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Solutions of two plasticity problems by the deformation and incremental theories

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SOLUTIONS OF TWO PLASTICITY PROBLEMS BY THE DEFORMATION
AND INCREMENTAL THEORIES

by

Roger Sandberg Hanson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

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Approved:

Signature was redacted for privacy.

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LIST OF SYMBOLS

$A_1, A_2$ constants defined in text

$C_1, C_2, C_3, C_4$ constants defined in text

$\cos(i, \eta)$ cosine of angle between $i$ axis and $\eta$ axis

$\text{dep} = \sqrt{\frac{2}{3}} (de_{11}^p + de_{22}^p + de_{33}^p)$

equivalent plastic strain increment

$E$ Young's Modulus

$e_{ep} = \frac{\sqrt{2}}{3} \sqrt{(e_{11}^p - e_{22}^p)^2 + (e_{22}^p - e_{33}^p)^2 + (e_{33}^p - e_{11}^p)^2}$

equivalent plastic strain

$e_{et} = \frac{\sqrt{2}}{3} \sqrt{(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2}$

equivalent total strain

$e_{ij}$ strain tensor

$e_{ij}^p$ plastic and elastic components of strain tensor

$f$ yield function

$G_{ij}$ tensor function of stress components

$g$ plastic potential

$h$ scalar function of 2nd invariant of stress deviation tensor

$J_1 = S_x + S_y + S_z = S_{11} + S_{22} + S_{33}$

first invariant of stress tensor

$J_2 = -(S_x S_y + S_y S_z + S_z S_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)$

second invariant of stress tensor
\[
J_3 = S_x S_y S_z + 2 \tau_{xy} \tau_{yz} \tau_{zx} - S_x \tau_{yz}^2 - S_y \tau_{zx}^2 - S_z \tau_{xy}^2
\]
third invariant of stress tensor

\[
J_1', J_2', J_3'
\]
first, second, and third invariants of stress deviation tensor

\[K_1, K_2\]
constants defined in text

\[r, t, z\]
radial, tangential, and axial coordinates

\[
\bar{S} = \frac{1}{\sqrt{2}} \sqrt{(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2}
\]
equivalent stress

\[
S = \frac{1}{3} S_{ii} = \frac{1}{3} (S_{11} + S_{22} + S_{33})
\]
mean normal stress

\[
S_{ij} = \begin{pmatrix}
S_{11} S_{12} S_{13} \\
S_{21} S_{22} S_{23} \\
S_{31} S_{32} S_{33}
\end{pmatrix}
= \begin{pmatrix}
S_x \tau_{xy} \tau_{xz} \\
\tau_{yx} S_y \tau_{yz} \\
\tau_{zx} \tau_{zy} S_z
\end{pmatrix}
\]
stress tensor

\[S_{ij}\]
stress deviation tensor

\[S_n\]
normal stress vector

\[T\]
temperature above an arbitrary level

\[w\]
work per unit volume

\[\alpha\]
coefficient of thermal expansion

\[\delta_{ij}\]
Kronecker delta

\[\theta, \psi\]
angles defined in text

\[\lambda\]
factor of proportionality

\[\mu\]
Poisson's ratio (when defined as such in text)
\[ v \]

\[ \mu, \nu \quad \text{Lode's variables (when defined as such in text)} \]

\[ \rho \quad \text{density of material} \]

\[ \omega \quad \text{angular velocity} \]
I. INTRODUCTION AND REVIEW OF LITERATURE

The calculation of stresses in deformable media in which account is taken of plastic flow is currently of great interest in order to take full advantage of the load carrying capacity of available materials. Numerous papers have been published wherein the solution of various plastic flow problems are undertaken. Many of these papers present methods that are applicable only to the problem at hand, and many employ rather complex mathematical techniques, beyond the scope of training usually given the average practicing engineer. It seems little attention has been directed at providing relatively simple, direct methods which can be applied by the engineer toward the solution of practical plastic flow problems.

In the mathematical theory of plasticity there are two widely known theories that may be utilized to solve a plastic flow problem, the incremental, or "flow" theory, and the deformation, or "total" theory. The deformation theory has as its postulate that the state of strain existing in a deformable medium in the plastic range is determined and influenced only by the existing, current state of stress. On the other hand, the incremental theory postulates that the state of strain existing at any given time is dependent on, and influenced by the complete history of the loading program that has been applied to the medium from the first occurrence of plastic flow at any point. Another way of stating the difference is that the deformation theory assumes a unique relation between stress and strain, while on the other hand, the incremental theory assumes that the rate at which the strains are changing is uniquely determined by the stresses. The deformation theory has as a further stipulation that prior loading
history of the material does not include removal of stresses after plastic
deformation has taken place, that is to say, the material does not unload.

A. Gleyzal has obtained a solution to the problem of a circular dia-
phragm under pressure by the deformation theory. S. S. Manson has pre-
sented by deformation theory a solution, and a general method for other
solutions of the problem of the rotating disk of arbitrary contour with a
radial temperature distribution, in the plastic range. Manson, along
with M. B. Millenson has also attacked the problem of creep associated
with high speed rotating disks in gas turbines, by deformation theory.

One may find a number of other applications of deformation theory, the
above being only a small sample, although the deformation theory is gen-
erally recognized to be theoretically unacceptable for the solution of
plastic flow problems.

In a classic often quoted paper, Rodney Hill, E. H. Lee, and S. J.
Tupper have solved the problem of the thick tube under internal pressure
by the incremental theory, using the method of the characteristics of the
differential equations, a technique not commonly known to the average prac-
ticing engineer. In this paper, an ideal plastic solid was assumed to
simplify the calculations. It is clear from a survey of the literature
that the number of problems solved by the deformation theory as compared
to the incremental theory is quite large. The reason is, of course, in
general the deformation theory is simpler to apply.

The objectives of this dissertation are the following:

1. The development of a relatively simple mathematical means of solv-
ing plastic deformation problems by either the incremental theory
or the deformation theory.
2. Application of both theories to two practical boundary value problems to illustrate the technique, and to attempt to answer the question of the amount of difference existing between solutions as obtained by both theories.

To the author's knowledge, the application of both theories to the same boundary value problem has not been done and presented in a simple manner that may be of use to practicing engineers and designers. It may be seen that the method given will be of use to designers of high speed rotating disks, cylinders, and other problems of axial symmetry, in the region of plastic deformation.

Two boundary value problems are solved. They are:

1. the generalized plane strain problem of the infinitely long circular cylinder that is maintained at a uniform temperature of 1000° F, and quenched quickly in order to induce plastic flow, and

2. the plane stress problem of the thin circular disk that is heated by induction at the rim quickly to induce plastic flow.

Both problems are solved by the deformation and incremental theories, and the results of both solutions are presented in graphical form. Experimental work is presented for the plane stress problem of the thin circular disk in the form of measurement of residual stresses, and is compared to the results predicted by the theoretical calculations.

Although no material is truly isotropic both before and after deformation, the simplifying assumption of constant isotropy is often made, and is made in this dissertation, in order to enable calculations to be made easily. The classic experiments of Sir Geoffrey Taylor and H. Quinney on the plastic distortion of thin walled tubes proved quite
conclusively the deviation from isotropy of materials, and even those specimens that seemed to show a high degree of isotropy showed deviation in yielding from the popular Von Mises' yield criteria.

The two most widely used yield criteria are those of Von Mises and Tresca. In the case of plane stress, the non-uniqueness of the normal to the yield surface in the Tresca criterion presents a serious objection to its use, as well as the fact that the ordering of the principal stresses must be known in advance of the solution in order to use it. In view of these two objections, the Von Mises' criterion is utilized.

Daniel C. Drucker\textsuperscript{6} has presented numerous stress-strain relations that can be used to correlate data obtained from experimental tests on material that deviates from isotropy and homogeneity, and exhibits a Bauschinger effect, among other imperfections. However, the mathematical complexity of these flow rules and yield criteria raise a serious objection to their consideration, as well as the expensive experimental program necessary to decide which of them to use, and to evaluate the constants appearing in them. Others who have contributed to this approach are Davis and Parker\textsuperscript{5}, Dorn and Latter\textsuperscript{6}, and Drucker\textsuperscript{7}.

Mathematical discussions have been given by F. Edelman\textsuperscript{9}, Prager\textsuperscript{26,27}, and White and Drucker\textsuperscript{32}.

The paper by Alexander Mendelson and S. S. Manson\textsuperscript{22} served as a background for this dissertation. In this paper the authors attacked by deformation theory four boundary value problems at one specified condition in the loading program, and obtained numerical results by the use of a desk calculator.
II. THE DEFORMATION THEORY OF PLASTICITY VERSUS THE INCREMENTAL THEORY OF PLASTICITY

It is well known that in the Theory of Elasticity, the state of strain existing in a deformable medium is determined uniquely by the accompanying state of stress. Furthermore, one could remove the existing state of stress, returning the body to its initial unstrained state, and then subject it to exactly the same state of stress, and this would produce the identical state of strain encountered originally. Finally, if one began with an existing state of stress, with its accompanying state of strain, and followed any arbitrary loading program, no matter how complicated, finally returning to exactly the original stress state, the conditions would be identical to those in the beginning of this loading program.

The basic feature of the mathematical theory of plasticity which distinguishes it from the mathematical theory of elasticity is the irreversibility of plastic action. This means that the work done by the deforming forces is not recoverable. A plastically deformed body may of course be restored to its original shape by additional plastic action, but in the cycle work is continually dissipated except when only elastic changes in strain take place. Another way of phrasing the distinction between elasticity and plasticity is that in the plastic range the state of strain is not determined by the state of stress but depends upon the prior history of stress, that is, the program of prior loading, or the path of loading. That this must be true can be shown by a simple example, but before this is done, it is necessary to establish a certain foundation for the discussion.
In the mathematical theory of plasticity there exists two theories, the incremental, or "flow" theory, and the deformation, or "total" theory. The deformation theory of plasticity, like the mathematical theory of elasticity, postulates that the state of strain existing in a plastically deformed medium is determined only by the existing accompanying state of stress. What path of loading this deformable medium has suffered prior to the state under consideration is of no consequence to this theory. In contrast to this theory, the incremental theory of plasticity takes into account the prior history of loading, and thus postulates that the state of strain existing in a plastically deformed medium is determined not only by the existing state of stress, but is also a consequence of the complete history of loading that the medium has undergone, before reaching the current state of stress and strain. In reality, deformation theory is not truly a plasticity theory, because it is not path dependent. It shall be shown that one could expect to obtain the same results to the same problem when solved by the two theories only under very special circumstances, that is, when the program of loading satisfies a certain relationship.

When a deformable medium is subjected to a loading program from the unstressed and unstrained state, the condition is first in general of elastic behavior, then elastic-plastic, and perhaps finally completely in the plastic realm. As is well known, in the theory of elasticity, Hooke's Law is usually taken as the relationship between stress and strain. However, in the theory of plasticity no such relatively simple and universal law is assumed; down through the decades since Barre de St. Venant founded the theory of plasticity around 1870, many stress-strain relations have been
postulated, and as many criterions for the transition from elastic action to plastic action have been advanced.

The majority of these stress-strain relations for the plastic range as well as the criteria for transition from the elastic to the plastic regions have been discarded for two reasons; they have been found to disagree with experimental evidence of the plastic behavior of materials, or they have been too mathematically complicated, rendering them too unwieldy to use.

In order to discuss effectively the criteria for transition from the elastic realm to the plastic, it is helpful to review some of the fundamentals of a three-dimensional state of uniform stress in a deformable medium. Consider the elementary tetrahedron shown in figure 1. In general, on the inclined face, the stress vector $S_n$ is neither perpendicular to the inclined face, nor lies in it. However, for simplicity, and without loss of generality for the purpose at hand, it will be assumed that the inclined surface is oriented in such a manner as to make the stress vector $S_n$ perpendicular to the surface; it is evident then, that $S_n$ is one of the three principal stresses, and that the inclined surface is a plane of principal stress. The fact that the principal stresses are normal stresses permits writing a set of three equations for each principal stress. These are the equilibrium equations. Denoting the principal stresses by $S_i$, $i = 1, 2, 3$, one finds

$$S_x \cos(i,x) + \tau_{xy} \cos(i,y) + \tau_{xz} \cos(i,z) = S_i \cos(i,x)$$

$$\tau_{yx} \cos(i,x) + S_y \cos(i,y) + \tau_{yz} \cos(i,z) = S_i \cos(i,y) \quad (1)$$

$$\tau_{zx} \cos(i,x) + \tau_{zy} \cos(i,y) + S_z \cos(i,z) = S_i \cos(i,z)$$
Figure 1. Stresses existing on an elementary tetrahedron, with the plane ABC oriented as a principal stress plane.
or, transposing and rearranging terms,

\[(S_x - S_i) \cos(i,x) + \tau_{xy} \cos(i,y) + \tau_{xz} \cos(i,z) = 0\]

\[\tau_{yx} \cos(i,x) + (S_y - S_i) \cos(i,y) + \tau_{yz} \cos(i,z) = 0\]  \hspace{1cm} (2)

\[\tau_{zx} \cos(i,x) + \tau_{zy} \cos(i,y) + (S_z - S_i) \cos(i,z) = 0\]

This is a system of three homogeneous linear equations that combined with the known relation

\[\cos^2(i,x) + \cos^2(i,y) + \cos^2(i,z) = 1\]  \hspace{1cm} (3)

can be solved for the unknown direction cosines. The trivial solution to the set of equations (2), that is, all the direction cosines zero, would violate equation (3), hence, it can be ruled out. Non-trivial solutions can exist only if the determinant of the coefficients of (2) vanishes, or

\[
\begin{vmatrix}
S_x - S_i & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & S_y - S_i & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & S_z - S_i
\end{vmatrix} = 0
\]  \hspace{1cm} (4)

Expanding this one finds

\[S_1^3 - (S_x + S_y + S_z)S_1^2 + (S_xS_y + S_yS_z + S_zS_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2)S_1 - \]

\[S_xS_yS_z - 2\tau_{xy}\tau_{yz}\tau_{zx} + S_x\tau_{yz}^2 + S_y\tau_{zx}^2 + S_z\tau_{xy}^2 = 0\]  \hspace{1cm} (5)

This is a cubic equation, the three roots of which are the three principal stresses. For a given state of stress, the cubic equation (5) furnishes the same principal stresses, regardless of the assumed Cartesian reference frame; this requires that the coefficients of the second, first, and zero degree terms be independent of the selection of such a reference frame. The three coefficients are the so-called invariants of the stress tensor and are denoted as follows:
\[ J_1 = S_x + S_y + S_z \]
\[ J_2 = -(S_x S_y + S_y S_z + S_z S_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2) \]
\[ J_3 = S_x S_y S_z + 2\tau_{xy} \tau_{yz} \tau_{zx} - S_x \tau_{yz}^2 - S_y \tau_{zx}^2 - S_z \tau_{xy}^2 \]

The same values can be obtained by using the principal axes as a reference frame.

\[ J_1 = S_1 + S_2 + S_3 \]
\[ J_2 = -(S_1 S_2 + S_2 S_3 + S_3 S_1) \]
\[ J_3 = S_1 S_2 S_3 \]

Thus (5) can be written as follows:

\[ S_i^3 - J_i S_i^2 - J_2 S_i - J_3 = 0 \] 

It has been mentioned previously that two necessary foundations for any theory of plasticity are the rule by which one can ascertain when a material undergoes the transition from the elastic to the plastic state, and the stress-strain relations for the plastic domain. A law defining the limit of elasticity under any possible combination of stresses is known as a yield criterion. For example, if a thin cylindrical tube has been uniformly strained in tension into the plastic range and then partially unloaded, one might inquire as to what torque must be added to cause a further permanent distortion.

The following development parallels closely Chapter 2 of reference 12. It is supposed for the present that the material is isotropic, and since then plastic yielding can depend only on the magnitudes of the three principal applied stresses, and not on their directions, any yield criterion is expressible in the form

\[ f(J_1, J_2, J_3) = 0 \]
where as has been stated, \( J_1, J_2, \) and \( J_3 \) are the first three invariants of the stress tensor.

An immediate simplification of (9) can be obtained by using the experimental fact that the yielding of a metal is, to a first approximation, unaffected by a moderate hydrostatic pressure or tension, either applied alone or superimposed on some state of combined stress. Assuming this to be true for the material under consideration, it follows that yielding depends upon only the principal components \((S_1', S_2', S_3')\) of the deviatoric, or reduced, stress tensor

\[
S_{ij}' = S_{ij} - \delta_{ij}S
\]

where \( S = \frac{1}{3} S_{ii} \) is the hydrostatic component of the stress. Here the summation convention is used, where repeated literal subscripts indicate summation over the digits 1, 2, and 3. That is, \( S = \frac{1}{3} (S_{11} + S_{22} + S_{33}) = \frac{1}{3} S_{ii} \). The reader is referred to appendix 1 of reference 12. \( \delta_{ij} \) is the Kronecker delta, defined as follows:

\[
\delta_{ij} = 1 \text{ if } i = j \\
\delta_{ij} = 0 \text{ if } i \neq j
\]

Physically, this means that if one adds or subtracts the same constant stress from all components of the stress tensor, the yielding is not affected. One could clearly subtract or add any constant, not just the one termed \( S \). It becomes evident that this particular choice is a convenient one to use. If one now denotes by \( J_1', J_2', \) and \( J_3' \) the three invariants of the stress deviation tensor, it is clear that the yield criterion may be written as

\[
f(J_1', J_2', J_3') = 0
\]
However, the usefulness of the particular choice of $S$ as the constant stress to be subtracted asserts itself, for under the definition of $J_1'$, it is seen that $J_1' = 0$. Hence, a further simplification may be obtained, the yield criterion becoming

$$f(J_2', J_3') = 0$$

(12)

where

$$J_1' = S_1' + S_2' + S_3' = 0$$

$$J_2' = -(S_1'S_2' + S_1'S_3' + S_2'S_3')$$

$$J_3' = S_1'S_2'S_3'$$

It may be easily verified that $J_2'$ may equally well be written as

$$J_2' = \frac{1}{2} \left( S_1'^2 + S_2'^2 + S_3'^2 \right) = \frac{1}{2} S_{ij}' S_{ij}'$$

and

$$J_2 = \frac{1}{6} \left[ (S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2 \right]$$

$$= \frac{1}{6} \left[ (S_x - S_y)^2 + (S_y - S_z)^2 + (S_z - S_x)^2 \right]$$

$$+ 6 [(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]$$

$$J_2' = \frac{1}{6} \left[ (S_1' - S_2')^2 + (S_2' - S_3')^2 + (S_3' - S_1')^2 \right]$$

as given on page XIV-3 of reference 8.

