List-coloring and sum-list-coloring problems on graphs

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List-coloring and sum-list-coloring problems on graphs

by

Michelle Anne Lastrina

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Mathematics

Program of Study Committee:
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Iowa State University
Ames, Iowa
2012

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DEDICATION

I would like to dedicate this thesis to my family and friends without whose support I would not have been able to complete my graduate studies. I would also like to dedicate this thesis to the many wonderful math instructors I encountered throughout my mathematical education. Without you, I would not have been inspired to pursue mathematics.
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ABSTRACT

Graph coloring is a well-known and well-studied area of graph theory that has many applications. In this dissertation, we look at two generalizations of graph coloring known as list-coloring and sum-list-coloring. In both of these types of colorings, one seeks to first assign palettes of colors to vertices and then choose a color from the corresponding palette for each vertex so that a proper coloring is obtained.

A celebrated result of Thomassen states that every planar graph can be properly colored from any arbitrarily assigned palettes of five colors. This result is known as 5-list-colorability of planar graphs. Albertson asked whether Thomassen’s theorem can be extended by precoloring some vertices which are at a large enough distance apart. Hutchinson asked whether Thomassen’s theorem can be extended by allowing certain vertices to have palettes of size less than five assigned to them. In this dissertation, we explore both of these questions and answer them in the affirmative for various classes of graphs.

We also provide a catalog of small configurations with palettes of different prescribed sizes and determine whether or not they can always be colored from palettes of such sizes. These small configurations can be useful in reducing certain planar graphs to obtain more information about their structure.

Additionally, we look at the newer notion of sum-list-coloring where the sum choice number $\chi_{SC}$ is the parameter of interest. In sum-list-coloring, we seek to minimize the sum of varying sizes of palettes of colors assigned the vertices of a graph. We compute $\chi_{SC}$ for all graphs on at most five vertices, present some general results about sum-list-coloring, and determine $\chi_{SC}$ for certain graphs made up of cycles.
CHAPTER 1. GENERAL INTRODUCTION

Graph coloring is a well-known and well-studied area of graph theory with many applications. In this thesis, we will consider two generalizations of graph coloring. In particular, list-coloring and sum-list-coloring.

We begin by defining a graph and the different types of colorings explored in this dissertation.

Definition 1.1. A graph $G$ is an ordered pair $(V,E)$ where elements of $V$ are called vertices and elements of $E$ are two element subsets of $V$ called edges. If $x,y \in V$ and $\{x,y\} \in E$, then it is said that $x$ and $y$ are adjacent, denoted $x \sim y$.

For simplicity of notation, we will use $xy$ to denote an edge $\{x,y\} \in E$. In this dissertation we will only be looking at connected simple graphs, those which contain no loops or multiple edges.

Definition 1.2. For a graph $G = (V,E)$, an assignment $c : V \rightarrow \mathbb{N}$ is a coloring of $G$. Furthermore, this coloring is proper if $c(u) \neq c(v)$ for all $uv \in E$. If $c$ uses only the colors $\{1,2,\ldots,k\}$, then $c$ is a $k$-coloring. When such a proper $k$-coloring exists, $G$ is said to be $k$-colorable.

Throughout this dissertation, we will look at a generalization of coloring called list-coloring.

Definition 1.3. For a graph $G = (V,E)$, let $L : V \rightarrow 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$. A coloring $c : V \rightarrow \mathbb{N}$ is an $L$-coloring or list-coloring of $G$ if $c(v) \in L(v)$ for all $v \in V$. Furthermore, this coloring is proper if $c(u) \neq c(v)$ for all $uv \in E$. When such an $L$-coloring exists, $G$ is said to be $L$-colorable.

It will be assumed for the remainder of this thesis that all colorings are proper, unless otherwise noted.
Definition 1.4. Let $G = (V, E)$ be a graph on $n$ vertices and $f : V \to \mathbb{N}$ be a size function that assigns to each vertex of $G$ a list size. Let an $f$-assignment $L : V \to 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ such that $|L(v)| = f(v)$ for all $v \in V$. The graph $G$ is said to be $f$-choosable if $G$ is $L$-colorable for every $f$-assignment $L$. A choosable size function is called a choice function.

For example, let $G$ be the 3-cycle $(u, v, w)$. Let $f \equiv 2$ be a size function for $G$, then $G$ is not $f$-choosable. This is because if $L$ is an $f$-assignment where the lists assigned to each vertex are identical, then the graph is not $L$-colorable. However, let $g$ be a size function for $G$ such that $g(u) = g(v) = 2$, $g(w) = 3$. Then $G$ is $g$-choosable. The 3-cycle can always be colored from lists of these sizes as follows: First choose a color from $L(u)$ to assign to $u$. Next there is at least one color in $L(v)$ to assign to $v$, so assign such a color to $v$. Finally, there is at least one color in $L(w)$ that may be assigned to $w$, so assign such a color to $w$. This will always yield a proper $L$-coloring of the 3-cycle.

With respect to the colorings defined above, there are some graph parameters that are utilized.

Definition 1.5. The minimum value of $k$ for which a graph $G$ is $k$-colorable is the chromatic number $\chi(G)$.

Definition 1.6. A graph $G$ is said to be $k$-list-colorable if $f \equiv k$ is a choice function for $G$. For a choice function $f$ define $\max(f) := \max_{v \in V(G)} f(v)$. The list chromatic number $\chi_l(G)$, or choice number $\text{ch}(G)$, is the minimum of $\max(f)$ over all choice functions for $G$.

Definition 1.7. For a choice function $f$ define $\text{size}(f) := \sum_{v \in V(G)} f(v)$. The sum choice number $\chi_{\text{SC}}(G)$ is the minimum of $\text{size}(f)$ over all choice functions for $G$.

One of the problems we will be looking at in this dissertation involves assigning some lists of size 1 to certain vertices of a graph.

Definition 1.8. For a graph $G = (V, E)$ and a subset $P \subset V$ of vertices, let $f : V \to \mathbb{N}$ be a size function for $G$. If $f(v) = 1$ for all $v \in P$, $f(v) = k$ for all $v \in V - P$, and $f$ is a choice
function for $G$, then it is said that a precoloring of $P$ is extendable to a $k$-list-coloring of $G$.

For a graph $G = (V, E)$ on $n$ vertices ordered $v_1, v_2, \ldots, v_n$, we may write a size function $f : V \to \mathbb{N}$ for $G$ as $(G; f(v_1), f(v_2), \ldots, f(v_n))$ or the vector $(f(v_1), f(v_2), \ldots, f(v_n))$ when $G$ is clear.

We say that $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is good if $f$ is a choice function for $G$. If $G$ is not $f$-choosable, then we say that $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is bad.

Given two size functions $f$ and $f'$ for $G$, if $f(v_i) \leq f'(v_i)$ for all $i = 1, 2, \ldots, n$, then we say that $f \leq f'$. This inequality is strict if $f(v_i) < f'(v_i)$ for some $i$. If $G$ is $f$-choosable and $f \leq f'$, then $G$ is also $f'$-choosable. Similarly, if $G$ is not $f'$-choosable and $f \leq f'$, then $G$ is not $f$-choosable. If $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is good and $f \leq f'$, then $(G; f'(v_1), f'(v_2), \ldots, f'(v_n))$ is also good. Also, if $(G; f'(v_1), f'(v_2), \ldots, f'(v_n))$ is bad and $f \leq f'$, then $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is also bad.

A fundamental problem in coloring the vertices of a graph is determining the optimal choice function for a given graph. In this case, we consider optimality in the sense of $\min \|f\|_{\infty}$ and $\min \|f\|_1$. In other words, minimizing the $L^\infty$ and $L^1$ norms, $\|f\|_{\infty} = \max_{i=1,\ldots,n} |f(v_i)|$ and $\|f\|_1 = \sum_{i=1}^n |f(v_i)|$, respectively.

The graph parameter $\chi_l(G)$ corresponds to $\min \|f\|_{\infty}$ over all choice functions $f$ for $G$. The newer graph parameter $\chi_{SC}(G)$ corresponds to $\min \|f\|_1$ over all choice functions $f$ for $G$. When a choice function $f$ is such that $f(v) = 1$ for some vertices $v$ in $G$, then this corresponds to the coloring extension problem mentioned earlier. In this thesis, we investigate coloring extension problems on planar graphs and sum-list-coloring.

### 1.1 Coloring

Of the various ways to color the vertices of a graph, the most well-studied is the traditional notion of graph coloring. Some of the first problems in graph coloring date back to the late 1800s and the Four Color Theorem.

**Theorem 1.9** (Four Color Theorem). Any planar graph is 4-colorable.
The graph $K_4$ is an example of a planar graph for which $\chi(K_4) = 4$. This shows that for an arbitrary planar graph, three colors are not enough. There are many results on planar graphs that are 3-colorable if they do not contain cycles of certain lengths. See [49, 32, 22, 1, 15, 47, 76, 48, 67, 79, 68, 70, 23, 18, 77, 21, 19, 17, 42] for more on these results. The Four Color Theorem was originally posed in 1852 by Francis Guthrie, and ultimately proved by Appel and Haken [9]. The proof of the Four Color Theorem has a long and storied past and the proof itself is very involved. It involves showing that a minimal counterexample to the theorem does not exist. This is done, in part, by providing an unavoidable set of configurations along with a set of reducible configurations. The proof also relies heavily on computers. See [6, 7, 8, 9] for more on the proof of the Four Color Theorem. See also [54, 53] for more on a newer proof the uses the same techniques, but is more efficient.

In Chapter 3 we will look at list-precoloring extensions. This study arises from similar questions asked about precoloring extensions. See [3, 10, 39, 64, 65, 51, 52, 45] for more on this topic.

### 1.2 List-coloring

List-coloring was first introduced by Vizing [61] and independently by Erdős, Rubin, and Taylor [28]. In graph coloring, one seeks to minimize the number of colors used. Similarly, in list-coloring, one seeks to minimize the list size.

Erdős et al. [28] came up with the notion of list-coloring in an attempt to solve a problem of Jeffrey Dinitz posed at the Tenth Southeastern Conference on Combinatorics, Graph Theory, and Computing at Boca Raton in April 1979 [26]. The problem was stated as follows:

**Question 1.10.** Given an $m \times m$ array of $m$-sets, is it always possible to choose one element from each set, keeping the chosen elements distinct in every row, and distinct in every column?

This problem can be stated in terms of list-coloring as follows:

**Question 1.11.** Let $G = (V, E)$ be a graph on $m^2$ vertices. Let $V = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq m\}$ and let $E$ be defined so that $v_{i,j} \sim v_{i',j'}$ if $i = i'$ or $j = j'$. To each vertex, assign an arbitrary set of $m$ colors. Can $G$ always be colored from the assigned lists?
So this question is asking whether or not $\chi_l(K_m \times K_m) = m$, where $K_m \times K_m$ is the Cartesian product (see Definition 2.14).

For some general results on list-coloring see [5, 75] and [59].

While it is difficult to compute $\chi_l(G)$ for an arbitrary graph $G$, there is an upper bound on $\chi_l(G)$ based on the maximum degree $\Delta(G)$.

**Lemma 1.12.** $\chi_l(G) \leq \Delta(G) + 1$.

**Proof.** Let $\Delta := \Delta(G)$ and assign arbitrary lists $L$ of size $\Delta + 1$ to each vertex of $G$. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of $G$. Use this ordering to $L$-color the vertices of $G$. This will provide a proper $L$-coloring of $G$ because each $v_i$ is adjacent to at most $\Delta$ vertices and at least one element of $L(v_i)$ will be available to assign to $v_i$. \qed

One of the most celebrated results in list-coloring is the following theorem of Thomassen which shows that there exist graphs with arbitrarily large maximum degree that are 5-list-colorable.

**Theorem 1.13** (Thomassen's 5-list-coloring theorem, [56, 57]). Let $G = (V, E)$ be a plane graph, let $C$ be the cycle that corresponds to the boundary of a face of $G$, and let $u, v \in V(C)$ such that $u \sim v$. Let $L : V \rightarrow 2^N$ be an assignment of lists of colors to vertices of $G$ such that $|L(u)| = |L(v)| = 1$ and $L(u) \neq L(v)$; $|L(w)| = 3$ for all $w \in V(C) - \{u, v\}$; and $|L(w)| = 5$ for all $w \in V - V(C)$. Then $G$ is $L$-colorable.

This result will be used in both chapters 3 and 4, as a tool within proofs of new results and as the inspiration for other questions to explore. If a planar graph does not contain cycles of certain lengths, then it is 4-list-colorable. See [20, 29] for more on these results. There are also similar results for determining planar graphs that are 3-list-colorable. See [46, 72, 71, 27, 78] for more on these results. See also [69].

For extensive literature on list-colorings of planar graphs we refer the reader to [28, 50, 56, 58, 61, 62, 63] and [55, 73].

Here, we briefly discuss Thomassen’s 5-list-coloring theorem and some related results. One thing that Thomassen’s 5-list-coloring theorem tells us is that planar graphs are 5-list-colorable.
For this reason, the result can be thought of as the list-coloring version of the famous Four Color Theorem.

While the proof of the Four Color Theorem is quite long and relies heavily on the use of computers, the proof of Thomassen’s 5-list-coloring theorem is a short induction argument (see [56]).

Additionally, for planar graphs, lists of size 4 are not enough. There exist multiple examples of planar graphs that are not 4-list-colorable. One of the first examples was constructed by Voigt [62] and had 238 vertices. This was improved in the years that followed. Gutner [33] and Voigt and Wirth [66] both came up with constructions of examples with 75 vertices, and Mirzakhani [50] presented an example with only 63 vertices. Each of these constructions uses multiple copies of a smaller graph as a building block to create a counterexample.

1.3 The relationship between coloring and list-coloring

Graph coloring is a special case of list-coloring where the lists assigned to each vertex are identical. For this reason, $\chi(G) \leq \chi_l(G)$ for all graphs $G$. In other words, if $G$ is $k$-list-colorable, then $G$ is $k$-colorable. The converse, however, is not true. There are graphs that are $k$-colorable, but not $k$-list-colorable. For example, the graph $K_{3,3}$ is bipartite and hence 2-colorable, but it is not 2-list-colorable. It is known that $K_{3,3}$ is 3-list-colorable [43].

