LIMITED DATA TOMOGRAPHY USING SUPPORT MINIMIZATION
WITH A PRIORI DATA

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INTRODUCTION
This paper addresses the reconstruction of incomplete projection data such as obtained in limited angle X-ray tomography, including cases where a priori object geometry information is available. A variational approach is taken in which the missing projection data is determined through optimization of a functional measure of non-physical or undesirable solution (image) attributes. The focus of the research is on the prescription of appropriate functional measures, and in particular on the application of a functional which minimizes the support (i.e. area) of the solution image. The connection between functional attributes and algorithm behaviors is discussed, and examples of application of various functionals are assessed. A scheme for incorporation of a priori geometry information in the variational reconstruction is demonstrated. It is shown that the minimal support functional is effective in reconstructing compact, high contrast features, and is therefore likely to be useful in applications such as dimensional analysis and the imaging of cracks. This paper presents a summary of the continuation of the work reported last year [1], hence concepts discussed in detail in ref. [1] will only be stated briefly in this writing.

PROBLEM STATEMENT
This discussion will be limited to measurements represented by Radon projections, such as projection radiography, although the concepts can in principle be applied to, say, measurements modeled by diffracting wavefields. Consideration is restricted to two dimensions, in which case the inverse mapping can be written
\[ p(r, \phi) = \int_{0}^{\pi} \int_{-\infty}^{\infty} h(r \cos(\theta - \phi) - u) m(\theta, u) du \] (1)
where, in an X-ray measurement, \( \mu(r, \phi) \) is the attenuation coefficient, \( m(\theta, u) \) represents the measured projection data and \( h(s) \) is a filter function whose Fourier transform \( h^*(k) \) approximates \( |k| \). In the examples of this paper, data is assumed to be collected on an equally spaced rectangular grid over the \( \theta-u \) measurement plane (i.e. parallel beam tomography). The discretized implementation of eq.(1) used in this work is a standard convolutional back projection algorithm employing a small amount of Gaussian filtering. The discretized algorithm is represented as a linear mapping by

\[ \mu_j = \sum_i e_{ij} m_i \]  

where \( i, j \) refer to discrete measurement points and image points in the \( \theta-u \) and \( r-\phi \) planes, respectively. It is assumed that the density of the measurement point spacing in the \( \theta-u \) plane is sufficient to yield an acceptable reconstruction when accurate projection data is available at all discrete measurement points. For the present purposes, a limited data set is defined as one for which data at some of the discrete measurement points are unknown, or "missing". These data are denoted \( m_k \). The reconstruction problem in this case is clearly under-determined, since random values assigned to the \( m_k \) will produce a reconstructed image which is completely consistent with the available data. The problem is therefore to determine a set \( m_k \) which yields an image most consistent with \textit{a priori} information of the reconstructed image. Stated in the abstract, assume a functional exists \( F(\mu) \) which measures image attributes for which there are \textit{a priori} information, such that the minimum of \( F(\mu) \) represents closest compliance with the \textit{a priori} information. It could be argued that the desired reconstruction is the one which optimizes this functional, i.e. the desired reconstruction is the solution to

\[ \frac{\partial F(\mu)}{\partial m_k} = 0 \]  

The objective of the present research is to determine such functionals \( F \) appropriate for applications of interest to QNDE.

FUNCTIONAL DEFINITION

The functionals examined in this work are integral operators of the form

\[ F = \int P(\mu(x)) \, dx \]  

where \( P(\mu(x)) \) is a scalar quantity determined by

\[ P(\mu(x)) = L_x(\mu(x)) \]  

where \( L_x(\mu) \) is a differential operator. Hence \( P \) represents the measure of a local image property, whereas \( F \) represents the global sum of these measures.

The development of primary importance in this work is the implementation of a functional \( F(\mu) \) which measures the \textit{support} of the reconstructed image. The motivation for this development was the repeated empirical observation that, when non-zero projection values \( m_i \) are set to zero within some arbitrary region of the \( \theta-u \) plane, the resulting reconstructed image displays an increase in the number of non-zero-valued pixels, i.e., the non-zero support of the image increases. This observation suggested that an effective limited data reconstruction scheme might be one which minimizes a measure of the non-zero support of the image. The practical implementation of this notion requires not the measure of non-zero support, but rather the measure of support above some prescribed noise level \( \varepsilon \). Hence, the \textit{penalty} \( P(\mu) \) should be one which approximates the behavior

\[ P(\mu) = 1, \quad |\mu(x)| > \varepsilon \]  

\[ P(\mu) = 0, \quad |\mu(x)| < \varepsilon \]  

