorously, the incident field
must be zero or outward propagating at the boundary of the solution domain $D$). The basic
procedure to be utilized in practice is obtained by applying the discrete version of eq.(2) to
eq(9), to yield
\[
\sum_{\alpha \in \chi_{\alpha}} v_{i \alpha} K_{k \beta} = \sum_{\alpha \in \chi_{\alpha}} v_{i \alpha} K_{k \beta} \chi
\]
where $\chi$ is an appropriately chosen starting point, such as the center of the incident beam.
The iterative procedure proceeds for eq.(10) as described for eq.(9). Thus, it is not neces-
sary to explicitly evaluate and retain the scattering matrix $M_{i \alpha \chi}$. This result is a primary
goal of this work.

The final point to be mentioned regards the reflection of surface fields arising from small
radii of curvature surface features. The non-iterative procedure discussed in the develop-
ment of eq.(9) requires that no energy be reflected back into the inner domain $I$. This con-
dition can be relaxed when applying the iterative technique via eq.(9) or (10). The iterative
technique requires only that the field be outward propagating at the boundary of the total
solution domain $D$. This, of course, is not a procedural problem, since the solution will as-
sume that no reflections occur outside $D$ in any case. Reflecting surface features positioned
completely within $D$ will not pose a problem. It was observed that the zeroth-order iteration
will largely determine the forward propagating fields. The first-order reflected fields will
first appear in the first-order iteration, along with corrections to the forward propagating
fields. Higher-order multiple reflections will first appear in the corresponding higher-order
iteration.

EXAMPLE OF APPLICATION

The iterative boundary integral solution is currently being applied to a number of two
and three dimensional test cases. Work studying the convergence properties of the tech-
nique have concentrated on two-dimensional problems, since the scattering matrix gener-
ated in these cases can be explicitly inverted for validation comparisons. Work has exam-
ined moderately curved surfaces, surfaces containing sharp angles (e.g. quarter plane), and
surfaces containing steps or notches. The technique has been applied to the corresponding
three-dimensional problems to assure that the convergence observed in the two dimensional
problems carries over to three dimensions. For brevity, examples here are limited to two
dimensions for which validation by explicit matrix inversion is possible.

The first example consider a hyperbolic tangent surface given by
\[
x_2(x_1) = A \tanh(\alpha x_1)
\]
where $A$ is the profile height and $\alpha$ controls the width of the step. Two cases are consid-
ered, depicted in fig.(1a). The first case considers a highly-focused Gaussian beam imping-
ing on a plane aluminum surface ($A=0$). The beam width in the focal zone is $1.68 \lambda_w$,
where the beam width is defined as the distance between the half-amplitude points of the
Gaussian profile. The the incident pressure profile is shown in fig.(2a) as a function of
arclength over the surface. The iterative solution uses a $2 \lambda_w$ radius inner solution domain.
Fig. (3a) compares the magnitude of the tangential solid surface displacement after the ze-
roth, first, and second iterations. Fig. (3b) compares the second iteration solution of fig.(3a)

125
with the explicit matrix inversion solution. The focused beam generates significant amplitude leaky quasi-transverse motion (Rayleigh) and quasi-compressional motion surface waves, which interfere to form the observed oscillation in the wavefield envelope. Fig.(3) shows that the zeroth iteration is a reasonable approximation, and the first iteration is essentially correct. The calculation of the iterated solution required only a fraction of the time needed for the explicit matrix inversion, and did not require retention of a scattering matrix.

The next example considers the curved step shown in fig.(1a). The step height and width are approximately $10 \lambda_w$ and $5 \lambda_w$, respectively. Fig. (4a) compares the magnitude of the tangential solid surface displacement after the zeroth, first, and second iterations for the same incident field as fig.(3), perpendicularly incident in the center of the step (the incident pressure as a function of surface arclength is visually identical to fig(2a), and hence is not plotted). Fig.(4b) compares the second iteration solution with the explicit matrix inversion. Note the effect of the different surface curvatures in the left- and right-hand directions. In the left-hand direction, the concave surface curvature sheds the surface wave motion into bulk wave motion as the profile is traversed, resulting in a faster surface wave decay than seen in fig.(2). In the right-hand direction, the convex surface profile not only retains the surface wave motion, but continuously re-reflects near-surface-skimming wave motion into like motion, resulting in a slower decay of surface motion as the profile is traversed than for the plane surface. Again, the zeroth iteration is a reasonable approximation, and the first iteration is essentially correct.

The final example considers the quarter plane geometry shown in fig.(1b). The sharp corner requires appropriate modifications of eqs.(1f-h) to account for the discontinuous displacement gradients at the corner.[1] An incident beam $4.2 \lambda_w$ wide is assumed incident on the top surface at 30 degrees, near the Rayleigh angle. The real part of the incident pressure field is plotted in fig(2b) as a function of surface arclength. Note that the incident field crosses the surface on both the top and side surfaces. The total surface pressure field predicted by the zeroth, first, and second iterations are shown in fig.(5a). The zeroth iteration predicts the forward propagating surface wave field, but significant errors are seen in the reflected fields. The reflected surface wave fields are substantially corrected in first iteration. Little additional correction is seen in the second iteration. The total surface pressure field obtained by an explicit matrix inversion is compared to the second iteration result in fig.(5b).

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Fig. 1. Profiles of water-solid interface. a) flat, hyperbolic tangent, b) quarter space.
Fig. 2. Real part of incident pressure. Horizontal axis is surface arclength in mm.  
a) flat surface,  b) quarter space (corner is in center of plot).

Fig. 3. Magnitude of tangential surface displacements on flat surface. Horizontal axis is surface arclength in mm.  
a) iterations 0, 1, and 2,  b) iteration 2 and explicit matrix inversion.

Fig. 4. Magnitude of tangential surface displacements on tanh surface. Horizontal axis is surface arclength in mm.  
a) iterations 0, 1, and 2,  b) iteration 2 and explicit matrix inversion.
Fig. 5. Magnitude of total surface pressure for quarter space. Horizontal axis is surface arclength in mm (corner is in center of plot). a) iterations 0, 1, and 2, b) iteration 2 and explicit matrix inversion

SUMMARY

An alternative means of solving the boundary integral equation governing ultrasonic beam transmission at a curved fluid-solid interface has been demonstrated. The technique is based on appropriate modifications of the Neumann series, which assure convergence and reduce computation. Results were shown demonstrating the convergence of the technique when applied to two dimensional problems. The technique has allowed the solution of previously intractable three dimensional problems in a workstation environment. Ongoing work is developing yet more efficient means of performing the three dimensional calculations. Application to three dimensional problems will be presented in detail at a future date.

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REFERENCE