Shape of the water table where steady rainfall seepage into drains is reinforced by artesian water

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SHAPE OF THE WATER TABLE WHERE STEADY RAINFALL
SEEPAE INTO DRAINS IS REINFORCED BY ARTESIAN WATER

by

Thomas Daniel Hinesly

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
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DOCTOR OF PHILOSOPHY

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INTRODUCTION

This problem was prompted by the need of methods for determining drain tile spacing and depth where there is both rainfall and artesian water to be removed.

Underground aquifers under hydrostatic pressure may often be a source of water to be dealt with in the design of a drainage system for the agricultural development of valley land. Peterson [1957, p. 198] says that "in the alluvial valleys of the irrigated West, increasing hydrostatic head with depth is more the rule than the exception, so that upward leakage of some magnitude is common".

Artesian pressures are produced by the confining of water in a very permeable layer of stratum of geological material, such as the artesian gravel indicated in Figure 1. The artesian gravel serves as a conduit by which water may be transmitted over long distances from the catchment area, where replenishment may be by rainfall, irrigation water, canals, rivers, surface reservoirs, etc. Due to the upward seepage of water through the slowly permeable soil, the water table in the valley soils would eventually stand above the soil surface if the water in the artesian gravel stands higher than the soil surface of the valley. The valley could be completely inundated unless some drainage system exists or unless evaporation could remove the upward seepage.

Rainfall or excess irrigation water may add to the water requiring disposal. In the following it is to be understood that "rainfall" or
Figure 1. Representation of origin of artesian water.
RAIN

PIEZOMETRIC SURFACE OF WATER

SOIL SURFACE

SLOWLY PERMEABLE SOIL

ARTESIAN GRAVEL

IMPERVIOUS STRATUM
"rain water" can mean excess irrigation water, that is, water which requires removal by drainage.

Our objective is to theoretically investigate several examples of drainage systems with different geometric conditions and different ratios of rainfall rate to upward seepage rate of artesian water. More specifically, we are interested in the maximum height of rise of the water table midway between drains of different sizes and spacings for various rainfall-artesian conditions. This maximum height of rise is important to know in drainage design in order that the drains may be placed at a great enough depth to insure that the cultivated plants will have a well aerated root feeding zone.
REVIEW OF LITERATURE

One of the earliest theoretical treatments of artesian water control is that of Farr and Gardner [1933] who found an approximate solution for determining drain spacings by means of combining the radial flow equation to a sink with that of a uniform vertical flow from the artesian gravel. This original solution was improved (see Muskat [1946, p. 356]) by a consideration of an infinite array of drains and their images in the "y" plane. Further refinement was given to the solution by Kirkham [1940] by the incorporation of a system of multiple image-arrays. Later Kirkham [1945] employed an infinite system of images in a theoretical analysis of the case where there is both artesian water and surface water flowing into the drains. None of the above analyses, however, deal with the curved free-water table, although maximum height of rise of the water table midway between drains was approximated. Also, none of the work considered simultaneous rainfall and artesian water.

Since the water table is a basic consideration in the design of any drainage system, a great deal of study has been devoted to this problem. Childs [1943, 1945a, 1945b, 1946, 1947; Childs and O'Donnell, 1951] has employed the electric analogue method for a series of water table investigations. More recently Youngs [1959] used this method to study the effect on water table heights when the soil to be drained overlies a more permeable layer. Although these studies have contributed valuable information, they do not have the applicability to a wide
range of different situations or variables that the mathematical approach affords.

Probably the most exact mathematical attack on this problem is that of Van Deemter [1950] who utilized the hodograph method. He considered both rainfall and artesian water, but the latter had to originate at an infinite depth for his mathematical analysis to apply. Furthermore, because of the complexity and number of conformal transformations that had to be made in order to attack the problem, only limited use has been made of Van Deemter's work.

Although based on a simplifying physical approximation, Kirkham's [1958] mathematical treatment of the seepage of rainfall into drains underlain by an impermeable layer has a great deal more usefulness, especially since results have been presented in the form of a nomograph [Toksoz and Kirkham, 1961] from which the solution to a wide range of conditions can be easily computed. Kirkham [1958] cites Hooghoudt for the basic physical approximation which is "the loss of hydraulic head in the arched region (of the water table) is negligible compared to the loss of head for the remainder of the flow region". Kirkham [1961] has recently justified this assumption and, as he says, has put it on firmer ground by the derivation of a factor which upon insertion by multiplication into his original equation gives an upper limit for the height of water table rise between drains. His equation has been field tested.
All of the works cited in this review which are dated prior to 1957 and many others which may be relative to this problem are discussed in detail in *Drainage of Agricultural Lands* [Luthin, 1957].
DERIVATION OF THE CONSTANT FLUX EQUATION

In the derivation which follows, a constant downward flux of rain water and a constant upward flux of water seeping through a slowly permeable layer which overlies an aquifer is assumed; although, in the artesian problem there is ordinarily an equipotential surface at the aquifer. Later we shall see that a constant upward seepage flux from the aquifer may or may not correspond to the existence of an equipotential surface. It will be seen that, if the artesian water originates at a great enough depth, the assumed condition of a constant upward flux is equivalent to the existence of an equipotential surface.

Basic Assumptions

The assumptions made in the development of this two dimensional solution are a reiteration of those normally assumed in drainage problems. Some of the particularly pertinent assumptions are given here for clarity.

Attention is directed first to Figure 2. Here we have represented the cross section of drained land cut perpendicular to equally spaced parallel tile drains, D, which are of such great length in comparison to the distance between adjacent tiles that conditions are assumed to be uniform in the tile length direction. Thus we reduce the problem to a two dimensional one of Figure 3 having only to consider now the x and y coordinates in a plane perpendicular to the drains. Next we note that, if we draw a line dd from the lowest point of the water table through the center of the drain to the surface of the aquifer and another line bb from the water table at its highest point midway
Figure 2. Representation of the problem of drainage of rainfall and artesian water.
RAINFALL

SOIL SURFACE

UNSATURATED ZONE

WATER TABLE

ZONE OF GROUND WATER

AQUIFER
between drains to the surface of the aquifer, we can isolate a section which with its mirror-image is repeated throughout the medium. Since, because of symmetry, the flow net of the mirror-image would just be a reflection of the isolated section we have only to consider a solution for the latter. Figure 3 is a representation of the section singled out of Figure 2, except that instead of a circular drain tube being shown at the left a slit drain (of height b - a) is represented. Also, Figure 3 is a representation of the special case where the downward rainfall and the upward artesian flux are equal.

We turn now to a consideration of the evenly distributed rainfall pattern at the surface of soil. It is assumed that the water which infiltrates across the soil surface percolates through the soil in straight vertical streamlines and crosses the water table into the isolated section of ground water in the same evenly distributed pattern that it had upon entering the soil. To support this assumption we seek support from the work of Childs [1945b] who says "It is shown that the streamlines above the capillary fringe in soil with sensibly uniform pore sizes are truly vertical, justifying assumptions made in previous work. Stream pictures obtained by the method of electric analogues show that, in more usual soil types, the streamlines above the capillary fringe do not depart very much from the vertical, and even where the departure is the maximum which can commonly occur in practice, the error introduced into drainage calculations by assuming them to be vertical is not serious, and is in any case on the safe side".
Figure 3. Geometry for the problem of drainage of rainfall and artesian water when the rainfall and upward seepage are equal. Compare section dd bb of Figure 2, noting that the drain tube there is replaced by a slit drain of height \((b - a)\) here. Also see text.
As for the capillary fringe, which we assume can be neglected, one realizes that the assumption is not altogether well grounded, since the only place that it would actually be of negligible thickness would be in coarse textured soils, whereas in finer textured soils it may range in thickness from 12 to 18 inches. However, most drainage workers have reasoned that the additional height increases the resistance to flow which offsets any addition to the potential, leaving the water table unchanged, but not the surface of saturation. Childs [1959] recently used the hodograph method to investigate the capillary fringe when there was both rainfall and artesian water. Since his solution was based on Van Deemter's [1950] solution, discussed earlier, it also applies only when the artesian water originates at a great depth. From this study he states that "thick fringes are accommodated mostly above a water table which is not proportionately much depressed below that which is appropriate to an absence of a fringe".

Even though one does neglect the capillary fringe in some theoretical developments, in actual practice this water saturated or near water saturated zone could be taken into account by solving for a water table height, and hence drain depth, calculated on the basis of zero fringe height, but placing the drains at a deeper depth corresponding to the thickness of the capillary fringe.

The most fundamental assumption in our analysis, and the one which makes the solution an approximate solution, although it will be mathematically exact, is the same assumption as that made by Kirkham [1958]
as quoted from Hooghoudt. The assumption is made that the area between
the hydraulic head reference level (Figure 3) and the water table, at
atmospheric pressure, contains an infinite number of infinitesimally
thin impermeable strips extending from the surface to the reference
level, these strips being filled with a conducting medium of infinite
permeability so that the loss of hydraulic head by water vertically
transversing this area can be neglected. The assumption amounts to
saying that there is a constant flux entering the surface of the
reference level, which is the same assumption we have made with respect
to the upper surface of the aquifer. For a lucid explanation and the
justification of this assumption the reader is referred again to
Kirkham [1961].

As in all steady-state solutions, it is assumed that enough time
has elapsed so that the total flux of water into the flow area under
consideration in Figure 3 is just equal to the flow out of the drain­
age facility. Furthermore, it is assumed that the rainfall rate, \( R \),
the upward artesian flux, \( F \), and the soil hydraulic conductivity, \( k \),
aré such that a water table will form entirely in the soil. If \( R \) is
not less than \( k \), such a water table would not form. Instead, satura­
tion would occur to the surface, forming ponded water because physically
the inequality \( R/k > 1 \) means that the soil cannot accept the water as
fast as it rains.

Again as in steady-state solutions, the applicability of the
generalized form of Darcy's law,
\[ q = -k \nabla \phi, \quad (1) \]

is assumed where \( q \) is the flux vector, \( k \) the hydraulic conductivity and \( \phi \) is the scalar potential; here the hydraulic head. Combining this equation with the equation of continuity for the steady-state flow of an incompressible fluid, which in vector notation is

\[ \nabla \cdot q = 0, \quad (2) \]

we find our assumption of Darcy's law leads to

\[ \nabla \cdot (k \nabla \phi) = 0 \quad (3) \]

which in two dimensional cartesian coordinates yields

\[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad (4) \]

where \( \phi \) is defined by the expression

\[ \phi = k\phi. \quad (5) \]

Making use of the well known Cauchy-Riemann relations which are

\[ \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad (6) \]

\[ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \quad (7) \]

we see that we can also write Laplace's equation for the stream function as

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad (8) \]
Consequently, because of this relationship between the potential and stream functions, we have a choice of finding a solution for our problem in terms of either one or the other, just so long as it satisfies the respective Laplace's equation and the specified conditions at the boundaries of the flow region.

The Stream Function

Laplace's equation has been the starting point for the solution of such a large number of problems in various branches of physics that one can often find a clue to the development of a solution for the problem at hand. With this in mind, we refer to Figure 3 again which is the same as Kirkham's [1958] Figure 2a except that the impermeable layer has been replaced by an aquifer. Also, we shall consider that the upward and downward flux need not be equal as Figure 3 shows.

If one considers, as Kirkham [1958] did, that the streamline dropping below the point \( x = 0 \), of our Figure 3 or his Figure 2a, into the sink is the zero streamline, then for our case the boundary conditions (see Roman numerals on Figure 3) for the stream function, \( \psi \), are:

For condition I, at \( y = 0 \), one has \( \psi = \frac{x}{s} Rs \) from \( x = 0 \) to \( s \) where \( R \) is the rate of downward flux density of the rain water, in the positive \( y \) direction, and \( s \) is the midpoint distance between drains.

For condition II, at \( x = s \), one has \( \psi = Rs \) from \( y = 0 \) to \( h \). This is true because a streamline going downward at \( x = s \) and \( y = 0 \) will meet another one coming upward in the negative \( y \) direction from \( x = s \) and \( y = h \). The point at which they meet is called the stagnation point.
point, P. Here the upward and downward streamlines merge into the same streamline. For the special case shown in Figure 3, the merged streamline has the value $\phi = 0.5$.

For condition III, at $y = h$, one has $\phi = Rs + \frac{(e-x)}{s} Fs$ from $x = 0$ to $s$ where $F$ is the upward flux density of artesian water;

For condition IV, at $x = 0$, one has $\phi = Rs + Fs$ from $y = b$ to $h$;

For condition V, at $x = 0$, one has

$$\phi = \frac{(y-a)}{(b-a)} (Rs + Fs) \text{ from } y = a \text{ to } b;$$

For condition VI, at $x = 0$, one has $\phi = 0$ from $y = 0$ to $a$.