It is evident that the yield function may be plotted as a surface in three dimensional space. It is helpful to consider a diagram of the yield surface at this time. For convenience, and again without loss of generality for the purpose at hand, one can restrict the discussion to the principal directions. Any stress state may be represented by a bound vector in a three-dimensional space where the three principal stresses are taken
as Cartesian coordinates. In figure 2 \( \mathbf{OS} \) is the vector representing the stress \((S_1, S_2, S_3)\). \( \mathbf{OP} \) is the vector representing the deviatoric stress \((S'_1, S'_2, S'_3)\). \( \mathbf{OP} \) always lies in the plane II whose equation is 

\[(S_1 + S_2 + S_3) = 0.\] 

Plane II is perpendicular to the normal vector to it whose direction cosines with the three principal stress axes are \((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\). \( \mathbf{PS} \), representing the hydrostatic component of stress \((S,S,S)\), is perpendicular to plane II and likewise has direction cosines \((\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\).

In summary, we have in vector notation,

\[
\begin{align*}
\mathbf{OS} &= S_1\mathbf{i} + S_2\mathbf{j} + S_3\mathbf{k} \\
\mathbf{OP} &= S'_1\mathbf{i} + S'_2\mathbf{j} + S'_3\mathbf{k} \\
\mathbf{PS} &= S_1 + S_2 + S_3
\end{align*}
\]

where \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) are the unit vectors in the 1, 2, and 3 directions.

By the rules for addition of vectors, one may write

\[
\mathbf{OS} = \mathbf{OP} + \mathbf{PS}
\]

and one sees that this is in harmony with the definition of the mean normal stress and the deviatoric stress.

As has been stated, the yield criterion, for a particular state of the deformable medium, can be regarded as a surface in this space. Since the yielding is independent of the hydrostatic component of stress, it is evident that the surface is a cylinder with generators perpendicular to the plane II and cutting it in some curve, \( C \). It is sufficient to discuss possible forms of the curve \( C \), and to consider only stress states whose hydrostatic component is zero, in view of the fact that the cylinder is the same for any value of the hydrostatic stress.
Figure 2. The yield surface in three-dimensional stress space.
It is helpful to look down into the cylinder, toward the origin, parallel to the axis of the cylinder. In figure 3, II is in the plane of the paper; the yield locus C and the orthogonal projections of the axes of reference are shown. The locus may be concave or convex to the origin, but obviously not in such a manner that the radius vector cuts it twice, as this makes no sense physically.

Assuming here an isotropic medium, if \((S_1, S_2, S_3)\) is a stress state on the yield surface, so also is \((S_2, S_1, S_3)\). Thus, the yield surface must be

Figure 3. View parallel to the axis of the yield cylinder.
symmetrical about the $S_3$ axis. In the same manner, one concludes that it must also be symmetrical about the $S_1$ and $S_2$ axes. Essentially, this is equivalent to saying that the yield criterion is a function of the stress invariants. If one assumes no Bauschinger effect, it follows that if for any point on the loading surface, a radius is drawn through the origin, and the material is unloaded meeting the yield surface at the diametrically opposite position, it must meet this yield surface at the same distance from the origin. One is forced to conclude, not only is the yield surface symmetrical about the three principal stress axes, but it is also symmetrical about the three diameters orthogonal to them. Another way of saying this is that the yield surface must be the same in each of the twelve segments of the plane $II$ as cut by the six aforementioned diameters.

One notes the immense simplification obtained from these conclusions, for if one were conducting an experimental determination of the yield surface, one could restrain the study to states of stress that lie in any one of the twelve segments.

Now, in the diagram of figure 3, one must talk about the projected values of $(S_1, S_2, S_3)$ on the plane $II$. If $P$ is the point where the vector from the origin strikes the yield surface ($P$ is coincidental with $S$ in this figure), let $x$ and $y$ designate the coordinates of $P$ as measured in the plane $II$, with respect to the dotted axes $\theta = 0$ and $S_3$. $\theta$ designates the angular position of the radius vector to the point.

Let

- $S_{III}$ designate the projected value of $S_3$
- $S_{II}$ designate the projected value of $S_2$
- $S_{I}$ designate the projected value of $S_1$
From the figure it is evident that

\[
x = \frac{\sqrt{3}}{2} S_{2\Pi} - \frac{\sqrt{3}}{2} S_{1\Pi}
\]

\[
y = S_{3\Pi} - \frac{1}{2} S_{1\Pi} - \frac{1}{2} S_{2\Pi}
\]

By considering figure 4, one finds \( \cos \phi = \frac{\sqrt{2}}{\sqrt{3}} \) and since

\[ S_{i\Pi} = S_i \frac{\sqrt{2}}{\sqrt{3}}, \quad i = 1, 2, 3, \]

it is found that

\[
x = \frac{S_2 - S_1}{\sqrt{2}}
\]

\[
y = \frac{2S_3 - S_1 - S_2}{\sqrt{6}}
\]

Since

\[
\frac{y}{x} = \tan \theta = \frac{2S_3 - S_1 - S_2}{\sqrt{3}(S_2 - S_1)}
\]

\[
= - \frac{\mu}{\sqrt{3}}
\]

where

\[
\mu = \frac{2S_3 - S_1 - S_2}{S_1 - S_2}
\]

is one of Lode's variables.

Figure 4. View parallel to plane II.
That is,

\[ \mu = \tan \theta \left( -\sqrt{3} \right) \]

Since \( 0^\circ \leq \theta \leq 30^\circ \), it follows \(-1 \leq \mu \leq 0\). Repeating a previous argument, a simplification is achieved in an experimental probe of the yield surface, for only values of stress that produce \(-1 \leq \mu \leq 0\) need be considered. Returning to the geometric concept of the yield surface, one may distinguish among three distinctly different movements of the vector from the origin to point P. The vector, after having once been on the yield surface, may fall short of the yield surface, due to a change in the stress field acting on the deformable medium, that is, geometrically, its tip may lie "inside" the yield cylinder. Physically, all changes in the medium are now of elastic nature, and the material loads and unloads elastically. Secondly, the vector, having been once on the yield surface, may move about the yield surface due to a change in the stress field. Physically, this constitutes neither loading nor unloading of the medium, but is termed neutral behavior. Lastly, the vector may attempt to pierce the yield surface, but in reality does not, causing the yield surface to expand and move out away from its original position. Physically, this constitutes loading of the medium, causing additional plastic flow, establishing a new yield surface, and maintaining the tip of the vector on the new yield surface, until the material unloads elastically from the new yield surface again. One should not assume that the new yield surface is merely an expanded version of the initial one, although ideally it would be.

During the loading of a deformable medium, work is done on the medium by the deforming forces. Energy is stored in the medium, some of which is returned by the medium upon removal of the deforming forces. This is the
elastic portion of the work. It is convenient to resolve the total strain increment \( \Delta e_{ij} \) into two components, the elastic component, which is recovered upon unloading, and the permanent plastic component. That is,

\[
\Delta e_{ij} = \Delta e_{ije} + \Delta e_{ijp}
\]  

(20)

The external work \( dW \) per unit volume done on an element of the medium during a differential strain \( \Delta e_{ij} \) is

\[
dW = S_{ij} \Delta e_{ij}
\]

of which a part

\[
dW = S_{ij} \Delta e_{ije}
\]

is the aforementioned recoverable elastic energy. The remainder is called the plastic work per unit volume, and it is therefore

\[
dW_p = dW - dW_e = S_{ij}(\Delta e_{ij} - \Delta e_{ije}) = S_{ij} \Delta e_{ijp}
\]  

(21)

where \( \Delta e_{ijp} = \Delta e_{ij} - \Delta e_{ije} \) is the plastic strain increment. Thus the total plastic work is \( W_p = \int S_{ij} \Delta e_{ijp} \), the integral being evaluated over the actual strain path of the loading program.

One of the most important hypotheses of the mathematical theory of plasticity is that the expansion of the yield surface is a function of the plastic work for a work hardening material (page 26, reference 12). For example, if the yield surface were a right circular cylinder of radius \( \sqrt{2k} \), where \( k \) is a parameter, the curve could be expressed in equation form as

\[
S_1'^2 + S_2'^2 + S_3'^2 = 2k^2
\]

or

\[
\frac{1}{2} (S_1'^2 + S_2'^2 + S_3'^2) = k^2
\]  

(22)
or

\[ J_2 = k^2 \]

In the one dimensional stress field, the simple stress-strain diagram is well known as a means of evaluating such characteristics of a deformable medium as Young's Modulus, the proportional limit, and the yield point. The elastic limit is a quantity that in principle is difficult to establish, but with painstaking experimental study, it too can be approximated for the one-dimensional case. When one considers two and three dimensional stress fields, the question arises as to what means one could utilize to obtain an analogy to the stress-strain curve. What is usually sought is a quantity termed the equivalent stress, and this is plotted vs. a quantity termed the equivalent strain. In this dissertation, as in many places in the literature, the quantity

\[ S = \frac{1}{\sqrt{2}} \sqrt{(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2} \]  

(23)

will be taken as the definition of the equivalent stress (page 46, reference 17). It is evident that the equivalent stress is \( J_2 \) multiplied by a constant. It is evident that in the choice of the value \( \frac{1}{\sqrt{2}} \), the equivalent stress reduces to the actual nominal stress in the uniaxial case. In defining the equivalent plastic strain, \( \varepsilon^p \), one must make sure the equality

\[ W_p = \int S_{ij} d\varepsilon^p_{ij} = \int S d\varepsilon^p \]  

(24)

will hold. It will be shown in the Appendix A, that if one takes the above definition of the equivalent stress, the equivalent plastic strain increment becomes
\[ de_p = \sqrt[3]{\frac{2}{3}} \left( \delta e_{11}^p + \delta e_{22}^p + \delta e_{33}^p \right) \] (25)

referred to the principal directions (reference 12, page 30).

Experimental evidence has shown, and it is usually assumed in the literature, that there is no change of volume in the plastic region of strain, that is, (reference 17, page 50),

\[ \delta e_{11}^p + \delta e_{22}^p + \delta e_{33}^p = 0 \] (26)

In view of this fact, and due to symmetry in the uniaxial tensile test of say, a circular bar, one has

\[ \delta e_{22}^p = \delta e_{33}^p = -\frac{\delta e_{11}^p}{2} \] (27)

It is clear, then, that likewise the equivalent plastic strain increment reduces to the uniaxial case. Thus the definition of the equivalent stress, and its consequence, the expression for the equivalent plastic strain increment, are compatible when specialized to the one dimensional stress field. Clearly one can then see a means of obtaining the equivalent stress vs. equivalent strain curve; it can be obtained from a simple tensile test. Under the definition of the equivalent stress and the equivalent plastic strain increment, the quantities are the same under any other stress field. It is helpful to define the equivalent plastic strain as

\[ e_{ep} = \frac{\sqrt{2}}{3} \sqrt{(e_{11}^p - e_{22}^p)^2 + (e_{22}^p - e_{33}^p)^2 + (e_{33}^p - e_{11}^p)^2} \] (28)

which will be used in the deformation theory (reference 22, page 29). It is shown in Appendix B that the definition of the equivalent plastic strain is compatible with the definition of the equivalent plastic strain.
increment when the former is differentiated, and reduced to the one dimen-
sional case.

One may now begin a discussion that will lead ultimately to the in-
cremental stress-strain relations, those of Prandtl-Reuss. The development
continues to follow Chapter 2 of reference 12. It has been mentioned that
since the beginning of research in the field of mathematical plasticity,
many yield criteria have been advanced. Two that have satisfied the re-
quirements of fitting experimental data and being sufficiently mathemat-
ically simple are those of Von Mises and Tresca. Von Mises suggested that
yielding occurred when $J_2^*$ reached a critical value, or, in other words,
that the function $f = f(J_2^*, J_3^*)$ did not involve $J_3^*$. His criterion can be
written in the alternate forms

$$2J_2^* = S_1^2 + S_2^2 + S_3^2 = 2k^2$$

or

$$(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2 = 2k^2$$  (29)

or

$$(S_x - S_y)^2 + (S_y - S_z)^2 + (S_z - S_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 6k^2$$

$k$ is a parameter depending on the amount of pre-strain. Many physical
interpretations of Von Mises' law can be given, such as that of elastic
energy of distortion, as pointed out by Hencky, or the shear stress over
the octahedral plane, as pointed out by Nadai (page 15, reference 17).
These two supporting physical correspondences help to make the Von Mises
criterion more attractive from a theoretical view point.

Tresca's criterion for yielding states that yielding occurs under any
stress field when the maximum shearing stress reaches the value that the
same material can withstand in a simple tensile test before yield takes place. It can be written

\[ S_1 - S_3 = \text{constant} \]  \hspace{1cm} (30)

where

\[ S_1 > S_2 > S_3 \]

It is evident that in order to apply the Tresca criterion to the solution of any practical problem, one must know in advance the ordering of the principal stresses. This often makes the Tresca criterion unsatisfactory. It also has other objectionable traits, such as non-uniqueness of the normal to the yield curve at certain points. In this dissertation, the Von Mises criterion will be used, because of the above two reasons.

It has been pointed out that the function \( f = f(J_2, J_3) \) which will be simplified to \( f = f(J_2) \) determines when yielding begins. That is, if \( df \) is greater than zero, physically the material is loading plastically, and plastic flow is increasing. On the other hand, if \( df \) is less than or equal to zero, the material is behaving elastically, or is in a neutral state, respectively. An incremental equation must satisfy this criterion. A general incremental equation that does satisfy this is (reference 12, page 33)

\[ \Delta e_{ij}^P = G_{ij} df, \quad df > 0 \]  \hspace{1cm} (31)

where \( G_{ij} \) is a symmetric tensor function of the stress components, and possibly the previous strain history, but not of the stress increments. Essentially this means that the ratios of the plastic strain increments are functions of the current stress, but not of the strain increment. It is evident that the functions \( G_{ij} \) must satisfy two necessary conditions, namely, \( G_{ii} \) must equal zero, for
\[ \frac{\partial \nu_1}{\partial S_{ii}} = \frac{\partial \nu_1}{\partial S_{ii}} \]

The left-hand side equals zero in view of the volume constancy, and in general \( df \) does not equal zero, so the above necessity is forced on \( G_{ij} \).

Secondly, the principal axes of the plastic strain increment tensor, and so of \( G_{ij} \), must coincide with the principal stress axes, since the element is isotropic. These conditions may be met by choosing

\[ G_{ij} = h \frac{\partial g}{\partial S_{ij}} \]  

(33)

where \( g \) and \( h \) are scalar functions of the second invariant of the stress deviation tensor, \( J_2 \). It is clear that the second condition is met, and the first one is also, for

\[ g = g(J_2) \]

\[ \frac{\partial g}{\partial S_{ii}} = \frac{\partial g}{\partial S_{ij}} \frac{\partial J_2}{\partial S_{ij}} \]

\[ = \frac{\partial g}{\partial J_2} \frac{\partial J_2}{\partial S_{ii}} \frac{\partial S_{ii}}{\partial S_{ii}} = 0 \]  

(34)

since

\[ S_{ii} = 0 \]

It should be noted that a rather marked simplicity is obtained by the observation that \( G_{ij} \) may be taken as \( h \frac{\partial g}{\partial S_{ij}} \); for \( G_{ij} \) was restricted in its nature as stated above, while \( g \) may be any scalar function of \( J_2 \).

