A MODEL FOR THE EFFECTS OF ABERRATIONS ON REFRACTED ULTRASONIC FIELDS

R. B. Thompson and E. F. Lopes
Ames Laboratory
Iowa State University
Ames, IA 50011

INTRODUCTION

When an ultrasonic beam refracts through a liquid-solid interface, two physical processes limit the degree of energy concentration that can be realized by focussing techniques. Diffraction prevents energy from being concentrated into a region with transverse dimension of much less than a wavelength. However, even if the wavelength is made arbitrarily small, unlimited energy concentrations cannot be realized because of aberrations caused by the fact that all focussed rays do not pass through a common point.

Theoretical techniques are available to treat each of these cases individually. Ray tracing [1] is routinely used to evaluate particular inspection configurations. These will reveal the presence of aberrations [2], but do not describe beam spreading effects due to diffraction. If one neglects aberrations, simple formulae have been developed which predict the effects of diffraction on the axial fields of piston sources [3] and on the full fields of Gaussian sources [4] after refraction through a planar or cylindrically curved interface. For the Gaussian case, a figure-of-merit has been proposed to indicate when the neglect of aberrations is a reasonable approximation and when it should lead to major errors [4,5]. However, there has been little systematic study of problems in which the diffraction and aberration limits on beam size are of comparable magnitude. This paper reports the development of a model to treat such cases.

GENERAL FRAMEWORK OF THE MODEL

Figure 1 illustrates the geometry of the computation. Consider a cylindrically curved liquid-solid interface such as is shown by the bold arc. Suppose that this is illuminated by a beam, whose central ray follows the path of the central dashed line. Define incident and transmitted planes as those planes, each passing through the intersection of the central ray and the surface, which are perpendicular to the incident and transmitted portions of the central ray, respectively.

The strategy is then as follows. A set of virtual fields are first defined. The virtual incident fields are those fields which would exist on the incident plane, were it in the fluid. These are fully specified
by the transducer radiation pattern, uninfluenced by the solid. The virtual transmitted fields are those fields which would be required on the transmitted plane, were it fully embedded in the solid, in order to produce the actual radiation into the solid. A ray analysis is then used to relate the complex amplitudes of this pair of virtual fields. Consider the dashed ray paths shown on either side of the central ray in Fig. 1. It is straightforward to determine virtual points at which extensions of these rays would strike the incident plane (if it lay in water) and the transmitted plane (if it lay in the solid). The virtual paths are indicated by dotted lines. The fields on the two planes are then related by arguments regarding the conservation of energy in flux tubes bounded by sets of rays, as will be discussed in more detail in the following section.

A rigorous justification for this hybrid approach is not yet available. However, the following heuristic arguments have motivated its development. The portion of the propagation path at which aberrations are introduced is in the vicinity of the interface, where the nonlinear relationship between incident and refracted angle, dictated by Snell's law, and rapidly varying complex interface transmission coefficients, can significantly change the beam profile. Hence, a ray model which can incorporate these effects appears desirable in this region. If a ray model were used in all space, the consequences of diffraction, such as beam spread, would be neglected. To avoid this, wave propagation theories are used before the incident plane and after the transmitted plane. The largest source of error in the approach would appear to be the neglect of diffraction in the region treated by the ray analysis. Since this is generally a small portion of the total path, it is believed that this will be an acceptable error in many practical situations.

GAUSSIAN-HERMITE MODEL FOR BEAM PROPAGATION

In the regions between the transducer and the incident plane and beyond the transmitted plane, the fields are represented as a superposition of a complete set of orthonormal Gaussian-Hermite functions. The use of such solutions for transducer radiation problems has been proposed previously by Cook and Arnoult [6] and numerical computations for circular transducers using the closely related, axially symmetric, Gaussian-Laguerre functions, have been reported by the same group [7].
In the Gaussian-Hermite formalism, one expresses the velocity potential, \( \phi \), as the sum of eigenfunctions

\[
\phi(x,y,z) = \sum_{mn} C_{mn} \psi_m(x,z) \psi_n(y,z) e^{j(\omega t-kz)}
\]  

(1)

where \( C_{mn} \) are constant coefficients, the eigenfunction have the form

\[
\psi_m(x,z) = \left[ \frac{1}{\sqrt{\pi} 2^m} \right]^{1/2} a_x(z) H_m \left[ a_x(z)x \right] e^{-j k x^2 / 2 q_x(z)} e^{j [(2m+1) \beta_x(z)]}
\]

(2)

and the parameters \( a, \), \( q, \) and \( \beta \) are related by

\[
a_x(z) = \{-k \text{Im}[q^{-1}(z)]\}^{1/2}
\]

(3)

\[
\beta_x(z) = 1/2 \left\{ \frac{\pi}{2} - \tan^{-1} \frac{\text{Im}[q_x(z)]}{\text{Re}[q_x(z)]} \right\}
\]

(4)

\[
q_x(z) = q_x(0) + z.
\]

(5)

These eigenfunctions satisfy a reduced wave equation in which terms of the order \( d^2 \psi / dz^2 \) are neglected with respect to \( k d\psi / dz \) [6]. This should be a good approximation for the well collimated beams usually used in ultrasonic NDE.

