EFFECT OF A DAMPING LAYER ON HARMONIC WAVE MOTION
IN A LAMINATED PLATE

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INTRODUCTION

This paper is concerned with investigating the difference in response between a three-ply elastic plate of infinite lateral extent and a similar plate in which the core elastic layer is replaced by a layer of viscoelastic damping material. The transient response of the plate to an impulsive loading is governed by the dispersion equations for infinite trains of harmonic waves propagating under traction free conditions on the plate surfaces (see e.g. Achenbach [1]). These dispersion equations express the wavelength and the damping factor as functions of the frequency and they possess an infinite set of roots, corresponding to different modes of wave motion. In addition to their relevance to the impulsive loading problem, the dispersion curves of the various modes, have a significance for the experimental determination of the properties of laminates by ultrasonic techniques (see e.g. Mal and Bar Cohen [2]).

The motivation for this study of damping effects in laminated plates arises from two different situations. The first of these is a consequence of the increasing use of lightweight fiber composite laminates in high risk load bearing situations. The very low inherent damping characteristics of these materials makes the incorporation of viscoelastic internal damping layers a desirable feature (see [3]) and a theoretical analysis of the effects of such layers provides the basic information to the design engineer. The second motivation arises from the nature of the fabrication process for multi-ply laminates of fiber composite material. In particular, the autoclave process of fabrication from sheets of prepreg gives rise to resin rich layers between the plies and these have the viscoelastic damping characteristic of the resin material [4].

The Rayleigh-Lamb dispersion curves for a single plate of isotropic elastic material have been examined in considerable detail by Mindlin [5] whilst Tanaka and Kon-No [6] have presented results for an isotropic viscoelastic plate. A list of references to work on wave propagation in laminates is to be found in the review article by Kapania and Raciti [7]. The work presented here is restricted to the study of shear waves (SH motion) in a symmetric three layered plate consisting of an inner core of one isotropic material, bounded by two identical layers of another isotropic material. The material of the outer layers is assumed to be elastic, whilst the core material may be elastic or viscoelastic with the viscoelastic material being modelled as a standard linear solid. Attention is restricted to shear motion since it affords the simplest situation, whilst yet exhibiting the phenomena of interest. The more physically significant case of coupled P, SV motion is examined in the thesis by Nkemzi [8].

This investigation covers two different aspects of the damping phenomena. One of these is the effect of varying the creep and relaxation times of the standard linear solid which models the viscoelastic damping layer. The other is the effect of changing the ratio of core thickness to overall plate thickness. For the purpose of evaluating the effect of changing the characteristic times it is
It is convenient to compare these with a reference unit. This is chosen as the time taken for the shear wave in the outer elastic material to travel through a distance of one half the overall plate thickness. For viscoelastic material whose creep and relaxation times are large in comparison with this reference unit, the dispersion curves bear a strong similarity to those for the plate with an elastic core. Where the characteristic times of the viscoelastic material are of the order of the reference time unit or less, the dispersion curves fall into two distinct sets. One set is associated with a virtually independent motion of the two elastic outer layers, with little or no damping. The other set involves high damping and is associated with a motion which is essentially confined to the internal viscoelastic layer. The number of curves of each set within the first ten roots depends on the relative thickness of the core and the outer layers.

DISPERSION EQUATION

Consider a symmetric sandwich plate of overall thickness $2H$ in which the viscoelastic core has thickness $2vH$ ($0 < v < 1$) and the thickness of each of the elastic facings is $(1-v)H$. The materials are assumed to be perfectly bonded together so that the displacements and tractions are continuous across the interfaces between the core and the facings. In terms of a system of rectangular coordinates $Oxyz$ with origin in the mid-plane of the plate, $Oz$ along the normal to this plane and $Ox$ in the direction of wave propagation, the displacement components, $u, v, w$ in each layer are assumed to be of the form

$$u = w = 0, \quad v = V(z) e^{i(\omega t-Kx)}.$$  \hspace{1cm} (1)