More generally, the penalty function should be essentially zero for \( \mu(x) < \varepsilon \), and should asymptotically approach a constant value for \( \mu \to \infty \), i.e., the penalty function should be \textit{bounded}. Clearly, such a penalty function will result in a functional \( F \) which is non-linear in \( \mu \). The system of equations represented by eq.(3) will therefore be non-linear in the \( m_k \). In the work presented here, a conjugate gradient algorithm is used to solve this non-linear system, hence it is important that the penalty function have the additional property of
P'(\mu) > 0 , \mu > 0 \\
P'(\mu) < 0 , \mu < 0  

so as to avoid local minima or plateaus in the gradient search. The function chosen to approximate eq.(6) is

\[ P(\mu) = (1 + (\epsilon / |\mu|)^\eta)^{-1} , \eta, \epsilon > 0 \]  

Equation (8) approaches eq.(6) as \eta \to \infty, and also satisfies eq.(7). In practice, it is desirable to set \eta small (say \eta=2) at the initiation of the iteration, so as to provide a larger gradient at large \mu (i.e., P' not too small). This will accelerate convergence to an approximate result. At the later stages of the optimization, \eta can be increased (say \eta=6) to improve the support minimization.

Other measures of image properties are often desired. In examples which follow, a measure of image negativity is imposed by the following modification of eq.(8)

\[ P(/.l) = [1 + (\epsilon / I')^\eta]^{-1} , I' > 0 \]  

Equation (9) applies a quadratic penalty to negative pixel values. Note that a negative pixel value is not always non-physical. For example, if a priori information of a component has been subtracted from a data set, then a void within the component will have a negative attenuation in the difference image. Thus eq.(9) is not applied universally, but rather only in cases where a negative pixel indicates the generation of energy.

In examples which follow, it will be seen that the minimal support penalty allows considerable freedom in the specification of pixel values which are well above the support threshold \epsilon, i.e., once the pixel value is significantly above the threshold value, the actual value of that pixel is of little consequence to the overall measured support. For this reason, additional penalty functions can be applied which operate only on pixel values well over the support threshold, thereby having negligible effect on the total measured support. A simple penalty function of this type which enforces a maximum pixel value is obtained by the following modification of eq.(9)

\[ P(\mu) = (\mu - \mu_0)^2 + (1 + (\epsilon / \mu_0)^\eta)^{-1} , \mu > \mu_0 \]  

\[ P(\mu) = (1 + (\epsilon / \mu)^\eta)^{-1} , \mu_0 \geq \mu \geq 0 \]  

\[ P(\mu) = \mu^2 , \mu < 0 \]  

This penalty will discourage pixels above the threshold \mu_0. An example of the effective use of such a penalty will be shown. A second type of penalty applied to pixel values well above the support threshold provides a measure of image smoothness through the use of second derivative information. This is accomplished by adding the following operator to either eqs.(8,9, or 10)

\[ Q = \gamma M(\mu) [H(\partial_x^2 \mu)(\partial_x^2 \mu)^2 + H(\partial_y^2 \mu)(\partial_y^2 \mu)^2] \]  

\[ M(\mu) = (1 + (\epsilon / (\mu - \mu_c))^\eta)^{-1} , \mu \geq \mu_c \]  

\[ M(\mu) = 0 , \mu < \mu_c \]  

where H(s) is the unit step function. The factor M(\mu) "turns on" the operator Q when \mu exceeds a threshold \mu_c, which is set appreciably larger than the noise threshold \epsilon, say at 50% of the maximum image value. The operator Q equals zero when the second derivatives of the image are both zero or less. The operator is seen to take on non-zero values when \mu > \mu_c, and either \mu_{xx} > 0 or \mu_{yy} > 0. This operator allows image line profiles to rise sharply,
level off, and fall sharply without penalty, but will penalize any oscillations occurring above the threshold $\mu_c$. Thus this penalty is, more accurately, a measure of "concavity" rather than smoothness. The observed effect of this constraint, however, is to smooth the image, and is therefore referred to as such. The relative strength of this penalty is controlled by the constant factor $\gamma$.

In addition to the functional measures applied to the reconstructed image, an additional penalty is available which penalizes negative values of interpolated projection data $m_k^m$

$$W = \beta \sum_k H(-m_k^m)(m_k^m)^2$$

(12)

As with eq.(9), eq.(12) is applied in those cases where negative projection values would indicate the generation of energy by the object. The relative strength of this penalty is controlled by the parameter $\beta$.

Comparisons will be made with reconstructions obtained using the maximum entropy functional, obtained through the use of the penalty function

$$P(\mu) = \mu \ln |\mu / \mu_c|$$

(13)

The parameter $\mu_c$ controls the width of the negative excursion of the penalty curve.[2] This penalty is seen to be unbounded as $\mu \to \infty$.