At this point, one could arbitrarily select \( \nu, g, \) and \( h \) as any arbitrary functions of the principal stresses, and one would have a set of incremental stress-strain relations. Whether they would be useful in the solution of any problem, or whether they would check any experimental
information is doubtful, but nevertheless, they would constitute a set of incremental stress-strain equations. Thus one has

\[ d e_{ij}^P = h \frac{\partial g}{\partial S_{ij}} df = h \left( \frac{\partial g}{\partial J_2} \frac{\partial S_{ij}}{S_{ij}} \right) df = \left( h \frac{\partial g}{\partial J_2} \frac{\partial S_{ij}}{\partial S_{ij}} \right) df \]

\[ = \left( h \frac{\partial g}{\partial J_2} S_{ij}^1 \right) df \]  

(35)

Since \( g \) and \( h \) may not both be chosen arbitrarily, there exists a certain relationship between them. If a new stress state is imposed on the deformable medium that causes an incremental plastic flow, a new state of hardening is reached, represented by a slightly different yield locus for a work hardening material. Since this yield locus must naturally pass through this new stress point, \( h \) must be some function of \( g \), or vice-versa. As was pointed out, one of the fundamental assumptions of the mathematical theory of plasticity is that the state of hardening, or the size or position of the yield cylinder, is a function of the total plastic work. That is, the radius of the cylinder, which is a function of \( J_2^1 \), may be written

\[ f(J_2^1) = F \left( \int S_{ij} d e_{ij}^P \right) = F \left( \int S d e_{ij}^P \right) \]  

(36)

Then \( df = F' S_{kl} d e_{kl}^P \), where \( F' \) is the derivative of \( F \) with respect to its argument. One finds

\[ d e_{ij}^P = h F' \frac{\partial g}{\partial S_{ij}} (S_{kl} d e_{kl}^P) \]  

(37)

Multiplying through by \( S_{ij} \) and summing, one finds

\[ S_{ij} d e_{ij}^P = h F' S_{ij} \frac{\partial g}{\partial S_{ij}} (S_{kl} d e_{kl}^P) \]  

(38)
It will now be assumed that $g$ is a homogeneous function of the principal stresses of degree $N$. Mathematically, this means that if

$$g = g(S_1, S_2, S_3)$$

then

$$g(\lambda S_1, \lambda S_2, \lambda S_3) = \lambda^N g(S_1, S_2, S_3) \quad (39)$$

For example, the function

$$z = ax^2 + bxy + cy^2 \quad (40)$$

is a homogeneous function of degree two in $x$ and $y$ for $z(vx, vy) = av^2x^2 + bv^2xy + cv^2y^2 = v^2(ax^2 + bxy + cy^2)$. If now one differentiates both sides with respect to $\lambda$, one finds

$$\frac{\partial g}{\partial (\lambda S_1)} \frac{\partial (\lambda S_1)}{\partial (\lambda)} + \frac{\partial g}{\partial (\lambda S_2)} \frac{\partial (\lambda S_2)}{\partial (\lambda)} + \frac{\partial g}{\partial (\lambda S_3)} \frac{\partial (\lambda S_3)}{\partial (\lambda)} = N\lambda^{N-1} g \quad (41)$$

or more simply

$$S_1 \frac{\partial g}{\partial S_1} + S_2 \frac{\partial g}{\partial S_2} + S_3 \frac{\partial g}{\partial S_3} = N\lambda^{N-1} g \quad (42)$$

Choosing $\lambda$ equal to one, one has

$$S_1 \frac{\partial g}{\partial S_1} + S_2 \frac{\partial g}{\partial S_2} + S_3 \frac{\partial g}{\partial S_3} = Ng \quad (43)$$

or more compactly by the tensor notation,

$$S_{ij} \frac{\partial g}{\partial S_{ij}} = Ng \quad (44)$$

This is the well-known Euler theorem. Thus one may write

$$S_{ij} \delta F' \delta e_{ij} = hF'^* Ng \quad S_{k'k} \delta e_{kl}^p = S_{k'k} \delta e_{kl}^p \quad (45)$$

Hence, one concludes

$$hF'^* Ng = 1 \quad (46)$$
or

\[ h = \frac{1}{NF'g} \quad (47) \]

This is the relationship that must exist between \( h \) and \( g \). There is no inaccuracy in assuming that \( g \) is a homogeneous function of the stresses if later when \( g \) is taken, it is thus specified. This will be so. One now has

\[
d e_{ij}^P = \frac{1}{NF'g} \frac{\partial g}{\partial S_{ij}} df
\]

\[ = \frac{1}{Ng} \frac{\partial g}{\partial S_{ij}} \frac{df}{F'} (df > 0) \quad (48)\]

It should be noted that at this point, the functions \( f, g, \) and \( F \) can be arbitrarily prescribed, and one would have a set of incremental stress-strain equations. These equations would be valid when \( df \) is greater than or equal to zero, and would be supplemented with the incremental form of Hooke's Law for the elastic portion of the strain. If the element unloads, that is, if \( df \) is less than zero, these equations do not apply at all, and again the incremental form of Hooke's Law would be the governing relationship between stress and strain.

At this time, before the equations above are specialized to the incremental equations of Prandtl-Reuss, it is well to consider the geometric representation of the plastic-strain increment.

On page 14 is illustrated the three dimensional plot in principal stress space of the yield cylinder. \( g = g(S_1, S_2, S_3) \) has been assumed a homogeneous function of the principal stresses. It is evident that under the assumption of isotropy in the material under consideration, \( g \) may be plotted on the same diagram, about the same axis, as the yield cylinder.
g is often called the plastic potential, as the strains in the plastic region are obtained from it by differentiating it. Now the plastic strain increment can also be plotted in this stress space, if one modifies each of its components by multiplying them by some function having the dimensions of stress. Here, for convenience, that factor will be 2G. Thus in the stress space one has the vector

\[ 2G\varepsilon^i_1 + 2G\varepsilon^j_2 + 2G\varepsilon^k_3 \]

superimposed on the same plot of both the yield surface and the plastic potential surface. It is evident that the plastic potential surface and the yield surface may be made to agree in at least one line parallel to the axes of the yield cylinder, and that the incremental plastic strain vector lies in the plane \( \Pi \) since \( \varepsilon^p_1 + \varepsilon^p_2 + \varepsilon^p_3 = 0 \). From the equations \( \varepsilon^p_{ij} = h \frac{\partial g}{\partial \varepsilon^p_{ij}} df \), one may conclude that the plastic strain increment vector is perpendicular to the surface of the plastic potential, for this may be written \( \varepsilon^p_{ij} = h(\text{Grad } g)df \).

By arguments of symmetry analogous to the ones advanced for the yield surface under isotropy, one may conclude that the plastic potential surface is symmetrical with respect to the six lines of figure 3, and that one may restrict the discussion of the space to one of the twelve segments. Let \( \Gamma \) designate the curve formed by the intersection of the plastic potential surface with the plane \( \Pi \).
The absolute size of $\Gamma$ is immaterial. $RQ'$ is the plastic strain increment vector and is parallel to the normal at the point $R$ where the stress vector $OP$ meets $\Gamma$. ($RQ'$ is a free vector and can be placed anywhere in the plane $II$.) Let $\psi$ denote the angle between $RQ'$ and the radius $\mu = 0$. It is evident that by projecting values of $\delta e^{P}_{ij}$ onto the plane $II$ as was done with the stresses, one can define the relationship

$$\nu = -\sqrt{3} \tan \psi = \frac{2\delta e^{P}_{33} - \delta e^{P}_{11} - \delta e^{P}_{22}}{\delta e^{P}_{11} - \delta e^{P}_{22}}$$

(50)

This definition is analogous to the one for $\mu$, and was also introduced into the literature by Lode. It is evident that the curve $\Gamma$ must cut orthogonally the radii bounding the $30^\circ$ segments, but between them could conceivably be of any shape, with the restriction, of course, for reasons of uniqueness that normals at two points in the same segment are not parallel. A few interesting properties may be obtained from figure 5. One has
\[ \delta W_p = S_1 \delta e_1^p + S_2 \delta e_2^p + S_3 \delta e_3^p \]

\[ = \frac{O P \cdot R Q'}{2G} \quad (51) \]

But

\[ |O P| = \sqrt{S_1^2 + S_2^2 + S_3^2} = \sqrt{\frac{2}{3} S} \quad (52) \]

\[ |R Q'| = 2G \sqrt{\delta e_{11}^p + \delta e_{22}^p + \delta e_{33}^p} \]

\[ = 2G \sqrt{\frac{5}{2} \delta e_p} \quad (53) \]

Since

\[ O P \cdot R Q' = |O P| |R Q'| \cos (\psi - \theta) \quad (54) \]

one has

\[ \delta W_p = \delta e_p \cos (\psi - \theta) \quad (55) \]

Theoretical speculation about the relation between stress and strain originated in 1870, as has been mentioned, with St. Venant's treatment.

With great physical insight, St. Venant assumed that the principal axes of the strain increment (and not the total strain) coincided with the axes of principal stress. A general relationship between the ratios of the strain increment and the stress ratios was first suggested by Levy in 1871. Levy's work remained largely unknown outside his own country (France), and it was not until the same equations were suggested independently by Von Mises (1913) that they became widely used as a basis for plasticity theory. The Levy-Mises equations, as they are known, may be expressed as (reference 12, page 38)

\[ \frac{d e_x}{S_x} = \frac{d e_y}{S_y} = \frac{d e_z}{S_z} = \frac{d \tau_{xy}}{\tau_{xy}} = \frac{d \tau_{yz}}{\tau_{yz}} = \frac{d \tau_{zx}}{\tau_{zx}} \quad (56) \]
or more compactly by tensor notation as $\text{de}_{ij} = S_{ij}d\lambda$, where $d\lambda$ is a factor of proportionality. It is evident that these equations are strictly applicable only to a fictitious material in which the elastic strains are zero, inasmuch as Levy and Mises used the total strain increment and not the plastic strain increment. Another way of saying this is that they are applicable to a material for which the Young's Modulus is infinite.

The extension of these equations to allow for the elastic component of strain was carried out in 1924 by Ludwig Prandtl for the plane problem, and by A. Reuss in 1930 for the general problem. Reuss assumed that (reference 12, page 39)

$$\text{de}_{ij}^P = S_{ij}d\lambda$$

(57)

If one substitutes this term for term into the expression for

$$v = -\sqrt{3} \tan \psi = \frac{2\text{de}_{55}^P - \text{de}_{11}^P - \text{de}_{22}^P}{\text{de}_{11}^P - \text{de}_{22}^P}$$

(58)

it is seen that this reduces, upon cancelling the proportionality factor, to $\mu = v$. It is evident that this is equivalent to $\tan \theta = \tan \psi$; thus $RQ'$ is parallel to $OP$, and the curve $\Gamma$ is a circle. This means $g = J_2^i$, and one has then

$$\text{de}_{ij}^P = \frac{1}{4g} \frac{\partial g}{\partial S_{ij}^P} \frac{df}{F'}$$

(59)

$$= \frac{1}{2J_2} S_{ij}^P \frac{df}{F'}$$

(60)

Recalling that $f' = F' \left( \int S_{ij} \text{de}_{ij}^P \right) = F' \left( \int \text{de}_{ij}^P \right)$, one finds

$$\frac{df}{F'} = \text{de}_{ij}^P = \sqrt{3J'} \text{de}_{ij}^P$$

(61)
so
\[ \overline{S}^2 = 3 J''_2 \quad (62) \]
and
\[ \text{de}^p_{ij} = \frac{3}{2S^2} S'_{ij} \overline{S} \text{de}^p \quad (63) \]
or
\[ \text{de}^p_{ii} = \frac{3 S'_{ii}}{2S} \text{de}^p \quad (64) \]

These are the celebrated Prandtl-Reuss incremental theory equations, and they will be used to carry the step-by-step solution of the problems through the plastic range of loading by the incremental theory of plasticity. Inasmuch as it will be seen that both problems treated have only principal stresses, and both lend themselves to solution most advantageously in cylindrical coordinates, it will be helpful to expand these equations for the \( r \), \( t \), and \( z \)-directions. One finds
\[
\begin{align*}
\text{de}^p_r &= \frac{2S_r - S_t - S_z}{2S} \text{de}^p \\
\text{de}^p_t &= \frac{2S_t - S_r - S_z}{2S} \text{de}^p \\
\text{de}^p_z &= \frac{2S_z - S_r - S_t}{2S} \text{de}^p
\end{align*}
(65)
\]
where
\[ \overline{S} = \frac{1}{\sqrt{2}} \sqrt{(S_r - S_t)^2 + (S_t - S_z)^2 + (S_z - S_r)^2} \quad (66) \]
and
\[ \text{de}^p = \sqrt{\frac{2}{3} (\text{de}^p_r^2 + \text{de}^p_t^2 + \text{de}^p_z^2)} \quad (67) \]
From this it is clear that

\[
\frac{de_r^p - de_t^p}{S_r - S_t} = \frac{de_t^p - de_z^p}{S_t - S_z} = \frac{de_z^p - de_r^p}{S_z - S_r} = \frac{3de_p^p}{2S}
\]

(68)

Recalling that \( g \) was assumed to a homogeneous function at one place in the development, it is now possible to justify that assumption, as \( g \) was found to be equal to \( J_2' \). \( J_2' \) is, of course, a homogeneous function of degree two in the principal stresses.

It is usual to resolve the total strain increment into the elastic and the plastic parts, that is,

\[
de_{ij} = de_{ij}^e + de_{ij}^p
\]

(69)

where \( de_{ij}^e = \frac{\partial s_{ij}}{2G} + (1 - 2\mu) \delta_{ij} \frac{\partial s}{E} \). In the shorthand notation, one has

\[
de_{ij}^e = \frac{\partial s_{ij}}{2G} + \frac{3\partial s_{ij}}{2S} de_p^p
\]

(70)

Writing this out for the case of cylindrical coordinates, one finds:

\[
de_r = \frac{1}{E} [\partial s_r - \mu (\partial s_t + \partial s_z)] + \frac{2S_t - S_t - S_z}{2S} de_p^p
\]

\[
de_t = \frac{1}{E} [\partial s_t - \mu (\partial s_r + \partial s_z)] + \frac{2S_r - S_z - S_r}{2S} de_p^p
\]

\[
de_z = \frac{1}{E} [\partial s_z - \mu (\partial s_r + \partial s_t)] + \frac{2S_z - S_t - S_r}{2S} de_p^p
\]

(71)

Before turning to the solution of the actual boundary value problems of this dissertation, it is well to consider the corresponding equations of the deformation theory of mathematical plasticity.

Essentially three analogous assumptions are made in the deformation theory, namely, (reference 22, page 29)
1. Constancy of volume in the plastic realm, that is,
\[ e_{rp} + e_{tp} + e_{zp} = 0 \] (72)

2. The directions of the principal strains coincide with the directions of the principal stresses.

3. The ratios of the principal shear strains coincide with the ratios of the principal shear stresses, that is,
\[ \frac{e_t - e_{zt}}{S_t - S_{zt}} = \frac{e_{zt} - e_z}{S_{zt} - S_z} = K_1 \] (73)

As in the incremental theory, it is convenient to resolve the total strain into an elastic part and a plastic part,
\[ e_{ij} = e_{ij}^e + e_{ij}^p \] (74)

For the elastic part, one has Hooke's Law.

In this dissertation, temperature dependent problems will be solved, hence it is convenient to take as Hooke's Law for the elastic range the three equations
\[
\begin{align*}
    e_t^e &= \frac{1}{E} \left[ S_t - \mu (S_t + S_z) \right] + \alpha T \\
e_z^e &= \frac{1}{E} \left[ S_z - \mu (S_z + S_t) \right] + \alpha T \\
e_r^e &= \frac{1}{E} \left[ S_r - \mu (S_r + S_t) \right] + \alpha T
\end{align*}
\] (75)

where \( \alpha \) is the coefficient of thermal expansion and \( T \) is the increase in temperature above an arbitrary level. Then the complete stress-strain relations become:
\[ e_r = \frac{1}{E} \left[ S_r - \mu(S_t + S_z) \right] + \alpha T + e^p_r \]
\[ e_t = \frac{1}{E} \left[ S_t - \mu(S_r + S_z) \right] + \alpha T + e^p_t \]
\[ e_z = \frac{1}{E} \left[ S_z - \mu(S_r + S_t) \right] + \alpha T + e^p_z \]

By substituting equations (76) into equations (73) one finds
\[ \frac{e_{rp} - e_{tp}}{S_r - S_t} = \frac{e_{tp} - e_{zp}}{S_t - S_z} = \frac{e_{zp} - e_{rp}}{S_z - S_r} = K_2 \]

where
\[ K_1 = K_2 + \frac{1 + \mu}{E} \]

It is illuminating to compare equations (77) with equations (68). One notes the set of deformation equations (77) has the difference of TOTAL plastic strains in the numerators, while the set of incremental equations (68) has the difference of INFINITESIMAL, or INCREMENTAL strains in the numerators. Both have the difference of stresses in the denominators.

Once again it will be found convenient to define an equivalent stress as
\[ \sigma = \frac{1}{\sqrt{2}} \sqrt{(S_r - S_t)^2 + (S_t - S_z)^2 + (S_z - S_r)^2} \]

and an equivalent plastic strain as
\[ e_{ep} = \frac{\sqrt{2}}{\sqrt{3}} \sqrt{(e^p_r - e^p_t)^2 + (e^p_t - e^p_z)^2 + (e^p_z - e^p_r)^2} \]

which becomes, in view of equation (72),
\[ e_{ep} = \frac{2}{\sqrt{3}} \sqrt{e^p_r^2 + e^p_t^2 + e^p_z^2} \]

It is seen that the definition of the equivalent plastic strain is consistent with the simple tensile test in the plastic range. Likewise one
can define the equivalent total strain as

\[ e_{et} = \frac{\sqrt{2}}{3} \sqrt{(e_r - e_t)^2 + (e_t - e_z)^2 + (e_z - e_r)^2} \]  

By squaring equations (73) and substituting them in equations (82), and squaring equations (77) and substituting them in equation (80), one may obtain the relations

\[
\begin{align*}
K_1 &= \frac{3}{2} \frac{e_{et}}{S} \\
K_2 &= \frac{3}{2} \frac{e_{ep}}{S}
\end{align*}
\]  

(83)

Substituting these into result (78) one finds

\[ \frac{3}{2} \frac{e_{et}}{S} = \frac{3}{2} \frac{e_{ep}}{S} + \frac{1 + \mu}{E} \]  

(84)

or

\[ e_{et} = e_{ep} + \frac{2}{3} \frac{1 + \mu}{E} S \]  

(85)

Upon dividing equations (77) by equations (73) and applying equations (83) one finds:

\[ \frac{e_{rp} - e_{tp}}{e_r - e_t} = \frac{e_{tp} - e_{zp}}{e_t - e_z} = \frac{e_{zp} - e_{rp}}{e_z - e_r} = \frac{K_2}{K_1} \]  

(86)

\[ = \frac{e_{ep}}{e_{et}} \]

Or if one writes

\[ e_{rp} - e_{tp} = \frac{K_2}{K_1} (e_r - e_t) \]  

(87)

\[ e_{tp} - e_{zp} = \frac{K_2}{K_1} (e_t - e_z) \]  

(88)
and adds (87) and (88) the result is

\[ 2e_{rp} - e_{tp} - e_{zp} = \frac{K_0}{K_1} (2e_r - e_t - e_z) \]  

(89)

which becomes in view of equation (72)

\[ 3e_{rp} = (2e_r - e_t - e_z) \frac{e_{ep}}{e_{et}} \]  

(90)

or

\[ e_{rp} = \frac{e_{ep}}{e_{et}} \left(\frac{2e_r - e_t - e_z}{3}\right) \]  

(91)

and likewise for the \( t \) and \( z \) directions. In summary,

\[
\begin{align*}
e_{rp} &= \frac{e_{ep}}{e_{et}} \left(\frac{2e_r - e_t - e_z}{3}\right) \\
e_{tp} &= \frac{e_{ep}}{e_{et}} \left(\frac{2e_t - e_z - e_r}{3}\right) \\
e_{zp} &= \frac{e_{ep}}{e_{et}} \left(\frac{2e_z - e_r - e_t}{3}\right)
\end{align*}
\]  

(92)

These equations may be changed to a slightly different form

\[
\begin{align*}
e_{rp} &= \frac{2S_r - S_t - S_z}{2S} e_{ep} \\
e_{tp} &= \frac{2S_t - S_z - S_r}{2S} e_{ep} \\
e_{zp} &= \frac{2S_z - S_r - S_t}{2S} e_{ep}
\end{align*}
\]  

(93)

by use of equations (73) and equations (83). The equations (93) for the deformation theory of plasticity are to be contrasted with the set of equations (65) for the incremental theory of plasticity (reference 22).

