2011

problems in graph theory and probability

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Problems in graph theory and probability

by

Jihyeok Choi

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
Maria Axenovich, Co-major Professor
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Iowa State University
Ames, Iowa

2011

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DEDICATION

To my sister in heaven
# TABLE OF CONTENTS

LIST OF FIGURES ......................................................... v
ACKNOWLEDGEMENTS ....................................................... vi
INTRODUCTION ........................................................... vii

PART I  Time-dependent preferential attachment models 1
INTRODUCTION .......................................................... 2
BIBLIOGRAPHY .......................................................... 5

CHAPTER 1. Large deviations analysis of time-dependent preferential attachment schemes ................................................. 7
Abstract ........................................................................... 7
1.1 Introduction ............................................................ 7
1.1.1 Model .............................................................. 10
1.1.2 Main Results ..................................................... 14
1.2 Proof of Theorem 1.1.1 ............................................. 20
1.2.1 Upper bound ..................................................... 24
1.2.2 Lower bound ..................................................... 27
1.3 Proof of Theorem 1.1.2 ............................................. 37
1.4 Proof of Theorem 1.1.3 ............................................. 40
1.5 Proof of Theorem 1.1.4 ............................................. 41
PART II Graph coloring

INTRODUCTION ................................................................. 48

BIBLIOGRAPHY ............................................................... 50

CHAPTER 1. A note on monotonicity of mixed Ramsey numbers ...... 51

Abstract ................................................................. 51

1.1 Introduction .......................................................... 51

1.2 Definitions and proofs of main results .......................... 53

BIBLIOGRAPHY ............................................................... 58

CHAPTER 2. On colorings avoiding a rainbow cycle and a fixed monochromatic subgraph .................................................. 60

Abstract ................................................................. 60

2.1 Introduction and main results .................................. 60

2.2 Definitions and preliminary results .............................. 62

2.3 Proof of Theorem 2.1.2 ............................................. 63

2.4 More precise results for $C_4$ ...................................... 74

BIBLIOGRAPHY ............................................................... 77

CHAPTER 3. A short proof of anti-Ramsey number for cycles ............ 79

Abstract ................................................................. 79

3.1 Introduction .......................................................... 79

3.2 Definitions and proofs of main results .......................... 80

BIBLIOGRAPHY ............................................................... 84

CHAPTER 4. General Conclusions ........................................... 85

4.1 Future improvements and other directions ...................... 85
LIST OF FIGURES

Figure 1.1 Numerical solutions of Euler equations ........................................ 16
Figure 1.2 Degree distribution for a step function ........................................ 19
Figure 1.3 Evolution of a degree distribution .................................................. 20
Figure 2.1 Tails ............................................................................................... 65
Figure 2.2 A rainbow cycle in Claim 1.3 ......................................................... 67
Figure 2.3 Representing graphs ....................................................................... 68
Figure 2.4 A rainbow cycle in Claim 2.1 and Claim 2.2-1.(1) ......................... 69
Figure 2.5 A rainbow cycle in Claim 2.2-1.(2) ............................................... 69
Figure 2.6 Rainbow cycles in Claim 2.2-1.(3) ............................................... 70
Figure 2.7 Rainbow cycles in Claim 2.2-2.(1) ............................................... 70
Figure 2.8 Rainbow cycles in Claim 2.2-2.(2) ............................................... 71
Figure 3.1 A rainbow cycle \( (v_2, v_i, v_{i-1}, \ldots, v_3, v_{i+1}, v_{i+2}, \ldots, v_{k+1}, v_2) \) ........................................ 83
ACKNOWLEDGEMENTS

I am grateful to my advisors, Maria Axenovich and Sunder Sethuraman, for their support and guidance over the past years. I also thank my committee members, Drs. Ryan Martin, Eric Weber and Sung-Yell Song for their help, guidance and patience.
INTRODUCTION

In this dissertation, our main object of study is a (hyper) graph, a pair of a set and a collection of its subsets. Although by definition a graph is purely set theoretic, it has a very nice representation, when one considers only 2-element subsets, as a diagram consisting of a set of points, called vertices, together with lines, called edges, joining certain pairs of these points. Many real-world situations are expressed in terms of graphs in this manner. For example, for social systems, in the simplest description, we may think of people as vertices where the (undirected) edge between two people exists if and only if they know each other. Another example is the world wide web in which the vertices are the web pages available at the website and a (directed) edge from page $A$ to page $B$ exists if and only if $A$ contains a link to $B$. Further the first example may be thought as a (edge) coloring of a graph, namely given $n$ people and $\binom{n}{2}$ edges between all possible pairs of them, we label an edge (two people) by a color blue if they know each other, or by a color red otherwise.

Graph theory finds many applications in other fields including biology, chemistry, computer science, linguistics, physics and sociology. Further, with connections to other branches of mathematics, many various tools are being employed to considerable effect from algebra, analysis, geometry, number theory, probability, and topology. In particular in this thesis, we use mainly two methods, probabilistic and structural, on some problems on graphs.

This dissertation is based on journal papers (published, submitted, or in preparation), and organized as follows. In Part I, containing the paper “Large deviations analysis of time-dependent preferential attachment schemes” in preparation, we study a random graph model associated with ‘scale-free’ networks, focused on the degree structure, by means of a probabilistic tool. Here a degree of a vertex is the number of vertices connected to the vertex by an edge. In Part II, with structural methods, we study colorings of graphs in Ramsey and anti-Ramsey theories. It contains three papers “A note on monotonicity of mixed Ramsey num-
bers” submitted to *Discrete Mathematics*, “On colorings avoiding a rainbow cycle and a fixed monochromatic subgraph” published in the *Electronic journal of Combinatorics*, and “A short proof of Anti-Ramsey number for cycles” in preparation. Each part has its own introduction containing more details on problems. The last chapter (Chapter 4) is for general conclusions, where future research work is presented.
PART I

Time-dependent preferential attachment models
INTRODUCTION

The study of random graphs, probability distributions on graphs, dates back to the 1950s (cf. Bollobás and Riordan [3] for an interesting survey). The best known and most studied is $G(n, p)$ model, in which, with $n$ labeled vertices, edges are presented independently with probability $p$ from every other edges. The main question on random graphs is that what asymptotic behaviors the random graphs possess with high probability. For instance, how does the degree distribution, the fraction of vertices with degree $k$, denoted by $p_k$, evolve over time? As each vertex is connected to other vertices with probability $p$ independently, in the $G(n, p)$ model, the degree distribution has a binomial distribution, which is asymptotically Poisson distribution with mean $np$ as $n \to \infty$. Therefore the number of vertices with degree $k$ is decreasing faster than exponentially.

In contrast to the rapid decaying tail of the degree distribution for $G(n, p)$, the recent discoveries in the field of the network evolution is that a number of large growing networks including the internet, the world wide web and some social networks, are “scale-free”, that is the degree distribution of their nodes are of a power-law form. In 1999, Barabási and Albert [2] proposed a random graph model as a mathematical model to explain this phenomenon. The basic idea of their model, somewhat generalized version of it, is that, starting from a number of vertices, each time one adds a new vertex, and connects it to one of the existing vertices with probabilities proportional to their “weight”. Here, the weight is usually a function, $w(k)$, of the degree $k$. Usually the weight function is nondecreasing, which is why this model is called a preferential attachment graph.

It has been shown that power-law distribution is only for a linear weight function, namely when $w(k) = k + \alpha$ with constant $\alpha > -1$, in which case $p_k \sim k^{-(3+\alpha)}$ (Bollobás et al [4], Mori [8], Athreya et al [1]). For $w(k) \sim k^{\gamma}$, when $0 < \gamma < 1$, $p_k \sim \frac{\mu\gamma}{k^{\gamma}}\exp(-ck^{1-\gamma})$, which decays faster than any power but not exponentially fast (Rudas et al [11]). When $\gamma > 1$, then there is
one vertex with degree of order \( \sim n \), but all the others have \( O(1) \) degrees (Oliveira and Spencer [9]).

The degree structure of preferential attachment graphs can be captured in urn models, namely every new connection that a vertex gains can be represented by a new ball added to a corresponding urn. In particular, Chung, Handjani and Jungreis [6] considered a generalized Pólya’s urn model such that at each time, with probability \( p \), a new urn with one ball is created; or with probability \( 1 - p \), a new ball is put in one of existing urns with probability proportional to a function of its size. They obtained analogous results to preferential attachment graphs. In particular, for a linear weight function, they proved that the empirical urn size distribution follows a power law, namely the fraction of urns with \( k \) balls is approximately \( k^{-1 - \frac{1}{1-p}} \).

Asymptotic behavior of the degree distribution is a consequence of the law of large numbers, an analysis of typical behavior, by showing the convergence of \( p_k \) to some constant \( c_k \) as \( k \to \infty \). Then we may ask how atypical degree distributions occur? In other words,

**Question 1** What is the probability that the degree distribution deviates to an atypical distribution?

The study of the asymptotic behavior of rare events of sequences of probability distributions is called the *large deviation theory*. Bryc et al [5] studied large deviations of a class of Markov chains, which covers the evolution of the vertices with degree 1 in Barabási-Albert model with a linear weight and the random number of edges added each time. Dereich and Mörters [7] gave large deviations bound of a general degree distribution for sublinear weights on some version of Barabási-Albert model.

All the results mentioned above are based on time-homogeneous preferential attachment models, in which the weight functions do not depend on time. However real-world networks are time-dependent. Then we may also ask the following question:

**Question 2** When the (linear) weight function depends also on time, does the power law behavior still appear?

A time-dependent urn was introduced in Pemantle [10], where, in a fixed (finite) number of urns, the number of balls added at each time is dependent of time given by a deterministic sequence of positive real numbers. Other time-dependent analysis of preferential attachment
graphs, in particular for models with time-dependent weight functions, are mostly formal in
the physics literature.

In Chapter 1, we try to answer two questions above. We obtain large deviation bounds for
the empirical degree structure in some time-dependent urn model, when the weight function,
\( w(k, t) = k + \beta(t) \), is linear in ‘degree’ \( k \) and \( \beta(t) \) is a function of time \( t \). This model includes
the evolving degree structure of the Barabási-Albert preferential attachment model, and also
the Pólya urn model of Chung-Handjani-Jungreis. For fixed \( n \geq 1 \) (which will be sent to
infinity later on), we consider the paths \( X^n(t) := \{ \frac{1}{n}(Z^n_{0}([nt]), Z^n_{1}([nt]), \ldots) : 0 \leq t \leq 1 \} \),
where \( Z_k^n(j) \) denotes the number of urns with \( k \) balls at time \( j \leq n \), then prove that, roughly
speaking, the probability that the path \( X^n(t) \) deviates to other paths \( \varphi \) is, for large enough \( n \),

\[
P\{X^n(\cdot) \sim \varphi(\cdot)\} \sim \exp\{-nI(\varphi)\},
\]

where \( I \) is the rate function, or the cost of achieving a degree structure given by \( \varphi \). As a
consequence of large deviations bound, we obtain the law of large numbers, namely, as \( n \to \infty \),
\( \frac{Z_k([nt])}{n} \to \zeta_k(t) \) a.s., where \( \zeta(t) = (\zeta_0(t), \zeta_1(t), \ldots) \) is the solution of a coupled system of ODE’s.

We then show a polynomial decay of \( \zeta_k(t) \) in terms of bounds, which shows that time-dependent
models are also scale-free in some sense. However, from analyzing the rate function, we see
that the process can deviate to a variety of distributions, including those with finite support,
with finite cost.
Bibliography


CHAPTER 1. Large deviations analysis of time-dependent preferential attachment schemes

Jihyeok Choi and Sunder Sethuraman

abstract

Preferential attachment schemes where the selection mechanism is time-dependent are considered, and an infinite dimensional large deviation principle for the sample path evolution of the empirical degree distribution is found by Dupuis-Ellis type methods. Interestingly, the rate function, which can be evaluated, includes a term which accounts for the cost of assigning a fraction of the total degree to an ‘infinite’ degree component.

As a consequence of the large deviation results, a sample path a.s. law of large numbers for the degree structure is deduced in terms of a coupled system of ODE’s. In addition, power law bounds for the limiting degree distribution are identified. However, from analyzing the rate function, one can see that the process can deviate to a variety of distributions, including those with finite support, with finite cost.

1.1 Introduction

We study a preferential attachment model of time-dependent Pólya’s urns, where at each step a new urn is created, and a new ball is added to it or an existing urn with probability depending on the urn’s size and the time of this addition (see Section 1.1.1 for a precise description). This general class includes the evolving degree structure of the Barabási-Albert preferential attachment model, and also the Pólya urn model of Chung-Handjani-Jungreis.
In Barabási-Albert [4], a preferential attachment scheme was proposed as a model for various types of complex real-world networks which, starting from an initial (finite) graph, evolve by “preferential attachment”, that is by attaching new vertices to existing ones with probabilities proportional to a function of its “weight”. An important property of such networks, when the weight function is linear, which can be proved is “scale-freeness”, that is the proportions of vertices with degrees 1, 2, . . ., e.g. the degree distribution of the network, at large times is in power-law form (see [8], [27]), as opposed to when the distribution decays super-exponentially as in Erdős-Rényi graphs. That “scale-freeness” can be widely observed in natural and man-made systems, including the internet, the world wide web, citation networks, and some social networks (see [1], [29], [26], [11], [22] and references therein), is in some sense a main reason for the popularity of the model.

Since Barabási and Albert’s work, much work has been done on versions of these graphs to understand their structure. A partial selection of this large literature includes: growth and location of the maximum degree [28], [3], [16]; form of the degree distribution under non-linear weight functions [30], [31]; spectral gap and cover time of a random walk on the graph [25], [13]; width and diameter [24], [7], [15]; graph limits [6], [5].

On the other hand, the degree structure of preferential attachment graphs can be captured in urn models, namely every new connection that a vertex gains can be represented by a new ball added to a corresponding urn in a collection of urns. A few years ago, Chung-Handjani-Jungreis [10] considered a generalized Pólya’s urn model such that at each time, with probability $p$, a new urn with one ball is created, and with probability $1 - p$, a new ball is put in one of existing urns with probability proportional to a function of its size. They proved, among other results, when the size function is linear, analogous to preferential attachment graphs, that the empirical urn size distribution is in the form of a power law.

As mentioned in [18], understanding preferential attachment or urn models when the weight function depends on time allows for more realistic models since real world networks are time-dependent. However, work on such time-dependent schemes is mostly formal in the physics
literature, e.g. Dorogovtsev-Mendes [17, Section E] where master equations are used to analyze continuous-time time-dependent models. We remark, on the other hand, in [12] and [3], a law of large numbers (LLN) for the empirical degree distribution is proved for a type of ‘time-dependent’ scheme where the weight function is fixed, but the process adds a random number of edges at each time.

Given this literature, detailing the large deviation behavior of the empirical degree/size distribution in time-dependent preferential attachment schemes is a natural problem which gives much understanding of typical and in particular atypical evolutions. We remark, even in the usual time-homogeneous models, large deviations of the full degree distribution is an open question. Previous large deviation work in preferential attachment models have focused on one dimensional objects, for instance the number of leaves in time-homogeneous processes [9], or the degree growth of a single vertex with respect to a type of ‘mean-field’ dynamics (where any vertex may attach to a newly added vertex with a small chance) [16]. See references therein for other large deviations literature on random trees, and balls-in-bins models.

In this context, our main work on a generalized preferential attachment model includes an infinite dimensional sample path large deviation principle (LDP) for the joint empirical distribution of the numbers of urns containing 0, 1, 2, ... balls, that is the ‘empirical degree structure’, when the initial configuration, not necessarily fixed, satisfies a limit condition (Theorem 1.1.2). As in many cases, when the object of interest are the numbers of urns with 0, 1, ... d balls when d < ∞, we state corresponding finite-dimensional LDP’s, with respect to explicit rate functions, which allow for instance variational analysis to find optimal trajectories for the sample path to achieve a given empirical distribution (Theorem 1.1.1). As a consequence of the large deviations results, we obtain an a.s. sample path LLN for the urn counts in terms of a system of coupled ODE’s (Theorem 1.1.3). Finally, the LLN limit trajectories are shown to have power law-type behavior in terms of bounds (Theorem 1.1.4), although it is argued through numerical studies that the general behavior can interpolate between these bounds (Fig. 1.2).

Interestingly, through calculation with the infinite-dimensional rate function, one can answer the questions: Which degree/size distributions can the process deviate to with finite cost?
For instance, naively, one might ask must the finite cost distributions by fully supported on the non-negative integers? Can some part of the total degree/size be lost in a finite cost distribution? The answers turn out to be NO and YES. Moreover, non-power law distributions can be achieved with finite rate. See the remarks after Theorem 1.1.2 for more discussion.

We also remark the large deviations and other work are with respect to the process starting from either ‘small’ or ‘large’ initial configurations, that is when the initial urn collection has $o(n)$ balls (for instance, finite), or when the size of the collection is on order $n$ respectively. It appears that such general initial configurations, which enter into all result statements, have not been considered before.

The main idea for the results is to extend a variational control problem/weak convergence approach of Dupuis and Ellis (cf. [19]) to establish finite-dimensional LDP’s in the time-dependent setting. Then, a projective limit approach, and some analysis to identify the rate function, is used to obtain the infinite-dimensional LDP. For the LLN and power-law results, a coupled system of ODE’s, which governs the typical degree distribution evolution, is identified, and analyzed.

In the next two subsections, we specify more carefully our model and results.

1.1.1 Model

Let $p(t) : [0, 1] \to [0, 1]$ and $\beta(t) : [0, 1] \to [0, \infty)$ be given functions. An urn configuration $U = \{b_x\}$ is a finite list of non-negative integers $b_x$, representing the number of balls in urn $x$. For $n \geq 0$ and an initial urn configuration $U_0^n$, we define a growing sequence $\{U_j^n\}_{0 \leq j \leq n}$ of urn configurations by the following time-dependent iterative scheme:

- Start at step 0, with the initial urn configuration $U_0^n$.
- At step $j + 1 \leq n$, to form a new urn configuration $U_{j+1}^n$, we first create a new urn with no ball. Then,
  - with probability $p(j/n)$, we place a new ball in that urn;
– with probability $1 - p(j/n)$, we place a new ball in one of other urns with probability proportional to

$$\frac{b_x + \beta(j/n)}{\sum_{y \in U_j^n}(b_y + \beta(j/n))},$$

This scheme recovers the degree distributions in the following models.

(1) ‘Classical’ Barabási-Albert processes. When $p(t) \equiv 0$ and $\beta(t) \equiv 1$, an urn with $k \geq 0$ balls corresponds to a vertex with degree $k + 1 = m \geq 1$.

(2) ‘Offset’ Barabási-Albert processes. When $p(t) \equiv 0$ and $\beta(t) \equiv \beta \geq 0$, an urn with $k \geq 0$ balls has weight $k + \beta$ which corresponds to a vertex with degree $k + 1 = m \geq 1$ and weight $m + (\beta - 1)$ in the BA process with offset $\beta - 1$.

(3) Chung-Handjani-Jungreis model of Pólya urns. When $p(t) \equiv p$ and $\beta(t) \equiv 0$, the number of urns of size $k \geq 1$ is recovered in the CHJ model. [We note, however, in our model, the number of “empty” urns, which have no weight when $\beta(t) \equiv 0$, is also kept track of.]

Let now $b_{U_0^n}$ be the total number of balls in $U_0^n$ and let $|U_0^n|$ be the number of urns in $U_0^n$. Then total number of balls in $U_j^n$ with $|U_0^n| + j$ urns is $\sum_{y \in U_j^n} b_y = b_{U_0^n} + j$. The total weight of the configuration after the $j$-th step is

$$s^*_j := \sum_{y \in U_j^n} (b_y + \beta(j/n)) = (1 + \beta(j/n))j + b_{U_0^n} + \beta(j/n)|U_0^n|.$$ (1.1)

Let $Z_i^n(j)$ be the number of urns with $i$ balls at time $j \leq n$ and, for $d \geq 0$, let $Z_{d+1}^n(j)$ denote the number of urns with more than $d$ balls at time $j \leq n$. These quantities satisfy

$$\sum_{i=0}^{d+1} Z_i^n(j) = |U_0^n| + j, \quad \text{and} \quad \sum_{i=0}^{d+1} iZ_i^n(j) \leq b_{U_0^n} + j.$$ (1.2)

Define now vectors in $\mathbb{R}^{d+2}$,

$$f^d_0 := \langle 0, 1, 0, \ldots, 0 \rangle,$$

$$f^d_i := \langle 1, 0, \ldots, 0, -1, 1, 0 \ldots, 0 \rangle, \quad \text{where} \ -1 \ \text{is at the} \ (i+1)\text{th position for} \ 1 \leq i \leq d$$

$$f^d_{d+1} := \langle 1, 0, \ldots, 0 \rangle.$$
For \( y = \langle y_0, y_1, \ldots, y_{d+1} \rangle \in \mathbb{R}^{d+2} \) and \( 0 \leq i \leq d + 1 \), denote
\[
[y]_i := \sum_{l=0}^{i} y_l.
\]

Note that
\[
0 \leq [r^d]_i \leq 1 \quad \text{for} \quad 0 \leq i \leq d, \quad [r^d]_{d+1} = 1, \quad \text{and} \quad 0 \leq \sum_{i=0}^{d+1} (1 - [r^d]_i) \leq 1. \tag{1.3}
\]

Consider now the truncated degree distribution
\[
\{ Z^{n,d}(j) := \langle Z_0^n(j), Z_1^n(j), \ldots, Z_d^n(j), Z_{d+1}^n(j) \rangle | 0 \leq j \leq n \},
\]
where \( Z_{d+1}^n(j) = \sum_{k=d+1}^{\infty} Z_k^n(j) = j + b_{U_0^n} - \sum_{k=0}^{d} Z_k^n(j) \), which forms a discrete time Markov chain with initial state \( Z^{n,d}(0) \) corresponding to the initial urn configuration \( U_0^n \) and one-step transition property,
\[
Z^{n,d}(j+1) - Z^{n,d}(j) = \begin{cases} f^d_0, & \text{with prob. } p(j/n) + (1 - p(j/n)) \frac{\beta(j/n) Z_0^n(j)}{s_j}, \text{ for } i = 0 \\ f^d_i, & \text{with prob. } (1 - p(j/n)) \frac{(i+\beta(j/n)) Z_i^n(j)}{s_j}, \text{ for } 1 \leq i \leq d, \\ f^d_{d+1}, & \text{with prob. } (1 - p(j/n)) \left(1 - \sum_{i=0}^{d} (i+\beta(j/n)) Z_i^n(j)\right). 
\end{cases} \tag{1.4}
\]

We also define the full degree distribution \( \{ Z^{n,\infty}(j) := \langle Z_0^n(j), Z_1^n(j), \ldots \rangle | 0 \leq j \leq n \} \) which is also a Markov chain on \( \mathbb{R}^{\infty} \) with increments
\[
Z^{n,\infty}(j+1) - Z^{n,\infty}(j) = \begin{cases} f^\infty_0, & \text{with prob. } p(j/n) + (1 - p(j/n)) \frac{\beta(j/n) Z_0^{n,\infty}(j)}{s_j}, \text{ for } i = 0 \\ f^\infty_i, & \text{with prob. } (1 - p(j/n)) \frac{(i+\beta(j/n)) Z_i^{n,\infty}(j)}{s_j}, \text{ for } i \geq 1 
\end{cases} \tag{1.5}
\]
where \( f_0^\infty = \langle 0, 1, 0, \ldots, 0, \ldots \rangle \) and \( f_i^\infty = \langle 1, 0, \ldots, 0, -1, 1, 0, \ldots, 0, \ldots \rangle \) with the ‘−1’ being in the \((i + 1)\)th place.

We will assume throughout the following initial condition.

(LIM) With respect to constants \( c_i \in [0, 1] \) for \( i \geq 0 \), initially
\[
\lim_{n \to \infty} \frac{1}{n} Z_i^n(0) =: c_i, \quad \text{and} \quad \tilde{c} := \sum_{i \geq 0} i c_i < \infty.
\]
Define also
\[ c := \sum_{i \geq 0} c_i, \quad \bar{c}^d := \sum_{i \geq d+1} c_i, \quad \text{and} \quad c^d := \langle c_0, c_1, \ldots, c_d, \bar{c}^d \rangle. \]

We remark one can classify the initial configurations depending on when \( c_i \equiv 0 \) or when \( c_i > 0 \) for some \( i \geq 0 \).

- **(Small Configuration)** \( c_i \equiv 0 \) for any \( i \geq 0 \). Here, the initial urn configurations are in a sense small in that their size is \( o(n) \). This is the case when the initial configurations do not depend on \( n \) for instance.

- **(Large Configuration)** \( c_i > 0 \) for some \( i \geq 0 \). In this case, the initial state is already a well-developed configuration whose size is of order \( n \).

