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Models of unsteady state flow in porous media applied to soil drainage

James Teddie Ligon
Iowa State University

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MODELS OF UNSTEADY STATE FLOW IN POROUS MEDIA
APPLIED TO SOIL DRAINAGE

by

James Teddie Ligon

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Dean of Graduate College

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Of Science and Technology
Ames, Iowa
1961
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INTRODUCTION AND OBJECTIVES

Artificial removal of excess water from within the soil to produce a more desirable medium for plant growth, to allow working of the soil, or to improve the soil condition for other purposes has been practiced in various parts of the world for many hundreds of years. In particular, drainage of land for crop production has been widespread since often the most fertile soils have poor natural drainage. In earlier days drainage was accomplished in most cases by the use of open ditches. Today buried tile drains are more common.

In Iowa artificial drainage of farm land is a widespread practice. However, there remains much additional land in the state which is so wet at certain seasons and in certain years that farming operations are hampered and crop yields reduced. As a result additional thousands of acres are being brought under subsurface drainage each year.

Too often the layout of drainage systems has been accomplished with little knowledge of the principles governing flow of water through the soil to the drainage structures. If the drainage system were designed on any rational basis at all it has usually been on the basis of the designer's experience with other systems in the same area and in similar soils.

In recent years an increasing amount of attention has been
given by soil physicists and agricultural engineers to the basic laws and principles underlying soil water movement. The volume of literature published in this area in recent years is quite large. Many particular soil drainage problems have been solved exactly mathematically on the basis of various simplifying physical assumptions. This is especially true for steady-state flow problems. The problem of the falling water table under drainage conditions is a more formidable one mathematically and when it has been approached at all it has been on the basis of both physical and mathematical assumptions and approximations. In most cases these underlying assumptions and approximations have not been checked for validity and thus the accuracy of the answers obtained is often unknown. The scarcity of field experimental data supporting the theoretical solutions is well known. Such data is difficult and expensive to obtain.

A situation has developed in which many soil drainage problems have been solved theoretically for idealized cases but in which the practical designer of drainage systems still relies largely on past practice and experience. This situation is due to the practical designer's distrust of theoretical solutions which have not been corroborated by actual drainage data. In view of this situation and the difficulty experienced in obtaining field data, models offer possibilities for at least partially bridging the gap between the two extremes which exist
in drainage work.

The ideal drainage model could be set up in the laboratory and operated as desired without regard to weather, crops, etc., and would yield results which could be used to accurately predict the operation of field drainage installations. At the same time the results could be compared with theory to test the validity of the assumptions and approximations made in arriving at theoretical solutions.

As will be discussed in the literature review, models have been utilized by other investigators for studying drainage problems. In most cases, however, the main concern has been the obtaining of geometrical similarity between model and prototype rather than a complete consideration of all the variables involved.

The objectives of this study were as follows:

1. To investigate the application of the theory of similitude to the modeling of unsteady-state soil drainage problems, particularly the problem of flow to open ditches under falling water table conditions,
2. To construct and operate such a model if feasible,
3. To obtain data from the model which could be used in field drainage design, and
4. To compare the results of the model tests with the results of certain theoretical investigations.
REVIEW OF LITERATURE

Drainage Theory

General

The volume of literature which treats theoretically the problem of soil drainage is considerable. The methods of approaching the problem have been many and varied. No attempt will be made here to present a complete review of the entire literature of drainage theory, but a few of the more widely recognized contributions will be presented, in particular those pertaining to the problem of unsteady flow in the soil.

Schmid (49) outlined the general types of problems of potential flow of water through the soil and methods of solving these problems as follows:

A. Steady-state flow with geometrically prescribed boundaries
   1. Analytical solutions
   2. Analogue solutions
   3. Graphical solutions (flow nets)
   4. Mechanical models

B. Steady-state flow with a free boundary
   1. Analytical solutions
      a) Complex representation and the hodograph transformation
      b) Dupuit's assumptions
2. Analogue solutions using electrical conducting paper
3. Graphical solutions
C. Unsteady-state flow.

Schmid did not suggest means of handling the problem of unsteady-state flow. He did suggest that the problem of steady-state flow with geometrically prescribed boundaries could be solved by direct model tests provided the prescribed geometric relations were maintained.

Muskat (42) presented methods for approaching many different types of problems of flow through porous media, particularly steady-state problems with both fixed boundaries and free surface boundaries. His book is still considered one of the best references on the subject.

Scheidegger (47) in a more recent volume considered in detail the physics of flow through porous media. He placed emphasis mainly on the laws underlying flow through porous media rather than on the solution of specific problems.

In one of a series of monographs prepared by the American Society of Agronomy, Luthin (31) edited the contributions of several authorities on the general subject of agricultural drainage. Physics of flow through porous media, mathematical solutions of drainage problems, engineering design of drainage
systems, and crop and soil relationships as affected by drainage were discussed in considerable detail.

van Schilfgaarde, Kirkham and Prevert (53) reviewed most of the theories which have been advanced concerning tile and ditch drainage. They attempted to evaluate these theories on the basis of physical correctness and their applicability in drainage design. Comparisons with some field data were made. The reader desiring a rather complete review of drainage theories useful in design is referred to this bulletin.

The steady-state problem with prescribed boundaries

In 1854, a French hydraulic engineer named Darcy (7) discovered a law which is now recognized as the basic law in practically all problems of flow of liquids through porous media. This has become known as Darcy's Law and is written as

\[ Q = -K_iA, \]

where \( Q \) is the rate of discharge of fluid, \( A \) is the constant cross sectional area of the flow region, \( i \) is the hydraulic gradient, and \( K \) is a constant called the hydraulic conductivity which is dependent upon the properties of the porous medium and the characteristics of the fluid. According to Philip (44) this law is applicable whenever flow through the pores is laminar and the inertia of the moving fluid is negligible.

Slichter (50) showed that the flow of groundwater takes
place according to Laplace's equation. He derived Laplace's equation from Darcy's Law and the equation of continuity. He pointed out that a problem in the steady motion of groundwater is thus analogous to a problem in the steady flow of heat or electricity or the steady motion of a perfect fluid. After pointing out that the solution of a steady flow problem must be a solution of Laplace's equation fitted to the particular boundary conditions of the problem, Slichter solved a few simple problems.

In a series of papers Kirkham (21, 22, 23, 24) presented exact solutions for several steady state drainage problems involving flow to tile and ditches from a horizontal water table or a ponded surface. These solutions are in the form of infinite series, but convergence of the series is quite rapid in most cases so that evaluation of only two or three terms is necessary to obtain a satisfactorily accurate numerical answer.

Gustafsson (12) investigated steady flow to tile drains from horizontal and constantly sloped water tables. While these steady state problems investigated by Kirkham and Gustafsson are rather ideal cases and probably rarely occur in practice, their solutions are of interest because in most cases they represent the extreme conditions which would be met in practice.
The steady-state problem with a free surface

The problem of steady state flow through soil under conditions where a free water surface or water table exists has been investigated theoretically by several authors. This is the basic problem reviewed by van Schilfgaarde et al. (53) in the bulletin mentioned earlier.

Shortly after the appearance of Darcy's work, Dupuit (9) examined certain problems involving gravity flow to shallow drains. His studies were based on the assumptions (now known as the Dupuit-Forcheimer assumptions) that all streamlines in the flow system are horizontal, and that the velocity along these streamlines is proportional to the slope of the free surface but independent of the depth. These assumptions have been used by a great many other investigators and under certain conditions of very flat water tables may lead to valid results.

Gustafsson (12) investigated the steady flow problem where a water table exists below the soil surface and flow to the tiles takes place from a gravel layer below. He utilized the hodograph method for solving the problems and his solutions are in the form of elliptic functions.

Breitenöder (1) discussed the solution of various groundwater flow problems where a water table exists in the soil.
Conformal mapping techniques, particularly the hodograph method, were used extensively.

Polubarinova-Kochina and Falkovich (45) presented methods for solving various groundwater flow problems, particularly those involving flow under and through dams where free surfaces exist. They also included an extensive bibliography of literature, mostly Russian, pertaining to seepage flow.

Kirkham (25) considered the general problem of flow to drains where the drains are underlain by an impermeable layer at some depth and where a water table exists in the soil. The existence of the water table is due to rain falling at a constant rate. The basic assumption in his analysis is that if a horizontal line be drawn through the lowest point of the water table, then the loss of hydraulic head in the arched region above this line and beneath the water table is negligible compared to the loss in the remainder of the flow region. He imagines this arched region to be composed of gravel placed between thin vertical membranes which direct the water vertically downward with no loss of head. This implies that the streamlines will be equally spaced along the horizontal line below the arched region, and on this assumption Kirkham obtained his solution to the problem. He first obtained a solution for the general problem of flow to a slit located randomly between the impermeable layer and the water table. He then
considered specialized cases for tile and slits located at various points. The solutions are in the form of infinite series involving circular, hyperbolic, and exponential functions. The height of the water table is determined by evaluating the potential along the horizontal line passing through the drain center.

Kirkham (26) later developed a correction factor for the approximate height of the water table as determined above which takes into account the head loss in the region beneath the arched water table. This correction factor is of the form $rac{1}{1-R/K}$, where $R$ is the steady rainfall rate and $K$ is the hydraulic conductivity of the soil. The water table height obtained in the previous paper is multiplied by this factor to obtain closer approximation to the actual water table height. Kirkham observed that for most practical cases the value $R/K$ is very small so that the correction makes relatively little difference in the height of the calculated water surface.

Hinesly (14) obtained a mathematical solution for a somewhat more general problem in which the steady rainfall is reinforced by artesian water from below.

The unsteady-state problem

The problem involving flow through porous media where the locations of one or more boundaries of the flow region are time
dependent is an extremely difficult one to handle mathematically. The attempts to solve such a problem have been based usually on several simplifying assumptions which may or may not be valid in practice.

The most common assumption has been that the falling water table may be treated as a succession of steady states. On the basis of this assumption Kirkham and Gaskell (28) used relaxation techniques to locate the falling water table under tile and ditch drainage starting from completely saturated conditions. Even though a fairly large mesh was used for the relaxation net the calculations were reported to be quite laborious. The large mesh also produced fairly large errors in the calculated results.

Isherwood (15) programmed a similar tile drainage problem for solution by a digital computer using iterative procedures. The general procedure used by Kirkham and Gaskell was modified only slightly in programming for the computer. The tile problem computed by Kirkham and Gaskell (28) was recomputed on the machine. Mainly due to smaller residuals in the machine computation, the rate of fall of the water table midway between tiles was found to be up to 18% greater than that found by Kirkham and Gaskell.

Luthin (33) derived a formula for the spacing of drain tile as a function of the desired rate of fall of the water
table. The assumptions upon which he based his analysis are as follows:

1. Rate of flow into the tile line is directly proportional to water table height midway between tile lines and to hydraulic conductivity.
2. Rate of flow is independent of spacing between drains.
3. Rate of flow is independent of drain diameter.
4. Water table is flat between drains.

Luthin and Worstell (35) investigated the above listed assumptions. They determined that if the impermeable layer is more than 2 feet below the drains then the first assumption holds. Otherwise the relationship between rate of outflow and water table height is nonlinear. The second assumption was found to be only approximately true. They reported that field data indicated that doubling the drain spacing reduces the rate of fall of the water table by about 40% rather than 50%. Finally they reported that observed water tables were essentially flat except very near the drain. The curvature was said to extend further out from the drain for soils of low hydraulic conductivity.

Maasland (36) studied the movement of the water table between ditch drains under intermittent applications of water to the soil surface. His basic solution was obtained using the Dupuit-Forchheimer assumptions, that is, that flow occurs
horizontally, the velocity is constant across any cross sec-
tional area, and this velocity is proportional to the slope of
the water surface. The solutions were given in the form of
combinations of periodic functions and time transient functions.

DeWiest (8) studied a problem which is somewhat the re-
verse of the soil drainage problem in that the water table
moves toward the unsaturated region rather than away from it.
This was the problem of seepage through an earth dam when the
water level above the dam is slowly raised. He considered the
unsteady free surface to be a perturbation of the steady free
surface, that is, that it is displaced from the steady free
surface by an amount which decrements exponentially with time,
and that the potential at a point at a given time is a per-
turbation of the steady state potential at that point. The
analysis was considered valid for small displacement of the un-
steady surface from the steady free surface.

In his analysis DeWiest drew on work in the field of un-
steady free flow of fluids, such as an analysis of an outward
springing jet. Such a problem is somewhat analogous to the
problem of unsteady flow through porous media, and there appears
to be the possibility that similar techniques may be used in
the solution of the two problems.

Most theoretical solutions for unsteady flow problems are
based on the assumption that the drainable pore space of the soil can be considered a constant at a particular point and is not dependent upon time or proximity of the point to the free surface. In effect this implies that the porous material at a point goes from a state of complete saturation to a state of complete drainage at the moment the water table passes that point. Some authors have taken issue with this assumption. Taylor (52) discussed the dependence of the drainable pore space \( f \) on the water table depth \( Z \), and upon the time \( t \) if drawdown does not take place very slowly, that is, where \( f = f(Z,t) \). He showed that for very slow drawdown conditions in a column of homogeneous soil, \( f = f(Z) \), and that the value of this relationship can be determined experimentally by lowering the water table by increments, allowing conditions to reach equilibrium, and measuring drawdown and volume of fluid discharged. He suggested that a similar procedure might be used to determine \( f(Z) \) from field data.

Jacob (17) also recognized that complete drainage of pores does not take place instantly. He observed that the moisture content of even a coarse sand is a function of the tension placed on it and that this moisture content is reached only after a period of time.
Models and Their Use in Drainage Studies

Models of various types have been used rather extensively in studying the flow of water through soils. In most cases these have taken the form of dissimilar models (electrical analogues for example), which may not resemble the prototype but whose functionings are based on similar principles, and adequate models, which do not meet all the requirements of a true model but still yield certain information concerning some phase of operation. An example of the latter is the use of sand tank models to obtain streamline patterns.

Sand tank models

Wyckoff, Botset, and Muskat (54) studied flow to a well under the action of gravity using a pie shaped sand section. They found that flow in the capillary zone above the piezometric surface was appreciable and that it was necessary to take steps to reduce the error thus incurred in the results.