Searching now for a solution of Laplace's equation in terms of the stream function which will satisfy the above boundary conditions, it seems logical that one should be able to add a term to Kirkham's [1958] general solution to satisfy the conditions at the boundary contiguous to the aquifer. Thus, the following general solution of Laplace's equation will be shown to be one which will satisfy the boundary conditions:

$$\phi = \sum_{m=1}^{\infty} A_m \sin \frac{nx}{s} \frac{\sinh[nx(h-y)/s]}{\sinh(nhx/s)}$$

$$+ \sum_{m=1}^{\infty} B_m \sin \frac{nx(h-y)}{h} \frac{\sinh[nx(s-x)/h]}{\sinh(nhx/h)}$$

$$+ \sum_{m=1}^{\infty} C_m \sin \frac{nx}{s} \frac{\sinh(nxy/s)}{\sinh(nhx/s)}$$

$$+ Rs$$

(9)
where $A_m$, $B_m$ and $C_m$ with $m = 1, 2, 3, \ldots$ are arbitrary constants. That this is a solution of Laplace's equation can be verified by differentiating and substituting into Equation 8.

To obtain the values of the arbitrary constants, $A_m$, $B_m$ and $C_m$, we consider the value of $\phi$ of Equation 9 as found on the boundaries. Along boundary I where $y = 0$ and $0 < x < s$, the value of $\phi$ is

$$\phi = \sum_{m=1}^{\infty} A_m \sin \frac{\pi x}{s} + Rs.$$  

Along boundary II where $x = s$ and $0 \leq y \leq h$, one has

$$\phi = Rs.$$  

Along boundary III where $y = h$ and $0 < x < s$, one finds

$$\phi = \sum_{m=1}^{\infty} C_m \sin \frac{\pi x}{s} + Rs.$$  

Along boundaries IV, V and VI where $x = 0$ and $0 < y < h$, one finds

$$\phi = \sum_{m=1}^{\infty} B_m \sin \frac{\pi m (h - y)}{h} + Rs.$$  

In order to determine $A_m$, one can equate the value of $\phi$ from condition I to the solution of $\phi$ along boundary I; specifically one has

$$Rs = \sum_{m=1}^{\infty} A_m \sin \frac{\pi x}{s} + Rs.$$  

The value of the constant $A_m$ now follows from a Fourier sine series for the interval $x = 0$ to $s$, with the result
\[ A_m = \frac{2}{s} \int_0^s - \left( \frac{s-x}{s} \right) R \sin \frac{mx}{s} \, dx. \]

That is
\[ A_m = -\frac{2R_0}{m}. \quad (10) \]

In passing it should be noted that the solution along boundary II, namely, \( \psi = R_0 \) when \( x = s \) for all values of \( y \), satisfied boundary condition II identically.

Skipping \( B_m \) for the present time to obtain the value for \( C_m \), we equate the condition for boundary III with the solution along this boundary in the following manner;

\[ R_0 + \left( \frac{s-x}{s} \right) F_s = \sum_{1}^{\infty} C_m \sin \frac{mx}{s} + R_0 \]

where \( 0 < x < s \). Then by using a Fourier sine series again for the interval \( x = 0 \) to \( s \) one finds

\[ C_m = \frac{2}{s} \int_0^s \left( \frac{s-x}{s} \right) F_s \sin \frac{mx}{s} \, dx \]

or after integrating one arrives at
\[ C_m = \frac{2F_0}{m}. \quad (11) \]

Determining the value of \( B_m \) is more complicated because in this case one must make comparison of the three boundary conditions IV, V, and VI with the solution along this boundary where \( x = 0 \). One can proceed in the following way:
First let there be some function of $y$ such that we have

$$f(y) = 0 \text{ where } 0 < y < a$$

$$= \left(\frac{y - a}{b - a}\right) (R_s + F_s) \text{ where } a < y < b$$

$$= R_s + F_s \text{ where } b < y < h$$

$$= \sum_{n=1}^{\infty} \frac{B_n \sin \frac{\pi n (h - y)}{h}}{n} + R_s \text{ where } 0 < y < h.\]

Next assume another function of $y$ such as $g(y) = f(y) - R_s$; then from above we have

$$g(y) = -R_s \text{ where } 0 < y < a,$$

$$g(y) = -R_s + \left(\frac{y - a}{b - a}\right) (R_s + F_s) \text{ where } a < y < b$$

$$g(y) = F_s \text{ where } b < y < h, \text{ and}$$

$$g(y) = \sum_{n=1}^{\infty} B_n \sin \frac{\pi n (h - y)}{h} \text{ where } 0 < y < h.$$

Converting to a new function of $z$ by letting $h - y = z$ or $y = h - z$ and substituting the equivalent for $y$ into the above equations and inequalities, one has, upon rewriting with a change of sign in the inequalities,

$$h(z) = g(y) - R_s \text{ where } (h - a) < z < h,$$

$$h(z) = g(y) = -R_s + \left(\frac{h - z - a}{b - a}\right) (R_s + F_s) \text{ where }$$

$$(h - b) < z < h - a,$$

$$h(z) = g(y) = F_s \text{ where } 0 < z < h - b$$

$$h(z) = g(y) = \sum_{n=1}^{\infty} B_n \sin \frac{\pi n (h - z)}{h} \text{ where } 0 < z < h.$$
Thus, one obtains

\[ B_m = \frac{2}{h} \int_0^h h(s) \sin \frac{\pi s}{h} \, ds \]

by employing the Fourier sine series again over the interval \( s = 0 \) to \( h \). Substituting the various values of \( h(s) \) into this last equation with their respective intervals over the boundary gives

\[ B_m = \frac{2}{h} \int_0^h F_s \sin \frac{\pi s}{h} \, ds \]

\[ + \frac{2}{h} \int_0^h \left[ \frac{\pi (h - s - a)}{h} \right] \left[ (R_s + F_s) - R_s \right] \sin \frac{\pi s}{h} \, ds \]

\[ - \frac{2}{h} \int_0^h R_s \sin \frac{\pi s}{h} \, ds, \]

which upon integrating yields

\[ B_m = \frac{2}{\pi} \left\{ [F_s + (-1)^n R_s] - \left[ \frac{(F_s + R_s)h}{\pi(b - a)} \right] \left[ \sin \frac{\pi(a - h)}{h} - \sin \frac{\pi(b - h)}{h} \right] \right\} \]

or finally by trigonometric substitution one arrives at

\[ B_m = \frac{2}{\pi} \left\{ [F_s + (-1)^n R_s] + \left[ \frac{(-1)^n(F_s + R_s)h}{\pi(b - a)} \right] \left[ \sin \frac{\pi a}{h} - \sin \frac{\pi b}{h} \right] \right\} \tag{12} \]

We have now only to insert the constants into Equation 9 to complete the solution for the stream function. However, we will not do this at the present; the reason will appear later when we show the procedure for the calculation of the flow-net.

The Potential Function

Since a good test of any drainage solution is obtained by plotting the flow-net and since the value of \( \phi = \frac{F}{k} \) for the line
$y = 0$ in Figure 3 is the main desired result for determining the height of the water table above the reference level, which in all cases will be our $x$ axis, we next obtain the potential function which is the conjugate of the stream function just developed.

Utilizing the Cauchy-Riemann relations, Equations 6 and 7, we can by differentiation and integration obtain

$$
\mathcal{F} = \sum_{m=1}^{\infty} A_m \cos \frac{m \pi (h - y)}{s} \cosh \left[ \frac{m \pi (s - x)}{h} \right] + \sum_{m=1}^{\infty} B_m \cos \frac{m \pi (h - y)}{s} \cosh \left[ \frac{m \pi (s - x)}{h} \right] + C
$$

where $m = 1, 2, 3, \ldots$ and $A_m$, $B_m$, and $C_m$ are the constants determined for the stream function. That Equation 13 is also a solution of Laplace's equation can be shown by performing the prescribed differentiation and substituting into Equation 4.

The $C$ in Equation 13 is not related to the constant $C_m$, but is to be chosen such that for the lowest point on the water table we will have $\mathcal{F} = k \varphi_1 + C = 0$ where $k \varphi_1$ is defined as the right hand side of Equation 13 less the constant $C$. The evaluation of $C$ will be made clear after we have accumulated a table of potential values for points throughout the flow region of an example.
Procedure for Calculating the Flow-net

Seldom, if at all, does one find in drainage literature a detailed procedure for doing the numerical calculations prerequisites to plotting the flow-net. For that matter, one seldom, if ever, finds detailed methods of plotting flow-nets. Therefore, some rudiments of the actual mechanics are presented here. However, first some auxiliary equations are derived to facilitate the manipulation of Equation 9 and Equation 13.

Auxiliary equations for the stream function

Now we are ready to substitute the values determined for the arbitrary constants into their respective summation equations of the stream function Equation 9. At the same time, let us label the individual summation equations in such a manner that, after factoring a \( a \) from each of the constants, we can write Equation 9 as

\[ \pi(\phi - \text{Re}) = S_1 + S_2 + S_3 \]

where we have

\[ S_1 = -2\text{Re} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{\text{wx}}{s} \frac{\sinh \left( \frac{w(h - y)}{s} \right)}{\sinh \left( \frac{wh}{s} \right)} \]

\[ S_2 = 2 \sum_{m=1}^{\infty} \left\{ \frac{F_s + (-1)^m \text{Re}}{m} \sinh \left( \frac{w(s - x)}{s} \right) \right\} \frac{\sin \frac{wx}{h} - \sin \frac{wsh}{h}}{\sinh \frac{wsh}{h}} \]

\[ \cdot \left\{ \sin \frac{w(h - y)}{h} \frac{\sinh \left( \frac{w(s - x)}{s} \right)}{\sinh \left( \frac{wsh}{s} \right)} \right\} \]

\[ S_3 = 2F_s \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{\text{wx}}{s} \frac{\sinh \left( \frac{wxy}{s} \right)}{\sinh \left( \frac{wsh}{s} \right)} \]
**Auxiliary equations for the potential function**

Reverting our attention to the potential function Equation 13, we will, after substituting the values of the constants $A$, $B$, and $C$ into Equation 13, factor $a \times$ from each constant and write it in the abbreviated form as

\[ \pi(g - c) = T_1 + T_2 + T_3 \] (18)

where we have

\[ T_1 = -2Rs \sum_{l=1}^{\infty} \cos \frac{\pi x}{s} \frac{\cosh[\pi h(y - x)/s]}{\sinh[\pi x h/s]} \] (19)

\[ T_2 = 2 \pi \sum_{l=1}^{\infty} \left[ \frac{F_l + (-1)^lRs}{s} \right] + \left[ \frac{(-1)^l(F_l + Rs)h}{s} \right] \left[ \sin \frac{\pi x a}{h} \sin \frac{\pi x b}{h} \right] \] (20)

\[ T_3 = -2Fs \sum_{l=1}^{\infty} \cos \frac{\pi x}{s} \frac{\cosh[\pi y/s]}{\sinh[\pi x h/s]} \] (21)

**Investigation of the slit sink, as an example, using the constant flux solution**

For convenience in this example, let us take the rain-water flux density equal to the aquifer-water flux density in order that we may write

\[ q = (F + R)s = (2R)s \]
where \( q \) is the total flux. Furthermore we will choose \( a = h = 10 \) feet in Figure 3, so that if we assume a rainfall density of 0.05 cubic feet per square foot of soil surface per day, then the total flux \( q \) is unity. This assumption will simplify our example calculation considerably. Also in Figure 3, let us take \( a = 3 \) feet and \( b = 7 \) feet so that we have a slit drainage facility 4 feet deep.

It should be emphasized here that we do not have to use feet for our unit of measure; we could just as well have adopted some other standard of measurement.

Upon rewriting Equations 15, 16 and 17 to contain the chosen dimensions we have

\[
S_1 = \sum_{l=1}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{\sinh \lambda(10 - y)}{\sinh \lambda x}
\]

\[
S_2 = \sum_{l=1}^{\infty} \left\{ \left[ \frac{1}{\lambda} \left( \frac{-1}{\lambda} \right)^{\frac{2}{\lambda}} \right] + \left[ \frac{5}{\lambda} \left( \frac{-1}{\lambda} \right)^{\frac{2}{\lambda}} \right] \left( \sin \lambda 3x - \sin \lambda 2x \right) \right\}
\]

\[
S_3 = \sum_{l=1}^{\infty} \frac{1}{\lambda} \sin \lambda \frac{\sinh \lambda xy}{\sinh \lambda x}
\]

where a quantity \( \frac{q}{2} = F_s = R_s = \frac{1}{2} \) no longer will appear in any of the auxiliary equations for this example solution.