APPLICATION TO X-RAY DATA

The limited data inversion technique discussed in the preceding section will now be demonstrated using synthetic and experimental data. Results showing the behavior of the various penalty functions are presented using limited projection data derived from the simple object shown in fig.(1). The image of fig.(1a) displays the reconstruction of a synthetic "complete" data set consisting of 101 translational points and projections at 5 degree intervals. For purposes of demonstration, half the projections (covering 90 out of 180 degrees) were set equal to zero. The resulting reconstruction is shown in fig.(1b). The negative projection penalty was incorporated in figs.(1-3).

The result of the functional optimization using the penalty function of eq.(9) is shown in fig.(2a). A threshold $\varepsilon=0.05 \mu_{\text{max}}$ was used, where $\mu_{\text{max}}$ is the maximum value of the complete data reconstructed image of fig.(1a), and $\eta=4$. The use of the minimal support minimization has essentially recovered the image support. However, the image remains quite non-uniform, and the maximum pixel magnitude exceeds the that of fig.(1a) by 37% (maximum pixel magnitudes are .767 and 1.06 in figs.(1a) and (2a), respectively). This is due to the relative insensitivity of the image support to the large-valued pixels. We are essentially

![Fig. 1. Simple geometry reconstructed from a) complete and b) incomplete projection sets.](image-url)
free to optimize separate measures over the large-amplitude pixels. As an example of this, the penalty function of eq.(10), which employs a maximum amplitude threshold, was applied in conjunction with the smoothness operator of eq.(11). During the optimization, the threshold $\mu_0$ was initially set at a very large number, then was incrementally decreased at the conclusion of each of repeated intermediate optimizations. The decreasing amplitude constraint was observed to decrease the maximum amplitude of the resulting image, but was seen to have negligible effect on the image support as long as $\mu_0$ was above the maximum pixel intensity of fig.(1a), $\mu_{\text{max}}$. However, once the threshold fell below $\mu_{\text{max}}$, the support of the image began to severely distort. Results are compared in figs.(2b,c) for $\mu_{\text{max}}$ = .8, and .2, resulting in maximum pixel magnitudes of .806 and .737, respectively. Figure(2b) is a nearly correct reconstruction, whereas fig.(2c) is quite erroneous. In fig(2b), no pixel exceeds $\mu_0$, whereas in fig.(2c) a large percentage of the pixels exceed $\mu_0$. This implies that the available data requires pixel values above the threshold in figs.(2c), but not in fig(2b), which in turn indicates a measure of the information content of the available data, and suggests a methodology for determining the actual magnitude of the reconstructed object. The success of the algorithm applied in fig.(2b) is due to the fact that the penalty assigned to large amplitude pixels is more sensitive to local smoothness than absolute pixel magnitude.

Note that, when $\mu_0 = 0.2$, fig(2c), the penalty function is nearly quadratic, i.e., the reconstruction is essentially a minimum $(L_2)^2$-norm reconstruction. The penalty function in this case is unbounded with a monotonically increasing slope. Numerical experimentation performed to date suggests that any penalty function displaying this qualitative behavior will yield similar results. As an example, a reconstruction using the maximum entropy penalty function of eq.(13) with $\mu_e = 0.05 \mu_{\text{max}}$ (i.e. $\mu_e = \epsilon$) is shown in fig.(2d). Note the similarity between figs.(2c and d). Both penalty functions are sacrificing the uniformity of the image and compactness of the image boundary so as to reduce the number of pixels in the largest range of magnitude.

Fig. 2. Application of support minimization a) alone, b), c) with smoothing and maximum amplitude penalty at b) .8, c) .2, d) maximum entropy penalty.
It is important to stress the fact that all the images of figs.(1 and 2) are completely consistent with the available data: there is no grounds to prefer any one of the images over the others based on agreement with the available projection data. Our preference must be based on other objective and subjective measures.

The functional optimization technique is next applied to experimental data. The object examined is the metal band attaching the eraser to the end of a wood pencil. Monoenergetic projection data was obtained at 2 degree intervals using 101 translational points. The complete data reconstruction of this set is shown in fig.(3a). An incomplete data set was fabricated by removing a third of the projections (60 out of 180 degrees). Reconstruction of the incomplete data set is shown in fig.(3b). The reconstruction obtained through the application of the penalty function of eq.(9) with $\eta=6$ and $\varepsilon=0.17 \mu_{\text{max}}$ is shown in fig.(3c).