It is known that bipartite graphs are 2-colorable because all the vertices in each partite set can be assigned the same color. However, there exist bipartite graphs whose list chromatic number is arbitrarily large. For example, if $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not $k$-list-colorable: let $K_{m,m} = (A \cup B, E)$ and assume $K_{m,m}$ is $k$-list-colorable. In both partite sets $A$ and $B$ assign to each vertex one of the $m$ distinct possible $k$-subsets of $\{1, 2, \ldots, 2k - 1\}$ as the list of available colors for that vertex. Any coloring of the vertices in $A$ from the lists of colors assigned to them must use $k$ distinct colors. Otherwise, there would be a vertex in $A$ with no color assigned to it. This is because there does not exist a subset of $k - 1$ colors of which at least one of these colors appears in every $k$-set assigned to the vertices of $A$. Thus, there is a vertex in $B$ which cannot be colored because its list is identical to the set of $k$ colors assigned to all of the vertices of $A$. This is a contradiction which implies that $K_{m,m}$ is not $k$-list-colorable.
and $\chi_l(K_{m,m}) > k$.

1.4 Sum-list-coloring

Sum-list-coloring was introduced by Isaak [40, 41] in 2002. It is a fairly new topic in graph theory, so there is much to be discovered. For more on sum-list-coloring see also [35, 36, 37, 13, 30]. In particular, [36] is a survey of all sum-list-coloring results up to 2007. In sum-list-coloring, the list sizes are allowed to vary and one seeks to minimize the sum of list sizes over all vertices.

For any graph $G$, the sum choice number is bounded above by

$$\chi_{SC}(G) \leq |V(G)| + |E(G)|,$$

as provided by a greedy coloring. See Lemma 1.14 for a proof of this result. When equality holds in the previous inequality, $G$ is said to be sc-greedy. The sum $|V(G)| + |E(G)|$ is called the greedy bound and denoted by $GB(G)$, or $GB$ when $G$ is implied.

**Lemma 1.14.** For any graph $G$, $\chi_{SC}(G) \leq |V(G)| + |E(G)|$.

**Proof.** Let $v_1, \ldots, v_n$ be an ordering of the vertices. Let $f(v_i) = 1 + |\{v_j : j < i \text{ and } v_iv_j \in E(G)\}|$. A greedy coloring using this ordering and arbitrary lists of the prescribed sizes provides a proper coloring for any such list assignment.

Observe that list-coloring and the list chromatic number, or choice number, $\chi_l(G)$ are related to sum-list-coloring and the sum choice number: $\chi_{SC}(G)/n \leq \chi_l(G)$. Moreover, for some graphs $G$, it is the case that $\chi_{SC}(G)/n$ is significantly smaller than $\chi_l(G)$. In particular, Füredi and Kantor [30] proved the following:

**Theorem 1.15** ([30]). There exist constants $c_1, c_2$ such that for all $m \geq 4$ and $n \geq 50m^2 \log m$

$$2n + c_1m\sqrt{n \log m} \leq \chi_{SC}(K_{m,n}) \leq 2n + c_2m\sqrt{n \log m}.$$

This implies there exists a choice function $f$ for such a $K_{m,n}$ whose average list size does not necessarily increase with the average degree. Note that as $n$ approaches infinity, the average degree approaches $2m$. Furthermore,

$$\lim_{m \to \infty, n \gg m^2 \log m} \frac{|E(K_{m,n})|}{m + n} = \infty, \quad \lim_{m \to \infty, n \gg m^2 \log m} \frac{\chi_{SC}(K_{m,n})}{m + n} = 2,$$
where the first limit looks at the average degree and the second looks at the average list size. See [30] for more on this result. Alon [4] showed that $\chi_l$ is bounded below by a function of the average degree:

**Theorem 1.16** (Alon [4]). For some constant $c$ and a graph $G$ with average degree $d$,

$$\chi_l(G) \geq c \frac{\log d}{\log \log d}$$

It can thus be observed that when the list sizes are allowed to vary, this result no longer holds.

To show that $\chi_{SC}(G) = m$, one must provide a choice function $f$ of size $m$ for $G$ and show that for each size function $g$ of size $m - 1$, there is a $g$-assignment that does not have a proper coloring. Chapter 6 will provide examples of certain graphs that are sc-greedy and determine information about the sum choice number of other graphs.

### 1.5 Computational complexity

We take a moment to comment on the complexity of coloring, list-coloring, and sum-list-coloring. The complexity of sum-list-coloring has yet to be determined. Coloring is hard in that it is NP-complete to determine if a given graph is $k$-colorable for $k \geq 3$. Furthermore, it is NP-hard to compute the chromatic number of a graph. In particular, it is NP-complete to determine if a 4-regular planar graph is 3-colorable [24].

With respect to the complexity of list-coloring, one main problem has been explored [33, 34]: determine whether a given graph is $k$-list-colorable. If $k \geq 3$, then it is NP-complete for bipartite graphs and if $k = 4$, then it is NP-complete for planar graphs. See also [31] for a discussion of the complexity of some variations of list-coloring.

### 1.6 Organization of thesis and main results

The organization of this dissertation will be as follows. In Chapter 1 we describe the main topics and results of this thesis and provide some general background. In Chapter 2 we define important terminology and give some preliminary results.
Chapter 3 contains the results of the paper “List precoloring extension in planar graphs” [11], which appears in Discrete Math. This paper, co-written with Maria Axenovich and Joan P. Hutchinson, explores a question posed in Albertson [2] about extending precolorings of vertices with respect to list-colorings of planar graphs. More specifically, in Chapter 3 we will discuss the following question of Albertson [2]:

Let $G$ be a plane graph. Is there a $d > 0$ such that whenever $P \subset V$ is such that the distance between every pair of vertices of $P$ is at least $d$, then every precoloring of $P$ extends to a 5-list-coloring of $G$?

This question was posed in response to a similar question with respect to coloring was originally asked by Thomassen [57]:

**Question 1.17.** Let $G = (V, E)$ be a planar graph and $P \subset V$ such that the distance between every pair of vertices of $P$ is at least 100. Can a 5-coloring of $P$ be extended to a 5-coloring of $G$?

Albertson was able to show the answer to the above question is yes, even when the distance is at least 4. See Chapter 3 for more details on this.

In Chapter 3, we show that a precoloring is extendable to a 5-list-coloring when there are certain distance conditions imposed on the precolored vertices and conditions on so-called “separating” structures. The main theorem, Theorem 3.4, proves that if, after reductions, the precolored vertices are far away and there are no 3 or 4-cycles separating them, then this precoloring extends to a proper 5-list-coloring. We also show that if all of the precolored vertices lie on the boundary of one face of a graph and the vertices satisfy either a distance condition or are not adjacent to certain “exceptional” vertices, then a precoloring will extend to a proper 5-list-coloring.

Chapter 4 looks at a conjecture of Hutchinson [38] which states that planar graphs are $\{2, 2\}$-extendable. This is a modification of Thomassen’s 5-list-coloring theorem which allows for two nonadjacent vertices that lie on the boundary of the unbounded face to have lists of size 2, instead of having two adjacent vertices precolored.
The main theorem, Theorem 4.3, in Chapter 4 proves that, after reductions, if the vertices with lists of size 2 are close enough or if the graph contains a special “skeleton” as a subgraph, then it can be list-colored. We also prove that outerplane graphs and wheels are \( \{2,2\} \)-extendable and look at the properties of a minimal counterexample to Hutchinson’s conjecture.

In Chapter 5, we present a catalog of some small graphs and corresponding choice functions. These graphs can be used to reduce larger graphs so that they do not contain certain subgraphs. This idea is used in Chapters 3 and 4.

Chapter 6 contains some results from work with Michael Young and Steve Butler. In this chapter we present results on the newer notion of sum-list-coloring. In particular, we find the sum choice number of all graphs on at most five vertices and prove that paths of cycles are sc-greedy, as are certain trees of cycles.

Finally, in Chapter 7, we provide general conclusions and discuss future work that can be done.

### 1.7 Applications of list-coloring

There are many applications of list-coloring. We will look at a connection to radio channel assignment \[12\] and the popular Sudoku puzzles. There are also applications to scheduling.

#### 1.7.1 An application of list-coloring

Consider a wireless network. Assume that due to hardware restrictions, each radio in the network has a limited set of frequencies through which it can communicate. Also assume radios within a certain distance of each other cannot operate on the same frequency without their transmissions interfering with each other. This problem can be modeled in terms of list-coloring as follows:

- let the vertices of the graph represent the wireless radios in the network,
- let two vertices be adjacent if their corresponding radios are within a certain distance of each other, and
• assign lists to each vertex according to the available frequencies for the corresponding radio.

### 1.7.2 An application of list-coloring extension

Here we provide an application of precoloring extensions to $k$-list-colorings. The popular Sudoku puzzles can actually be described in terms of a list-coloring extension problem. Consider an $k \times k$ Sudoku puzzle, typically $k = 9$. There is a graph $G$ and list-assignment $L$ that corresponds to the given puzzle.

- The vertices of $G$ correspond to the entries of the $k \times k$ grid. There will be $k^2$ vertices in $G$.
- There is an edge between two vertices in $G$ if their corresponding entries in the grid lie in the same row, column, or subgrid of the puzzle. If $k = 9$, there will be nine $3 \times 3$ subgrids.
- Assign lists $L$ that are subsets of $\{1, 2, \ldots, k\}$ to the vertices of $G$ as follows: if the entry corresponding to a vertex contains a number $l$, then that vertex is precolored $l$. Otherwise, the vertex is assigned the list $\{1, \ldots, k\}$.

This graph $G$ is isomorphic to the graph $K_k \boxtimes K_k$, the strong product of $K_k$ with itself. An $L$-coloring of this graph with yield a solution to the given Sudoku puzzle.

We illustrate this idea with a small example of a $4 \times 4$ Sudoku puzzle. Consider the Sudoku puzzle in Figure 1.1a. This Sudoku puzzle can be represented as the graph $K_4 \boxtimes K_4$ with the lists as illustrated in Figure 1.1c. Figure 1.1d provides a coloring of the graph. This coloring corresponds to the solution of the given Sudoku puzzle shown in Figure 1.1b.
Figure 1.1: Sudoku puzzle and corresponding graph.
CHAPTER 2. DEFINITIONS AND PRELIMINARY RESULTS

2.1 Definitions

In this dissertation, we assume a general knowledge of basic graph theory terminology and results. See [74] and [25] for notation and definitions that are not included here.

For a vertex $v$ and a set of vertices $X$, we write $v \sim X$ if $v$ is adjacent to all vertices in $X$.

**Definition 2.1.** For two vertices $x$ and $y$ in a graph $G$, the distance $\text{dist}(x, y)$ is the length of, or number of edges in, a shortest path between them.

**Definition 2.2.** For a subset $P$ of vertices in a graph $G$, the distance $\text{dist}(P) = \text{dist}(P, G)$ is defined as

$$\text{dist}(P) := \min_{x,y \in P} \text{dist}(x, y).$$

For any vertex $v \in V(G)$, let $N(v) = N(v, G)$ denote the neighborhood of $v$ in $G$. In other words, $N(v, G) = \{u \in V(G) : u \sim v\}$. This definition can also be restricted to the neighborhood of $v$ in a subgraph of $G$. When the graph $G$ is implied, the subscript $G$ will often be omitted. For a vertex set $X$ in $G$, let $N(X) = N(G[X])$ be the set of neighbors of vertices from $X$ not in $X$.

Let $H$ be a subgraph of $G$ and $c$ be a vertex coloring of $H$. For $v \notin V(H)$, let $c(v, H) = \{c(u) : u \in N(v) \cap V(H)\}$ be the set of colors used on neighbors of $v$ in $H$.

For an induced subgraph $H$ of $G$ and $v \in V(G) - V(H)$, let $L_c(v, H) = L(v) - c(v, H)$. When the subgraph $H$ is clear, we use $L_c(v)$. We say $H$ is colored **nicely** by a coloring $c$ with respect to lists $L$ if $c$ is an $L$-coloring of $H$ and for every vertex $v \in N(H)$, $|L_c(v, H)| \geq 3$. We also say $c$ is a **nice** coloring of $H$ in this case.
Let $H$ be a proper induced subgraph of $G$. For a vertex $v \in V(G - H)$, let $d(v, H) := |N(v) \cap V(H)|$ be the size of the neighborhood of $v$ in $H$. Similarly, for a vertex $v \in V(H)$, let $d(v, G - H) := |N(v) \cap V(G - H)|$ be the size of the neighborhood of $v$ outside of $H$.

A vertex from $N(H)$ adjacent to at least three vertices in $H$ is called a three-neighbor, or simply 3-neighbor, of $H$. We denote the set of 3-neighbors of $H$ by $N_3(H)$.

**Definition 2.3.** Let $Q(H) = G[H \cup N_3(H)]$ be the subgraph of $G$ induced by vertices of $H$ and its 3-neighbors.

For a path $S = v_0v_1 \ldots v_m$, and two vertices $v_i, v_j$ of $S$ we write $v_iSv_j$ to denote the subpath $v_iv_{i+1} \ldots v_{j-1}v_j$ of $S$.

Many of the results presented here will deal with graphs that can be drawn in the plane so that no edges cross.

**Definition 2.4.** An embedding of a graph $G = (V, E)$ is a map onto $\mathbb{R}^2$ such that

1. vertices are distinct,

2. edges are paths between the vertices, and

3. edges only intersect at vertices.

A graph is **planar** if it can be embedded in the plane.

**Definition 2.5.** A plane graph is a fixed embedding of a (planar) graph where the arcs representing the edges do not intersect other points of the embedding except at the endpoints.

**Definition 2.6.** A face $f$ of a plane graph $G$ is a connected component of $\mathbb{R}^2 - G$, the regions in the plane that are not covered by the plane graph $G$. The boundary of a face $f$ are the edges and vertices that separate $f$ from the other faces of $G$.