In equations (1), $i^2 = -1$, the angular frequency $\omega$ is real and positive and the wave number $K = k - i\delta$, where $k$ and $\delta$ are real and positive. These displacements represent an anti-plane shearing wave propagating in the positive $x$ direction with speed $c = \omega/k$ and with spatial attenuation coefficient $\delta$. The only non-zero stress components associated with the displacements (1) are

$$\sigma_{xy} = -iK\mu(\omega) V(z) e^{i(\omega t-Kx)}, \quad \sigma_{yz} = \mu(\omega) \frac{dV}{dz} e^{i(\omega t-Kx)}, \hspace{1cm} (2)$$

where $\mu(\omega)$ is the complex shear modulus of the material in the layer. For the outer elastic material this shear modulus is a constant, $\mu_v$, independent of $\omega$, whereas for the core material, modelled as a standard linear solid with creep time $\tau_1$ and stress relaxation time $\tau_2$, the complex shear modulus is $\mu(\omega) = \mu_0 (1+i\omega\tau_1)/(1+i\omega\tau_2)$, where $\mu_0$ is a constant. Here, the constant $\mu_0$ is the long time shear modulus of the core material (corresponding to $\omega = 0$) and the instantaneous shear modulus is $\mu_0\tau_1/\tau_2$ (corresponding to $\omega \rightarrow \infty$). Let $\rho_0$ denote the density of the core material and $\rho_1$ the density of the outer material, then it is convenient to introduce the dimensionless quantities $\bar{x} = x/H$, $\tau = c_1\bar{t}/H$, $\Omega = \omega H/c_1$, $K = \bar{x}H = kH-i\delta H$ where $c_1 = (\mu_1/\rho_1)^{1/2}$ is the shear wave speed in the outer material.

Because of the symmetry of the plate lay-up, the dispersion equation factorises into two distinct factors, one associated with a displacement which is symmetric with respect to the plate mid-surface and the other associated with anti-symmetric displacement. The dispersion equations for the symmetric motion is

$$G(\Omega)p \sin pv \cos q(1-v) + \mu q \cos pv \sin q(1-v) = 0 \hspace{1cm} (3)$$

and that for the anti-symmetric motion is

$$G(\Omega)p \cos pv \cos q(1-v) - \mu q \sin pv \sin q(1-v) = 0. \hspace{1cm} (4)$$

The parameters appearing in these equations are defined by

$$p = \frac{\alpha^2\Omega^2 - K^2}{G(\Omega)}, \quad q = \Omega^2 - K^2.$$  \hspace{1cm} (5)

$$\mu = \mu_1/\mu_0, \quad \alpha^2 = \rho_0\mu_1/\rho_1\mu_0, \quad G(\Omega) = (1+i\Omega\tau_1)/(1+i\Omega\tau_2).$$

Each of equations (3) and (4) is complex valued and each leads to expressions for the wave number $k$ and damping factor $\delta$ as functions of $\omega$.