INCORPORATION OF A PRIORI DATA

The primary problem encountered in the incorporation of a priori geometry and material information in a reconstruction algorithm is the registration of the data with the measured data both spatially and in magnitude. Six variables are identified as controlling the registration: x-y translation ($r_0$, $\omega_0$ in polar coordinates), rotation ($\phi_0$), spatial magnification ($S_0$), mean intensity (brightness) ($I_0$), and intensity magnification (contrast) ($A_0$). The approach to data registration taken in this work is to treat these six registration variables as additional unknowns to be optimized. An algorithm was developed which fits the projections of the a priori data to the measured data projections. A projection set $D(\theta,u)$ is defined as the difference between the measured projections $m(\theta,u)$ (including the missing projection data) and the projection of the registered a priori data $m_0(\theta,u)$

$$D(\theta,u) = I_0 + A_0 m_0(\theta - \theta_0, (u - c) S_0 + c_0 + r_0 \cos(\theta - \omega_0)) - m(\theta,u)$$

where $c$ and $c_0$ are the centers of rotation of the measured and a priori projection sets, respectively, on the translation $u$ axis. The reconstruction proceeds by applying the variational methods demonstrated in the previous section to the difference projection $D(\theta,u)$. However, the solution vector includes the six registration variables in addition to the discrete missing data points $m_k^m$. Numerical difficulties arose in the initial implementation of this scheme due to potential orders of magnitude differences in the sensitivity of the optimized functional to the registration variables compared to the missing projection data. A normalization process was implemented in the algorithm which, when sensing numerical problems, automatically re-scales the registration variables to produce approximately uniform sensitivity to all variables. This modification resulted in a robust algorithm which has thus far proven free from local minima traps.

Fig. 3. Application to experimental data. a) complete data set, b) incomplete data set, c) incomplete data set with minimal support reconstruction.
The registration algorithm is demonstrated using a synthetic projection set derived from a turbine blade CAD drawing. The data set consists of projections at 5 degree intervals using 201 translational points, shown reconstructed in fig.(4). This data set represents the unregistered \textit{a priori} data. An "experimental" data set was constructed by removing every other translational point, rotating the image 90 degrees, then seeding the data with an additional hole to simulate an "unknown" flaw. The reconstruction of this "experimental" data set is shown in fig.(5a). An incomplete data set was then formed by removing half the data (90 out of 180 degrees), resulting in the reconstruction shown in fig.(5b).

![Fig. 4. Reconstruction of \textit{a priori} turbine blade data.](image)

![Fig. 5. "Experimental" data set seeded with flaw. a) complete data, b) incomplete data, c) minimal support reconstruction, d) quadratic registration of \textit{a priori} data, e) minimal support reconstruction with \textit{a priori} data, f) image of penalty function of e.](image)
The reconstruction was initiated by applying the variational reconstruction algorithm to the incomplete data set alone (i.e., the a priori data was not used) using eq.(9) with $\eta=4$, $\varepsilon=.05\mu_{\text{max}}$ and eq.(12), resulting in fig.(5c). The reconstruction has improved significantly, but the holes in the thin portion of the blade are not sharply defined. Next, the registration algorithm is applied to the difference projection, eq.(14), using eq.(8). The penalty on negative valued pixels is not imposed on the difference image. To initiate the registration, a quadratic fit was performed (i.e. least integrated squared error) by setting $\eta=2$ and $\varepsilon \gg \mu_{\text{max}}$. This significantly accelerates the initial alignment of the a priori and measured data. Figure (5d) shows the reconstruction of the "experimental" data at the conclusion of the initial registration. Finally, the threshold is lowered to $\varepsilon=.05\mu_{\text{max}}$, and $\eta=4$. The algorithm then fine-tunes the data alignment, and adjusts the missing projection points so as to minimize the support of the difference image. The final reconstruction of the "experimental" data is shown in fig(5e). Comparison with fig.(5a) indicates good algorithm performance. The penalty function $P(\mu(x))$ is displayed as an image in fig.(5f). Note that the algorithm has reduced the support of the penalty function to a few pixels in the vicinity of the "unknown" flaw. Figure (5f) suggests the use of the penalty function image as a means for flaw detection, since in the absence of a flaw, the penalty function would be uniformly zero.

SUMMARY

A means of compensating for missing projection data in CT reconstruction has been developed based on the optimization of measures of image properties, and in particular on the minimization of a measure of the image support. Implementation of these concepts has taken the form of a variational algorithm which displays robust convergence properties. The behavior of the reconstruction algorithm has been related to the general qualitative shape of the penalty curve used in the optimization. A functional measure has been demonstrated which allows the simultaneous, and relatively independent, optimization of image support and image smoothness. A comparison to a conventional maximum entropy method demonstrated the superiority of the minimal support approach in the reconstruction of a compact, high-contrast object. A means for the variational registration of a priori data has been incorporated into the reconstruction process. The algorithm proceeds by minimizing the support of the difference between the reconstructed image and the a priori data, resulting in a reconstruction with unknown flaw data displayed in an optimally compact form. It is conceivable that such an algorithm could have particular utility in the area of dimensional analysis, due to the compact, high-contrast nature of the differences between the object and a priori data.

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REFERENCES