For further detail in the procedure of calculation, we will calculate the stream function for the point in the flow region where \( x = 3 \) and \( y = 3 \). Substituting these values for \( x \) and \( y \) we have
\[ S_1 = -\sum_{n=1}^{\infty} \frac{1}{n} \sin 0.3\pi n \frac{\sinh 0.7\pi n}{\sinh \pi n} \]

\[ S_2 = \sum_{n=1}^{\infty} \left\{ \frac{1 + (-1)^n}{n} + \frac{5}{2} \frac{(-1)^n}{n^2} \right\} [\sin 0.3\pi n - \sin 0.7\pi n] \cdot \frac{\sin 0.7\pi n}{\sinh \pi n} \]

\[ S_3 = \sum_{n=1}^{\infty} \frac{1}{n} \sin 0.3\pi n \frac{\sinh 0.7\pi n}{\sinh \pi n} . \]

Since too much detail in the actual summation of these equations above would be repetitious as far as procedure is concerned, let us center our attention only on the more complex auxiliary equation \( S_2 \).

In the last equation for \( S_2 \), let the quantity \( B' \) be defined by

\[ B' = \left\{ \frac{1 + (-1)^n}{n} + \frac{5}{2} \frac{(-1)^n}{n^2} \right\} (\sin 0.3\pi n - \sin 0.7\pi n) \]

then one has

\[ S_2 = \sum_{n=1}^{\infty} B' \frac{\sin 0.7\pi n}{\sinh \pi n} \frac{\sinh 0.7\pi n}{\sinh \pi n} . \]

We now devise a scheme that can be used to obtain the constant \( B' \) to as great summation indices as is needed to sum the \( S_2 \) auxiliary equation to four decimal place accuracy at any coordinate point in the flow region.

The following scheme (which does not give every computational step) is one which can be used to obtain \( B' \).
Scheme for Obtaining the Constant $B'_m$

$$\left[ \frac{5 \ (-1)^m}{x^2} \right] \cdot \sin 0.3m$$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\frac{1 + (-1)^m}{m}$</th>
<th>$\sin 0.3m$</th>
<th>$- \sin 0.7m$</th>
<th>$B'_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0000</td>
<td>+</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>1.0000</td>
<td>+</td>
<td>0.7569</td>
<td>+1.7569</td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
<td>+</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>4</td>
<td>0.5000</td>
<td>-</td>
<td>0.1169</td>
<td>+0.3831</td>
</tr>
<tr>
<td>5</td>
<td>0.0000</td>
<td>+</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>7</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The $\frac{1}{m}$ values can be copied from tables present in any late edition of the *Handbook of Chemistry and Physics* to save calculation time.

For the sine values it is convenient to have tables available such as Table 1. Let us, for example, use Table 1 to obtain the values of $\sin 0.3m$. For $m = 1$, we look in Table 1 for $3 \times 1 = 3m$ where we find $\sin 0.3x = 0.809017$; for $m = 2$, we look for $3 \times 2 = 6m$ where we find $\sin 0.6x = 0.951057$; we continue this process to as many $m$ values as are wanted, for we see immediately that the values are cyclic so that we can extend the $m$ column of Table 1 to as great a length as needed. In the scheme for the constant $B'_m$, we have for simplicity only used four decimals.

It might be well to mention here that one will need other tables of sine and cosine similar to that of Table 1, when the dimensionless
Table 1. Table of sines and cosines of argument 0.1mm

<table>
<thead>
<tr>
<th>cos (0.1mm)</th>
<th>sin (0.1mm)</th>
<th>m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.951057</td>
<td>0.309017</td>
<td>1</td>
</tr>
<tr>
<td>0.809017</td>
<td>0.587785</td>
<td>2</td>
</tr>
<tr>
<td>0.587785</td>
<td>0.809107</td>
<td>3</td>
</tr>
<tr>
<td>0.309017</td>
<td>0.951057</td>
<td>4</td>
</tr>
<tr>
<td>ZERO</td>
<td>1.000000</td>
<td>5</td>
</tr>
<tr>
<td>-0.309017</td>
<td>0.951057</td>
<td>6</td>
</tr>
<tr>
<td>-0.587785</td>
<td>0.809017</td>
<td>7</td>
</tr>
<tr>
<td>-0.809017</td>
<td>0.587785</td>
<td>8</td>
</tr>
<tr>
<td>-0.951057</td>
<td>0.309017</td>
<td>9</td>
</tr>
<tr>
<td>-1.000000</td>
<td>ZERO</td>
<td>10</td>
</tr>
<tr>
<td>-0.951057</td>
<td>-0.309017</td>
<td>11</td>
</tr>
<tr>
<td>-0.809017</td>
<td>-0.587785</td>
<td>12</td>
</tr>
<tr>
<td>-0.587785</td>
<td>-0.809017</td>
<td>13</td>
</tr>
<tr>
<td>-0.309017</td>
<td>-0.951057</td>
<td>14</td>
</tr>
<tr>
<td>ZERO</td>
<td>-1.000000</td>
<td>15</td>
</tr>
<tr>
<td>0.309017</td>
<td>-0.951057</td>
<td>16</td>
</tr>
<tr>
<td>0.587785</td>
<td>-0.809017</td>
<td>17</td>
</tr>
<tr>
<td>0.809017</td>
<td>-0.587785</td>
<td>18</td>
</tr>
<tr>
<td>0.951057</td>
<td>-0.309017</td>
<td>19</td>
</tr>
<tr>
<td>1.000000</td>
<td>ZERO</td>
<td>20</td>
</tr>
</tbody>
</table>
ratio of \( s \) to \( h \) in the auxiliary equations is different from \( \frac{s}{h} = 1 \) in other solutions. One can compile tables of sines and cosines of arguments \( 0.05m, 0.025m, 0.005m \), and others, as has been done in the Soil Physics Section of Agronomy at Iowa State University [Kirkham, 1957], by using the Applied Mathematics Series [U. S. Dept. of Commerce, 1949] tables of sine and cosine in decimal fractions of a degree.

Now that we have tables of the constant \( \frac{1}{m} \) and a table for obtaining values of sine in tenths of a \( \pi \) unit for \( m = 1, 2, 3, \ldots \), we lack only values of the hyperbolic sine quotients to sum the \( S_2 \) auxiliary equation for our example.

To obtain the hyperbolic sine quotients we substitute successive values of \( m \) into the dividend and divisor, ascertain the values of the hyperbolic sines from appropriate tables (see A Guide to Mathematical Tables [Lebesew and Fedorova, 1960] for various kinds of tables available) [U. S. Dept. of Commerce, 1955; Becker and Van Orstrand, 1942] and perform the division for each value of \( m \) through \( m = 4 \). For \( m = 5 \) we will write

\[
\frac{\sinh (5)(0.7) \pi}{\sinh 5 \pi} = \frac{\sinh 3.5 \pi}{\sinh 5 \pi} = \frac{\frac{1}{2}(e^{3.5 \pi} - e^{-3.5 \pi})}{\frac{1}{2}(e^{5 \pi} - e^{-5 \pi})} = \frac{e^{3.5 \pi} - e^{-3.5 \pi}}{e^{5 \pi} - e^{-5 \pi}}.
\]

We observe that to at least four decimal places

\[
e^{-3.5 \pi} = e^{-5 \pi} = 0.0000
\]

so that without loss of exactness in the first four decimal places
we have
\[
\frac{e^{3.5x} - e^{-3.5x}}{e^{5x} - e^{-5x}} = (e^{3.5x})(e^{-5x}) = e^{-1.5x}.
\]

Similarly, for \(m \geq 5\) we write
\[
\frac{\sinh 0.7xm}{\sinh xm} = e^{-0.3xm}.
\]

We can now very quickly enumerate as many \(m\) values of the hyperbolic sine quotient as we need by reading them from a table of descending exponentials [U. S. Dept. of Commerce, 1951; U. S. Dept. of Commerce, 1955] when \(m \geq 5\).

After this hurried explanation of how the various parts of the expression for \(S_2\) are obtained, let us now put the parts in a schematic form below, for calculating the sum, by letting

\[
B_m = A_m \sin 0.7xm = B, \quad \text{and} \quad \frac{\sinh 0.7xm}{\sinh xm} = C
\]

**Scheme for summing the auxiliary equations**

<table>
<thead>
<tr>
<th>(m)</th>
<th>(A)</th>
<th>(B)</th>
<th>(A \times B)</th>
<th>(C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.8090</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>+1.7569</td>
<td>-0.9511</td>
<td>-1.6710</td>
<td>0.1521</td>
</tr>
<tr>
<td>3</td>
<td>0.0</td>
<td>0.3090</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>+0.3831</td>
<td>+0.5878</td>
<td>+0.2252</td>
<td>0.0231</td>
</tr>
<tr>
<td>5</td>
<td>0.0</td>
<td>-1.0000</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>6</td>
<td>+0.2814</td>
<td>+0.5878</td>
<td>+0.1654</td>
<td>0.0035</td>
</tr>
<tr>
<td>7</td>
<td>0.0</td>
<td>+0.3090</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>8</td>
<td>+0.2973</td>
<td>-0.9511</td>
<td>-0.2828</td>
<td>0.005</td>
</tr>
<tr>
<td>9</td>
<td>0.0</td>
<td>+0.8090</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>10</td>
<td>+0.2000</td>
<td>0.0000</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>11</td>
<td>0.0</td>
<td>-0.8090</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>12</td>
<td>+0.1457</td>
<td>+0.9511</td>
<td>+0.1386</td>
<td>0.00001</td>
</tr>
</tbody>
</table>
If we now utilize the accumulative and negative multiplier on the calculating machine in the final multiplication of \( A \times B \times C \), we will have the sum needed for \( S_2 \) which is

\[
S_2 = -0.2485
\]

In the same schematic fashion, we arrive at the sums for \( S_1 \) and \( S_3 \), which are

\[
S_1 = -0.3849
\]

and

\[
S_3 = +0.0820
\]

thus for \( x = 3 \) and \( y = 3 \),

\[
\pi[\phi(3,3) - R_s] = -0.3849 - 0.2485 + 0.0820
\]

\[
\phi(3,3) = -\frac{0.5514}{\pi} + 0.5000
\]

\[
= +0.3245,
\]

recalling that \( R_s = 0.5000 \) for our example.

We now have calculated the stream function for the point \( x = 3 \) and \( y = 3 \) to four decimal places; that is to say, if we took any number of additional \( m \) terms in the sums for \( S_1 \), \( S_2 \) and \( S_3 \) the sums would remain in each case unchanged in the fourth decimal place.

Following the procedure just outlined, stream function values are obtained for every one foot intersection of the flow region and tabulated in Table 2.
Table 2. Stream function values calculated for the slit drain example.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0000</td>
<td>0.0500</td>
<td>0.1000</td>
<td>0.1500</td>
<td>0.2000</td>
<td>0.2500</td>
<td>0.3000</td>
<td>0.3500</td>
<td>0.4000</td>
<td>0.4500</td>
</tr>
<tr>
<td>1</td>
<td>0.0000</td>
<td>0.0760</td>
<td>0.1429</td>
<td>0.2027</td>
<td>0.2489</td>
<td>0.2941</td>
<td>0.3375</td>
<td>0.3784</td>
<td>0.4193</td>
<td>0.4598</td>
</tr>
<tr>
<td>2</td>
<td>0.0000</td>
<td>0.1138</td>
<td>0.1965</td>
<td>0.2556</td>
<td>0.3018</td>
<td>0.3401</td>
<td>0.3749</td>
<td>0.4073</td>
<td>0.4384</td>
<td>0.4695</td>
</tr>
<tr>
<td>3</td>
<td>0.0000</td>
<td>0.1814</td>
<td>0.2727</td>
<td>0.3245</td>
<td>0.3613</td>
<td>0.3907</td>
<td>0.4148</td>
<td>0.4373</td>
<td>0.4584</td>
<td>0.4795</td>
</tr>
<tr>
<td>4</td>
<td>0.2500</td>
<td>0.3274</td>
<td>0.3773</td>
<td>0.4077</td>
<td>0.4284</td>
<td>0.4439</td>
<td>0.4568</td>
<td>0.4683</td>
<td>0.4790</td>
<td>0.4897</td>
</tr>
<tr>
<td>5</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.5000</td>
</tr>
<tr>
<td>6</td>
<td>0.7500</td>
<td>0.6726</td>
<td>0.6227</td>
<td>0.5923</td>
<td>0.5716</td>
<td>0.5561</td>
<td>0.5432</td>
<td>0.5317</td>
<td>0.5210</td>
<td>0.5121</td>
</tr>
<tr>
<td>7</td>
<td>1.000</td>
<td>0.8186</td>
<td>0.7274</td>
<td>0.6755</td>
<td>0.6387</td>
<td>0.6093</td>
<td>0.5852</td>
<td>0.5627</td>
<td>0.5416</td>
<td>0.5205</td>
</tr>
<tr>
<td>8</td>
<td>1.000</td>
<td>0.8863</td>
<td>0.8035</td>
<td>0.7444</td>
<td>0.6982</td>
<td>0.6599</td>
<td>0.6251</td>
<td>0.5927</td>
<td>0.5616</td>
<td>0.5305</td>
</tr>
<tr>
<td>9</td>
<td>1.000</td>
<td>0.9240</td>
<td>0.8571</td>
<td>0.8073</td>
<td>0.7511</td>
<td>0.7059</td>
<td>0.6625</td>
<td>0.6216</td>
<td>0.5897</td>
<td>0.5402</td>
</tr>
<tr>
<td>10</td>
<td>1.000</td>
<td>0.9500</td>
<td>0.9000</td>
<td>0.8500</td>
<td>0.8000</td>
<td>0.7500</td>
<td>0.7000</td>
<td>0.6500</td>
<td>0.6000</td>
<td>0.5500</td>
</tr>
</tbody>
</table>
Assuming the same conditions and dimensions as those specified for the stream function with regard to R, F, q, s and h we can now rewrite the equations for $T_1^*$, $T_2^*$ and $T_3^*$ for the slit drainage problem at hand as

$$
T_1 = - \sum_{l=1}^{\infty} \frac{1}{l^2} \cos m0.1xx \frac{\cosh m0.1x(10 - y)}{\sinh mx}
$$

(23)

$$
T_2 = + \sum_{l=1}^{\infty} \frac{1}{l} \cos m0.1x(10 - y) \frac{\cosh m0.1x(10 - x)}{\sinh mx}
$$

(24)

$$
T_3 = - \sum_{l=1}^{\infty} \frac{1}{l} \cos m0.1xx \frac{\cosh m0.1xy}{\sinh mx}
$$

(25)

and these expressions may be handled as were the expressions for $S_1^*$, $S_2^*$ and $S_3^*$.