Kirkham (19, 20) used a sand tank model to test the hypothesis that for steady flow through homogeneous soil the streamline pattern is not affected by the value of the hydraulic conductivity. He used coarse sand on one side of the tile drain and fine sand on the other side and observed the streamline pattern when dye was introduced into the flow. The pattern was the same on
either side and therefore the hypothesis was confirmed.

Using a soil tank model, Harding and Wood (13) studied streamline patterns for steady flow to tile drains. They studied cases of artesian flow and saturated flow from a ponded surface through uniform soil and through soil with layers of differing hydraulic conductivity. The results were presented as photographs of the streamline patterns.

Gustafsson (12) tested his theoretical solutions in models. He used a sand model to test theories for flow to tile from a ponded surface.

Luthin and Worstell (34) studied the falling water table in a sand tank model and compared the results with the relaxation solutions of Kirkham and Gaskell (28). They concluded that Kirkham and Gaskell's solutions gave the location of the surface of saturation rather than the location of the free surface.

Using a rather large scale model, Keller (18) conducted a study of the effectiveness of interceptor drains located on sloping areas. Robinson (46) also reported on this work. A sloping flume filled with coarse sand was supported by jacks which could be raised or lowered to vary the slope of the flume. The system represented a layer of uniformly permeable soil overlying an impermeable layer on a slope. Tiles were placed
at three separate levels near the downstream end of the flume to simulate interceptor drains. Each tile could be operated separately while headwater depths were varied in steps and water surface profiles and discharges studied. Dimensional analysis was employed to reduce the amount of experimental work necessary to obtain an overall picture of the phenomena and to facilitate the presentation of results.

**Electrical analogues**

Childs (5) studied the falling water table under tile drainage using an electrical analogue constructed of conducting paper. An initial surface was set up to represent the steady rainfall case. The surface was then lowered by increments, the new surface being located each time by a process of trial and error until the prescribed surface conditions were met. Childs found from this study that following the cessation of steady rainfall, and until the water table very nearly reaches the drain level, the water table falls as a whole without appreciable changes of shape. Afterward it falls more slowly near the drain than further out. The water table approaches the drain level asymptotically but not necessarily exponentially. Childs also observed that when the soil is initially saturated to the surface and rainfall ceases, the water table initially falls more rapidly over the drain but eventually movement repeats that described above. This implies that starting from
an initially saturated condition, a surface corresponding to that for steady rainfall is eventually reached.

Childs also studied the effect of a capillary fringe. He found that if a capillary fringe is present, the initial rate of fall of the water table is reduced but complete drawdown (to the height of the capillary fringe above the drain) comes more quickly since the capillary fringe provides additional flow area when drawdown is nearing completion.

Meisel (37), Collis-George and Youngs (6), and Youngs (55) have also reported on electrical analogue studies of drainage problems. Meisel studied the falling water table for wide tile drain spacings. Collis-George and Youngs studied the effect of location of the impermeable layer and rate of rainfall on the height of the water table between circular drains. They found that if the distance between the impermeable layer and the drains were greater than about 0.3 times the drain half-spacing, the location of the impermeable layer had little effect on the height of the water table. They also obtained some data for unsteady state conditions and observed that the points obtained fit the steady state curves very closely.

Luthin (32) described an electrical resistance network designed for solving LaPlace's equation to find patterns of flow to drains. This was considered most applicable to flat
water table systems, but with modifications could be used for steady, curved water table conditions.

**Viscous fluid models**

Grover (10) used a glassbead-glycerol model to study unsteady water tables with tile drainage. Glass beads were used to represent the soil and glycerol to represent water. The model was two dimensional and was scaled geometrically. Rates of fall of the water table and discharge rates were obtained under various conditions of tile depth and spacing, and location of the impermeable layer. The effect of the drainable pore space was not considered. The results of the study were presented in the form of dimensionless graphical plots. Grover, Ligon and Kirkham (11) reported on the use of this model to study the operational characteristics of laterals near the edge of a drainage system.

Zanker (56) reported on a study of the falling water table over tile drains using a so-called Hele-Shaw model. This is a model employing a viscous fluid flowing between two closely spaced plexiglas sheets. The effect of variations in tile spacing and depth, depth to the impermeable layer, and conductivity were studied. The results were presented in the form of curves. The writer reported that the capillary fringe in the model caused some error in the results, the magnitude of the error depending on the scale of the model. Gustafsson
(12) also used a Hele-Shaw type model for studying the case of artesian flow to tile with a water table standing beneath the soil surface.

**Dimensional analysis**

The majority of the workers conducting model studies of soil drainage problems to date have not considered the dimensional relationships among the variables involved nor made use of these relationships in the design of their models and interpretation of the results. However, the theory of dimensional analysis and its application in model studies has been developed to a point where it appears that its consideration in drainage model studies would be fruitful.

Several authors have discussed extensively the application of dimensional analysis to the construction and operation of models of physical systems and to the interpretation of results obtained therefrom. Buckingham (3) and Bridgman (2) limited their discussions to systems where true models were feasible. Murphy (41) extended the theory to distorted models, that is, to situations where it is impossible to satisfy in model and prototype the true model requirement of absolute equality of all dimensionless terms involved. Langhaar (30) discussed many problems which can be effectively studied using models, and numerous articles have appeared in the literature of engineering research concerning studies carried out on models designed and operated on the basis of dimensional analysis.
As mentioned earlier Keller (18) used dimensional analysis in his model study of interceptor drains. Grover (10) presented his data in dimensionless form but did not otherwise utilize dimensional analysis techniques. Elsewhere in the literature of soil science Miller and Miller (38, 39, 40) concerned themselves with the problem of modeling unsaturated flow problems in which hydraulic conductivity and moisture content are a function of moisture tension. They developed a theory for utilizing "reduced variables" developed from the moisture-tension and conductivity-tension relationships. These "reduced variables" are essentially dimensionless combinations of variables (or "system parameters" as the authors call them). They are similar to the "pi terms" of Buckingham (3) and Murphy (41) except that it appears that they are not always mutually independent. Two characteristic length terms were considered in the Miller theory. One represents the microscopic particle size and spacing and the other is a term representing the macroscopic geometry of the system. By equalizing the "reduced variables" in two systems Miller and Miller found it possible to obtain similar results from the systems.

Klute and Wilkinson (29) reported the results of some tests of the Miller theory. A sand was sieved into five fractions and each fraction was packed in a standard manner to obtain equal bulk densities. Particle densities were essentially the same for all fractions. The distribution of
particle sizes was determined by microscopic examination and it was found that the fractions could be considered similar in this respect. Moisture-tension and conductivity-tension curves were determined for each fraction. The moisture-tension relationships in the form of "reduced variables" were then plotted on a common graph, as were the conductivity-tension relationships. It was found in each case that a single curve could be drawn to fit the data within the limits of experimental error.

Field Drainage Studies

Relatively little information is available from field experiments to substantiate or reject the results of theoretical and model studies of soil drainage. Before 1920, Schlick (48) reported on several years of drainage investigations on six Iowa farm tile drainage systems. He measured rainfall, tile discharge, and water table heights. However, he did not report any measurements of hydraulic conductivity or porosity of the soil. Schlick suggested that the drainage system should be planned so as to give the desired control of the groundwater level and that the drainage coefficient used should be one which would accomplish this.

Kirkham and DeZeeuw (27) reported field water table measurements made in the Netherlands over a period of time in both tile and ditch drained areas. Although individual measurements
varied considerably, enough data was taken so that the authors were able to treat it statistically and obtain relatively smooth curves. Kirkham (25) later compared this data with his theory for the steady rainfall case and found quite good agreement.

Isherwood and Pillsbury (16) obtained discharge and water table height curves for an area under tile drainage following applications of irrigation water. They found that the water table height decreased exponentially with time.

Talsma and Haskew (51) reported field measurements of water tables and comparisons with various drainage theories. They observed that the field data agreed favorably with most of the theories considered whenever the assumptions underlying the theoretical solutions were met.

A fairly large volume of tile drainage data has been accumulated for Shallow Chin soils on the Bow River Project near Vauxhall, Alberta (4). Measurements of hydraulic conductivity and drainable porosity were made along with water table observations and discharge measurements after irrigation. To this writer's knowledge no attempt has been made to date to compare this data with any theoretical solutions.
Dimensional Analysis, Design, and Construction of a Model of a Ditch Drainage System

**Dimensional analysis**

When this writer first began to investigate the application of dimensional analysis to the problem of unsteady state flow through porous media, a list of variables was set down which was believed to include all variables which might be pertinent to the problem. These included properties of the fluid and porous medium involved in viscous flow such as the dynamic viscosity, fluid density, acceleration of gravity, and permeability of the medium. Also included were variables related to the capillary rise of a fluid within a porous medium. These were the surface tension of the fluid and the wetting angle or angle of contact between fluid and particle surface at the air-fluid interface. The formulation of dimensionless pi terms from this set of variables revealed several which could not be simultaneously set equal in model and prototype. Three and perhaps four distortion factors would have been involved. It is well known by those who have worked with distorted models that evaluating the effect of such a number of distortion factors is indeed a formidable if not impossible task. Therefore it was decided that an attempt should be made to eliminate some variables by making certain simplifying
The basic assumptions made in the second analysis were as follows:

1. The effect of the capillary fringe in the model can be eliminated, and

2. All the effects of fluid characteristics, acceleration of gravity, and characteristics of the porous material are taken into account by the hydraulic conductivity $K$ of the system and the drainable porosity $f$ of the porous medium.

The capillary fringe in the model was made very small by using fairly large glass beads and by treating the beads with a silicone material. The second assumption appears to be valid where laminar flow occurs in the medium, that is, where the rate of flow is linearly proportional to the hydraulic gradient.

If we are interested in the discharge to the ditches from the soil, the variables considered pertinent to the problem under the above assumption along with their dimensions and the symbols used to represent them are as follows:

$q$ - rate of discharge per unit length of ditch \((L^2T^{-1})\)

$t$ - time measured from the moment drainage begins at complete saturation of the porous medium \((T)\)

* See Appendix A for complete list of symbols used in this dissertation.
a - ditch spacing (L)
d - ditch depth (L)
b - height of ditch bottom above an impermeable layer (L)
h - depth of fluid standing in ditch (L)
K - hydraulic conductivity of the porous medium-fluid system (LT\(^{-1}\))
f - drainable porosity of the porous medium (-)

Discharge per unit length of ditch can be written as a function of the remaining variables in the form

\[ q = F(t, a, d, b, h, K, f). \]

We have eight variables involving two basic dimensions. Thus, according to the Buckingham Pi Theorem (3) we can write six dimensionless and independent pi terms. One possible set is as follows, with the term involving discharge written as a function of the remaining terms:

\[ q/Kb = G(a/b, d/b, h/b, Kt/b, f). \]

If we are interested in being able to predict drawdown of the water table at its highest point (which will be midway between drains for our symmetrical, homogeneous system) we can write a similar list of pertinent variables but with discharge q replaced by drawdown Z. A possible set of pi terms in this case will be

\[ Z/b = G(a/b, d/b, h/b, Kt/b, f). \]

Thus we have the pi terms \( Z/b \) and \( q/Kb \) as functions of a common set of five independent terms.
Model design

In order to obtain a true model of the system the five pi terms on the right side of the equations must each be set equal in model and prototype. The first three of these terms imply that geometrical similarity must exist between model and prototype. The fourth term merely determines a time scale, that is (subscript m indicates values of the variables in the model),

\[
\frac{K_m t_m}{b_m} = \frac{K t}{b}
\]

or

\[
\frac{t}{t_m} = \frac{K_m b}{K b_m}
\]

or

\[
\frac{t}{t_m} = n \frac{K_m}{K}
\]

where \(n\) is the length scale. It is interesting to note that this places no limitation on the value of the hydraulic conductivity \(K\) of the model system.

The fifth dimensionless term implies that the drainable porosity must be the same in model and prototype. In order to reduce or eliminate capillary effects in the model it was desirable to use fairly large diameter glass beads as the porous medium. These beads have a drainable porosity of about .38 as compared to a drainable porosity of around .05 to .10 in most soils. Obviously if these beads were to be used in the model,
then the pi term \( f \) would be distorted. Ordinarily, this would mean that the effect of the distortion on the prediction of \( q/Kb \) and \( Z/b \) would have to be evaluated. This would require that \( f \) be varied in the model without varying the other variables at the same time. This would involve considerable difficulty. Fortunately, by studying the drainage of a porous medium with a simple geometry it was possible to determine that the pi term \( f \) combines with the pi term \( Kt/b \) in the form \( Kt/bf \). This study will be reported later.

The final result of the dimensional analysis was to require that the model be geometrically similar to the field system in every respect except particle size. If this requirement were met, then the results of model tests should be valid for the field situation, that is, curves of \( q/Kb \) versus \( Kt/bf \) and \( Z/b \) versus \( Kt/bf \), with the other dimensionless terms as parameters, obtained from the model could be used in designing field systems. Thus it was necessary to be able to vary \( a/b \), \( d/b \), and \( h/b \) over a practical range in the model and, after starting drainage from complete saturation, to determine discharge and drawdown with time for each geometry. It was also necessary to be able to determine the values of hydraulic conductivity \( K \) and the drainable porosity \( f \) for each run.

The situation where the ditches penetrate completely to an impermeable layer and no water stands in the ditches was
treated as a separate problem. In this case b and h are no longer pertinent variables and we can write

\[ \frac{q}{K_d} = G(\frac{a}{d}, \frac{K_t}{d'}) \]

and

\[ \frac{z}{d} = G'(\frac{a}{d}, \frac{K_t}{d'}). \]  

It was possible to use the same model with slight modifications for the case in which the ditches completely penetrate to the impermeable layer and for the case in which the ditches only partially penetrate to the impermeable layer.

**Model construction**

In most tile and ditch drainage problems flow is considered to occur in a plane perpendicular to the longitudinal axes of the drains and is therefore two-dimensional in nature. This was assumed to hold true for the cases to be studied, and thus the model, as constructed, represented a two-dimensional flow system.