This brings to a close the development of the equations relating stress to strain in the two theories. In summary, it may be said that for
the Von-Mises flow criterion, the two theories produce equations that look
amazingly similar. On page 6 the statement was made that it can be shown
that the two theories agree for a certain relationship existing among the
loading components. This is readily seen at this time, for if one stipu-
lates that as the stresses acting on the deformable medium increase from
the zero values and the unstrained state, they increase in a constant ratio
to each other, that is for the cylindrical coordinates employed here,

\[
\begin{align*}
S_t &= A_1 S_r \\
S_z &= A_2 S_r
\end{align*}
\]  

(94)

where \( A_1 \) and \( A_2 \) are constants during the loading program, then in
equations (65), the term multiplying \( \dot{\varepsilon}_p^{\text{eq}} \) is a constant and the three
equations may be integrated directly giving equations (93). It is evident
from the definition of the equivalent plastic strain in equation (80) and
the definition of the equivalent plastic strain increment in equation (25),
that the two are not equivalent; that is, one could not differentiate (80)
and obtain (25), but that is of no consequence here. It is shown in
Appendix B that they agree for the one dimensional case, which may be ob-
tained from the uniaxial stress strain diagram, and that is the link that
one is looking for.

It is appropriate at this time to show by simple examples the need
for the incremental theory, that is, to show how the deformation theory
is theoretically unacceptable for the solution of plastic flow problems.

In reference 29 are described the classic experiments of Taylor and
Quinney on experimental verification of the yield criteria by means of
combined loading on thin-walled tubes. It will now be shown, with the aid
of tests such as those described in reference 29, the need for incremental theory, as the deformation theory fails in certain cases.

Equations (72) and (73) list the fundamental assumptions of the deformation theory of plasticity. The Von Mises criteria may be written as

$$f(S_{11}, S_{22}, S_{33}) = 0$$

(95)

It is physically sound to expect, and to require, of a mathematical theory for a work hardening material, that a small increase in $f$, say $df$, produces a small change in the plastic strain components, $d\varepsilon_{ij}^p$. In the limit, the derivative, or the ratio of the change of any component of plastic strain to the change in $f$ must be finite, that is,

$$\frac{d\varepsilon_{ij}^p}{df} \neq \infty$$

(96)

However, this infinite behavior may take place under a deformation theory analysis, as may be seen in two simple examples (reference 8, page V-6).

A. The Inconsistency for Rotation of Principal Axes

Consider figure 6. A thin-walled tube is illustrated, with an axial tensile force and a torque applied.

Figure 6. Combined loading on a thin-walled tube.
Another physical system that serves well for the following argument would be a thin-walled pressure vessel, subjected to torque. This would give a more general two dimensional stress field than figure 6. It is evident that the two dimensional stress field discussed in the second physical system may be represented on an element as in figure 7.

Figure 7. General two dimensional stress field, produced in a thin-walled tube by internal pressure and torque. The radial stress, small in comparison to the axial and tangential, is neglected in reducing the actual three dimensional to a two dimensional stress field.

It is shown by Glenn Murphy in reference 24, page 25, that the principal stresses are

\[
S_1 = \frac{S_x + S_y}{2} + \sqrt{\left(\frac{S_x - S_y}{2}\right)^2 + \tau^2} \\
S_2 = \frac{S_x + S_y}{2} - \sqrt{\left(\frac{S_x - S_y}{2}\right)^2 + \tau^2}
\]

(97)
and the angle of orientation of the principal stress planes is

$$\theta = \frac{1}{2} \tan^{-1} \frac{-2\pi}{S_x - S_y}$$  \hspace{1cm} (98)$$

It is evident that the state of stress can be rotated without changing the magnitudes of the principal stresses. The addition of torque and the proper variation of the internal pressure and axial pull may easily be calculated. If the tension is increased infinitesimally during rotation, the isotropic deformation theory predicts that the permanent strain will rotate completely with the stress so that when the stress is a pure circumferential tension, the diameter will be permanently stretched and there will be a permanent axial contraction due to the Poisson ratio effect. The deformation theory leads to the result that for infinitesimal loading a plastic axial elongation has been converted to a contraction.

B. The Inconsistency for Constant Principal Directions

Consider the case of plane stress, with the z direction normal to the tube. Equations (73) reduce to

$$\frac{S_x - S_y}{e_p^x - e_p^y} = \frac{S_y}{e_p^y - e_p^z} = \frac{-S_x}{e_p^z - e_p^x}$$ \hspace{1cm} (99)$$

and the Von Mises criteria to

$$f = S_x^2 - S_x S_y + S_y^2$$ \hspace{1cm} (100)$$

where the coefficients and the square root have been dropped, not being necessary for the purpose at hand.

Suppose a state of simple tension to exist in the tube, with the principal stresses becoming \((A, 0, 0)\) and the plastic strains becoming \(e_p^x = a, e_p^y = \frac{-b}{2} = e_p^z\). Substituting into (100), \(f = A^2\). Now change the
magnitudes of $S_x$ and $S_y$ in such a manner to keep $f$ constant. If $f$ is increased infinitesimally, then the change in permanent strain should also be infinitesimal by all physical reasoning. Equation (98) gives the value of $S_y$ for any arbitrary change of $S_x$. Arbitrarily, increase $S_x$ from $A$ to $\frac{2A}{\sqrt{3}}$. From (100), for constant $f$, $S_y = \frac{A}{\sqrt{3}}$. Taking the permanent volume change equal zero, equations (97) give $e^P_x = e^P_2$, and $e^P_3 = 0$. Despite the unchanged Von Mises criteria, the theory predicts an increase in the plastic circumferential strain of $\frac{a}{2}$.

The above two examples, simple but illuminating, show inconsistencies in the deformation theory.
III. THE INFINITELY LONG CIRCULAR CYLINDER AT UNIFORM TEMPERATURE
SUBJECTED TO A RAPID QUIENCH TO INDUCE PLASTIC FLOW

It is well known from the theory of elasticity or strength of materials that a homogeneous isotropic circular cylinder when heated uniformly above a certain reference temperature is stress free and it is simple to compute the elastic strains resulting. For more generality, the problem of generalized plane strain will be considered, that is, there exists a uniform axial strain which is constant (at any time, but varying, of course, with time), and in general different from zero, at all radial points (plane sections are assumed to remain plane sections at all times in the loading program.) In Figure 8 is shown the cylinder, and in Figure 9 is shown a small element extracted from the cylinder, with all existing stresses on it. Due to symmetry, of course, no shearing stresses are present. Thus the principal directions are $r$, $t$, and $z$.

Figure 8. The infinitely long circular cylinder.
As has been pointed out, the total strain may be decomposed into an elastic component and a plastic component, the elastic component coming from Hooke's Law. One has

\[
\begin{align*}
  e_r &= \frac{1}{E} [S_r - \mu (S_T + S_z)] + \alpha T + e_{rp} \\
  e_t &= \frac{1}{E} [S_t - \mu (S_r + S_z)] + \alpha T + e_{tp} \\
  e_z &= \frac{1}{E} [S_z - \mu (S_r + S_t)] + \alpha T - (e_{rp} + e_{tp})
\end{align*}
\]
where in (101) use is made of the incompressibility relationship for the plastic range, equation (72).

From conditions of symmetry, it is evident that the tangential stress is not a function of angular position, being constant at any radial position. The equilibrium equations in the axial and circumferential directions are satisfied automatically and the only one producing useful results is in the radial direction. Summing forces in this direction one finds:

\[
(S_r + \frac{dS_r}{dr} dr)(r + dr) \theta dz - S_r r \theta dz - 2S_\theta dz \sin \frac{\theta}{2} = 0 \quad (102)
\]

Since the angle is small, \( \sin \frac{\theta}{2} \) is approximately equal to \( \frac{\Delta \theta}{2} \) and the equation simplifies to

\[
rS_r + r \frac{dS_r}{dr} dr + S_\theta dr + \frac{dS_r}{dr} (dr)^2 - rS_r - S_\theta dr = 0 \quad (103)
\]

and upon neglecting higher order differentials this becomes

\[
\frac{dS_r}{dr} + \frac{S_r - S_\theta}{r} = 0 \quad (104)
\]

Assuming small strains, one has for the three total strains

\[
\begin{align*}
e_r &= \frac{dU}{dr} \\
e_t &= \frac{U}{r} \\
e_z &= e_z
\end{align*}
\]

where \( U = U(r) \) is the total radial displacement of any point. That a point in the cylinder suffers only radial displacement is apparent from symmetry.

It is now convenient to solve equations (101) for the radial and the tangential stresses. One finds that
\[
S_r = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\varepsilon_r + \mu(\varepsilon_t + \varepsilon_z) - (1 - 2\mu)\varepsilon_{rp} - (1 + \mu)\alpha T \right]
\]
\[
S_t = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\varepsilon_t + \mu(\varepsilon_r + \varepsilon_z) - (1 - 2\mu)\varepsilon_{tp} - (1 + \mu)\alpha T \right];
\]

or, making use of equations (105),

\[
S_r = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\frac{dU}{dr} + \mu\left(\frac{U}{r} + \varepsilon_z\right) - (1 - 2\mu)\varepsilon_{tp} - (1 + \mu)\alpha T \right]
\]
\[
S_t = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\frac{U}{r} + \mu\left(\frac{dU}{dr} + \varepsilon_z\right) - (1 - 2\mu)\varepsilon_{tp} - (1 + \mu)\alpha T \right]
\]

Now substituting (107) into the equilibrium equation (104), and taking \(E\) and \(\alpha\) constant, one finds

\[
\frac{d^2U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{U}{r^2} = \frac{1 + \mu}{1 - \mu} \alpha \frac{dt}{dr} + \frac{1 - 2\mu}{1 - \mu} \left(\frac{\varepsilon_{rp} + \varepsilon_{tp}}{r}\right)
\]

as the second order differential equation that is to be solved for \(U\). If this can be done, it is evident then that the strains may be obtained from (105) and the stresses from (107). Rewriting the left-hand side of this equation as

\[
\frac{d}{dr} \frac{1}{r} \left[ \frac{d}{dr} (rU) \right],
\]

(108) becomes

\[
\frac{d}{dr} \frac{1}{r} \left[ \frac{d}{dr} (rU) \right] = \frac{1 + \mu}{1 - \mu} \alpha \frac{dt}{dr} + \frac{1 - 2\mu}{1 - \mu} \left[ \frac{\varepsilon_{rp} + \varepsilon_{tp}}{r}\right]
\]

Upon integrating once, one has

\[
\frac{1}{r} \left[ \frac{d}{dr} (rU) \right] = \frac{1 + \mu}{1 - \mu} \alpha T + \frac{1 - 2\mu}{1 - \mu} \varepsilon_{rp} + \frac{1 - 2\mu}{1 - \mu} \int_0^r \frac{\varepsilon_{rp} - \varepsilon_{tp}}{r} dr + C_1
\]

(111)
Multiplying both sides by $r$ gives

$$\frac{d(rU)}{dr} = \frac{1 + \mu}{1 - \mu} r\ln r + \frac{1 - 2\mu}{1 - \mu} \int_0^r \frac{e^{rU} - e^{rU}}{r} dr + C_1 r$$

(112)

Integrating once more yields

$$rU = \frac{1 + \mu}{1 - \mu} \int_0^r \ln r dr + \frac{1 - 2\mu}{1 - \mu} \int_0^r \int_0^r \frac{e^{rU} - e^{rU}}{r} dr dr + \frac{C_1 r^2}{2} + C_2$$

(113)

Lastly, dividing by $r$ gives

$$U = \frac{1 + \mu}{1 - \mu} \frac{1}{r} \int_0^r \ln r dr + \frac{1 - 2\mu}{1 - \mu} \frac{1}{r} \int_0^r \int_0^r \frac{e^{rU} - e^{rU}}{r} dr dr + \frac{C_1 r}{r} + C_2$$

(114)

Consider

$$\int_0^r r \int_0^r \frac{e^{rU} - e^{rU}}{r} dr dr$$

Integrating by parts with

$$u = \int_0^r \frac{e^{rU} - e^{rU}}{r} dr \quad du = \frac{e^{rU} - e^{rU}}{r} dr$$

$$v = \frac{r^2}{2} \quad dv = rdr,$$

the result becomes

$$\int_0^r \int_0^r \frac{e^{rU} - e^{rU}}{r} dr dr = \frac{r^2}{2} \int_0^r \frac{e^{rU} - e^{rU}}{r} dr - \frac{1}{2} \int_0^r r(e^{rU} - e^{rU}) dr$$

(115)
so that

\[ U = \frac{1 + \mu}{1 - \mu} \int_0^r \alpha Tr \, dr + \frac{1 - 2\mu}{1 - \mu} \frac{1}{r} \int_0^r \varepsilon \varepsilon_p \, r \, dr + \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r} \int_0^r \frac{\varepsilon \varepsilon_p - \varepsilon \varepsilon_p}{r} \, dr - \]

\[ \frac{(1 - 2\mu)}{2(1 - \mu)} \frac{1}{r} \int_0^r \varepsilon \varepsilon_p \, r \, dr + \frac{C_1 r}{2} + \frac{C_2}{r} \]

(117)

\[ = \frac{1 + \mu}{1 - \mu} \frac{1}{r} \int_0^r \alpha Tr \, dr + \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r} \int_0^r \varepsilon \varepsilon_p + \varepsilon \varepsilon_p \, dr + \]

\[ \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r} \int_0^r \frac{\varepsilon \varepsilon_p - \varepsilon \varepsilon_p}{r} \, dr + \frac{C_1 r}{2} + \frac{C_2}{r} \]

(118)

Imposing the condition that the displacements must be finite as \( r \) approaches zero, \( C_2 \) must be zero.