The main results will be on the family of stochastic processes \({X^n, d(t) \mid 0 \leq t \leq 1}\) and \({X^n, \infty(t) \mid 0 \leq t \leq 1}\) obtained by linear interpolation of the discrete-time Markov chains \( \frac{1}{n} Z^{n, d}(j) \) and \( \frac{1}{n} Z^{n, \infty}(j) \) respectively. For \( t \geq 0 \), let

\[
X^{n, d}(t) := \frac{1}{n} Z^{n, d}([nt]) + \frac{nt - \lfloor nt \rfloor}{n} \left( Z^{n, d}(\lfloor nt \rfloor + 1) - Z^{n, d}([nt]) \right),
\]
\[
X^{n, \infty}(t) := \frac{1}{n} Z^{n, \infty}([nt]) + \frac{nt - \lfloor nt \rfloor}{n} \left( Z^{n, \infty}([nt] + 1) - Z^{n, \infty}([nt]) \right).
\]

The trajectories \( X^{n, d}(t) \) lie in \( C([0, 1]; \mathbb{R}^{d+2}) \), and are Lipschitz, with constant at most 1, satisfying \( X^{n, d}(0) = \frac{1}{n} Z^{n, d}(0) \). On the other hand, \( X^{n, \infty}(t) \in \prod_{i=1}^{\infty} C([0, 1]; \mathbb{R}) \), considered with the weak topology, where \( X^{n, \infty}(0) = \frac{1}{n} Z^{n, \infty}(0) \).

We now specify the assumptions on \( p(t) \) and \( \beta(t) \) used for the main results.

- **(ND)** \( p \) and \( \beta \) are piecewise continuous and, for some constants \( 0 \leq p_0 < 1, \beta_0 > 0 \),

\[
0 \leq p(\cdot) \leq p_0 \quad \text{and} \quad \beta_0 \leq \beta(\cdot) < \infty. \tag{1.6}
\]

We discuss more on (ND) in the remark after Theorem 1.1.1.

We note, throughout the article, we use conventions

\[
0 \log 0 = 0 \log(0/0) = 0, \quad x/0 = \infty \quad \text{for} \quad x > 0, \quad \text{and} \quad E[X; \mathbb{A}] = \int_{\mathbb{A}} X \, dP. \tag{1.7}
\]
1.1.2 Main Results

We now recall the statement of a large deviation principle (LDP). A sequence \( \{X^n\} \) of random variables taking values in a complete separable metric space \( X \) satisfies the LDP with rate \( n \) and good rate function \( J : X \to [0, \infty] \) if for each \( M < \infty \), the level set \( \{ x \in X | J(x) \leq M \} \) is a compact subset of \( X \), i.e. \( J \) has compact level sets, and if the following two conditions hold.

(i) Large deviation upper bound. For each closed subset \( F \) of \( X \)
\[
\limsup_{n \to \infty} \frac{1}{n} \log P\{X^n \in F\} \leq - \inf_{x \in F} J(x).
\]

(ii) Large deviation lower bound. For each open subset \( G \) of \( X \)
\[
\liminf_{n \to \infty} \frac{1}{n} \log P\{X^n \in G\} \geq - \inf_{x \in G} J(x).
\]

1.1.2.1 Empirical degree distribution

For \( d \geq 1 \), we now state the LDP for \( \{X^{n,d}(t) | 0 \leq t \leq 1\} \). Let \( I_d : C([0,1]; \mathbb{R}^{d+2}) \to [0, \infty] \) be the function given by
\[
I_d(\varphi) = \int_0^1 \left(1 - [\dot{\varphi}(t)]_0\right) \log \frac{1 - [\dot{\varphi}(t)]_0}{p(t) + (1 - p(t)) \beta(t) \varphi(t)} \left(\frac{\beta(t) \varphi(t)}{1 + \beta(t) \varphi(t)}\right) dt + \sum_{i=1}^d \left(1 - [\dot{\varphi}(t)]_i\right) \log \frac{1 - [\dot{\varphi}(t)]_i}{(1 - p(t)) (1 + \beta(t) \varphi(t))} \left(\frac{(1 + \beta(t) \varphi(t)) \varphi(t)}{1 + \beta(t) \varphi(t)}\right) dt + \left(1 - \sum_{i=0}^d (1 - [\dot{\varphi}(t)]_i)\right) \log \frac{1 - \sum_{i=0}^d (1 - [\dot{\varphi}(t)]_i)}{(1 - p(t)) (1 - \sum_{i=0}^d (1 + \beta(t) \varphi(t)) \varphi(t))} dt,
\]
where \( \varphi(0) = e^d, \varphi_i \geq 0 \) is Lipschitz with constant 1 such that \( 0 \leq [\dot{\varphi}]_i \leq 1 \) for \( 0 \leq i \leq d \), \( \sum_{i=0}^d [\dot{\varphi}(t)]_i = 1, \sum_{i=0}^d i [\dot{\varphi}(t)]_i \leq 1 \) for almost all \( t \), and the integral converges; otherwise, \( I_d(\varphi) = \infty \). It will turn out that \( I_d \) is convex and is a good rate function.

**Theorem 1.1.1 (Sample path LDP).** Under assumption (ND), the sequence \( \{X^{n,d}\} \) of \( C([0,1]; \mathbb{R}^{d+2}) \)-valued random variables satisfies an LDP with rate \( n \) and convex, good rate function \( I_d \).

**Remark 1.** We now comment on the assumption (ND).
(A) In some sense, it specifies that the process considered is ‘non-degenerate’. (ND) does not cover some ‘boundary’ cases, for instance, when \( p(t) \equiv 1 \), the process is deterministic in that at each time, one places a new ball in a new urn. Also, when \( \beta(t) \equiv 0 \) and \( p(t) \equiv 0 \), urns without a ball have no weight, and all new balls are placed into urns in the initial configuration. Although one should still obtain an LDP in these cases, and other boundary cases which are less ‘degenerate’, the form of the rate function may differ in that some increments may not be possible.

(B) On the other hand, assumption (ND) is a natural condition with respect to the convergence estimates needed for the proof of the lower bound in the LDP. However, the LDP upperbound holds without the assumption (ND).

Remark 2. One can recover the LDP at a fixed time, say \( t = 1 \), by the contraction principle with respect to continuous function \( F : C([0, 1]; \mathbb{R}^{d+2}) \rightarrow \mathbb{R}^{d+2} \) defined by \( F(\varphi) = \varphi(1) \), so that \( F(X^{n,d}) = X^{n,d}(1) = \frac{1}{n} Z^{n,d}(n) \). Then, Theorem 1.1.1 implies the LDP for \( \frac{1}{n} Z^{n,d}(n) \) with rate function given by the variational expression

\[
K(x) = \inf \left\{ I_d(\varphi) \mid \varphi(0) = c^d, \varphi(1) = x \right\}.
\]

In Figure 1.1, we consider optimal trajectories for the number of empty urns, \( \{ \frac{1}{n} Z^0_0(\lfloor nt \rfloor) : 0 \leq t \leq 1 \} \), when \( d = 0 \), given that \( \frac{1}{n} Z^0_0(n) = x \) for various values of \( x \).

We now extend the finite-dimensional LDP results to the infinite dimensional case \( (d = \infty) \).

Define for \( \xi \in \prod_{i=0}^{\infty} C([0, 1]; \mathbb{R}) \) the function

\[
I^\infty(\xi) = \int_0^1 \lim_{d \to \infty} \left[ (1 - [\xi(t)]_0) \log \frac{1 - [\xi(t)]_0}{p(t) + (1 - p(t))} \frac{\beta(t) [\xi(t)]_0}{(1 + \beta(t))^2 + c\beta(t)} \right. \\
+ \left. \sum_{i=1}^{d} (1 - [\xi(t)]_i) \log \frac{1 - [\xi(t)]_i}{(1 - p(t)) \xi(t)} \frac{\beta(t) [\xi(t)]_i}{(1 + \beta(t))^2 + c\beta(t)} \right] dt \\
+ \left( 1 - \sum_{i=0}^{d} (1 - [\xi(t)]_i) \right) \log \frac{1 - \sum_{i=0}^{d} (1 - [\xi(t)]_i)}{(1 - p(t))(1 - \sum_{i=0}^{d} \xi(t))} \right] dt
\]

where \( \xi_i(0) = c_i, \xi_i(t) \geq 0 \) is Lipschitz with constant 1, \( 0 \leq [\xi_i(t)]_i \leq 1 \) for \( i \geq 0 \), \( \sum_{i=0}^{d} \xi_i(t) = 1 \) almost all \( t \), and the integral converges; otherwise \( I^\infty(\xi) = \infty \). It will turn out through a projective limit approach (cf. [14, Section 4.6]) that \( I^\infty \) is well-defined, convex and a good rate
Figure 1.1 The red curves are numerical solutions of the Euler equations with respect to (1.9) for $\frac{1}{n}Z_n^0(n)$ with $p(t) = t, \beta(t) = 1, Z_n^k(0) = \frac{1}{2^k-1}n$ for $k \geq 0$ when $x = 0.6, 0.7, 0.8, 0.9, 1$. The blue curve is the the LLN line for which $I_0(\varphi) = 0$

function, and the limit in the integrand exists because the expressions in square brackets are increasing in $d$.

**Theorem 1.1.2** (Sample path LDP - infinite dimension). *Given assumption (ND), the sequence $\{X_n\}_{n=0}^{\infty}$ of $\prod_{i=0}^{\infty} C([0,1]; \mathbb{R})$-valued random variables satisfies an LDP with rate $n$ and convex, good rate function $I^\infty$.*

**Remark 3.** From the result, one can see that degree sequences not fully supported on the non-negative integers, that is when $\sum_{i \geq 0} \dot{\varphi}_i(\cdot) < 1$, cannot be achieved with finite cost. Intuitively, one can understand this as follows: The $(\text{urns})$ with size larger than $A$ at time $n$ is bounded by $(\text{balls at time } n)/A$, and so the proportion of these urns is on order $A^{-1}$, which vanishes as $A \uparrow \infty$.

On the other hand, it seems some of the total weight can indeed be lost, that is $\lim_{d \rightarrow \infty} \sum_{i=0}^{d} \frac{(i+\beta(t))\xi_i(t)}{(1+\beta(t))t+c\beta(t)} < 1$ with finite rate. The interpretation is that in the growth evolution, it is possible to put a large number of balls into a few very large urns with finite cost.

To give an example, let $c_i \equiv 0$, $\beta(t) = 1$, $p(t) \equiv 0$, corresponding to the ‘classical’ Barabási-Albert model, and consider $\xi_i(t) = t(\alpha-1)/\alpha^{i+1}$ for $i \geq 0$ and $\alpha \geq 2$. Then, $[\dot{\xi}_i](t) = 1-\alpha^{-(i+1)}$, ...
\[ \sum_{i=1}^{\infty} \xi_i(t) = 1, \text{ and } \sum_{i=0}^{\infty} i \xi_i(t) = (\alpha - 1)^{-1} \leq 1. \] 

One may compute that the rate
\[ I^\infty(\xi) = -\sum_{i=0}^{\infty} \frac{1}{\alpha^{i+1}} \log \frac{(\alpha - 1)(i + 1)}{2} + \left(1 - \frac{1}{\alpha - 1}\right) \log 2. \]

The second term can be thought of as the cost given to ‘increment’ \( \langle 1, 0, \ldots, 0, \ldots \rangle \) corresponding to when the dynamics puts balls into very large sized urns, informally urns with size infinity, or in other words when new vertices attach to very large hubs.

If now \( \alpha \uparrow \infty \), one is computing the cost of mostly evolving according to increment \( \langle 1, 0, \ldots, 0, \ldots \rangle \), which is \( \log 2 \). An evolution which achieves this is the growing ‘star’ configuration where all new vertices connect to the same vertex. When initially, there are only two vertices with a single edge between them, this configuration has probability \( 2^{-n} \) of occurring at time \( n \).

One also points out, deviations to non-power law degree sequences such that \( \sum_{i \geq 0} i \phi_i(\cdot) = 1 \), when all the weight is on urns with finite size, is possible with finite rate, e.g. when \( \alpha = 2 \) above.

We now turn to the LLN behavior of the \( d + 2 \)-dimensional model which corresponds to the “zero-cost” trajectory on which the rate function vanishes. Consider the system of ODEs for \( \varphi^d = \varphi \), with initial condition \( \varphi(0) = c^d \):

\[
\begin{align*}
\dot{\varphi}_0(t) & = 1 - p(t) - (1 - p(t)) \frac{\beta(t)\varphi_0(t)}{(1 + \beta(t))t + \tilde{c} + c\beta(t)}, \\
\dot{\varphi}_1(t) & = p(t) + (1 - p(t)) \frac{\beta(t)\varphi_0(t)}{(1 + \beta(t))t + \tilde{c} + c\beta(t)} - (1 - p(t)) \frac{(1 + \beta(t))\varphi_1(t)}{(1 + \beta(t))t + \tilde{c} + c\beta(t)}, \\
\dot{\varphi}_i(t) & = (1 - p(t)) \frac{(i - 1 + \beta(t))\varphi_{i-1}(t)}{(1 + \beta(t))t + \tilde{c} + c\beta(t)} - (1 - p(t)) \frac{(i + \beta(t))\varphi_i(t)}{(1 + \beta(t))t + \tilde{c} + c\beta(t)}, \quad \text{for } 2 \leq i \leq d \text{ and} \\
\dot{\varphi}_{d+1}(t) & = 1 - \sum_{i=0}^{d} \dot{\varphi}_i(t). 
\end{align*}
\]

For \( t \in [0, 1] \), define
\[ \zeta^d(t) = \langle \zeta_0(t), \zeta_1(t), \ldots, \zeta_{d+1}(t) \rangle \]
by

\[
\begin{align*}
\zeta_0(t) &:= \frac{1}{M_0(t)} \left[ c_0 + \int_0^t (1 - p(s)) M_0(s) \, ds \right], \\
\zeta_1(t) &:= \frac{1}{M_1(t)} \left[ c_1 + \int_0^t \left( p(s) + (1 - p(s)) \frac{\beta(s) \zeta_0(s)}{(1 + \beta(s)) s + \bar{c} + c\beta(s)} \right) M_1(s) \, ds \right], \\
\zeta_i(t) &:= \frac{1}{M_i(t)} \left[ c_i + \int_0^t (1 - p(s)) \frac{(i - 1 + \beta(s)) \zeta_{i-1}(s)}{(1 + \beta(s)) s + \bar{c} + c\beta(s)} M_i(s) \, ds \right], \quad \text{for } 2 \leq i \leq d, \\
\tilde{\zeta}_{d+1}(t) &:= t + c - \sum_{i=0}^d \zeta_i(t) = \bar{c}d + \int_0^t (1 - p(s)) \frac{(d + \beta(s)) \zeta_d(s)}{(1 + \beta(s)) s + \bar{c} + c\beta(s)} \, ds.
\end{align*}
\]

where

\[
M_i(t) := \begin{cases} 
\exp \left[ - \int_0^t (1 - p(u)) \frac{i + \beta(u)}{(1 + \beta(u)) u + \bar{c} + c\beta(u)} \, du \right], & \text{if } c_i = 0 \\
\exp \left[ \int_0^t (1 - p(u)) \frac{i + \beta(u)}{(1 + \beta(u)) u + \bar{c} + c\beta(u)} \, du \right], & \text{if } c_i \neq 0
\end{cases}
\]

for \(0 \leq i \leq d\).

Define also the infinite distribution \(\zeta^\infty := \langle \zeta_0(t), \zeta_1(t), \ldots \rangle \in \prod_{i=0}^\infty C([0, 1]; \mathbb{R})\).

We now state a LLN for \(X_{n,d}\), as a consequence of the LDP upper bound.

**Theorem 1.1.3 (LLN).** Let \(d \geq 0\) be fixed. We have \(\zeta^d\) is the unique solution to the system (1.10), with the initial condition \(\varphi(0) = \zeta^d\). Also, in the sup topology on \(C([0, 1]; \mathbb{R}^{d+2})\),

\[X_{n,d}(\cdot) \rightarrow \zeta^d(\cdot) \quad \text{a.s.}\]

As a consequence, we have in the product topology that \(X_{n,\infty}(\cdot) \rightarrow \zeta^{\infty}(\cdot)\).

Here, \(\zeta^d\), and \(\zeta^{\infty}\) are the limiting “urn-size” distributions of the \(d+2\) and infinite-dimensional processes. We now consider its “scale-freeness.” Although it seems difficult to control each \(\zeta_i\), under some assumptions, nevertheless \(\zeta^d\) has “power law” behavior, in terms of bounds on \([\zeta^d]\). In general, it appears \(\zeta^d\) can interpolate between the bounds (cf. Figure 1.2\(^1\)). Moreover, analyzing \(\zeta^d\) allows to follow the typical evolution of the continuum graph (cf. Figure 1.3).

**Theorem 1.1.4 (Power Law).** Assume \(0 \leq p_{\min} \leq p(\cdot) \leq p_0 =: p_{\max} < 1\), and \(0 < \beta_0 =: \beta_{\min} \leq \beta(\cdot) \leq \beta_{\max} < \infty\). Then \(\zeta^d\) may be bounded between two power laws:

1. When the initial configuration is small, i.e. \(c_i \equiv 0\), we have for \(0 \leq i \leq d\) and \(t \geq 0\) that

\[
[n]^i t \leq [\zeta^d(t)]_i \leq [n]_i t.
\]

\(^1\)As a curiosity, we note a very similar figure is found in [23] with respect to Facebook social network data.
The thick curve is the (numerical) degree distribution at time $t = 1$ using the LLN path with $d = 10000$, $p(t) = 0$, $\beta(t) = 8$ for $t < 1/100$, 1 for $t \geq 1/100$ and $c_k = 0$. The lines have slopes $-3$ and $-10$. The plot uses log-log scale.

2. When the initial configuration is large, i.e. $c_i > 0$ for some $i \geq 0$, we have, for $0 \leq i \leq d$ and as $t \uparrow \infty$, that

$$[\eta'_i](t + o(1)) \leq [\zeta^d(t)]_i \leq [\eta_i](t + o(1)).$$

Here

$$\eta'_i := \frac{C'}{t^{1 + \frac{1 + \beta_{\min}}{1 - p_{\min}}}}(1 + o(1)), \quad \text{and} \quad \eta_i := \frac{C}{t^{1 + \frac{1 + \beta_{\max}}{1 - p_{\max}}}}(1 + o(1)),$$

and $C, C'$ are positive constants depending on $p$ and $\beta$.

The outline of the paper is now as follows. In Sections 1.2 and Section 1.3, we prove the finite and infinite dimensional LDP's, Theorems 1.1.1 and 1.1.2. In Section 1.4, we prove the law of large numbers (Theorem 1.1.3). Finally, in Section 1.5, we discuss power-law behavior (Theorem 1.1.4).
Figure 1.3 Thick curves are numerical solutions for the degree distribution using the LLN path with $d = 10000$, $p(t) = 0$, $\beta(t) = 1$ and $c_k = (k + 1)^{-10}$ for each time $t = 1, 10, 100, 10000$. The lines have slopes $-3$ and $-10$. All plots use log-log scale. It shows the distribution is moving from the slope $-10$ to the slope $-3$, which are from the initial condition $c_k$, and the power law exponent from $p$ and $\beta$, respectively.

1.2 Proof of Theorem 1.1.1

We follow the method and notation of Dupuis-Ellis [19]. Some steps are similar to those in [9] where the “leaves” in a more simplified graph scheme is considered. However, as many things differ in our model, in the upper bound, and especially the lower bound proof, we present the full argument.

We now fix $0 \leq d < \infty$ and equip $\mathbb{R}^{d+2}$ with the $L_1$-norm denoted by $|\cdot|$. Recall, from assumption (LIM),

$$
c^{n,d} = (c_0^{n,d}, c_1^{n,d}, \ldots, c^{n,d}) := \frac{1}{n} Z^{n,d}(0) \to c^d.
$$

(1.12)
Let
\[ A := \sup_n |c^{n,d} - c^d|. \]

Let also
\[ c^{n,d} := \sum_{i \geq 0} c_i^{n,d}, \quad \text{and} \quad \tilde{c}^{n,d} := \sum_{i \geq 0} ic_i^{n,d}. \]

Denote also
\[ \vec{\xi}(n,t) := (p_n(t), \beta_n(t), \sigma_n(t)), \quad (1.13) \]
where
\[
\begin{align*}
p_n(t) &:= p([nt]/n), \\
\beta_n(t) &:= \beta([nt]/n), \\
\sigma_n(t) &:= \frac{1}{n} s_n^{nt} = (1 + \beta_n(t))\frac{[nt]}{n} + \tilde{c}^{n,d} + c^{n,d}\beta_n(t).
\end{align*}
\]

Let
\[
\begin{align*}
\sigma(t) &:= (1 + \beta(t))t + \tilde{c} + c\beta(t), \\
\vec{\xi}(t) &:= (p(t), \beta(t), \sigma(t)).
\end{align*}
\]

We note that, as \( n \to \infty \), as \( p(t) \) and \( \beta(t) \) are piecewise continuous,
\[
\vec{\xi}(n,t) \to \vec{\xi}(t) \quad \text{for almost all } t. \quad (1.14)
\]

In the following we will drop the superscript \( d \) to save on notation. Define
\[ X^n_j := \frac{1}{n} Z^n_{i,j}. \]
Then, recall \( X^n_0 = c^{n,d} \) and \( X^n_{j+1} = X^n_j + \frac{1}{n} v^n_j(X^n_j) \), where \( v^n_j(x) \) has a distribution \( \rho_{\vec{\xi}(n,j/n),x} \).

Here, for \( x = (x_0, x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+2} \) such that \( x_i \geq 0 \) for \( 0 \leq i \leq d+1 \), \( \sum_{i=0}^{d+1} (i + \beta_n(t))x_i \leq \sigma_n(t) \), and \( A \subset \mathbb{R}^{d+2} \),
\[
\rho_{\vec{\xi}(n,t),x}(A) := \left(p_n(t) + (1 - p_n(t))\frac{\beta_n(t)x_0}{\sigma_n(t)}\right)\delta_{f_0}(A) + \sum_{i=1}^{d} \left(1 - p_n(t)\right)\frac{(i + \beta_n(t))x_i}{\sigma_n(t)}\delta_{f_i}(A)
\]
\[ + \left(1 - p_n(t)\right)\left(1 - \sum_{i=0}^{d}(i + \beta_n(t))x_i\right)\delta_{f_{d+1}}(A). \]
From (1.3), the paths $X^n(t) = X^{n,d}(t)$ belong to

$$
\Gamma_{d,A} := \left\{ \varphi \in C([0,1]; \mathbb{R}^{d+2}) \mid ||\varphi(0) - e^d|| \leq A, \ \varphi_i \text{ is Lipschitz with constant } 1, \\
\sum_{i=0}^{d+1} \varphi_i(t) = 1, \sum_{i=0}^{d+1} i\varphi_i(t) \leq 1, \ 0 \leq [\varphi]_i \leq 1 \ \text{for } 0 \leq i \leq d+1 \right\}.
$$

(1.15)

Here, we equip $C([0,1]; \mathbb{R}^{d+2})$ with the supremum norm, i.e. for $\varphi \in C([0,1]; \mathbb{R}^{d+2})$,

$$
||\varphi||_{\infty} := \sup_{t \in [0,1]} |\varphi(t)| = \sup_{t \in [0,1]} \sum_{i=0}^{d+1} |\varphi_i(t)|.
$$

For probability measures $\mu$ and $\nu$, let $R(\mu || \nu)$ denote the relative entropy of $\mu$ with respect to $\nu$, i.e. $R(\mu || \nu) := \int \log(\frac{d\mu}{d\nu}) d\mu$ if $\mu \ll \nu$, $\infty$ otherwise. Let $h : C([0,1]; \mathbb{R}^{d+2}) \to \mathbb{R}$ be a bounded continuous function. Let also

$$
W^n := -\frac{1}{n} \log E\{\exp[-nh(X^n)]\}.
$$

To prove Theorem 1.1.1, we need to establish Laplace principle upper and lower bounds (cf. [19, Section 1.2]), namely upper bound

$$
\liminf_{n \to \infty} W^n \geq \inf_{\varphi \in C([0,1]; \mathbb{R}^{d+2})} \{I_d(\varphi) + h(\varphi)\}
$$

for a good rate function $I_d$, and lower bound

$$
\limsup_{n \to \infty} W^n \leq \inf_{\varphi \in C([0,1]; \mathbb{R}^{d+2})} \{I_d(\varphi) + h(\varphi)\}.
$$

Define, for $0 \leq j \leq n$, that

$$
W^n(j, \{x_0, \ldots, x_j\}) := -\frac{1}{n} \log E\{\exp[-nh(X^n)] \mid X^n_0 = x_0, \ldots, X^n_j = x_j\},
$$

and

$$
W^n := W^n(0, \emptyset) = -\frac{1}{n} \log E\{\exp[-nh(X^n)]\}.
$$

The Dupuis-Ellis method stems from the following discussion. From the Markov property, for $0 \leq j \leq n - 1$,

$$
e^{-nW^n(j, \{x_0, \ldots, x_j\})} = E\{e^{-nh(X^n)} \mid X^n_0 = x_0, \ldots, X^n_j = x_j\} = E\{E\{e^{-nh(X^n)} \mid X^n_0 = x_0, \ldots, X^n_{j+1}\} \mid X^n_0 = x_0, \ldots, X^n_j = x_j\} = E\{e^{-nW^n(j+1, \{x_0, \ldots, x_n, x_{j+1}\})} \mid X^n_0 = x_0, \ldots, X^n_j = x_j\}
$$

$$
= \int_{\mathbb{R}^{d+2}} e^{-nW^n(j+1, \{x_0, \ldots, x_n, x_{j+1}\} + \frac{1}{n} dy)} \rho_{\zeta(n,j/n), x_j} (dy).
$$
Then, by the variational formula for relative entropy (cf. [19, Proposition 1.4.2]), for $0 \leq j \leq n - 1$,

$$W^n(j, \{x_0, \ldots, x_j\}) = -\frac{1}{n} \log \int_{\mathbb{R}^{d+2}} e^{-nW^n(j+1, \{x_0, \ldots, x_j, y\})} \rho_{\bar{\xi}(n,j/n),x_j}(dy)$$

$$= \inf_{\mu} \left\{ \frac{1}{n} R(\mu||\rho_{\bar{\xi}(n,j/n),x_j}) + \int_{\mathbb{R}^{d+2}} W^n(j+1, \{x_0, \ldots, x_j, y\}) \mu(dy) \right\}.$$ 

We also have a terminal condition $W^n(n, \{x_0, \ldots, x_n\}) = h(x)$, where $x$ is the linear interpolated path connecting $\{(j/n, x_j)\}_{0 \leq j \leq n}$.