Note that the entire z-dependence of \( \psi_m \) is determined by the linear variation of \( q \) with \( z \). From the transverse variations of several of the \( \psi_m \), whose magnitudes are shown in Fig. 2, it will be recognized that the beam is represented as a superposition of transversely oscillating, bound beams. The first of these is a beam of Gaussian cross-section. From previous discussion of the propagation of such beams [4], it can be seen that the beam width is equal to \( \sqrt{2}/a \) and the radius of wavefront curvature is \( \{\text{Re}[q^{-1}(z)]\} \). Since \( a, \beta, q \) are independent of \( m \), these comments qualitatively apply to the higher order eigenfunctions as well and the transverse scale of all the eigenfunctions changes with propagation distance \( z \) as dictated by \( [q(z)] \).

If the potential \( \phi \) is known on some plane, than orthogonality relationships [8] can be used to derive the expression for the coefficients

\[
C_{pq} = e^{-j(\omega t-kz)} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \psi^*_p(x,z) \psi^*_q(y,z) \phi(x,y,z)
\]  

(6)

In evaluating this, one should select complex initial values for \( q_x \) and \( q_y \), consistent with the initial width and curvature (e.g., focusing) of the beam. In principle, arbitrary values may be chosen, but the rate of convergence appears to be maximized by particular values [7].

In a solid, it is well known that the solution of the elastic wave equation cannot be rigorously represented by a scalar potential. Nevertheless, that approximation will be employed here with separate potentials defined for the transverse and longitudinal wave fields.
Fig. 2. Transverse variations of $\psi_m(x)$.

Ray Tracing Through Interface

The amplitudes of the fields at the intersections of rays with the incident and transmitted planes are related in the model by the principle of conservation of energy. Consider a flux tube bound by four rays. If one assumes that energy travels down this tube without loss, then it follows that the amplitudes of the fields must vary inversely as the square root of the tube cross-sectional area. When this idea is generalized to include the transmission loss at the liquid-solid interface, the resulting expression is

$$\phi_T = \phi_I T_{01} \frac{\cos \theta_I}{\cos \theta_0} \left\{ \frac{dA_I}{dA_T} \right\}^{1/2} \exp\left( \frac{\gamma dA_I}{dA_T} \right)^{1/2}$$

(7)

where the subscripts I and T refer to the incident and transmitted planes, respectively, $T_{01}$ is the interface transmission coefficient relating particle displacements (or velocities) on the two sides of the interface, $\theta_0$ and $\theta_I$ are the angles of incidence and refraction, respectively, $\gamma$ is a phase shift associated with the propagation delay between the two planes, and $dA$ is an element of cross-sectional area associated with a flux tube.

As defined, the interface transmission coefficient $T_{01}$ can assume values greater than unity, even though energy is lost due to interface reflections. These values occur because of the geometrical increase in density of rays when the refracted angle is greater than the incident angle. The cosine ratio in Eq. (7) is required as a normalization to avoid the double counting of this effect, which is also included in the flux tube area ratio.

Figure 3a illustrates, in greater detail, the tracing of several adjacent rays through a curved interface. In Fig. 3b, the coordinate of a ray intersection with the transmitted plane is plotted as a function of its interception with the incident plane. Note that this is not
Fig. 3. Details of ray tracing
   a) Behavior of five rays
   b) Intersection of a ray with transmitted plane versus intersection with incident plane.

necessarily a monotonic function since it is possible for more than one ray to pass through the same point in the transmitted plane. From Fig. 3b, it will also be noted that \( \frac{dy_r}{dy_i} = 0 \) occurs at the point of zero slope. Equation (7) then implies that \( \phi_T \) is unbounded. Although a more sophisticated analysis would be required to rigorously eliminate the singularity, preliminary analysis suggests that it is an integrable, square root singularity. From the computational points-of-view, it does not appear to pose a problem. When Eqs. (6) and (7) are combined to evaluate the coefficients in the solid, the result may be placed in the formal form,

\[
C_{pq} = e^{-j\omega t} \int \int \psi_p \psi_q \phi_T \left( \frac{\cos \theta_i}{\cos \theta_o} \right)^{1/2} e^{j\gamma (dA_I dA_T)^{1/2}}
\]

Although the differential area \((dA_I dA_T)^{1/2}\) may not be familiar, it has a well defined meaning when evaluating the integral numerically.

**Numerical Results**

Figure 4 shows illustrative results for the case of the illumination of a cylindrical interface having a 7.62 cm (3 in.) radius of curvature with a 5 MHz, 1.25 cm (0.5 in.) diameter unfocussed piston transducer. The probe face was set off from the interface by a distance of 5.6 cm (2.2 in.) and was oriented such that the refracted central ray traveled in the solid at an angle of 60° with respect to the local normal. The solid was taken to have longitudinal and transverse wave velocities of 0.6 and 0.3 cm/\(\mu\)sec, respectively.
In Fig. 4a, a ray diagram is shown. The set of rays leaving the transducer has been chosen to be slightly divergent to reproduce to the divergence of an "equivalent" Gaussian beam [4] as it strikes the interface. These rays are not explicitly used in the calculation. However, the asymmetry in the diagram gives a qualitative indication of the significance of aberrations in this case.