The procedure for dealing with the cosine and hyperbolic cosine terms are the same as those given for the sine and hyperbolic sine. Even the $B_m$ values tabulated for use in the sums for $S_2$ can also be used in the sums for $T_2$. There is, however, one big difference to consider. In the summation of $S_1$, $S_2$ and $S_3$ for the stream function we encounter no difficulty at the boundaries because the solution was developed to satisfy conditions at the boundaries with respect to the stream function. On the other hand, an inspection of Equations 23, 24 and 25 reveals the fact that some of the sums will converge very slowly at the boundaries in their present form. Let us now direct our attention to the alleviation of this difficulty of convergence.

Let us first consider the boundary where $y = 0$ and $0 < x < s$ (see Figure 3). On this boundary $T_1^*$, $T_2^*$ and $T_3^*$ for our example are
Here it is seen that the sums for $T_2$ and $T_3$ converge rapidly.

To make the sum for $T_1$ converge rapidly we observe from Dwight [1957, formula 654.5] that we can write

$$\frac{\cosh \frac{wh}{s}}{\sinh \frac{wh}{s}} = \frac{e^{-wh/s}}{\sinh \frac{wh}{s}} + 1,$$

so that the expression for $T_1$ becomes

$$T_1 = - \sum_{l=1}^{\infty} \frac{1}{l} \cos \frac{w_{.1}x}{\sinh \frac{wh}{s}} - \sum_{l=1}^{\infty} \frac{1}{l} \cos \frac{w_{.1}x}{\sinh \frac{wh}{s}}$$

(26)

which is now separated into two parts; the first part converges rapidly because of the exp ($- \varphi x$) and the infinite cosine sum can be obtained in closed form by employing a revision of Dwight's [1957] formula 603.2 which is

$$\sum_{l=1}^{\infty} \frac{\cos \frac{w_{.1}x}{s}}{s} = - \ln(2 \sin \frac{\varphi x}{2s})$$

when our cosine argument is substituted into the equation. But more convenient, and used extensively throughout this study, is a table of sums of infinite trigonometric series [Kitower, 1948]. Those parts
of the tables used to such a great advantage in this study have been excerpted and included as Table 6 in the appendix of this thesis. To obtain the value of the infinite cosine term one has only to change radian arguments to degrees and read the value directly from Table 6 under the column designated as $F_1$.

Referring back to Equations 23, 24 and 25 we see that for $y = h = 10$, where $0 < x < s$, we have the same identical situation for $T_3$ that we have just discussed for $T_1$.

Let us look now at the situation along the boundary where $y = 0$ and $x = s$.

When $y = 0$ and $x = s$ Equations 23, 24 and 25 reduce to

$$T_1 = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{\cosh mx}{\sinh mx}$$

$$T_2 = \sum_{m=1}^{\infty} B_m (-1)^m \frac{1}{m} \frac{1}{\sinh mx}$$

$$T_3 = -\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{1}{\sinh mx}. $$

Now we make use of a formula [Kirkham, 1958 - Eq. 99] derived by expanding and regrouping terms, which is

$$\sum_{m=1}^{\infty} (-1)^m \frac{f(m)}{m} = -\sum_{m=1}^{\infty} \frac{f(m)}{m} + \sum_{m=1}^{\infty} \frac{f(2m)}{m}. $$

We will apply this formula, at the same time we multiply $B_m$ term by term by all the other quantities in $T_2$, and rewrite all the above equations as
When we add the auxiliary equations together we have

\[ T_1 + T_2 + T_3 = \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh \pi n}{\sinh \pi n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh 2\pi n}{\sinh 2\pi n} \]

\[ + \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sinh \pi n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sinh 2\pi n} \]

\[ + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \sin \pi n - \sin 2\pi n \right] \cdot \frac{1}{\sinh \pi n} \cdot \frac{1}{\sinh 2\pi n} \]

Then from Dwight [1957, formula 653.8] we find

\[ \frac{\cosh \pi n + 1}{\sinh \pi n} = \frac{\cosh \pi n}{\sinh \pi n} \]

Therefore, we have

\[ T_1 + T_2 + T_3 = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sinh \pi n} + \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\sinh 2\pi n} \]

\[ + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \sin \pi n - \sin 2\pi n \right] \cdot \frac{1}{\sinh \pi n} \cdot \frac{1}{\sinh 2\pi n} \]

An inspection of equations 23, 24 and 25 for this example reveals that when \( y = h = 10 \) and \( x = s \) we arrive at the same solution, at this point on the boundary, as we derived above for \( y = 0 \) and \( x = s \).
Going further, let us investigate Equations 23, 24 and 25 for our example when \( x = 0 \) and \( 0 < y < s \).

On this boundary we see that \( T_1 \) and \( T_3 \) still sum as rapidly as they did in the interior. Hence, we will be concerned only with \( T_2 \) of Equation 24, in which we now substitute the value of \( B_m \) given by Equation 22 and expand the result to find

\[
T_2 = \sum_{m=1}^{\infty} \frac{1}{m} \cos m0.1x(10 - y) \frac{\cosh m\pi}{\sinh m\pi} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cos m0.1x(10 - y) \frac{\cosh m\pi}{\sinh m\pi} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin 0.3x m \cos m0.1x(10 - y) \frac{\cosh m\pi}{\sinh m\pi} - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin 0.7x m \cos m0.1x(10 - y) \frac{\cosh m\pi}{\sinh m\pi}
\]

to which we can apply formula 401.04, as given by Dwight [1957] to the cosine terms in \( T_2 \) and rewrite as

\[
T_2 = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \cos m0.1xy \frac{\cosh m\pi}{\sinh m\pi} + \sum_{m=1}^{\infty} \frac{1}{m} \cos m0.1xy \frac{\cosh m\pi}{\sinh m\pi} + \sum_{m=1}^{\infty} \frac{1}{m} \sin 0.3x m \cos m0.1xy \frac{\cosh m\pi}{\sinh m\pi} + \sum_{m=1}^{\infty} \frac{1}{m} \sin 0.7x m \cos m0.1xy \frac{\cosh m\pi}{\sinh m\pi}
\]
Now we use a trigonometric substitution [Dwight, 1957; formula 401.05], Kirkham's [1958] Equation 99, and formula 654.5 from Dwight [1935] to obtain finally

\[ T_2 = \sum_{m=1}^{\infty} \frac{1}{m} \cos \frac{m2x}{\sinh m2x} \]

\[ + \frac{5}{2x} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[ \sin \left(0.3x + 0.1my\right) + \sin \left(0.3x - 0.1my\right) \right] \]

\[ + \frac{5}{2x} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[ \sin \left(0.7x + 0.1my\right) + \sin \left(0.7x - 0.1my\right) \right] \]

\[ + \frac{5}{2x} \sum_{m=1}^{\infty} \frac{1}{m} \left[ \sin \left(0.3x + 0.1my\right) + \sin \left(0.3x - 0.1my\right) \right] \]

\[ + \frac{5}{2x} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[ \sin \left(0.7x + 0.1my\right) + \sin \left(0.7x - 0.7my\right) \right] \].

Clearly all of the terms in the above \( T_2 \) equation containing the \( \exp \left(-mx\right) \) converge rapidly. For the others, we can read the infinite cosine values from Table 6 from the column labeled \( F_{\lambda} \) and the infinite sine values from the same table from the column labeled \( F_{\theta} \).

Let us now consider the point \( x = 0 \) and \( y = 0 \) for our example.

For this point Equations 23, 24 and 25 reduce to

\[ T_1 = - \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cosh mx}{\sinh mx} \]

\[ T_2 = \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{\cosh mx}{\sinh mx} + \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cosh mx}{\sinh mx} \]

\[ + \frac{5}{2x} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[ \sin \left(0.3x + 0.7mx\right) - \sin \left(0.7mx\right) \right] \frac{\cosh mx}{\sinh mx} \]

\[ T_3 = - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\sinh mx}. \]
Transforming the first term of $T_2$ [Kirkham, 1958; Eq. 99] and adding the equations we have

$$T_1 + T_2 + T_3 = -\sum_{m}^{\infty} \frac{1}{m \sinh mx} + \sum_{m}^{\infty} \frac{1}{m \sinh 2mx}$$

$$+ \frac{5}{\pi} \sum_{m}^{\infty} \frac{1}{2} \left[ \sin m0.3x - \sin m0.7x \right] \frac{\cosh mx}{\sinh mx}$$

$$- \sum_{m}^{\infty} \frac{1}{m \sinh mx}$$

Applying again formula 654.5 from Dwight [1957] we write

$$T_1 + T_2 + T_3 = -\sum_{m}^{\infty} \frac{1}{m \sinh mx} + \sum_{m}^{\infty} \frac{1}{m \sinh 2mx}$$

$$+ \frac{5}{\pi} \sum_{m}^{\infty} \frac{1}{2} \left[ \sin m0.3x - \sin m0.7x \right]$$

$$+ \frac{5}{\pi} \sum_{m}^{\infty} \frac{1}{2} \left[ \sin m0.3x - \sin m0.7x \right] \frac{\cosh mx}{\sinh mx}$$

$$- \sum_{m}^{\infty} \frac{1}{m \sinh mx}$$

In the above, the sum of the infinite sine term can be obtained from column $P_2$ of Table 6, whereas we note that the others converge rapidly.

Here again we notice that at the point $y = h$ and $x = 0$, we would arrive at the same final summation equation as that just derived for the point $y = 0$, $x = 0$; the reason being the interchangeability of the Equations 23 and 25 for the two points.
Having derived the methods for calculating the potential $k\phi_1$ at
the boundaries of our flow region (and everywhere in the flow region),
we are now in a position to consider the potential function values
$k\phi_1$ accumulated for the whole flow region of our example and these are
shown in Table 3.

Referring to Table 3 and recalling that earlier we said (below
Equation 13) that we wanted the lowest point on the water table to
have a zero hydraulic head (denoted in Figure 3 as the reference level),
we, therefore, must have

$$\phi(0,0) = k\phi_1 + C;$$ \hspace{1cm} (27)

that is,

$$0 = -0.0500 + C$$

$$C = 0.0500 .$$

It is clear now from Equation 27, with our value $C = 0.0500$
known, that a table of values of $\phi = k\phi$ may be formed by adding 0.0500
to each value in Table 3 and this has been done to form Table 4.

We now come to further important specifications for our example
which are: (1) we wish the drain to be running full at zero back
pressure and at zero suction; and (2) we wish the drain to have its
upper most point at a depth exactly 3 feet below our specified lowest
water table level; that is at $y = 3$ feet. These specifications require
that
Table 3. Values of $k_{f1}$. See below Equation 13 and above Equation 27. The numbers $0, 1, 2, ... 10$ reading from left to right are values of $x$ in $k_{f1}(x,y)$ and the numbers $0, 1, 2, ... 10$ reading from top to bottom are the values of $y$ in $k_{f1}(x,y)$.

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Table 4. Values of $\tilde{f}$, obtained by adding the value of $C = 0.0500$ to Table 3.