We may interpret these equations and terminal condition in terms of dynamic programming and a particular stochastic control problem. Define (i) $\mathcal{L}_j = (\mathbb{R}^{d+2})^j$, the state space on which $W^n(j, \cdot)$ is defined; (ii) $\mathcal{U} = \mathcal{P}(\mathbb{R}^{d+2})$, the space of probability measures on $\mathbb{R}^{d+2}$ which is the control space on which the infimum is taken; (iii) for $j = 0, \ldots, n - 1$, “control” $v^n_j(dy) = v^n_j(dy|x_0, \ldots, x_j)$ which is a stochastic kernel on $\mathbb{R}^{d+2}$ given $(\mathbb{R}^{d+2})^j$; (iv) $\{\bar{X}^n_j; 0 \leq j \leq n\}$, the “controlled” process which is the adapted path satisfying $\bar{X}^n_0 = c$ and $\bar{X}^n_{j+1} = \bar{X}^n_j + \frac{1}{n} \bar{Y}^n_j$ for $0 \leq j \leq n - 1$, where $\bar{Y}^n_j$, conditional on $(\bar{X}^n_0, \ldots, \bar{X}^n_j)$ has distribution $v^n_j(\cdot)$ (e.g. $\bar{P}\{\bar{Y}^n_j \in dy | \bar{X}^n_0, \ldots, \bar{X}^n_j\} = v^n_j(dy | \bar{X}^n_0, \ldots, \bar{X}^n_j)$), and $\bar{X}^n$ is the piecewise linear interpolation of these controlled random variables; (v) “running costs” $C_j(v) = \frac{1}{n} R(\nu||\rho)$ for $v \in \mathcal{P}(\mathbb{R}^{d+2})$; and (vi) “terminal cost” equal to the function $h$.

Then $W^n(j, \{x_0, \ldots, x_j\})$ satisfies a control problem whose solution is for $0 \leq j \leq n - 1$ (cf. [19, Section 3.2]),

$$V^n(j, \{x_0, \ldots, x_j\}) = \inf_{\{v^n_i\}} \bar{E}_{j,x_0,\ldots,x_j} \left\{ \frac{1}{n} \sum_{i=j}^{n-1} R(v^n_i(\cdot)||\rho_{\bar{\xi}(n,i/n),x_i}) + h(\bar{X}^n_i) \right\},$$

where $v^n_i(\cdot) = v^n_i(\cdot | \bar{X}^n_0, \ldots, \bar{X}^n_j)$, and the infimum is taken over all control sequences $\{v^n_i\}$. Here, $\bar{E}_{j,x_0,\ldots,x_j}$ denotes expectation, with respect to the adapted process $\bar{X}^n$ associated to $\{v^n_i\}$, conditioned on $\bar{X}^n_0 = x_0, \ldots, \bar{X}^n_j = x_j$.

The boundary conditions are $V^n(n, \{x_0, \ldots, x_n\}) = h(x)$ and

$$V^n := V^n(0, \emptyset) = \inf_{\{v^n_i\}} \bar{E} \left\{ \frac{1}{n} \sum_{j=0}^{n-1} R(v^n_j(\cdot)||\rho_{\bar{\xi}(n,j/n),x_j}) + h(\bar{X}^n) \right\}.$$

(1.16)

In particular, by [19, Corollary 5.2.1],

$$W^n = -\frac{1}{n} \log E\{\exp[-nh(\bar{X}^n)]\} = V^n.$$

(1.17)
1.2.1 Upper bound

To prove the upper bound, it will be helpful to put the controls \{v^n\} into continuous-time paths. Let \(v^n(dy|t) := v^n_j(dy)\) for \(t \in [j/n, (j+1)/n)\), \(j = 0, \ldots, n - 1\), and \(v^n(dy|1) := v^n_{n-1}\). Define
\[
v^n(A \times B) := \int_B v^n(A|t)dt
\]
for Borel \(A \subset \mathbb{R}^{d+2}\) and \(B \subset [0,1]\). Also define the piecewise constant path \(\tilde{X}^n(t) := \tilde{X}^n_j\) for \(t \in [j/n, (j+1)/n), 0 \leq j \leq n - 1\), and \(\tilde{X}^n(1) = \tilde{X}^n_{n-1}\). Then
\[
W^n = V^n = \inf_{\{v^n\}} \mathbb{E} \left\{ \int_0^1 R(v^n(\cdot|t)||\rho_{\tilde{\xi}(n,t),\tilde{X}^n(t)})dt + h(\tilde{X}^n) \right\}.
\]
Given \(\rho_{\tilde{\xi},\bar{X}}\) is supported on \(K := \{f_0, f_1, \ldots, f_{d+1}\}\), if \(\{v^n_j\}\) is not supported on \(K\), then \(R(v^n||\rho_{\tilde{\xi},\bar{X}}) = \infty\). Since \(|V^n| \leq ||h||_\infty < \infty\) and \(K \subset \mathbb{R}^{d+2}\) is compact, for each \(n\), there is \(\{v^n_j\}\) supported on \(K\) and corresponding \(v^n(dy \times dt) = v^n(dy|t) \times dt\) such that, for \(\varepsilon > 0\),
\[
W^n + \varepsilon = V^n + \varepsilon \geq \mathbb{E} \left\{ \int_0^1 R(v^n(\cdot|t)||\rho_{\tilde{\xi}(n,t),\tilde{X}^n(t)})dt + h(\tilde{X}^n) \right\}.
\]
(1.18)
Recall that \(\bar{X}^n\) takes values in \(\Gamma_{d,A}\). Since \(\Gamma_{d,A}\) is compact, and \(\{v^n_j\}\) is tight, by Prokhorov’s Theorem, given any subsequence of \(\{v^n, \bar{X}^n\}\), there is a further subsubsequence, a probability space \((\bar{\Omega}, \mathcal{F}, \bar{P})\), a stochastic kernel \(v\) on \(K \times [0,1]\) given \(\bar{\Omega}\), and a random variable \(\bar{X}\) mapping \(\bar{\Omega}\) into \(\Gamma_{d,A}\) such that the subsubsequence converges in distribution to \((v, \bar{X})\). In particular, since \(\bar{X}^n(0) = c^{n,d} \to c^d\) as \(n \to \infty\), we have \(\bar{X}\) belongs to
\[\Gamma_d := \Gamma_{d,0}, \text{ those functions such that } \varphi(0) = c^d \text{ (cf.}(1.15)).\]
Then, [19, Lemma 3.3.1] shows that \(v\) is a subsequential weak limit of \(v^n\), and there exists a stochastic kernel \(v(dy|t,\omega)\) on \(K\) given \([0,1] \times \bar{\Omega}\) such that \(\bar{P}\)-a.s. for \(\omega \in \bar{\Omega}\),
\[
v(A \times B | \omega) = \int_B v(A|t,\omega)dt.
\]
Now, the same proof given for [19, Lemma 5.3.5] shows that \((v^n, \bar{X}^n, \tilde{X}^n)\) has a subsequential weak limit \((v, \bar{X}, \tilde{X})\), where the last coordinate is with respect to \(D([0,1]:\mathbb{R}^{d+2})\), and \(\bar{P}\)-a.s. for \(t \in [0,1]\), and
\[
\bar{X}(t) = \int_{\mathbb{R}^{d+2} \times [0,t]} yv(dy \times ds) = \int_0^t \left( \int_K yv(dy|s) \right) ds
\]
\[
\tilde{X}(t) = \int_K yv(dy|t).
\]
By Skorokhod Representation Theorem, we may take that \((v, \bar{X}^n, \bar{X})\) converges to \((v, \bar{X}, \bar{X})\) a.s.. In particular, \(\bar{X}^n \to \bar{X}\) uniformly a.s., and as \(\bar{X}\) is continuous, it follows that also \(\bar{X}^n \to \bar{X}\) uniformly a.s. (cf. [19, Theorem A.6.5]).

Let \(\lambda\) denote Lebesgue measure on \([0, 1]\) and \(\rho \times \lambda\) product measure on \(K \times [0, 1]\). Then [19, Lemma 1.4.3(f)] yields

\[
\int_0^1 R(v^n(\cdot | t)||\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t)) dt = R(v^n(\cdot | t) \times \lambda(dt)||\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t) \times \lambda(dt)).
\]

We now evaluate the limit inferior of \(W^n\) using formula (1.18), along a subsequence as above:

\[
\liminf_{n \to \infty} V^n + \varepsilon \geq \liminf_{n \to \infty} E \left\{ \int_0^1 R(v^n(\cdot | t)||\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t)) dt + h(\bar{X}^n) \right\}
\]

\[
= \liminf_{n \to \infty} E \left\{ R(v^n(\cdot | t) \times \lambda(dt)||\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t) \times \lambda(dt)) + h(\bar{X}^n) \right\}
\]

\[
\geq E \left\{ R(v(\cdot | t) \times \lambda(dt)||\rho_{\bar{\xi}(t)}, \bar{X}(t) \times \lambda(dt)) + h(\bar{X}) \right\}
\]

\[
= E \left\{ \int_0^1 R(v(\cdot | t)||\rho_{\bar{\xi}(t)}, \bar{X}(t)) dt + h(\bar{X}) \right\}.
\]

In the second inequality, we used Fatou’s lemma with (i) \(v^n(dy|dt) \times \lambda(dt) \to v(dy|dt) \times \lambda(dt)\) a.s. as \(v^n \to v\) a.s.; (ii) \(\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t) \Rightarrow \rho_{\bar{\xi}(t)}, \bar{X}(t)\) as \(p_n(t) \to p(t), \beta_n(t) \to \beta(t), \sigma_n(t) \to \sigma(t), \bar{X}^n(t) \to \bar{X}(t)\) uniformly on \([0, 1]\) a.s., and \(\rho_{\bar{\xi}(n,t), \bar{X}}\) is continuous in its arguments; (iii) \(\liminf_{n \to \infty} R(v^n(dy|dt) \times \lambda(dt)||\rho_{\bar{\xi}(n,t)}, \bar{X}^n(t) \times \lambda(dt)) \geq R(v(dy|dt) \times \lambda(dt)||\rho_{\bar{\xi}(t)}, \bar{X}(t) \times \lambda(dt))\) a.s. as \(R\) is lower semi-continuous. (iv) \(h(\bar{X}^n) \to h(\bar{X})\) a.s. as \(h\) is continuous and \(\bar{X}^n \to \bar{X}\) uniformly on \([0, 1]\) a.s.

By [19, Lemma 3.3.3(c)],

\[
R(v(\cdot | t)||\rho_{\bar{\xi}(t)}, \bar{X}(t)) \geq L\left(\bar{\xi}(t), \bar{X}(t), \int_K zv(dz|t)\right),
\]

where

\[
L(\bar{\xi}(t), x, y) := \sup \left\{ \langle \theta, y \rangle - \log \int_K \exp\langle \theta, z \rangle \rho_{\bar{\xi}(t), x}(dz) \mid \theta \in \mathbb{R}^{d+2} \right\} \quad (1.19)
\]

\[
= \inf \left\{ R(v(\cdot | t)||\rho_{\bar{\xi}(t), x}) \mid v(\cdot | t) \in \mathcal{P}(K), \int_K zv(dz|t) = y \right\}. \quad (1.20)
\]
We note, in (1.20), the infimum is attained at some $\nu_0 \in \mathcal{P}(K)$ as the relative entropy is convex and lower semicontinuous (cf. [19, Lemma 1.4.3(b)]). Since $\int zv(dz|t) = \dot{X}(t)$, we have

$$\liminf_{n \to \infty} V^n \geq \bar{E} \left\{ \int_0^1 L(\tilde{\xi}(t), \bar{X}(t), \dot{\bar{X}}(t)) dt + h(\varphi) \right\}.$$

As $\bar{X} \in \Gamma_d$, we have

$$\liminf_{n \to \infty} V^n \geq \inf_{\varphi \in \Gamma_d} \int_0^1 L(\tilde{\xi}(t), \varphi(t), \dot{\varphi}(t)) dt + h(\varphi).$$

For $\varphi \in \Gamma_d$, we can evaluate the minimizer $\nu_0$ in the definition of $L(\tilde{\xi}(t), \varphi(t), \dot{\varphi}(t))$ uniquely:

Recall that $[\dot{\varphi}(t)]_i := \sum_{i=0}^j \dot{\varphi}_i(t)$. Then, as $\sum_{i=0}^{d+1} \nu_0(\xi_i|t) = (\dot{\varphi}_0(t), \ldots, \dot{\varphi}_{d+1}(t))$, a calculation gives

$$\nu_0(\dot{\varphi}(t)|t) = \sum_{i=0}^d (1 - [\dot{\varphi}(t)]_i) \delta_i + \left( \sum_{i=0}^d [\dot{\varphi}(t)]_i - d \right) \delta_{d+1}.$$ 

Since $\nu_0$ is absolutely continuous with respect to $\rho_{\tilde{\xi}(t), \varphi(t)}$,

$$L(\tilde{\xi}(t), \varphi(t), \dot{\varphi}(t)) = R(\nu_0(\dot{\varphi}(t)|t) \| \rho_{\tilde{\xi}(t), \varphi(t)}) = \left(1 - [\dot{\varphi}(t)]_0\right) \log \frac{1 - [\dot{\varphi}(t)]_0}{\rho(t) + (1 - \rho(t))^{(1 + \beta(t))} + c \beta(t)}$$

$$+ \sum_{i=1}^d (1 - [\dot{\varphi}(t)]_i) \log \frac{1 - [\dot{\varphi}(t)]_i}{\rho(t) + (1 - \rho(t))^{(1 + \beta(t))} + c \beta(t)} + \left(1 - \sum_{i=0}^d (1 - [\dot{\varphi}(t)]_i) \right) \log \frac{1 - \sum_{i=0}^d (1 - [\dot{\varphi}(t)]_i)}{(1 - \rho(t)) \left(1 - \sum_{i=0}^d (1 + \beta(t)) \rho_i(t) \right)},$$

interpreted under our conventions listed at the end of the introduction.

Finally, define

$$I_d(\varphi) := \int_0^1 L(\tilde{\xi}(t), \varphi(t), \dot{\varphi}(t)) dt,$$

when $\varphi \in \Gamma_d$, and $I_d(\varphi) = \infty$ otherwise. Since $L$ is convex, $I_d$ is convex, and also $I_d$ has compact level sets by the proof of [19, Proposition 6.2.4], and so is a good rate function. Hence, the Laplace principle upper bound holds with respect to $I_d$.

We will need the following result for the proof of the lower bound in the next section. Principally, assumption (ND) is needed here to show that linear functions have finite rate.

**Lemma 1.2.1.** Let $\ell(t) = et + c^d$ be a linear function, where $e = (e_0, e_1, \ldots, e_{d+1})$ is such that $e_i > 0$ for $i \geq 0$, $\sum_{i=0}^{d+1} e_i = 1$, and $\sum_{i=0}^{d+1} ie_i \leq 1$. Then, under (ND), $I_d(\ell(t)) < \infty$. 


Proof. Noting $\sum_{i=0}^{d}(1 - [e]_i) = \sum_{i=0}^{d+1} ie_i \leq 1$, explicitly

$$I_d(\ell(t)) = \int_{0}^{1} (1 - [e]_0) \log \frac{1 - [e]_0}{p(t) + (1 - p(t)) \frac{\beta(t)(e_0t + c_0)}{(1 + \beta(t))t + c + c\beta(t)}}$$

$$+ \sum_{i=1}^{d} (1 - [e]_i) \log \frac{1 - [e]_i}{(1 - p(t)) \frac{(1 + \beta(t))(e_i t + c_i)}{(1 + \beta(t))(t + c + c\beta(t))}}$$

$$+ \left(1 - \sum_{i=0}^{d} (1 - [e]_i)\right) \log \frac{1 - \sum_{i=0}^{d} (1 - [e]_i)}{(1 - p(t)) \frac{1}{(1 + \beta(t))(t + c + c\beta(t))}} dt$$

is bounded under assumption (ND).

\(\square\)

### 1.2.2 Lower bound

Fix a bounded, continuous function $h : C([0, 1]; \mathbb{R}^{d+2}) \to \mathbb{R}$, and $\varphi^* \in \Gamma_d$ such that $I_d(\varphi^*) < \infty$. To show the lower bound, it suffices to prove, for each $\varepsilon > 0$, that

$$\limsup_{n \to \infty} V^n \leq I(\varphi^*) + h(\varphi^*) + 8\varepsilon. \quad (1.24)$$

The main idea of the argument is to construct from $\varphi^*$ a sequence of control measures suitable to evaluate formulas for $V^n$.

Note in the following, to make some expressions simpler, we often take $c_{d+1} := \bar{c}^d$.

**Step 1. Convex combination and Regularization.** Rather than work directly with $\varphi^*$, we consider a convex combination of paths with better regularity: For $0 \leq \theta \leq 1$, let

$$\varphi_\theta(t) = (1 - \theta)\varphi^*(t) + \theta \ell(t), \quad (1.25)$$

where $\ell(t) = et + c^d$ is a linear function such that $e$ satisfies the assumptions of Lemma 1.2.1, say $e = (\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{d+1}}, \frac{1}{2^{d+1}})$.

**Lemma 1.2.2.** As $\theta \downarrow 0$, we have

$$|I_d(\varphi_\theta) - I_d(\varphi^*)| \to 0, \quad \text{and} \quad |h(\varphi_\theta) - h(\varphi^*)| \to 0.$$
Proof. By convexity of $I_d$, and finiteness of $I_d(\ell(t))$ from Lemma 1.2.1,

$$I_d(\varphi_{\theta}) \leq (1 - \theta)I_d(\varphi^*) + \theta I_d(\ell).$$

On the other hand, since $\|\varphi_{\theta} - \varphi^*\|_{\infty} < |2\theta 1| = 2\theta(d + 2) \downarrow 0$, by lower semi-continuity of $I_d$, we have

$$\liminf_{\theta \downarrow 0} I_d(\varphi_{\theta}) \geq I_d(\varphi^*).$$

Also, as $h$ is continuous, we have that $|h(\varphi_{\theta}) - h(\varphi^*)| \to 0$.

Now, fix $\theta > 0$ such that

$$I_d(\varphi_{\theta}) \leq I_d(\varphi^*) + \varepsilon \quad \text{and} \quad h(\varphi_{\theta}) \leq h(\varphi^*) + \varepsilon.$$

Next, for $\kappa \in \mathbb{N}$ and $t \in [0, 1]$, define

$$\psi_{\kappa}(t) = \int_0^t \gamma_\kappa(s) \, ds + c^d,$$

where

$$\gamma_\kappa(t) = \kappa \int_{i/\kappa}^{(i+1)/\kappa} \varphi_{\theta}(s) \, ds$$

for $t \in [i/\kappa, (i + 1)/\kappa)$, $0 \leq i \leq \kappa - 1$, and $\gamma_\kappa(1) = \gamma_\kappa(1 - 1/\kappa)$. Note that $\psi_\kappa \in \Gamma_d$, and on $[i/\kappa, (i + 1)/\kappa)$ for $0 \leq i \leq \kappa - 1$, $\dot{\psi}_\kappa(t)$ equals the constant vector $\gamma_\kappa(i/\kappa)$. In particular, $\dot{\psi}_\kappa$ is a step function.

Lemma 1.2.3. For $0 \leq i \leq d + 1$,

$$\psi_{\kappa,i}(t) \geq \theta(c_i t + c_i),$$

$$\sum_{i=0}^{d} (1 - [\psi_{\kappa}(t)]_i) \leq 1 - \theta c_{d+1}.$$
Lastly, (1.28) follows as, noting that \( \sum_{i=0}^{d} (1 - [e]_i) = \sum_{i=0}^{d+1} i e_i = 1 - c_{d+1} \),

\[
\sum_{i=0}^{d} (1 - [\psi_\kappa(t)]_i) = (1 - \theta) \sum_{i=0}^{d} (1 - [\varphi^*(t)]_i) + \theta \sum_{i=0}^{d} (1 - [\ell(t)]_i),
\]

\[
\leq 1 - \theta + \theta \sum_{i=0}^{d} (1 - [e]_i) = 1 - \theta e_{d+1}.
\]

Lemma 1.2.4. For large enough \( \kappa \), we have

\[
h(\psi_\kappa) \leq h(\varphi^*) + 2\varepsilon, \quad \text{and} \quad I(\psi_\kappa) \leq I_d(\varphi^*) + 2\varepsilon.
\] (1.29)

Proof. Since

\[
\lim_{\kappa \to \infty} \sup_{t \in [0,1]} |\varphi_\theta(t) - \psi_\kappa(t)| = 0,
\]

the inequality with respect to \( h \) follows from continuity of \( h \) and choosing \( \kappa \) in terms of \( \theta \). We also note that a.s. in \( t \),

\[
\dot{\psi}_\kappa(t) = \gamma_\kappa(t) = \kappa \int_{[\tau \kappa]/\kappa}^{([\tau \kappa] + 1)/\kappa} \dot{\varphi}_\theta(s) \, ds \to \varphi_\theta(t) \text{ as } \kappa \uparrow \infty.
\]

Then, by the form of \( L \) (cf. (1.23)), bounds in Lemma 1.2.3, and assumption (ND), we have, as \( \kappa \uparrow \infty \), that \( L(\xi(t), \psi_\kappa(t), \dot{\psi}_\kappa(t)) \to L(\xi(t), \varphi_\theta(t), \dot{\varphi}_\theta(t)) \) for almost all \( t \in [0,1] \).

Also, we can dominate \( L(\xi(t), \psi_\kappa(t), \dot{\psi}_\kappa(t)) \) as follows: First bound, using \( x \log x \leq 0 \) for \( 0 \leq x \leq 1 \), that

\[
L(\xi(t), \psi_\kappa(t), \dot{\psi}_\kappa(t)) \leq (1 - [\psi_\kappa(t)]_0) \log \left( p(t) + (1 - p(t)) \frac{\beta(t) \psi_{\kappa,0}(t)}{(1 + \beta(t)) t + \bar{c} + c\beta(t)} \right) - \sum_{i=1}^{d} (1 - [\psi_\kappa(t)]_i) \log \left( (1 - p(t)) \frac{(i + \beta(t)) \psi_{\kappa,i}(t)}{(1 + \beta(t)) t + \bar{c} + c\beta(t)} \right) - \left( 1 - \sum_{i=0}^{d} (1 - [\psi_\kappa(t)]_i) \right) \log \left( (1 - p(t))(1 - \sum_{i=0}^{d} (i + \beta(t)) \psi_{\kappa,i}(t)) \frac{(1 + \beta(t)) t + \bar{c} + c\beta(t)}{(1 + \beta(t)) t + \bar{c} + c\beta(t)} \right).
\]

Now, as \( 0 \leq [\psi_\kappa]_i \leq 1 \) and \( 0 \leq \sum_{i=0}^{d} (1 - [\psi_\kappa]_i) \leq 1 \), we have the further upperbound, using
\[(1.27), \]
\[- \log \left( p(t) + (1 - p(t)) \frac{\beta(t) \theta(e_0 t + c_0)}{(1 + \beta(t)) t + \bar{c} + c \beta(t)} \right) \]
\[- \sum_{i=1}^{d} \log \left( (1 - p(t)) \frac{(i + \beta(t)) \theta(e_i t + c_i)}{(1 + \beta(t)) t + \bar{c} + c \beta(t)} \right) \]
\[- \log \left( (1 - p(t)) \frac{(d + 1 + \beta(t)) \theta(e_{d+1} t + \bar{c}^d)}{(1 + \beta(t)) t + \bar{c} + c \beta(t)} \right) \]
\[\leq - \sum_{i=0}^{d} \log \left( (1 - p(t)) \frac{i + \beta(t)}{1 + \beta(t)} \cdot \frac{\theta(e_i t + c_i)}{i + \max\{\bar{c}, c\}} \right) \]
\[- \log \left( (1 - p(t)) \frac{d + 1 + \beta(t)}{1 + \beta(t)} \cdot \frac{\theta(e_{d+1} t + \bar{c}^d)}{i + \max\{\bar{c}, c\}} \right), \]

which is integrable on \([0, 1]\) under assumption (ND).