Dividing both sides by \( r \), the tangential strain becomes

\[ \varepsilon_t = \frac{1 + \mu}{1 - \mu} \frac{1}{r^2} \int_0^r \alpha Tr \, dr + \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r^2} \int_0^r \varepsilon \varepsilon_p + \varepsilon \varepsilon_p \, dr + \]

\[ \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r} \int_0^r \frac{\varepsilon \varepsilon_p - \varepsilon \varepsilon_p}{r} \, dr + \frac{C_1}{2} \]

(119)

Differentiating both sides of (118) produces the radial strain

\[ \varepsilon_r = \frac{dU}{dr} = \frac{1 + \mu}{1 - \mu} \alpha T - \frac{(1 + \mu)}{1 - \mu} \frac{1}{r^2} \int_0^r \alpha Tr \, dr - \]

\[ \frac{(1 - 2\mu)}{2(1 - \mu)} \frac{1}{r^2} \int_0^r \varepsilon \varepsilon_p + \varepsilon \varepsilon_p \, dr + \frac{1 - 2\mu}{1 - \mu} \varepsilon \varepsilon_p + \]

\[ \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r} \int_0^r \frac{\varepsilon \varepsilon_p - \varepsilon \varepsilon_p}{r} \, dr + \frac{C_1}{2} \]

(120)
Now since the cylinder is free to expand in the axial direction, there must be no resultant axial force, that is,

\[ \int_{\text{Area}} S_z dA = 0 \quad (121) \]

or

\[ \int_0^{2\pi} \int_0^R S_z r dr d\theta = 0 \quad (122) \]

which says

\[ \int_0^R S_z r dr = 0 \quad (123) \]

Since the axial stress is

\[ S_z = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\varepsilon_z + \mu(\varepsilon_r + \varepsilon_t) + (1 - 2\mu)(\varepsilon_{\tau p} + \varepsilon_{\tau p}) - (1 + \mu)\alpha T \right] \quad (124) \]

equations (119) and (120) may be used to obtain

\[ \varepsilon_z = \frac{2(1 - 2\mu)(1 + \mu)}{(1 - \mu)^2} \left[ \frac{1}{R^2} \int_0^R \alpha T r dr - \frac{(2 - \mu)(1 - 2\mu)}{(1 - \mu)^2} \frac{1}{R^2} \int_0^R (\varepsilon_{\tau p} + \varepsilon_{\tau p}) r dr - \frac{\mu(1 - 2\mu)}{(1 - \mu)^2} \int_0^R \frac{\varepsilon_{\tau p} - \varepsilon_{\tau p}}{r} dr - \frac{\mu C_1}{1 - \mu} \right] \quad (125) \]

Imposing the boundary condition that the radial stress must vanish at the outer radius, \( S_r \) becomes

\[ S_r = \frac{E}{(1 - 2\mu)(1 + \mu)} \left[ (1 - \mu)\varepsilon_r + \mu(\varepsilon_t + \varepsilon_z) - (1 - 2\mu)e_{\tau p} - (1 + \mu)\alpha T \right] = 0 \quad (126) \]

at \( r = R \). Equations (119) and (120) may be used to evaluate \( C_1 \):
\[
C_1 = \frac{2(1 - 3\mu)}{1 - \mu} \frac{1}{R^2} \int_0^R \alpha T r d r + \frac{1}{1 - \mu} \frac{1}{R^2} \int_0^R (e_{tp} + e_{tp}) r d r - \\
\frac{(1 - 2\mu)}{1 - \mu} \int_0^R \frac{e_{tp} - e_{tp}}{r} d r;
\]

(127)

upon substituting this result into (120) one finds that

\[
e_z = \frac{2}{R^2} \left[ \int_0^R \alpha T r d r - \int_0^R (e_{tp} + e_{tp}) r d r \right]
\]

(128)

In summary, for the strains one has

\[
e_r = \frac{1 + \mu}{1 - \mu} \alpha T - \frac{(1 + \mu)}{1 - \mu} \frac{1}{r^2} \int_0^R \alpha T r d r + \frac{1 - 2\mu}{1 - \mu} e_{tp} + \\
\frac{1 - 2\mu}{2(1 - \mu)} \int_0^R \frac{e_{tp} - e_{tp}}{r} d r - \\
\frac{(1 - 2\mu)}{2(1 - \mu)} \frac{1}{r^2} \int_0^R (e_{tp} + e_{tp}) r d r + C_1 \frac{1}{2}
\]

(129)

\[
e_t = \frac{1 + \mu}{1 - \mu} \frac{1}{r^2} \int_0^R \alpha T r d r + \frac{1 - 2\mu}{2(1 - \mu)} \frac{1}{r^2} \int_0^R (e_{tp} + e_{tp}) r d r + \\
\frac{1 - 2\mu}{2(1 - \mu)} \int_0^R \frac{e_{tp} - e_{tp}}{r} d r + C_1 \frac{1}{2}
\]

\[
e_z = \frac{2}{R^2} \left[ \int_0^R \alpha T r d r - \int_0^R (e_{tp} + e_{tp}) r d r \right]
\]

where

\[
C_1 = \frac{2(1 - 3\mu)}{1 - \mu} \frac{1}{R^2} \int_0^R \alpha T r d r + \frac{1}{1 - \mu} \frac{1}{R^2} \int_0^R (e_{tp} + e_{tp}) r d r - \\
\frac{(1 - 2\mu)}{1 - \mu} \int_0^R \frac{e_{tp} - e_{tp}}{r} d r
\]
Now upon substituting the results summarized in equation (129) into (107) and (124) the stresses may be obtained as

\[
S_r = \frac{E}{(1 - \mu)(1 + \mu)} \left[ - \frac{(1 + \mu)}{r^2} \int_0^r \alpha Tr\,dr - \frac{(1 - 2\mu)}{r^2} \int_0^r (e_{rp} + e_{tp})r\,dr + \frac{1}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 + \mu}{R^2} \int_0^R \alpha Tr\,dr - \frac{1}{2} \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 - 2\mu}{R^2} \int_0^R (e_{rp} + e_{tp})r\,dr \right]
\]

(130)

\[
S_t = \frac{E}{(1 - \mu)(1 + \mu)} \left[ \frac{1 + \mu}{r^2} \int_0^r \alpha Tr\,dr + \frac{1 - 2\mu}{r^2} \int_0^r (e_{rp} + e_{tp})r\,dr + \frac{1}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr - (1 + \mu)\alpha T + \mu e_{rp} - (1 - \mu)e_{tp} + \frac{1 + \mu}{R^2} \int_0^R \alpha Tr\,dr + \frac{1 - 2\mu}{R^2} \int_0^R (e_{rp} + e_{tp})r\,dr - \frac{1}{2} \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr \right]
\]

\[
S_z = \frac{E}{(1 + \mu)(1 - \mu)} \left[ -(1 + \mu)\alpha T + (1 - \mu)e_{tp} + e_{rp} + \mu \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{2(1 + \mu)}{R^2} \int_0^r \alpha Tr\,dr - \mu \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr - \frac{(2 - \mu)}{R^2} \int_0^R (e_{rp} + e_{tp})r\,dr \right]
\]
IV. THE PLANE STRESS PROBLEM OF THE CIRCULAR DISK WHEN HEATED BY
INDUCTION FIELDS TO HIGH RIM TEMPERATURE TO INDUCE PLASTIC FLOW

Following much of the development of the problem of the infinitely
long circular cylinder, the problem of plane stress in a circular disk
will now be developed. Figure 10 shows this disk.

![Diagram of a circular disk](image)

**Figure 10. The plane stress problem of the circular disk.**

As a free body diagram of an element of that disk, figure 9 applies,
with the omission in this case of the stress in the axial direction. As
is usual, the weight of the small element is omitted also. The equilibrium
equation (104) likewise applies without modification. The Hooke's Law
equations are somewhat simplified, with the axial stress equal zero, to

\[
\begin{align*}
e_r &= \frac{1}{E} (s_r - \mu s_t) + \alpha T + e_{rp} \\
e_t &= \frac{1}{E} (s_t - \mu s_r) + \alpha T + e_{tp} \\
e_z &= -\frac{1}{E} (s_r + s_t) + \alpha T - (e_{rp} + e_{tp})
\end{align*}
\]  

(131)
Equation (105) for the strains applies without change. It was

\[
e_r = \frac{dU}{dr}, \quad e_t = \frac{U}{r}, \quad e_z = e_z
\]

(132)

where \( U = U(r) \) is the total radial displacement of any point. It is evident here also that a point may suffer only a radial displacement due to symmetry. Solving (131) for the stresses in terms of the strains one obtains

\[
S_r = \frac{E}{1 - \mu} \left[ e_r + \mu e_t - (e_{rp} + \mu e_{tp}) - (1 + \mu)\sigma T \right]
\]

\[
S_t = \frac{E}{1 - \mu} \left[ e_t + \mu e_r - (e_{tp} + \mu e_{rp}) - (1 + \mu)\sigma T \right];
\]

(133)

or, making use of equations (132),

\[
S_r = \frac{E}{1 - \mu} \left[ \frac{dU}{dr} + \mu \frac{U}{r} - (e_{rp} + \mu e_{tp}) - (1 + \mu)\sigma T \right]
\]

\[
S_t = \frac{E}{1 - \mu} \left[ \frac{U}{r} + \mu \frac{dU}{dr} - (e_{tp} + \mu e_{rp}) - (1 + \mu)\sigma T \right]
\]

(134)

Now substituting (134) into (104), one finds, that

\[
\frac{1}{r} \frac{d}{dr} \left( \frac{U}{r} \right) = \frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{U}{r^2} = (1 + \mu)\sigma T + \frac{de_{rp}}{dr} + \mu \frac{de_{tp}}{dr} +
\]

\[
(1 - \mu) \left( \frac{e_{rp} - e_{tp}}{r} \right) = \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (rU) \right]
\]

(135)

Upon integrating once,

\[
\frac{1}{r} \frac{d}{dr} (rU) = (1 + \mu)\sigma T + e_{rp} + \mu e_{tp} + (1 - \mu) \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + C_3
\]

(136)
Multiplying both sides by $r$, one sees that

$$\frac{d}{dr} (rU) = (1 + \mu) \alpha T_r + e_{r\theta}r + \mu e_{t\phi} r + (1 - \mu) r \int_0^r \frac{e_{r\phi} - e_{t\phi}}{r^2} dr + C_3 r$$

(Integrating a second time, one finds)

$$rU = (1 + \mu) \int_0^r \alpha T_r dr + \int_0^r e_{r\theta} r dr + \mu \int_0^r e_{t\phi} r dr +$$

$$+ (1 - \mu) \int_0^r r \int_0^r \frac{e_{r\phi} - e_{t\phi}}{r^2} dr dr + C_3 r^2 + C_4$$

(137)

And finally dividing by $r$,

$$U = \frac{1 + \mu}{r} \int_0^r \alpha T_r dr + \frac{1}{r} \int_0^r e_{r\theta} r dr + \mu \int_0^r e_{t\phi} r dr +$$

$$+ \frac{1 - \mu}{r} \int_0^r r \int_0^r \frac{e_{r\phi} - e_{t\phi}}{r^2} dr dr + \frac{C_3 r^2 + C_4}{r}$$

(139)

It is evident that $C_4$ must be zero to insure a finite displacement as $r$ approaches zero.

Consider

$$\int_0^r r \int_0^r \frac{e_{r\phi} - e_{t\phi}}{r} dr dr$$

Let

$$u = \int_0^r \frac{e_{r\phi} - e_{t\phi}}{r} dr \quad \text{and} \quad \dot{u} = \frac{e_{r\phi} - e_{t\phi}}{r} dr$$

$$dv = r dr \quad \text{and} \quad v = \frac{r^2}{2}$$

(140)
Making use of integration by parts,
\[
\int_0^r \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr \, dr = \frac{r^2}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr - \frac{1}{2} \int_0^r (e_{rp} - e_{tp}) \, dr
\]
(141)

Thus \( U \) simplifies to
\[
U = \frac{1 + \mu}{r} \int_0^r \alpha T \, dr + \frac{1 - \mu}{2} r \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 + \mu}{2} \int_0^r (e_{rp} + e_{tp}) \, dr + \frac{C_3 r}{2}
\]
(142)

and the strains become
\[
e_t = \frac{U}{r} = \frac{1 + \mu}{r^2} \int_0^r \alpha T \, dr + \frac{1 - \mu}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 + \mu}{2} \int_0^r (e_{rp} + e_{tp}) \, dr + \frac{C_3}{2}
\]
(143)

\[
e_r = \frac{dU}{dr} = (1 + \mu) \alpha T - \frac{1 + \mu}{r^2} \int_0^r \alpha T \, dr + \frac{1 + \mu}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{(1 + \mu)}{2} \int_0^r \frac{e_{rp} - e_{tp} - e_{rp} - e_{tp}}{r} \, dr + \frac{C_3}{2}
\]
(144)

From the last of equations (131), the axial strain is
\[
e_z = \frac{-\mu}{1 - \mu} (e_r + e_t) - \frac{1 - 2\mu}{1 - \mu} (e_{rp} + e_{tp}) + \frac{1 + \mu}{1 - \mu} \alpha T
\]
(145)

To evaluate the constant \( C_3 \), use is made of the fact that the radial stress must be zero when \( r = R \). Thus one finds
\[ C_3 = \frac{2(1 - \mu)}{R^2} \int_0^R \alpha \tau dr - (1 - \mu) \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr + \]
\[ \frac{1 - \mu}{R^2} \int_0^R (e_{rp} + e_{tp}) \, r \, dr \]  
(146)

Evaluating the stresses from equations (133) one gets

\[ S_r = \frac{E}{1 - \mu} \left[ \frac{1 - \mu^2}{2} \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr - \frac{(1 - \mu^2)}{2r^2} \int_0^R (e_{rp} + e_{tp}) \, r \, dr + \right] \]
\[ \frac{1 + \mu}{2} C_3 - \frac{(1 - \mu^2)}{r^2} \int_0^R \alpha \tau \, dr \]  
(147)

\[ S_z = 0 \]  
(148)

\[ S_t = \frac{E}{1 - \mu} \left[ \frac{1 - \mu^2}{2} \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 - \mu^2}{2r^2} \int_0^R (e_{rp} + e_{tp}) \, r \, dr + \right] \]
\[ \frac{1 + \mu}{2} C_3 - (1 - \mu^2) \alpha T - (1 - \mu^2) e_{tp} + \frac{1 - \mu^2}{r^2} \int_0^R \alpha \tau \, dr \]  
(149)

where \( C_3 = \)

\[ \frac{2(1 - \mu)}{R^2} \int_0^R \alpha \tau dr - (1 - \mu) \int_0^R \frac{e_{rp} - e_{tp}}{r} \, dr + \frac{1 - \mu}{R^2} \int_0^R (e_{rp} + e_{tp}) \, r \, dr \]  
(150)

With the axial stress equal to zero, it is clear that the equivalent stress simplifies to

\[ \bar{S} = \frac{1}{\sqrt{2}} \sqrt{S_r^2 - S_r S_t + S_t^2} \]  
(151)
V. METHOD OF SOLUTION OF THE EQUATIONS

It is evident that the equations (129) and (130), for the stresses and strains pose a problem in their solution, for in the initial analysis of them, the only quantity known is the temperature variation as a function of radial position and time. This temperature variation would be measured experimentally, computed theoretically, or for that matter, arbitrarily taken. Even if the temperature distribution was not known as a simple mathematical function, it is apparent that the integrals involving the temperature could be evaluated numerically by any one of the standard techniques such as Simpson's rule or the trapezoidal rule. The difficulty comes in the integrals involving the plastic strain components in the radial and the tangential directions, which of course, are not known. However, the task may be accomplished as follows:

Initially the problem is one of completely elastic behavior. Then it is true that all the integrals involving the plastic strains vanish, and a solution may be obtained for the stresses, and in turn the strains by equations (101). These three values for the stresses may be substituted into equation (79) and the result compared with the yield strength of the material involved. If the value obtained is less than the yield strength at all points in the problem, the result is correct, and the problem is indeed one of elastic nature. However, if the result is greater than the yield strength at certain points, then plastic flow has taken place at these points, and the result obtained is in error, perhaps markedly so. For the incremental theory of plasticity, one would begin at the start of the loading program, and follow that loading program through to completion in small
steps. Hence it would be known just when the material flowed plastically for the first time at any position; and thus any calculation made showing plastic flow for the first time would not be very much in error. In the deformation theory, it is quite possible that the initial calculation would be highly in error. The discussion will now be restricted to the method of attack on the deformation theory solution, followed by the incremental theory.

Consider the set of equations (129). Knowing the temperature distribution as a function of radius at the time under consideration, assume the plastic strain components zero. The total strains can then be computed by equations (129), which are in error. One may compute the total equivalent strain by equation (82) and the stresses by equations (106) and (124). Then the equivalent stress may be computed by equation (79). Using equation (85), the equivalent plastic strain may be computed. Lastly, using equations (92) the values for the plastic strains in the three directions may be evaluated. These values may be substituted into equations (129), holding the integrals involving the temperatures constant. Now closer values to the total strains are obtained, and the result may be run through the above explained sequence again, until the values converge. Convergence may be determined by a number of means. In this dissertation it is obtained by comparing the values of two successive calculations for the equivalent stress at the rim and making the differences less than 0.1 psi. The attack is a standard one in the field of integral equations, and is discussed by E. L. Ince, (reference 18, page 63). The method is shown to be mathematically convergent in this reference. For ease of computation, the equation (85) may be approximated by a linear equation in the
equivalent strain (plastic) and the equivalent strain (total) when it is plotted using these two variables as the coordinates. For machine calculation, this greatly simplifies matters. This method described is rapidly convergent, requiring some times as few as five iterations, and occasionally as high as ten to twelve, which of course, can be negligible on an electronic computer such as International Business Machine's model 653, which was used in this work. The reason for the rapidity of convergence is explained by reference to the stress-strain diagram, figure 11.

![Figure 11. Sketch of typical stress-strain diagram.](image)

The ordinary stress-strain diagram, in the range of plastic flow, is quite insensitive to values of stress as determined from values of strain. What this means is, if one selects, or computes a value of strain that is highly in error, the corresponding stress, or the corresponding equivalent stress is not so highly in error, due to the horizontal trend of the stress-strain curve in the plastic region. However, it is also noted that
if one approached the problem from the other viewpoint, difficulty could result. For instance, imposing a certain temperature distribution that would cause plastic flow of a fairly large amount would correspond to an elastic solution that would give an equivalent stress much above the stress-strain diagram, thus missing it entirely, and mathematically, making the problem unsolvable. This is not so difficult if one is solving the problem with, for example, a desk computer, as one can watch the curve, and see what is happening after every computation, but is evident that in high speed machine calculations where all data are placed into the machine, divergent solutions may result if this approach is used. It is better to compute strains initially, and then solve for the stresses. This approach is made possible by the assumptions of the deformation theory, allowing relations (72) through and inclusive (93) to be obtained.

If one now considers the incremental theory equations (65), and attempts to pursue an analogous development along the lines of equations (72) through (93), it is evident that the attack is not as fruitful. This is because, of course, the incremental relationship of equation (68) has INCREMENTALS in the numerators, while the deformation relationship of equation (77) has TOTAL plastic strains in the numerator. Hence an attempt to transform equations (65) into an incremental form corresponding to (93) does not meet with success.

The method used for solving the incremental theory problem is a modification of the method used by Millenson and Manson. It has been pointed out that for high enough plastic strains, the elastic solution may miss the stress-strain curve entirely. Hence, one must locate some point on the
stress strain curve from which to begin the convergence process. This may readily be discussed with the aid of figure 12.