**Definition 2.7.** A graph is a triangulation if every face has three edges on its boundary. A graph is a near-triangulation if the boundary of every face except the outer face has three edges on its boundary.
Definition 2.8. An graph is outerplanar if it is a planar graph that can be embedded in the plane so that all of the vertices belong to the boundary of the unbounded face. An outerplane graph is such an embedding of an outerplanar graph.

Definition 2.9. In a cycle $C$, a chord is an edge between two nonadjacent vertices of the chord which itself is not an edge of the cycle.

In this dissertation, we will use the term chord when referring to an edge between two nonadjacent vertices that lie on the cycle that corresponds to the boundary of the unbounded face of a plane graph $G$.

Definition 2.10. Let $G = (V, E)$ be a planar graph. The dual of $G$ is the graph $G' = (V', E')$ whose vertex set $V'$ is made up of vertices $v'$, each of which corresponds to exactly one plane region of $G$. The edge set $E'$ of $G'$ is made up of edges $e'$, each of which exists if and only if there is a corresponding edge $e$ joining two adjacent regions in a fixed plane embedding of $G$.

Note that dual graphs are not unique because they depend on the particular embedding used. In this dissertation, we will refer to the notion of a weak dual.

Definition 2.11. The weak dual of $G$ is the induced subgraph $G'' = (V'', E'')$ of the dual $G'$ whose vertices $v''$ correspond to the bounded faces of $G$.

It can be observed that the dual and weak dual of a graph will not necessarily be a simple graph.

Definition 2.12. The triple $(x_1, x_2, x_3)$ is a triangle in $G$ if $x_1, x_2, x_3 \in V(G)$ and $x_1x_2, x_2x_3, x_1x_3 \in E(G)$.

Definition 2.13. Let $G = (V, E)$ be a near-triangulation. Assume the non-triangular face is the cycle $C = v_1v_2\ldots v_kv_1$. Fix vertices $v_1, v_2, v_k$.

1. If $V = V(C) \cup \{v\}$ and $E = E(C) \cup \{vv_1, vv_2, \ldots, vv_k\}$, then $G$ is said to be a wheel $W_k$ with center $v$.

2. If $V = V(C)$ and $E = E(C) \cup \{v_1v_3, v_1v_4, \ldots, v_1v_{k-1}\}$, then $G$ is said to be a broken wheel $BW_{k-1}$. (This graph is also referred to as a fan $F_{k-1}$.)
3. A **generalized wheel** is a graph that is either a wheel, broken wheel, or one of the following two types of graphs:

   (a) $V = V(C) \cup \{u, v\}$ and $E = E(C) \cup \{v_1v_i, vv_{i+1}, \ldots, vv_k, uv_1uv_2, \ldots, uv_i\}$, or

   (b) $V = V(C) \cup \{v\}$ and $E = E(C) \cup \{v_1v_3, v_1v_4, \ldots, v_1v_k, vv_i, vv_{i+1}, \ldots, vv_k\}$.

For a generalized wheel, $v_kv_1v_2$ is the **principal path**, $v_1$ is the **major vertex**, and the edges $v_1v_2, v_1v_k$ are **principal edges**. See Figure 2.1 for an illustration of each of the types of generalized wheels described above.

![Figure 2.1: Examples of generalized wheels.](image)

**Definition 2.14.** The **Cartesian product** $G \times H$ is the graph with vertex set $V(G) \times V(H)$ where any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$.

**Definition 2.15.** The **theta graph** $\Theta_{k_1,k_2,k_3}$ is the union of three internally disjoint paths with $k_1, k_2, k_3$ internal vertices, respectively, that have the same two distinct end vertices.

**Definition 2.16.** The graph $G^k$, the $k$th **power** of a graph $G$, is the graph with the same vertex set as $G$ and an edge between vertices $u$ and $v$ if and only if there is a path of length at most $k$ between $u$ and $v$ in $G$.

We define the following graph that is obtained by laying cycles of arbitrary and varying lengths greater than 3 end to end so that they share an edge.

**Definition 2.17.** A graph $G$ is called a **path of $k$ cycles**, or **path of cycles**, if $G = \bigcup_{i=1}^{k} G_i$ where each $G_i$ is a cycle of length $a_i \geq 4$ for $i = 1, \ldots, k$, $V(G_{i-1}) \cap V(G_i) = \{t_i, b_i\}$ and
$E(G_{i-1}) \cap E(G_i) = \{t_ib_i\}$ for all $i = 2, \ldots, k$, and $t_i, b_i \notin V(G_j)$ for all $j \neq i - 1, i$. If $w_i \in V(G_i) - \{t_i, b_i, t_{i+1}, b_{i+1}\}$, then $w_i \notin V(G_j)$ for all $j \neq i$. Furthermore, $G$ can be drawn in the plane so that the weak dual of $G$ is a path.

See Figure 2.2 for an example of a path of cycles. We also define a graph that is obtained by laying cycles of arbitrary and varying lengths greater than 3 along a special tree-like structure so they share an edge.

**Definition 2.18.** A graph $G$ is called a **tree of $k$ cycles**, or **tree of cycles**, if $G = \bigcup_{i=1}^{k} G_i$ where each $G_i$ is a cycle of length $a_i \geq 4$ for $i = 1, \ldots, k$ and for all pairs $i, j$, it must be that $V(G_i) \cap V(G_j) = \emptyset$ or $V(G_i) \cap V(G_j) = \{u,v\}$ for two adjacent vertices $u, v \in V(G)$. If $V(G_i) \cap V(G_j) = \{u,v\}$, then $u, v \notin V(G_l)$ for all $l \neq i, j$. If $w_i \in V(G_i)$, then $w_i \notin V(G_j)$ for all $j \neq i$ unless $V(G_i) \cap V(G_j) = \{u,v\}$ and $w_i \in \{u, v\}$. Furthermore, $G$ can be drawn in the plane so that the weak dual of $G$ is a tree and $G$ can be drawn in the plane so that in the dual of $G$, the vertex corresponding to the unbounded face of $G$ is adjacent to all other vertices.

See Figure 2.3 for an example of a tree of cycles. Note that a path of cycles is a special case of a tree of cycles that occurs when the underlying tree-like structure is a path.

### 2.2 Reductions

When looking at plane graphs in Chapters 3 and 4, all graphs considered will be assumed to be 2-connected. Assume otherwise, then it is possible to add edges between some pairs of vertices on the boundary of the unbounded face so that the modified graph is 2-connected and all vertices on the boundary of the unbounded face of the original graph are on the boundary.
Figure 2.3: An example of a tree of cycles.

of the unbounded face of the modified graph, see Lemma 2.19 below. Any list-coloring of this new graph will then provide a list-coloring of the original graph.

**Lemma 2.19.** Let \( G = (V, E) \) be a plane graph that is not 2-connected. Let \( F \) be the set of vertices on the boundary of the unbounded face of \( G \). Then it is possible to add edges between some pairs of vertices in \( F \) so that the modified graph \( G' = (V, E') \) is 2-connected and all vertices of \( F \) lie on the boundary of the unbounded face of \( G' \).

**Proof.** Since \( G \) is not 2-connected, there is at least one vertex \( v \in V \) that is a cut-vertex of \( G \), otherwise \( G \) will be disconnected. An edge \( e \) must be added to \( G \) so that in the resulting graph, call it \( G' \), the vertex \( v \) is not a cut vertex. Additionally, we must add this edge in such a way that all vertices of \( F \) lie on the boundary of the unbounded face of \( G' \). Let \( W = vv_2 \ldots v_mv \) be a closed walk through all the vertices of \( F \) along the boundary of the unbounded face of \( G \). Since \( G \) is not 2-connected, this closed walk is not a cycle and some of the vertices, perhaps edges also, are repeated but not in the same order. In particular, \( v \) is repeated. Add the edge \( v_mv_2 \) to \( G \) so that it now lies on the boundary of the unbounded face of \( G' \) instead of \( v_mvv_2 \). It is important to note that the order matters here. If \( vv_mv_2 \) was part of \( W \), it will also be part of the corresponding closed walk through all of the vertices along the boundary of the unbounded face of \( G' \). The vertex \( v \) is not a cut vertex in the graph \( G' \). This process may be
repeated with each remaining cut vertex so that the modified graph does not contain any cut vertices.

Now that we are assuming all graphs in consideration are 2-connected, we may also employ the following result, see [25].

**Proposition 2.20.** Let $G$ be a 2-connected plane graph. Then every edge of $G$ belongs to the boundary of exactly two faces and the boundary of every face is a cycle.

In particular, it can be assumed that the boundary of the face corresponding to the unbounded face of $G$ is a cycle.

In Chapters 3 and 4 we use two different notions of a reduced graph. This allows for us to get information about certain forbidden subgraphs of such graphs. We define the necessary terminology here and then define these reductions. In Chapters 3 and 4 we will show why such reductions can be made.

**Definition 2.21.** Let $X \subset V$ be a subset of vertices in a connected graph $G = (V, E)$. If $G - X$ contains at least two connected components, then $X$ is said to be a **separating set**. If $X$ is also an $i$-vertex set spanning an $i$-cycle, then $X$ is a **separating $i$-cycle**.

**Definition 2.22.** Let $X \subset V$ be a separating set in a graph $G = (V, E)$. If $P \subset V$ and there are at least two vertices of $P$ in distinct connected components of $G - X$, then $X$ is a **$P$-separating set**. If $X$ is also a separating $i$-cycle, then $X$ is said to be a **$P$-separating $i$-cycle**.

In particular, if $X$ consists of two adjacent vertices $u, v$ and $X$ is $P$-separating in $G$, then $uv$ is an **$P$-separating edge**. If an edge does not have this property, it is called a **non-$P$-separating edge**. When looking at a plane graph $G$, we will use the following:

**Definition 2.23.** Let $G$ be a plane graph and let $C$ be the cycle that corresponds to the boundary of the unbounded face of $G$. If the edge $uv$ is a chord in $G$ with endpoints in $C$ and a $P$-separating edge, then $uv$ is called a **$P$-separating chord**.

**Definition 2.24.** We say that a set $X$ of four vertices of degree at most 5 in a graph $G$ forms the configuration $D = D(X)$ if $G[X]$ is isomorphic to $K_4 - e$. 
See Figure 2.4a for an illustration of $D$.

**Definition 2.25.** We say that a set $X$ of seven vertices of degree at most 6 in a graph $G$ forms the configuration $W = W(X)$ if $G[X]$ induces a 6-wheel, formed from a central vertex $w$ adjacent to a 6-cycle $x_1x_2x_3x_4x_5x_6x_1$ such that $x_2$, $x_3$, $x_5$ and $x_6$ have degree at most 5 in $G$.

See Figure 2.4b for an illustration of $W$.

![Configuration D and W](image)

(a) Configuration $D$. (b) Configuration $W$.

**Figure 2.4:** Reducible configurations.

We now have all of the terminology needed to define the two types of reductions that will be used in this thesis. The first type will be used in Chapter 3 and the second type will be used in Chapter 4.

**Definition 2.26.** For a graph $G$ and a set of vertices $P$, let $R(G) = R(G,P)$, a **Type I reduction of $G$ with respect to $P$**, be a graph obtained by performing one of the following operations on $G$:

1. for a separating 3-cycle or 4-cycle that does not separate $P$, remove from $G$ the vertices and edges in the region that is bounded by the separating 3-cycle or 4-cycle and that does not contain any vertices of $P$,

2. for a configuration $D = D(X)$ such that $P \cap X = \emptyset$, remove $X$ from $G$, or

3. for a configuration $W = W(X)$ such that $P \cap X = \emptyset$, remove $X$ from $G$.

If none of these operations can be carried out, let $R(G) = G$. 
Consider a sequence of graphs $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_m$ such that $G_i = R(G_{i-1}, P)$ for $i = 1, \ldots, m$ and $R(G_m) = G_m$. Call such a graph $G_m$ a Type I reduced graph of $G$. A Type I reduced graph does not have a separating 3-cycle or 4-cycle that does not separate $P$ and it contains no configurations $D(X)$ or $W(X)$ with $P \cap X = \emptyset$. We shall show in Chapter 3 that if a Type I reduced graph of $G$ has a coloring extension of $P$, then so does $G$.

**Definition 2.27.** Let $G$ be a plane graph and let $C$ be the cycle that corresponds to the boundary of the unbounded face of $G$. Let $x, y \in V(C)$. Let $R(G) = R(G, \{x, y\})$, a Type II reduction of $G$ with respect to $x, y$, be a graph obtained by performing one of the following operations on $G$:

1. If $X$ is a set of vertices in $G$ that induces a separating 3-cycle or a separating 4-cycle and $X'$ is the vertex set of the connected component of $G - X$ which contains neither $x$ nor $y$, then let $R(G) = G - X'$.

2. If there is a non-$\{x, y\}$-separating chord $uv$ in $G$ that splits $G$ into two graphs $G_A$ and $G_B$ such that $G = G_A \cup G_B$, $V(G_A) \cap V(G_B) = \{u, v\}$, and $x, y \in V(G_A)$, then let $R(G) = G_A$.

If neither of these operations can be carried out, then $R(G) = G$.

Consider a sequence of graphs $G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_m$ such that $G_i = R(G_{i-1}, \{x, y\})$ for $i = 1, \ldots, m$ and $R(G_m) = G_m$. Call such a graph $G_m$ a Type II reduced graph of $G$. Note that a Type II reduced graph does not contain any separating 3-cycles, separating 4-cycles, nor any non-$\{x, y\}$-separating chords. Observe also that if $G$ is 2-connected, then $R(G)$ is 2-connected.