A schematic diagram of the model set up for studying the case where the ditches partially penetrate to the impermeable layer is shown in Figure 1. The model was designed with a study section 80 inches wide, 10 inches deep and 3/4 inch in thickness. It was constructed of ¼ inch Plexiglas II, a transparent and craze-resistant plastic material. The front and back of the model were each formed from a single sheet of Plexiglas. A model thickness of 3/4 inch was obtained by in-
Figure 1. Schematic diagram of the ditch drainage model.
serting triple thickness spacers of the same plexiglas material. These spacers were continuous strips of 1 inch width along the sides of the model and along the bottom between ditch outlets. Three-quarter inch square spacers were placed every 6 inches along the top of the model. One-eighth inch screw head bolts were placed in holes drilled through front, spacers, and back on 2 inch centers along the bottom and sides of the model and on 6 inch centers along the top. Nuts were placed loosely on these bolts, acetone was injected between the sheets and spacers, and then the nuts were tightened. This resulted in a tight and almost perfectly transparent seal.

The back and edges of the model were painted with a flat finish black paint to produce a dark background which accentuated the appearance of the surface of saturation in the porous medium. A grid was drawn on the front of the model with thin black lines inked on a 2 inch horizontal spacing and a 1 inch vertical spacing.

The model was first used to study the case where the ditches penetrate completely to an impermeable layer and no water stands in the ditches. For this case a 1 inch gap was left in the spacers every 20 inches along the bottom of the model and a ½ inch gap was left at each end. A plastic plate drilled with a 3/4 inch hole was glued in place to cover each gap. A short section of rigid tubing was then attached perpendicularly
to the plate and a length of flexible tubing slipped over the rigid tubing. Finally, C-clamps with especially prepared wide contacting surfaces were placed on the flexible tubing to serve as valves for the outlets. Any of the outlets could be kept closed while others were opened as the ditch spacing was varied.

The ditch walls were formed from 18-mesh brass screening. Those used at either end were 1 inch in width (representing half-ditches) and the remainder were 2 inches in width. The ditch walls were formed to give a snug fit when in the model but could be easily removed to provide for varying ditch spacings. When a ditch was removed, a small section of screening was placed on the bottom of the model to cover that particular outlet to keep the porous material from falling into the closed outlet tube.

The model was revised for studying the case where the ditches only partially penetrate to the impermeable layer (see Figure 1) by closing the bottom outlet tubes and placing a section of screening over each outlet. The same ditches were used but were provided with bottoms made of the brass screening material. Since it was necessary to vary only the dimensionless terms $a/b$, $d/b$, and $h/b$, it was possible to conduct the entire study of this case with only one value of $b$. A value of 2 inches was selected. Outlet holes were drilled in
the back of the model at the 2 inch level (bottom of the ditches). A small plastic transition box was attached to the back of the model over each hole. To this box a section of flexible tubing was attached. The discharge end of this tubing could be raised or lowered to give the desired fluid level in the ditch during drainage.

As mentioned earlier glass spheres approximately 2 mm in diameter were used as the porous medium. These were treated with a silicone material, "General Electric SC-87 Dri-Film," to minimize capillary effects.

Glycerol was used as the model fluid. It was desirable because of its relatively high viscosity which slowed the drainage process and insured laminar flow in the model. The water content of the glycerol varied slightly during the time the experiments were being run but was approximately 10 percent.

Investigations with the Ditch Drainage Model

General procedure

Preliminary work It was necessary to know the hydraulic conductivity $K$ and the drainable porosity $f$ in the model for each run. However, it was possible to make certain preliminary measurements on these quantities so as to reduce the amount of work necessary during each run.
According to Muskat (42), hydraulic conductivity $K$ is related to the fluid viscosity $\mu$ by the equation

$$K = \frac{\rho g k}{\mu},$$

where $\rho$ is the fluid density, $g$ is the acceleration of gravity, and $k$ is the permeability of the porous medium (independent of the fluid properties). We can assume for practical purposes that in the range of temperatures and water contents encountered in our experiments the density of the glycerol did not change. If the bead packing were the same in each case, $k$ would not vary, and of course the acceleration of gravity did not change. Therefore, knowing the value of the hydraulic conductivity $K_1$ and the viscosity $\mu_1$, for a particular run it was possible to determine the hydraulic conductivity $K$ for any subsequent run by determining the viscosity $\mu$ for that run and substituting into the equation

$$K = \frac{\mu_1 K_1}{\mu}.$$ 

The value of the factor $\mu_1 K_1$ was determined initially by setting up the model with only the ditches at either end in place. These ditches rested on the bottom of the model. However, all the bottom outlets were closed. Glass beads were added to a depth of 6 inches and saturated with glycerol. Precautions were taken to remove air bubbles and to obtain uniform packing as in any regular run. A siphon tube was inserted in one ditch and the outlet adjusted so as to maintain the fluid level in the ditch at the same height as the bead.
surface at the ditch. The opposite end of the model was then raised to give a slope to the flow region. The fluid level in the upstream ditch was maintained at the bead surface by periodically adding fluid from a beaker. Figure 2 shows the model set up for a hydraulic conductivity determination.

After steady flow conditions had been reached, the slope of the model was measured and three separate discharge measurements were made. These three discharge measurements differed by less than four per cent in each case. The slope of the model was then increased and the appropriate quantities measured again. The hydraulic conductivity was obtained from the equation

$$K_1 = \frac{Q}{iA},$$

where $Q$ was the average rate of discharge through the model for a given slope, $i$ was the slope, and $A$ the cross-sectional area of the flow region. The conductivities determined for the two slopes agreed closely and were averaged.

Two determinations of $K_1$ were made according to the procedure just described. The first was made using beads which had been alternately saturated and drained for several cycles so that they might have become "dirty". The second was made using freshly washed and dried beads. The difference between the values obtained for $K_1$ was less than four per cent, and it was felt that this difference was of the same order as the
Figure 2. The ditch drainage model set up for a hydraulic conductivity determination.

Figure 3. The ditch drainage model in operation during a typical run.
errors encountered in measuring the discharge and slope. Therefore, an average of the two values was taken and multiplied by the measured viscosity of the fluid to obtain the value for the factor $\mu_1 K_1$. The hydraulic conductivity $K$ for each particular run was then determined by measuring the viscosity $\mu$ of the fluid and dividing the value into the factor $\mu_1 K_1$.

The fluid viscosity itself was measured with previously calibrated large-bore Ostwalt type viscosimeters. Duplicate measurements were made in separate viscosimeters for each run and an average of the two readings taken.

The drainable porosity of the beads in the model was determined at the time of each conductivity measurement. After the conductivity measurement had been completed the model was brought back to a level position and the drains at each end opened. As soon as the ditches were emptied, beakers were placed to catch the discharge. When drainage was virtually complete, the total discharge was measured and the total volume of beads drained was determined. The drainable porosity was then determined as the ratio of the volume of fluid discharged to the volume of beads drained. Two determinations of the drainable porosity agreed within one per cent, and their average value was used as the value for each run.

After 15 runs had been made with the model, it was necessary to add some new beads in order to obtain the desired
depths. These new beads, while listed by the manufacturer as being the same average diameter as the original beads, were in fact about \( \frac{1}{4} \) mm smaller in diameter. Therefore, when these beads were mixed in with the old ones, new determinations of \( \mu_1 k_1 \) and \( f \) had to be made. However, the values obtained varied only slightly from those originally determined. Table 1 gives the values of \( \mu_1 k_1 \) and \( f \) before and after the new beads were added.

Table 1. Values of the hydraulic conductivity-viscosity factor and the drainable pore space

<table>
<thead>
<tr>
<th>Runs</th>
<th>( \mu_1 k_1 )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 15</td>
<td>116</td>
<td>.37</td>
</tr>
<tr>
<td>16 - 43</td>
<td>114</td>
<td>.38</td>
</tr>
</tbody>
</table>

The model was operated in a room in which the temperature and relative humidity could be held within ranges of about plus or minus one degree Fahrenheit and five per cent, respectively. Thus the fluid viscosity did not change appreciably during any particular run. However, over a longer period of time there was some change due to the fact that the water content of the glycerol was not in exact equilibrium with the humidity in the room. This was the reason for the measurement of viscosity for each separate run.
**Operation of the model**  

Figure 3 shows the model in operation. It was mounted on a stand which was placed on top of a table in such a way that the model extended slightly in front of the table, allowing the outlet tubes from the bottom of the model to extend vertically downward. The glycerol draining from the ditch outlets was caught in collectors and returned to a carboy reservoir on the floor beneath the model.

After the ditches were positioned in the model as desired for the particular run, beads were poured in to the desired depth. Glycerol was then added to saturate the beads. As far as practical the beads were saturated from the bottom up. This considerably reduced the amount of entrapped air which had to be removed. Enough excess glycerol was added so that the bead-glycerol mixture could be stirred to remove air bubbles. After these air bubbles had been removed the excess glycerol was drained away and the beads were tamped with a flat plastic tamper on the end of a rod. This procedure seemed to result in a reproducible packing of the beads.

When thorough saturation of the beads had been attained and the model leveled carefully, the run was ready to begin. It was started by opening the outlet tubes from the ditches and allowing them to drain freely or by adjusting the discharge end of the tubes so as to maintain the desired fluid level in the ditches while drainage took place. Outlet tubes were used which were of sufficiently large diameter so that
the ditches were emptied down to the desired level of operation almost instantaneously.

A record of the downward movement of the surface of saturation was obtained by taking photographs at time intervals determined by the rate of fall of the water table. In general eight to ten photographs recorded a particular drawdown sequence. The camera, a press-type camera equipped with a wide angle lens, was mounted on a concrete block stand for steadiness. From the exposed film, $3\frac{1}{2} \times 4\frac{1}{2}$ inch contact prints were made. A timer, started at the moment the outlets were opened, and a card bearing the run number were included in the pictures. A permanent record was thus obtained of the run number, time, and location of the surface of saturation. Figure 3 is an example of the photographs which were taken. Figure 4 shows a sequence of photographs for a single run. The individual photographs have been trimmed to show only the model itself.

After the photographs had been printed, the desired drawdown data was picked from them. The level of the surface of saturation could be read to an accuracy of approximately 0.1 inch.

Discharge was determined by measuring the outflow from a ditch for a certain time interval in a graduated beaker and recording the time, time interval, and amount of effluent.

The determination of hydraulic conductivity and drainable
Figure 4. Sequence of photographs showing successive water table locations during a typical run. The individual photographs have been trimmed to show only the model proper.
Study of the effect of capillarity in the model

By using 2 mm diameter beads as the porous medium and treating these beads with a silicone material, capillary rise of fluid in the beads above the free surface or surface of atmospheric pressure was considerably reduced. However, when the fluid surface was moving downward there still existed a small fringe of about 0.3 inch which had the characteristics of a true capillary fringe. Grover (10) observed this fringe in his model and called it a "pseudo-capillary fringe" because he noted that it had a positive value when the beads were draining but had a negative value when the beads were being wetted.

Since one of the basic assumptions in the analysis was that capillary effects in the model could be eliminated, it was necessary to know whether the existing small capillary fringe would appreciably affect the results obtained. The case in which the ditches penetrate to the impermeable layer and no fluid stands in the ditches was ideal for studying this. For this case the functional relationships

\[ \frac{q}{Kd} = G\left( \frac{a}{d}, \frac{Kt}{d^2} \right) \]

and

\[ \frac{z}{d} = G'\left( \frac{a}{d}, \frac{Kt}{d^2} \right) \]
were previously determined.

By holding the dimensionless term \( a/d \) constant but varying the values of \( a \) and \( d \), the scale of the model could be varied. If \( a/d \) were held constant at 20 but one run was made with \( d = 2 \), \( a = 40 \) and another with \( d = 4 \), \( a = 80 \), one setup could be thought of as a model of the other, or a "model of a model" so to speak. In such a case, if \( q/Kd \) and \( Z/d \) were plotted as functions of \( Kt/df \), the same curves should result regardless of the particular values of \( a \) and \( d \). If there were an appreciable effect of the capillary fringe on the dependent \( \pi \) terms then the data would not be expected to fit a common curve because the relative effect of the fringe would vary depending upon the value of the total bead depth and ditch depth \( d \). For a greater bead depth the capillary fringe would constitute a smaller portion of the total flow area.

To test the capillary effects runs were made with \( a/d \) held at values of 10, 20, and 40 while the values of \( a \) and \( d \) themselves were varied. The drawdown results are shown in Figures 5, 6, and 7. Different symbols were used for plotting the data for the different values of \( a \) and \( d \) so that the data points can be easily identified. For \( a/d = 10 \), runs were made with \( d = 2 \), \( a = 20 \); \( d = 4 \), \( a = 40 \); and \( d = 8 \), \( a = 80 \). As plotted in Figure 5 there is some scatter of the points about a common curve. However, there is no consistent change in
Figure 5. Dimensionless drawdown for $a/d = 10$, but with $a$ and $d$ varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER

NO FLUID STANDING IN DITCHES

\[
\frac{a}{d} = 10
\]

SYMBOL \hspace{1cm} d \hspace{1cm} a

- \( \circ \) 2.0 20
- \( \square \) 4.0 40
- \( \triangle \) 8.0 80
Figure 6. Dimensionless drawdown for $a/d = 20$, but with $a$ and $d$ varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ \alpha/d = 20 \]

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>d</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>○</td>
<td>1.0</td>
<td>20</td>
</tr>
<tr>
<td>□</td>
<td>2.0</td>
<td>40</td>
</tr>
<tr>
<td>△</td>
<td>4.0</td>
<td>80</td>
</tr>
<tr>
<td>●</td>
<td>8.0</td>
<td>160</td>
</tr>
</tbody>
</table>

\[ Kt/df \]
Figure 7. Dimensionless drawdown for $a/d = 40$, but with $a$ and $d$ varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER

NO FLUID STANDING IN DITCHES

\( a/d = 40 \)

SYMBOL

\( \begin{array}{cc}
\circ & 2.0 \ 80 \\
\square & 4.0 \ 160 \\
\end{array} \)
Z/d with increasing values of a and d. Figure 6 shows the data for a/d = 20. The values for d = 2, a = 40; d = 4, a = 80; and d = 8, a = 160 follow a common curve quite closely. However, for the run with d = 1, a = 20, the points deviate from this common curve for values of Z/d greater than about 0.2 as indicated by the dotted curve. Figure 7 shows the data for a/d = 40 for two runs, one with d = 2, a = 80 and another with d = 4, a = 160. The deviations of the points from a common curve are neither large nor consistent.