By dominated convergence, we obtain \(\lim_{\kappa} I(\psi_\kappa) = I(\varphi_\theta)\), and therefore the other inequality with respect to \(I_d\).

Let now \(\kappa\) be such that (1.29) holds. Finally, we modify \(\psi_\kappa\) on the interval \([0, \delta]\), for a small enough \(\delta > 0\) to be chosen later.

Define \(t_i := \delta - \sum_{l=i}^{d} (\delta + |c_l| t - [\psi_\kappa(t)])\) for \(0 \leq i \leq d\), and \(t_{d+1} := \delta\); set also \(t_{-1} := 0\). Let also
\[\psi^*(t) = \int_0^t \gamma^*(s) \, ds + c^d \quad (1.30)\]
where
\[\gamma^*(t) = \begin{cases} 
    f_{d+1}, & \text{when } 0 \leq t < t_0, \\
    f_i, & \text{when } t_i \leq t < t_{i+1}, \ 0 \leq i \leq d, \\
    \gamma_\kappa(t), & \text{when } t \geq \delta.
\end{cases}\]

Note that \(\gamma^*\) may not be defined at some endpoints as possibly \(t_i = t_{i+1}\) for some \(i\).

By inspection, \(\psi^* \in \Gamma_d\), and \(\psi_0^*(t) = t + c_0\) when \(0 \leq t \leq t_0\). Also, \(\dot{\psi}^*(t) = f_{d+1} \) when \(0 \leq t < t_0\) and \(\dot{\psi}^*(t) = f_i \) when \(t_i \leq t < t_{i+1}\) for \(0 \leq i \leq d\). Moreover, we have the following properties.
Lemma 1.2.5. We have $\psi^*(\delta) = \psi_\kappa(\delta)$ and $t_0 \geq \theta e_{d+1}\delta$. Also,

$$
\psi_0^*(t) = t + c_0 \quad \text{and} \quad \psi_j^*(t) = c_j \quad \text{for} \quad 1 \leq j \leq d + 1 \quad \text{when} \quad 0 \leq t < t_0,
$$

$$
\psi_0^*(t) \geq \theta e_{d+1}\delta + c_0 \quad \text{when} \quad t_0 < t < t_1,
$$

$$
\psi_i^*(t) \geq \theta(e_i\delta + c_i) \quad \text{when} \quad t_i < t < t_{i+1} \quad \text{and} \quad 1 \leq i \leq d,
$$

Proof. The lower bound for $t_0$ follows from the integration of both sides in (1.28) and the definition of $t_0$. Now, we note that $\dot{\psi}_0^*(t) = 0$ if $t_0 \leq t \leq t_1$, and 1 otherwise. Also, note that for $1 \leq i \leq d$, $\dot{\psi}_i^*(t) = 1$ if $t_{i-1} < t < t_i$, $\dot{\psi}_i^*(t) = -1$ if $t_i < t < t_{i+1}$, and $\dot{\psi}_i^*(t) = 0$ otherwise.

Thus

$$
\psi_0^*(\delta) = \int_0^\delta \gamma_0^*(s) \, ds + c_0 = \delta - (t_1 - t_0) + c_0 = \psi_\kappa,0(\delta),
$$

and for $1 \leq i \leq d$,

$$
\psi_i^*(\delta) = \int_0^\delta \gamma_i^*(s) \, ds + c_i = (t_i - t_{i-1}) - (t_{i+1} - t_i) + c_i = \psi_\kappa,i(\delta),
$$

which proves that $\psi^*(\delta) = \psi_\kappa(\delta)$. Since $\psi_0^*(t)$ is nondecreasing, for $t \geq t_0$, $\psi_0^*(t) \geq \psi_0^*(t_0) = t_0 + c_0 \geq \theta e_{d+1}\delta + c_i$. For $1 \leq i \leq d$, for $t_i < t < t_{i+1}$, $\psi_i^*(t)$ decreases to its final value $\psi_\kappa,i(\delta) \geq \theta(e_i\delta + c_i)$ by (1.27).

\[\square\]

Step 2. More properties of $\psi^*$. We now show the rate of $\psi^*$ up to time $\delta$ does not contribute too much.

Lemma 1.2.6. For small enough $\delta > 0$,

$$
\int_0^\delta L(\bar{\xi}(t), \psi^*(t), \dot{\psi}^*(t)) \, dt \leq \varepsilon \quad \text{and} \quad \|\psi^* - \psi_\kappa\|_\infty < \varepsilon.
$$

In particular, $h(\psi^*) < h(\varphi^*) + 3\varepsilon$ and $I_d(\psi^*) < I_d(\varphi^*) + 3\varepsilon$. 

Proof. Write, for $0 \leq t \leq \delta$,

\[
L(\bar{\xi}(t), \psi^*(t), \dot{\psi}^*(t)) = R(\delta_{t+1}, ||\rho_{\bar{\xi}(t), \psi^*(t)}||)1(0 < t < t_0) + \sum_{i=0}^{d} R(\delta_i, ||\rho_{\bar{\xi}(t), \psi^*(t)}||)1(t_i < t < t_{i+1})
\]

\[
= - \log \left( (1 - p(t)) \left(1 - \sum_{i=0}^{d} (l + \beta(t)) \psi_i^*(t) \right) \right)1(0 < t < t_0)
\]

\[
- \log \left( p(t) + (1 - p(t)) \left( \beta(t) \psi_0^*(t) \right) \right)1(t_0 < t < t_1)
\]

\[
- \sum_{i=1}^{d} \log \left( (1 - p(t)) \left( i + \beta(t) \right) \psi_i^*(t) \right)1(t_i < t < t_{i+1}).
\]

By Lemma 1.2.5 and the assumption (ND), this expression is integrable for $0 \leq t \leq \delta$. [It would be bounded unless $c^d = 0$ and $c \neq 0$, in which case the first term in the expression involves $- \log t$.] Hence, the first statement follows for small enough $\delta > 0$. Also, the second statement holds as $\|\psi^* - \psi_\kappa\|_\infty = \sup_{0 \leq t < \delta} |\psi^* - \psi_\kappa| \leq 2\delta(d + 2)$. The last statement is a consequence now of (1.29).

We will take $\delta > 0$ small enough so that the bounds in the above lemma hold.

**Lemma 1.2.7.** We have

\[
\lim_{n \to \infty} \sup_{0 \leq j \leq n} \left| \psi^*(j/n) - \frac{1}{n} \sum_{l=0}^{j-1} \dot{\psi}^*(l/n) - c^d \right| = 0. \tag{1.31}
\]

Also, for $j \geq \lfloor \delta n \rfloor$ and $0 \leq i \leq d + 1$,

\[
\frac{1}{n} \sum_{i=0}^{j-1} \dot{\psi}_i^*(l/n) + c_i \geq \frac{\theta}{2} \left( \frac{e_i j}{n} + c_i \right), \tag{1.32}
\]

Proof. Since $\dot{\psi}^*$ is piecewise constant, $|\dot{\psi}^*(s) - \dot{\psi}^*(l/n)| \neq 0$ for at most $\kappa$ subintervals (cf. (1.26) and (1.30)), and is also bounded by $|2 \cdot 1| = 2(d + 2)$. Hence,

\[
\left| \psi^*(j/n) - \frac{1}{n} \sum_{l=0}^{j-1} \dot{\psi}^*(l/n) - c^d \right| = \left| \sum_{l=0}^{j-1} \int_{l/n}^{(l+1)/n} \left( \dot{\psi}^*(s) - \dot{\psi}^*(l/n) \right) ds \right| \leq \frac{2(d + 2)}{n} \kappa.
\]

The last statement follows from (1.27).
Step 3. Admissible control measures and convergence. We now build a sequence of controls based on $\psi^*$. Define $\nu_0 = \nu_0(\psi^*(j/n)|j/n)$ using (1.21). Define

$$v^*_j(dy; x_0, \ldots, x_j) = \begin{cases} 
\nu_0(\psi^*(j/n)|j/n) & \text{when } 0 \leq j \leq [\delta n] \\
\rho_{\xi_j/n}, x_j & \text{or when } j \geq [\delta n] \text{ and } x_{j,i} \geq \frac{\theta}{4}(e_i \delta + c_i) \text{ for } 0 \leq i \leq d + 1,
\end{cases}$$

Define also $\bar{X}^n_0 = c^d$, and $\bar{X}^n_{j+1} = \bar{X}^n_j + \frac{1}{n} \bar{Y}^n_j$ for $j \geq 0$ where

$$\bar{P}(\bar{Y}^n_j \in dy|\bar{X}^n_0, \ldots, \bar{X}^n_j) = v^*_j(dy; \bar{X}^n_0, \ldots, \bar{X}^n_j).$$

Thus, for $j \geq 0$, $\bar{X}^n_j = \frac{1}{n} \sum_{l=0}^{j-1} \bar{Y}^n_l + c^d$. In particular, for $0 \leq j \leq [\delta n]$, $\bar{X}^n_j$ is deterministic and $\bar{X}^n_j = \frac{1}{n} \sum_{l=0}^{j-1} \psi^*(l/n) + c^d$.

Define now, for each $n \geq 1$, that

$$M^n_j := \frac{1}{n} \sum_{l=0}^{j-1} (\bar{Y}^n_l - \bar{E} \left( \bar{Y}^n_l | \bar{X}^n_l \right)) = \bar{X}^n_j - \frac{1}{n} \sum_{l=0}^{j-1} \bar{E} \left( \bar{Y}^n_l | \bar{X}^n_l \right) - c^d \tag{1.33}$$

is a martingale sequence for $0 \leq j \leq n$. Let

$$\tau_n := n \wedge \min \left\{ [\delta n] \leq l \leq n : \bar{X}^n_{l,i} < \frac{\theta}{4}(e_i \delta + c_i) \text{ for some } i \right\}.$$

Then, $\tau_n$ is a stopping time and the corresponding stopped process $\{M^n_{j \wedge \tau_n}\}$ is also a martingale for $0 \leq j \leq n$.

Let now

$$\mathcal{A}_n := \left\{ \sup_{0 \leq j \leq n} |M^n_{j \wedge \tau_n}| > \frac{\theta e_{d+1}}{4n^{1/8}} \right\}.$$

Lemma 1.2.8. For $n \geq \delta^{-8}$, on the set $\mathcal{A}^c_n$, we have $\tau_n = n$.

Proof. For $j \geq [\delta n]$, from the definition of $\{v^*_j\}$ and $\tau_n$, and (1.32), we have

$$\bar{X}^n_{j \wedge \tau_n,i} \geq c_i + \frac{1}{n} \sum_{l=0}^{j \wedge \tau_n-1} E(\bar{Y}^n_l | \bar{X}^n_l) - \frac{\theta e_{d+1}}{4n^{1/8}}$$

$$= c_i + \frac{1}{n} \sum_{l=0}^{j \wedge \tau_n-1} \psi^*(l/n) - \frac{\theta e_{d+1}}{4n^{1/8}} \geq \frac{\theta}{2} \left( \frac{e_i (j \wedge \tau_n)}{n} + c_i \right) - \frac{\theta e_{d+1}}{4n^{1/8}} \geq \frac{\theta}{4}(e_i \delta + c_i).$$

Hence, $\tau_n = n$. \qed
We now observe, by Doob’s martingale inequality and bounds, in terms of constants $C = C_d$, that

$$
P[\mathbb{A}_n] \leq C n^{1/2} E \left| M_{j \wedge \tau_n}^n \right|^4 = C n^{-7/2} E \left| \sum_{l=0}^{j \wedge \tau_n - 1} (\bar{Y}_l^n - \bar{Y}_l^n | X_l^n) \right|^4 \leq C n^{-7/2} n^2 = C n^{-3/2}. \quad (1.34)$$

We now state the following almost sure convergence.

**Lemma 1.2.9.**

$$\lim_{n \to \infty} \sup_{0 \leq j \leq n} \left| \bar{X}_n^j - \frac{1}{n} \sum_{l=0}^{j-1} \psi^\ast(l/n) - c \right| = 0 \quad \text{a.s.} \quad (1.35)$$

**Proof.** First, by (1.34) and Borel-Cantelli lemma, $P(\lim \sup \mathbb{A}_n) = 0$. On the other hand, on the full measure set $\cup_{n \geq 1} \cap_{k \geq n} \mathbb{A}_k^c$, since $\tau_n = n$ on $\mathbb{A}_n^c$ by Lemma 1.2.8, the desired convergence holds. \qed

**Step 4.** We now argue the lower bound through representation (1.16). The sum in (1.16) equals

$$
\tilde{E}\left[ \frac{1}{n} \sum_{j=0}^{n-1} R(v^n_j || \rho_{\xi(j/n)}^j, X_j^n) \right] = \tilde{E}\left[ \frac{1}{n} \sum_{j=0}^{\lfloor \delta n \rfloor} R(v^n_j || \rho_{\xi(j/n)}^j, X_j^n) \right] + \tilde{E}\left[ \frac{1}{n} \sum_{j=\lceil \delta n \rceil}^{n-1} R(v^n_j || \rho_{\xi(j/n)}^j, X_j^n) \right] \\
+ \tilde{E}\left[ \frac{1}{n} \sum_{j=\lceil \delta n \rceil}^{n-1} R(v^n_j || \rho_{\xi(j/n)}^j, X_j^n) \right] = A_1 + A_2 + A_3. \quad (1.36)
$$

**Step 4.1** We first treat the second term $A_2$ in (1.36). Recall, for $j \geq 0$, that

$$
\sigma(j/n) = (1 + \beta(j/n)) \frac{j}{n} + \tilde{c} + \beta(j/n). \quad (1.37)
$$
For $j \geq \lceil \delta n \rceil$,

\[
R(v^n_j || \rho \xi_{(j/n)} \bar{X}_j^n) = R(v_0(\psi^*(j/n)) || \rho \xi_{(j/n)} \bar{X}^n_{j,d}) (\bar{X}^n_{j,i} \geq (\theta/4)(\epsilon_i \delta + c_i) \text{ for } 0 \leq i \leq d + 1).
\]

Noting (1.23), this is bounded above, using $x \log x \leq 0$ for $0 \leq x \leq 1$, by

\[
\left[ - \left( 1 - \left[ \psi^* \left( \frac{j}{n} \right) \right]_0 \right) \log \left( \frac{(1 - p(j/n)) \beta(j/n) \bar{X}^n_{j,0}}{\sigma(j/n)} \right) - \sum_{i=1}^{d} \left( 1 - \left[ \psi^* \left( \frac{j}{n} \right) \right]_i \right) \log \left( \frac{(1 - p(j/n)) \bar{X}^n_{j,i}}{\sigma(j/n)} \right) - \left( \sum_{i=0}^{d} \left[ \psi^* \left( \frac{j}{n} \right) \right]_i - d \right) \log \left( (1 - p(j/n)) \left( 1 - \frac{\sum_{i=0}^{d} (i + \beta(j/n)) \bar{X}^n_{j,i}}{\sigma(j/n)} \right) \right) \times 1(\bar{X}^n_{j,i} \geq (\theta/4)(\epsilon_i \delta + c_i) \text{ for } 0 \leq i \leq d + 1) \right]
\]

Further, given assumption (ND), as $0 \leq \left[ \psi^* \right]_i \leq 1$, $d \leq \sum_{i=1}^{d} \left[ \psi^* \right]_i \leq d + 1$ and

\[
\sum_{i=0}^{d} (i + \beta(j/n)) \bar{X}^n_{j,i} \leq \sigma(j/n) - (d + 1 + \beta(j/n)) \bar{X}^n_{j,d+1} \leq \sigma(j/n) - (d + 1 + \beta(j/n)) \cdot (\theta/4)(\epsilon_{d+1} \delta + c_{d+1}),
\]

this expression is bounded by a constant $C_d$. Thus, we obtain, for large $n$,

\[
A_2 = \bar{E} \left[ \frac{1}{n} \sum_{j=1}^{n} R(v^n_j \parallel \rho \xi_{(j/n)} \bar{X}_j^n) ; \bar{A}_n \right] \leq C_d \cdot \bar{P} \left[ \sup_{0 \leq j \leq n} |M^n_{j \land \tau_n}| > \frac{\theta \epsilon_{d+1}}{4n^{1/8}} \right] < \varepsilon. \hspace{1cm} (1.38)
\]

**Step 4.2.** Now, for the first term $A_1$ in (1.37), we recall for $j \leq \lfloor \delta n \rfloor$ that $\bar{X}^n = \frac{1}{n} \sum_{l=0}^{j-1} \psi^*(l/n) + c^d$ is deterministic. Also note, for $0 \leq i \leq d$, that $\psi^*(t) = f_i$ on $t_i < t < t_{i+1}$, and $\psi^*(t) = f_{d+1}$ on $0 = t_{-1} \leq t \leq t_0$ (cf. near Lemma 1.2.5). Thus, for $0 \leq j \leq \lfloor \delta n \rfloor$, denoting $f_{-1} = f_{d+1}$, we
may write, as in the proof of Lemma 1.2.6,

\[
R(v^n_j\|\bar{\xi}_j^{n/j/n})(\bar{\bar{X}}_n^s) = \mathcal{L}\left(\bar{\xi}(\frac{j}{n}), \frac{1}{n}\sum_{t=0}^{j-1} \psi^*(\frac{t}{n}) + \epsilon^d, \psi^*(\frac{j}{n})\right)
\]

\[
= -\log\left((1 - p(j/n))(1 - \frac{\sum_{i=0}^{d} i + 2\beta(j/n)}{\sigma(j/n)} \frac{\sum_{m=0}^{j-1} \psi^*(m/n) + \epsilon^d}{\sigma(j/n)})\right) 1(0 < j < [t_0n])
\]

\[
- \log\left((p(j/n) + (1 - p(j/n)) \frac{\beta(j/n)}{\sigma(j/n)} (\frac{1}{n}\sum_{m=0}^{j-1} \psi^*(m/n) + \epsilon^d)\right) 1([t_0n] < j < [t_1n])
\]

\[
- \sum_{i=1}^{d} \log\left((1 - p(j/n)) (i + 2\beta(j/n)) \frac{1}{\sigma(j/n)} (\frac{1}{n}\sum_{m=0}^{j-1} \psi^*(m/n) + \epsilon^d)\right) 1([t_1n] < j < [t_{i+1}n])
\]

This expression is summable for 0 \leq j \leq \lfloor \delta n \rfloor - the only possible unbounded term is of the form \(-1/n \sum_{j=1}^{\lfloor \delta n \rfloor} \log(j/n) \leq \int_{0}^{\delta} \log(t)dt. [Again, the expression is bounded unless \epsilon^d = 0 and \epsilon \neq 0.] Hence,

\[
A_1 = \mathbb{E}\left[\frac{1}{n} \sum_{j=0}^{\lfloor \delta n \rfloor} R(v^n_j\|\bar{\xi}_j^{n/j/n})(\bar{\bar{X}}_n^s)\right] < \epsilon(\delta), \quad (1.39)
\]

where \epsilon(\delta) \to 0 as \delta \to 0.

**Step 4.3.** We now estimate the last term A_3 in (1.36). For \(n \geq \delta^{-8}\), by Lemma 1.2.8,

\[
A_3 \leq \mathbb{E}\left[\frac{1}{n} \sum_{j=\lfloor \delta n \rfloor}^{n-1} R(v^n_j\|\bar{\xi}_j^{n/j/n})(\bar{\bar{X}}_n^s) ; A_n^c \cap \{\tau_n = n\}\right]
\]

\[
= \mathbb{E}\left[\frac{1}{n} \sum_{j=\lfloor \delta n \rfloor}^{n-1} L(\bar{\xi}(\frac{j}{n}), \bar{\bar{X}}_n^s, \psi^*(\frac{j}{n}) ; A_n^c \cap \{\tau_n = n\}\right]
\]

\[
\leq \mathbb{E}\left[\frac{1}{n} \int_{\delta}^{1} L(\bar{\xi}(\frac{nt}{n}), \bar{\bar{X}}_{nt}^s, \psi^*(\frac{nt}{n}) ; A_n^c \cap B_n]\right]
\]

where \(B_n = \{\bar{\bar{X}}_{j+n}^s \geq (\theta/4)(c_i \delta + c_i)\} \text{ for } 0 \leq i \leq d + 1, j \geq \lfloor \delta n \rfloor\}. \text{ On the event } A_n^c \cap B_n, \text{ we know that } \mathbb{E}(Y^n_l\|\bar{\bar{X}}_n^s) = \psi^*(l/n) \text{ for } l \geq 0, \text{ and so } |\bar{\bar{X}}_{nt}^s - \psi^*(\frac{nt}{n})| \to 0 \text{ from (1.31)}.

Also, from the form of \(L(1.23)\), and bounds and piecewise continuity of \(p\) and \(\beta\) given in assumption (ND), on this event, \(L\) is dominated by a constant \(C_d\) as in Step 4.1, and converges to \(L(\bar{\xi}(t), \psi^*(t), \psi^*(t))\) for almost all \(t\). Hence, by bounded convergence theorem,

\[
\limsup_{n \to \infty} A_3 \leq \int_{\delta}^{1} L(\bar{\xi}(t), \psi^*(t), \psi^*(t))dt. \quad (1.40)
\]
Step 5. Finally, by (1.31) and (1.35), in the sup topology, \( \lim_{n \to \infty} h(\overline{X}^n) = h(\psi^*(\cdot)) \).

We now combine all bounds to conclude the proof of (1.24). By (1.16), bounds (1.39), (1.38), (1.40), and nonnegativity of \( L \), we have

\[
\limsup_{n \to \infty} V^n \leq \limsup_{n \to \infty} E\left[\frac{1}{n} \sum_{j=0}^{n-1} R(\psi^n_j, [\rho_{\overline{X}^n}(n,j/n), \overline{X}^n_j]) + h(\overline{X}^n)\right]
\]

\[
\leq 2 \varepsilon + \int_0^1 L(\tilde{\xi}(t), \psi^*(t), \dot{\psi}^*(t))\, dt + h(\psi^*).
\]

Then, by Lemma 1.2.6, we obtain (1.24). \( \square \)

1.3 Proof of Theorem 1.1.2

The proof of Theorem 1.1.2 follows from the following two propositions, and is given below.

We first recall the projective limit approach, following notation in [14, Section 4.6]. Let \( J = \mathbb{N}, \mathcal{Y}_j = C([0,1]; \mathbb{R}^{d+2}) \), and define, for \( 0 \leq i \leq j \), \( p_{ij} : C([0,1]; \mathbb{R}^{j+2}) \to C([0,1]; \mathbb{R}^{i+2}) \) by \( \langle \phi_0, \ldots, \phi_{j+1} \rangle \mapsto \langle \phi_0, \ldots, \phi_i, \sum_{l=i+1}^{j+1} \phi_l \rangle \). Also define \( \varprojlim_{j} \mathcal{Y}_j \subset \prod_{i \geq 0} \mathcal{Y}_i \) as the subset of elements \( x = \langle x^0, x^1, \ldots \rangle \) such that \( p_{ij}x^j = x^i \), equipped with the product topology. Let also \( p_j : \varprojlim_{j} \mathcal{Y}_j \to \mathcal{Y}_j \) be the canonical projection, \( p_jx = x^j \).

Since \( I_d \) are convex, good rate functions on \( C([0,1]; \mathbb{R}^{d+2}) \), by the LDP’s Theorem 1.1.1 and [14, Theorem 4.6.1], we obtain the following proposition. Recall the notation in Theorem 1.1.1. For \( n \geq 1 \), let \( X^{n,\infty} = \langle X^{n,0}, X^{n,1}, \ldots, X^{n,d}, \ldots \rangle \).