Let $f^{G-H}(v) = f(v) - d(v, G - H)$ for all $v \in V(G - H)$. The subgraph $H$ is called a reducible configuration with respect to $f$ if $H$ is $f^{G-H}$-choosable. We say that $H$ is reducible because if $H$ satisfies these conditions and $G - H$ is $f_{G-H}$-choosable, then $G$ will be $f$-choosable. Thus, to show that $G$ is $f$-choosable, we can reduce the problem to showing that $G - H$ is $f_{G-H}$-choosable for any such $H$. In many cases, this will simplify the work that needs to be done. This is because it allows for the assumption that no such reducible
configuration exists as a subgraph. Such an assumption will often lead to a contradiction. The good configurations of Chapter 5 can be considered reducible configurations as subgraphs of a planar graph with 5-lists assigned to its vertices.
CHAPTER 3. LIST PRECOLORING EXTENSIONS ON PLANAR GRAPHS

Based on a paper published in *Discrete Mathematics* [11]
with Maria Axenovich and Joan P. Hutchinson

3.1 Introduction

In this chapter we will explore the following question posed by Albertson [2]:

**Question 3.1.** Let $G$ be a plane graph. Is there a $d > 0$ such that whenever $P \subseteq V$ is such that $\text{dist}(P) \geq d$, then every precoloring of $P$ extends to a 5-list-coloring of $G$?

Tuza and Voigt [60], see also [66], showed that the condition of a large distance between precolored vertices is essential by finding a planar graph $G$ with a set of precolored vertices $P$ with $\text{dist}(P) \geq 4$ such that the precoloring is not extendable to a 5-list-coloring of $G$. So, the distance $d$ in the above question should be at least 5. Does this question have a positive answer if $d \geq 1000$? The original theorem of Thomassen [56] implies that if there are two adjacent precolored vertices assigned distinct colors, then the precoloring is extendable to a 5-list-coloring of $G$. Böhme, Mohar, and Stiebitz [14] described when the precoloring of vertices on a short face with at most six vertices can be extended to a 5-list-coloring of a planar graph.

In this chapter, we introduce a technique using shortest paths in planar graphs which allows us to answer Albertson’s question for a wide class of planar graphs. We prove that a proper precoloring of a pair of vertices can always be extended to a 5-list-coloring of a planar graph provided they are not separated by 3- or 4-cycles. We also provide results about extensions of precolorings of vertices on one face. Finally, we answer Albertson’s question in the case where
there are no 3- or 4-cycles separating precolored vertices and there is a special tree containing all of the precolored vertices.

To state our main results in all their generality, we need to define some additional notions. For a path \( S \), with endpoints \( u \) and \( v \), we say a vertex \( w \) is central if the distances in \( S \) from \( w \) to \( u \) and from \( w \) to \( v \) differ by at most 1. Note there are at most two central vertices in \( S \). For graph theoretic terminology not defined here, we refer the reader to [74]. By Lemma 2.19, we assume all graphs in consideration are 2-connected. Where necessary, if the application of this lemma causes the distance between a pair of vertices to decrease, we assume the graph the result is applied to is the graph obtained by applying Lemma 2.19. This is important for the results that have a lower bound on a distance constraint.

**Definition 3.2.** Let \( G \) be a planar graph, \( P \) a subset of vertices of \( G \). Fix a positive integer \( d \). Let \( T \) be a tree with \( P \subseteq V(T) \). Let the set of special vertices be the union of \( P \) and the set of vertices of degree either 1 or at least 3 in \( T \). A path in \( T \) with special vertices as endpoints and containing no other special vertices is called a branch of \( T \). We say a tree \( T \) is \((P,d)\)-Steiner if

1. every branch has length at least \( 2d \),
2. every branch is a shortest (in \( G \)) path between its endpoints,
3. if \( v_c \) is a center of a branch of \( T \), then a shortest (in \( G \)) path between \( v_c \) and every vertex in another branch has length at least \( d \), and
4. no two vertices of \( T \) from distinct branches have a common neighbor outside of \( T \) nor are they adjacent.

For example, when \( P = \{u,v\} \) is a set of two vertices at distance 30 from each other, a shortest \((u,v)\)-path is a \((P,15)\)-Steiner tree with a single branch.

Next, we define some terms that will be used to simplify the statement of one of the main theorems, as well as a previous result of Böhme et al. [14].

**Definition 3.3.** Let \( G = (V,E) \) be a plane graph and let \( C \) be the cycle that corresponds to the boundary of a face of \( G \). Let \( P = \{x_0,x_1,\ldots,x_{k-1}\} \subseteq C \), where the vertices of \( P \) are labeled cyclically around \( C \).
1. The vertex \( u \in V - P \) is called a **bad vertex** if \( u \) is adjacent to at least five vertices of \( C \).

2. The edge \( u_0u_1 \in E, u_0, u_1 \in V(G) - P \), is called a **bad edge** if \( k = 6 \) and \( u_i \sim \{x_{3i+1}, x_{3i+2}, x_{3i+3}, x_{3i+4}\} \) for \( i = 0, 1 \), where addition of indices is modulo 6.

3. The triangle \( (u_0, u_1, u_2) \), \( u_0, u_1, u_2 \in V - P \), is called a **bad triangle** if \( k = 6 \) and the vertex \( u_i \sim \{x_{2i+2}, x_{2i+3}, x_{2i+4}\} \) for \( i = 0, 1, 2 \), where addition of indices is modulo 6.

If a vertex is a bad vertex or part of a bad edge or a bad triangle, it is called an **exceptional vertex**.

See Figure 3.1 for examples of a bad edge and a bad triangle.

![Exceptional vertices](a) A bad edge \( u_0u_1 \). (b) A bad triangle \( (u_0, u_1, u_2) \).

Figure 3.1: Exceptional vertices \( u_i \).

We now state the main results of this chapter.

**Theorem 3.4.** Let \( G \) be a plane graph, let \( P \) be a set of vertices such that there is no \( P \)-separating 3-cycle or 4-cycle in \( G \). If there is a Type I reduced graph of \( G \) that has a \((P, 45)\)-Steiner tree, then every precoloring of \( P \) is extendable to a proper 5-list-coloring of \( G \).

**Theorem 3.5.** Let \( G \) be a plane graph and \( u, v \in V(G) \). If \( G \) has no \( \{u, v\} \)-separating 3-cycle or 4-cycle, then every proper precoloring of \( \{u, v\} \) is extendable to a proper 5-list-coloring of \( G \).
Theorem 3.6. Let $G = (V, E)$ be a plane graph and let $C$ be the cycle that corresponds to the boundary of a face of $G$. Let $P = \{v_0, v_1, \ldots, v_{k-1}\} \subseteq V(C)$, where the vertices of $P$ are labeled cyclically around $C$. Then every proper precoloring of $P$ is extendable to a 5-list-coloring of $G$ if one of the following conditions holds:

1. $G[P]$ consists of disjoint vertices and edges with pairwise distance at least 3,

2. $k \leq 6$ and none of the following occurs:

   (a) $G$ contains a bad vertex $u \in V - P$ and $L(u)$ consists of exactly five of the colors assigned to five of the neighbors of $u$ in $P$.

   (b) $k = 6$, $G$ contains a bad edge $u_1u_2$ and there is a color $\alpha$ such that, for $i = 1, 2$,

      $L(u_i)$ consists of $\alpha$ and the colors assigned to the four neighbors of $u_i$ in $P$.

   (c) $k = 6$, $G$ contains a bad triangle $(u_1, u_2, u_3)$ and there are colors $\alpha, \beta$ such that, for $i = 1, 2, 3$,

      $L(u_i)$ consists of $\alpha, \beta$ and the colors assigned to the three neighbors $u_i$ in $P$.

Theorem 3.7. Let $P$ be a set of vertices in a plane graph $G$, $\text{dist}(P) \geq 3$, such that there are two faces $F_1, F_2$ where the vertices of $P$ lie on the boundaries of $F_1$ and $F_2$. Assume $G$ contains no $P$-separating 3-cycle or separating 4-cycle. Then every precoloring of $P$ is extendable to a proper 5-list-coloring of $G$.

The rest of the chapter is organized as follows. In Section 3.2 we describe the origin of Albertson’s question and related results, state known results mentioned above in detail, and prove some technical lemmas. We prove all of the theorems in Section 3.3. Finally, we state open problems and comments in Section 3.4.

## 3.2 Preliminaries

As mentioned earlier, Albertson [2] was able to answer Thomassen’s question about precoloring extensions on planar graphs.

Theorem 3.8 (Albertson [2]). Let $G = (V, E)$ be a planar graph and $P \subset V$ such that $\text{dist}(P) \geq 4$. Then any 5-coloring of $P$ can be extended to a 5-coloring of $G$. 
Proof. Let \( c \) be a 5-coloring of \( P \) using colors \( \{\alpha, \beta, \gamma, \delta, \epsilon\} \) and \( c' \) be an arbitrary 4-coloring of \( G \) using colors \( \{\alpha, \beta, \gamma, \delta\} \). The goal is to modify \( c \) so that it agrees with \( c' \). If there is a vertex \( v \in P \) for which \( c(v) \neq c'(v) \), recolor all vertices adjacent to \( v \) that were assigned color \( c(v) \) so that they are now assigned color \( \epsilon \). Now redefine \( c' \) so that \( c'(v) = c(v) \). Since \( \text{dist}(P) \geq 4 \), no two adjacent vertices of \( G \) are both colored \( \epsilon \). Thus, \( c' \) is a proper 5-coloring of \( G \) extended from a 5-coloring of \( P \).

If 5-coloring is replaced with 6-coloring in Thomassen’s question, then it works for \( \text{dist}(P) \geq 3 \).

**Theorem 3.9** (Albertson [2]). Let \( G = (V, E) \) be a planar graph and \( P \subset V \) such that \( \text{dist}(P) \geq 3 \). Then any 6-coloring of \( P \) can be extended to a 6-coloring of \( G \).

Proof. Let \( c \) be an arbitrary 6-coloring of \( P \) and let \( G' := G - P \). Let \( L \) be an assignment of lists of size 5 to the vertices of \( G' \), all of which are subsets of the list \( \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\} \). Additionally, for a vertex \( v \) in \( G' \), make sure that \( L(v) \) does not contain a color assigned to a vertex of \( P \) that is adjacent to \( v \) in \( G \). Since \( \text{dist}(P) \geq 3 \), each vertex of \( G' \) is adjacent to at most one vertex of \( P \), so there are at least 5 colors available for \( L(v) \). By Thomassen’s 5-list-coloring theorem, \( G' \) is \( L \)-colorable and this provides a 6-coloring of \( G \) extended from a 6-coloring of \( P \).

Note that in the proofs of the previous two results, the planarity of \( G \) is only used in the fact that planar graphs are 4-colorable and 5-list-colorable. Thus, the previous two results can easily be generalized to the following which do not require planarity of the graphs in consideration:

**Theorem 3.10** (Albertson [2]). Let \( G = (V, E) \) be a graph that is \( k \)-colorable and let \( P \subset V \) such that \( \text{dist}(P) \geq 4 \). Then any \( (k+1) \)-coloring of \( P \) is extendable to a \( (k+1) \)-coloring of \( G \).

**Theorem 3.11** (Albertson [2]). Let \( G = (V, E) \) be a graph that is \( k \)-list-colorable and let \( P \subset V \) such that \( \text{dist}(P) \geq 3 \). Then any \( (k+1) \)-coloring of \( P \) is extendable to a \( (k+1) \)-coloring of \( G \).

As mentioned previously, if there is a positive answer to Albertson’s question, then \( d > 4 \).

**Theorem 3.12** (Tuza & Voigt [60]). There is a planar graph \( G = (V, E) \) and a set \( P \subset V \) of vertices such that \( \text{dist}(P) \geq 4 \) and a list assignment \( L : V \to 2^N \) such that \( |L(v)| = 3 \) for all \( v \in P \) and \( |L(v)| = 5 \) for all \( v \in V - P \) such that \( G \) is not \( L \)-colorable.
As mentioned previously, Böhme et al. described when a precoloring of vertices on a small face is extendable.

**Theorem 3.13 (Böhme et al. [14])**. Let $G = (V, E)$ be a plane graph. Let $C = v_0v_1 \ldots v_{k-1}v_0$, $k \leq 6$, be the cycle the corresponds to the boundary of a face of $G$. If $|L(v)| = 1$ for $v \in V(C)$, $|L(v)| = 5$ for $v \in V - V(C)$, and $G[V(C)]$ is $L$-colorable, then $G$ is $L$-colorable unless one of the following occurs:

1. $G$ contains a bad vertex $u$ and $L(u)$ consists of exactly five of the colors assigned to five of the neighbors of $u$ in $V(C)$.

2. $k = 6$, $G$ contains a bad edge $u_1u_2$ and there is a color $\alpha$ such that, for $i = 1, 2$, $L(u_i)$ consists of $\alpha$ and the colors assigned to the four neighbors of $u_i$ in $V(C)$.

3. $k = 6$, $G$ contains a bad triangle $(u_1, u_2, u_3)$ and there are colors $\alpha, \beta$ such that, for $i = 1, 2, 3$, $L(u_i)$ consists of $\alpha, \beta$ and the colors assigned to the three neighbors $u_i$ in $V(C)$.

Here we observe that if one of the exceptional cases of Theorem 3.13 occurs, it is the only such exceptional case.

**Lemma 3.14.** Let $G = (V, E)$ be a planar graph and $C = x_1 \ldots x_kx_1$, $k \leq 6$, be the cycle that corresponds to the boundary of a face of $G$. Then $G$ contains at most one bad vertex, bad edge or bad triangle.

**Proof.** First note that $|V(C)| \in \{5, 6\}$, otherwise $G$ does not contain any exceptional vertices. Note that if $S$ is a vertex set of a bad edge or bad triangle, then $V(C) \subseteq N(S)$ and $|V(C)| = 6$.