The discharge data from part of this same series of runs is shown in Figures 8 and 9. In Figure 8 the data for a/d = 40 is shown. Separate curves were fitted to the points for d = 2, a = 80 and for d = 4, a = 160. A linear equation was fitted to the logarithms of the values by the least squares method and the regression coefficients for the two sets of data were compared. The regression coefficients were found to be not significantly different at the five per cent confidence level.

Figure 9 shows the discharge data for a/d = 20. The points for d = 2, a = 40; d = 4, a = 80; and d = 8, a = 160 deviate only slightly from a common curve. However the data for d = 1, a = 20, appears to follow a completely different curve. Data from two separate runs at this depth and spacing are shown and the two sets of data agree quite well.

From the above observations it can be concluded that the
Figure 8. Dimensionless discharge for $a/d = 40$, but with $a$ and $d$ varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ a/d = 40 \]

**SYMBOL**

- **d**
  - 2.0
  - 4.0

- **a**
  - 80
  - 160

**Regression Coefficients**

- \( q/Kd = 0.919 (Kt/df)^{-0.594} \)
- \( q/Kd = 0.791 (Kt/df)^{-0.568} \)

REGRESSION COEFFICIENTS
NOT SIGNIFICANTLY DIFFERENT
Figure 9. Dimensionless discharge for a/d = 20, but with a and d varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ a/d = 20 \]

**SYMBOL**

- \( * \) \( d = 2.0 \), \( a = 40 \)
- \( o \) \( d = 4.0 \), \( a = 80 \)
- \( x \) \( d = 8.0 \), \( a = 160 \)
- \( \triangle \) \( d = 1.0 \), \( a = 20 \)
- \( \diamond \) \( d = 1.0 \), \( a = 20 \)

\[ \frac{q}{K_d} \text{ vs. } \frac{K_t}{df} \]
small capillary fringe in the model did not appreciably affect the dimensionless drawdown or discharge curves as long as the model was operated with a bead depth of 2 inches or greater. Therefore, all subsequent runs were conducted with depths of at least 2 inches.

Study of the effect of variation of the hydraulic conductivity

The hydraulic conductivity $K$ of the porous medium-fluid system was assumed to be a suitable variable for representing the effects of all the fluid characteristics and effects of the media on flow through the porous media. The hydraulic conductivity of the model system was of the order of 1 in/min, whereas a typical field value might be of the order of .05 in/min. It was desirable to know whether the assumption was correct and thus to know whether the large change in hydraulic conductivity encountered in going from model to field might render the model results inapplicable to the field situation.

An experiment was conducted in an attempt to determine whether varying the hydraulic conductivity alone would affect the model results. This again was conducted for the case of the ditches penetrating to the impermeable layer and having no water standing in them. With $d = 2,$ $a = 80$ and thus with $a/d = 40,$ three separate runs were made with different values of the hydraulic conductivity $K$ in each case. These values
were $K = 1.00$ in/min., $K = 0.57$ in/min., and $K = 0.33$ in/min. The different values for hydraulic conductivity were obtained by varying the temperature in the room in which the model was operated. This in turn varied the viscosity of the glycerol and hence the hydraulic conductivity.

The dimensionless drawdown data for the three runs is presented in Figure 10. The variations of $Z/d$ between runs are no greater than the possible errors due to reading drawdown values from the photographs.

The dimensionless discharge data is presented in Figure 11. Linear equations were fitted to the logarithms of the values by the method of least squares for each of the three runs. The regression coefficients were compared statistically and found to be not significantly different at the five per cent level of confidence.

The results of this experiment indicate that the data from the three runs belong to the same population. No effect of the variation of hydraulic conductivity $K$ on the relationship between the dimensionless terms $Z/d$ and $Kt/df$, or $q/Kd$ and $Kt/df$ was apparent.

**Accumulation of drainage data with the model**

Runs were conducted with the model set up in such a manner as to obtain drawdown and discharge information which would be useful in the design of field drainage systems. For the case
Figure 10.  Dimensionless drawdown for $a/d = 40$ with the hydraulic conductivity $K$ varied between runs.
60

DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ d = 2.0, \quad a = 80 \]

**SYMBOL**

\[
\begin{align*}
\Delta & : 0.33 \\
\circ & : 0.57 \\
\square & : 1.00
\end{align*}
\]

\[ K \text{ (in/min)} \]
Figure 11. Dimensionless discharge for a/d = 40 with the hydraulic conductivity K varied.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ d = 2.0, \quad \alpha = 80 \]

SYMBOL \quad K (\text{in/min})

\[
\begin{array}{cc}
\square & 1.00 \\
\circ & 0.57 \\
\triangle & 0.33 \\
\end{array}
\]

\[ q/Kd = 0.951 (Kt/df)^{-0.615} \]

\[ q/Kd = 0.919 (Kt/df)^{-0.594} \]

\[ q/Kd = 0.792 (Kt/df)^{-0.588} \]

REGRESSION COEFFICIENTS
NOT SIGNIFICANTLY DIFFERENT
in which the ditches penetrate to an impermeable layer and no water stands in the ditches, the dimensionless parameter \( \frac{a}{d} \) was varied between values of 4.0 and 80 and discharge and drawdown were measured as a function of time for each value. From this data the relationships between the dimensionless terms \( \frac{Z}{d} \) and \( \frac{Kt}{df} \) and between \( \frac{q}{Kd} \) and \( \frac{Kt}{df} \) for each value of \( \frac{a}{d} \) were plotted.

For the case in which the ditches do not penetrate to the impermeable layer and in which water may stand at different levels in the ditches, runs were made with each of the dimensionless terms \( \frac{a}{b}, \frac{d}{b}, \) and \( \frac{h}{b} \) varied in turn while the others were held constant. The term \( \frac{a}{b} \) was varied between 5.0 and 80; \( \frac{d}{b} \) between 1.0 and 4.0. The model was operated with ditches practically empty \( (\frac{h}{b} = 0.075) \) and with ditches half full \( (\frac{h}{b} = \frac{1}{2} \frac{d}{b}) \). Here again drawdown and discharge were measured as functions of time and the relationships between \( \frac{Z}{b} \) and \( \frac{Kt}{bf} \) and between \( \frac{q}{Kb} \) and \( \frac{Kt}{bf} \) for each particular geometry were plotted.

Table 2 lists all the individual runs made with the model for the case of the completely penetrating ditches and the values of all the pertinent variables and dimensionless terms for each run. Table 3 lists similar information for all runs conducted for the case of the ditches partially penetrating to the impermeable layer.
Table 2. Values of pertinent variables and dimensionless \( \pi \) terms for all runs in which the ditches penetrated to the impermeable layer

<table>
<thead>
<tr>
<th>Run no.</th>
<th>( a ) (in)</th>
<th>( d ) (in)</th>
<th>( K ) (in/min)</th>
<th>( f )</th>
<th>( a/d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>2.0</td>
<td>0.57</td>
<td>0.37</td>
<td>40</td>
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<tr>
<td>2</td>
<td>80</td>
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<td>0.58</td>
<td>0.37</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
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<td>0.59</td>
<td>0.37</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
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<tr>
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Table 3. Values of pertinent variables and dimensionless pi terms for all runs in which the ditches only partially penetrated to the impermeable layer

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<th>Run no.</th>
<th>a (in)</th>
<th>d (in)</th>
<th>h (in)</th>
<th>b (in)</th>
<th>K (in/min)</th>
<th>f</th>
<th>a/b</th>
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Drainage from a Vertical Porous Medium Column

Theoretical

As mentioned earlier, an investigation of drainage from a porous medium having a relatively simple geometry led
to the combination of the pi terms $f$ and $Kt/b$ in the form $Kt/bf$. This investigation will now be reported.

The system studied consists of a vertically oriented, cylindrical, porous-medium column of length $L$ confined within impermeable walls. Flow within the column takes place in the vertical direction only. The column may be thought of as having its lower end immersed in a pail of fluid to a depth $\delta$ so that the head acting to produce drainage of the fluid from the column is reduced by the amount $\delta$ as compared to a freely draining column. The manner in which the column was actually set up for experimental study is shown schematically in Figure 12.

The column is considered to be initially saturated to the surface and at time $t = 0$ drainage is allowed to begin. The items of ultimate interest are $Z$, the drawdown of the surface of saturation, and $q_A$, the discharge per unit area of column cross section. The height of the surface of saturation above the fluid level in the piezometer tube during drainage is called $y$, and the thickness of the capillary fringe is $\varepsilon$.

As the basis of the theoretical development, we can write two completely independent expressions for $q_A$, the rate of discharge of fluid per unit of cross sectional area at any given time. The first expression is merely Darcy's Law,

$$q_A = K \frac{y-\varepsilon}{y+\delta}.$$
The second expression states that the rate of discharge is equal to the rate of fall of the surface of saturation, multiplied by the drainable porosity,

\[ q_A = -f \frac{dy}{dt} \quad \text{11} \]

that is, the rate of discharge is equal to the rate of removal of fluid from the porous medium.

In order to solve for \( y \), and ultimately \( z \), we equate Eqs. 10 and 11,

\[ -f \frac{dy}{dt} = K \frac{y-\varepsilon}{y+\delta} \]

Separating variables we have

\[ \frac{y+\delta}{y-\varepsilon} \, dy = - \frac{K}{f} \, dt, \]

or

\[ \frac{y \, dy}{y-\varepsilon} + \delta \frac{dy}{y-\varepsilon} = - \frac{K}{f} \, dt. \quad \text{12} \]

Integrating both sides of the equation,

\[ \int \frac{y \, dy}{y-\varepsilon} + \delta \int \frac{dy}{y-\varepsilon} = - \frac{K}{f} \int dt + C_1, \]

we get [See Eqs. 28 and 31 of Peirce (43)]

\[ y-\varepsilon + \varepsilon \ln(y-\varepsilon) + \delta \ln(y-\varepsilon) = - \frac{Kt}{f} + C_1, \]

or upon rearranging terms

\[ y + (\delta + \varepsilon) \ln(y-\varepsilon) = - \frac{Kt}{f} + \varepsilon + C_1. \quad \text{13} \]

To evaluate \( C_1 \), note that when \( t = 0, y = L-\delta \), and thus

\[ C_1 = (L-\delta) + (\delta + \varepsilon) \ln (L - \delta - \varepsilon) - \varepsilon, \]
Substituting this value for $C_1$ back into Eq. 13 gives

$$y + (\xi + \varepsilon) \ln(y - \zeta) = -\frac{Kt}{f} + (L - \xi) + (\xi + \varepsilon) \ln(L - \xi - \zeta).$$

After rearranging terms and combining logarithms we get

$$(\xi + \varepsilon) \ln \frac{y - \zeta}{L - \xi - \zeta} = L - \xi - \frac{Kt}{f} - y.$$  

Taking the antilogarithms of both sides of the equation yields

$$\left[\frac{y - \zeta}{L - \xi - \zeta}\right]^{\xi + \varepsilon} = e^{L - \xi - \frac{Kt}{f} - y}$$

or, after collecting $y$ terms on the left hand side,

$$\left[\frac{y - \zeta}{L - \xi - \zeta}\right]^{\xi + \varepsilon} e^y = e^{L - \xi - \frac{Kt}{f}}.$$

We note (See Figure 12) that

$$y = L - \xi - Z.$$  

Substituting Eq. 18 into Eq. 17 and simplifying we get

$$\left[1 - \frac{Z}{L - \xi - \zeta}\right]^{\xi + \varepsilon} e^{-Z} = e^{-\frac{Kt}{f}},$$

which is an equation expressing drawdown $Z$ as a function of time $t$ but which cannot be solved explicitly for $Z$.

In order to solve for discharge $q_A$, again consider Eqs. 10 and 11. We will get $dy/dt$ from Eq. 10 and substitute into Eq. 11. Eq. 10 can be written

$$q_A (y + \xi) = K(y - \zeta)$$

or
\[ q_A y - Ky = - \oint q_A - K\epsilon \]

or

\[ y(K-q_A) = K\epsilon + \oint q_A. \]

Solving for \( y \) gives

\[ y = \frac{K\epsilon + \oint q_A}{K - q_A}. \tag{20} \]

Recalling that both \( y \) and \( q_A \) are functions of \( t \), we get, after differentiating Eq. 20,

\[ \frac{dy}{dt} = \frac{K(\delta + \epsilon)}{(K-q_A)^2} \frac{dq_A}{dt}. \tag{21} \]

Substituting Eq. 21 into Eq. 11 yields

\[ q_A = -\frac{Kf(\delta + \epsilon)}{(K-q_A)^2} \frac{dq_A}{dt}. \tag{22} \]

Separating variables we have

\[ \frac{dq_A}{q_A(K-q_A)^2} = -\frac{dt}{Kf(\delta + \epsilon)}, \]

and integrating we get [See Eq. 38 of Peirce (43)]

\[ \frac{1}{K(K-q_A)} - \frac{1}{K^2} \ln \frac{K-q_A}{q_A} = -\frac{t}{Kf(\delta + \epsilon)} + C_2. \tag{23} \]

Now we know from Darcy's Law that when \( t = 0 \), then

\[ q_A = q_0 = K\left(\frac{L-\delta}{L} - \epsilon\right). \]

Therefore

\[ C_2 = \frac{L}{K^2(\delta + \epsilon)} - \frac{1}{K^2} \ln \frac{\delta + \epsilon}{L-(\delta + \epsilon)}. \tag{24} \]
Substituting Eq. 24 into Eq. 23 gives
\[
\frac{1}{K(K-q_A)} - \frac{1}{K^2} \ln \frac{K-q_A}{q_A} = -\frac{t}{Kf(\delta+\epsilon)} + \frac{L}{K^2(\delta+\epsilon)} - \frac{1}{K^2 \ln L}(\delta+\epsilon) \cdot 25
\]

Multiplying through by \(K^2\), rearranging, and combining logarithms, we have
\[
\ln \frac{\delta+\epsilon}{L-(\delta+\epsilon)} = \frac{L}{\delta+\epsilon} - \frac{K}{K-q_A} - \frac{Kt}{f(\delta+\epsilon)} . \quad 26
\]

Taking the antilogarithms of both sides, we get
\[
\frac{\delta+\epsilon}{L-(\delta+\epsilon)} = e^{\frac{L}{\delta+\epsilon}} - \frac{K}{K-q_A} - \frac{Kt}{f(\delta+\epsilon)}
\]

or, after rearranging,
\[
\frac{q_A}{K-q_A} e^{\frac{K}{K-q_A}} = \frac{L - \delta - \epsilon}{\delta + \epsilon} e^{\frac{L}{\delta+\epsilon}} - \frac{Kt}{f(\delta+\epsilon)} , \quad 27
\]

which cannot be solved explicitly for \(q_A\).