RESULTS AND DISCUSSION

For the numerical results which are reported here, the values chosen for $\mu$ and $\alpha$ are $\mu = 50$, $\alpha = 5$ and the values of the parameters $\tau_1$ and $\tau_2$ are allowed to vary whilst keeping the ratio $\tau_1/\tau_2 = 5$. The choice $\tau_1 = \tau_2 = 0$, corresponds to an elastic core material and gives $G(\Omega) = 1$, for
which equation (3) reduces to
\[ p \sin pv \cos q(1-v) + 50q \cos pv \sin q(1-v) = 0, \] (6)
with \( p \) and \( q \) being given by
\[ p^2 = 25\Omega^2 - K^2, \quad q^2 = \Omega^2 - K^2. \] (7)
It is instructive to examine the solutions of equation (6) in some detail. The solutions for \( \Omega = 0 \) are
\[ K = 0 \quad \text{or} \quad K = -i\delta_r H, \quad r = 1, 2, \ldots, \]
where \( \delta_r H \) are the roots of the equation
\[ \tan v \delta_r H + 50 \tan(1-v) \delta_r H = 0. \] (8)
The first five roots of this equation for \( v = 0.2 \) are shown below and the remaining roots are obtained from these by adding integer multiples of \( 5\pi \).
\[
\begin{array}{cccccc}
  r & 1 & 2 & 3 & 4 & 5 \\
\delta_r H & 3.9022 & 7.5054 & 8.2026 & 11.8057 & 15.7080 \\
\end{array}
\]
Of these roots, the first and fourth correspond to values close to the roots of \( \sin(1-v) \delta_r H = 0 \), the second and third correspond to values close to the roots of \( \cos \delta_r H = 0 \) and the fifth root occurs when both \( \sin(1-v) \delta_r H = 0 \) and \( \sin v \delta_r H = 0 \). As \( \Omega \) increases from zero, it is evident from equation (6) that for moderate values of \( p \) and \( q \) the roots of the dispersion equation must be close to the roots of one or other of the equations
\[ \cos pv = 0, \quad \sin q(1-v) = 0. \] (9)
It is also evident that the dispersion curves pass through the common roots of the equations
\[ \sin pv = 0, \quad \sin q(1-v) = 0 \]
and that they also pass through the common roots of the equations
\[ \cos pv = 0, \quad \cos q(1-v) = 0. \]
With this information it is possible to trace the behaviour of any branch of the dispersion equation (6), starting from \( \Omega = 0 \). Details of this tracing technique are given elsewhere [8]. Here it is noted that each branch alternatively follows close to a root of one of the other of the equations (9) until a point is reached for which \( q = 0 \) and \( \sin pv = 0 \). Thereafter, \( q \) becomes pure imaginary and for increasing values of \( k \) the solution of equation (6) approaches a root of \( \cos pv = 0 \), corresponding to a portion of the hyperbola
\[ 25\Omega^2 - k^2 H^2 = (r + \frac{1}{2})^2 \pi^2 \]
where \( r \) is some integer. This behaviour is clearly shown in Figure 1 where the solid curves are the branches of the symmetric dispersion equation (6) with the vertical axis showing the non-dimensional frequency \( \Omega \), the horizontal axis to the left showing the imaginary part of \( K(\delta H) \) and the horizontal axis to the right shows the real part of \( K(kH) \). The dotted curves in the same figure represent the antisymmetric modes whose equation is obtained from (4) and (5) on setting \( G(\Omega) = 1 \). A similar analysis of the solution of this equation shows that the dispersion curves follow close to the roots of one or other of the equations
\[ \sin q(1-v) = 0, \quad \sin pv = 0, \] (10)
in an alternating fashion as \( \Omega \) increases from zero, until \( q \) reaches the value \( q = 0 \) after which the solution tends to a root of \( \sin pv = 0 \) as \( kH \to \infty \).

It is pertinent to remark that the equation
\[ \sin q(1-v) = 0 \]
is the dispersion equation for symmetric SH waves propagating under traction free conditions in an elastic plate made of the outer material and having an overall depth \( 2(1-v)H \). The equation \( \cos pv = 0 \) may be interpreted as the dispersion equation for symmetric SH wave propagation in an elastic plate of overall depth \( 2vH \) made of the core material, under the conditions of zero displacement at the outer boundaries. Similarly the equation \( \sin pv = 0 \) may be interpreted as the dispersion equation in the core under zero displacement conditions for anti-symmetric SH waves.
For non-zero values of $\tau_1$ and $\tau_2$ equations (3) and (4) are complex valued but the solutions for $\Omega = 0$ are still given by $K = -i\delta_\nu H$ where $\delta_\nu H$ are the roots of equation (8). The modulus of the ratio $G(\Omega)/\mu$ is still small as $\Omega$ increases from zero and the solution of equation (4) for the symmetric motion must be close to a root of one or other of the equations

$$\cos p\nu = 0, \quad \sin q(1-\nu) = 0,$$

where $p$ and $q$ are given by equations (5). The second of these equations is identical with that in the elastic sandwich plate ($G(\Omega) = 1$) and it has the same interpretation as before. The first equation is now complex valued and yields expressions for both the real part and the imaginary part of $K$ as functions of $\Omega$. Of these, the real part $kH$ increases from zero as $\Omega = 0$ and tends asymptotically to the value $5\Omega \tau_2/\tau_1$ as $\Omega \to \infty$ whilst the imaginary part $\delta H$ starts at the value $(r+\frac{1}{2})\pi/\nu$ at $\Omega = 0$ and tends asymptotically to the constant value $2/\sqrt{\tau_1\tau_2}$ as $\Omega \to \infty$. This solution may be interpreted as the dispersion equation for symmetric SH wave propagation in a plate of depth $2\nu H$ made of the viscoelastic core material, under conditions of zero displacements on its surface.