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<td>0.2808</td>
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<td>0.2997</td>
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\( f(0,3) = k(-3). \)

That is, using Table 4 we must have

\[-.3589 = k(-3).\]

That is

\[ k = 0.11963 \text{ ft/day,} \] (28)

if feet and days are the units used.

In reviewing the way we have obtained Equation 27 and 28, we see that we can calculate the hydraulic head, \( \phi \), at any point in the flow region by dividing the value in Table 4 for that point by the hydraulic conductivity, \( k \), determined above. For example, let us determine the hydraulic head midway between drains, where \( y = 0 \) and \( x = s \), that is, where in Table 4 \( \phi(0,s) = 0.2997 \). At this point we have

\[ H = \frac{0.2997}{0.11963} = 2.5052 \]

where we let \( H \) denote the maximum height of rise of the water table midway between drains. This means that with the rainfall-artesian flux conditions imposed on our solution, with the drain running full of water with no back pressure, the water table midway between drains stands 2.5052 feet, or approximately 2.51 feet above the reference level (as shown in Figure 3). In the same way we determine \( H \), we can determine the \( Z \) of the water table above the reference level (Figure 3) for all values of \( x \) from \( x = 0 \) to \( x = s \). By connecting points of \( Z \) from \( x = 0 \) to \( x = s \), we obtain the shape of the water table. At \( x = \)
Figure 4. Auxiliary curves for plotting stream functions.
x or y value. At this intersection, we read the coordinate opposite
to the x or y labeled line on the abscissas below; that is, if the line
is labeled as a y we read x below. This procedure gives us the
coordinate values to be plotted on the flow-net. for a given percent
increment of the total \( \psi \). For example let us obtain the coordinates
for the 20 percent streamline of our example. At \( \psi = 0.2 \) we read to
the right in Figure 4, to obtain the coordinate values: at \( x = 4, y =
0 \); at \( x = 3, y = 1.10 \); at \( y = 3, x = 1.16 \); at \( x = 2, y = 2.03 \); etc.
After a point is entered in the flow-net for each set of coordinates
determined in this manner, the line connecting these points is labeled
on the flow-net as the 20 percent streamline.

By the same procedure that we used to obtain Figure 4, from Table
2, for the stream function, \( \psi \), we obtain Figure 5, from Table 4, for
the potential function, \( \varphi \). Figure 5 is then used to find the coordinate
points for a given percent of the total potential difference between
the potential of greatest value in the flow region and the potential
existing at the drain. Returning to Table 4 we see that the greatest
potential for our example solution exists at the points \( y = 0, x = s \);
and \( y = h, x = s \), the value of which is \( \varphi(s,0) = \varphi(s,h) = 2.997 \).
We see also that at the top of the drain where \( y = 3 \), and \( x = 0 \), there
is a negative potential of 0.3589. Therefore, the potential difference
between these two points is

\[
0.2997 - (-0.3589) = 0.6586
\]
Figure 5. Auxiliary curves for plotting potential functions.
When equipotential surfaces are plotted percentage wise on the flow-net the drain radius is taken to be at zero potential. Let us suppose that we want to plot the 20 percent equipotential surface. Then 20 percent of the total potential difference is

\[(0.20)(0.6586) = 0.1317.\]

Therefore, taking the potential at the drain radius of \(-0.3589\) (Table 4) to be the zero equipotential surface we have

\[-0.3589 + 0.1317 = -0.2272.\]

From Figure 5, beginning with the potential value on the ordinate axis of \(-0.2272\), we read to the right determining the coordinate points to be plotted on the flow-net for the 20 percent equipotential surface in the same manner that we used Figure 4 to obtain the location of the points for the streamline.

In concluding the discussion on the procedure for plotting flow-nets, it should be said that our example used here, because of the symmetry that it possesses, is a comparatively simple one to handle. Nevertheless, it involves all of the fundamental steps in plotting flow-nets except for the method of determining the stagnation point. Because of the symmetry of the flow region of our example it was not necessary to determine the stagnation point, but in most of the cases investigated later in this thesis this is an essential step in the
plotting of the streamlines and the method of determination for the point will be discussed.

**Discussion of the slit sink flow-net obtained with the constant flux solution**

In the preceding section, we discussed in detail the procedure for plotting flow-nets and used, in this discussion, as an example, the problem of a slit drainage facility. The flow-net for the slit drainage facility is shown in Figure 6. Probably, the most outstanding feature of Figure 6 is the lack of a constant potential at the surface of the aquifer. In view of the way that the equal potential surfaces form perfect symmetry around the drainage facility, it is obvious that the hydraulic head at the surface of the aquifer varies in exactly the same way as, and equal to, the hydraulic head at the reference level for all values of \( x \). That is, if we were able to slide the piezometer, that is located at the extreme right of Figure 6 with its end on the aquifer surface, to the left, we would see that the water in the piezometer would stand just to the height of the water table at all points along the way.

At the mid point between drains where \( s = 10 \), a stagnation point \( P \) is located directly opposite the center of the drainage facility. The stagnation point, emphasized in Figure 6 with a circle, is the point at which the streamline directed downwardly from the surface meets an upwardly directed one from the aquifer. This is called a
Figure 6. Flow-net for the slit sink example of symmetry having equal and constant flux from rainfall and from the aquifer.
\[ F = R \]

\[ q = \frac{F + R}{k} = \frac{2R}{k} = 0.836 \]
stagnation point because at this point the velocity is zero. The streamline that results from the merging of the two oppositely directed streamlines cuts the flow region in equal vertical halves from the stagnation point to the center of the drain and in this particular case it is the 50 percent streamline.

Several piezometers are inserted into the flow-net to more clearly point out some of the physical aspects. First we observe the increasing dissipation of hydraulic head from the mid point between drains toward the drain. The two piezometers with ends touching the 80 percent equipotential surface illustrate the constancy of the hydraulic head (the water height in the piezometers) along such a surface. The water in each of these piezometers will stand, as is clear from Figure 6, at the height \[(3 + 2.51) \times 0.80 - 3\] = 1.408 feet or approximately 1.4 feet as shown.

It is not very easy to relate the problem of the slit sink to a practical situation; therefore, for our next application of the constant flux solution we adjust the dimensions of the problem to more realistic circumstances.
Investigation of the Point Sink Problem

Using the Constant Flux Solution

Let us now consider the more realistic case of a point sink. This problem presents a situation more easily related to actual conditions than did the slit sink example for several reasons: first, because we will consider flow to drain tubes instead of a slit; secondly, we will consider the situation where the aquifer is located at a depth several times that of the drain radius; finally, the drain tube will not be located in a symmetrical position with respect to the flow region.

In order to apply our solution which was developed by the consideration of a slit sink, we must first modify it. Digressing, it is recalled that in the development of the solution the value of the arbitrary constant $B_m$ was determined with reference only to the boundary containing the drainage slit. This single constant (Equation 12) is the only part of the overall solution of the problem in which the dimensions of the drainage facility are involved.

In order to modify our solution in such a manner that it applies to the point sink problem, we refer to Kirkham [1958]. Let us restate here his procedure for making this modification, since we will want to refer to this process at least once more in this thesis.

If we refer to our Figure 3 (compare his Figure 2a) and suppose that we divide the slit drain of length $b = a$ into equal half-lengths so that we can write
or \( b - a = 2\Delta \) from which we see that with respect to the dimension \( c = \frac{(b + a)}{2} \) we have \( b = c + \Delta \) and \( a = c - \Delta \). Now factoring that part of the second term of \( B_m \) (Equation 12) with which we will be concerned in the modification, we have

\[
- \frac{h}{\pi(b - a)} \left( \sin \frac{\pi b}{h} - \sin \frac{\pi a}{h} \right),
\]

In this quantity we substitute the equivalents of \( b \) and \( a \) from above and find

\[
- \frac{h}{\pi(b - a)} \left( \sin \frac{\pi b}{h} - \sin \frac{\pi a}{h} \right) = - \frac{h}{2\pi\Delta} \left[ \sin \frac{\pi(c + \Delta)}{h} - \sin \frac{\pi(c - \Delta)}{h} \right].
\]

Using a trigonometric substitution [Dwight, 1957; formula 401.05] yields

\[
- \frac{h}{2\pi\Delta} \left[ \sin \frac{\pi(c + \Delta)}{h} - \sin \frac{\pi(c - \Delta)}{h} \right] = - \frac{h}{\pi\Delta} \sin \frac{\pi\Delta}{h} \cos \frac{\pi c}{h}.
\]

Now let \( \Delta \) in the right hand side approach zero so that \( \sin(\pi\Delta/h) \rightarrow \pi\Delta/h \); whence we obtain

\[
- \frac{h}{\pi\Delta} \sin \frac{\pi\Delta}{h} \cos \frac{\pi c}{h} = - \frac{h}{\pi} \cos \frac{\pi c}{h}.
\]

Upon substituting the result of the last expression in \( B_m \) (Equation 12) to replace that portion of the equation which was extracted for
modification, we now designate this modified constant as $D_m$ and write

$$D_m = 2 \{ \frac{[F_s + (-1)^n R_s]}{m} - \frac{[(-1)^n(F_s + R_s) \cos \frac{\pi x}{h}]}{m} \} \cdot (29)$$

With the new constant $D_m$ above, substituted into the auxiliary equations $S_2$, Equation 16, and $T_2$, Equation 20, we are ready to attack the new problem of a point sink solution. We observe that the auxiliary Equations 15, 17, 19 and 21 remain unchanged.

With the replacement of $B_m$ (Equation 12) with its modified version $D_m$, Equation 29, in the auxiliary Equations 16 and 20 the result is

$$S_2 = 2 \sum_{l=1}^{\infty} \left\{ \frac{[F_s + (-1)^n R_s]}{m} - \frac{[(-1)^n(F_s + R_s) \cos \frac{\pi x}{h}]}{m} \right\} \cdot \left\{ \sin \frac{\pi x(h - y)}{h} \frac{\sinh[\pi x(s - x)/h]}{\sinh(\pi x s/h)} \right\} \cdot (30)$$

and

$$T_2 = 2 \sum_{l=1}^{\infty} \left\{ \frac{[F_s + (-1)^n R_s]}{m} - \frac{[(-1)^n(F_s + R_s) \cos \frac{\pi x}{h}]}{m} \right\} \cdot \left\{ \cos \frac{\pi x(h - y)}{h} \frac{\cosh[\pi x(s - x)/h]}{\sinh(\pi x s/h)} \right\} \cdot (31)$$

For the point sink solution we take $s = 10$ feet, $h = 20$ feet, and $c = 4$ feet. We assume the same conditions with respect to $q$, the total flux, that was assumed in the slit sink problem. With these dimensions and assumed $q$ value we can write the auxiliary equations for the stream
function employing Equations 15, 30 and 17, and also, the auxiliary equations for the potential function using Equations 19, 31 and 21. With the auxiliary equations set down, the calculation procedure is much the same as that explained in detail for the slit sink. However there are some differences that should be discussed for the procedure of obtaining flow-nets for the point sink problem.

In the slit sink example a value for the potential function could be obtained for every point on the sink but the point sink potential value is negative infinity. Therefore, one calculates potential values as near to the point sink as practical (in this problem, where c = 4) in order to obtain as accurately as possible some plots of radial equipotentials around the point. Then we choose one of the radial equipotentials to be the zero equipotential or we might say it is chosen to be the drain tube surface. Here, we chose the drain tube surface to pass through the point \((x,y) = (0,3.75)\) so that the drain tube would have a radius of approximately 3 inches. Having made the choice of equipotential surface to be the drain radius, we then go through a procedure similar to that outlined for getting Equations 27 and 28 for the slit sink problem, to adjust the constant \(C\) in the potential function Equation 13, and to determine the hydraulic conductivity \(k\).

We meet with another difference in the plotting of the streamlines. We must accurately locate the stagnation point, since a new streamline arises here and the point is essential for an accurate plot. This
can be accomplished by recalling that at the stagnation point the velocity is zero; therefore, we have

$$\frac{\partial \phi}{\partial y} = 0 \quad (32)$$

at this point.

Since the procedure for calculating a stagnation point for the general case of a slit sink example was not given, we will show the procedure used to calculate the point for the point sink problem by writing

$$\vec{q} = k \phi = -\frac{1}{c} \left[ \sum_{m=1}^{\infty} \cos m0.1 \pi \frac{\cosh m0.1 \pi (20 - y)}{\sinh 2x} \right]$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} \cos m0.5 \pi \frac{\cosh m0.05 \pi (10 - x)}{\sinh 0.5 \pi}$$

$$- \frac{1}{2} \sum_{m=1}^{\infty} \cos m0.1 \pi \frac{\cosh m0.1 \pi y}{\sinh 2x}$$

$$+ C,$$

where $\vec{q}$ here is the complete potential function equation for the point sink problem. After we differentiate the above potential function equation with respect to $y$, we have

$$\frac{\partial \phi}{\partial y} = \sum_{m=1}^{\infty} \frac{(-1)^m \sinh m0.1 \pi (20 - y)}{\sinh 2x}$$

$$- \frac{1}{2} \sum_{m=1}^{\infty} \left[ (-1)^m + 1 - 2 \cos 0.2 \pi m \right] \frac{\sin m0.05 \pi y}{\sinh 0.5 \pi}$$

$$- \sum_{m=1}^{\infty} \frac{(-1)^m \sinh m0.1 \pi y}{\sinh 2x} = 0$$
at the boundary where $x = s$ and with all factorable quantities removed.