For the draining porous medium column we can write
\[
Z = P(L, \delta, \epsilon, K, f)
\]
and
\[
q_A = P'(L, \delta, \epsilon, K, f).
\]

Dimensional analysis gives
\[
\frac{Z}{\delta} = G(\frac{L}{\delta}, \frac{\epsilon}{\delta}, \frac{Kt}{\delta}, f) \quad 28
\]

and
First, consider Eq. 28. This indicates that \( \frac{Z}{\delta} \) is functionally related to \( \frac{L}{\delta} \), \( \frac{\xi}{\delta} \), \( \frac{Kt}{\delta} \), and \( f \). If we knew nothing about the theory, we could perhaps still determine the functional relationship. This would be accomplished by determining experimentally the relationship between \( \frac{Z}{\delta} \) and each of the independent pi terms on the right side and then determining how the terms combine to give the general function. However, we have the theoretical relationship as given in Eq. 19. Converting this to an equation in dimensionless terms corresponding to those of Eq. 28, we have

\[
\left[ 1 - \frac{Z}{\delta} \right] \left[ 1 - \frac{L}{\delta} - \frac{\xi}{\delta} - 1 \right] \frac{1 + \frac{\xi}{\delta}}{1 - qA} \frac{1 + \frac{L}{\delta} Kt}{1 + \frac{qA}{K}} = e^{-\frac{Z}{\delta}} e^{-\frac{Kt}{\delta f}}.
\]

In a similar manner we put Eq. 27 into the form of the dimensionless terms of Eq. 29,

\[
\left[ \frac{qA}{K} \right] \left[ 1 + \frac{\xi}{\delta} \right] e^{\frac{1 + \frac{L}{\delta} Kt}{1 + \frac{qA}{K}}} \left[ \frac{1 + \frac{qA}{K}}{1 + \frac{\xi}{\delta} + 1} \right] = e^{\frac{1 + \frac{L}{\delta} Kt}{\delta f}}.
\]

In Eqs. 30 and 31 the dimensionless terms \( \frac{Kt}{\delta f} \) and \( f \) are combined in the form \( \frac{Kt}{\delta f} \). Thus we have theoretical solutions which indicate the manner in which the two pi terms combine. If experimental data agrees with the theoretical solutions then it should be possible to assume that this combination is correct.
To determine how the results would be affected by failure to consider the capillary fringe, the theoretical development was also carried through without consideration of the term $\mathcal{C}$. The resulting equations were

$$\left[1 - \frac{Z}{L} \right] e^{-\frac{Z}{\delta}} = e^{-\frac{Kt}{\delta f}}$$

and

$$\frac{qA}{K} e^{\frac{1}{1 - \frac{qA}{K}}} = \left[\frac{L}{\delta} - 1\right] e^{\frac{L}{\delta} - \frac{Kt}{\delta f}}.$$

**Experimental**

To check the theoretical solutions three experimental runs were conducted with the column as shown schematically in Figure 12. Table 4 gives the values of the pertinent variables and dimensionless pi terms for each run. In addition the diameter of the column was 4.0 in. in all cases. All three runs were made with the same geometrical relationships for the column itself. However, the first two runs were made using 2 mm diameter beads as the porous medium while in the third run 2 mm beads were mixed with 0.5 mm diameter beads to reduce the drainable porosity. The reduction in pore size resulted in an increased thickness of the capillary fringe.

Before each actual drainage run was made the hydraulic conductivity $K$ of the bead-fluid system was determined. This...
Figure 12. Schematic diagram of the apparatus used to study drainage from a vertical porous medium column.
Table 4. Values of the pertinent variables and dimensionless pi terms for each drainage run with the porous medium column

<table>
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<tr>
<th>Run no.</th>
<th>L (in)</th>
<th>δ (in)</th>
<th>ε (in)</th>
<th>K (in/min)</th>
<th>f</th>
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<th>ε/δ</th>
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<td>0.218</td>
<td>0.23</td>
<td>3.0</td>
<td>0.075</td>
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</table>

was accomplished by maintaining ponded fluid on the surface, measuring the rate of discharge, head differential, and length of column, and computing K from Darcy's Law. The drainable porosity f was determined by measuring the total amount of fluid discharged during the drainage run and dividing by the volume of the medium drained.

The bead column was initially saturated just to the surface. The run was started by unclamping the discharge tube and adjusting the outlet so that the fluid stood to the height δ in the piezometer. The outlet was adjusted from time to time during the run so as to maintain a constant fluid level in the piezometer tube. The rate of discharge was determined by catching the fluid in a graduated beaker for certain time intervals. A scale was attached to the wall of the column and the level of the surface of saturation was read from this scale. Time was determined by observing a timer which was started at the moment drainage began.
Comparison of theory and experiment

Eqs. 30, 31, 32, and 33 cannot be solved explicitly for \( Z/\delta \) and \( q_A/K \) by any means known to this writer. However, a graphical method of solution was used to obtain the theoretical curves of \( Z/\delta \) versus \( \frac{Kt}{\delta T} \) and \( q_A/K \) versus \( \frac{Kt}{\delta T} \) for the particular values of \( L/\delta \) and \( \epsilon/\delta \) studied. Eq. 30 will serve as an example.

First, the left hand side of Eq. 30 was plotted as a function of \( Z/\delta \). This was accomplished by selecting values of \( Z/\delta \) in the range of interest, evaluating the left hand side of the equation for each, and plotting the results on graph paper with the left hand side of the equation as the ordinate and \( Z/\delta \) as the abscissa. Next, values of \( \frac{Kt}{\delta T} \) were selected and the right hand side of Eq. 30 was evaluated for each. Since the two sides of the equation are equal, the value of \( Z/\delta \) corresponding to a given value of \( \frac{Kt}{\delta T} \) was found by entering the graph with the computed value of the right hand side of the equation and reading off the value of \( Z/\delta \). The values of \( Z/\delta \) were then plotted against the values of \( \frac{Kt}{\delta T} \). A similar procedure was followed in obtaining the curves showing \( q_A/K \) as a function of \( \frac{Kt}{\delta T} \).

In Figures 13 through 16 the theoretical curves of dimensionless drawdown and discharge for the bead column have been compared with the experimental data. The solid lines show
Figure 13. Dimensionless drawdown in the porous medium column with 2 mm diameter beads and drainable porosity $f = .33$. 
DIMENSIONLESS DRAWDOWN IN BEAD COLUMN
(2 mm. DIAMETER BEADS)

\[ \frac{L}{\delta} = 3 \]
\[ \frac{\epsilon}{\delta} = 0.0375 \]
\[ f = 0.33 \]

---

---

THEORETICAL (capillary fringe not considered)

THEORETICAL (capillary fringe considered)

○ △ EXPERIMENTAL (2 runs)

\[
1 - \frac{Z/\delta}{L/\delta - \epsilon/\delta - 1} \left[ 1 + \frac{\epsilon/\delta}{Z/\delta} \right] = e^{-Z/\delta} = e^{-Kt/\delta f}
\]
Figure 14. Dimensionless discharge from the porous medium column with 2 mm diameter beads and drainable porosity $f = .33$. 
DIMENSIONLESS DISCHARGE FROM BEAD COLUMN
(2 mm DIAMETER BEADS)

\[ L/\delta = 3 \]
\[ \epsilon/\delta = 0.0375 \]
\[ f = 0.33 \]

--- THEORETICAL (capillary fringe not considered)
--- THEORETICAL (capillary fringe considered)
○△ EXPERIMENTAL (2 runs)

\[ \frac{q_A}{K} = \frac{1 + \epsilon/\delta}{1 - q_A/K} e^{\frac{L/\delta}{1+\epsilon/\delta}} \]

\[ e^{-Kt/\delta f} \]
Figure 15. Dimensionless drawdown in the porous medium column with mixed 2 mm and 0.5 mm diameter beads, and with drainable porosity $f = 0.23$. 
DIMENSIONLESS DRAWDOWN IN BEAD COLUMN
(MIXED 2 mm AND 0.5 mm BEADS)

\[ L/\delta = 3 \]
\[ \epsilon/\delta = 0.075 \]
\[ f = 0.23 \]

- THEORETICAL
- EXPERIMENTAL

\[
\frac{Z/\delta}{e^{-Z/\delta} - e^{-Z/\delta\cdot f}} \cdot \frac{1 + \epsilon/\delta}{e^{-Z/\delta\cdot f}}
\]

\[ Kt/\delta f \]
Figure 16. Dimensionless discharge from the porous medium column with mixed 2 mm and 0.5 mm diameter beads, and with drainable porosity $f = .23$. 
DIMENSIONLESS DISCHARGE FROM BEAD COLUMN
(MIXED 2 mm. AND 0.5 mm. BEADS)

- \( L/\delta = 3 \)
- \( \epsilon/\delta = 0.075 \)
- \( f = 0.23 \)

--- THEORETICAL

--- EXPERIMENTAL

\[
\frac{q_A/K}{1-q_A/K}^{1+\epsilon/\delta} \cdot \frac{1+\epsilon/\delta}{e^{1-q_A/K}} = e^{L/\delta - Kt/\delta f}
\]
the theoretical relationships with capillary fringe considered, the dotted lines show the theoretical relationships with capillary fringe not considered, and the experimental points are indicated by small circles and triangles.

Figure 13 shows $Z/\delta$ as a function of $\frac{Kt}{\delta_f}$ for the 2 mm diameter beads. There were two experimental runs for this case and the two separate sets of data agree quite well. There is also good agreement between the experimental data and the theory for which the capillary fringe was taken into account, the maximum deviation being about two per cent.

Figure 14 shows the relationship between $q_A/K$ and $\frac{Kt}{\delta_f}$ for the 2 mm beads. Here also the two experimental runs show close agreement with each other and with the theory for which the capillary fringe was considered.

Figures 13 and 14 indicate that no very great error would have been incurred in the theory if the capillary fringe had not been considered. However, the capillary fringe was rather small in this case and a greater error would result with a larger fringe.

Figures 15 and 16 show the dimensionless drawdown and discharge for the case of a reduced drainable porosity. The agreement between theory and experiment is not as good as before. It is believed that the discrepancy is due to the fact
that not all of the fluid is drained from the smaller pores immediately as the surface of saturation passes, but that some continues to drain after the surface has passed. In the theoretical development it was implicitly assumed that all pores are completely drained immediately as the surface of saturation passes.

Generally good agreement was found between theory and experiment for both dimensionless drawdown and discharge in the porous medium column. Therefore, it was assumed that the combination of the pi terms $\frac{Kt}{S}$ and $f$ in the form $\frac{Kt}{Sf}$ is correct. The assumption of a similar combination of terms in the ditch drainage problem was based on the fact that this combined dimensionless term is a parameter having to do with the movement of the surface of saturation through the porous medium. As such it should appear in any problem in which a porous medium is being drained or wetted.
RESULTS

Data Obtained for Ditches Penetrating to an Impermeable Layer

**Drawdown and discharge curves**

For the case of the ditches penetrating to the impermeable layer and no water standing in the ditches, it was determined that

\[ \frac{q}{K_d} = G\left(\frac{a}{d}, \frac{K_t}{d_f}\right) \]

and

\[ \frac{z}{d} = G'\left(\frac{a}{d}, \frac{K_t}{d_f}\right) \]

Runs were made with the model for various values of \(a/d\). There was some duplication of runs since this case was also used to study the effect of capillarity in the model, and thus runs were made at constant \(a/d\) but with \(a\) and \(d\) varied.

At first \(Z/d\) was plotted as a function of \(\frac{K_t}{d_f}\) for each value of \(a/d\). However, it was observed that on semi-logarithmic paper the curves were roughly parallel. This indicated the possibility of combining the curves so as to obtain a single functional relationship. Investigation revealed that if the values of \(\frac{K_t}{d_f}\) were multiplied by the square of the reciprocal of \(a/d\), that is \((d/a)^2\), then the data would all fall quite close together when plotted on the same set of axes.

Figure 17 shows the drawdown data plotted with \(Z/d\) as a
Figure 17. Drawdown data obtained from the model for the case of ditches penetrating to the impermeable layer and no fluid standing in the ditches.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

SYMBOL a/d  NUMBER OF RUNS
  o  5    1
  □ 10   3
  △ 20   3
  • 40   2
  x 80   1

\[ Z/d = \sqrt{\frac{K}{d/a}} \]

\[ Z/d = 0.80 + 0.23 \ln \left[ \frac{K_l}{df_x d/a} \right] \]

\((0.012 < Z/d < 0.76)\)
function of \( \frac{Kt}{df} \left( \frac{d}{a} \right)^2 \). The solid line with dashed tail is the curve of observed best fit to this data. The individual points exhibit some scatter about this curve. However, the maximum deviation is little more than would be expected when one considers the accuracy with which data could be read from the photographs. There is no consistent trend in deviations with different values of a/d.

An equation was derived for the solid portion of the curve of best fit. This represents the portion of the curve which is a straight line on the semi-logarithmic plot. The equation is

\[
\frac{Z}{d} = 0.80 + 0.23 \ln \left[ \left( \frac{Kt}{df} \right) \left( \frac{d}{a} \right)^2 \right]
\]

and is valid for values of \( \frac{Z}{d} \) between 0.12 and 0.76. For practical purposes this is the portion of the curve which would be of interest. A designer would probably only rarely be interested in values of \( \frac{Z}{d} \) outside this range.

The completely dashed curve of Figure 17 has the equation

\[
\frac{Z}{d} = \left[ \left( \frac{Kt}{df} \right) \left( \frac{d}{a} \right)^2 \right]^{\frac{1}{3}}.
\]

Rearranging terms we obtain a relatively simple ditch spacing formula,

\[
a = \frac{d}{Z} \sqrt{\frac{Kd}{f}}.
\]

It is obvious that for practical design purposes this formula would be much easier to work with than would Eq. 34. Its use
would result in a slightly over-designed system. However, in many cases of variable soil conditions this would not be undesirable.