**Claim A.** If $G$ contains a bad vertex $u$, then it cannot contain any other exceptional vertices.

Assume an additional exceptional vertex $v$ exists. Consider the set of edges of $G$ corresponding to $C$ and the edges between $u$ and $V(C)$. Deleting these edges from the plane creates five or six bounded regions and one unbounded region. Each of these bounded regions contains at most three vertices of $V(C)$ on its boundary. The vertex $v$ must be in one of these bounded
regions. Since $v$ is an exceptional vertex, it must have at least three neighbors in $V(C)$. Since there are at most three vertices of $V(C)$ in that region’s boundary, $v$ has exactly three neighbors in $V(C)$. Thus, $v$ is part of a bad triangle $(u, v, v')$ and $u$ is adjacent to five vertices of $V(C)$. The vertex $v'$ from that bad triangle must lie in the same region as $v$. The vertices $v$ and $v'$ are each adjacent to three vertices of $V(C)$ and have one common neighbor in $V(C)$. So together, they must be adjacent to a total of five vertices of $V(C)$, a contradiction. See Figure 3.2a for verification. Thus, given a bad vertex, $G$ cannot contain any additional exceptional vertices.

![Figure 3.2: Exceptional vertices $u_i$ for Lemma 3.14.](image)

**Claim B.** If $G$ contains a bad edge $u_1u_2$, then the only exceptional vertices of $G$ are $u_1$ and $u_2$.

Assume an additional exceptional vertex $v$ exists. Consider the set of edges of $G$ corresponding to $C$ and the edges between $u_1, u_2$ and $F$. Deleting these edges from the plane creates regions each with at most two vertices of $V(C)$ on their boundary. Since $v$ is an exceptional vertex, it must have at least three neighbors in $V(C)$. Therefore, there does not exist a region in which $v$ could lie without contradicting the planarity of $G$.

**Claim C.** If $G$ contains a bad triangle $(u_1, u_2, u_3)$, then the only exceptional vertices of $G$ are $u_1, u_2$ and $u_3$.

The proof of this statement is similar to that of Claim B and left to the reader. The lemma follows by the three claims above. □
The following proposition is almost identical to Theorem 5.3 of [57], with the added condition that \( H \) contains all precolored vertices. The proof is included for completeness.

**Proposition 3.15.** Let \( G \) be a planar graph and \( P \) a set of vertices. Let \( L \) be an assignment of lists of colors such that \(|L(v)| = 1\) for \( v \in P \) and \(|L(v)| = 5\) for \( v \in V(G) - P \). If there is an induced connected subgraph \( H \) of \( G \) containing all vertices from \( P \) such that it can be nicely colored with respect to \( L \), then \( G \) is \( L \)-colorable.

Note if \( d(v, H) \leq 2 \) for each \( v \not\in V(H) \) then every proper coloring of \( H \) is a nice coloring.

**Proof.** Consider a nice coloring \( c \) of \( H \). Then \(|L_c(v, H)| \geq 3\) for all \( v \in N(H) \) and \(|L_c(v, H)| = 5\) for all \( v \in G - V(H) \). Therefore, by Thomassen’s 5-list-coloring theorem, \( G - V(H) \) is \( L_c \)-colorable. Together with the coloring \( c \) of \( H \), this gives a proper \( L \)-coloring of \( G \) as \( L_c(v) \subset L(v) \) for all \( v \in G - V(H) \).

**Lemma 3.16.** Let \( S \) be a shortest \((u,v)\)-path in a planar graph \( G \), where \( S = v_0v_1...v_m \) with \( u = v_0, v = v_m \). Then the following properties hold:

1. for all \( w \in N(S) \), \( d(w, S) \leq 3 \),
2. for every \( x, y \in V(S) \), \( x \not\sim y \) in \( G \) unless \( \{x, y\} = \{v_i, v_{i+1}\} \), for \( i = 0, \ldots, m - 1 \),
3. if \( d(w, S) = 3 \) for some \( w \in N(S) \), then \( w \sim \{v_i, v_{i+1}, v_{i+2}\} \), for \( i = 0, 1, \ldots, m - 2 \),
4. if there is no separating 3-cycle or 4-cycle in \( G \), then for each \( i \) with \( i = 0, 1, \ldots, m - 2 \) there is at most one vertex \( w \in N(S) \) such that \( w \sim \{v_i, v_{i+1}, v_{i+2}\} \).

**Proof.** Items (1)-(3) hold because \( S \) is a shortest \((u,v)\)-path. To see the validity of item (4), assume there are two vertices adjacent to \( v_i, v_{i+1}, v_{i+2} \). Then it is easy to verify that there is either a separating 3-cycle or a separating 4-cycle in \( G \).
3-cycles or 4-cycles in $G$, the vertices of $Q(S) - S$ form an independent set. We call a block of $Q(S)$ with $i$ vertices of $S$ an **i-block**, $i = 2, 3, 4, \ldots$. See Figure 3.3 for examples of blocks in $Q(S)$. Note also that the block-cut-vertex tree of $Q(S)$ is a path. Note that if $Q(S)$ has a cut-edge, that edge is in $S$, and if $Q(S)$ has a cut-vertex, that vertex is in $S$. We shall need a notion of a nontrivial block which will allow us to focus on subpaths of $S$ and not worry about the boundary conditions. For a shortest $(u', v')$-path $T'$, we say an edge $e$ is a **nontrivial cut-edge** of $Q(T')$ if $e$ is a cut-edge not incident to either $u'$ or $v'$; we say $B$ is a **nontrivial block** of $Q(T')$ if $B$ is a block that does not contain $u'$ or $v'$. We say a block $B$ is a **remote nontrivial block** of $Q(T')$ if $|V(B) \cap V(B_1)| = |V(B) \cap V(B_2)| = 1$ where $B_1$ and $B_2$ are distinct nontrivial blocks of $Q(T')$. Let $u', v' \in V(S)$ and let $T' = u'Sv'$. If $e$ is a nontrivial cut-edge in $Q(T')$, then it is easy to see that $e$ is a cut-edge in $Q(S)$; if $B$ is a nontrivial block of $Q(T')$, then $B$ is a block of $Q(S)$.

**Lemma 3.17.** Let $S$ be a shortest $(u, v)$-path in a planar graph $G$, where $S = v_0v_1 \ldots v_m$ with $u = v_0$ and $v = v_m$. Let $G$ have no separating 3-cycle and no separating 4-cycle. Let $L : V(G) \to 2^N$ be an assignment of lists of colors to the vertices of $G$ where $|L(u)| = |L(v_1)| = 1$ with $L(u) \neq L(v_1)$ and $|L(x)| = 5$ for $x \in S \cup N(S) - \{u, v_1\}$. Then $S$ can be nicely colored with respect to $L$.

**Proof.** Assume $v_0Sv_{i+1}$, where $i+2 \leq m$, has been colored nicely by $c$ and $c(v_i) = 1, c(v_{i+1}) = 2$. If there is no $w \in N(S)$ such that $w \sim \{v_i, v_{i+1}, v_{i+2}\}$, then color $v_{i+2}$ arbitrarily from its list so that $c(v_{i+2}) \neq c(v_{i+2})$. If there is a $w \in N(S)$ such that $w \sim \{v_i, v_{i+1}, v_{i+2}\}$, choose a color for $v_{i+2}$ more carefully. If 1 or 2 is not in $L(w)$, then choose $c(v_{i+2})$ from $L(v_{i+2}) - \{2\}$. Otherwise, $L(w) = \{1, 2, \alpha, \beta, \gamma\}$, for some colors $\alpha, \beta, \gamma$. If $1 \in L(v_{i+2})$, let $c(v_{i+2}) = 1$. If $1 \not\in L(v_{i+2})$,
then there is \( a \in L(v_{i+2}) - L(w) \). Let \( c(v_{i+2}) = a \). In each case, we have constructed a nice coloring of \( v_0Sv_{i+2} \). Since \( |L(v_j)| = 5 \) for \( j = 2, \ldots, m \), the above argument may be applied along \( S \) up through \( v \) so that \( S \) is nicely colored.

\[ \square \]

**Lemma 3.18.** Let \( S \) be a shortest \((u, v)\)-path in a planar graph \( G \), where \( S = v_0v_1v_2 \ldots v_m \) with \( u = v_0 \) and \( v = v_m \). Let \( G \) have no separating 3-cycle and no separating 4-cycle. Let \( L : V(G) \to 2^\mathbb{N} \) be an assignment of lists of colors to the vertices of \( G \) where \( |L(u)| = |L(v)| = 4 \) and \( |L(x)| = 5 \) for \( x \in S \cup N(S) - \{u, v\} \). Then \( S \) can be nicely colored with respect to \( L \).

**Proof.** The proof is by induction on \(|V(S)|\).

If \(|V(S)| \leq 2\), the statement follows trivially. If \(|V(S)| = 3\), we can assume that there is a vertex \( w \), where \( w \sim \{v_0, v_1, v_2\} \), otherwise color \( S \) properly from \( L \). If \( c_0 \in L(v_0) \cap L(v_2) \) for some \( c_0 \), let \( c(v_0) = c(v_2) = c_0 \) and color \( v_1 \) arbitrarily from \( L(v_1) - \{c_0\} \). If \( L(v_0) \cap L(v_2) = \emptyset \), then \( |L(v_0) \cup L(v_2)| = 8 \) and there is a color \( c_0 \in (L(v_0) \cup L(v_2)) - L(w) \). Assume without loss of generality that \( c_0 \in L(v_0) \). Then let \( c(v_0) = c_0 \), and color \( v_1, v_2 \) arbitrarily from their lists so the path \( v_0v_1v_2 \) is properly colored. As a result \(|L_c(w)| \geq 3\).

Now assume the result holds for shortest paths on fewer than \( m + 1 \) vertices. Let \(|V(S)| = m + 1\). Color \( v_0Sv_{m-1} \) nicely with a coloring \( c \). If there is no vertex outside of \( S \) adjacent to \( v_{m-2}, v_{m-1} \) and \( v \), then choose \( c(v) \) from \( L(v) - \{c(v_{m-1})\} \). This gives a nice coloring of \( S \). So assume there is a vertex \( w \in N(S) \) such that \( w \sim \{v_{m-2}, v_{m-1}, v\} \).

If \( c(v_{m-1}) \not\in L(w) \) or \( c(v_{m-2}) \not\in L(w) \), then let \( c(v) \in L(v) - \{c(v_{m-1})\} \). If \( c(v_{m-1}) \in L(w) - L(v) \) and \( c(v_{m-2}) \in L(w) - L(v) \), then \( L(v) \) contains a color not in \( L(w) \). Assign this color to \( v \) to obtain a nice coloring of \( S \). So we can assume \( c(v_{m-2}) \in L(v) - L(w) \). If \( c(v_{m-2}) \in L(v) \), let \( c(v) = c(v_{m-2}) \) providing a nice coloring of \( S \). Thus we can assume \( L(v) = \{c_1, c_2, c_3, c_4\} \), \( a = c(v_{m-2}) \neq c_i \) for all \( i = 1, 2, 3, 4 \), and \( L(w) = \{a, c_1, c_2, c_3, c_4\} \).

Apply induction to \( v_0Sv_{m-2} \) in the graph \( G' \) induced in \( G \) by this path and its neighbors, with a new list \( L(v_{m-2}) - \{a\} \) assigned to \( v_{m-2} \) and all other old lists. There is a nice coloring \( c' \) of \( v_0Sv_{m-2} \) in \( G' \). Note that it is a nice coloring of \( v_0Sv_{m-2} \) in \( G \). We either have \( c'(v_{m-2}) = c_i \), for some \( i = 1, 2, 3, 4 \), or \( c'(v_{m-2}) \not\in L(w) \). If \( c'(v_{m-2}) \not\in L(w) \), color \( v_{m-1} \) first so that if there is \( w' \sim \{v_{m-3}, v_{m-2}, v_{m-1}\} \), then \(|L_{c'}(w')| \geq 3\). Then let \( c'(v) \in L(v) - \{c'(v_{m-1})\} \). If
vertices of $T$.

**Lemma 3.20.** Let $S$ be a shortest $(u, v)$-path in a planar graph $G$, where $S = v_0v_1\ldots v_m$ with $u = v_0$ and $v = v_m$. Let $G$ have no separating 3-cycles and no separating 4-cycles. Let $L : V(G) \to 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ with $|L(u)| = |L(v)| = 1$ and $|L(x)| = 5$ for $x \in S \cup N(S) - \{u, v\}$. Assume $Q(S)$ has at least two cut-edges. Then $S$ can be nicely colored with respect to $L$.

**Proof.** Let $v_kv_{k+1}$ and $v_{l}v_{l+1}$ be two cut-edges of $Q(S)$, where $0 \leq k < l < m$. Using Lemma 3.17 color $v_0Sv_k$ and $v_{l+1}Sv_m$ nicely with a coloring $c$. If $v_{k+1} = v_l$, we are done by giving $v_l$ a color different from $c(v_k)$ and $c(v_{l+1})$. Otherwise, delete $c(v_k)$ from $L(v_{k+1})$, delete $c(v_{l+1})$ from $L(v_l)$, and color $v_{k+1}Sv_l$ nicely from the updated lists using Lemma 3.18. Since $v_0Sv_k$, $v_{k+1}Sv_l$, and $v_{l+1}Sv_m$ do not have pairwise common neighbors in $N_3(S)$, this gives a nicely colored $S$.

The next lemma is a key lemma in this chapter, stating that either a given shortest path between two precolored vertices could be nicely colored, or another subgraph that is close to that path could be nicely colored.

For a path $T'$, a center $v_c$ of $T'$, and an even positive integer $d \leq |V(T')| - 1$, we call the two vertices of $T'$ at distance (in $T'$) $\frac{1}{2}d$ from $v_c$ the $d$-tag vertices with respect to $v_c$, or simply tag vertices, of $T'$.

**Lemma 3.21.** Let $S$ be a shortest $(u, v)$-path in a planar graph $G$ with a center $v_c$ and 40-tag vertices $u^*, v^*$ with respect to $v_c$, where $S = v_0v_1\ldots v_m$ with $v_0 = u$ and $v_m = v$. Assume $G$ contains no separating 3-cycle or 4-cycle and no configuration $D(X)$ or $W(X)$ with $\{u, v\} \cap X = \emptyset$. Let $L : V(G) \to 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ such that $|L(u)| = |L(v)| = 1$ and $|L(x)| = 5$ for $x \in V(G) - \{u, v\}$. Then there is a connected graph $H = H(S, u, v) = uSu^* \cup H' \cup v^*Sv$ such that every vertex of $H'$ is at distance at most 21 from $v_c$, and $H$ can be nicely colored from $L$. 