Figure 2 shows a number of branches of the dispersion equations (3) and (4) corresponding to anti-symmetric and symmetric motion of the viscoelastic sandwich plate. These solutions relate to the choice $\nu = 0.2$ and $\tau_2 = 0.1$ and the solid curves (symmetric motion) clearly separate into two groups. Members of the group with the larger number of branches appear to follow very closely the roots of the equation $\sin q(1-\nu) = 0$, whereas those of the other group appear to match the solutions of the equation $\cos p\nu = 0$. A similar effect is displayed by the anti-symmetric dispersion curves (dotted curves) and the group of three members which follow the roots of the equation $\sin p\nu = 0$ are clearly visible in Figure 2. In this case, the larger group is virtually indistinguishable from that for the symmetric motion and the member curves again correspond closely to the solutions of $\sin q(1-\nu) = 0$. 

Figure 1. Dispersion curves for sandwich plate with elastic core $\nu = 0.2$, $\tau_1 = 0$, $\tau_2 = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dispersion_curves.png}
\caption{Dispersion curves for sandwich plate with elastic core $\nu = 0.2$, $\tau_1 = 0$, $\tau_2 = 0$.}
\end{figure}
Figure 2. Dispersion curves for sandwich plate with elastic core $\nu = 0.2$, $\tau_1 = 0.5$, $\tau_2 = 0.1$.

Figures 3 and 4 show the solutions of equations (3) and (4) for $\nu = 0.2$ and for $\tau_2 = 0.01$ and $\tau_2 = 10.0$ respectively. The grouping into two sets of curves is again evident in Figure 3 with the smaller group of both symmetric and antisymmetric curves having an imaginary part which increases rapidly with $\Omega$. This is consistent with an imaginary part tending towards the limiting value $\delta H = 89.4$ which is the appropriate limit derived from the solution $\cos \nu = 0$ for this case. In Figure 4 on the other hand there is no evidence of these two distinct groups and the dispersion curves for both symmetric and antisymmetric motion bear much closer resemblance to those for the elastic sandwich, which are shown in Figure 1.

The effect of varying the core thickness is displayed in Figures 5 and 6. These figures show some of the symmetric mode dispersion curves for the plate where the core material has time constant $\tau_2 = 0.1$ and for the values $\nu = 0.5$ and $\nu = 0.8$ respectively. Here the solid curves correspond to those branches which have the low damping, elastic type, behaviour and the dotted curves relate to the group with viscoelastic, high damping type behaviour. These figures are to be compared with the solid curves (symmetric modes) in Figure 2, for which $\tau_2 = 0.1$ and $\nu = 0.2$. From these it may be seen that the increase in core thickness leads to an increasing number of high damping branches with a corresponding drop in the number of elastic type branches.

Finally, Figure 7(a) shows the variation of displacements through one half of the plate for two branches of each group at dimensionless frequency $\Omega = 10$ and for core properties $\tau_2 = 0.1$ and $\nu = 0.2$. The corresponding plots for variation of the shear stress are shown in Figure 7(b). It is clear from these that the displacement associated with the second (heavily damped) group of branches (dotted curves) is consistent with a plate motion in which the core alone vibrates under zero displacement conditions at its interfaces. A more detailed study of these results is to be found in [8].
Figure 3. Dispersion curves for sandwich plate with elastic core $\nu = 0.2$, $t_1 = 0.05$, $t_2 = 0.01$.

Figure 4. Dispersion curves for sandwich plate with elastic core $\nu = 0.2$, $t_1 = 50$, $t_2 = 10$. 
Figure 5. Dispersion curves for sandwich plate with elastic core $v = 0.5$, $\tau_1 = 0.5$, $\tau_2 = 0.1$.

Figure 6. Dispersion curves for sandwich plate with elastic core $v = 0.8$, $\tau_1 = 0.5$, $\tau_2 = 0.1$. 
Figure 7. Variation of (a) displacement and (b) shear stress through one half of viscoelastic sandwich plate, $v = 0.2, \tau_1 = 0.5, \tau_2 = 0.1, \Omega = 10$.
Solid curves, first two branches of elastic type modes.
Dotted curves, first two branches of viscoelastic type modes.

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