In Figure 18 the discharge data for the completely penetrating ditch case is presented. On this logarithmic plot of \( q/K_d \) versus \( K_t/df \), the data for each value of \( a/d \) follows a common straight line for a portion of the run. The dimensionless discharge decreases more rapidly later on, however. The points at which the curves begin to deviate from the straight line are dependent upon the value of the parameter \( a/d \). The dots in this figure represent data points for all values of \( a/d \) for which runs were conducted. These points were used in the calculation of a general equation. The circles represent data points deviating from this straight line. The values of \( a/d \) for the deviating curves are encircled.

The general equation calculated by the method of least squares to fit the dotted points is

\[
\frac{q}{K_d} = 0.671 \left( \frac{K_t}{df} \right)^{-0.516}.
\]

The value of \( K_t/df \) at which the curves begin to deviate from the straight line represented by the above equation are plotted in Figure 19 against the corresponding values of \( a/d \). A surprisingly good correlation was found between these quantities. The equation relating the point of deviation to the value of \( a/d \) was found to be

\[
\left( \frac{K_t}{df} \right)_D = 0.064 (a/d)^{2.08}.
\]
Figure 18. Discharge data obtained from the model for the case of ditches penetrating to the impermeable layer and no fluid standing in the ditches.
DITCHES PENETRATING TO IMPERMEABLE LAYER
NO FLUID STANDING IN DITCHES

\[ \frac{q}{Kd} = 0.671 \left( \frac{Kt}{df} \right)^{-0.16} \]

- Points used in calculating general equation
- Points deviating from the straight line for which equation was determined

\[ \frac{a}{d} = 4 \]
Figure 19. Point of deviation of the data of Figure 18 from a straight line, plotted as a function of a/d.
\[(Kt/df)_D = 0.64 \times (a/d)^{2.08}\]

Data taken from previous figure.
Comparison with work of Kirkham and Gaskell

Kirkham and Gaskell (28) utilized relaxation techniques to determine the falling water table location between ditch drains which penetrated to an impermeable layer. They calculated water table drawdown for five foot ditches spaced at 20 feet, 40 feet, and 80 feet. Considering a model scale of one inch to one foot, the model runs with a/d = 4.0, 8.0, and 16 corresponded to the cases studied by Kirkham and Gaskell.

Kirkham and Gaskell presented water table location as a function of the parameter Kt/f. Figure 20 shows their predicted water table location for three values of Kt/f for a ditch spacing of 40 feet. Also on this figure are the water table locations for three values of Kt/f for the corresponding case in the model. Corresponding values of Kt/f were not available for theory and model but the water table locations may still be compared. Midway between ditches the agreement appears to be good with the water table in the model falling just a little more rapidly than predicted by the theory. At the ditch wall, however, the experimental water table remained considerably higher than that obtained by the relaxation technique. This was probably due to the fact that Kirkham and Gaskell did not consider the existence of a surface of seepage.

Figure 21 shows the value of the parameter Kt/f for ½ foot and 1 foot of water table drawdown midway between ditches as a
Figure 20. Location of the water table as a function of the parameter $Kt/f$ for $a/d = 8$. NUMBERS REPRESENT VALUES OF THE PARAMETER $Kt/f$ FOR THE CURVES SHOWN.
Figure 21. Values of the parameter $Kt/f$ for $\frac{1}{2}$ ft. and 1 ft. of water table drawdown midway between ditches for three ditch spacings.
I UNIT OF DRAWDOWN MIDWAY BETWEEN DITCHES

- - - KIRKHAM AND GASKELL
- - FROM MODEL

1/2 UNIT OF DRAWDOWN MIDWAY BETWEEN DITCHES

- - - KIRKHAM AND GASKELL
- - FROM MODEL

DISTANCE BETWEEN DITCHES (UNITS)
function of the ditch spacing. The agreement between the model results and Kirkham and Gaskell's results is quite close, particularly for the closer spacings.

An attempt was made by this writer to obtain a mathematical solution for water table height for the case of the ditches penetrating to the impermeable layer and with the surface of seepage considered. This attempt was only partially successful as a complete solution was not obtained. However, the work is presented in Appendix B.

Data Obtained for Ditches Partially Penetrating to an Impermeable Layer

**Drawdown and discharge curves**

The drawdown data for the case of the ditches partially penetrating to the impermeable layer and only a small depth of water standing in the ditches is presented in Figure 22. In a manner similar to that followed for the case of the ditches penetrating completely to an impermeable layer, the dimensionless drawdown $Z/b$ is plotted as a function of $\left(\frac{K_t}{d_f}\right)\left(\frac{b}{a}\right)^2$. When plotted in this manner the data for a particular value of $d/b$ seems to follow the same relationship regardless of the value of $a/b$. The scatter of the points about curves drawn for visual best fit through the data is no more than would be expected when the accuracy with which the data
Figure 22. Drawdown data obtained from the model for the case of ditches partially penetrating to an impermeable layer and only an insignificant amount of water standing in the ditches.
DITCHES PARTIALLY PENETRATING IMPERMEABLE LAYER

$h/b = 0.75$ (DITCHES ESSENTIALLY EMPTY)

$Z/b$ vs. $(Kt/bf)(b/a)^2$

**Symbol Key:**
- $a/b$: 10
- $a/b$: 20
- $a/b$: 40
- $a/b$: 80

**Key Values:**
- $d/b = 4.0$
- $d/b = 3.0$
- $d/b = 2.0$
- $d/b = 1.0$
could be read from the photographs is considered. Attempts made to further consolidate the data and obtain a general expression for drawdown $Z$ were unsuccessful.

In Figure 23 the drawdown data for the case of partially penetrating ditches running half full is presented. Since the model was operated with the ditches half full of fluid in each of these runs, the value of the pi term $h/b$ varies with $d/b$, that is, $h/b = \frac{1}{2}d/b$. Here again $Z/b$ was plotted versus $(\frac{Kt}{b^2})(\frac{b}{a})^2$ with the result that data for different values of $a/b$ was clustered together in such a way that a single curve could be drawn in for each value of $d/b$.

It should be noted in both Figures 22 and 23 that the drawdown curves do not pass through the origin. In every case when starting from a completely saturated condition, drawdown did not begin midway between ditches until some drainage had taken place from the corners near the ditches.

Figures 22 and 23 can be used in drainage design as will be illustrated later. Figure 22 represents a rather idealized condition in which the ditches have practically no water in them. Figure 23 represents a more severe drainage condition in which the ditches are running half full of water.

The discharge data for the case of the essentially empty ditches is shown in Figure 24. On the logarithmic scale $q/Kb$
Figure 23. Drawdown data obtained from the model for the case of ditches partially penetrating to an impermeable layer and the ditches running half full of water.
DITCHES PARTIALLY PENETRATING TO IMPERMEABLE LAYER
(DITCHES RUNNING HALF FULL)

SYMBOL a/b
• 10
□ 20
△ 40
○ 80

d/b = 4.0
h/b = 2.0

d/b = 3.0
h/b = 1.5

d/b = 2.0
h/b = 1.0

(Kt/bf)(b/a)^2
Figure 24. Discharge data obtained from the model for the case of ditches partially penetrating to an impermeable layer and only an insignificant amount of water standing in the ditches.
is a linear function of $\frac{Kt}{bf}$ for a portion of each run but decreases more rapidly toward the end of the run. For a given value of $d/b$ the straight line portions of the curves fall essentially on top of one another for different values of $a/b$. However, the point at which the curves start to deviate from the straight line is a function of $a/b$.

The straight lines of best fit to the data of Figure 24 had slopes of every nearly $-0.50$. Therefore, the slopes were all forced to be exactly $-0.50$ and the equations were determined on that basis. Thus the curves for different values of $d/b$ were made parallel. In Figure 25 values of $q/Kb$ are plotted as a function of $d/b$ at constant values of $\frac{Kt}{bf}$. These values were taken from Figure 24. The equations relating $q/Kb$ to $a/d$ at constant values of $\frac{Kt}{bf}$ are shown.

According to Murphy (41), when logarithmic plots of one pi term as a function of a second with a third held constant are parallel lines, then the functions relating $\pi_1$ to $\pi_2$ and $\pi_1$ to $\pi_3$ may be combined by multiplication to give a general relationship for $\pi_1$ as a function of $\pi_2$ and $\pi_3$. On this basis the general expression

$$\frac{q}{Kb} = 1.09(d)^{1.28} (\frac{Kt}{bf})^{-0.50}$$

was determined.

This general expression is not applicable after the curves of Figure 24 begin to deviate from the linear logarith-
Figure 25. $q/K_b$ versus $d/b$ for constant values of $K_t/b$. Data taken from Figure 24.
DITCHES PARTIALLY PENETRATING TO IMPERMEABLE LAYER

\[ h/b = 0.075 \quad Kt/bf = 1.0 \]

\[ \frac{q}{Kb} = 1.09 \left( \frac{d}{b} \right)^{1.28} \]

\[ \frac{q}{Kb} = 0.34 \left( \frac{d}{b} \right)^{1.28} \]

\[ \frac{q}{Kb} = 0.107 \left( \frac{d}{b} \right)^{1.28} \]

\( d/b \) vs. \( q/Kb \)
Figure 26. Points of deviation of the data of Figure 24 from the straight lines, plotted as a function of a/b.
DITCHES ESSENTIALLY EMPTY

(h/b) = .075

SYMBOL  d/b
  •  1.0
  △  2.0
  □  3.0
  ○  4.0

\[(Kt/bf)_D = 0.025(a/b)^{1.94}\]
mic relationship. The points at which these curves begin to deviate from the straight line are plotted in Figure 26 against $a/b$. A single relationship,

$$\left[ \frac{Kt}{bf} \right]_D = 0.025 \left( \frac{a}{b} \right)^{1.94},$$

was determined relating the point of deviation to the value of $a/b$ for all values of $d/b$ studied.

Eq. 39 represents an upper limit for the dimensionless discharge term. For values of $\frac{Kt}{bf}$ beyond the range of applicability of the equation for a particular value of $a/b$, the value of $q/KB$ will be less than that given by the equation. This is evident from the manner in which the curves of Figure 24 deviate from the linear logarithmic relationship.

The discharge data for the case of the ditches running half full is presented in Figure 27. In Figure 28, $q/KB$ is plotted as a function of $d/b$ for constant values of $\frac{Kt}{bf}$. The data was treated in a manner similar to that for the previous case and the equation

$$\frac{q}{KB} = 0.62 \left( \frac{d}{b} \right)^{1.45} \left( \frac{Kt}{bf} \right)^{-0.54}$$

was determined.

In Figure 29 the points of deviation of the curves of Figure 27 from the linear logarithmic relationships are plotted versus the corresponding values of $a/b$. The equation

$$\left[ \frac{Kt}{bf} \right]_D = 0.019 \left( \frac{a}{b} \right)^{1.99}$$
Figure 27. Discharge data obtained from the model for the case of ditches partially penetrating to an impermeable layer and the ditches running half full of water.
Figure 28. \( \frac{q}{K_b} \) versus \( \frac{d}{b} \) for constant values of \( \frac{K_t}{b_f} \). Data taken from Figure 27.
DITCHES PARTIALLY PENETRATING TO IMPERMEABLE LAYER
DITCHES RUNNING HALF FULL

\[ \frac{q}{Kb} = 1.0 \]

\[ \frac{q}{Kb} = 0.62 \left( \frac{d}{b} \right)^{1.45} \]

\[ \frac{q}{Kb} = 0.17 \left( \frac{d}{b} \right)^{1.45} \]

\[ \frac{q}{Kb} = 0.048 \left( \frac{d}{b} \right)^{1.45} \]
Figure 29. Points of deviation of the curves of Figure 27 from the straight lines, plotted as a function of $a/b$. 
DITCHES RUNNING HALF FULL

SYMBOL
△ 2.0
□ 3.0
○ 4.0

\[(Kt/bf)_D = 0.019 (a/b)^{1.99}\]
was determined from this data. Eq. 41 is applicable for values of $\frac{Kt}{bf}$ less than those obtained from Eq. 42 for a particular value of $a/b$.

Comparison with results of the extension of the Kirkham steady state theory to the falling water table case

In Appendix C of this dissertation an extension of the Kirkham (25, 26) theory for water table heights between parallel drains is presented. The original theory of Kirkham was for the case where the water table is maintained in a fixed position by rain falling at the steady rate $R$. The development given in Appendix C, which is also due to Dr. Kirkham, extends this theory to the falling water table case.

The equation derived in Appendix C for the height $H$ of the falling water table midway between ditches is

$$H_0 e^{\frac{Kt}{f}} = H_0 e^{\frac{1}{SF(s,0)}(H_0 - \frac{Kt}{f})}$$

where $H_0$ is the height at time $T = 0$, $s$ is the half spacing of the ditches ($s = a/2$), and $F(s,0)$ is $2/\pi$ times the bracketed portion of Eq. 61 with $x = s$. It is assumed in the derivation of this equation that the water table at time $T = 0$ is due to a steady rate of rainfall.

It occurred to this writer that the water table position in the model could be checked against this theory provided that
the time of beginning of drawdown was taken as the moment at which drawdown began midway between ditches, that is, after the arched water table had developed. This should be true provided that the falling water table does reach, in a reasonably short time, a shape which can be adequately represented by a steady state water table.

To check on the agreement between the shape of the falling water table at various times and the shape of the theoretical steady state water table, Figure 30 was constructed. The case for comparison was that with a ditch depth of 8 in. with water standing in the ditch to a depth of 4 in. The depth below the ditch bottom to the impermeable layer was 2 in. and the ditch spacing was 80 in.

The observed falling water table is shown as a dotted line in Figure 30 for three different values of the parameter $\frac{Kt}{Df}$. The theoretical steady state water table, computed from Eq. 61 (See Appendix C) to coincide with the observed water table midway between ditches, is shown as a solid line. It is obvious from this figure that, starting shortly after drawdown begins midway between ditches, the falling water table can be represented by a steady state water table with very little error. The error that does exist in the second and third cases of the figure appears to be due to the neglect of the surface of seepage in the theoretical development.
The curves presented in Figure 30 indicate that the falling water table can be represented by a series of steady state water tables throughout most of the drawdown cycle. Therefore, the observed drawdown as a function of time at the midpoint between ditches was compared with the theoretical drawdown computed using Eq. 73 and the fact that

\[ Z = d - h - H. \]

Eq. 73 was solved for \( H \) as a function of \( T \) by a graphical method similar to that employed in the solution of Eq. 30. (See page 75.)