Suppose point A corresponds to an elastic solution to the problem at some point in the loading program. The equivalent plastic strain corresponding to this stress is the distance AC. However, if point A is too high, the horizontal projection from A may miss the stress-strain curve completely; it is necessary therefore to have an alternate means of obtaining the actual solution. It must be remembered that the value of equivalent stress given by point A is completely fictitious, the actual value lying somewhere on the stress-strain curve at a lower stress and to the right at a higher strain. Point A is (essentially a fictitious) elastic solution. Assuming completely elastic behavior, there must occur, of course, some plastic strain, which necessitates the true value lying both below and to the right of A somewhere on the curve. To accomplish this, one may project vertically down from point A, intersecting the curve at point B. Corresponding to this value of stress at point B is the equivalent plastic strain represented by BD. This is a lower plastic strain than is actually occurring. With this value of plastic strain, and the stresses corresponding to point A (obtained from equations (130) with either the plastic strain components zero or held constant from the previous iteration) equations (65) give new values for the plastic strain increments. (The cylinder problem will serve well for discussion purposes.) At this time, it is helpful to think of equations (130) as being written as equations (152) which appear on page 63. Now, in this set of equations, hold all integrals involving the total plastic strains and the temperatures constant. By substituting the results just obtained
Figure 12. Equivalent stress vs. strain.
\[ S_r = \frac{E}{(1 + \mu)(1 - \mu)} \left[ - \frac{(1 + \mu)}{r^2} \int_0^r aTrdr - \frac{(1 - 2\mu)}{2} \int_0^r \left( e_{rp} + e_{tp} \right) rdr - \right. \\
\left. \frac{1}{2} \int_0^r \frac{(\Delta e_{rp} + \Delta e_{tp})}{r} rdr + \frac{1}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} dr + \right. \\
\left. \frac{1}{2} \int_0^r \frac{\Delta e_{rp} - \Delta e_{tp}}{r} dr + \frac{1 + \mu}{R^2} \int_0^R aTrdr - \frac{1}{2} \int_0^R \frac{e_{rp} - e_{tp}}{r} dr - \right. \\
\left. \frac{1}{2} \int_0^R \frac{\Delta e_{rp} - \Delta e_{tp}}{r} dr + \frac{1 - 2\mu}{2} \frac{1}{R^2} \int_0^R \left( e_{rp} + e_{tp} \right) rdr + \right. \\
\left. \frac{1}{2} \int_0^R \frac{\Delta e_{rp} + \Delta e_{tp}}{r} rdr \right] \\
(152)
\]

\[ S_t = \frac{E}{(1 + \mu)(1 - \mu)} \left[ \frac{1 + \mu}{2} \int_0^r aTrdr + \frac{1 - 2\mu}{2} \frac{1}{r^2} \int_0^r \left( e_{rp} + e_{tp} \right) rdr + \right. \\
\left. \frac{1}{2} \int_0^r \frac{(\Delta e_{rp} + \Delta e_{tp})}{r} rdr + \frac{1}{2} \int_0^r \frac{e_{rp} - e_{tp}}{r} dr + \right. \\
\left. \frac{1}{2} \int_0^r \frac{\Delta e_{rp} - \Delta e_{tp}}{r} dr + \frac{1 + \mu}{R^2} \int_0^R aTrdr - (1 + \mu)aT + \mu e_{rp} + \mu \Delta e_{rp} - \right. \\
\left. (1 - \mu) e_{tp} - (1 - \mu) \Delta e_{tp} + \frac{1 - 2\mu}{2} \frac{1}{R^2} \int_0^R \left( e_{rp} + e_{tp} \right) rdr + \right. \\
\left. \int_0^R \frac{\Delta e_{rp} + \Delta e_{tp}}{r} rdr \right] \\
(152)
for the increments in plastic strain from (65) into (152) in the integrals involving the respective increments in plastic strain, new values for the stresses may be obtained. These values may be substituted into equation (79) and a new value of the equivalent stress is obtained. Since this value of equivalent stress was obtained with a more realistic value of plastic strains, it produces a closer approximation to the correct answer to the problem at this stage in the loading program. It thus corresponds to an equivalent stress intermediate to that of either A or D, say at E. Since this point corresponds to an equivalent stress of E and an equivalent plastic strain of \( DB = EF \), it locates the point F, which is still not on the stress-strain curve. One would then drop vertically to point G, on
the curve, giving an equivalent plastic strain of $E_G$ and an equivalent stress of $H$. Using the stresses obtained just previously, and the equivalent strain $G_H$, by means of (65), new increments for the plastic strains are obtained. These are substituted into (152), replacing the previous increments in those integrals, giving again a more correct value for the stresses, and determining a new equivalent stress, this time intermediate to both $E$ and $H$, say $J$. This last process determines the new point $K$, still not on the stress-strain curve, but much closer. By repeating this process, it is evident one may approach the stress-strain curve as closely as desired. Let point $L$ be that point eventually obtained by this process.

If one now advances the solution by a change in the temperature profile (in other words, by the next time increment), this would locate a point on the line through $L$ parallel to the original elastic line, say $M$, whereupon the entire process would be repeated until convergence was once again obtained, and so on through the loading program. One must watch for unloading at any point in the program. For example, if one were at the converged point $N$, and unloading took place, the point may assume the position $O$, on a line parallel to the original stress-strain curve. At this point in the problem, from that time on, the problem would be of elastic nature, and all changes involving stress and strain would take place along the line $NO$ and its extension, until the equivalent stress had again exceeded the value $N$, if that ever occurred. This feature was built into the IBM computing program, and was used with success.

Since a work hardening material was used, it is impossible to find the exact relationship, in general, between stress and strain, so these data were used in table look up form in the machine program. The data used
were values of equivalent stress vs. the equivalent plastic strain corresponding to values obtained by dropping vertically to the curve as from A to B and then reading the value BD, as well as the equivalent stress at D. If this was used, it should be evident to the reader that some additional feature was necessary to carry out the solution. Referring to Figure 12, after dropping from A to B, and reading BD, which is in the program in the tables, it was shown that this locates point F. Now, however, it is impossible with the information on hand, to drop down from F to the point G, because in order to drop to G from some point above, that point must be on the original elastic line, point P, directly above F and G. It is evident that in the parallelogram AFBP, AP is parallel to BF, and equals it, and in the parallelogram EPBD, ED is parallel and equal to BF. Hence the distance in terms of equivalent stress between A and P equals the distance in equivalent stress between E and D, which is known. Therefore, adding the distance ED to the equivalent stress at A locates P, and then the machine program can drop from P to G.

In the problem of the thin circular disk, intervals of 4 seconds each were selected for the computations for the first two minutes, then every minute until twenty minutes elapsed, finally the forty minute interval, and the six hour interval in conclusion. This procedure was followed for both the deformation and incremental theory.

In the problem of the circular cylinder, intervals of dimensionless time of 0.001 were selected from the beginning until 0.30, and then every 0.10 until the conclusion at 1.500.

The IBM program was designed to punch answers out on the strip chart after every conversion with the information given in the graphs. In this
procedure, the problem could be watched, the data plotted as it was being computed, and a better means of evaluating the progress was obtained than if the information was left on the punched card until the conclusion.
VI. RESULTS OF THEORETICAL ANALYSIS

A. The Infinitely Long Circular Cylinder Subjected to Rapid Quench

Figures 13 to 42 pertain to results obtained from a solution of the problem of quenching a circular cylinder. This solution was carried out with a stress-strain curve for 18-8 Stainless Steel obtained from reference 22, page 7, and reproduced as figure 14. The time-position-temperature data was obtained from reference 2, pages 120-138 by a solution of the boundary value problem in terms of Bessel Functions, as given by Ruel V. Churchill in reference 4, pages 143-174, for example. In order to make the given data applicable to many quenching tests, the time is presented as a "dimensionless time," defined as the product of the thermal diffusivity and the actual time divided by the square of the radius. This time ratio therefore takes account of differences in size and diffusivity. This is explained in detail on page 120 of reference 2.

In reference 16, page 2063, the coefficient of thermal expansion of stainless steel is given as $9.5(10)^{-6}$ per °F and increases with temperature increase to $9.8(10)^{-6}$ per °F at a temperature of 400° F. In the solutions of this dissertation, the lower value is taken, and assumed constant as a simplification in the solution.

Young's Modulus (E) is assumed constant at $30(10)^6$ psi, as may be confirmed in any standard reference on mechanics of materials or theory of elasticity. Poisson's ratio is assumed constant at 0.3.

The time-position-temperature profile for the quench of the cylinder is independent upon the Nusselt number. In order to carry out a specific solution, a Nusselt number of 10 was arbitrarily assumed. This represents a rather severe quench. This data for the Nusselt number of 10 can be
obtained from reference 2 on page 137. The time-temperature-position data is given graphically in figure 13.

It will be noted that rather extensive information is given in the set of figures 16 to 42, the first portion being the radial, tangential, and axial stresses, with cross plots of stress vs. time at selected radial positions, and stress vs. radial position at selected time. Certain radial positions were selected and both theories plotted on the same graph for ease in comparison. It is noted that not all positions given are compared in this manner, the reason being to avoid confusion by adding more lines to the many already presented.

Equivalent stress as a function of position and time is also presented, enabling the reader to determine when the plastic flow began and ended at any time and position.

The radial, tangential, and axial strain is presented for both theories and certain curves from both theories are plotted on a separate graph for comparison.

The ratios of radial stress to axial stress and tangential stress to axial stress as a function of time and position is presented, for it was pointed out that one criterion for the coincidence of the two theories is the constancy of these ratios during the program of loading and unloading.

In general, from studying the results, it is evident that the two theories give results in agreement during the loading, and indicate more and more disagreement as the unloading progresses.

It is seen from figures 37 to 42 that the ratios are essentially constant, diverging quickly to infinite values at the time the axial stress tends toward zero and changes sign, that is, from tension to
compression, or vice-versa. These curves in general show again an almost constant behavior of the ratios after this divergence has taken place, until the end of the test at a dimensionless time of 1.5.

B. The Thin Circular Disk Heated by Induction

Figures 43 to 81 pertain to the results obtained from a solution of the problem of heating a circular disk by induction to induce plastic flow. Figures 43 and 44 present the time-position-temperature data as obtained by the author experimentally and described in detail under the heading of Determination of Temperature Profile and Method of Inducing Plastic Flow. Figure 45 shows the results of obtaining the stress-strain curves, and the resulting scatter of these curves. Although some of the inconsistency could certainly be ascribed to experimental deviation, it is evident that the material had a preferred orientation and a lack of homogeneity. Figure 46 is an average of the curves of figure 45, and is the one actually used in the solution of the problem. Figure 47 shows the variation of the equivalent plastic strain with the total equivalent strain, based on the stress-strain curve of figure 46.

The subsequent figures follow largely the order of the circular cylinder data, first the radial, tangential, and equivalent stresses as a function of time and position, and then the radial, tangential, and axial strains as functions of position and time. The ratio of the radial stress to the axial stress is presented as a function of time at 6 of the radial positions.

In studying these curves, it may be concluded that the two theories are in agreement during the loading program, and show increasing deviation
Figure 13. - Temperature against radial position at selected dimensionless time. Quenched cylinder.
Figure 14. - Typical equivalent stress against strain for 18-8 stainless-steel. Data used for quenched cylinder.
Figure 15. - Variation of total equivalent strain with equivalent plastic strain, based on stress-strain curve of figure 32. Quenched cylinder.
Figure 16. - Radial stress against radial position at selected times. Incremental theory; quenched cylinder.
Figure 17. - Radial stress against dimensionless time. Incremental theory; quenched cylinder.
Figure 18. - Radial stress against dimensionless time. Deformation theory; quenched cylinder.
Figure 19. - Comparison of radial stress against dimensionless time at two selected radial positions. Incremental and deformation theory; quenched cylinder.
Figure 20: Tangential stress against radial position at selected dimensionless time. Incremental theory; quenched cylinder.
Figure 21. - Tangential stress against dimensionless time. Incremental theory; quenched cylinder.
Figure 22. - Tangential stress against dimensionless time. Deformation theory; quenched cylinder.
Figure 23. - Comparison of tangential stress against dimensionless time at two selected radial positions. Incremental and deformation theory; quenched cylinder.
Figure 24. - Axial stress against radial position at selected time. Incremental theory; quenched cylinder.
Figure 25. - Axial stress against dimensionless time. Incremental theory; quenched cylinder.
Figure 26. Axial stress against dimensionless time. Deformation theory; quenched cylinder.
Figure 27. - Comparison of axial stress against dimensionless time at two selected radial positions. Incremental and deformation theory; quenched cylinder.
Figure 26. - Equivalent stress against dimensionless time at selected radial positions. Incremental theory; quenched cylinder.
Figure 29. - Equivalent stress against dimensionless time at selected radial positions. Deformation theory, quenched cylinder.
Figure 30. - Equivalent stress against dimensionless time at two selected radial positions. Deformation and incremental theory; quenched cylinder.
Radial strain against radial position at selected times. Incremental theory; quenched cylinder.
Figure 32. Comparison of radial strain against dimensionless time at six selected radial positions. Incremental and deformation theory; quenched cylinder.
Figure 33. - Tangential strain against radial position at selected times. Incremental theory; quenched cylinder.
Figure 34. - Comparison of tangential strain against dimensionless time at six selected radial positions. Incremental and deformation theory; quenched cylinder.
Figure 35. - Axial strain against radial position at selected times. Incremental theory; quenched cylinder.
Figure 36. - Comparison of axial strain against dimensionless time. Incremental and deformation theory; quenched cylinder.
Figure 37. - Radial stress and Tangential stress against dimensionless time at $r = 0$

during loading program. Incremental theory; quenched cylinder.
Figure 38. Radial stress and Tangential stress against dimensionless time at $r = 0.2$
Axial stress
inch during loading program. Incremental theory; quenched cylinder.
Dimensionless time, $\tau$

Figure 39. Radial stress and Tangential stress against dimensionless time at $r = 0.4$ inch during loading program. Incremental theory, quenched cylinder.
Figure 40. Radial stress and tangential stress against dimensionless time at \( r = 0.6 \) inch during loading program. Incremental theory, quenched cylinder.
Figure 41. - Radial stress and Tangential stress against dimensionless time at $r = 0.8$ inch during loading program. Incremental theory; quenched cylinder.
Figure 42. Radial stress and Tangential stress against dimensionless time at \( r = 1 \) inch during loading program. Incremental theory; quenched cylinder.
during the unloading program. In figures 76 to 81, it may be seen that
the constancy of the stress ratios prevails until the tangential stress
changes sign, and then the ratio diverges rapidly to infinity. Following
the divergence, the ratio again becomes essentially constant. This of
course is presented only for the incremental theory, as this is the only
one of the two where such data are meaningful.

The Young's Modulus (E) obtained from the material experimentally was
found by an average of the separate curves to be 28.8(10)^6 psi. In this
problem, as in the cylinder, the value for the coefficient of thermal ex­
pansion was taken to be 9.5(10)^-6 per °F, and Poisson's ratio was taken to
be 0.3.
Figure 4b. Heating curves for circular disk temperature, $\theta^*$ against dimensionless radius at selected times during application of induction heating.
Figure 44. - Cooling curves for circular disk temperature, °F against dimensionless radius at selected time following application of induction heating.
Figure 45. Results of determination of stress-strain curves.
Figure 46. Average stress-strain curve obtained from figure 45.
Figure 47. Total equivalent strain against equivalent plastic strain based on stress-strain diagram of figure 67 18-8 type 347 stainless steel.
Figure 46. - Radial stress against dimensionless radius at selected time. Incremental theory; heated disk.
Figure 49. Radial stress against dimensionless radius at selected time. Deformation theory; heated disk.
Figure 50. - Radial stress against time at selected radial positions. Incremental theory; heated disk.
Figure 51. - Radial stress against time at selected radial positions. Deformation theory; heated disk.
Figure 52. - Comparison of radial stress against time at two selected radial positions. Incremental and deformation theory; heated disk.
Figure 53. - Tangential stress against dimensionless radius at selected time. Incremental theory; heated disk.
Figure 54. - Tangential stress against dimensionless radius at selected time. Deformation theory; heated disk.
Figure 55 - Tangential stress against time at selected radial positions. Incremental theory; heated disk.
Figure 56. - Tangential stress against time at selected radial positions. Deformation theory; heated disk.
Figure 57. - Comparison of tangential stress against time at two selected radial positions. Incremental and deformation theory; heated disk.
Figure 58. - Equivalent stress against time at selected radial positions. Incremental theory; heated disk.
Figure 59. - Equivalent stress against time at selected radial positions. Deformation theory; heated disk.
Figure 60. - Comparison of equivalent stress against time at two selected radial positions. Incremental and deformation theory; heated disk.
Figure 61. - Radial strain against dimensionless radius at selected time. Incremental theory; heated disk.
Figure 62. - Radial strain against dimensionless radius at selected times. Deformation theory; heated disk.
Figure 53. - Radial strain against time at selected radial positions. Incremental theory; heated disk.
Figure 64. - Radial strain against time at selected radial positions. Deformation theory; heated disk.
Figure 65. - Radial strain against time at two radial positions. Incremental and deformation theory; heated disk.
Figure 66. - Tangential strain against dimensionless radius at selected time. Incremental theory; heated disk.
Figure 67. - Tangential strain against dimensionless radius at selected time. Deformation theory; heated disk.
Figure 68. - Tangential strain against time at selected radial positions. Incremental theory; heated disk.
Figure 69. - Tangential strain against time at selected radial positions. Deformation theory; heated disk.
Figure 70. Comparison of tangential strain against time at two selected radial positions. Incremental and deformation theory; heated disk.
Figure 71. - Axial strain against dimensionless radius at selected time. Incremental theory; heated disk.
Figure 72. Axial strain against dimensionless radius at selected time. Deformation theory; heated disk.
Figure 73. - Axial strain against time at selected radial positions. Incremental theory; heated disk.
Figure 74. - Axial strain against time at selected radial positions. Deformation theory; heated disk.
Figure 75. - Comparison of axial strain against time at two selected radial positions. Deformation and incremental theory; heated disk.
Figure 76. - $\frac{\text{Radial stress}}{\text{Tangential stress}}$ against time during loading program at $r = 0$. Incremental theory; heated disk.
Figure 77. - $\frac{\text{Radial stress}}{\text{Tangential stress}}$ against time during loading program at $r = 0.2$. Incremental theory; heated disk.
Figure 78. Radial stress and tangential stress against time during loading program at $r = 0.4$. Incremental theory; heated disk.
Figure 79. - Radial stress over Tangential stress against time during loading program at $r = 0.6$. Incremental theory; heated disk.
Figure 80. $\frac{\text{Radial stress}}{\text{Tangential stress}}$ against time during loading program at $r = 0.8$. Incremental theory; heated disk.
Figure 61. - $\frac{\text{Radial stress}}{\text{Tangential stress}}$ against time during loading program at $r = 1.0$. Incremental theory; heated disk.
VII. EXPERIMENTAL PROGRAM

A. Materials

1. General description of material used in experimental tests

The material used in experimental work in this dissertation was 18-8 Type 347 Stainless Steel, which was supplied to the author in the form of a circular billet 15 inches in length and 13 inches in diameter. From this billet circular disks three-quarters of an inch thick and 12 inches in diameter were cut and machined to a 32 finish. Photographs of these disks are given in this dissertation as figures 94, 95, 96, and 97.