$c'(v_m-2) = c_i$, without loss of generality say $c'(v_m-2) = c_1$, then let $c'(v) = c_1$ and color $v_m-1$ so that $|L_{c'}(w')| \geq 3$. 

$\square$
Proof. Recall that every vertex in \( V - V(S) \) is adjacent to at most three vertices in \( S \), and if a vertex from \( Q(S) - S \) is adjacent to vertices in \( S \), these vertices in \( S \) must be consecutive.

**Observation 1.** If \( Q(S) \) has a \( p \)-block \( B \), for a \( p \geq 6 \), then there is a shortest \((u, v)\)-path \( S' \) such that \( Q(S') \) has a nontrivial cut-edge.

Let \( B \) contain \( v_i, v_{i+1}, \ldots, v_{i+5} \) and vertices \( w_{i+k} \) not in \( V(S) \), where for \( k = 1, 2, 3, 4 \) \( w_{i+k} \sim \{v_{i+k-1}, v_{i+k}, v_{i+k+1}\} \). Consider the shortest \((u, v)\)-path

\[
S' = v_0v_1 \ldots v_i w_{i+1}v_{i+2}v_{i+3}w_{i+4}v_{i+5} \ldots v_m.
\]

Then it is a routine check to see that \( v_{i+2}v_{i+3} \) is a nontrivial cut-edge in \( Q(S') \), as shown in Figure 3.4.

![Figure 3.4](image)

**Observation 2.** We can assume at least one of the following holds:

1. for every shortest \((u^*, v_c)\)-path \( S' \), each nontrivial block of \( Q(S') \) is either a 3-, 4-, or 5-block,
2. for every shortest \((v_c, v^*)\)-path \( S' \), each nontrivial block of \( Q(S') \) is either a 3-, 4-, or 5-block.

If there is a shortest \((u^*, v_c)\)-path \( T' \) such that \( Q(T') \) has a nontrivial cut-edge and there is a shortest \((v_c, v^*)\)-path \( T'' \) such that \( Q(T'') \) has a nontrivial cut-edge, then Lemma 3.19 implies \( uSu^*T'v_cT''v^*Sv \) can be nicely colored. Assume, without loss of generality that for every shortest \((u^*, v_c)\)-path \( S' \), \( Q(S') \) has no nontrivial cut-edges. Then Observation 1 implies there is no \( p \)-block of \( Q(S') \) with \( p \geq 6 \).

Assume that part (1) of Observation 2 holds. Let \( u' = u^* \), \( v' = v_c \). Let \( T \) be a shortest \((u', v')\)-path with the largest number of 3-neighbors.
Observation 3. If, for some shortest \((u', v')\)-path \(T\) with maximum number of 3-neighbors, \(Q(T)\) has a nontrivial 3-block, then there is a graph \(H(S, u, v)\) satisfying the conditions of the lemma.

If such a block \(B\) were to exist, say with consecutive vertices \(x_i, x_{i+1}, x_{i+2}\) of \(T\) and \(w \notin V(T)\), \(w \sim \{x_i, x_{i+1}, x_{i+2}\}\), then there is no vertex \(w' \notin S \cup T\) such that \(w'\) is adjacent to \(w\) and two other vertices of \(T\), otherwise there is a shortest \((u', v')\)-path with more 3-neighbors than \(T\). Let \(H(S, u, v)\) be the graph induced by vertices of \(uSu'Tv'Sv\) and \(w\). Nicely color \(uSu'Tx_i\) and nicely color \(x_{i+2}Tv'Sv\), then properly color \(w\) and \(x_{i+1}\) from remaining available colors in their lists. Since there is no 3-neighbor of \(H(S, u, v)\) adjacent to \(x_{i+1}\) and there is no such 3-neighbor adjacent to \(w\), this coloring is a nice coloring of \(H(S, u, v)\).

Thus, we can assume that all nontrivial blocks of \(Q(T)\) are 4- or 5-blocks. Since \(\text{dist}(u', v') = 20\), there are remote nontrivial blocks in \(Q(T)\).

Observation 4. If \(Q(T)\) has a remote 5-block for some shortest \((u', v')\)-path \(T\) with maximum number of 3-neighbors, then there is a graph \(H(S, u, v)\) satisfying the conditions of the lemma.

Assume there is such a block \(B\) with consecutive vertices \(x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\) of \(T\) and vertices \(w_{i+1}, w_{i+2}, w_{i+3}\) not in \(T\) such that \(w_k \sim \{x_{k-1}, x_k, x_{k+1}\}\), for \(k = i + 1, i + 2, i + 3\). From Observation 3, we can assume that every nontrivial block of \(Q(T)\) is either a 4- or a 5-block.

Note first that there is no vertex \(w\) adjacent to \(w_{i+1}\) and two vertices of \(T\), and there is no vertex \(w\) adjacent to \(w_{i+3}\) and two vertices of \(T\). Indeed, assume otherwise that there is a vertex \(w\) adjacent to \(w_{i+1}\) and two vertices of \(T\). Then \(w \sim \{x_{i-1}, x_i, w_{i+1}\}\). Since all nontrivial blocks of \(T\) have at least four vertices of \(T\), and there are nontrivial blocks \(B_1\) and \(B_2\) of \(Q(T)\) such that \(|V(B) \cap V(B_1)| = |V(B) \cap V(B_2)| = 1\), we see there is a vertex \(w_{i-1}\) adjacent to \(\{x_{i-2}, x_{i-1}, x_i\}\) and there is a vertex \(w_{i-2}\) adjacent to \(\{x_{i-3}, x_{i-2}, x_{i-1}\}\), as shown in Figure 3.5. Then \(u'Tx_{i-3}x_{i-2}x_{i-1}x_iw_{i+1}x_{i+2}Tv'\) is a shortest \((u', v')\)-path \(T''\) with a block in \(Q(T'')\) having at least six vertices of \(T''\), as can be seen in Figure 3.5. This is a contradiction to Observation 2. Similarly, it is impossible to have a vertex \(w\) adjacent to \(w_{i+3}\) and two vertices
Figure 3.5: Example corresponding to a case of Observation 4.

Assume now that there is no vertex \( w \) adjacent to \( w_{i+1} \) and \( w_{i+3} \) and a vertex of \( T \). Let \( H(S, u, v) \) be a graph induced by vertices of \( uSuv'Tv'Sv \) and \( w_{i+1}, w_{i+3} \), as shown in Figure 3.6. Note that while \( w_{i+2} \) is shown in the figure, it is not a vertex in the graph \( H(S, u, v) \). To color \( H(S, u, v) \) nicely, first color \( uSu'Tx_i \) and \( x_{i+4}Tv'Sv \) nicely, then color \( x_{i+1}Tx_{i+3} \) properly so \( w_{i+2} \) has at least three colors remaining in its list after the removal of colors used on adjacent vertices, and finally color \( w_{i+1} \) and \( w_{i+3} \) using available colors.

**FACT.** We can assume for every shortest \((u', v')\)-path \( T = y_0y_1 \ldots y_l \), where \( y_0 = u' \), \( y_l = v' \), with maximum number of 3-neighbors and for every remote nontrivial 5-block \( B \) of \( T \) with vertices \( y_i, y_{i+1}, y_{i+2}, y_{i+3}, y_{i+4} \) of \( T \) and \( w_j \sim \{y_{j-1}, y_j, y_{j+1}\} \), for \( j = i + 1, i + 2, i + 3 \), there is a vertex \( w \sim \{w_{i+1}, y_{i+2}, w_{i+3}\} \).

Consider the shortest \((u', v')\)-path \( \tilde{T}_1 = u'Ty_1y_{i+1}w_{i+2}y_{i+3}y_{i+4}Tv' \). There must be a vertex \( w' \sim \{y_i, y_{i+1}, w_{i+2}\} \), otherwise \( y_iy_{i+1} \) is a cut-edge in \( Q(\tilde{T}_1) \). There must also be a vertex \( w'' \)
such that \( w'' \sim \{w_{i+2}, y_{i+3}, y_{i+4}\} \), otherwise \( y_{i+3}y_{i+4} \) is a cut-edge in \( Q(\tilde{T}_1) \).

Next, consider the shortest \((u', v')\)-path \( \tilde{T}_2 = u'Ty_iiw_{i+1}ww_{i+3}y_{i+4}Tv' \). There must be a vertex \( x \sim \{y_i, w_{i+1}, w\} \), otherwise \( y_iw_{i+1} \) is a cut-edge in \( Q(\tilde{T}_2) \). There must also be a vertex \( x' \sim \{w, w_{i+3}, y_{i+4}\} \), otherwise \( w_{i+3}y_{i+4} \) is a cut-edge in \( Q(\tilde{T}_2) \). Finally, consider the shortest \((u', v')\)-paths

\[
\tilde{T}_3 = u'Tyi_iy_{i+1}ww_{i+2}y_{i+3}y_{i+4}Tv' \quad \text{and} \quad \tilde{T}_4 = u'Tyi_iy_{i+1}ww_{i+3}y_{i+4}Tv'.
\]

By the fact above, there must be vertices \( z \) and \( z' \) such that \( z \sim \{w', w_{i+2}, w''\} \) and \( z' \sim \{x, w, x'\} \). Thus, \( G[X] \) where \( X = \{y_{i+1}, y_{i+2}, y_{i+3}, w_{i+1}, w_{i+2}, w_{i+3}, w\} \) corresponds to the configuration \( W(X) \) in \( G \), as seen in Figure 3.7, where the bold vertices represent \( X \). This completes the proof of Observation 4.

To summarize, we know that for any shortest \((u', v')\)-path \( T \), every nontrivial block of \( Q(T) \) is a 3-, 4-, or 5-block. Moreover, if \( T \) has the largest number of 3-neighbors among all such shortest \((u', v')\)-paths, then every remote nontrivial block of \( Q(T) \) is a 4-block.

To conclude the proof of Lemma 3.20, let \( T \) be a shortest \((u', v')\)-path with the largest number of 3-neighbors among all such shortest \((u', v')\)-paths. Consider a remote nontrivial block of \( Q(T) \) with consecutive vertices \( x_i, x_{i+1}, x_{i+2}, x_{i+3} \) of \( T \) and vertices \( w_{i+1}, w_{i+2} \) not in \( T \) such that \( w_k \sim \{x_{k-1}, x_k, x_{k+1}\} \) for \( k = i + 1, i + 2 \).

Case 1. There is no \( w \) adjacent to \( w_{i+1} \) and two vertices of \( T \), and there is no vertex \( w \) adjacent to \( w_{i+2} \) and two vertices of \( T \).
Let $H(S, u, v)$ be the graph induced by vertices of $uSv'Tv'Sv$ and $w_{i+1}, w_{i+2}$. To color $H(S, u, v)$ nicely, first color $uSv'Tx_i$ and $x_i+3Tv'Sv$ nicely, then color $G[x_{i+1}, x_{i+2}, w_{i+1}, w_{i+2}]$ properly.

Case 2. There is, without loss of generality, a vertex $w$ adjacent to $w_{i+1}$ and two vertices of $T$.

If $w' \sim \{x_{i-1}, x_i, w_{i+1}\}$ for some vertex $w'$, then consider the path $T' = u'Tx_iw_{i+1}x_i+2Tv'$. Then in $Q(T')$ there is a $p$-block with $p \geq 6$, a contradiction as shown on the left in Figure 3.8. So assume $w \sim \{w_i, x_{i+2}, x_{i+3}\}$. Consider a path $T'' = u'Tx_iw_{i+1}wx_{i+3}Tv'$. Observe that the edge $x_iw_{i+1}$ is a nontrivial cut-edge in $Q(T'')$ unless there is a vertex $w'$ adjacent to $x_i, w_{i+1}$ and another vertex of $T''$. This third vertex is either $x_{i-1}$ or $w$. It could not be $x_{i-1}$ as shown before. Thus, $w' \sim \{x_i, w_{i+1}, w\}$. See the right hand side of Figure 3.8. Similarly,

![Figure 3.8: Observe why there is no $w$ such that $w \sim \{w_{i+1}, x_{i+2}, x_{i+3}\}$.](image)

by considering the path $u'Tx_iw_{i+1}w_{i+2}x_{i+3}Tv'$, we see there is a vertex $w'' \sim \{x_i, x_{i+1}, w_{i+2}\}$. Finally, by considering the path $u'Tx_iw''w_{i+2}x_{i+3}Tv'$, we have a vertex $w''' \sim \{w'', w_{i+2}, x_{i+3}\}$.

But now the graph $G[X]$, where $X = \{x_{i+1}, x_{i+2}, w_{i+1}, w_{i+2}\}$, gives the configuration $D(X)$ in $G$, as seen in Figure 3.8 where the bold vertices represent $X$.

We see now, that a graph $H(S, u, v)$ in all the cases above was constructed by taking the union of $uSv', v'Sv$, and a graph $H'$ induced by a shortest $(u', v')$-path $T$ (of length 20) and, perhaps some vertices at distance 1 from $T$. Thus, any vertex of $H'$ is at distance at most 21 to $v' = v_c$. \qed
3.3 Proofs of Theorems

Proof of Theorem 3.5. Note if the two precolored vertices are adjacent, then the coloring is extendable by Thomassen’s 5-list-coloring theorem. In general, we use induction on $|V(G)|$ where the base case is precolored $u$ and $v$ connected by an edge. Assume $G$ is connected, otherwise the result follows trivially by induction.

Claim. $G$ has no separating 3-cycle or 4-cycle. Let $U$ be a vertex set of such a separating cycle. By the assumption of the theorem, $U$ does not separate $\{u,v\}$. Let $V_1$ and $V_2$ be the vertex sets of disconnected plane graphs obtained by removing $G[U]$ from $G$, such that $\{u,v\} \subseteq V_1 \cup U$. By induction, color $G[V_1 \cup U]$ from $L$. This gives a proper coloring $c$ of $U$. Now, in $G[V_2 \cup U]$, there is a face with vertex set $U$ having color lists of size 1 and all other vertices have color lists of size 5. Thus, by Theorem 3.13, $G[V_2 \cup U]$ is colorable from the corresponding lists.