Figure 31 compares the theoretical and observed drawdown curves for the same case as for the water table comparisons of Figure 30. In Figures 32 and 33 the theoretical and observed drawdown curves are compared for two other cases. These cases are the same as that of Figure 31 except that for Figure 32, \( a = 40 \) in. and for Figure 33, \( a = 160 \) in. As mentioned earlier, the time \( T \) was measured from the moment drawdown began midway between ditches. Therefore, in all cases \( T \) has values smaller than \( t \), the time measured from the moment drainage began at complete saturation.

In all three of the cases for which comparisons were made the observed drawdown took place a little more rapidly than predicted by the theory. However, the difference in drawdown values was rather small and was in no case greater than about
Figure 30. Comparison of observed falling water table positions with theoretical steady state water table positions. The theoretical surfaces were computed so as to coincide with the observed surfaces midway between the ditches.
UNSTEADY SURFACE IN MODEL
STEADY SURFACE FROM KIRKHAM THEORY

Kt/bf = 3.34

Kt/bf = 21.8

Kt/bf = 53.5

DISTANCE FROM DITCH CENTER (in)

HEIGHT ABOVE DITCH BOTTOM (in)
Figure 31. Comparison of theoretical and observed drawdown curves for the geometry indicated. Geometry same as that for previous figure.
DRAWDOWN MIDWAY BETWEEN DITCHES

\[ o = 80 \text{ in} \]
\[ K = 1.27 \text{ in/min} \]
\[ f = 0.38 \]

\[ b = 2 \text{ in} \]
\[ d = 8 \text{ in} \]
\[ h = 4 \text{ in} \]
\[ a = 80 \text{ in} \]

---

- **FROM MODEL**
- **FROM EXTENSION OF STEADY STATE THEORY OF KIRKHAM**
Figure 32. Comparison of theoretical and observed drawdown for the geometry indicated.
DRAWDOWN MIDWAY BETWEEN DITCHES

\[ b = 2 \text{ in} \]
\[ d = 8 \text{ in} \]
\[ h = 4 \text{ in} \]
\[ a = 40 \text{ in} \]
\[ K = 1.25 \text{ in/min} \]
\[ f = .38 \]
Figure 33. Comparison of theoretical and observed drawdown curves for the geometry indicated.
DRAWDOWN MIDWAY BETWEEN DITCHES

b = 2 in

d = 8 in

h = 4 in

a = 160 in

K = 1.23 in/min

f = .38

--- FROM MODEL ---

--- FROM EXTENSION OF STEADY STATE THEORY OF KIRKHAM ---
fifteen percent.

Use of results in a drainage design problem

Let us consider an example to show how the data obtained from drainage runs made with the model can be used in drainage design. The case is hypothetical but the values of the variables are in a practical range.

Suppose we wish to drain a field by constructing parallel open ditches. These ditches are to discharge into a collector ditch in which water backs up and stands for several days after heavy rains. Hydraulic conductivity measurements in the field indicate a fairly uniform soil with an average hydraulic conductivity of \( K = 4 \text{ ft/day} \) underlain at a depth of 5.5 ft. by a relatively impermeable layer. The average drainable porosity at a tension of 60 cm. of water is \( f = 0.06 \). The crop to be grown on the area will not be damaged if the water table is lowered to a depth \( Z \) of at least 1 ft. in 36 hrs. (1\frac{1}{2} \text{ days}) after a heavy rain.

Tentatively select a ditch depth \( d \) of 4 ft. We will assume that it may be necessary to obtain the desired drainage with water standing in the ditch to within 2 ft. of the surface. Thus for a 4 ft. ditch, the ditch will be half full of water. We have \( b = 5.5 - 4 = 1.5 \text{ ft.} \), and \( d/b = 2.67 \). For a drawdown \( Z \) of 1 ft., \( Z/b = 0.67 \). Enter Figure 23 with \( Z/b = 0.67 \), read
across to the interpolated position of the curve for \( d/b = 2.67 \), and then read down to find the value

\[
\frac{Kt}{bf} \left( \frac{b}{a} \right)^2 = 0.039.
\]

Solving for \( a \), we find

\[
a = \sqrt{\frac{Ktb}{0.039f}}.
\]

Substituting the appropriate values for \( K \), \( t \), \( b \), and \( f \), we find the spacing,

\[
a = 62 \text{ feet}.
\]

Other ditch depths might be tried until the most economical combination of depth and spacing were found. If the discharge were desired this could then be found from Eq. 41, subject to the limitation imposed by Eq. 42.
DISCUSSION

Effect of Pore Size on Drainage Characteristics

It appears that one of the major problems in the study of soil drainage using a model of the type described in this dissertation may lie in the size of the pores. Large particles must almost certainly be used in a small scale model unless some surface treatment can be found to completely eliminate capillary rise of the fluid. In the consequent large pores, drainage takes place practically instantly and completely as the surface of saturation moves downward. However, in the soil drainage situation, where pores are quite small, the amount of water removed is dependent upon time and upon location with respect to the free surface. With a downward moving water table, water may continue to drain from a given pore location for some time after the water table itself has passed. Such a phenomenon apparently caused some discrepancy between theoretical and experimental results for the porous medium column when smaller pores were utilized.

A drainage system designed on the basis of results of tests in models using relatively large size particles would probably be somewhat over designed. Although drainage would not be instantly complete as the water table passed, the water table itself would actually be expected to fall more rapidly than predicted by the model. Thus plant roots would receive
some aeration more quickly than would be expected from the model results.

Significance of the Parameter $\frac{Kt}{bf}$

The dimensionless term $\frac{Kt}{bf}$, or a similar one incorporating some other length term besides b, appears to be a very important parameter in unsteady state flow in porous media. It would probably be important in a situation in which the medium was being saturated as well as in one in which drainage was taking place. The significance of the parameter becomes apparent if one examines the variables which make it up. The hydraulic conductivity $K$ is the approach velocity or "Darcy velocity" under a unit hydraulic gradient. Division by the drainable porosity $f$ gives $\frac{K}{f}$ which is the velocity of movement of the fluid surface through the porous medium under a unit hydraulic gradient normal to the surface. $\frac{Kt}{f}$ is the distance the surface would move in time $t$ under a unit normal hydraulic gradient. Division by some length term such as $b$ then yields $\frac{Kt}{bf}$ which is the ratio of the distance the free surface would move in time $t$ under a unit hydraulic gradient to some constant length of the system.

Applicability of the Results Obtained from Model Studies to Field Conditions

Certain characteristics of the ditch drainage model are such that the value of the results and their applicability to
field conditions might be further examined. First of all, open ditch drainage systems are not common, particularly in this country. The open ditch system was chosen for study because it lends itself somewhat better to a model investigation and also to theoretical investigation. The first objective of this study was to investigate the application of the theory of similitude to the modeling of unsteady state soil drainage problems; the system for study was chosen with this in mind. Furthermore, there are indications that in many cases tile drainage systems perform very much like ditch systems because of the higher permeability of the tile trench backfill, and in such cases the data presented herein might be applicable.

Next, the shape of the ditch cross section in the model was strictly rectangular. Such a shape would rarely occur in the field. The usual field ditch would have a rounded bottom and sloping sides. Although no data is available, it is this writer's opinion that the effect on drainage characteristics of this difference in ditch shape would be slight.

Finally, the drainage data obtained from the model and presented in this dissertation does not necessarily cover the entire range of geometries that would be encountered in practice. In particular more data is needed for the case of relatively shallow ditches in a deep permeable layer; that is, for smaller values of d/b.
SUMMARY AND CONCLUSIONS

The purposes of the study were to investigate the application of the theory of similitude to the modeling of problems involving unsteady state soil drainage, to construct and operate a model, and to compare results of the model tests with certain theories.

A group of variables which is believed to adequately describe the system involving drainage to parallel open ditches under falling water table conditions was selected. These variables were combined to form dimensionless pi terms and a model was designed on the basis of these dimensionless terms. Two cases were considered separately. One was the case in which the ditches penetrate to an impermeable layer and no water stands in the ditches, and the other was the case in which the ditches only partially penetrate to an impermeable layer and water may stand at various levels in the ditches.

A two-dimensional model of the parallel ditch drainage system was constructed of plexiglas. Glass beads were used as the porous medium and glycerol as the viscous fluid. The capillary fringe in the model was reduced by treating the glass beads with a silicone material. Tests were run to determine whether the remaining capillary fringe would appreciably affect the drawdown and discharge results obtained. No effect was apparent when a minimum depth of two inches of beads was used in
the model. The effect of varying the hydraulic conductivity of the porous medium was also studied. It was found that the relationships between the dimensionless terms were not affected by variations in the hydraulic conductivity.

One of the dimensionless parameters in any unsteady state drainage problem is the drainable porosity of the porous medium. In order to examine how this parameter might be combined with another dimensionless parameter to avoid the necessity for evaluating the effect of a distortion factor, the drainage from a vertical porous medium column was studied. A theoretical solution for the drawdown of the surface of saturation and the discharge from the column was derived. Experimental data was obtained which agreed well with the theory when the assumptions made in deriving the theory were met. From this study it was determined that the drainable porosity \( f \) could be combined with another dimensionless term to form the parameter \( Kt/bf \), where \( K \) is the hydraulic conductivity, \( t \) is the time, and \( b \) is a length term.

Experimental runs were made with the ditch drainage model for a variety of geometrical relationships. The results of these runs were presented as dimensionless plots of discharge and drawdown of the surface of saturation. Where possible the data was consolidated and general expressions for drawdown and discharge were obtained. An example was presented which
indicates that the use of the model data in drainage design leads to reasonable results.

The water table drawdown observed in the model was compared with the theoretical work of Kirkham and Gaskell (28) and with an extension of the Kirkham (25, 26) theory for the water table position under steady rainfall. Fairly good agreement between theory and experiment was found in each case.

From this study it can be concluded that the application of the theory of similitude in the modeling of systems involving flow through porous media is feasible. It appears that further investigations in this area would be profitable.
SELECTED BIBLIOGRAPHY


48. Schlick, W. J. The spacing and depth of laterals in Iowa underdrainage systems and the rate of runoff from them with data from investigations. Iowa State University of Science and Technology. Engr. Exp. Sta. Bul. 52. 1918.


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## APPENDIX A. LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$(L^2)$</td>
</tr>
<tr>
<td>$A_n, B_n, \ldots$</td>
<td>- Fourier constants</td>
</tr>
<tr>
<td>$a$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$b$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$c_1, c_2, \ldots$</td>
<td>- constants of integration</td>
</tr>
<tr>
<td>$d$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$e$</td>
<td>$(-)$</td>
</tr>
<tr>
<td>$f$</td>
<td>$(-)$</td>
</tr>
<tr>
<td>$g$</td>
<td>$(LT^{-2})$</td>
</tr>
<tr>
<td>$H$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$h$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$i$</td>
<td>$(-)$</td>
</tr>
<tr>
<td>$K$</td>
<td>$(LT^{-1})$</td>
</tr>
<tr>
<td>$k$</td>
<td>$(L^2)$</td>
</tr>
<tr>
<td>$L$</td>
<td>$(L)$</td>
</tr>
<tr>
<td>$ln$</td>
<td>$(-)$</td>
</tr>
<tr>
<td>$m, n, \ldots$</td>
<td>- integers</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(L^3T^{-1})$</td>
</tr>
<tr>
<td>$q$</td>
<td>$(L^2T^{-1})$</td>
</tr>
</tbody>
</table>
\( q_A \) - rate of discharge per unit of cross-sectional area of the porous medium column \((LT^{-1})\)

\( R \) - rate of rainfall \((LT^{-1})\)

\( r \) - semi-width of a ditch \((L)\)

\( s \) - ditch semi-spacing \((L)\)

\( T \) - time measured from the moment drawdown begins midway between drains \((T)\)

\( t \) - time measured from the moment drainage begins at complete saturation \((T)\)

\( x \) - horizontal component of rectangular coordinates \((L)\)

\( y \) - vertical component of rectangular coordinates \((L)\)

\( z \) - minimum drawdown of surface of saturation below the surface of a porous medium \((L)\)

\( z \) - height of the water table above its lowest point at a given value of \( x \) and at a given time \((L)\)

\( \delta \) - portion of the length of the porous medium column not contributing to the potential which produces drainage \((L)\)

\( \varepsilon \) - thickness of the capillary fringe \((L)\)

\( \mu \) - dynamic fluid viscosity \((ML^{-1}T^{-1})\)

\( \pi \) - 3.1416.... \((-)\)

\( \pi_1, \pi_2, \ldots \) - dimensionless pi terms \((-)\)

\( \phi \) - fluid density \((ML^{-3})\)

\( \sigma \) - height of the upper limit of the surface of seepage above the ditch bottom \((L)\)

\( \Phi \) - velocity potential \((L^2T^{-1})\)

\( \phi \) - hydraulic head or potential \((L)\)

\( \psi \) - stream function \((L^2T^{-1})\)
APPENDIX B. A PARTIAL THEORETICAL SOLUTION FOR THE WATER TABLE HEIGHT BETWEEN DITCHES PENETRATING TO AN IMPERMEABLE LAYER WITH A SURFACE OF SEEPAGE CONSIDERED

An attempt was made to obtain a mathematical solution for the water table height between ditches penetrating to an impermeable layer. A surface of seepage was taken into account. This has not been done in most such problems to date. The attempt was not completely successful since the solution obtained is a function of the unknown height of the surface of seepage. The work which was done in carrying the solution of the problem to this point will be presented here.