2. Method of stress relief of the circular disks

In order to relieve as much as possible the residual stresses that may have existed in the material after it had been forged, cut with a circular saw, and machined to a smooth 32 finish, the disks were placed in an electric furnace at room temperature, and the temperature increased to $1500^\circ F$ and held at that level for 3 hours. The door of the furnace was then opened, the power discontinued, and the disks were allowed to cool in the furnace. Figure 82 is a plot of the furnace temperature near the disk against time as determined by an automatically recording Minneapolis-Honeywell thermocouple, an integral part of the furnace regulating equipment.

This treatment is recommended as a stress relief and stabilizing treatment on page 109 of reference 1, specifications for stabilization of type 347 stainless steel. The ASTM designation is A 182-55 T.

Upon cutting the disk thus treated after SR-4 strain gages were attached, negligible stresses were found, and therefore the disks were
Figure 82. Furnace temperature against time. Heat treatment used for stress relief of stainless-steel disks.
assumed to be completely stress-free originally in the subsequent heating by the induction coil.

3. Determination of stress-strain curve

The assumptions made in this dissertation involving properties of the material are constancy of $E$, Young's Modulus, $\alpha$, the coefficient of thermal expansion, $\mu$, Poisson's ratio, and the yield strength, all with changing temperature.

The stress-strain characteristics were determined at room temperature, and assumed to be constant for higher temperatures.

The $J_{2}^{I}$ or Von Mises flow criterion was utilized, which postulates a homogeneous, isotropic medium. It is therefore desirable to obtain the stress-strain characteristics at various positions and directions in the material. This method enables one to determine if the material is isotropic and homogeneous, and to average the data should scatter result in the data. It should be noted that one stress-strain curve is necessary to give information for the application of the Von Mises criteria.

One of the disks cut from the billet was cut into 14 blanks approximately three and one-half inches long and seven-eighths inches square. These blanks were machined into tensile specimens conforming to figure 84. The locations of these specimens, and their accompanying numbers, which correspond to the numbers found on the stress-strain curves of figure 45, are shown in figure 85.

The tensile tests were performed on a Riehle hydraulic testing machine with Baldwin strain recording equipment.
Figure 83. Locations of the fourteen tensile specimens cut from circular disk. The numbers conform to the numbers found on the stress-strain curves of figure 45.
The different positions selected should give a good indication of the correctness, or degree of deviation from, of the assumption of both two-dimensional homogeneity and isotropy.

![Diagram of tensile specimen used to obtain stress-strain data for stainless steel disk.](image)

**Figure 84.** Diagram of tensile specimen used to obtain stress-strain data for stainless steel disk.

**B. Experimental Procedure**

1. Determination of temperature-position-time data and method of inducing plastic flow in the circular disks

In order to induce plastic flow in the stainless steel thin disks, an induction coil was designed that would heat the rim of the disk to 500°F in 72 seconds, using the maximum voltage source available, 400 volts. This coil was manufactured at NACA Lewis laboratory. A motor generator set manufactured by the Tocco Process Equipment Company of Cleveland, Ohio, was used as a source of excitation, and the power was supplied at a frequency of 10,000 cps. The coil is shown in figures 85, 87, and 88. The coil was
made of brass, and had a cooling passage around the circumference through which water was circulated during the test to avoid excessive heat. Originally the disk was placed in a horizontal position concentric with the coil on a small wood pedestal, supporting it only at its center. The disk remained stationary during the test. However, it was found that the disk did not heat uniformly at all points, there being as much as 60° F difference at opposite points of one diameter. Therefore it was decided to rotate the disk during the test to smooth out the heating cycle.

A Boston Gear turntable, model VIW 13, with a step down ratio of 150-1 was selected, and driven through a universal joint coupling by a one-eighth horsepower General Electric single phase induction motor, rated speed, 1800 rpm. Thus the turntable rotated at 12 rpm. Figure 89 shows the driving apparatus. A two-inch thick, twelve-inch diameter maple disk was mounted on the turntable, upon which was attached several sheets of asbestos, one-eighth inch thick, to insulate the disk and prevent the maple from being burned. The disk was placed on top this insulation. It was found that gravity would hold the disk in place, each disk weighing about 25 pounds. Figure 89 shows the disk mounted on the turntable. The top side of the disk was covered with more insulation, and a similar maple disk with a small hole through its center to admit the thermocouple wires was placed on top the asbestos insulation. Inasmuch as the disk rotated, slip rings were necessary in order to transmit the signal from the thermocouples to the recording equipment. An integral slip ring unit, manufactured by the Trombetta Solenoid Company of Milwaukee, Wisconsin, shown in figure 86 was selected. The slip rings were rotated by means of a vertical shaft connecting the top maple disk with the slip ring shaft, as
Figure 85. Power source, induction coil, driving mechanism, and slip rings for experimental work.
Figure 86. Close up view of slip rings and solenoid used to close circuit.
Figure 87. Close up view of thermocouples and induction coil.
Figure 88. Close up view of maple disk with junction terminals and induction coil.
Figure 89. Close up view of driving mechanism and lower maple disk.
shown in figure 85. This shaft was hollow, and contained the thermocouple wires. In order to facilitate dismantling the apparatus, a junction was made at posts mounted on the top of the maple disk as seen in figure 88. The signal from the slip rings was then transmitted directly to a Minneapolis-Honeywell multichannel strip chart recorder, model 5401-N, which had a moving strip chart to record directly in degrees Fahrenheit.

In order to determine the temperature profile as a function of radial position and time, thermocouples were mounted on the disk at the positions indicated in figure 90 and seen in figure 87 of the disk. It should be noted that they were placed at 90° positions on the circumference and at a quarter of an inch in from the circumference, in order to check that the disk was being heated uniformly in the tangential direction at any given radius. Readings taken at these quarter radial positions did verify that the disk was heating uniformly. Thermocouples were mounted along one radius, with a progressively smaller interval between the thermocouples the closer to the rim in order to evaluate the steeper temperature gradient existing at the rim. Thermocouples were sunk into the center plane of the disk to evaluate the temperature on this plane and to determine if the disk was heating uniformly through the thickness. Readings obtained from a sunken couple versus readings obtained from a surface couple at the same radial position were in agreement.

As mentioned previously, the rim of the disk could be heated to 500°F above room temperature in 72 seconds, at which time the power was discontinued, and the disk allowed to cool. The temperatures were evaluated for the first 40 minutes, at which time the disk had assumed an approximately uniform temperature of 80°F. Since no more plastic flow would take place
Figure 90. Locations of thermocouples used to determine temperature profile. Heating of thin disk by induction.

- location of surface couple
- x location of sunken couple
Figure 91. - Millivolt output against temperature difference between hot and cold junction; Iron-Constantan thermocouple. (From Pyrometry, Minneapolis-Honeywell Instruments Manual.)
at this uniform temperature, the test was concluded. The disk required approximately 6 hours to cool to room temperature, because of the heavy insulation and large mass.

It was found that the data taken could be repeated almost exactly during every subsequent test, the applied voltage being maintained at 400 volts and the frequency at 10,000 cps. Thus one disk could be rather extensively instrumented with thermocouples and used again upon cooling to room temperature.

The timing of the test was accomplished by means of an ordinary sweep second hand stop watch and the moving strip chart, set to move at one-half inch per second.

The results of the time-temperature-position curves is given in figures 43 and 44, the former the heating, the latter the cooling, in order to minimize confusion.

After a disk was used in this manner, it was cleaned and SR-4 electric strain gages were mounted on the disk to measure residual stresses in a manner described in the following chapter.

Iron-constantan thermocouples were used, and figure 91 illustrates the millivolt output as a function of temperature difference for these two materials.

2. Determination of residual stresses

Residual stresses in the tangential direction were measured by the technique of the strain gage oriented in such a direction to the principal stresses as to measure one of them directly. The theory of such an application of the SR-4 strain gage is covered by Herbert R. Lissner and
C. C. Perry. It is shown in that paper and in condensed form in appendix E that if the gage is mounted at an angle $\phi$ to one of the principal stresses, where $\phi$ is $\tan^{-1}\sqrt{\mu}$, then the one principal stress may be determined very simply from the formula

$$S_t = \frac{E}{1 - \mu} \tan \phi$$

where $E$ is Young's Modulus and $\mu$ is Poisson's ratio. Figure 92 shows the orientation of the gage.

Figure 92. Method of mounting the SR-4 strain gage as a principal stress gage.

Since the equilibrium equation is

$$\frac{dS_r}{dr} + \frac{S_r - S_t}{r} = 0$$

which may be rewritten as

$$\frac{d}{dr} (rS_r) = S_t$$

it follows by integration that the radial stress may be found if the tangential stress is measured. One finds

$$S_r = \frac{1}{r} \int_0^r S_t \, dr$$
The idea is essentially that if the gage is oriented to measure one of the principal stresses, then when that surface is made stress free, the gage will record the change in stress from the original level of the residual stress to the final level of zero stress. For example, in the thin circular disk under consideration, under a symmetrical heating cycle, it is known that the radial and tangential directions are the principal directions. Hence, the gages are mounted in such a way as to record the tangential stress across one diameter, and the disk is cut in half across this diameter, which exposes the gage to the free surface where no tangential stress may remain. Hence, this change is recorded on the gage.

The disk was cut in such a manner as to make the pattern of cutting as symmetrical as possible, and to make the original cuts as far away from the active gages as possible. Thus, the original value of the residual stress is lowered, and lessens the possibility of the cutting procedure causing additional plastic flow in the region of the gage. Hence the disk was cut according to figure 93, the last cut being the one across the diameter. By watching the strain change during the cutting procedures labeled one through eight, it was seen that the majority of the stress relief had taken place before the final diametral cut was made. Photographs listed as figures 94, 95, and 96 show how the SR-4 strain gages were attached to the stainless steel disks after these disks were caused to flow plastically by induction heating in the manner described in Chapter 7, part B-1. These gages were type AB-11 manufactured by Baldwin-Lima-Hamilton. After being attached they were coated with a glyptol solution to waterproof them, as later during the cutting procedure, oil was constantly applied to the disk to maintain a constant temperature. In
order to determine if the disks were originally stress free, one disk was instrumented with ten gages and cut before induction heating was applied. All gages showed insignificant magnitudes of stress. Two disks were then subjected to exactly the same induction heating treatment, and each disk had twenty-one gages applied, at equal intervals across one diameter, to insure continuity in the readings. Figure 97 shows one of these disks in the process of being cut. A band saw was used with an automatic feed which allowed the disk to be cut at about a half-inch per minute. Reference 14 lists techniques useful in the cutting procedure, which were used in this determination.

To determine the strains, a strain analyzer manufactured by the strain gage laboratory of the NACA Lewis laboratory was utilized. This is shown in figure 98. The circuit diagram for this instrument is shown in figure 99. It may be seen that it operates on the principal of the Wheatstone
Figure 94. Stainless steel disk with SR-4 strain gages attached.
Figure 95. Stainless steel disk with SR-4 strain gages attached.
Figure 96. Stainless steel disk after being cut.
Bridge. The nulling unit used was a Minneapolis-Honeywell Doelcam. The instrument was calibrated to read unit strain directly on a microammeter. As in the usual technique of measuring with this type of instrument, the Bridge Circuit was nulled out at equilibrium conditions, the disks were cut, and final readings taken at equilibrium conditions. The strain indicated by each gage was then computed as final reading minus initial reading. These results for the two disks are given in tables one and two, as well as the radial and tangential stresses as determined from the above formulas. These results are plotted as figures 100 and 101, along with the residual stresses as determined by the two theories of mathematical plasticity.
Figure 97. Stainless steel disk in process of being cut to relieve residual stresses.
Figure 98. NACA strain gage analyzer.
Figure 99. - Circuit diagram for NACA strain analyzer.
VIII. EXPERIMENTAL RESULTS

The experimental results obtained by measuring the residual stresses in the two disks are presented in tables one and two, as well as the radial and tangential stresses as determined by the formulas of Appendix E. These results are given graphically as figures 100 and 101, along with the residual stresses as determined by the two theories of mathematical plasticity.

In observing these two figures, it is noted that both the experimental disks give results in close agreement, the largest percentage error being at the rim, of magnitude 30 percent. In this region results are likely to be in error, due to the sharp stress gradient at this position.

In general it may be said that both tests indicate lower residual stresses than predicted by either theory, but both gives results closer to the incremental theory than the deformation theory.

It should be noted in studying the results that:

1. both disks, when subjected to the same treatment, give quite close results,

2. an excellent check is given by the determination of the radial stress from the equilibrium equation, that is, the integral of the tangential stress across the cross section of the disk must theoretically be zero. Or it may be said equivalently, the radial stress found by integrating the tangential stress across the cross section must be zero. It is seen that this is nearly so.

3. residual stresses were of low magnitudes except near the rim. Hence an experimental error of a few thousand psi would show a glaring percentage error.
Ideally, it would seem that an experimental procedure that would introduce residual stresses of upwards of fifty or sixty thousand psi would give much more reliable results. It should be pointed out that these magnitudes are difficult to obtain. When the disk is subjected to a high temperature gradient to cause plastic flow, the edge attains relatively high stresses, and the remaining portion of the cross section relatively low stresses. The above difficulty is true in a quench test also. The author felt that by not subjecting the disk to a temperature higher than 500° F, the Bauschinger effect in the material could be minimized, for it may be seen that the disk flows plastically with stresses of opposite sign when cooling. This behavior could be prevented by heating the disk to only 250° F, but this level would leave residual stresses of about half the magnitudes remaining at the 500° F level.

The assumption of constancy of Young's Modulus with temperature is valid at lower temperatures, and this was a further motivation to keep the temperature at lower values.

In view of the above points, the author feels that the experimental data obtained by the inclined gage is quite satisfactory; it does not clearly indicate which of the two mathematical theories is "correct," but it does show that the two theories predict the trend of behavior of the residual stresses.
Table 1. Results of residual stress measurements on stainless steel disks heated by induction

<table>
<thead>
<tr>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gage number</td>
<td>$r$ radius</td>
<td>Initial strain, microinches per inch</td>
<td>Final strain, microinches per inch</td>
<td>Net strain, microinches per inch</td>
<td>Tangential stress, psi $^a$</td>
<td>Radial stress, psi $^a$</td>
</tr>
<tr>
<td>1</td>
<td>-1.0</td>
<td>5</td>
<td>-980</td>
<td>-985</td>
<td>29,750 T</td>
<td>442 C</td>
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<tr>
<td>2</td>
<td>-0.9</td>
<td>10</td>
<td>-375</td>
<td>-385</td>
<td>16,600 T</td>
<td>3070 C</td>
</tr>
<tr>
<td>3</td>
<td>-0.8</td>
<td>15</td>
<td>-20</td>
<td>-35</td>
<td>1,510 T</td>
<td>4510 C</td>
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<tr>
<td>4</td>
<td>-0.7</td>
<td>13</td>
<td>120</td>
<td>107</td>
<td>4,620 C</td>
<td>4970 C</td>
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<tr>
<td>5</td>
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<td>138</td>
<td>5,960 C</td>
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<tr>
<td>6</td>
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<td>128</td>
<td>115</td>
<td>4,970 C</td>
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</tr>
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<td>7</td>
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<td>4740 C</td>
</tr>
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<td>4,620 C</td>
<td>4780 C</td>
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<td>4,440 C</td>
<td>4900 C</td>
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<td>5160 C</td>
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<td>127</td>
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<td>5480 C</td>
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<tr>
<td>12</td>
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<td>140</td>
<td>133</td>
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<td>5625 C</td>
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<td>3195 C</td>
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<td>-991</td>
<td>30,100 T</td>
<td>565 C</td>
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</table>

$^a_T$ denotes tension, $C$ denotes compression.
Table 2. Results of residual stress measurements on stainless steel disks heated by induction

<table>
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<td>(1) Gage number</td>
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<td>19</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>21</td>
</tr>
</tbody>
</table>

$^a$T denotes tension, C denotes compression.
Figure 100. Residual radial stress against dimensionless radius as determined by incremental and deformation theory and by experimental means.
Figure 101. - Residual tangential stress against dimensionless radius as determined by incremental and deformation theory and by experimental means.
IX. CONCLUSIONS

In this program of research, two boundary value problems have been solved, each by two theories of mathematical plasticity, the deformation theory and the incremental theory. Experimental evidence has been presented for the problem of the heating of a stainless steel disk by induction.

It has been seen that the two theories agree quite closely until the material unloads. If one knew a priori when this would take place, it would suffice to solve a few deformation problems in the neighborhood where the unloading was taking place to obtain the stress pattern. However, since as a rule this is not known a priori, it is better to use the incremental theory.