Proposition 1.3.1. Under assumption (ND), the sequence \( \{X^{n,\infty}\} \subset \varprojlim_{j} \mathcal{Y}_j \) satisfies an LDP with rate \( n \) and convex, good rate function \( J^{\infty} \),

\[
J^{\infty}(\varphi) = \sup_{d} \{I_d(p_d(\varphi))\}.
\]

To establish Theorem 1.1.2, it remains to identify more \( J^{\infty} \), which is done in the next proposition. Recall \( \Gamma_d \subset C([0,1]; \mathbb{R}^{d+2}) \) are those elements \( \varphi = \langle \varphi_0, \ldots, \varphi_d, \varphi_{d+1} \rangle \) such that
\( \varphi(0) = c^d \), each \( \varphi_i \geq 0 \) is Lipschitz with constant 1 such that \( 0 \leq [\dot{\varphi}(t)]_i \leq 1 \) for \( 0 \leq i \leq d \), 
\[ \sum_{i=0}^{d+1} \dot{\varphi}_i(t) = 1, \text{ and } \sum_{i=0}^{d+1} i \dot{\varphi}_i(t) \leq 1 \text{ for almost all } t. \]

Let also \( \Gamma^* \subset \lim_{t \to \infty} \mathcal{Y}_j \) be those elements \( \varphi = (\varphi^0, \varphi^1, \ldots) \) such that 
\[ \varphi^d \in \Gamma_d \text{ for } d \geq 0, \text{ and } \lim_{t \to \infty} \dot{\varphi}^d_{d+1}(t) = 0 \text{ (or } \lim_{t \to \infty} [\dot{\varphi}^d(t)]_d = 1) \text{ for almost all } t. \]

Define 
\[ L_d(p_d(\varphi(t))) = (1 - [\dot{\varphi}^d(t)]_0) \log \frac{1 - [\dot{\varphi}^d(t)]_0}{p(t) + (1 - p(t))} \frac{\beta(t) \varphi^d_0(t)}{(1 + \beta(t)) + \beta(t)} \]
\[ + \sum_{i=1}^{d} (1 - [\dot{\varphi}^d(t)]_i) \log \frac{1 - [\dot{\varphi}^d(t)]_i}{p(t)} \frac{\beta(t) \varphi^d_i(t)}{(1 + \beta(t)) + \beta(t)} \]
\[ + \sum_{i=0}^{d} (1 - [\dot{\varphi}^d(t)]_i) \log \frac{1 - \sum_{i=0}^{d} (1 - [\dot{\varphi}^d(t)]_i)}{(1 - p(t))} \frac{1}{1 - \sum_{i=0}^{d} (1 + \beta(t)) \varphi^d_i(t)} \frac{\beta(t)}{(1 + \beta(t)) + \beta(t)} \] 

**Proposition 1.3.2.** Under assumption (ND), the rate function \( J^\infty(\varphi) \) diverges when \( \varphi \not\in \Gamma^* \).

However, for \( \varphi \in \Gamma^* \) and almost all \( t \), \( \lim_{t \to \infty} L_d(p_d(\varphi(t))) \) exists, and we can evaluate 
\[ J^\infty(\varphi) = \int_0^1 \lim_{t \to \infty} L_d(p_d(\varphi(t))) \, dt. \]

**Proof.** First, \( J^\infty(\varphi) \) diverges unless \( \varphi \in \prod_{d \geq 0} \Gamma_d \). However, for \( \varphi \in \prod_{d \geq 0} \Gamma_d \) such that \( J^\infty(\varphi) < \infty \), since \( \sup_d \sum_{i=0}^{d+1} i \dot{\varphi}^d_i(t) \leq 1 \text{ almost all } t \), we must have \( \lim_{t \to \infty} \sum_{i=0}^{d} (1 - [\dot{\varphi}^d(t)]_i) = \lim_{t \to \infty} \sum_{i=0}^{d+1} i \dot{\varphi}^d_i(t) < \infty \). Hence, \( \lim_{t \to \infty} 1 - [\dot{\varphi}^d(t)]_d = 0 \text{ almost all } t, \text{ and } \varphi \in \Gamma^* \).

Now, for \( \varphi \in \Gamma^* \) and almost all \( t \), we argue 
\[ L_r(p_r(\varphi(t))) \leq L_s(p_s(\varphi(t))) \text{ when } r < s. \quad (1.41) \]

It will be enough to show from the form of the rates, the following: 
\[ \left( 1 - \sum_{i=0}^{r} (1 - [\dot{\varphi}^s(t)]_i) \right) \log \frac{1 - \sum_{i=0}^{r} (1 - [\dot{\varphi}^s(t)]_i)}{(1 - p(t))} \frac{1}{1 - \sum_{i=0}^{r} (1 + \beta(t)) \varphi^s_i(t)} \frac{\beta(t)}{(1 + \beta(t)) + \beta(t)} \]
\[ \leq \sum_{i=r+1}^{s} (1 - [\dot{\varphi}^s(t)]_i) \log \frac{1 - [\dot{\varphi}^s(t)]_i}{(1 - p(t))} \frac{1 + \beta(t)}{(1 + \beta(t)) + \beta(t)} \]
\[ + \sum_{i=0}^{s} (1 - [\dot{\varphi}^s(t)]_i) \log \frac{1 - \sum_{i=0}^{s} (1 - [\dot{\varphi}^s(t)]_i)}{(1 - p(t))} \frac{1}{1 - \sum_{i=0}^{s} (1 + \beta(t)) \varphi^s_i(t)} \frac{\beta(t)}{(1 + \beta(t)) + \beta(t)}. \]
Consider now $h(x) = x \log x$ which is convex for $x \geq 0$. Under conventions (1.7), for non-negative numbers, $a_i$ and $b_i$, we have

$$\sum_{i=p}^q a_i \log \frac{\sum_{i=p}^q a_i}{\sum_{i=p}^q b_i} = h\left(\frac{\sum_{i=p}^q a_i}{\sum_{i=p}^q b_i}\right) = h\left(\frac{\sum_{i=p}^q b_i}{\sum_{i=p}^q b_i}\right) \leq \sum_{i=p}^q b_i \frac{a_i}{b_i} h\left(\frac{a_i}{b_i}\right) = \frac{\sum_{i=p}^q a_i \log(a_i/b_i)}{\sum_{i=p}^q b_i}.$$

We now finish the proof of (1.41) by applying the last sequence, with $p = r + 1$ and $q = s + 1$, to

$$a_j = \begin{cases} 1 - [\varphi^t(t)]_j & \text{for } r + 1 \leq j \leq s \\ 1 - \sum_{i=0}^s 1 - [\varphi^t(t)]_i & \text{for } j = s + 1 \end{cases}$$

$$b_j = \begin{cases} (1 - p(t)) \frac{(j+\beta(t))\varphi^t(t)}{(1+\beta(t))(t+\epsilon+c\beta(t))} & \text{for } r + 1 \leq j \leq s \\ (1 - p(t)) \left(1 - \sum_{i=0}^s \frac{(j+\beta(t))\varphi^t(t)}{(1+\beta(t))(t+\epsilon+c\beta(t))}\right) & \text{for } j = s + 1. \end{cases}$$

Finally, given $L_d(p_d(\varphi(t))) \geq 0$ is increasing in $d$, the identification of $J^\infty$ in the display of the proposition follows from monotone convergence.

We now give the proof of Theorem 1.1.2.

**Proof of Theorem 1.1.2.** Let $\Gamma^\infty \subset \prod_{i \geq 0} C([0,1];\mathbb{R})$, endowed with the product topology, be those elements $\xi = \langle \xi_0, \xi_1, \ldots \rangle$ such that

$$\xi_i(0) = c_i, \xi_i(t) \geq 0 \text{ is Lipschitz with constant } 1, 0 \leq [\xi(t)]_i \leq 1 \text{ for } i \geq 0, \text{ and } \sum_{i \geq 0} \dot{\xi}_i(t) = 1 \text{ and } \sum_{i \geq 0} i \dot{\xi}_i(t) \leq 1 \text{ for almost all } t.$$

The spaces $\Gamma^\infty$ and $\Gamma^s$ are homeomorphic. Define $F : \Gamma^\infty \to \Gamma^s$ by $F(\xi) = \langle \xi^0, \ldots, \xi^d, \ldots \rangle$ where $\xi^d = \langle \xi_0, \ldots, \xi_d, \sum_{l=d+1}^\infty \xi_l \rangle$. It is not difficult to see that $F$ is a bi-continuous bijection.

Now, $X^{n,\infty} \in \Gamma^\infty$ for $n \geq 1$. Hence, through the action of $F$, the LDP for $X^{n,\infty}$ translates to the LDP for $X_X^{n,\infty}$. We now identify the rate function. Given Propositions 1.3.1 and 1.3.2, for a degree distribution $\xi \in \Gamma^\infty$, we identify its rate as $I^\infty(\xi) = J^\infty(F(\xi))$. Since distributions $\xi \notin \Gamma^\infty$ can never be attained, we set $I^\infty(\xi) = \infty$ in this case.
1.4 Proof of Theorem 1.1.3

Let \( d < \infty \) be fixed. We first give some properties of \( \zeta^d(\cdot) = \langle \zeta_0(\cdot), \zeta_1(\cdot), \ldots, \zeta_{d+1}(\cdot) \rangle \) (cf. (1.11)).

**Lemma 1.4.1** (The zero-cost trajectory). We have \( \zeta^d \in \Gamma_d \). In particular,

\[
\sum_{i=0}^{d} \zeta_i(t) + \tilde{\zeta}_{d+1}(t) = t + \zeta, \quad \sum_{i=0}^{d} i\zeta_i(t) + (d+1)\tilde{\zeta}_{d+1} \leq t + \tilde{\zeta}, \tag{1.42}
\]

and \( I_d(\zeta^d) = 0 \). Moreover \( \zeta^d \) is the unique path with zero cost.

**Proof.** From the definition of \( \zeta^d \) and simple computations, we get (1.42).

For the uniqueness, suppose \( \zeta^{(1)}, \zeta^{(2)} \) are solutions to the system of ODEs (1.10) with initial condition \( \zeta^{(0)}(0) = \mathbf{c}^d \) also satisfying (1.42). We show \( \zeta^{(1)} = \zeta^{(2)} \) using induction on \( 0 \leq i \leq d \).

For \( i = 0 \), suppose \( \zeta^{(1)}_0 \neq \zeta^{(2)}_0 \). Then, by continuity of \( \zeta^{(1)}_0 \) and \( \zeta^{(2)}_0 \), we may assume that \( \zeta^{(1)}_0(t) > \zeta^{(2)}_0(t) \) for \( t \in [t_0, t_1] \) for some \( 0 \leq t_0 < t_1 \leq 1 \). By the mean value theorem, there is a \( t' \in (t_0, t_1) \) such that \( \zeta^{(1)}_0(t') - \zeta^{(2)}_0(t') > 0 \). But the ODE gives

\[
\dot{\zeta}^{(1)}_0(t') - \dot{\zeta}^{(2)}_0(t') = -(1 - p(t')) \frac{\beta(t')}{(1 + \beta(t'))t' + \tilde{c} + c\beta(t')}(\zeta^{(1)}_0(t') - \zeta^{(2)}_0(t')) < 0,
\]

a contradiction. Now let \( i \geq 1 \). Assume \( \zeta^{(1)}_k = \zeta^{(2)}_k \) for \( k < i \). Suppose \( \zeta^{(1)}_i \neq \zeta^{(2)}_i \), say \( \zeta^{(1)}_i(t) > \zeta^{(2)}_i(t) \) for some \( t \in [0, 1] \). As \( \zeta^{(1)}_i \) and \( \zeta^{(2)}_i \) are continuous, there exist \( 0 \leq t_0 < t_1 \leq 1 \) such that \( \zeta^{(1)}_i(t) > \zeta^{(2)}_i(t) \) for \( t \in [t_0, t_1] \). From the mean value theorem, there exists a \( t'' \in (t_0, t_1) \) such that \( \zeta^{(1)}_i(t'') - \zeta^{(2)}_i(t'') > 0 \). On the other hand, by the induction hypothesis \( \zeta^{(1)}_{i-1} = \zeta^{(2)}_{i-1} \), from the ODE (1.10), we have

\[
\dot{\zeta}^{(1)}_i(t'') - \dot{\zeta}^{(2)}_i(t'') = -(1 - p(t'')) \frac{i + \beta(t'')}{(1 + \beta(t''))t'' + \tilde{c} + c\beta(t'')}(\zeta^{(1)}_i(t'') - \zeta^{(2)}_i(t'')) < 0,
\]

a contradiction. Now \( \zeta^{(1)}_{d+1} = \zeta^{(2)}_{d+1} \) from (1.42).

From the form of \( I_d \), any zero cost trajectory must satisfy the ODEs (1.10).

**Proof of Theorem 1.1.3.** Since the rate function \( I_d \) has a unique minimizer, there exists a \( \delta > 0 \) such that \( I(\varphi) > \epsilon > 0 \) for any \( \varphi \in B\bar{\zeta}^d(c^d) = \{ \varphi \in C([0, 1]; \mathbb{R}^{d+2}) \mid \| \varphi - \zeta^d \| \geq \delta \} \). The LDP upper bound in Theorem 1.1.1 gives

\[
\limsup_{n \to \infty} \frac{1}{n} \log P \{ \| X^n - \zeta \| \geq \delta \} \leq - \inf_{\varphi \in B\bar{\zeta}^d(\zeta)} I(\varphi),
\]
and the desired result follows from Borel-Cantelli Lemma.

\[ \square \]

### 1.5 Proof of Theorem 1.1.4

The proof of the theorem follows from the next lemma. Define, for positive real numbers \(o_1, o_2, o_3, o_4, o_5\), the system of ODEs, \(O(o_1, o_2, o_3, o_4, o_5)\): With initial condition \(\varphi(0) = c^d\)

\[
\varphi_0(t) = 1 - o_1 - (1 - o_2) \frac{o_3}{1 + o_4} \cdot \frac{\varphi_0(t)}{t + o_5}
\]

\[
[\dot{\varphi}(t)]_i = 1 - (1 - o_2) \frac{i + o_3}{1 + o_4} \cdot \frac{\varphi_i(t)}{t + o_5} \quad \text{for} \quad 1 \leq i \leq d.
\]

One can readily check that, when \(0 < o_2 < 1\), \(\chi(t)\) is the solution to \(O(o_1, o_2, o_3, o_4, o_5)\) above where

\[
\chi_i(t) = b_1(t + o_5) + \sum_{\ell=0}^{i} a_{i,\ell} \left( \frac{o_5}{t + o_5} \right)^{\frac{(1 - o_2) \ell + o_3}{1 + o_4}} \quad \text{for} \quad 0 \leq i \leq d + 1.
\]

Here, for given \((o_1, o_2, o_3, o_4, o_5)\), the sequences \(b_1 = b_1(o_1, o_2, o_3, o_4, o_5)\) and \(a_{i,\ell} = a_{i,\ell}(o_1, o_2, o_3, o_4, o_5)\) are defined as \(b_0 = \frac{1 - o_1}{1 + (1 - o_2) \frac{o_3}{1 + o_4}}, b_1 = \frac{o_1 + (1 - o_2) \frac{o_3}{1 + o_4} - b_0}{1 + (1 - o_2) \frac{o_3}{1 + o_4}}\), and for \(i \geq 0\)

\[
b_i = b_1 \prod_{\ell=2}^{i} \frac{(1 - o_2) \frac{\ell - 1 + o_3}{\ell + o_4}}{1 + (1 - o_2) \frac{\ell + o_3}{1 + o_4}} = b_1 \frac{\Gamma(2 + o_3 + \frac{1 + o_3}{1 - o_2})}{\Gamma(1 + o_3)} \frac{\Gamma(i + o_3)}{\Gamma(i + 1 + o_3 + \frac{1 + o_4}{1 - o_2})} \sim \frac{1}{i + \frac{1 + o_4}{1 - o_2}},
\]

\[
a_{i,\ell} = \frac{i - 1 + o_3}{i - \ell} a_{i-1,\ell} \quad \text{where} \quad 0 \leq \ell < i,
\]

and \(a_{i,i} = c - b_1 o_5 - \sum_{\ell=0}^{i-1} a_{i,\ell}\).

Recall the assumption for Theorem 1.1.4:

\[
0 \leq p_{\text{min}} \leq p(\cdot) \leq p_{\text{max}} < 1, \quad \text{and} \quad 0 < \beta_{\text{min}} \leq \beta(\cdot) \leq \beta_{\text{max}} < \infty.
\]

**Lemma 1.5.1 (Comparison Lemma).** We have \(\tilde{\zeta}, \tilde{\zeta}\) are unique solutions of systems of ODEs, \(O(p_{\text{min}}, p_{\text{max}}, \beta_{\text{min}}, \beta_{\text{max}}, \max\{\tilde{c}, c\}), O(p_{\text{max}}, p_{\text{min}}, \beta_{\text{max}}, \beta_{\text{min}}, \min\{\tilde{c}, c\})\), respectively. Moreover, for \(0 \leq i \leq d\) and \(t \in [0, 1]\), with respect to the zero-cost trajectory \(\zeta^d(t)\) in Theorem 1.1.3 with initial condition \(\zeta^d(0) = c^d\), we have

\[
[\tilde{\zeta}(t)]_i \leq [\zeta^d(t)]_i \leq [\tilde{\zeta}(t)]_i,
\]

(1.43)
Proof. The proof that \( \tilde{\zeta} \) and \( \hat{\zeta} \) are the unique solutions uses a similar argument to that used in the proof of Lemma 1.4.1.

We now establish the inequality in the display with respect to \( \tilde{\zeta} \) as an analogous proof works for \( \hat{\zeta} \). We use induction to see that \( |\tilde{\zeta}|_i \geq |\zeta|_i \) for \( 0 \leq i \leq d \).

Since \( \tilde{\zeta}(0) = \zeta(0) = e^d \), from the ODEs, \( O(p_{\min}, p_{\max}, \beta_{\min}, \beta_{\max}, \max\{\hat{c}, c\}) \) and (1.10), we have

\[
\tilde{\zeta}_0(t) - \dot{\zeta}_0(t) = p(t) - p_{\min} \\
+ (1 - p(t)) \frac{\beta(t)\zeta_0(t)}{(1 + \beta(t))t + \hat{c} + c\beta(t)} - (1 - p_{\max}) \frac{\beta_{\min}\tilde{\zeta}_0(t)}{(1 + \beta_{\max})(t + \max\{\hat{c}, c\})}
\]

\[
\tilde{\zeta}(t)_i - [\zeta(t)]_i = (1 - p(t)) \frac{(i + \beta(t))\zeta_i(t)}{(1 + \beta(t))t + \hat{c} + c\beta(t)} - (1 - p_{\max}) \frac{(i + \beta_{\min})\tilde{\zeta}_i(t)}{(1 + \beta_{\max})(t + \max\{\hat{c}, c\})}.
\]

For \( i = 0 \), suppose \( \tilde{\zeta}_0(t) < \zeta_0(t) \) for some \( t \). Then, by continuity, we may assume that \( \tilde{\zeta}_0(t) < \zeta_0(t) \) for all \( t \in [t_0, t_1] \) for some \( 0 \leq t_0 < t_1 \leq 1 \). From the mean value theorem, we find a \( t' \in (t_0, t_1) \) such that \( \tilde{\zeta}_0(t') < \zeta_0(t') \), which contradicts the ODE (1.44) as it gives \( \tilde{\zeta}_0(t') - \zeta_0(t') > 0 \). Therefore, \( \tilde{\zeta}_0 \geq \zeta_0 \).

Now, for \( 1 \leq i \leq d \), suppose \( |\tilde{\zeta}(t)|_i < |\zeta(t)|_i \) for some \( t \). By induction hypothesis (\( |\tilde{\zeta}(\cdot)|_{i-1} \geq |\zeta(\cdot)|_{i-1} \)), we must have \( \tilde{\zeta}_i(t) < \zeta_i(t) \). Since \( |\tilde{\zeta}(\cdot)|_i, |\zeta(\cdot)|_i, \tilde{\zeta}(\cdot) \) and \( \zeta(\cdot) \) are continuous functions, as for the case \( i = 0 \), we may assume that \( |\tilde{\zeta}(t)|_i < |\zeta(t)|_i \) and \( \tilde{\zeta}_i(t) < \zeta_i(t) \) for all \( t \in [t_0, t_1] \) for some \( 0 \leq t_0 < t_1 \leq 1 \). By the mean value theorem for \( \tilde{\zeta}(t)_i - [\zeta(t)]_i \), there is \( t' \in (t_0, t_1) \) such that \( |\tilde{\zeta}(t')|_i < |\zeta(t')|_i \). But (1.45) gives \( |\tilde{\zeta}(t')|_i - [\zeta(t')]|_i > 0 \), a contradiction. Therefore \( |\tilde{\zeta}|_i \geq |\zeta|_i \).

\[ \square \]

**Proof of Theorem 1.1.4.** Given Lemma 1.5.1, we need only detail the solutions \( \tilde{\zeta} \) and \( \hat{\zeta} \) when the initial configuration is ‘small’ and ‘large’ respectively. To this end, when the initial configuration is ‘small’ \( (e_i \equiv 0) \), \( \tilde{\zeta}, \hat{\zeta} \) are linear, namely

\[
\tilde{\zeta}_i(t) = \bar{b}_i t, \quad \text{and} \quad \hat{\zeta}_i(t) = \hat{b}_i t,
\]

where \( \bar{b}_i := b_i(p_{\min}, p_{\max}, \beta_{\min}, \beta_{\max}, \max\{\hat{c}, c\}) \) and \( \hat{b}_i := \hat{b}_i(p_{\max}, p_{\min}, \beta_{\max}, \beta_{\min}, \min\{\hat{c}, c\}) \).
On the other hand, when the initial configuration is 'large' \((c_i > 0 \text{ for some } 0 \leq i \leq d + 1)\), \(\tilde{\zeta}, \hat{\zeta}\) are almost linear, namely as \(t \uparrow \infty\),

\[
\tilde{\zeta}_i(t) = (\tilde{b}_i + o(1))t, \quad \text{and} \quad \hat{\zeta}_i(t) = (\hat{b}_i + o(1))t.
\]

\(\square\)
Bibliography


PART II

Graph coloring
INTRODUCTION

An edge-colored graph is called monochromatic if all its edges have the same color. An edge-colored graph is called rainbow or totally multicolored if all its edges have distinct colors. For a coloring of the edges of a graph, classical Ramsey type problems involve determining whether a given monochromatic subgraph exists. On the other hand, classical anti-Ramsey problems see if a given rainbow subgraph exists. More precisely, the classical multicolor Ramsey function $R_k(G)$ is defined to be the smallest $n$ such that any edge-coloring of a complete graph on $n$ vertices, $K_n$, in $k$ colors contains a monochromatic copy of $G$, and the classical anti-Ramsey function $AR(n; H)$ is defined to be the largest number of colors in an edge-coloring of $K_n$ not containing a rainbow copy of $H$. The Ramsey and anti-Ramsey functions were introduced by Ramsey [7] and by Erdős, Simonovits and Sós [2], respectively. These functions have found applications in many other branches of mathematics and computer science (cf. Rosta [8]). Since exact results are very difficult to find, a number of generalizations have sprung up over the years (see, for example, Graham, Rothschild and Spencer [4], Fujita, Magnant and Ozeki [3]). One of the generalizations is to investigate Ramsey properties involving a monochromatic subgraph and a rainbow subgraph at the same time.

Let $G$ and $H$ be two graphs on fixed number of vertices. An edge coloring of a complete graph, $K_n$, is called $(G, H)$-good if there is no monochromatic copy of $G$ and no rainbow copy of $H$ in this coloring. As shown by Jamison and West [5], a $(G, H)$-good coloring of an arbitrarily large complete graph exists unless either $G$ is a star or $H$ is a forest. Let $S(n; G, H)$ be the set of the number of colors, $k$, such that there is a $(G, H)$-good coloring of $K_n$ with $k$ colors. We call $S(n; G, H)$ a spectrum. Let $\max S(n; G, H)$, $\min S(n; G, H)$ be the maximum, minimum number in $S(n; G, H)$, respectively.

In Chapter 1, we study the following question:
Question 1  Given \( n, G \) and \( H \), is \( S(n; G, H) \) interval? I.e. for any \( k \) with \( \min S(n; G, H) \leq k \leq \max S(n; G, H) \), is there a \( (G, H) \)-good coloring using \( k \) colors?

We show that the answer is YES for any \( n \) if \( G \) is not a star and \( H \) does not have a degree 1 vertex, or leaf. However we find that for some graphs \( G, H \) and some values of \( n \), \( S(n; G, H) \) can have a gap.

The function \( \max S(n; G, H) \) is closely related to the classical anti-Ramsey function \( AR(n; H) \).

For graphs \( H \) which can not be vertex-partitioned into at most two induced forests, \( \max S(n; G, H) \) has been determined asymptotically by Axenovich and Iverson [1]. Determining \( \max S(n; G, H) \) is challenging for other graphs \( H \), in particular for bipartite graphs or even for cycles.