Let $S = v_0v_1 \ldots v_m$ be a shortest $(u,v)$-path in $G$, with $v_0 = u$ and $v_m = v$, for $m \geq 2$. By Lemma 3.17 there is a nice coloring $c$ of $v_0v_1 \ldots v_{m-2}$. By Lemma 3.16 (4) there is at most one vertex adjacent to $v_{m-2}, v_{m-1}, v_m$ and at most one vertex adjacent to $v_{m-3}, v_{m-2}, v_{m-1}$, if $m \geq 3$. Let $c(v_{m-1}) \in L(v_{m-1}) - (\{c(v_{m-2})\} \cup L(v_m))$.

If there is no vertex $x$, with $x \sim \{v_{m-2}, v_{m-1}, v_m\}$, and no vertex $x$, with $x \sim \{v_{m-3}, v_{m-2}, v_{m-1}\}$, then $c$ is a nice coloring of $S$.

Assume that there is a vertex $y$, with $y \sim \{v_{m-3}, v_{m-2}, v_{m-1}\}$, and there is no vertex $x$, with $x \sim \{v_{m-2}, v_{m-1}, v_m\}$, or, the other way around, there is no vertex $x$, with $x \sim \{v_{m-3}, v_{m-2}, v_{m-1}\}$ and there is a vertex $y$, with $y \sim \{v_{m-2}, v_{m-1}, v_m\}$. Then $c$ is a proper coloring of $S$ such that $|L_c(p)| \geq 3$ for every $p \in N(S) - \{y\}$, and $|L_c(y)| \geq 2$. Deleting $S$ and the corresponding colors from the lists of their neighbors in $G - S$ produces a list assignment where all vertices in a face containing $N(S)$ have lists of size at least 3 (except for $y$), and all other vertices have lists of size 5. Using Thomassen’s 5-list-coloring theorem, $G - S$ can be colored from these lists. Together with the coloring $c$ of $S$, it gives a proper $L$-coloring of $G$.

Finally, assume there is a vertex $x$, with $x \sim \{v_{m-3}, v_{m-2}, v_{m-1}\}$, and there is a vertex $w$, with $w \sim \{v_{m-2}, v_{m-1}, v_m\}$. Note that there is at most one additional vertex adjacent
to $v_{m-1}$ and $v_m$, call it $z$ if it exists. Delete $S$ from $G$ and add two new adjacent vertices $t$ and $s$ in the resulting face, also add edges $xt, ws, tz, sz, ty_i$, where $y_i \in N(v_{m-1})$ and $sx_i$, where $x_i \in N(v_m)$. Choose two new colors $\alpha$ and $\beta$ not used in any of the lists assigned to vertices of $G$. Let $L'(t) = \{\alpha\}$, $L'(s) = \{\beta\}$, $L'(y_i) = L_c(y_i) \cup \{\alpha\}$, $L'(x_i) = L_c(x_i) \cup \{\beta\}$, $L'(z) = L_c(z) \cup \{\alpha, \beta\}$, $L'(x) = L_c(x) \cup \{\alpha\}$, and $L'(w) = L_c(w) \cup \{\beta\}$. For every other vertex of this modified graph, let $L'$ be equal to $L_c$. See Figure 3.9 for an illustration of this process. Observe that $L'$ satisfies the conditions of Thomassen’s 5-list-coloring theorem, so there is a proper $L'$-coloring of this graph. Thus, there is a proper $L'$-coloring of $G - S$, where no vertex uses colors $\alpha$ or $\beta$. This is a proper $L_c$-coloring of $G - S$. Together with the coloring $c$ of $S$, it gives a proper $L$-coloring of $G$.

\begin{proof}[Proof of Theorem 3.4] Let $T$ be a $(P,45)$-Steiner tree in $G'$, a Type I reduced graph of $G$ satisfying the conditions of the theorem. Let $L$ be an assignment of lists of colors to vertices of $G$ such that $|L(v)| = 1$ for $v \in P$ and $|L(v)| = 5$ for $v \notin P$. We first color $G'$, then extend it to a proper $L$-coloring of $G$.

To color $G'$, first color special vertices of $T$ which are not in $P$ arbitrarily from their lists. Let $S$ be the set of branches in $T$ and let $S \in S$ with endpoints $u_S, v_S$. Let $H(S, u_S, v_S) = H(S)$ be the graph obtained by applying Lemma 3.20 to $S$ and $c_S$ be a nice coloring of $H(S)$ from the corresponding lists (see Figure 3.10). Finally, let $c$ be a coloring of $H = \cup_{S \in S} H(S)$, such that $c(v) = c_S(v)$ if $v \in H(S)$.

Claim 1. The coloring $c$ is a nice coloring of $H$.\end{proof}
Let $x, x'$ be two vertices of $H$ that do not belong to the same $H(S)$. We shall prove that $x$ and $x'$ do not have common neighbors outside of $H$ and they are not adjacent. Let $x \in H(S)$, $x' \in H(S')$, $S, S' \in S$, $S \neq S'$.

If $x, x' \in V(T)$, then $x$ and $x'$ do not have a common neighbor outside of $T$ and they are not adjacent by part (4) of the definition of a $(P, d)$-Steiner tree.

If $x \in V(T), x' \not\in V(T)$, then $x' \in V(H(S')) - V(S')$, thus $\text{dist}(x', v_{c'}) \leq 21$, where $v_{c'}$ is a center of $S'$, as follows from Lemma 3.20. From part (3) of the definition of a $(P, d)$-Steiner tree, we have that $\text{dist}(v_{c'}, x) \geq d$. Thus $\text{dist}(x, x') \geq d - 21 \geq 3$ when $d \geq 24$.

Finally if $x, x' \not\in V(T)$, then $x \in V(H(S)) - V(S)$ and $x' \in V(H(S')) - V(S')$. Thus $\text{dist}(x, v_{c}) \leq 42$ and $\text{dist}(x', v_{c'}) \leq 42$, where $v_{c}, v_{c'}$ are centers of $S$ and $S'$, respectively. Moreover $\text{dist}(v_{c}, v_{c'}) \geq d$. Thus $\text{dist}(x, x') \geq d - 42 \geq 3$ if $d \geq 45$.

It follows that $c$ is a proper coloring of $H$. To show that $c$ is nice, consider a vertex $v$ adjacent to $H$. We see that $v$ is adjacent to non-special vertices of $H(S)$ for at most one branch $S$ of $T$. Since $c$ is a nice coloring of $H(S)$, it follows that $|L_{c}(v)| \geq 3$.

To conclude the proof of Claim 1, recall that $H$ is a connected graph containing all vertices of $P$. Proposition 3.15 implies that $G'$ is $L$-colorable. To show that $G$ is $L$-colorable, it is sufficient to observe the following.

![Figure 3.10: An example of the graph $H$ obtained in the proof of Theorem 3.4.](image)

**Claim 2.** Let $F$ be a graph, $P$ be a set of vertices, and $L$ be an assignment of lists of size 5 to vertices of $V(G) - P$ and lists of size 1 to vertices of $P$. Let $F' = R(F)$ be a reduction of $F$. If $F'$ has a proper coloring from lists $L$ then $F$ has a proper coloring from lists $L$.

Let $c$ be a proper coloring of $F'$ from lists $L$. 
If \( F' \) was obtained from \( F \) by removing the vertices in a region separated by 3-cycle or 4-cycle, these vertices can be colored properly from \( L \) using Theorem 3.13.

If \( F' \) was obtained from \( F \) by removing the set \( X \) of 4 vertices, \( y_1, y_2, z_1, z_2 \) of configuration \( D \), we see that \( |L_c(y_i)| \geq 2 \), \( i = 1, 2 \) for the two vertices \( y_1, y_2 \) of degree two in \( F[X] \) and \( |L_c(z_i)| \geq 3 \), \( i = 1, 2 \), for the two vertices \( z_1, z_2 \) of degree three in \( F[X] \). In the subgraph \( F[X] \) each vertex has list size equal to its degree under list assignment \( L_c \). An \( L_c \)-coloring of \( F[X] \) can be found directly or by the results of [44, 57]. Thus \( F \) has a proper coloring from lists \( L \).

If \( F' \) was obtained from \( F \) by removing the set \( X \) of 7 vertices \( w, x_1, \ldots, x_6 \) of configuration \( W \), then we see that \( |L_c(x_1)|, |L_c(x_4)| \geq 2 \), \( |L_c(x_2)|, |L_c(x_3)|, |L_c(x_5)|, |L_c(x_6)| \geq 3 \), and \( |L_c(w)| = 5 \). Let \( \alpha \in L_c(w) - (L_c(x_1) \cup L_c(x_4)) \), so color \( w \) with \( \alpha \) and remove \( \alpha \) from \( L_c(x_2), L_c(x_3), L_c(x_5), L_c(x_6) \). What remains to be colored is a 6-cycle with vertices having lists of size at least 2, which is colorable by the classification of all 2-list-colorable graphs by Erdős et al. [28]. Since \( F[X] \) is properly colorable from lists \( L_c \), \( F \) is properly colorable from lists \( L \).

This proves Claim 2.

Since \( G' \) was obtained from \( G \) via a sequence of reductions, the theorem follows.

---

**Proof of Theorem 3.6.** (1) Let \( L \) be an assignment of lists of colors to vertices of \( G \) such that \( |L(x)| = 5 \) for all \( x \notin P \) and \( |L(v_i)| = 1 \) for all \( v_i \in P \). If \( P \) is a set of vertices and edges with pairwise distance at least 3, then for all \( x \notin P \), \( x \) is adjacent to at most two vertices of \( P \). Thus, for every proper coloring \( c \) of \( G[P] \) from the corresponding lists \( L \) and for all \( x \notin P \), we have \( |L_c(x, P)| \geq 3 \). Moreover, \( N(P) \) belongs to the frontier of a face in \( G - P \). Thus, by Proposition 3.15, \( G \) is colorable from lists \( L \).

(2) Without loss of generality, assume \( C \) is on the unbounded face of \( G \). Let \( P = \{v_0, v_1, \ldots, v_{k-1}\} \subseteq C \) be a set of at most six precolored vertices on the boundary of \( C \). Fix an assignment \( L \) of lists of colors to the vertices of \( G \) with \( |L(v)| = 5 \) for all \( v \in V(G) - P \) and \( |L(v_i)| = 1 \) for all \( v_i \in P \). We shall show that \( G \) is \( L \)-colorable provided the three forbidden configurations are not present.

We shall create a new graph \( G' \) on the vertex set of \( G \) with new lists \( L' \). Let \( c_0, \ldots, c_{k-1} \)
be distinct colors not present in $L(v)$ for any $v \in V(G)$. Let $L'$ be a new list assignment with $L'(v_i) := \{c_i\}$ for $i = 0, \ldots, k - 1$ and $L'(v) = L(v) - S_v \cup S'_v$ for each $v \in V(G) - P$, where $S_v$ is the set of colors used in lists $L$ of vertices in $P \cap N(v)$ and $S'_v$ is an arbitrary subset of the set of colors used in lists $L'$ of vertices of $P \cap N(v)$, such that $|S'_v| = |S_v|$. In creating $L'$ we simply replaced the colors originally assigned to $P$ with new distinct colors, and replaced the old colors in the lists of vertices in the neighborhood of vertices of $P$.

Let a new plane graph $G'$ be obtained from $G$ by removing the edges $v_iv_{i+1}$ for $i = 0, \ldots, k - 1$ that correspond to non-consecutive vertices of $C$, and adding all edges $v_iv_{i+1}$ for $i = 0, \ldots, k - 1$ in the unbounded face of $G$. The resulting graph has a new unbounded face with vertex set $P$, and, perhaps, some new edges. By Theorem 7, $G'$ is $L'$-colorable by a coloring $c$ provided the three forbidden configurations are not present. Moreover, for any $v \not\in P$, we have $c(v) \not\in \{c_0, \ldots, c_k\} \cup S_v$, so $c(v) \in L(v)$ and $c(v) \not\in L(v_i)$ if $v \sim v_i$. To create a proper $L$-coloring of $G$, replace the color $c_i$ with an element of $L(v_i)$ for $i = 0, \ldots, k - 1$.

\[\square\]

Proof of Theorem 3.7. Delete $P$ and the corresponding colors from the lists of adjacent vertices. There are at most two faces, $F'_1$ or $F'_2$ and $F'_3$, in the graph $G - P$ such that the vertices adjacent to $P$ in $G$ belong to the boundaries of these two faces. These vertices have lists of size at least 4, and all other vertices in $G - P$ have lists of size at least 5. Call the resulting lists $L'$. Add a vertex $v_i$ to the face $F'_i$ and make it adjacent to all vertices on $F'_i$, $i = 1$, or $i = 1, 2$. Let $\alpha$ be a color not used in any of the lists $L(v), v \in V$. Let $L''(v_1) = L''(v_2) = \{\alpha\}, L''(v) = L'(v) \cup \{\alpha\}$, if $v \in V(F'_1 \cup F'_2)$ and $|L'(v)| = 4$. For all other vertices, let $L''(v) = L'(v)$. Applying Theorem 3.5 to the resulting graph with lists $L''$ allows for this graph to be properly colored from these lists. We note here that it is not hard to see that this new graph does not contain any $\{v_1, v_2\}$-separating 3-cycles or 4-cycles because such a separating 3-cycle or 4-cycle would have to be made up of vertices and edges from the original graph and would have separated some of the precolored vertices of $G$, a contradiction. This coloring gives a proper coloring of $G - P$ from lists $L'$, and thus it gives a proper coloring of $G$ from lists $L$. \[\square\]
3.4 Conclusions

We proved the question of Albertson has a positive answer if there are no short cycles separating precolored vertices and there is a nice tree containing precolored vertices.