If one has plotted all of the streamline points for the streamline that intersects the boundary, at $x = s$, he will have a general idea as to the value of $y$ that must be substituted into the above expression to make the sum of the summation equations equal to zero.

With values of $y$ on each side of the general location of the stagnation point, one obtains numerical values of the sums. These sums obtained are then plotted against the $y$ values substituted in the above expression. Some lesser values of $y$ will yield negative sums, but eventually, as the next greater values of $y$ are substituted, the sums will become positive. From a graph constructed from the sums and values of $y$ used to obtain the sums, one can accurately locate the stagnation point. In this example the value of $y$ that makes the expression above equal to zero is $y = 4.76$.

Finally, we must discuss one other item of procedure in plotting the flow-net for the point sink problem that was not necessary in the slit sink problem. With the streamlines plotted in percentage increments and with all the streamlines directed toward the point on the boundary at which the potential is at negative infinity, the procedure is to lay off equiangular radial lines from this point of the same number as the number of streamlines.

With this brief discussion of the methods of obtaining the flow-net concluded, let us look now at Figure 7 for some of the interesting aspects of this point sink problem as exhibited by its flow-net.
Figure 7. Flow-net for the point sink example having nonsymmetry, but having equal and constant flux from rainfall and the aquifer.
\[ q = \frac{F + R}{k} = \frac{2R}{k} = 0.445 \]
One of the most outstanding features that one notices in Figure 7 is that the aquifer surface is almost an equipotential surface. The next most striking thing is the amount of hydraulic head necessary at the aquifer surface (as shown by the height of water in the piezometer) in order that half of the total flux may be contributed by the aquifer. With the six inch diameter drain tube located only 16 feet above the aquifer surface, we notice that water stands in the piezometer to a height above the reference level that is more than double the maximum height of the water table midway between drains. Another interesting observation one makes from Figure 7 is that the stagnation point on the boundary midway between drains is not located directly opposite the drain, but is actually slightly more than 9 inches below the center of the drain.

From what we have observed about the flow-net presented in Figure 6, it is evident that the constant flux solution for the rainfall-artesian conditions assumed does not give results that correspond very well to a real situation when the ratio of the depth of the aquifer to the drain size is small. On the other hand, Figure 7 shows that the assumed conditions correspond to remarkably realistic results when the ratio of aquifer depth to drain radius is large.
DERIVATION OF THE CONSTANT POTENTIAL SOLUTION

In the foregoing part of this thesis a solution was derived for
the drainage of agricultural land when artesian water seeping upward
from below the drain, and assumed at a constant flux distribution, adds
to the problem of disposing of surplus water from the surface. An
analysis of the solution, which was made by the application of the
solution to two flow regions with different geometries, showed that
the solution gave results that would represent real situations only
when the aquifer was at a depth several times the radius of the
drainage facility.

In the rest of this thesis we will be concerned with the derivation
of another solution and its application to problems of different
geometric and different rainfall-artesian water conditions, from which
the solution will be shown to present realistic results regardless of
the depth of the aquifer.

Basic Assumptions

The assumptions upon which the derivation of this solution is
based include all of those discussed for the first solution, except
that for this solution, to be derived, it is not assumed that the flux
of water from the aquifer surface is constant over the length s (see
Figures 3 or 8). Instead, it is assumed that the artesian gravel
surface is an equipotential surface. It is this equipotential surface
that the artesian gravel in Figure 8 denotes.
The Potential Function

In Figure 8 we notice the geometry is the same as that in Figure 3 except that in Figure 8 a piezometer has been installed with its end at the artesian gravel to draw attention to the hydraulic head \( G \), assumed constant, at the surface of the artesian gravel. The reference level for hydraulic head is at \( y = 0 \) as in Figure 3.

Defining symbols not shown on Figure 8, we let \( R \) represent rainfall flux density, \( \bar{F} \) represent the \textbf{average} upward flux density, \( k \), the hydraulic conductivity constant, \( L \), the thickness of porous medium (perpendicular to the plane of Figure 8) and \( q \), the total flux of water entering a length \( L \) of drain per unit time. The total magnitude of the flux is, for the situation indicated in Figure 8,

\[ q = (R + \bar{F})sL \]

where \( R \) is positive when downwardly directed (evaporation would be a negative \( R \) but we do not consider evaporation here) and where \( \bar{F} \) is positive when directed upward.

With special reference to the Roman numerals at the boundaries of Figure 8, we write the boundary condition for the potential function, \( \varphi \), which are:

At boundary I, one has \(- k \frac{\partial \varphi}{\partial y} = R \), when \( y = 0 \), and \( 0 < x < s \);

At boundary II, one has \(- k \frac{\partial \varphi}{\partial x} = 0 \), when \( x = s \), and \( 0 < y < h \);

At boundary III, one has \( \varphi = G \), when \( y = h \), and \( 0 < x < s \);

At boundary IV, one has \(- k \frac{\partial \varphi}{\partial x} = 0 \), when \( x = 0 \), and \( b < y < h \);
Figure 8. Representation of the drainage problem when rainfall is as before (Figure 3) but when a constant hydraulic head is assumed to exist at the base of the flow region (artesian gravel), rather than a constant flux density.
At boundary V, one has \(- k \frac{\partial \phi}{\partial x} = - \frac{q}{L(b-a)}\) when \(x = 0\), and \(a < y < b\).

At boundary VI, one has \(- k \frac{\partial \phi}{\partial x} = 0\) when \(x = 0\), and \(0 < y < a\).

With our boundary conditions having been set down, we look for a solution which will satisfy these and Equation 4.

In the development of the constant potential solution, we choose \(\phi = k \psi\) in such a form that it will reduce to the condition required by boundary condition III. Such a desired solution is

\[
\psi = \frac{A_0}{2} (h - y) + \sum_{m=1}^{\infty} A_m \frac{2h}{m \pi \cosh(mx(s-x)/2h)} \cos \frac{m\pi y}{2h} \sinh(mx/2h)
+ R(h - y) + kG
\]  

(33)
in which we must take an odd numbered summation index, \(m = 1, 3, 5, \ldots\) for boundary condition III to be satisfied. In the expression above \(\phi\) and \(G\) have the dimensions of length (hydraulic head) and \(A_0\) and \(A_m\) have the same dimensions as the hydraulic conductivity, \(k\); that is length per unit time.

Differentiating Equation 33, we find

\[
-k \frac{\partial \phi}{\partial x} = 0 + \sum_{m=1}^{\infty} A_m \frac{2h}{m \pi \sinh(mx(s-x)/2h)} \cos \frac{m\pi y}{2h} \sinh(mx/2h)
\]  

(34)

and

\[
-k \frac{\partial \phi}{\partial y} = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \frac{2h}{m \pi \cosh(mx(s-x)/2h)} \sin \frac{m\pi y}{2h} \sinh(mx/2h) + R
\]  

(35)

Now if \(A_0\) can be chosen to give

\[
A_0 = 0,
\]  

(36)
we see from Equation 35 that boundary condition I will be satisfied for all values of $A_m$.

We see from Equation 34 that boundary condition II is satisfied for all values of $A_m$ regardless of how we choose $A_o$.

Next, let us assume some function of $y$ which we define as

$$f(y) = -k \frac{\partial f}{\partial x}.$$  

With this definition of $f(y)$ and in view of the boundary conditions IV, V, and VI we can write

$$f(y) = 0 \text{ where } b < y < h, \quad (37)$$

$$f(y) = -\frac{q}{(b-a)L} \text{ where } a < y < b, \quad (38)$$

$$f(y) = 0 \text{ where } 0 < y < a \quad (39)$$

To determine the arbitrary constants in Equation 33, we now make use of Kirkham's [1957, Equation 16] quarter range odd index cosine series (derivable from a Fourier series), which is

$$f(y) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m \cos \frac{\pi y}{2h}, \quad m = 1, 3, 5\ldots \quad (40)$$

where

$$\frac{A_0}{2} = f(h) \quad (41)$$

$$A_m = -\frac{4}{\pi m} f(h) \sin \frac{\pi m}{2}$$

$$+ \frac{2}{h} \int_0^h f(y) \cos \frac{\pi y}{2h} \, dy \quad (42)$$
From Equation 37 we have

\[ f(h) = 0. \]  \hspace{1cm} (43)

Therefore, Equation 41 yields

\[ \frac{A_0}{2} = 0, \]

which shows that \( A_0 \) can be chosen as in Equation 36 without invalidating Equation 40 for use in the development here.

Now substituting Equations 37, 38, 39 and 43 into Equation 42 yields

\[
A_m = \frac{2}{h} \int_{0}^{b} \frac{q}{(b - a)L} \cos \frac{\text{m}y}{2h} \, dy
\]

\[
= \frac{2}{h} \int_{y=a}^{y=b} \frac{q}{(b - a)L} \cos \frac{\text{m}y}{2h} \cdot \frac{2h}{\text{m}x} \, dy
\]

\[
= - \frac{4q}{\text{m}(b - a)L} \int_{\alpha=\text{m}a/2h}^{\alpha=\text{m}b/2h} \cos \alpha \, d\alpha
\]

or

\[
A_m = - \frac{4q}{\pi(b - a)L} \sum_{1}^{\infty} \frac{1}{m} \left( \sin \frac{\text{m}b}{2h} - \sin \frac{\text{m}a}{2h} \right). \]  \hspace{1cm} (44)

Substituting Equations 44 and 36 into Equation 33 yields

\[
\hat{g} = k\hat{G} = - \frac{2ch}{\pi} \sum_{1}^{\infty} \frac{1}{m^2} \left( \sin \frac{\text{m}b}{2h} - \sin \frac{\text{m}a}{2h} \right)
\]

\[
\cdot \cos \frac{\text{m}y}{2h} \cosh \left[ \frac{\text{m}(s - x)}{2h} \right]
\]

\[
+ R(h - y) + kG,
\]  \hspace{1cm} (45)
which can be shown to be a solution of Laplace’s equation (by differentiating and substituting into Equation 4) and which satisfies all the boundary conditions.

The potential function, Equation 45, derived above, is the solution for the slit sink problem of Figure 8.

The Stream Function

With the potential function derived, it is a simple matter to obtain the stream function for the constant potential solution.

Using the Cauchy-Riemann relationships, Equations 6 and 7, one may derive the stream function which is

\[ \psi = \frac{8gh}{\pi^2(b-a)L} \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \sin \frac{\pi nb}{2h} - \sin \frac{\pi a}{2h} \right) \sin \frac{\pi y}{2h} \]

\[ \cdot \frac{\sinh[\frac{\pi L(s-x)}{2h}]}{\sinh(\frac{\pi s}{2h})} + Rx + C \]  

(46)

where C is an arbitrary constant which ordinarily may be taken (as in the following work) equal to zero.

Auxiliary Equations for Calculating Flow-nets

In calculating the flow-nets using the constant flux solution one has to know the hydraulic head G in the artesian gravel necessary to produce a given or assumed total flux. That is to say, that while one can assume a rainfall flux density and a total flux density, one cannot assume simultaneously a fixed value of G and a fixed value of \( \overline{F} \). In our case \( \overline{F} \) was chosen to be a certain value of q with q being related to R. The corresponding value of G was then computed.
Our purpose here will be to explore some relationships between
the various quantities of the solution. Some interesting auxiliary
equations emerge. Since we earlier discussed in some detail the actual
procedure for calculating and plotting flow-nets, and since the methods
are not materially different using the constant potential solution, we
will not repeat details here.