Figure 4b presents the predicted fields in the solid in the plane $z_1=0$ (the transmitted plane). Note the sharp truncation of the beam for negative values of $y$. No rays illuminated this region because the corresponding incident rays were beyond the critical angle and hence no energy is found. Note also that the fields shown for positive values of $y$ are the virtual fields that would be required in a solid to produce the subsequent radiation pattern.
Figure 4c shows the fields in the solid at \( z_1 = 1 \text{ cm} \). Here, diffraction has begun to smooth out the sharp beam edge and prominent side lobes are beginning to appear for positive \( y \). These general features are retained in Figs. 4d and 4e, corresponding to \( z_1 = 2 \text{ cm} \) and 5 cm, as the beam moves into the far field.

The predicted far-field radiation pattern has several interesting properties. The asymmetry in beam profile and side lobe structures has been predicted before for the related problem of a focussed beam illuminating a planar interface at an angle. Those calculations were made on the basis of a two-dimensional angular spectrum of plane waves approach [5]. It will also be noticed that the peak of the beam appears to be moving towards negative \( y \). This is not surprising in view of the distribution of rays in Fig. 4a. When one notes that the initial position of the beam was shifted to positive \( y \)-coordinates because of the loss of energy in the truncated rays, it becomes evident that the center of the beam is propagating at an angle about 10° less than the refracted angle of the nominal central ray.

To test this prediction, as well as the overall methodology further, the transmitted plane was rotated and translated so that a) the origin was at the center of mass of the translated rays and b) the linear component of phase variation was eliminated. The new transmitted plane (NTP) so defined is shown in Fig. 4a by a broken curve. When the Gaussian-Hermite expansion was performed in this plane, the predicted profiles analogous to those shown in Fig. 4b-e had the same shape but were translated so that their peaks fell on the origin. This result confirms the predicted change in the beam refracted angle and establishes the robustness of the Gaussian-Hermite expansion in the presence of small misorientations of the coordinate system.

The model has also been used to compute the fields in the \( x-z \) plane. Figure 5 presents the results obtained when \( z_1 = 0 \text{ cm} \) and \( z_1 = 5 \text{ cm} \). As would be expected, the fields are symmetric and do not show obvious aberration effects.

Fig. 5. Propagation of a 5 MHz, 1.27 cm diameter beam through a 7.6 cm radius cylindrical interface for nominal 60° refracted longitudinal wave angle. Fields are shown in \( x-z \) plane; a) hybrid model: \( z_1 = 0 \text{ cm} \); b) hybrid model: \( z_1 = 5 \text{ cm} \)
An important advantage of this computational approach is the ability to make full field maps of beam patterns. In a Green's function or related approach for radiation into a single medium, one must evaluate a new integral over the face of the transducer when each new field point is considered. For short wavelengths, the evaluation of each of these integrals can be quite time consuming. The problem becomes even more complex when considering radiation through an interface. In the present approach, considerable computational time is required to determine the coefficients, $C_{mn}$, of the expansion of the fields in the solid (approximately 30 minutes on a microvax I for a $65 \times 65$ set of coefficients using a research program not fully optimized for speed). However, these must only be determined once, since the same coefficients enter into the determination of the fields in all space. The fields at each new field point are then computed in a time of 2 seconds. If fewer than $65 \times 65$ coefficients are required in the expansion, the time is reduced accordingly. To illustrate the utility of full field computations, Fig. 6 shows perspective plots of the profiles of two beams. In each case, a 15 MHz, 1.27 cm (0.5 in.) diameter transducer is allowed to illuminate a cylindrical interface of 7.5 cm (3.0 in.) radius of curvature. The top plot shows the beam profile in the $y$-$z$ plane for normal incidence. The near field structure, the peak of the beam at the focal point, and the side lobes are all seen. It should be noted, however, that the near field structure very close to the transducer is not fully reproduced because of the Fresnel approximation implicit in the Gaussian-Hermite solution and the finite number of terms selected in the series.

The following two plots consider the case of illumination at an angle to produce a central longitudinal ray refracted at ±30°. These coincide to the same physical problem but allow one to see in perspective the behavior of both sides of the beam. The aforementioned asymmetric side lobe structure is clearly shown.

DISCUSSION

The Gaussian-Hermite formulation for predicting the effects of aberrations on refracted ultrasonic beams appears quite promising. Validation can be accomplished by either comparison to experiment or more exact computations. The former are reported elsewhere in this volume [9] and the latter are in progress. Applications include the direct modeling of various systems for detecting and characterizing flaws. The programs can either be used directly in the computations or as a check on the accuracy of simpler beam models [4]. It is also hoped that the understanding of aberration effects gained through the use of the models will provide insights to further simplifications in the future.

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REFERENCES

Fig. 6. Perspective plots of radiation of a 15 MHz, 1.27 cm diameter beam through a 7.5 cm radius cylindrical interface for nominal refracted angles of 0° and ±30°. The stand-off distance was 1.6 cm.