Question 2  What is the value of \( \max S(n; G, H) \) when \( H \) is bipartite?

The case when \( H \) is a cycle is studied in Chapter 2, where we show that for a large \( n \), \( G \) and a cycle of length \( k \), \( C_k \), \( \max S(n; G, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1) \) when \( G \) is either bipartite with ‘large’ parts, or a graph with chromatic number at least 3, and \( \max S(n; G, C_k) = n(\frac{s-2}{2} + \frac{1}{s-1}) + g \) when \( G \) is bipartite with a ‘small’ part of size \( s \), where \( g \) is a constant depending on \( G \) and \( k \), not depending on \( n \).

Similar to \( \max S(n; \cdot, H) \), determining the classical anti-Ramsey function \( AR(n; H) \) is hard when \( H \) is bipartite.

Question 3  What is the value of \( AR(n; H) \) when \( H \) is bipartite?

When \( H \) is a cycle of length \( k \), \( C_k \), Erdős, Simonovits and Sós [2] provided a rainbow \( C_k \)-free coloring of edges of \( K_n \) with \( n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1) \) colors and conjectured that this would be optimal. Since then, the conjecture had been verified for small values of \( k \) by a series of papers by Alon (1983), Schiermeyer (2001), Jiang and West (2003). Finally, Montellano-Ballesteros and Neumann-Lara [6] proved the conjecture completely. In Chapter 3, we give a short proof. We use the main technique used in [6] that proves each component of a graph representing the coloring is Hamiltonian if each vertex has enough ‘new’ colors.
Bibliography


CHAPTER 1. A note on monotonicity of mixed Ramsey numbers

A paper submitted to Discrete Mathematics

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Abstract

For two graphs, $G$, and $H$, an edge-coloring of a complete graph is $(G,H)$-good if there is no monochromatic subgraph isomorphic to $G$ and no rainbow subgraph isomorphic to $H$ in this coloring. The set of number of colors used by some $(G,H)$-colorings of $K_n$ is called a mixed-Ramsey spectrum. This note addresses a fundamental question of whether the spectrum is an interval. It is shown that the answer is “yes” if $G$ is not a star and $H$ does not contain a pendent edge.

1.1 Introduction

Let $G$ and $H$ be two graphs on fixed number of vertices. An edge coloring of a complete graph, $K_n$, is called $(G,H)$-good if there is no monochromatic copy of $G$ and no rainbow (totally multicolored) copy of $H$ in this coloring. This, sometimes called mixed-Ramsey coloring, is a hybrid of classical Ramsey and anti-Ramsey colorings, [18, 9]. As shown by Jamison and West [15], a $(G,H)$-good coloring of an arbitrarily large complete graph exists unless either $G$ is a star or $H$ is a forest.

Let $S(n;G,H)$ be the set of the number of colors, $k$, such that there is a $(G,H)$-good coloring of $K_n$ with $k$ colors. We call $S(n;G,H)$ a spectrum. Let $\max S(n;G,H)$, $\min S(n;G,H)$ be the maximum, minimum number in $S(n;G,H)$, respectively. The behavior of these functions was studied in [2], [8], [1] and others. Note that if there is no restriction on a graph $H$, $S(n;G,*)$ is
an interval \([k, \binom{n}{2}]\), where \(k\) is the largest number such that \(r_{k-1}(G) \leq n\), a classical multicolor Ramsey number.

The main question investigated in this note is whether the same behavior continues to hold for mixed Ramsey colorings. Specifically, for given integer \(n\) and graphs \(G\) and \(H\), is \(S(n; G, H)\) an interval? When \(G\) is not a star, for most graphs \(H\), we show that \(S(n; G, H)\) is an interval.

**Theorem 1.1.1.** Let \(G\) be a graph that is not a star, and let \(H\) be a graph with minimum degree at least 2. Then for any natural number \(n\), \(S(n; G, H)\) is an interval.

The simplest connected graph \(H\) which is not a tree and which has a vertex of degree 1 is \(K_3 + e\), a 4-vertex graph obtained by attaching a pendent edge to a triangle. We show that \(S(n; G, K_3 + e)\) could have a gap for some graphs \(G\) and some values of \(n\). However, when \(n\) is arbitrarily large, we do not have a single example of a graph \(G\) and a graph \(H\) for which \(S(n; G, H)\) is not an interval.

Specifically, the next theorem is a collection of results on \(S(n; G, K_3 + e)\). Here, \(\ell K_2\) is a matching of size \(\ell\), \(C_4\) is a 4-cycle, and \(P_4\) is a path on 4 vertices.

**Theorem 1.1.2.**

- \(S(n; \ell K_2, K_3) = S(n; \ell K_2, K_3 + e) = \left[\left\lceil \frac{n-2\ell+1}{\ell-1} \right\rceil + 1, n - 1\right], \quad n \geq 4,
- \(S(n; P_4, K_3) = S(n; P_4, K_3 + e) = [n - 2, n - 1], \quad n \geq 4,
- \(S(n; C_4, K_3) = S(n; C_4, K_3 + e) = [n - 3, n - 1], \quad n \geq r_3(C_4) = 11,
- \(S(n; K_3, K_3) = S(n; K_3, K_3 + e) = [c \log n, n - 1], \quad n \geq r_3(K_3) = 17,
- \(S(n; K_1,\ell, K_3) = S(n; K_1,\ell, K_3 + e) = \emptyset, \quad n \geq 3\ell.

- \(S(10; C_4, K_3 + e) = \{3, 7, 8, 9\}.

**Corollary 1.1.3.** If \(\ell \geq 2\) and \(n \geq \max\{17, 3\ell + 1\}\), then \(S(n; G, K_3 + e)\) is an interval for any \(G \in \{K_3, \ell K_2, C_4, P_4, K_1,\ell\}\). However, \(S(n; G, K_3 + e)\) is not an interval if \(n = 10\) and \(G = C_4\).
Open question. Are there graphs $G$ and $H$ such that for any natural number $N$ there is $n > N$ so that $S(n; G, H)$ is not an interval?

1.2 Definitions and proofs of main results

For an edge coloring $c$ of $K_n$ and a vertex $x \in V(K_n)$, let $N_c(x)$ be the set of colors used only on edges incident to $x$, and for $X \subseteq V(K_n)$ let $c(X)$ be the set of colors used on edges induced by $X$. Let $|c|$ denote the number of colors used in the coloring $c$. Then $|c| = |N_c(x)| + |c(V \setminus x)|$ for any $x \in V$.

Observation 1 If $G$ is not a star, and $A$ and $B$ are color classes which are stars with the same center in a $(G, H)$-good coloring $c$ of $K_n$ with $k$ colors, then replacing $A$ and $B$ in $c$ with a new color class $A \cup B$ gives a $(G, H)$-good coloring using $k - 1$ colors.

Observation 2 For any graphs $G$ and $H$,

$$\min S(n; G, H) \leq \min S(n + 1, G, H).$$

Proof. Consider a $(G, H)$-good coloring of $K_{n+1}$ with $k$ colors. Delete one vertex to get a $(G, H)$-good coloring of $K_n$ with $k' \leq k$ colors.

Observation 3 For $G \subseteq G'$ and $H \subseteq H'$,

$$S(n; G, H) \subseteq S(n; G', H) \subseteq S(n; G', H') \quad \text{and} \quad S(n; G, H) \subseteq S(n; G, H') \subseteq S(n; G', H').$$

Proof. If there is no monochromatic $G$ and no rainbow $H$ in a coloring of $E(K_n)$, then there is no monochromatic $G'$ and no rainbow $H'$ in this coloring.

Observation 4 If $G$ is not a star, $H$ has minimum degree at least 2, and $k \in S(n; G, H)$, then $k + 1 \in S(n + 1; G, H)$.
Proof. Consider a \((G, H)\)-good coloring of \(K_n\) with \(k\) colors. Add a new vertex \(x\), and color edges incident to \(x\) by a new color to get a \((G, H)\)-good coloring of \(K_{n+1}\) with \(k+1\) colors. 

\[ \]

Proof of Theorem 1.1.1.

We need to prove that \([\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)\). We use induction on \(n\). When \(n = 2\), any coloring uses one color. Let \(n \geq 3\). Consider the smallest \(k\) such that \([k, \max S(n; G, H)] \subseteq S(n; G, H)\). Observe that in any \((G, H)\)-good \(k\)-coloring of \(K_n\) and any vertex \(x\), we have \(|N_c(x)| \leq 1\), otherwise applying Observation 1 gives us a \((G, H)\)-good \((k-1)\)-coloring of \(K_n\) violating minimality of \(k\). Consider a \((G, H)\)-good \(k\)-coloring of \(K_n\) and any vertex \(x\), and delete it. Then we have a \((G, H)\)-good coloring of \(K_{n-1}\) with \(k\) or \(k-1\) colors. Here we note that \(\max S(n-1; G, H) \geq k-1\). By induction, \(S(n-1; G, H)\) is an interval, i.e., \([\min S(n-1; G, H), \max S(n-1; G, H)] = S(n-1; G, H)\). Then by Observation 4, \([\min S(n-1; G, H) + 1, \max S(n-1; G, H) + 1] \subseteq S(n; G, H)\). Since \(\min S(n; G, H) \geq \min S(n-1; G, H)\) from Observation 2, \([\min S(n; G, H), \max S(n-1; G, H) + 1] \subseteq S(n; G, H)\).

Since \(k \leq \max S(n-1; G, H) + 1\) and \([k, \max S(n; G, H)] \subseteq S(n; G, H)\) we finally have that \([\min S(n; G, H), \max S(n; G, H)] \subseteq S(n; G, H)\). 

For the proof of Theorem 1.1.2, we shall use the function 

\[ f(k; G, H) = \max\{n : \text{there is a } (G, H)\text{-good coloring of } K_n \text{ using exactly } k \text{ colors}\} \]

Note that if \(f(k; G, H) = n\), then \(\min S(n; G, H) \leq k\).

Observation 5 If \(f(k; G, H) = n\) and \(f(\tilde{k}; G, H) < n\) for any \(\tilde{k} < k\), then \(\min S(n; G, H) = k\).

In particular, if \(f\) is strictly increasing in \(k\), then \(f(k; G, H) = n\) implies \(\min S(n; G, H) = k\).

Proof of Theorem 1.1.2.

First observe that \(\max S(n; G, H) \leq AR(n, H)\), where \(AR(n, H)\) is the classical anti-Ramsey
number, the maximum number of colors in an edge-coloring of $K_n$ with no rainbow subgraphs isomorphic to $H$. If $G$ is not a star, max $S(n; G, K_3) = AR(n, K_3) = n - 1$, see [2]. Moreover, from Observation 3, we obtain that max $S(n; G, K_3) \leq max S(n; G, K_3 + e)$; and from [12], we know that $AR(n, K_3) = AR(n, K_3 + e)$. Thus, when $G$ is not a star, max $S(n; G, K_3) = max S(n; G, K_3 + e) = n - 1$ for $n \geq 4$.

Therefore if min $S(n; G, K_3) = min S(n, G, K_3 + e)$, and $G$ is not a star, we can conclude that $S(n; G, K_3 + e) = S(n; G, K_3)$, which is an interval by Theorem 1. Next, we shall analyze min $S(n, G, K_3 + e)$. We note that $f(k; G, H) + 1 \leq r_k(G)$, where $r_k(G)$ denotes the classical $k$-color Ramsey number for $G$. The equality holds if there is a $k$-coloring of $E(K_{r_k(G)-1})$ with no monochromatic $G$ and no rainbow $H$.

**Case 1.** $G = \ell K_2$

From [17], we have that $r_k(\ell K_2) = (k - 1)(\ell - 1) + 2\ell$. The extremal coloring providing this Ramsey number can be constructed as follows. Consider a complete graph on $2\ell - 1$ vertices colored entirely with color 1, add $\ell - 1$ vertices and color all edges incident to these vertices with color 2, then add another $\ell - 1$ vertices and color all edges incident to these vertices with color 3. Repeat this process until we get a $k$-coloring of a complete graph on $2\ell - 1 + (k - 1)(\ell - 1)$ vertices which contains no monochromatic $\ell K_2$. Note that this coloring contains no rainbow cycles, thus, it contains neither rainbow copy of $K_3$ nor rainbow copy of $K_3 + e$. Hence $f(k; \ell K_2, H) = f(k; \ell K_2, H + e) = (k - 1)(\ell - 1) + 2\ell - 1$ for any $H$, not a forest. By Observation 5, min $S(n; \ell K_2, H) = min S(n; \ell K_2, H + e)$. In particular for $\ell \geq 2$, min $S(n; \ell K_2, K_3) = min S(n; \ell K_2, K_3 + e) = \left\lceil \frac{n-2\ell+1}{2} \right\rceil + 1$.

**Case 2.** $G \in \{K_3, P_4, C_4\}$

From [5, 2, 13, 7, 8] we have that $f(k; K_3, K_3) = f(k; K_3, K_3 + e) = \lambda(k)$, for $k \geq 1$ and $k \neq 3$, $f(3; K_3, K_3) = 10$, and $f(3; K_3, K_3 + e) = r_3(K_3) - 1 = 16$, where $\lambda(k) = 5^{k/2}$ if $k$ is even, $2 \cdot 5^{(k-1)/2}$ if $k$ is odd; $f(k; P_4, K_3) = f(k; P_4, K_3 + e) = k + 2$ for $k \geq 1$; and $f(k; C_4, K_3) = f(k; C_4, K_3 + e) = k + 3$, for $k = 2$ or $k \geq 4$, $f(3; C_4, K_3) = 6$, and $f(k; C_4, K_3 + e) = r_3(C_4) - 1 = 10$. Therefore from Observation 5, min $S(n; P_4, K_3) = min S(n; P_4, K_3 + e) = n - 2$ for $n \geq 4$, min $S(n; C_4, K_3) = min S(n; C_4, K_3 + e) = n - 3$ for $n \geq r_3(C_4) = 11$, and min $S(n; K_3, K_3) =$
min S(n; K_3, K_3 + e) = c \log n \text{ for } n \geq r_3(K_3) = 17. \text{ Thus } min S(n; G, K_3) = min S(n; G, K_3 + e) \text{ for } G \in \{K_3, P_4, C_4\} \text{ and } n \geq r_3(G).

Case 3. G = K_{1,\ell}

In [14], it was shown that any coloring of E(K_n) with no rainbow triangles has a monochromatic star K_{1,2n/5}. Using this fact and the pigeonhole principle, we easily see that any coloring of E(K_n) with no rainbow K_3 + e has a monochromatic star K_{1,n/3}. Namely, let c be a coloring of E(K_n) with no rainbow K_3 + e. Since 2n/5 \geq n/3, we may assume there is a rainbow copy T of K_3. To avoid a rainbow K_3 + e in this coloring, the edges between V(T) and V(K_n) − V(T) have colors only presented on the edges of T, i.e., at most three colors. Then by pigeonhole principle, we can find a monochromatic star K_{1,s} with one vertex in V(T) and s \geq \frac{n-3}{3} vertices in V(K_n) − V(T). Considering other two vertices in V(T), we finally have a monochromatic star K_{1,n/3}. This is sharp as is seen in [8], namely there is a (K_{1,s}, K_3 + e)-good coloring of E(K_{3s-2}). Therefore S(n; K_{1,\ell}, K_3) = S(n; K_{1,\ell}, K_3 + e) = \emptyset \text{ if } n \geq 3\ell.

Summarizing 1), 2), and 3) we have that S(n; G, K_3) = S(n; G, K_3 + e) is an interval if G is one of \{\ell K_2, K_3, P_4, C_4, K_{1,\ell}\} \text{ and } n \geq N, \text{ where } N \text{ is a constant depending only on } G. \text{ This concludes the proof of the first part of the Theorem.}

Consider the case when G = C_4, H = K_3 + e \text{ and } n = 10. \text{ Since } r_2(C_4) = 6 < 10, \text{ we see that there is no } (C_4, K_3 + e)-\text{good coloring of } K_{10} \text{ in two colors. On the other hand, since } r_3(C_4) = 11, \text{ there is a } (C_4, K_3 + e)-\text{good coloring of } K_{10} \text{ in three colors. Thus } min S(10; C_4, K_3 + e) = 3. \text{ We also have that } max S(10; C_4, K_3 + e) = AR(10, K_3) = 9. \text{ Since } f(k; C_4, K_3 + e) = k + 3 < 10 \text{ for } 4 \leq k \leq 6, \text{ there is no } (C_4, K_3 + e)-\text{good coloring of } K_{10} \text{ with } 4, 5, \text{ or } 6 \text{ colors. To construct 8- and 7-colorings of } K_{10} \text{ with no rainbow } K_3 + e \text{ and no monochromatic } C_4, \text{ consider a vertex set } \{v_1, \ldots, v_{10}\}. \text{ Let } c(v_i v_j) = i, 1 \leq i \leq 7, i < j; c(v_8 v_9) = c(v_8 v_{10}) = c(v_9 v_{10}) = 8. \text{ Let } c'(v_i v_j) = i, 1 \leq i \leq 5, i < j; c'(v_6 v_7) = c'(v_7 v_8) = c'(v_8 v_9) = c'(v_9 v_{10}) = c'(v_{10} v_6) = 6, \text{ all other edges get color } 7 \text{ under } c'. \text{ Note that } c \text{ and } c' \text{ are 8- and 7-colorings, respectively, containing no rainbow } K_3 \text{ and no monochromatic } C_4. \text{ Thus } S(10; C_4, K_3 + e) = \{3, 7, 8, 9\}. \square
Acknowledgments: The authors thank the anonymous referee for careful reading and comments improving the manuscript.
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CHAPTER 2. On colorings avoiding a rainbow cycle and a fixed monochromatic subgraph

A paper published in the Electronic Journal of Combinatorics

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Abstract

Let $H$ and $G$ be two graphs on fixed number of vertices. An edge coloring of a complete graph is called $(H, G)$-good if there is no monochromatic copy of $G$ and no rainbow (totally multicolored) copy of $H$ in this coloring. As shown by Jamison and West, an $(H, G)$-good coloring of an arbitrarily large complete graph exists unless either $G$ is a star or $H$ is a forest. The largest number of colors in an $(H, G)$-good coloring of $K_n$ is denoted $\maxR(n; G, H)$. For graphs $H$ which can not be vertex-partitioned into at most two induced forests, $\maxR(n; G, H)$ has been determined asymptotically. Determining $\maxR(n; G, H)$ is challenging for other graphs $H$, in particular for bipartite graphs or even for cycles. This manuscript treats the case when $H$ is a cycle. The value of $\maxR(n; G, C_k)$ is determined for all graphs $G$ whose edges do not induce a star.

2.1 Introduction and main results

For two graphs $G$ and $H$, an edge coloring of a complete graph is called $(H, G)$-good if there is no monochromatic copy of $G$ and no rainbow (totally multicolored) copy of $H$ in this coloring. The mixed anti-Ramsey numbers, $\maxR(n; G, H)$, $\minR(n; G, H)$ are the maximum, minimum number of colors in an $(H, G)$-good coloring of $K_n$, respectively. The number $\maxR(n; G, H)$ is closely related to the classical anti-Ramsey number $AR(n, H)$, the largest number of colors in
an edge-coloring of $K_n$ with no rainbow copy of $H$ introduced by Erdős, Simonovits and Sós [9].

The number $\min R(n; G, H)$ is closely related to the classical multicolor Ramsey number $R_k(G)$, the largest $n$ such that there is a coloring of edges of $K_n$ with $k$ colors and no monochromatic copy of $G$. The mixed Ramsey number $\min R(n; G, H)$ has been investigated in [3, 13, 11].

This manuscript addresses $\max R(n; G, H)$. As shown by Jamison and West [14], an $(H, G)$-good coloring of an arbitrarily large complete graph exists unless either $G$ is a star or $H$ is a forest. Let $a(H)$ be the smallest number of induced forests vertex-partitioning the graph $H$. This parameter is called a vertex arboricity. Axenovich and Iverson [3] proved the following.

**Theorem 2.1.1.** Let $G$ be a graph whose edges do not induce a star and $H$ be a graph with $a(H) \geq 3$. Then $\max R(n; G, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$, where the equality is proven by Montellano-Ballesteros and Neumann-Lara [16]. We show that $\max R(n; G, C_k) = AR(n; C_k)$ when $G$ is either bipartite with large enough parts, or a graph with chromatic number at least 3. In case when $G$ is bipartite with a “small” part, $\max R(n; G, C_k)$ depends mostly on $G$, namely, on the size of the “small” part. Below is the exact formulation of the main result.

If a graph $G$ is bipartite, we let $s(G) = \min\{s : G \subseteq K_{s,r}, s \leq r \text{ for some } r\}$ and $t(G) = |V(G)| - s(G)$. I.e., $s(G)$ is the sum of the sizes of smaller parts over all components of $G$.

**Theorem 2.1.2.** Let $k \geq 3$ be an integer and $G$ be a graph whose edges do not induce a star. Let $s = s(G)$ and $t = t(G)$ if $G$ is bipartite. There are constants $n_0 = n_0(G, k)$ and $g = g(G, k)$ such that for all $n \geq n_0$

\[
\max R(n; G, C_k) = \begin{cases} 
  n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1), & \text{if } (\chi(G) = 2 \text{ and } s \geq k) \text{ or } (\chi(G) \geq 3) \\
  n\left(\frac{s-2}{2} + \frac{1}{s-1}\right) + g, & \text{otherwise}
\end{cases}
\]
Here \( g = g(G, k) = ER^2(s + t, 3sk + t + 1, k) \), where the number ER denotes the Erdős-Rado number stated in section 2. Note that it is sufficient to take \( g(G, k) = 2^{c\ell^2\log \ell} \), where \( \ell = 3sk + t + 1 \).

We give the definitions and some observations in section 2.2, the proof of the main theorem in section 2.3 and some more accurate bounds for the case when \( H = C_4 \) in the last section of the manuscript.

2.2 Definitions and preliminary results

First we shall define a few special edge colorings of a complete graph: lexical, weakly lexical, \( k \)-anticyclic, \( c^* \) and \( c^{**} \).

Let \( c : E(K_n) \to \mathbb{N} \) be an edge coloring of a complete graph on \( n \) vertices for some fixed \( n \).

We say that \( c \) is a weakly lexical coloring if the vertices can be ordered \( v_1, \ldots, v_n \), and the colors can be renamed such that there is a function \( \lambda : V(K_n) \to \mathbb{N} \), and \( c(v_i v_j) = \lambda(v_{\min\{i,j\}}) \), for \( 1 \leq i, j \leq n \). In particular, if \( \lambda \) is one to one, then \( c \) is called a lexical coloring.

We say that \( c \) is a \( k \)-anticyclic coloring if there is no rainbow copy of \( C_k \), and there is a partition of \( V(K_n) \) into sets \( V_0, V_1, \ldots, V_m \) with \( 0 \leq |V_0| < k - 1 \) and \( |V_1| = \cdots = |V_m| = k - 1 \), where \( m = \lfloor \frac{n}{k-1} \rfloor \), such that for \( i, j \) with \( 0 \leq i < j \leq m \), all edges between \( V_i \) and \( V_j \) have the same color, and the edges spanned by each \( V_i \), \( i = 0, \ldots, m \) have new distinct colors using pairwise disjoint sets of colors.

We denote a fixed coloring from the set of \( k \)-anticyclic colorings of \( K_n \) such that the color of any edges between \( V_i \) and \( V_j \) is \( \min\{i, j\} \) by \( c^* \).

Finally, we need one more coloring, \( c^{**} \), of \( K_n \). Let \( c^{**} \) be a fixed coloring from the set of the following colorings of \( E(K_n) \); let the vertex set \( V(K_n) \) be a disjoint union of \( V_0, V_1, \ldots, V_m \) with \( 0 \leq |V_0| < s - 1, |V_1| = \cdots = |V_{m-1}| = s - 1, \) and \( |V_m| = k - 1 \), where \( m - 1 = \lfloor \frac{s-k+1}{s-1} \rfloor \). Let the color of each edge between \( V_i \) and \( V_j \) for \( 0 \leq i < j \leq m \) be \( i \). Color the edges spanned by each \( V_i \), \( i = 0, \ldots, m \) with new distinct colors using pairwise disjoint sets of colors.

For a coloring \( c \), let the number of colors used by \( c \) be denoted by \( |c| \). Observe that \( c^* \) is a blow-up of a lexical coloring with parts inducing rainbow complete subgraphs. Any
monochromatic bipartite subgraph in \( c^* \) and \( c^{**} \) is a subgraph of \( K_{k-1,t} \) and \( K_{s-1,t} \) for some \( t \), respectively. Also we easily see that if \( c \) is \( k \)-anticyclic, then

\[
|c| \leq |c^*| = n \left( \frac{k-2}{2} + \frac{1}{k-1} \right) + O(1), \tag{2.2}
\]

\[
|c^{**}| = n \left( \frac{s-2}{2} + \frac{1}{s-1} \right) + O(1). \tag{2.3}
\]

Let \( K = K_n \). For disjoint sets \( X, Y \subseteq V \), let \( K[X] \) be the subgraph of \( K \) induced by \( X \), and let \( K[X,Y] \) be the bipartite subgraph of \( K \) induced by \( X \) and \( Y \). Let \( c(X) \) and \( c(X,Y) \) denote the sets of colors used in \( K[X] \) and \( K[X,Y] \), respectively by a coloring \( c \).