As an aid in manipulation, looking at Equations 45 and 46 we
define a quantity $E_m$ by the relation

$$E_m = \frac{4}{\pi} \left( \sin \frac{\pi h}{2h} - \sin \frac{\pi s}{2h} \right).$$

(47)

Therefore, Equation 45 can be written as

$$k\phi = -\frac{q}{(b - a)L} \sum_{m=1}^{\infty} \frac{2h}{\pi m} \cos \frac{\pi y}{2h} \cos \frac{\pi y}{2h} \frac{\cosh \frac{\pi (s - x)}{2h}}{\sinh \frac{\pi y}{2h}} + R(h - y) + kG.$$

(48)

Then from Equation 48 we can obtain, upon solving for $q$, the
result

$$q = \frac{(-k\phi + R(h - y) + kG)(b - a)L}{\sum_{m=1}^{\infty} \frac{2h}{\pi m} \cos \frac{\pi y}{2h} \cos \frac{\pi y}{2h} \frac{\cosh \frac{\pi (s - x)}{2h}}{\sinh \frac{\pi y}{2h}}}.$$

(49)

From this rearrangement of Equation 48, we see that if we know
all the quantities on the right hand side of Equation 49 we can
calculate the total flux $q$. 
Next, we will solve for the average flux density $\bar{F}$ from the artesian gravel where $\bar{F}$, as a matter of definition, is given by

$$
\bar{F} = \frac{1}{s} \int_{s=0}^{s=s} k \frac{\partial q}{\partial y} \bigg|_{y=h} \, dx .
$$

(50)

From Equation 48 we have

$$
k \frac{\partial q}{\partial y} \bigg|_{y=h} = - \frac{a}{(b-a)L} \sum_{m} \frac{2h}{2x} \left( - \frac{nx}{2h} \right) \sin \frac{nx}{2}
$$

(51)

Substituting Equation 51 into Equation 50 and integrating we obtain

$$
\bar{F} = \frac{2oh}{s(b-a)\pi L} \sum_{m} \frac{1}{m} \sin \frac{nx}{2} - R .
$$

(52)

Substituting the right hand side of Equation 49 for $q$ in Equation 52, then cancelling and rearranging quantities, finally yields

$$
\bar{F} + R = \frac{(- k\phi + B(h - y) + kG) \sum_{m} \frac{1}{m} \sin \frac{nx}{2}}{s \sum_{m} \frac{1}{m} \cos \frac{mx}{2h} \sinh \left(\frac{mx}{2h}\right) / \cosh \left(\frac{mx}{2h}\right)} ,
$$

(53)

from which the quantity $L$ has dropped out.

We now introduce a quantity $A$ to represent the summation terms in Equation 53, so that we have

$$
A = \frac{\sum_{m} \frac{1}{m} \sin \frac{nx}{2}}{\sum_{m} \frac{1}{m} \cos \frac{mx}{2h} \sinh \left(\frac{mx}{2h}\right) / \cosh \left(\frac{mx}{2h}\right)} .
$$

(54)
With the substitution of Equation 54 into Equation 53 we obtain

\[(F + R)s = (-k\phi + R(h - y) + kG)A\]. (55)

Use of Equation 55 provides us with the means to make the lowest point of the water table have a hydraulic head of zero.

It is seen from Equation 55 that if we set \(-k\phi\) equal to zero and substitute in Equation 54 the appropriate values of \(x\) and \(y\), say \(x\) and \(y\), for which we desire \(k\phi\) to be zero, we can then rearrange Equation 55 to give

\[kG = \frac{(F + R)s}{A(x, y)} - R(h - y)\] (56)

from which we can calculate the value of \(kG\) necessary to make the potential function zero for any value of \((F + R)s\). Here \((F + R)s\) is the total flux, \(q\), for a unit length of drainage facility. If we assume, as was done in this study for most of the problems considered, that \(F\) is equal to \(R\) we have

\[kG = \frac{(2R)s}{A(x, y)} - R(h - y)\]. (57)

With the above expression we end the discussion of calculation procedures. However, throughout the discussion of individual problems investigated, any noteworthy procedures which apply specifically to the particular problem will be considered at that time.

It should be remembered that in Equation 33 through 57 and in all remaining equations to be displaced, except for two unnumbered equations following Equation 63, that \(m = 1, 3, 5, \ldots\).
Investigation of the Slit Sink Problem

Using the Constant Potential Solution

The equations for the stream and potential functions of the slit sink problem are Equations 45 and 46. The dimensions and rainfall-artesian conditions assumed in this slit sink problem are the same as those used for the calculation of the slit sink flow-net (Figure 6) obtained by using the constant flux solution. These were chosen to be the same so that a direct comparison could be made between the two solutions: one with constant flux providing the upward seepage, and one with constant potential providing the upward seepage.

Figure 9 is our first flow-net with constant potential providing the upward seepage.

We observe that the piezometer standing with its end on the surface of the artesian gravel has a hydraulic head of 1.31 feet with respect to the reference level. This artesian gravel hydraulic head and all of the hydraulic head dimensions shown in Figure 9 are calculated by using the same hydraulic conductivity of 0.11963 feet per day as that determined for the slit sink problem earlier. Water will stand in the piezometer at 1.31 feet above the reference level regardless of where it is located on the surface of the artesian gravel.

For this problem, the 100 percent potential is located at \( x = s \) on the reference level, which results in a maximum water table height of 2.31 feet as compared to a maximum height of 2.51 feet for the same problem solved with the constant flux solution. Another feature wherein
Figure 9. Same as Figure 6 except that the upward seepage comes from a surface of constant potential (here 1.31 feet of head above the reference level) instead of from a constant flux source.
$F = R$

$q = \frac{F + R}{k} = \frac{2R}{k} = 0.836$
Figure 9 differs markedly from Figure 6 in the location of the stagnation point P which is lower in Figure 8 than in Figure 6. Also in Figure 8, the piezometer at the drain surface is at a slight suction; whereas in Figure 6 the drain was just running full (at zero-suction and zero pressure).

Investigation of the Point Sink Problem

Using the Constant Potential Solution

One can modify the slit sink Equations 45 and 46 to obtain a point sink solution in the same way that the constant flux, slit sink solution was modified earlier to give a solution for the point sink case. Specifically, the transformation shown below as

\[
\frac{h}{\pi(b-a)} \left( \sin \frac{\pi b}{2h} - \sin \frac{\pi a}{2h} \right) = \frac{m}{2} \cos \frac{\pi c}{2h}
\]

is the modification to be made in the summation terms of Equations 45 and 46 and is arrived at by the same reasoning process as that used in the derivation of Equation 29. Upon the substitution of the right hand side of Equation 58, for its equivalent, in Equations 45 and 46 we have

\[
\mathcal{F} = kG = - \frac{4G}{Lx} \sum_{l=1}^{\infty} \frac{1}{m} \cos \frac{\pi c}{2h} \cos \frac{\pi y}{2h} \frac{\cosh[\pi x(s-x)/2h]}{\sinh(\pi x/2h)} + R(h-y) + kG,
\]

and

\[
\psi = \frac{4G}{Lx} \sum_{l=1}^{\infty} \frac{1}{m} \cos \frac{\pi c}{2h} \sin \frac{\pi y}{2h} \frac{\sinh[\pi x(s-x)/2h]}{\sinh(\pi x/2h)} + Rx + C
\]
where $C$ is as before an arbitrary constant, where $m = 1, 3, 5, ...$

The first flow-net calculated by making use of Equations 59 and 60 has the same dimensions and rainfall-artesian flux conditions as that of the point sink problem which was worked out using the constant flux equation. The dimensions and conditions imposed on the problem were done so purposely, in order to compare the two types of derived solutions. That the comparison is remarkably good between the two solutions is attested to by the fact that, for all practical purposes, the flow-net calculated with the constant flux solution and exhibited as Figure 7 would be the same as the one worked out here using the constant potential solution. Because the two flow nets would be, for the scale plotted here, the same, the latter was not plotted. For example, when Equation 57 with $X = 0$ feet and $Y = 4$ was employed to obtain $kG$ for the problem it was found to have a value of 0.8132 as compared to the greatest potential value of 0.8177 at the point $x = 10$ and $y = 20$ for the constant flux solution. Dividing these two values by the hydraulic conductivity value of 0.2247, which we recall, was calculated from the constant flux solution for the problem, we find that the numerical value of the hydraulic head between the two varies by only 0.02 feet. Since this is the greatest difference that exist between numerical values of hydraulic head at any point throughout the flow-nets, it is seen that either of the two solutions could have been used to obtain Figure 7. However, the constant potential solution is preferred because it is simpler.
The next problem worked out, by using Equations 59 and 60, was that for the case where the lowest point of the water table just touched the top of a drain tube with a radius of 3 inches. For the conditions, in their respective units, of \( F = R = 0.05 \) and \( k = 2 \) it was found, utilising Equations 57 with \( X = 0 \) and \( Y = 0.25 \), that \( kG = 2.1151 \). That is, if units of feet are used, as they were here, then the hydraulic head \( G \) at the artesian gravel is \( 2.1151/2 = 1.06 \) feet above the reference level as shown in Figure 10. This hydraulic head at the artesian gravel just slightly exceeds the maximum height of the water table which is 1.03 feet, for the \( k \) value of 2 feet per day.

One can adapt the constant potential solution for the point sink to the most interesting case of tile drains running half full. When the drains are running half full then the lowest point of the water table will touch the reference level at a distance equal to the radius of the tile from the zero point of the coordinate axes (where the reference level is the \( x \) axis). To obtain the solution for the problem of drain tubes running half full, we merely let the dimension \( c \) in Equations 59 and 60 go to zero and write

\[
\bar{g} = k\phi = -\frac{4G}{Lx} \sum_{1}^{\infty} \frac{1}{m} \cos \frac{\pi y}{2h} \frac{\cosh[m\pi(s-x)/2h]}{\sinh(m\pi s/2h)} + R(h - y) + kG, \quad (61)
\]

and

\[
\phi = \frac{4G}{Lx} \sum_{1}^{\infty} \frac{1}{m} \sin \frac{\pi y}{2h} \frac{\sinh[m\pi(s-x)/2h]}{\sinh(m\pi s/2h)} + Rx + C, \quad (62)
\]

where \( m = 1, 3, 5, \ldots \)
Figure 10. Flow net for drain running full.
\[ F = R \]

\[ \frac{q}{k} = \frac{F + R}{k} \]

\[ = \frac{2R}{k} = 0.05 \]
Before we go further with the discussion of example problems worked out with this solution, let us note first that, when we substitute small values of $X$ in Equation 57 to solve for $kG$ at a specified or assumed condition for $q$, and also, when we substitute small values of $x$ in Equation 61, our summation term converges only slightly better than $\frac{1}{n}$. Let us eliminate this difficulty that would arise in the calculation of the potential function at $y = 0$ by writing Equation 61 as

$$
\psi = kG = \frac{-4a}{lx} \sum_{1}^{\infty} \frac{1}{m} \frac{\cosh[m(s-x)/2h]}{\sinh(mxs/2h)} + Rh + kG.
$$

Then considering only the summation term in the equation above, we can further write

$$
\sum_{1}^{\infty} \frac{1}{m} \frac{\cosh[m(s-x)/2h]}{\sinh(mxs/2h)} = \sum_{1}^{\infty} \frac{1}{m} e^{-2h} + \sum_{1}^{\infty} \frac{1}{m} e^{-2h} \frac{\cosh(mx/h)}{\sinh(mx/h)}
$$

by an identity [Kirkham, 1958]. In Equation 63, we see that the second term of the identity will sum rapidly because of the $\exp (-\frac{mx}{2h})$, but the first term $\exp (-\frac{mx}{2h})$ still sums very slowly because of the small value of $x$ near the drain tube. In order to get the first term in the right hand side of Equation 63 in a closed form, let us borrow a result of Kirkham [1958, Equation 97] which is

$$
\sum_{1}^{\infty} \frac{1}{m} \cos ma e^{-mb} = -\frac{1}{2} \ln[2e^{-B} \cosh B - \cos A] .
$$
We see from this expression that, if we let the cosine argument be zero, it yields

\[ \sum_{m=1}^{\infty} \frac{1}{m} e^{-mB} = -\frac{1}{2} \ln[2e^{-B}(\cosh B - 1)] \]

where \( m = 1, 2, 3, \ldots \). Now to change this last expression in order that it may be applicable for our odd number summation index, we expand and regroup terms such that we can finally write (now with \( m = 1, 3, 5, \ldots \))

\[ \sum_{m=1}^{\infty} \frac{1}{m} e^{2h} = -\frac{1}{2} \ln[2(\cosh \frac{2h}{h} - 1)] + \frac{1}{4} \ln[2(\cosh \frac{2h}{h} - 1)]. \] (64)

With the right hand side of Equation 64 replacing the first term in the right hand side of Equation 63 one can obtain accurate values of the potential function very close to the point sink, located at the zero point of the coordinate axes.

In the following discussion of the several problems worked out using Equations 61 and 62 all of them are dimensionally alike — only the rainfall-artesian conditions change. All of the problems are calculated for a drain radius of 3 inches.

Figure 11 is the flow-net for the same rainfall-artesian conditions as those assumed earlier for the tile running full. The latter flow-net was shown in Figure 10. Comparing Figures 10 and 11 one sees immediately that according to our solution, for a 3 inch radius drain tube, there is, except near the drain tubes, no discernible difference
Figure 11. Same as Figure 10 except that the drain runs half full, and with its axis at the zero reference level.
$q = \frac{F + R}{k} = \frac{2R}{k} = 0.05$
in the flow-nets; and there are no observed differences for the artesian and surface heads, 1.06 and 1.03 feet for the two figures. Evidently, from our solution, for the dimensions taken and conditions assumed for the problem, there is no practical difference in the water table or other points of the flow region regardless of whether the tile is running full or only half full.