The flow region under consideration is shown schematically in Figure 34. The region shown extends from one ditch to the midpoint between ditches. A mirror image of this region would extend to the next ditch to the right. As before, a is the ditch spacing and thus s is the semispacing of the ditches. The symbol d represents the ditch depth and for this case is also the thickness of the permeable layer. Water stands in the ditch to the height h. Steady rain falls at the rate R on the soil surface, seeps downward, and maintains a water table in the soil at the height $\sigma + z$ where $\sigma$ is the height of the upper edge of the surface of seepage. The length of the surface of seepage itself is thus $\sigma - h$. The origin of rectangular coordinates is taken at O, with y positive upward and x positive to the right.
The assumptions for this problem are basically the same as those made by Kirkham (25) except that a surface of seepage is considered. The head dissipated in the region beneath the water table and above the line BC is considered to be negligible in relation to that dissipated in the region OBCD. Therefore, the region above BC can be thought of as being filled with gravel in which no head is dissipated, but having closely spaced, infinitely thin vertical membranes which direct the steady rainfall vertically downward to the surface BC.

Under the above assumptions the problem becomes a boundary value problem in the rectangular region OBCD. The five boundaries are indicated in Figure 34 by the circled numbers. The conditions on these boundaries are as follows:

1. Constant potential along OA due to standing water in the ditch,
2. Potential proportional to elevation along the surface of seepage AB. Exact location of B is unknown since the value of σ is unknown,
3. Water crossing the surface BC at a constant rate due to the steady rainfall seeping downward. Thus the potential gradient normal to the surface is a constant,
4. No flow across CD due to the symmetry of the system; that is, the potential gradient normal to the surface is zero,
Figure 34. Schematic diagram of the flow region considered in Appendix B.
5. No flow across the impermeable surface DO, that is, the potential gradient normal to the surface is zero.

Expressed mathematically these boundary conditions, in terms of the potential $\phi$, are as follows:

1. $\phi = h; \quad 0 \leq y \leq h, \quad x = 0$
2. $\phi = y; \quad h \leq y \leq \sigma, \quad x = 0$
3. $\frac{\partial \phi}{\partial y} = -\frac{R}{k}; \quad y = \sigma, \quad 0 \leq x \leq s$
4. $\frac{\partial \phi}{\partial x} = 0; \quad 0 \leq y \leq \sigma, \quad x = s$
5. $\frac{\partial \phi}{\partial y} = 0; \quad y = 0, \quad 0 \leq x \leq s$

The problem now is to find a solution of Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

satisfying the boundary conditions listed above.

If we obtained a solution directly in the form of a Fourier series, boundary conditions 1 and 2 could be satisfied by proper evaluation of the Fourier constants. However, B.C. (boundary condition) 3 would be difficult to satisfy. Hence we form a new function,

$$f(x, y) = \phi(x, y) + Ax + By + C(x^2 - y^2),$$
which still satisfies Laplace's equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$  

We want B.C.'s 3, 4, and 5 to be homogeneous for \( f(x, y) \).

Thus we set

$$\frac{\partial f(x, 0)}{\partial y} = \frac{\partial \varphi(x, 0)}{\partial y} + B - 2 Cy = 0$$

$$= -\frac{R}{K} + B - 2 C_0 = 0, \quad 46$$

and

$$\frac{\partial f(s, y)}{\partial x} = \frac{\partial \varphi(s, y)}{\partial x} + A + 2 Cx = 0$$

$$= 0 + A + 2 C_S = 0, \quad 47$$

and

$$\frac{\partial f(x, 0)}{\partial y} = \frac{\partial \varphi(x, 0)}{\partial y} + B - 2 Cy = 0$$

$$= 0 + B + 0 = 0. \quad 48$$

Solving Eqs. 46, 47, and 48 simultaneously gives

$$B = 0, \quad C = \frac{R}{2\sigma K}, \quad \text{and} \quad A = \frac{R S}{\sigma K}.$$  

Thus we want a function \( f(x, y) \) satisfying Laplace's equation and the following boundary conditions:

1a. \( f(0, y) = h + \frac{R}{2\sigma k} y^2; \ 0 \leq y \leq h \)

2a. \( f(0, y) = y + \frac{R}{2\sigma k} y^2; \ h \leq y \leq \sigma \)

3a. \( \frac{\partial f(x, \sigma)}{\partial y} = 0 \)
4a. \( \frac{\partial f(s,y)}{\partial x} = 0 \)

5a. \( \frac{\partial f(x,0)}{\partial y} = 0 \)

When we have found \( f(x, y) \) then we can obtain \( \phi(x, y) \) from the equation

\[
\phi(x, y) = f(x, y) - \frac{Rs}{\sigma} x + \frac{R}{\sigma} (x^2 - y^2).
\]

We find \( f(x, y) \) by assuming a solution of the form

\[
f(x, y) = X(x) Y(y).
\]

Eq. 45 becomes

\[
X''(x)Y(y) + X(x) Y''(y) = 0.
\]

Separating variables we have

\[
\frac{Y''}{Y} = -\frac{X''}{X} = \alpha \text{ (a constant)}.
\]

Thus we must solve the two differential equations

\[
Y'' - \alpha Y = 0
\]

and

\[
X'' + \alpha X = 0,
\]

subject to the appropriate boundary conditions.

Utilizing B.C.'s 3a, 4a, and 5a we find an infinite number of solutions of the form

\[
f_n(x, y) = E_n \cos \frac{n\pi y}{\sigma} \cosh \frac{n\pi (s-x)}{\sigma}, \quad n = 0, 1, 2, \ldots
\]

In fact we can write a general solution of the form

\[
f(x, y) = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi y}{\sigma} \cosh \frac{n\pi (s-x)}{\sigma}.
\]
We still must satisfy B.C.'s 1a and 2a. From Eq. 55 we now have for these boundary conditions:

\[ f(0,y) = \frac{B_0}{2} + \sum_{n=1}^{\infty} E_n \cos \frac{n\pi y}{\sigma} = h + \frac{R}{2\sigma K} y^2, \quad 0 < y < h \]

\[ = y + \frac{R}{2\sigma K} y^2, \quad h < y < \sigma. \]

The left side of the above equation is a Fourier cosine series. Evaluating the constants we get

\[ E_0 = \sigma + \frac{h^2}{\sigma} + \frac{R\sigma}{3K} \]

and

\[ E_n = \frac{2\sigma}{\pi^2 n^2} \left[ (1 + \frac{R}{K})(-1)^n - \cos \frac{n\pi h}{\sigma} \right]. \]

Substituting these constants into Eq. 55, and Eq. 55 into Eq. 49, we get the potential function in the form

\[ \phi(x,y) = \frac{\sigma}{2} + \frac{h^2}{2\sigma} + \frac{\sigma R}{6 K} x + \frac{R s}{2\sigma K} (x^2 - y^2) \]

\[ + \frac{2\sigma}{\pi^2 n^2} \sum_{n=1}^{\infty} \left[ (1 + \frac{R}{K})(-1)^n \cos \frac{n\pi h}{\sigma} \right] \left[ \cos \frac{n\pi y}{\sigma} \cosh \frac{n\pi (s-x)}{\sigma} \right]. \]

In order to obtain the height \( z \) of the water table above the level \( y = \sigma \) (the BC level in Figure 34) we evaluate the potential along the surface BC. Since we have taken the point 0 as the reference level, we have

\[ z = \phi(x,\sigma) - \sigma, \]

or

\[ z = \frac{h^2}{2\sigma} - \frac{\sigma}{2} + \frac{R}{K} \left( \frac{x^2}{2\sigma} - \frac{s x}{\sigma^2} - \frac{\sigma}{3} \right) \]

\[ + \frac{2\sigma}{\pi^2 n^2} \sum_{n=1}^{\infty} \left[ (1 + \frac{R}{K})(-1)^n \cos \frac{n\pi h}{\sigma} \right] \cos \frac{n\pi (s-x)}{\sigma} \cosh \frac{n\pi s}{\sigma}. \]
For the special case where \( h = 0 \) (no water standing in the ditch) we have

\[
z = -\frac{g}{2} \left( \frac{R}{2\sigma} \left( \frac{x^2}{2\sigma} - \frac{s}{3} \right) + \frac{2g}{R} \right) 1 + \sqrt{\frac{1}{\frac{R}{K}} \cos \left( \frac{n\pi(s-x)}{\sigma} \right)}.
\]

Eqs. 58, 59, and 60 are not complete solutions of the steady flow problem because they are in terms of \( \sigma \), the value of which is unknown. If an expression, either theoretical or experimental, could be found relating \( \sigma \) to the other variables, \( K, R, s, \) and \( h \), the steady state problem could be solved completely. After that the theory could be extended to the falling water table case in a manner similar to that given in Appendix C.
Dr. Don Kirkham demonstrated to this writer a method by which the Kirkham (25) theory for the height of the water table under steady rainfall can be extended to the falling water table case. This has not yet been published but will be presented here since the results were compared with the results of model tests in this dissertation.

After making certain simplifying assumptions, Kirkham (25) was able to obtain a mathematical solution for the height of the water table between parallel drains under steady rainfall conditions. He specialized the solution for various cases of tile and ditch drainage. The case of interest here is that of ditch drains which only partially penetrate to an impermeable layer and which have water standing to some depth in them.

Figure 35 is a schematic diagram of the ditch drainage system. The shape of the ditch shown here is rectangular (corresponding to that in the model), whereas Kirkham’s ditch had a somewhat rounded bottom. However, this difference is considered relatively unimportant. Kirkham did not consider the existence of a surface of seepage.

The origin of the coordinate system is at the center of
ditch. The solution for the height $z$ of the water table above the water level in the ditch for steady rainfall conditions is (using Eqs. 41, 29, 31, 34, 38, 39, 73, and 81 of Kirkham and putting into terms used in this thesis)

$$z = \frac{2Rs}{\pi K} \left[ \ln \frac{\sin \frac{\pi x}{2s}}{\sin \frac{\pi r}{2s}} - \sum \frac{1}{n} \frac{e^{-\frac{1}{s} \sinh \left[ \frac{\pi r}{s} (b+d) \right]}}{\sinh \frac{\pi r}{s}} \left( \cos \frac{\pi x}{s} - \cos \frac{\pi r}{s} \right) \right]$$

$$+ \frac{\pi}{2(b+d)} (x-r) + \ln \frac{\sinh \frac{\pi r}{2(b+d)}}{\sinh \frac{\pi x}{2(b+d)}} + \sum \frac{1}{n} \frac{e^{-\frac{1}{b+d} \sinh \left[ \frac{\pi r}{b+d} \sinh \frac{\pi s}{b+d} \right]}}{\sinh \frac{\pi s}{b+d}} \left( \cosh \frac{\pi x}{b+d} - \cosh \frac{\pi r}{b+d} \right)$$

$$- \frac{b+d}{\pi h} \frac{1}{n^2} (\sin b+d)(e^{\frac{\pi x}{b+d}} - e^{-\frac{\pi r}{b+d}})$$

$$- \frac{b+d}{\pi h} \frac{1}{n^2} \sinh \frac{\pi s}{b+d} \frac{e^{-\frac{1}{b+d} \sinh \left[ \frac{\pi r}{b+d} \sinh \frac{\pi s}{b+d} \right]}}{\sinh \frac{\pi s}{b+d}} (\cosh \frac{\pi x}{b+d} - \cosh \frac{\pi r}{b+d})$$

$$\text{or } z = \frac{Rs}{K(1-R/K)} F(x,0), \quad 62$$

where $F(x,0) = \frac{2}{\pi}$ times the bracketed portion of Eq. 61.

After applying the correction factor developed by Kirkham (26) to take into account the head loss beneath the water table, we have the equation

$$z = \frac{Rs}{K(1-R/K)} F(x,0) \quad 63$$

for the upper limit of the water table under a steady rate of rainfall $R$.

Suppose now that the rainfall suddenly stops. At the instant rainfall stops the water table will fall at the rate

$$\frac{dz}{dT} = -\frac{R}{f}, \quad 64$$
Figure 35. Schematic diagram of the flow region considered by Kirkham for steady rainfall.
where \( f \) is the drainable porosity. If we assume that the falling water table can be represented by a series of steady state water tables, then at any subsequent time \( T \) after rainfall stops we can still say that

\[
\frac{dz}{dT} = -\frac{R}{f},
\]

where \( R \) now has some value less than before.

Solving Eq. 63 for \( R \) gives

\[
R = \frac{Kz}{sF(x,0)+z}.
\]

Substituting Eq. 66 into Eq. 65, we have

\[
\frac{dz}{dT} = -\frac{K}{f}\left(\frac{z}{sF(x,0)+z}\right).
\]

Separating variables we get

\[
\left[\frac{sF(x,0)}{z} + 1\right] \frac{dz}{z} = -\frac{K}{f} dT,
\]

and after integrating we have

\[
sF(x,0) \ln z + z = \frac{KT}{f} + C_1.
\]

Let \( z = z_0 \) when \( T = 0 \). Then

\[
C_1 = sF(x,0) \ln z_o + z_o.
\]

Substituting Eq. 69 into Eq. 68 gives

\[
\frac{sF(x,0) \ln z + z}{sF(x,0) \ln z_0 + z_0} = -\frac{KT}{f},
\]

or, after rearranging,

\[
\ln \frac{z}{z_0} = \frac{1}{sF(x,0)} \left( z_0 - z - \frac{KT}{f} \right).
\]
After taking the antilogarithms of both sides of Eq. 70 we obtain
\[ \frac{z}{z_0} = e^{\frac{1}{sF(x,0)} (z_0 - z - \frac{KT}{T})} , \]
or
\[ z = z_0 e^{\frac{1}{sF(x,0)} (z_0 - z - \frac{KT}{T})} . \]

Multiplying both sides of Eq. 71 by \( e^{\frac{z}{sF(x,0)}} \) yields
\[ \frac{z}{z_0} \frac{1}{e^{sF(x,0)}} = z_0 e^{\frac{1}{sF(x,0)} (z_0 - \frac{KT}{T})} . \]

Eq. 72 relates the water table height \( z \) to the time \( T \) measured from the moment the water table begins falling from its original steady rainfall position \( z_0 \). If we are interested only in the maximum height \( H \) of the water table (at the midpoint between drains) we then can write
\[ H e^{\frac{H}{sF(s,0)}} = H_0 e^{\frac{1}{sF(s,0)} (H_0 - \frac{KT}{T})} . \]

The function \( F(s,0) \) is a constant for a given geometry of the flow region.

Eqs. 72 and 73 cannot be solved explicitly for \( z \) and \( H \) as functions of \( T \). However, a graphical method of solution can be utilized.