Next, we state a canonical Ramsey theorem which is essential for our proofs.

**Theorem 2.2.1** (Deuber [7], Erdős-Rado [8]). For any integers \( m, l, r \), there is a smallest integer \( n = ER(m, l, r) \), such that any edge-coloring of \( K_n \) contains either a monochromatic copy of \( K_m \), a lexically colored copy of \( K_l \), or a rainbow copy of \( K_r \).

The number \( ER \) is typically referred to as Erdős-Rado number, with best bound in the symmetric case provided by Lefmann and Rödl [15], in the following form: \( 2^{c_1 \ell^2} \leq ER(\ell, \ell, \ell) \leq 2^{c_2 \ell^2 \log \ell} \), for some constants \( c_1, c_2 \).

### 2.3 Proof of Theorem 2.1.2

If \( G \) is a graph with chromatic number at least 3, then \( \max R(n; G, C_k) = n \left( \frac{k-2}{2} + \frac{1}{k-1} \right) + O(1) \) as was proven in [3].

For the rest of the proof we shall assume that \( G \) is a bipartite graph, not a star, with \( s = s(G) \), \( t = t(G) \), and \( G \subseteq K_{s,t} \). Note that \( 2 \leq s \leq t \). Let \( K = K_n \). If \( s \geq k \), then the lower bound on \( \max R(n; G, C_k) \) is given by \( c^* \), a special \( k \)-anticyclic coloring. The upper bound follows from (2.1).

Suppose \( s < k \). The lower bound is provided by a coloring \( c^{**} \). Since \( \max R(n; G, C_k) \leq \max R(n; K_{s,t}, C_k) \), in order to provide an upper bound on \( \max R(n; G, C_k) \), we shall be giving
an upper bound on $\max R(n; K_{s,t}, C_k)$.

The idea of the proof is as follows. We consider an edge coloring $c$ of $K = (V, E)$ with no monochromatic $K_{s,t}$ and no rainbow $C_k$, and estimate the number of colors in this coloring by analyzing specific vertex subsets: $L, A, B$, where $L$ is the vertex set of the largest weakly lexically colored complete subgraph, $A$ is the set of vertices in $V \setminus L$ which “disagrees” with coloring of $L$ on some edges incident to the initial part of $L$, and $B$ is the set of vertices in $V \setminus L$ which “disagrees” with coloring of $L$ on some edges incident to the terminal part of $L$. Let $V' = V \setminus L$. We are counting the colors in the following order: first colors induced by $V'$ which are not used on any edges incident to $L$ or any edges induced by $L$, then colors used on edges between $V'$ and $L$ which are not induced by $L$, finally colors induced by $L$.

Now, we provide a formal proof. Assume that $n$ is sufficiently large such that $n \geq ER(s + t, 3sk + t + 1, k)$. Let $c$ be a coloring of $E(K)$ with no monochromatic copy of $K_{s,t}$ and no rainbow copy of $C_k$, $c : E(K) \to \mathbb{N}$. Then there is a lexically colored copy of $K_{3sk+t+1}$ by the canonical Ramsey theorem. Let $L$ be a vertex set of a largest weakly lexically colored $K_q$, $q \geq 3sk + t + 1$, say $L = \{x_1, \ldots, x_q\}$ and $c(x_ix_j) = \lambda(x_i)$ for $1 \leq i < j \leq q$, for some function $\lambda : L \to \mathbb{N}$. If $X = \{x_{i_1}, \ldots, x_{i_l}\} \subseteq L$ and $\lambda(x_{i_1}) = \cdots = \lambda(x_{i_l}) = j$ for some $j$, then we denote $\lambda(X) = j$. We write, for $i \leq j$, $x_iLx_j := \{x_i, x_{i+1}, \ldots, x_j\}$, and for $i > j$, $x_iLx_j := \{x_i, x_{i-1}, \ldots, x_j\}$. We say that $x_i$ precedes $x_j$ if $i < j$.

Let $T_t, T_{sk+t}, T_{2sk+t}$, and $T_{3sk+t}$ be the tails of $L$ of size $t$, $sk + t$, $2sk + t$, and $3sk + t$ respectively, i.e.,

$$T_t := \{x_{q-t+1}, x_{q-t+2}, \ldots, x_q\},$$

$$T_{sk+t} := \{x_{q-sk-t+1}, x_{q-sk-t+2}, \ldots, x_q\},$$

$$T_{2sk+t} := \{x_{q-2sk-t+1}, x_{q-2sk-t+2}, \ldots, x_q\},$$

$$T_{3sk+t} := \{x_{q-3sk-t+1}, x_{q-3sk-t+2}, \ldots, x_q\},$$

see Figure 2.1.

We shall use these tails to count the number of colors: the common difference, $sk$, of sizes
of tails is from observations below (Claims 0.1–0.3). The first tail $T_t$ is used in Claims 0.1–0.3 and to find monochromatic copy of $K_{s,t}$. The third tail $T_{2sk+t}$ is the main tool used in Part 1, 2 of the proof, it helps finding rainbow copy of $C_k$. The other tails $T_{sk+t}$ and $T_{3sk+t}$ are for technical reasons used in Claim 2.1 and Claim 1.3, respectively. Note that the size of the fourth tail is used in the second parameter of Erdős-Rado number bounding $n$.

We start by splitting the vertices in $V \setminus L$ according to “agreement” or “disagreement” of a corresponding colors used in $L \setminus T_{2sk+t}$ and in edges between $L$ and $V \setminus L$. Formally, let $V' = V \setminus L$, and

$$A := \{v \in V' \mid \text{there exists } y \in L \setminus T_{2sk+t} \text{ such that } c(vy) \neq \lambda(y)\},$$

$$B := \{v \in V' \mid c(vx) = \lambda(x), x \in L \setminus T_{2sk+t},$$

and there exists $y \in T_{2sk+t} \setminus \{x_q\}$ such that $c(vy) \neq \lambda(y)\}.$

Note that $V' - A - B = \{v \in V' \mid c(vx) = \lambda(x), x \in L \setminus \{x_q\}\} = \emptyset$ since otherwise $L$ is not the largest weakly colored complete subgraph. Thus

$$V = L \cup A \cup B.$$

Let $c_0 := c(L) \cup c(V', L)$. In the first part of the proof we bound $|c(B) \cup c(B, A) \setminus c_0| + |c(B, L) \setminus c(L)|$, in the second part we bound $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)|$.

**Claim 0.1** Let $x \in L \setminus T_t$. Then $|\{y \in L \setminus T_t \mid \lambda(x) = \lambda(y)\}| \leq s - 1 < s$.

If this claim does not hold, the corresponding $y$’s and $T_t$ induce a monochromatic $K_{s,t}$.

**Claim 0.2** Let $y, y' \in L \setminus T_t$ such that $|yL_y'| > (s - 1)\ell + 1$ for some $\ell \geq 0$. Then $|c(yL_y')| \geq \ell + 1$. 

**Figure 2.1** $T_t, T_{sk+t}, T_{2sk+t}$, and $T_{3sk+t}$
It follows from Claim 0.1.

**Claim 0.3** Let \( v, v' \in V' \) and \( y, y' \in L \setminus T \) such that \( y \) precedes \( y' \). Let \( P \) be a rainbow path from \( v \) to \( v' \) in \( V' \) with \( 1 \leq |V(P)| \leq k - 2 \) and colors not from \( c_0 \). If \( c(vy) \neq \lambda(y) \), \( c(v'y') \notin \{c(vy), \lambda(y)\} \), and \( |yLy'| > (s-1)(k-|V(P)|)+1 \), then there is a rainbow \( C_k \) induced by \( V(P) \cup yLy' \).

Indeed, by Claim 0.2, \( |c(yLy')| \geq k - |V(P)| + 1 \). Hence \( |c(yLy') \setminus \{c(vy), c(v'y')\}| \geq k - |V(P)| - 1 \). So we can find a rainbow path on \( k - |V(P)| \) vertices in \( L \) with endpoints \( y \) and \( y' \) of colors from \( c(yLy') \setminus \{c(vy), c(v'y')\} \), which together with \( V(P) \) induce a rainbow \( C_k \) since colors of \( P \) are not from \( c_0 \).

**PART 1**

We shall show that \( \left| (c(B) \cup c(B, A)) \setminus c_0 \right| + |c(B, L) \setminus c(L)| \leq \text{const} = \text{const}(k, s, t) \).

**Claim 1.1** \( |B| < ER(s + t, 2sk + t + 1, k) \).

Suppose \( |B| \geq ER(s + t, 2sk + t + 1, k) \). Then there is a lexically colored copy of a complete subgraph on a vertex set \( Y \subseteq B \) of size \( 2sk + t + 1 \). Then \( (L \cup Y) \setminus T_{2sk+t} \) is weakly lexical, which contradicts the maximality of \( L \).

**Claim 1.2** \( |c(B, L) \setminus c(L)| \leq (2sk + t)|B| \).

\( |c(B, L) \setminus c(L)| \leq |c(B, T_{2sk+t})| \leq (2sk + t)|B| \) by the definition of \( B \).

**Claim 1.3** \( \left| (c(B) \cup c(B, A)) \setminus c_0 \right| < \left( \frac{ER(s+t,3sk+t+1,k)}{2} \right) \).

Let \( A = A^1 \cup A^2 \), where \( A^1 := \{v \in A \mid \text{there exists } y \in L \setminus T_{3sk+t} \text{ with } c(vy) \neq \lambda(y)\} \), and \( A^2 := A \setminus A^1 \).
Figure 2.2 A rainbow $C_k$ in Claim 1.3

First, we show that $c(B, A^1) \subseteq c_0$. Assume that $c(v'v) \not\in c_0$ for some $v \in A^1$ and $v' \in B$ with $c(vy) \neq \lambda(y)$ for some $y \in L \setminus T_{3sk+t}$ and $c(v'x) = \lambda(x)$ for any $x \in L \setminus T_{2sk+t}$. From Claim 0.1, we can find $y'$, one of the last $2s - 1$ elements in $T_{3sk+t} \setminus T_{2sk+t}$ such that $\lambda(y')$ is neither $c(vy)$ nor $\lambda(y)$. Since $\lambda(y') = c(v'y')$, we have that $c(v'y') \not\in \{c(vy), \lambda(y)\}$. Moreover we have $|yLy'| > (s - 1)(k - 2) + 1$. By Claim 0.3, there is a rainbow $C_k$ induced by $\{v, v'\} \cup yLy'$, see Figure 2.2.

Second, we shall observe that $|A^2 \cup B| < ER(s + t, 3sk + t + 1, k)$ by the argument similar to one used in Claim 1.1. We see that otherwise $A^2 \cup B$ contains a lexically colored complete subgraph on $3sk + t + 1$ vertices, which together with $L - T_{3sk+t}$ gives a larger than $L$ weakly lexically colored complete subgraph.

PART 2

We shall show that $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)| \leq n\left(\frac{s-3}{2} + \frac{1}{s-1}\right)$.

In order to count the number of colors in $A$ and $(A, L)$, we consider a representing graph of these colors as follows. First, consider a set $E'$ of edges from $K[A]$ having exactly one edge of each color from $c(A) \setminus c_0$. Second, consider a set of edges $E''$ from the bipartite graph $K[A, L]$ having exactly one edge of each color from $c(A, L) \setminus c(L)$. Let $G$ be a graph with edge-set $E' \cup E''$ spanning $A$. Then $|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| = |E(G)|$.

We need to estimate the number of edges in $G$. Let $A_1, \ldots, A_p$ be sets of vertices of the
connected components of $G[A]$. Let $L_1, \ldots, L_p$ be sets of the neighbors of $A_1, \ldots, A_p$ in $L$ respectively, i.e., for $1 \leq i \leq p$, $L_i := \{x \in L \mid \{x, y\} \in E(G) \text{ for some } y \in A_i\}$. Let

$$G_1 := \bigcup_{i : |E(G[A_i, L_i])| \leq 1} G[A_i],$$

$$G_2 := \bigcup_{i : |E(G[A_i, L_i])| \geq 2} G[A_i \cup L_i].$$

Let $G'_1, \ldots, G'_{p_1}$ be the connected components of $G_1$, and let $G''_1, \ldots, G''_{p_2}$ be the connected components of $G_2$. See Figure 2.3 for an example of $G_1$ and $G_2$.

Claim 2.1 We may assume that $V(G) \cap L \subseteq L \setminus T_{sk+t}$.

For a fixed $v \in A$, let $\omega$ be a color in $c(v, L) \setminus c(L)$, if such exists. Let $L(\omega) := \{x \in L \mid c(vx) = \omega\}$. Suppose $L(\omega) \subseteq T_{sk+t}$. Since $v \in A$, there exists $y \in L \setminus T_{sk+t}$ such that $c(vy) \neq \lambda(y)$. Let $y' \in L(\omega) \subseteq T_{sk+t}$. Then $c(vy') \notin \{c(vy), \lambda(y)\}$. Since $|yLy'| > (s-1)k + 1 > (s-1)(k-1) + 1$, there is a rainbow $C_k$ induced by $\{v\} \cup yLy'$ by Claim 0.3, see figure 4. Therefore $L(\omega) \cap (L \setminus T_{sk+t}) \neq \emptyset$. Hence we can choose edges for the edge set $E''$ of $G$ only from $K[A, L \setminus T_{sk+t}]$.

Claim 2.2 For every $i$, $1 \leq i \leq p$, $K[A_i, T_i]$ is monochromatic; for every $j$, $1 \leq j \leq p_2$, $K[V(G''_j), T_i]$ is monochromatic. In particular, for every $h$, $1 \leq h \leq p_1$, $K[V(G'_i), T_i]$ is monochromatic.

1. Fix $i$, $1 \leq i \leq p$. We show that $K[A_i, T_i]$ is monochromatic. Let $v \in A_i$ and $y \in L \setminus T_{2sk+t}$ with $c(vy) \neq \lambda(y)$.
(1) For any $y' \in T_{sk+t}$, $c(vy')$ is either $c(vy)$ or $\lambda(y)$. Indeed if $c(vy') \not\in \{c(vy), \lambda(y)\}$, then there is a rainbow $C_k$ induced by $\{v\} \cup yLy'$ by Claim 0.3, see Figure 2.4.

(2) $|c(v, T_t)| = 1$. Indeed, let $L^y = \{x \in T_{sk+t} \setminus T_t \mid \lambda(x) \neq c(vy) \text{ and } \lambda(x) \neq \lambda(y)\}$. Then by Claim 0.1, $|L^y| \geq |T_{sk+t} \setminus T_t| - 2(s - 1) + 1 > (s - 1)(k - 3) + 1$. Hence $|c(L^y)| \geq k - 2$ by Claim 0.2. Let $z$ be the vertex in $L^y$ preceding every other vertex in $L^y$. Suppose there is $x \in T_t$ such that $c(vx) \neq c(vz)$. Since $c(L^y) \subseteq c(zLx)$, there exists a rainbow path from $z$ to $x$ on $k - 1$ vertices in $T_{sk+t}$ of colors disjoint from $\{c(vy), \lambda(y)\}$. So there is a rainbow $C_k$ induced by $\{v\} \cup zLx$, see Figure 2.5. Therefore for any $x \in T_t$, $c(vx) = c(vz) \in \{c(vy), \lambda(y)\}$.

(3) For any neighbor $v'$ of $v$ in $G[A_i]$, if such exists, $c(v', T_t) = c(v, T_t)$. Indeed, we see that for any $y' \in T_{sk+t}$, $c(v'y') \in \{c(vy), \lambda(y)\}$, otherwise there is a rainbow $C_k$ induced by $\{v, v'\} \cup yLy'$ by Claim 0.3. Also we see that for any $x \in T_t$, $c(v'x) = c(vz) \in \{c(vy), \lambda(y)\}$, where $z$ is defined above; otherwise there is a rainbow $C_k$ induced by $\{v, v'\} \cup zLx$, see Figure 2.6. Therefore $c(v', T_t) = c(v, T_t)$.
Figure 2.6 Rainbow $C_k$’s in Claim 2.2-1.(3)

Figure 2.7 Rainbow $C_k$’s in Claim 2.2-2.(1): red when $|P| = k - 2$, green when $|P| < k - 2$.

(4) Since $G[A_i]$ is connected, $K[A_i, T_t]$ is monochromatic of color $c(vz)$.

Note that to avoid a monochromatic $K_{s,t}$, we must have that $|A_i| \leq s - 1 \leq k - 2$ for $1 \leq i \leq p$.

2. Fix $j$, $1 \leq j \leq p_2$. We show that $K[V(G''_j), T_t]$ is monochromatic.

(1) $K[V(G''_j) \cap L, T_t]$ is monochromatic. Indeed, since $G''_j$, a connected component of $G$, is a union of $G[A_i \cup L_i]$’s satisfying $|E(G[A_i, L_i])| \geq 2$, by the connectivity, it is enough to show that $\lambda(x) = \lambda(x')$ for any $x, x' \in L_i$ for $L_i$ in $G''_j$, where $x$ precedes $x'$. From Claim 2.1, we may assume that $x, x'$ are in $L \setminus T_{sk+t}$. Suppose $\lambda(x) \neq \lambda(x')$. Let $v, v' \in A_i$ such that $\{v, x\}$ and $\{v', x'\}$ are edges of $G$ (possibly $v = v'$). Let $P$ denote a set of vertices on a path from $v$ to $v'$ in $G[A_i]$. Then $1 \leq |P| \leq k - 2$ since $|A_i| \leq k - 2$. If $|P| = k - 2$, then $P \cup \{x, x'\}$ induces a rainbow $C_k$, otherwise so does $P \cup \{x\} \cup x'Lx_q$ from Claim 0.3, see Figure 2.7. Therefore $\lambda(x) = \lambda(x')$.

(2) $K[V(G''_j), T_t]$ is monochromatic. To prove this, consider $i$ such that $G[A_i, L_i] \subseteq G''_j$.

Observe first that $K[A_i, T_t]$ and $K[L_i, T_t]$ are monochromatic by 1.(4) and 2.(1). Next,
we shall show that \( c(A_i, T_i) = \lambda(L_i) \). Suppose \( c(A_i, T_i) \neq \lambda(L_i) \) for some \( i \) such that \( G[A_i \cup L_i] \subseteq G_j'' \). Let \( v, v' \in A_i \) and \( x, x' \in L_i \) such that \( \{v, x\} \) and \( \{v', x'\} \) are edges of \( G \) (possibly either \( v = v' \) or \( x = x' \)). Since \( |E(G[A_i, L_i])| \geq 2 \), we can find such vertices. So \( c(vx) \neq c(v'x') \) and \( \{c(vx), c(v'x')\} \cap c(L) = \emptyset \). We may assume that \( x, x' \in L \setminus T_{sk+t} \) by Claim 2.1. Since \( c(A_i, T_i) \neq \lambda(L_i) \), \( c(vx) = c(v'x') = c(A_i, T_i) \), otherwise there is a rainbow \( C_k \) induced by \( \{v\} \cup xLx_q \) or \( \{v'\} \cup x'Lx_q \) by Claim 0.3, see Figure 2.8. Then it contradicts the fact that \( c(vx) \neq c(v'x') \).

We have that for any \( i \) such that \( G[A_i, L_i] \subseteq G_j'' \), \( c(A_i, T_i) = \lambda(L_i) \). This implies that \( K[A_i \cup L_i, T_i] \) is monochromatic of color \( \lambda(L_i) \). Since \( G_j'' \) is connected and \( A_i \)s are disjoint, we have that for any \( i, i' \) such that \( G[A_i, L_i], G[A_{i'}, L_{i'}] \subseteq G_j'' \), \( L_i \cap L_{i'} \neq \emptyset \), so \( \lambda(L_i) = \lambda(L_{i'}) = \lambda \), for some \( \lambda \). Therefore \( K[V(G_j''), T_i] \) is monochromatic of color \( \lambda \).

**Claim 2.3** For \( 1 \leq i \leq p_1 \) and \( 1 \leq j \leq p_2 \), \( 1 \leq |V(G_j'')| \leq s - 1 \) and \( 1 \leq |V(G_j'')| \leq s - 1 \).

This claim now follows from the previous instantly.

The following claim deals with a small quadratic optimization problem we shall need.

**Claim 2.4** Let \( n, s \in \mathbb{N} \). Suppose \( n \) is sufficiently large and \( s \geq 2 \). Let \( \xi_1, \ldots, \xi_m \in \mathbb{N} \), \( 1 \leq \xi_i \leq s - 1 \) and \( \sum_{i=1}^{m} \xi_i \leq n \). Then
\[
\sum_{i=1}^{m} \left( \frac{\xi_i - 1}{2} \right) \leq n \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right).
\]
The equality holds if and only if \( m = \frac{n}{s - 1} \) and \( \xi_1 = \cdots = \xi_m = s - 1 \).

We use induction on \( m \). If \( m = 1 \), then
\[
\frac{(\xi - 1)(\xi - 2)}{2} \leq \frac{(s - 2)(s - 3)}{2} \leq n \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right), \text{ for any } n \geq s - 1,
\]
where the first inequality becomes equality iff \( \xi = s - 1 \), and the second does iff \( n = s - 1 \).

Suppose \( m \geq 2 \), \( \sum_{i=1}^{m} \xi_i \leq n \), and \( 1 \leq \xi_i \leq s - 1 \) for \( 1 \leq i \leq m \). Since \( \sum_{i=1}^{m-1} \xi_i \leq n - \xi_m \), by induction,
\[
\sum_{i=1}^{m-1} \left( \frac{\xi_i - 1}{2} \right) \leq (n - \xi_m) \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right),
\]
for any \( n \geq (m - 1)(s - 1) + \xi_m \), where the equality holds iff \( m - 1 = \frac{n - \xi_m}{s - 1} \) and \( \xi_1 = \cdots = \xi_{m-1} = s - 1 \). Hence it is enough to show that \( (n - \xi_m) \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right) + \left( \frac{\xi_m - 1}{2} \right) \leq n \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right) \) or equivalently \( \xi_m \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right) - \left( \frac{\xi_m - 1}{2} \right) \geq 0 \), and the equality holds iff \( \xi_m = s - 1 \). If \( \xi_m = 1 \), that is obvious. Assume \( \xi_m > 1 \), then
\[
\xi_m \left( \frac{s - 4}{2} + \frac{1}{s - 1} \right) - \left( \frac{\xi_m - 1}{2} \right) = \frac{1}{2} \left( -\xi_m^2 + \left( s - 1 + \frac{2}{s - 1} \right) \xi_m - 2 \right) = \frac{1}{2} \left( -\xi_m + \frac{2}{s - 1} \right) \left( \xi_m - (s - 1) \right) \geq 0,
\]
since \( 2 \leq \xi_m \leq s - 1 \).

**Claim 2.5** \( |c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)| = |E(G)| + |c(L)| \leq n \left( \frac{s - 2}{2} + \frac{1}{s - 1} \right) \).

We have that
\[
|E(G)| \leq (|E(G_1)| + p_1) + |E(G_2)| = \sum_{i=1}^{p_1} |E(G'_i)| + p_1 + \sum_{i=1}^{p_2} |E(G''_i)|.
\]
Moreover each component \( G''_i \) of \( G_2 \) contributes at most 1 to \( |c(L)| \) by Claim 2.2, and \( G_1 \) and \( G_2 \) are vertex disjoint. So
\[
|c(L)| \leq n - |V(G_1)| - |V(G_2)| + p_2 = n - \sum_{i=1}^{p_1} |V(G'_i)| - \sum_{i=1}^{p_2} |V(G''_i)| + p_2
\]
Hence we have

\[
|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)| = |E(G)| + |c(L)|
\]

\[
\leq \sum_{i=1}^{p_1} |E(G'_i)| + p_1 + \sum_{i=1}^{p_2} |E(G''_i)| + n - \sum_{i=1}^{p_1} |V(G'_i)| - \sum_{i=1}^{p_2} |V(G''_i)| + p_2
\]

\[
= \sum_{i=1}^{p_1} |E(G'_i)| + \sum_{i=1}^{p_2} |E(G''_i)| - \sum_{i=1}^{p_1} (|V(G'_i)| - 1) - \sum_{i=1}^{p_2} (|V(G''_i)| - 1) + n
\]

\[
\leq \sum_{i=1}^{p_1} \left( \frac{|V(G'_i)|}{2} \right) + \sum_{i=1}^{p_2} \left( \frac{|V(G''_i)|}{2} \right) - \sum_{i=1}^{p_1} (|V(G'_i)| - 1) - \sum_{i=1}^{p_2} (|V(G''_i)| - 1) + n
\]

\[
= \sum_{i=1}^{p_1} \left( \frac{|V(G'_i)| - 1}{2} \right) + \sum_{i=1}^{p_2} \left( \frac{|V(G''_i)| - 1}{2} \right) + n
\]

For \(1 \leq i \leq p_1 + p_2\), let

\[
\xi_i = \begin{cases} 
|V(G'_i)|, & \text{if } 1 \leq i \leq p_1 \\
|V(G''_{i-p_1})|, & \text{if } p_1 + 1 \leq i \leq p_1 + p_2
\end{cases}
\]

Then \(\sum_{i=1}^{p_1+p_2} \xi_i \leq n\) and \(1 \leq \xi_i \leq s - 1\) for \(1 \leq i \leq p_1 + p_2\) by Claim 2.3.