In Figure 12 is the flow-net for the condition of zero rainfall, with the total flux density maintained by the artesian water flux to the same value as that assumed in the problems for which Figures 10 and 11 were drawn. Figure 12 shows that a higher hydraulic head in the artesian gravel is needed to maintain a maximum water table height equal to that present when half of the water comes from the surface. Note also that the stagnation point is now on the reference level midway between drains.

Figure 13 is the flow-net for the condition that the rainfall is maintained at the same value of 0.05 feet per day as was assumed for the rainfall in the problems from which Figures 10 and 11 had their inception. However, in Figure 13 half of the water goes to the drains and the other half to the artesian gravel. To more clearly illustrate how this could be an actual phenomenon, attention is directed to Figure 14. In Figure 14, we see that, if a stream intersects the artesian layer, the height of the water table in the soil depends on the height of water in the stream. Therefore, additional water applied to the soil would not affect the water table height
Figure 12. Flow-net for which the conditions are the same as for Figure 11 except that here \( R = 0 \) (to result in a higher artesian head 1.31 feet compared with the value 1.06 feet of Figure 11, \( q/k \) being the same, 0.05, in Figures 11 and 12).
NO RAIN

SOIL SURFACE

WATER TABLE

REF. LEVEL

H = 1.03'

R = 0

\[ \frac{q}{k} = \frac{F + R}{k} \]

\( = \frac{F}{k} = 0.05 \)

DISTANCE IN FEET

DISTANCE IN FEET

ARTESIAN GRAVEL

2 4 6 8 10
Figure 13. Flow-net for a condition when half the rain water drains into the tube drain and half into the artesian gravel. See text.
\[
\begin{align*}
\bar{F} & = -\frac{R}{2} \\
\frac{q}{k} & = \frac{\bar{F} + R}{k} \\
& = \frac{-\frac{R}{2} + R}{k} \\
& = \frac{R}{2k} = 0.005
\end{align*}
\]
RAIN

STREAM

SOIL SURFACE

WATER TABLE

ARTESIAN GRAVEL

IMPERVIOUS STRATUM

TILE
Figure 14. Diagram illustrating how, in nature, drainage may occur to drain tubes and to a gravel stratum, simultaneously.
unless it was supplied in quantities great enough to increase the water level in the stream. If the water height in the stream is lower than the water table in the soil, the artesian gravel would serve as a drainage facility for all or part of the percolating surface water.

Figures 13 and 14 show that this could happen even when the hydraulic head in the artesian gravel is slightly higher than the reference level. The stagnation point in Figure 13 lies on the y axis below the drain, as indicated by P.

Figure 15 shows a more extreme case of the condition observed in Figure 13. When the rainfall flux density is maintained at a value of 0.05k and the artesian hydraulic head is such that $G = -h/2$.

Then, the situation shown by Figure 15 is one for which the drain tubes are supplying water at a calculated flux density of 1.715k. That is, the tubes are not drains in this case, but are in fact irrigating the soil. The negative heights in the piezometers in Figure 15 show that the soil water along the reference level is under a tension equal to the vertical distance from a point on the reference level to the tension line, T. With a wide stretch of the imagination one can conceive of the possibility that the stream, depicted in Figure 14, may produce an effect similar to that of a tension table apparatus when the stream stands at a low level. It should be noted in Figure 15 that with these conditions there is no stagnation point.
Figure 15. Flow-net when the "drain tube" and the rain both supply water to the gravel. See text.
Investigation of the Slit Sink Ditch Drain Problem

Using the Constant Potential Solution

By an inspection of Equation 45 and 46 one soon discovers that if he lets the value \( a \), present in the constant quantity of the summation term, go to zero he would have the solution for the ditch problem. With this let us write,

\[
\phi = k\varphi = - \frac{8oh}{2} \sum_{n=1,3,5,\ldots} \frac{1}{l} \sin \frac{mb}{2h} \cosh \frac{mX}{2h} + R(h - y) + kG
\]

(65)

\[
\psi = \frac{8oh}{2} \sum_{n=1,3,5,\ldots} \frac{1}{l} \sin \frac{mb}{2h} \sin \frac{mX}{2h} + Rx + C
\]

(66)

where \( m = 1, 3, 5, \ldots \)

The flow-net in Figure 16 was calculated by the Equations 65 and 66. For Figure 16, the value of \( A(X,Y) \) was taken as \( A(0.5,0) \) in Equation 57 in order to calculate the value of \( kG \) needed to make the lowest point of the water be at the point \( (X,Y) = (0.5,0) \). The value of \( b \) used in Equations 65 and 66 was 2 feet. The rest of the dimensions are those of the flow-net, Figure 16. The rainfall-artesian conditions assumed were the same as those for the problems represented by the flow-nets of Figures 10 and 11. It is interesting to note that the
Figure 16. Flow-net for a ditch of semi-width 0.5 foot and depth of approximately 2 feet. See text.
hydraulic head at the artesian gravel surface and the maximum height of the water table is less by almost one half for the ditch than it is in Figures 10 and 11 for the same rainfall-artesian flux conditions. It should be noted that although the depth \( b \) was chosen to be 2 feet, the equipotential (also ditch surface) that intersects the reference level at the lowest point on the water table does not intersect the \( y \) axis at 2 feet as chosen. It is in fact 0.2 foot shallower than \( b \) was chosen to be. Figure 16 does, however, give a good representation of a ditch. It was thought that perhaps this "pushing up" of the bottom of the ditch might be due to the proximity of the artesian gravel. In order to investigate a little further the effect of the artesian gravel on the shape of the ditch and also to see how the water table was affected, the potential function shown in Figure 17 was calculated and plotted. In Figure 17 the rainfall-artesian flux conditions are the same as for Figure 16, but \( b \) was chosen to be 4 feet deep. For \( A(X,Y) \), in Equation 57, the values are \((1,0)\) which yields a hydraulic head at the surface of the artesian gravel of approximately 0.62 feet above the reference level and which is almost double the maximum height of the water table. Looking at Figure 17, it is clear that taking the artesian gravel at a greater depth did not change the "pushing up" of the equipotential surface of the ditch at the bottom. In fact, the ditch is somewhat dished out on the sides in this case, but still an excellent representation of a ditch.

Figure 17 is the last of the flow-nets.
Figure 17. Same as Figure 16 except that ditch has semi-width 1.0 feet. See text.
RAIN
SOIL SURFACE

WATER TABLE
REF. LEVEL

H = 0.37'

DISTANCE IN FEET

ARTESIAN GRAVEL

\[ F = R \]
\[ \frac{q}{k} = \frac{F + R}{k} \]
\[ = \frac{2R}{k} = 0.05 \]
Computational Methods for Obtaining the Maximum Water Table Height

After having put our constant potential solution through the test of plotting a number of flow-nets, let us now concentrate our attention on obtaining a computational method from this solution which will aid the drainage worker in determining spacing and depth of placement of drains, when artesian water must be taken into account. Things which the designer usually knows, or can readily determine are (1) the rainfall flux density to be expected, (2) the hydraulic head on the artesian water source and (3) hydraulic conductivity of the soil. With the above three factors known, or at least well estimated, let us precede to develop an equation by which the designer can easily make the necessary computations for a drainage system.

Since the most practical design would be one for drain tubes running half full, (which would be, for all practical purposes, equivalent to the one for drains running full — compare Figures 10 and 11) let us begin with Equation 61 by writing

\[ \bar{q} = k \phi = \frac{-kG}{Lx} + \frac{1}{m} \int_{1}^{\infty} \frac{\cosh(\pi x/2h)}{\sinh(\pi x/2h)} + Rh + kG \]  

(67)

where we have taken \( y = 0 \) since we are only interested in the potential function at the reference level.

Now for any specified rainfall flux density \( R \), artesian hydraulic head \( G \), and hydraulic conductivity \( k \), we want the lowest point of the water table to intersect the drain surface at one-half the drain
height. If we take our origin of coordinates at the drain center with positive y in the downward direction and the x axis positive to the right as the reference level, then the lowest point of the water table will be on the reference level at a distance r from the origin of coordinates, where r is the drain tube radius. To this point we have just followed the reasoning used in obtaining Equations 56 and 57, except there x was used for r here because at the time we developed Equations 56 and 57 we wanted them to be applicable to types of drains which were not necessarily tubes.

Recalling that at the lowest point of the water table, k0 = 0, we have from Equation 67

\[ 0 = \frac{4g}{L} \sum_{m=1}^{\infty} \frac{1}{m} \cosh[m(s-r)/2h] + Rh + kG \]

where r, the drain radius has replaced x. With this expression we can solve for q to obtain

\[ q = \frac{4L}{L} \left( \frac{Rh + kG}{\sum_{m=1}^{\infty} \frac{1}{m} \cosh[m(s-r)/2h]} \right) \]  \( (68) \)

Returning to Equation 67 and writing it in the form for obtaining the potential midway between drains or where k0 = kH (H is the maximum hydraulic head which equals the height of water table above the reference level where x = s) we have
Substituting the value of $q$ from Equation 68 into Equation 69 we have

$$H = \frac{(R + kG)}{k} \left[ 1 - \frac{\sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\sinh(m\pi s/2h)} + R_G + kG}{1 - \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\cosh(m\pi (s-r)/2h)} \frac{1}{\sinh(m\pi s/2h)}} \right]$$

where $m = 1,3,5,...$

In Equation 70 let $g$ be defined by

$$g = \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{\sinh(m\pi s/2h)} \frac{1}{\cosh(m\pi (s-r)/2h)}$$

With the above definition we can now write Equation 70 as

$$H = \frac{(R + kG)}{k} [1 - g(\frac{2s}{h}, \frac{h}{2r})]$$

where $g$ is a function of $\frac{2s}{h}$ and $\frac{h}{2r}$.

As an aid in computation an extensive table of $g$ has been prepared and given in Table 5. The entries in the table of $g$ will be of more value once they are presented in the form of a nomograph similar to the nomograph for drainage of rainfall alone [Kirkham, 1958; Toksöz and Kirkham, 1961].
Table 5. Values of $g$ in which $g$ is considered to be a function of $\frac{2a}{h}$ and $\frac{h}{2r}$. 
The objective of this study was to theoretically analyze the problem of drainage (of agricultural lands) when the rainfall water seeping through a homogeneous soil medium to drains of various types is reinforced by upward seepage of artesian water.

During the investigation, two solutions were derived for the problem of dealing with various rainfall-artesian water conditions. The first solution considered a simultaneous constant upward flux density from an aquifer and a constant flux density downward of rain water. The second solution considered the downward flux density of rain water as constant, but the upward flux density of water from an artesian gravel originated from a constant equipotential surface.

Detailed procedure was presented for calculating and plotting flow-nets.

Flow-nets were plotted for both solutions, with some comparisons made between the two solutions.

It was found that the two solutions gave nearly identical results when the aquifer or artesian gravel was deep compared to the drain size. However, the constant flux solution did not give realistic results when the aquifer was near the drainage facility. The constant potential solution gave exceptionally realistic results regardless of the depth to the artesian gravel.

The constant potential solution (the superior solution) was used to derive an equation for the calculation of the maximum height of
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rise midway between drains. With the derived equation for calculating the maximum height of rise of the water table, a table of values of a function $g$ was calculated and presented in tabular form. The table of $g$ values saves time of calculation and should eventually be presented in the form of a nomograph.

In conclusion it should be added that the constant potential solution, because of its simplicity of calculation and excellent results obtained from the flow-nets has contributed materially to the soil drainage literature in which there is a real need for solutions dealing with simultaneous flux of water from the surface and artesian sources.
Table 6. Sums of infinite sine and cosine series, where \( \sum_{m=1}^{\infty} \cos \frac{mx}{m} = F_1, \sum_{m=1}^{\infty} \sin \frac{mx}{m} = F_2 \) for 

\( m = 1, 2, 3, \ldots, \) and also, \( \sum_{m=1}^{\infty} \frac{\cos \frac{mx}{m}}{m} = \frac{1}{2}[F_1 - F_1(\pi - x)] \) and \( \sum_{m=1}^{\infty} \frac{\sin \frac{mx}{m}}{m} = \frac{1}{2}[F_2 + F_2(\pi - x)] \) 

for \( m = 1, 3, 5, \ldots \)
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Kirkham, Don 1957a. Tables of sine and cosine of arguments 0.1mm, 0.05mm, and 0.025mm. (Mimeographed) Soil Physics Section, Agronomy Dept., Iowa State University of Science and Technology. Ames, Iowa.


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