A few years ago calculations of this type would have been out of the question, before the widespread use of high speed computing equipment. Now, inasmuch as most large academic and industrial organizations have this equipment, the computational time and expense is of little question. The author therefore feels that the use of the incremental theory is the better mode of approach. In the solution of the two boundary value problems given, radial loading took place essentially. However, no one can guarantee that this will always follow. Hence, conceivably, one could commit glaring errors in solving a plastic flow problem by the deformation theory, if the stresses did not conform to the radial loading behavior.

Many boundary value problems may be solved by the incremental method. It should be of use to designers of rotating disks, shafts, and large cylinders used in high speed turbines. There is no "guess and test," or
"trial and error" necessary in this method of approach, as is found in many solutions to plastic flow problems in the literature.

The author found that a problem of this scope could be solved in about ten to twelve hours on the International Business Machine Company's 653 digital computer, with the SOAP (Systematic Optimal Assembly Program). If one had access only to the IBM 650, for example, considerably more time would be necessary, but the solutions could be carried out just as readily.

It is seen that even though the material used did not conform to the assumptions of isotropy, homogeneity, and no Bauschinger effect, that creditable answers were obtained experimentally. Therefore, the author feels justified in making these assumptions. Indeed, if they were not made, an expensive time consuming experimental program would be necessary to enable one to proceed in the theoretical solution.
X. RECOMMENDATIONS FOR FURTHER STUDY

It should be noted that the solutions of both boundary value problems undertaken in this dissertation showed close agreement during the loading program, and this was further confirmed by the constancy of the stress ratios during the loading program. Hence, one would expect close agreement, when solved by both mathematical theories of plasticity.

It would seem desirable to solve a problem that does not possess the constancy of the stress ratios during the loading program, one that has a chance of producing differing solutions when solved by both the deformation and the incremental theory.

It is noted in both problems undertaken in this dissertation that one mechanism, namely a varying temperature, provided the impetus to the plastic flow.

It is shown in reference 31, page 72, that the elastic solution to the thin disk with a thermal gradient is

\[
S_r = \alpha E \left( \frac{1}{R^2} \int_0^R Trdr - \frac{1}{R^2} \int_0^R Trdr \right)
\]

\[
S_t = \alpha E \left( - T + \frac{1}{R^2} \int_0^R Trdr + \frac{1}{R^2} \int_0^R Trdr \right)
\]

For convenience in discussion, assume a simple temperature distribution, say \( T(r) = Kr^2 \), where \( K \) is a constant with respect to radial position but not time. It is easily verified that the elastic solution produces radial loading, that is, at any radial position, the ratio of the radial
stress to the tangential stress is a constant at all times for a time varying K.

Although this is certainly no proof, one might believe the plastic solution would tend to exhibit radial loading also, during the loading program.

It is also shown in reference 31, page 64, that the solution to the problem of the thin disk under rotation is:

\[ S_r = \frac{3 + \mu}{8} \omega^2 (R^2 - r^2) \]
\[ S_t = \frac{\omega^2}{8} \left[ (3 + \mu)r^2 - (1 + 3\mu) r^2 \right] \]

It is easily verified that the elastic solution is one of radial loading for varying angular velocity.

Again, although this is certainly no proof, one might believe the plastic solution to the rotating disk would tend to exhibit radial loading also, during the loading program, that is, for ever increasing angular velocity.

However, if the elastic solutions for the above two problems are superimposed, giving the problem of the rotating disk with a temperature gradient, it is easily verified that the ratio of the radial stress to the tangential stress at any given position is NOT constant at any selected radial position for variable temperature and angular velocity.

Emphasizing one more that this is no proof, one might believe that the plastic solution to the combination of temperature gradient and angular velocity would not be one of radial loading.
The author feels that problems involving two or more mechanisms for controlling the stresses would be a fruitful area to investigate coincidence of solutions by the two theories. The problem of the thin disk with a temperature gradient and rotation would be a good starting place. It is the extremely important problem of the steam turbine or gas turbine disk.
XI. APPENDIX A

A. Proof of the Expression for the Equivalent Plastic Strain Increment

if the Equivalent Plastic Stress is Taken to be

$$
\bar{S} = \frac{1}{\sqrt{2}} \sqrt{(s_1 - s_2)^2 + (s_2 - s_3)^2 + (s_3 - s_1)^2}
$$

The equivalent plastic strain increment, $\text{de}_p$ was defined by the relationship for the definition of the total plastic work, that is,

$$
\dot{W}_p = \int s_{ij} \text{de}^p_{ij} = \int \bar{S} \text{de}_p
$$

so that one has

$$
s_{ij} \text{de}^p_{ij} = \bar{S} \text{de}_p
$$

Hence, from equation (35),

$$
\bar{S} \text{de}_p = s_{ij} \text{de}^p_{ij} = s_{ij}^h \frac{\partial g}{\partial s_{ij}} \text{d}f
$$

$$
= s_{ij}^h \frac{\partial J_i^1}{\partial s_{ij}} \text{d}J_i^1
$$

$$
= s_{ij}^h \frac{\partial J_i^2}{\partial s_{ij}} \frac{\partial s_i^1}{\partial s_{ij}} \text{d}J_i^2
$$

$$
= s_{ij}^h s_i^1 \text{d}J_i^2
$$

Now $s_{ij} \text{de}^p_{ij} = s_{ij}^1 \text{de}^p_{ij}$, since $s_{ij}^1 = s_{ij} - \frac{1}{3} \delta_{ij} s_{kk}$ and therefore

$$
s_{ij} \text{de}^p_{ij} = s_{ij}^1 \text{de}^p_{ij} + \frac{1}{3} \delta_{ij} s_{kk} \text{de}^p_{ij}; \text{ the last term is always zero. This may be seen to be so, for if } i \neq j, \text{ then } \delta_{ij} \text{ is zero, and if } i = j, \text{ then } \text{de}^p_{ij} \text{ is zero.}$$
Therefore, $S_{de} = h s_{ij} s_{ij} dJ^2$. From $d_{ij} = h s_{ij} dJ^2$, one has

$$d_{ij} d_{ij} = h^2 s_{ij} s_{ij} (dJ^2)^2 = h^2 2J^2 (dJ^2)^2$$

so that one finds $\sqrt{d_{ij} d_{ij}} = h \sqrt{2J^2} dJ^2$

Therefore,

$$\bar{S}_{de} = h \sqrt{2J^2} \sqrt{2J^2} dJ^2 = h \sqrt{2J^2} dJ^2 \sqrt{2J^2}$$

$$= \sqrt{d_{ij} d_{ij}} \sqrt{2J^2} = \sqrt{\frac{2}{3} d_{ij} d_{ij} \sqrt{3J^2}}$$

$$= \bar{S} \sqrt{\frac{2}{3} d_{ij} d_{ij}}$$

Hence it may be concluded that

$$d_{ij} = \sqrt{\frac{2}{3} d_{ij} d_{ij}}$$

$$= \sqrt{\frac{2}{3} (d_{ij}^2 + d_{ii}^2 + d_{ij}^2)}$$

$$= \sqrt{\frac{2}{3} (d_{ii}^2 + d_{ii}^2 + d_{ij}^2)}$$

It is to be recalled that $J^2 = \frac{1}{2} s_{ij} s_{ij}$ and that

$$\bar{S} = \sqrt{3J^2}$$
A. Proof of Equivalence of Differential of Equivalent Plastic Strain to that Obtained in a Simple Tensile Test

The equivalent plastic strain was defined as

\[
e_{ep} = \frac{\sqrt{2}}{3} \sqrt{ (e_{lp} - e_{2p})^2 + (e_{2p} - e_{3p})^2 + (e_{3p} - e_{lp})^2}
\]

In view of the constancy of volume in the plastic domain, that is,

\[e_{lp} + e_{2p} + e_{3p} = 0\]

this may be shown to reduce to

\[
e_{ep} = \frac{2}{\sqrt{3}} \sqrt{e_{lp}^2 + e_{2p}^2 + e_{3p}^2}
\]

upon elimination of \(e_{3p}\). Differentiating this last expression, one finds

\[
de_{ep} = \frac{2}{\sqrt{3}} \frac{2e_{1p} de_{1p} + 2e_{2p} de_{2p} + e_{1p} de_{2p} + e_{2p} de_{1p}}{2\sqrt{e_{lp}^2 + e_{2p}^2 + e_{3p}^2}}
\]

In a simple tensile test,

\[e_{2p} = -\frac{1}{2} e_{lp}, \quad de_{2p} = -\frac{1}{2} de_{lp}\]

So that one finds

\[
de_{ep} = \frac{2}{\sqrt{3}} \frac{2e_{1p} de_{1p} + \frac{1}{2} e_{lp} de_{1p} - \frac{1}{2} e_{lp} de_{1p} - \frac{1}{2} e_{lp} de_{1p}}{2\sqrt{e_{lp}^2 - \frac{1}{2} e_{lp}^2 + \frac{1}{4} e_{lp}^2}}
\]

\[= \frac{2}{\sqrt{3}} \frac{\frac{3}{2} e_{lp} de_{1p}}{\frac{\sqrt{3}}{2} e_{lp}} = \frac{2}{\sqrt{3}} \cdot \frac{3}{2} \cdot \frac{1}{\sqrt{3}} \cdot de_{lp}\]

which says

\[de_{ep} = de_{lp}\]
A. Method of Evaluation of the Integrals Appearing in the Dissertation

Noting the integrals appearing throughout the thesis, it is helpful to consider in detail the method of integration. Since Simpson's Rule gives in general a much better approximation to the area under a given curve than for example does the trapezoidal rule, it was decided to use it. This is a straightforward task, with the exception of the behavior near zero of three of the integrals. They are:

1. \[ \frac{1}{r^2} \int_{0}^{r} \omega Tr dr \]

Here the integral is satisfactory, but the multiplication by \( \frac{1}{r^2} \) needs attention.

2. \[ \int_{0}^{r} \frac{e_{rp} - e_{tp}}{r} dr \]

Here the integrand appears to be infinite.

3. \[ \frac{1}{r^2} \int_{0}^{r} (e_{rp} + e_{tp}) rdr \]

This is similar to (1).

Investigating each separately, in (1),

\[ \lim_{r \to 0} \frac{1}{r^2} \int_{0}^{r} \omega Tr dr = \lim_{r \to 0} \frac{\omega Tr}{2r} = \frac{\alpha T}{2} \]

In (2)

\[ \lim_{r \to 0} \frac{e_{rp} - e_{tp}}{r} = \lim_{r \to 0} \frac{d e_{rp} - d e_{tp}}{1} = 0 \]

because the radial strain becomes coincident with the tangential strain at the origin.
In (3)
\[
\lim_{r \to 0} \int_0^r \frac{(e_{rp} + e_{tp})rdr}{r^2} = \lim_{r \to 0} \frac{(e_{rp} + e_{tp})r}{2r}
\]
\[
= \frac{e_{rp} + e_{tp}}{2} \bigg|_{r=0} = e_{rp} \bigg|_{r=0}
\]
by the preceding arguments.

Referring to page 17, of Timoshenko and Young's Advanced Dynamics, it is found that the development there enables one to find the area under, for example, the curve from A to C in terms of the ordinates at \( \eta_1, \eta_2, \eta_3 \), that is,
\[
S = \frac{\Delta \xi}{3} (\eta_1 + 4\eta_2 + \eta_3)
\]
where
\[
\Delta \xi_1 = \Delta \xi_2 = \Delta \xi_3 = \Delta \xi
\]

Figure 102. Numerical evaluation of integrals.
If one wishes to obtain only the area from A to D, that is, for \( t \) ranging over a width \( \Delta \xi \), the above development quoted does not apply. In view of the fact that integrals of the above type are desired, it becomes necessary to derive a special relationship.

As is well known, Simpson's Rule replaces the given curve by a parabola,

\[ \eta = A + Bt + Ct^2 \]

The constants A, B, and C can be chosen to insure the parabola fitting the three given coordinates. Suppose that the area under the curve from D to C is desired, in terms of values of ordinates \( \eta_2, \eta_3, \eta_4 \), that is, the two "enclosing" ordinates, and the next following. Taking the \( \eta \) axes as passing through C, one has

\[ D = (-\Delta \xi, \eta_2), \quad C = (0, \eta_3), \quad E = (\Delta \xi, \eta_4) \]

From

\[ \eta = A + Bt + Ct^2 \]
\[ \eta_2 = A - B(\Delta \xi) + C(\Delta \xi)^2 \]
\[ \eta_3 = A \]
\[ \eta_4 = A + B(\Delta \xi) + C(\Delta \xi)^2 \]

The area \( S = \int_{-\Delta \xi}^{0} \eta \, dt = \int_{-\Delta \xi}^{0} (A + Bt + Ct^2) \, dt = At + \frac{Bt^2}{2} + \frac{Ct^3}{3} \bigg|_{-\Delta \xi}^{0} = A(\Delta \xi) - \]

\[ \frac{B}{2} (\Delta \xi)^2 + \frac{C}{3} (\Delta \xi)^3 = \frac{\Delta \xi}{12} \left[ 12A - 6B(\Delta \xi) + 4C(\Delta \xi)^2 \right] \]

For an analogous expression to \( S = \frac{\Delta \xi}{3} (\eta_1 + 4\eta_2 + \eta_3) \) one thus has
\[ S = \frac{\Delta \xi}{12} \left[ (K\eta_2 + L\eta_3 + M\eta_4) \right] = \frac{\Delta \xi}{12} \left[ 12A - 6B(\Delta \xi) + 4C(\Delta \xi)^2 \right] \]

and upon substituting the above expressions for \( \eta_2, \eta_3, \) and \( \eta_4 \) into the preceding equality, it is found that

\[ K \left[ A - B(\Delta \xi) + C(\Delta \xi)^2 \right] + LA + M \left[ A + B(\Delta \xi) + C(\Delta \xi)^2 \right] = 12A - 6B(\Delta \xi) + 4C(\Delta \xi)^2 \]

Upon matching coefficients on the zero, first, and second degree terms,

\[ K + L + M = 12 \]
\[ -K + M = -6 \]
\[ K + M = 4 \]

From these three equations, one finds

\[ K = 5 \]
\[ L = 8 \]
\[ M = -1 \]

Hence,

\[ S(2) = \frac{\Delta \xi}{12} \left[ 5\eta_2 + 8\eta_3 - \eta_4 \right] \]

This relationship will evaluate the area from the left limit to within one \( \Delta \xi \) from the right limit. In a similar manner it can be shown that (referring to figure 102), \( S_8 = (-\eta_7 + 8\eta_8 + 5\eta_9) \frac{\Delta \xi}{12} \). This is necessary, for the first relationship derived will not give \( S_8 \), since there is no \( \eta_{10} \).
A. Relationship Between $J_2'$ and the Equivalent Stress

$J_2'$ was defined as $\frac{1}{2} s_{ij}^1 s_{ij}^1$, which in the case of the principal stress deviations expands as

$$J_2' = \frac{1}{2} (s_{11}^2 + s_{22}^2 + s_{33}^2)$$

Since $s_{ij}^1 = s_{ij} - \frac{1}{3} \delta_{ij} s_{kk}$, one finds

$$s_{11}^1 = \frac{2s_{11} - s_{22} - s_{33}}{3}$$
$$s_{22}^1 = \frac{2s_{22} - s_{11} - s_{33}}{3}$$
$$s_{33}^1 = \frac{2s_{33} - s_{22} - s_{11}}{3}$$

Upon substituting the above formulas for the deviatoric stress components into the expression for $J_2'$ and simplifying the algebra, one finds

$$J_2' = \frac{1}{6} \left[ (s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{33} - s_{11})^2 \right]$$

Since the equivalent stress was defined to be

$$\bar{s} = \frac{1}{\sqrt{2}} \sqrt{(s_{11} - s_{22})^2 + (s_{22} - s_{33})^2 + (s_{33} - s_{11})^2}$$

it is evident that the desired relation is

$$6J_2' = 2\bar{s}^2$$

or

$$\bar{s}^2 = 3J_2'$$
XV. APPENDIX E

A. Brief Summary of the Theory of the Stress Gage

For the convenience of the reader, important excerpts from the paper of Herbert R. Lissner and C. C. Perry\textsuperscript{20} are here presented.

The mechanism by which a strain gage indicates the principal stress when oriented at a particular angle from the principal axes can be readily understood from a review of stress and strain fundamentals.

Expressing the principal stresses in terms of the principal strains through Hooke's Law one has

$$ S_t = \frac{E}{1 - \mu} (e_t + \mu e_r) $$ (1)

This indicates that a strain gage must be so oriented in a biaxial field that it will be properly affected by both principal strains in order to produce an output proportional to a principal stress. The strain at any angle $\phi$ from a principal stress axis can be written

$$ e_\phi = e_t \cos^2 \phi + e_r \sin^2 \phi $$ (2)

which may be rewritten as

$$ e_\phi = \cos^2 \phi (e_t + \tan^2 \phi e_r) $$ (3)

This development is given by Glenn Murphy\textsuperscript{24} on pages 40-41. Equations (1), (2), and (3) are roughly similar, and can be made more so if $\tan^2 \phi = \mu$.

This makes $\cos^2 \phi = \frac{1}{1 + \mu}$. This substitution fixes the angle at which the strain gage must be mounted and results in

$$ e_\phi = \frac{1}{1 + \mu} (e_t + \mu e_r) $$ (4)
Dividing equation (1) by equation (4), one has

\[
\frac{S_t}{e_\phi} = \frac{E}{1 - \mu}
\]  

or upon rearranging, one finds

\[
S_t = \frac{E}{1 - \mu} e_\phi
\]

As a result, it is evident that if the strain gage is mounted at an angle \( \Phi = \tan^{-1}\sqrt{\mu} \), the gage output can be converted to the associated principal stress simply by use of (6) above. An interesting analysis of errors is given in reference 20, where the complete development of this theory may be found.
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