From Claim 2.4, we get

\[
|c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)|
\]

\[
\leq \sum_{i=1}^{p_1+p_2} \left( \frac{\xi_i - 1}{2} \right) + n \leq n \left( \frac{s - 2}{2} + \frac{1}{s - 1} \right).
\]

This concludes Part 2 of the proof.

Combining Parts 1 and 2, we see that the total number of colors is at most

\[
\left| (c(B) \cup c(B, A)) \setminus c_0 \right| + |c(B, L) \setminus c(L)| + |c(A) \setminus c_0| + |c(A, L) \setminus c(L)| + |c(L)|
\]

\[
< \left( \frac{ER(s + t, 3sk + t + 1, k)}{2} \right) + (2sk + t)ER(s + t, 2sk + t + 1, k) + n \left( \frac{s - 2}{2} + \frac{1}{s - 1} \right)
\]

\[
\leq g + n \left( \frac{s - 2}{2} + \frac{1}{s - 1} \right),
\]

where \(g = g(s, t, k) = ER^2(s + t, 3sk + t + 1, k)\).
2.4 More precise results for $C_4$

For a coloring $c$ of $E(K_n)$ and a vertex $v$, let $N_c(v)$ be the set of colors between $v$ and $V(K_n) \setminus \{v\}$, not used on edges spanned by $V(K_n) \setminus \{v\}$. Let $n_c(v) = |N_c(v)|$. Note that $c(uv) \in N_c(u) \cap N_c(v)$ if and only if the color $c(uv)$ is used only on the edge $uv$ in the coloring $c$. We call this color a unique color in $c$. For a path $P = v_1v_2 \cdots v_k$, we say that the path $P$ is good if $c(v_iv_{i+1}) \in N_c(v_i)$ for $i = 1, \ldots, k-1$.

**Lemma 2.4.1.** Let $c$ be an edge-coloring of $K_n$ with no rainbow $C_k$. If for all $v \in V(K_n)$, $n_c(v) \geq k-2$, then $(k-1) \mid n$ and $c$ is $k$-anticyclic.

**Proof.** Let $c$ be an edge-coloring of $K_n$ with no rainbow $C_k$. Suppose for all $v \in V(K_n)$, $n_c(v) \geq k-2$. Then for any $v \in V$, we can find a good path of length $k-2$ starting at $v$ by a greedy algorithm. Let this path be $v_1v_2 \cdots v_{k-1}$, and let $c(v_iv_{i+1}) = i$ for $i = 1, \ldots, k-2$. Let $V_0 = \{v_1, \ldots, v_{k-1}\}$.

**Claim 1** For any $u \in V \setminus V_0$, $c(uv_1) = 1$ or $c(uv_1) \notin N_c(v_1)$.

Assume that $c(uv_1) \in N_c(v_1)$. If $c(uv_1) \neq 1$ then $c(uv_{k-1})$ must be the same as $c(uv_1)$, otherwise $v_1 \cdots v_{k-1}uv_1$ is a rainbow $C_k$. Thus, if $c(uv_1) \neq 1$ then $c(uv_1) \notin N_c(v_1)$.

**Claim 2** $\{c(v_1v_i) \mid i = 2, \ldots, k-1\}$ is a set of distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

From Claim 1 we see that the colors from $N_c(v_1)$ not equal to 1 appear only on edges $v_1v_i$ for $i = 2, \ldots, k-1$. Since $n_c(v_1) \geq k-2$, all these edges have distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

**Claim 3** For any $u \in V \setminus V_0$, $c(uv_{k-1}) \notin N_c(v_{k-1})$.

Assume otherwise, then $v_2v_3 \cdots v_{k-1}u$ is a good path. Then $v_1v_3v_4 \cdots v_{k-1}uv_2v_1$ is a rainbow $C_k$ from Claim 2.

**Claim 4** $\{c(v_iv_{k-1}) \mid i = 1, \ldots, k-2\}$ is a set of distinct colors from $N_c(v_{k-1})$ and $n_c(v_{k-1}) = k-2$. 
By Claim 3, we see that all edges of colors from \( N_c(v_{k-1}) \) must occur on edges from \( \{v_i v_{k-1} : i = 1, \ldots, k - 2 \} \). Since \( n_c(v_{k-1}) \geq k - 2 \), edges \( v_i v_{k-1} \), \( i = 1, \ldots, k - 2 \) have distinct colors from \( N_c(v_{k-1}) \) and \( n_c(v_{k-1}) = k - 2 \).

**Claim 5** \( V_0 \) induces a rainbow complete subgraph with all colors unique in \( c \). Moreover, for each \( v_i \) and each \( u \notin V_0 \), \( c(uv_i) \) is not unique in \( c \).

This follows from the above claims since for \( i = 1, \ldots, k - 2 \), \( v_i v_{i+1} \cdots v_{k-1} v_1 v_2 \cdots v_{i-1} \) is a good path, and \( n_c(v_i) = k - 2 \).

Consider \( u \notin V_0 \) and a good path of length \( k - 2 \) starting at \( u \). Let the vertex set of this path be \( V_1 \). If \( V_0 \) and \( V_1 \) share a vertex, say \( v_i \), then \( v_i u \) has a unique color, a contradiction to Claim 5. Thus the graph is vertex-partitioned into copies of \( K_{k-1} \) each rainbow colored with unique colors. To avoid a rainbow \( C_k \), any edges between two fixed parts must have the same color. Therefore \((k - 1) | n \) and \( c \) is \( k \)-anticyclic.

By induction on \( n \) and the above lemma with \( k = 4 \), we have the following results.

**Corollary 2.4.2.** \( AR(n, C_4) = |c^*| = 4/3n + O(1) \).

**Proof.** We need to show that for any edge-coloring \( c \) of \( K_n \) with no rainbow \( C_4 \), \( |c| \leq |c^*| = 4/3n + O(1) \).

We use induction on \( n \). The statement trivially holds for \( n = 3 \). Let \( c \) be a coloring of \( E[K_n] \) with no rainbow \( C_4 \), \( n \geq 4 \). If for all \( v \in V(K_n) \), \( n_c(v) \geq 2 \), then by Lemma 2.4.1, \( c \) is 4-anticyclic. So \( |c| \leq |c^*| \). Suppose there is a \( v \in V(K_n) \) with \( n_c(v) \leq 1 \). Let \( G = K_n - v \). Let \( c' \) be the coloring of \( E(G) \) induced by \( c \). Then by induction hypothesis, \( |c'| \leq 4/3(n-1) + O(1) \). Hence \( |c| \leq |c'| + 1 \leq 4/3n + O(1) \).

**Theorem 2.4.3.** Let \( n \geq 3 \). Let \( G \) be a graph whose edges do not induce a star. Let \( s = s(G) \) and \( t = t(G) \) if \( G \) is bipartite.
maxR(n; G, C_4) = \begin{cases} 
\frac{4}{3}n + O(1), & \text{if } (\chi(G) = 2 \text{ and } s(G) \geq 4) \text{ or } (\chi(G) \geq 3) \\
n, & \text{otherwise}
\end{cases}

Proof. Suppose \((\chi(G) = 2 \text{ and } s(G) \geq 4) \text{ or } (\chi(G) \geq 3)\). For the lower bound, consider the 4-anticyclic coloring \(c^*\). Each color class of \(c^*\) is either \(K_{1,m}, K_{2,m},\) or \(K_{3,m}\) for some \(m \geq 1\), thus \(c^*\) contains no monochromatic copy of \(G\). The upper bound follows from Corollary 2.4.2.

Suppose \(G\) is bipartite and \(s(G) \leq 3\). We use induction on \(n\). The statement trivially holds for \(n = 3\). Let \(c\) be a coloring of \(E(K_n)\) with no monochromatic \(G\) and no rainbow \(C_4\). If \(n_c(v) \geq 2\) for all \(v \in V\), by Lemma 2.4.1 there is a color class of \(c\) that induces a \(K_{3,3}^m\) for some \(m \geq 1\), which contains \(G\). Hence we can find a \(v \in V\) with \(n_c(v) \leq 1\). Then by the induction hypothesis, \(\maxR(n; G, C_4) \leq n\). The lower bound is obtained from the coloring \(c^{**}\) with \(s = s(G)\) and \(k = 4\). Each color class of \(c^{**}\) is \(K_{1,m}\) if \(s(G) = 2\), either \(K_{1,m}\) or \(K_{2,m}\) if \(s(G) = 3\) for some \(m \geq 1\), thus \(c^{**}\) contains no monochromatic copy of \(G\). The total number of colors in either cases is \(n\).

Acknowledgments The authors thank the referee for a very careful reading and useful comments improving the presentation of the results.
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CHAPTER 3. A short proof of anti-Ramsey number for cycles

Jihyeok Choi

Abstract

This note contains a simplified proof of Anti-Ramsey theorem for cycles by J. J. Montellano-Ballesteros, V. Neumann-Lara [5], which was originally conjectured by P. Erdős, M. Simonovits, V. T. Sós [3].

3.1 Introduction

For a graph $H$, the classical anti-Ramsey number $AR(n, H)$ is the maximum number of colors in a coloring of edges of $K_n$ with no rainbow copy of $H$. It was introduced by Erdős, Simonovits and Sós [3]. When $H$ is a cycle of length $k$, $C_k$, they provided a rainbow $C_k$-free coloring of edges of $K_n$ with $n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1)$ colors and conjectured that this is optimal. They proved the conjecture for $k = 3$. Alon [1] proved the conjecture for $k = 4$ and derived an upper bound for general $k$. In [4], Jiang and West improved the general upper bound and mentioned that the conjecture has been proven for $k \leq 7$, see also Schiermeyer [7]. Finally, Montellano-Ballesteros and Neumann-Lara [5] proved the conjecture completely. The main technique used in [5] is a careful, detailed analysis of a graph representing the coloring, in particular, proving that each component of such a graph is Hamiltonian if each vertex has enough “new” colors. This paper uses the same idea as in [5], but shortens the proof.

Theorem 3.1.1. For $k \geq 3$ and $n \in \mathbb{N}$,

$$AR(n, C_k) \leq n \left(\frac{k-2}{2} + \frac{1}{k-1}\right) - 1.$$
3.2 Definitions and proofs of main results

Let $K = K_n$ for a fixed $n$. For an edge coloring $c$ of $K$, and a vertex $v \in V(K)$, let the set of new colors at $v$, $NEW_c(v)$, be the set of colors used on edges between $v$ and $V(K) \setminus \{v\}$, but not used on edges spanned by $V(K) \setminus \{v\}$. Let $new_c(v) = |NEW_c(v)|$. Then the number of colors used by $c$ on $K$, $|c|$, equals $new_c(v) + |c(K - v)|$, where for a subgraph $H$ of $K$, $|c(H)|$ denotes the number of colors used by $c$ on the edges of $H$. Here we simply have written $|c|$ instead of $|c(K)|$. For pairwise disjoint subsets $X, Y, Z$ of $V(K)$, let $K[X]$ be the subgraph induced by $X$, $K[X,Y]$ the bipartite subgraph induced by $X$ and $Y$, $K[X,Y,Z]$ the tripartite subgraph induced by $X$, $Y$, and $Z$. Then the corresponding sets of colors used in those subgraphs are denoted by $c(X)$, $c(X,Y)$, and $c(X,Y,Z)$ respectively. For a subgraph $H$ of a graph $G$ and a vertex $v$ of $G$, let $\deg_H(v) := |N_G(v) \cap V(H)|$.

We now state a version of the Dirac and Ore’s theorems for Hamiltonian cycle which is essential for our proofs.

Theorem 3.2.1 (Dirac[2], Ore[6]). Let $P = v_1, v_2, \ldots, v_m$, $m \geq 3$, be a path in a connected graph $G$. Suppose $\deg_P(v_1) + \deg_P(v_m) \geq m$.

(i) Then $V(P)$ contains a cycle of length $m$ in $G$.

(ii) If $P$ is a longest path in $G$, then $G$ is Hamiltonian.

We define a few special edge colorings of a complete graph with no rainbow $C_k$. We say that an edge-coloring $c$ of $K$ is weak $k$-anticyclic if there is a partition of $V(K)$ into sets $V_1, \ldots, V_t$ with $1 \leq |V_i| \leq k - 1, i = 1, \ldots, t$, such that (i) for any $i, j$ with $1 \leq i < j \leq t$, $|c(V_i, V_j)| = 1$; (ii) for any $i, j, \ell$ with $1 \leq i < j < \ell \leq t$, $|c(V_i, V_j, V_\ell)| \leq 2$; and (iii) $c$ has no rainbow $C_k$. In addition, if all but at most one of the parts of the partition are exactly of size $k - 1$ and the edges spanned by each $V_i$ have own colors (i.e., colors used only once), then $c$ is called $k$-anticyclic. We denote a fixed coloring from the set of $k$-anticyclic colorings of $K_n$ such that the color of any edge between $V_i$ and $V_j$ is $\min\{i, j\}$, by $c^*$. Then we easily see the following.
Lemma 3.2.2. If $c$ is weak $k$-anticyclic, then

$$|c| \leq |c^*| \leq n \left( \frac{k - 2}{2} + \frac{1}{k - 1} \right) - 1.$$ 

Next lemma is the main tool for the proof of the main theorem. It appears in a different form in [5, Lemma 9]. We include it here for completeness.

Lemma 3.2.3. Let $k \geq 4$. Let $c$ be an edge-coloring of $K$ with no rainbow $C_k$. If for all $x, y \in V(K)$ with $x \neq y$,

$$\text{new}_c(x) \geq 2 \text{ and } \text{new}_c(x) + \text{new}_c(y) \geq k - 1,$$ (3.1)

then $c$ is weak $k$-anticyclic.

Proof. Consider a representing graph $G$ of $c$ such that it spans $K$ and has exactly one edge of each color from $\{\text{NEW}_c(v) \mid v \in V(K)\}$. The hypothesis (3.1) gives a bound on degrees of vertices in $G$, namely the sum of degrees of two distinct vertices in $G$ is at least $k - 1$. In the following, $H$ denotes a connected component of $G$.

Claim 1 If there is a cycle of length $k - 1$ in $H$, then $|V(H)| = k - 1$.

Suppose not, i.e., there is a cycle, $(v_1, \ldots, v_{k-1}, v_1)$, and $V(H) \setminus \{v_1, \ldots, v_{k-1}\} \neq \emptyset$. Since $H$ is connected, some $u \in V(H) \setminus \{v_1, \ldots, v_{k-1}\}$ is adjacent to some vertex in $\{v_1, \ldots, v_{k-1}\}$, say $v_1$. If $c(u, v_1) \in \text{NEW}_c(v_1)$, then $c(u, v_2) = c(v_2, v_3)$; otherwise $(v_1, u, v_2, v_3, \ldots, v_{k-1}, v_1)$ in $K$ is a rainbow $C_k$. Similarly $c(u, v_3) = c(v_3, v_4), \ldots, c(u, v_{k-1}) = c(v_{k-1}, v_1)$, and eventually $c(u, v_1) = c(v_1, v_2)$, which contradicts that $uv_1$ and $v_1v_2$ are edges of $H$. Hence $c(u, v_1) \in \text{NEW}_c(u)$. By the similar argument as above, we have $c(u, \{v_1, \ldots, v_{k-1}\}) = c(u, v_1) \in \text{NEW}_c(u)$. Since we assumed $\text{new}_c(u) \geq 2$, there is $w \in V(H) \setminus \{v_1, \ldots, v_{k-1}, u\}$ with $c(u, w) \in \text{NEW}_c(u)$ and $c(u, w) \neq c(u, v_1)$. Considering cycles of length $k$ in $K, (v_{k-2}, u, w, v_1, v_2, \ldots, v_{k-2})$ and $(v_1, w, u, v_3, \ldots, v_{k-1}, v_1)$, we have $c(w, v_1) = c(v_1, v_2) = c(v_{k-1}, v_1)$, which contradicts that $v_1v_2$ and $v_{k-1}v_1$ are edges of $H$.

Claim 2 $\frac{k + 1}{2} \leq |V(H)| \leq k - 1$ (Hence $H$ is Hamiltonian from by (3.1) and Theorem 3.2.1 ).
The lower bound follows from (3.1). If in \( H \) every path has at most \( k - 1 \) vertices or there is a cycle of length \( k - 1 \), then from Theorem 3.2.1 and Claim 1, we have that the upper bound holds. Hence we may assume that in \( H \) there is a path on at least \( k \) vertices, but no \( C_{k-1} \). In particular, we can find a path, \( v_1, \ldots, v_k \) satisfying \( c(v_{k-1}, v_k) \in NEW_c(v_{k-1}) \) since (i) considering \( P_1 := v_1, \ldots, v_{k-1} \), to avoid \( C_{k-1} \), we have \( \deg_{P_1}(v_1) + \deg_{P_1}(v_{k-1}) < k - 1 \); (ii) from (3.1), without loss of generality we can find a \( v_k \in V(H) \setminus V(P_1) \) such that \( v_{k-1}v_k \) is an edge of \( H \) and \( c(v_{k-1}, v_k) \in NEW_c(v_{k-1}) \).

Let \( P_2 := v_2, \ldots, v_k \). Then

\[
\deg_{P_2}(v_2) \geq new_c(v_2), \tag{3.2}
\]

since otherwise there is \( x \in V(H) \setminus V(P_2) \) such that \( c(x, v_2) \in NEW_c(v_2) \) and \( c(x, v_2) \neq c(v_2, v_3) \), in which case we obtain a rainbow \( C_k \) in \( K \), namely \( (x, v_2, \ldots, v_k, x) \). Also we have \( \deg_{P_2}(v_k) < new_c(v_k) \) since otherwise together with (3.2), \( V(P_2) \) induces a cycle of length \( k - 1 \) in \( H \) by Theorem 3.2.1. Therefore we can find a \( v_{k+1} \in V(H) \setminus V(P_2) \) such that \( v_kv_{k+1} \) is an edge of \( H \) and \( c(v_k, v_{k+1}) \in NEW_c(v_k) \). Note that \( v_{k+1} \neq v_2 \) since otherwise \( (v_2, \ldots, v_k, v_2) \) is a rainbow \( C_{k-1} \) in \( H \).

Let \( P_3 := v_3, \ldots, v_k, v_{k+1} \). Then

\[
\deg_{P_3}(v_3) \geq new_c(v_3), \tag{3.3}
\]

since otherwise there is \( y \in V(H) \setminus V(P_3) \) such that \( c(y, v_3) \in NEW_c(v_3) \) and \( c(y, v_3) \neq c(v_3, v_4) \), so \( (y, v_3, \ldots, v_{k+1}, y) \) is a rainbow \( C_k \) in \( K \).

Now we note that \( c(v_2, v_{k+1}) = c(v_2, v_3) \) to avoid a rainbow \( C_k \) induced by \( \{v_2, \ldots, v_{k+1}\} \) in \( K \). Let \( S = \{i+1 \mid v_2v_i \in E(H), i = 3, \ldots, k - 1\} \) and \( T = \{j \mid v_3v_j \in E(H), j = 4, \ldots, k\} \). So \( S,T \subseteq \{4, \ldots, k\} \) and \( |S| + |T| \geq new_c(v_2) + new_c(v_3) \geq k - 1 \). Thus \( |S \cap T| \geq 2 \). Let \( i + 1 \in S \cap T \) where \( i \neq 3 \). Then \( (v_2, v_i, v_{i-1}, \ldots, v_3, v_{i+1}, v_{i+2}, \ldots, v_{k+1}, v_2) \) is a rainbow \( C_k \) (see Figure 3.1).

**Claim 3** For any two components \( H \) and \( H' \) of \( G \), \( |c(H, H')| = 1 \).
If there is an edge $e$ between $H$ and $H'$, incident to, say, some $v \in H$ of color from $NEW_c(v)$, then we can make $H$ and $H'$ connected by adding the edge $e$ and deleting some edge incident to $v$ of the same color as $e$ in $H$, so the resulting graph $\tilde{G}$ has a connected component of order $\geq 2(\frac{k+1}{2})$, which contradicts that every connected component is of order $\leq k - 1$. Hence the colors of edges between $H$ and $H'$ are not from $c(H)$ nor from $c(H')$. Since each component is Hamiltonian and of order $\geq \frac{k+1}{2}$, to avoid a rainbow $C_k$, by the same type of argument as in Claim 1, we must have that $|c(H, H')| = 1$.

**Claim 4** For any components $H$, $H'$, and $H''$ of $G$, $|c(H, H', H'')| \leq 2$.

By Claim 3, $|c(H, H')| = |c(H', H'')| = |c(H'', H)| = 1$. If $|c(H, H', H'')| = 3$, then we can easily find a rainbow $C_k$ since each component is Hamiltonian and of order $\geq \frac{k+1}{2}$.

**Proof of Theorem 3.1.1.** When $k = 3$, it is proved that $AR(n, C_3) = n - 1$ in [3]. Suppose $k \geq 4$. We use an induction on $n$. If $n = k - 1$, then trivial. If there is $v \in V(K)$ with $new_c(v) \leq \frac{k}{2} - 1$, then by induction,

$$|c| \leq \frac{k}{2} - 1 + |c(K - v)| \leq \frac{k}{2} - 1 + (n - 1) \left( \frac{k - 2}{2} + \frac{1}{k - 1} \right) - 1 = n \left( \frac{k - 2}{2} + \frac{1}{k - 1} \right) + \frac{k}{2} - 2 - \frac{k - 2}{2} - \frac{1}{k - 1} < n \left( \frac{k - 2}{2} + \frac{1}{k - 1} \right) - 1$$

If, for any $v \in V(K)$, $new_c(v) \leq \frac{k}{2} - 1$, then it now follows from Lemma 3.2.2 and 3.2.3.
Bibliography


CHAPTER 4. General Conclusions

In Part I, a random graph model associated with scale-free networks is studied. In particular, preferential attachment models with the time-dependent selection scheme are considered, and an infinite dimensional large deviations bound for the sample path evolution of the empirical degree distribution is found.

In Part II, some (edge) colorings of graphs in Ramsey and anti-Ramsey theories are studied. For two graphs, $G$ and $H$, an edge-coloring of a complete graph, $K_n$, is called $(G,H)$-good if there is no monochromatic subgraph isomorphic to $G$ and no rainbow (totally multicolored) subgraph isomorphic to $H$ in this coloring. The set of the number of colors used by $(G,H)$-good colorings of $K_n$, called a spectrum and denoted by $S(n;G,H)$, is considered. Then the maximum element in this set, when $H$ is a cycle, is investigated.

4.1 Future improvements and other directions

We plan to continue studying preferential attachment graphs and colorings of graphs:

- In the study of a random graph model (Part I, Chapter 1), we deal with the time-inhomogeneity of the form $j/n$, which is in an array structure. The method used there is not applicable directly to other time-inhomogeneous types. For example, if $\beta \equiv 0$ and $p(j/n) = 1/j$ in our model, then it is the Chinese restaurant process, where, with an infinite number of circular tables, each with infinite capacity, at each time $j \leq n$, a new customer chooses uniformly at random to sit at one of the following $j$ places: directly to the left of one of the $j - 1$ customers already sitting at an occupied table, or at a new, unoccupied circular table. Since this process is already in the ‘boundary’ in our model, LDP obtained gives only a trivial distribution. It would be of interest to understand nice
properties of other time-inhomogeneous models.

- In Part II, Chapter 1, we study the fundamental question of whether the spectrum, $S(n; G, H)$, is an interval for given $n, G$ and $H$, i.e. whether, for any $k$ with $\min S(n; G, H) \leq k \leq \max S(n; G, H)$, there is a $(G, H)$-good coloring of $K_n$ using $k$ colors. It is still open whether there are graphs $G$ and $H$ such that, for any natural number $N$, there exists $n > N$ so that $S(n; G, H)$ is not an interval. This is one of future research goals.

- The maximum element in a spectrum, $\max S(n; G, H)$, has already been determined asymptotically when $H$ is not vertex-partitioned into at most two induced forests. In this context, $\max S(n; G, H)$ is studied when $H$ is a cycle in Part II, Chapter 2. ‘One of the most intriguing open problems’ in this area is to determine $\max S(n; K_4, K_4)$. 