We note here that by the definition of a \((P, d)\)-Steiner tree, Theorem 3.4 can be applied to plane graphs with precolored vertices that are not far apart. For example, let \(G\) be a 100-cycle with vertices \(v_0, v_1, \ldots, v_{99}\) and \(P = \{v_1, v_{50}, v_{98}\}\). Then \(G\) contains a \((P, 48)\)-Steiner tree obtained from deleting \(v_0, v_{99}\) and incident edges. The centers of the branches are far apart, but \(\text{dist}(v_1, v_{98}) = 3\).

We believe that in a planar triangulation either such a tree could always be found, or there are small reducible configurations such as shown in Figure 2.4. The reducible configurations \(D\) and \(W\) are just two in a family of many reducible configurations of those types, see Chapter 5. Modifying the definition of a reduced graph to include the removal of every reducible \(K_4 - e\) and every reducible 6-wheel leads us to the following question.

**Question 3.21.** Is it the case that every reduced planar triangulation with a set \(P\) of precolored vertices with \(\text{dist}(P) \geq 1000\) contains a \((P, 45)\)-Steiner tree?

If the above question has a positive answer, then by Theorem 3.4, the precoloring of \(P\) extends to a 5-list-coloring of \(G\). We did not strive to improve the constants here. With more careful calculations, one could easily obtain smaller constants.

The condition of no separating short cycles seems to be essential. It is important to note that the condition in Thomassen’s 5-list-coloring theorem that the two precolored vertices must be adjacent is essential. See Figure 3.11a. Also, the distance condition in this problem cannot be eliminated, even for a small number of precolored vertices. See Figure 3.11b. However, we conjecture that a precoloring of two far-apart vertices is always extendable to a 5-list-coloring of a planar graph.
(a) Non-extendable precoloring of two vertices at distance 2 where other vertices have lists of size 3.

(b) Non-extendable precoloring of three vertices at distance 2.

Figure 3.11: Non-extendable precolorings.
CHAPTER 4. \( \{2, 2\}\)-EXTENDABILITY OF PLANAR GRAPHS

4.1 Introduction

In this chapter, the idea of assigning lists of varying sizes to vertices of a planar graph will be explored.

Thomassen’s 5-list-coloring theorem [56] states that plane graphs are list-colorable when two adjacent vertices on the boundary of the unbounded face are precolored, other vertices on the boundary of the unbounded face are assigned lists of size 3, and all other vertices of the graph are assigned lists of size 5. This can be thought of as being 2-extendable. Thomassen also defined an analogous property of 3-extendability, see [58] and Definition 4.8, which corresponds to having the vertices of a 3-path along the boundary of the unbounded face precolored. While every planar graph is 2-extendable, it is not the case that every planar graph is 3-extendable. The following section will describe this notion in more detail.

In [38], Hutchinson defines the following notion of \( \{i, j\}\)-extendability.

**Definition 4.1.** Let \( G = (V, E) \) be a plane graph and let \( C \) be the cycle that corresponds to the boundary of the unbounded face of \( G \). Let \( x, y \in V(C) \) be two nonadjacent vertices of \( C \). Let \( L : V \to 2^\mathbb{N} \) be an arbitrary assignment of lists of colors to the vertices of \( G \) such that \( |L(x)| = i \), \( |L(y)| = j \), \( |L(v)| = 3 \) for all \( v \in V(C) - \{x, y\} \), and \( |L(w)| = 5 \) for all \( w \in V - V(C) \). If \( G \) is \( L \)-colorable for all such list assignments \( L \), then \( G \) is said to be \( (i, j) \)-extendable with respect to \( (x, y) \). If \( G \) is \( (i, j) \)-extendable with respect to \( (x, y) \) for every pair of vertices \( x, y \in V(C) \), then \( G \) is said to be \( \{i, j\}\)-extendable.

In [38], Hutchinson characterized all \( \{1, 1\}\)- and \( \{1, 2\}\)-extendable outerplanar graphs and showed that every outerplanar graph is \( \{2, 2\}\)-extendable. Here an alternate proof of the \( \{2, 2\}\)-
extendability of outerplanar graphs will be presented and the following conjecture, posed by Hutchinson, will be explored.

**Conjecture 4.2.** Plane graphs are \(2, 2\)-extendable.

This chapter contains results that provide some types of planar graphs that are \(2, 2\)-extendable.

Let \(x, y\) be vertices on the boundary of the unbounded face of a plane graph \(G\), where \(C\) is the cycle that corresponds to the boundary of the unbounded face of \(G\). Let \(T\) be the set of endpoints of all chords in \(G\). The induced subgraph \(G[T \cup \{x, y\}]\) is said to be an \(\{x, y\}\)-**skeleton** if it is a tree; in this case it is said that \(G\) contains an \(\{x, y\}\)-skeleton. See Figure 4.1 for an example, where the \(\{x, y\}\)-skeleton is shown in bold.

**Theorem 4.3.** Let \(G\) be a plane graph, let \(C = x_1x_2 \ldots x_kx_1\) be the cycle that corresponds to the boundary of the unbounded face of \(G\). Let \(x = x_1\) and \(y = x_j\) for some \(j \in \{2, \ldots, k\}\). Let \(\tilde{G}\) be a Type II reduced graph of \(G\) and let \(\tilde{C}\) be the cycle that corresponds to the boundary of the unbounded face of \(\tilde{G}\). If one of the following holds:

1. the distance between \(x\) and \(y\) in \(\tilde{G}[V(\tilde{C})]\) is at most 3, or
2. \(\tilde{G}\) contains an \(\{x, y\}\)-skeleton,

then \(G\) is \((2, 2)\)-extendable with respect to \((x, y)\).

Note that Theorem 4.3 (1) implies the following corollary. This follows because if the unbounded face of \(G\) has at most six vertices, then the distance between \(x\) and \(y\) in \(G[V(C)]\) is at most 3.
Corollary 4.4. Let \( G \) be a plane graph and let \( \tilde{G} \) be a Type II reduced graph of \( G \). Let \( \tilde{C} \) be the cycle that corresponds to the boundary of the unbounded face of \( \tilde{G} \). If \( |V(\tilde{C})| \leq 6 \), then \( G \) is \( \{2,2\}\)-extendable.

Theorem 4.5. Outerplane graphs and wheels are \( \{2,2\}\)-extendable.

Let \( \{2,2,2\}\)-extendable be defined analogously to \( \{2,2\}\)-extendable, except three vertices instead of two vertices on the boundary of the unbounded face are assigned lists of size 2. Note that if Conjecture 4.2 is true, the result cannot be strengthened without additional restrictions. This is because not all planar graphs are \( \{1,2\}\)-extendable, see Figures 4.2a and 4.2c, and not all planar graphs are \( \{2,2,2\}\)-extendable, see Figures 4.2b and 4.2d, even if the vertices with lists of size 2 are arbitrarily far apart. Note also that the graphs in Figures 4.2a and 4.2c and Figures 4.2b and 4.2d belong to infinite families of planar graphs which are not \( \{1,2\}\)-extendable or \( \{2,2,2\}\)-extendable, respectively. These graphs must be such that the lengths of the paths along the boundary of the unbounded face between the vertices with lists of size smaller than 3 must be congruent to 2 \( \mod 3 \) to get a graph that is not \( \{1,2\}\)-extendable, and the lengths of the paths along the boundary of the unbounded face between the vertices with lists of size 2 and the inner triangle must be congruent to 1 \( \mod 3 \) to get a graph that is not \( \{2,2,2\}\)-extendable when assigning lists of these types. Recall from earlier that Hutchinson [38] classified all \( \{1,2\}\)-extendable outerplanar graphs, so it was already known that not all planar graphs are \( \{1,2\}\)-extendable.

4.2 Preliminaries

This section contains some previously known results and Section 4.3 contains some new results that will be used in proving the main theorems of this chapter. Sections 4.4 and 4.5 will prove the main theorems. In Section 4.6, various properties of a minimal counterexample to Conjecture 4.2, if such a counterexample exists, will be presented.

It is a known result proven by Erdős et al. [28] and Borodin [16] that a graph \( G \) is list-colorable if the size of the list assigned to a vertex is at least the degree of that vertex for each vertex in \( G \), unless \( G \) is a Gallai tree and the lists have special properties.
The following is a result of Böhme, Mohar and Stiebitz which gives a weaker version of \{2, 2\}-extendability for planar graphs, where the lists are of size 4 along a path of the unbounded face. Similar results can also be found in [11] and [38]. See Figure 4.3b for a reference to the list sizes in this theorem.

**Theorem 4.6** (Böhme et al. [14]). Let $G = (V, E)$ be a plane graph, let $C$ be cycle that corresponds to the boundary of the unbounded face of $G$, and let $P = v_1v_2\ldots v_{k-1}v_k$ be a subpath of $C$. Let $L : V \to 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ such that $|L(v_i)| = 2$ for $i = 1, k$; $|L(v_i)| = 4$ for all $i \in \{2, \ldots, k-1\}$; $|L(v)| = 3$ for all $v \in V(C) - V(P)$; and $|L(w)| = 5$ for all $w \in V(G) - V(C)$. Then $G$ is $L$-colorable.
Figure 4.3: List sizes that indicate $L$-colorability.

It is also known that if all of the vertices on a small face of a plane graph are precolored, then it is extendable to a 5-list-coloring of the graph. This result is stated more precisely in the following theorem.

**Theorem 4.7** (Thomassen [58]). Let $G = (V, E)$ be a plane graph and let $C = v_1v_2 \ldots v_kv_1$ be the cycle that corresponds to the boundary of the unbounded face of $G$. Assume $k \leq 5$. Let $L : V \rightarrow 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ such that $|L(v_i)| = 1$ for all $i = 1, \ldots, k$ and $|L(v)| = 5$ for all $v \in V - V(C)$. If $G[V(C)]$ is $L$-colorable, then $G$ is $L$-colorable unless $k = 5$, $L(v_1)$ is distinct for each $i = 1, \ldots, 5$, and there is a vertex $u \in V - V(C)$ such that $u \sim v_i$ for $i = 1, \ldots, 5$ and $L(u) = L(v_1) \cup \ldots \cup L(v_5)$.

**Definition 4.8.** Let $G = (V, E)$ be a planar graph and $C = v_1v_2 \ldots v_kv_1$ be the cycle corresponding to one face of $G$. It is said that $G$ is 3-extendable with respect to the path $v_kv_1v_2$ if $G$ is $L$-colorable for any assignment $L$ of lists of colors to the vertices of $G$ in which $|L(v_i)| = 1$ for $i = 1, 2, k$; $|L(v_i)| \geq 3$ for $i = 3, \ldots, k - 1$; $|L(v)| \geq 5$ for $v \in V - V(C)$, and $G[v_k, v_1, v_2]$ is $L$-colorable.

As described in the following theorem, Thomassen showed that a planar graph $G$ is 3-extendable provided it does not have a subgraph that is a generalized wheel for which the boundary of its unbounded face is made up of vertices that lie on the boundary of the unbounded face of $G$. See Figure 4.3c for a reference to the list sizes in the following theorem.

**Theorem 4.9** (Thomassen [58]). Let $G$ be a near-triangulation and $C = v_1v_2 \ldots v_kv_1$ be the cycle that corresponds to the boundary of the unbounded face of $G$. Then $G$ is 3-extendable with
respect to $v_kv_1v_2$ unless there is a subgraph $G'$ of $G$ that is a generalized wheel with principal path $v_kv_1v_2$ and all other vertices that lie on the boundary of the unbounded face of $G'$ are elements of $V(C)$. Furthermore, if such a subgraph $G'$ exists and $G'$ is not a broken wheel, then for each list assignment $L$, there is at most one proper coloring of $G[v_1, v_2, v_k]$ for which $G$ is not 3-extendable with respect to $v_kv_1v_2$.

Assume $(c_k, c_1, c_2)$ is the unique proper precoloring of $v_kv_1v_2$ that is not 3-extendable, given that the obstruction is not a broken wheel. Call the triple $(c_k, c_1, c_2)$ the bad coloring of $v_kv_1v_2$ with respect to $(G, L)$ and call $c_i$ the bad color of $v_i$, for $i = 1, 2, k$, with respect to the corresponding bad coloring of $v_kv_1v_2$, $G$, and $L$. For convenience, given a path $P = v_kv_1v_2$, let $C_P = C_{v_kv_1v_2} = C_{v_kv_1v_2}(G, L)$ denote the ordered triple that is the bad coloring of $v_kv_1v_2$ with respect to $(G, L)$.

**Definition 4.10.** It is said that a coloring $c$ of $P$ avoids $C_P$ if, given $P = v_kv_1v_2$ and $C_P = (c_k, c_1, c_2)$, then $c(v_i) \neq c_i$ for some $i \in \{1, 2, k\}$. Additionally, it is said that, for some $i \in \{1, 2, k\}$, a color $c$ of $v_i$ avoids $C_P$ if $c(v_i) \neq c_i$.

If $G$ is an odd wheel, then Figure 4.4 illustrates the list assignment that corresponds to the bad coloring of $v_kv_1v_2$ that is not 3-extendable for $W_5$. This list assignment may be generalized for any odd wheel by assigning the list \{a, d, e\} to any additional vertices.

![Figure 4.4: Unique non-3-extendable precoloring of an odd wheel.](image)

### 4.3 New results

**Lemma 4.11.** Consider a triangle $(x_1, x_2, x_3)$ with lists $L$ of sizes 2, 3, 3 assigned to $x_1, x_2, x_3$, respectively. Then there are at least three $L$-colorings of this triangle such that the ordered pairs of colors assigned to $x_2, x_3$ are distinct.
Proof. Let $G$ be the triangle $(x_1, x_2, x_3)$ and assume $L(x_1) = \{\alpha, \beta\}$. Let
\[
S = \{(c(x_1), c(x_2), c(x_3)) : c \text{ is a proper } L\text{-coloring of } G\}.
\]
Let $\{\gamma, \gamma'\} \subseteq L(x_2) - \{\beta\}$, then $S' = \{(\beta, \gamma, q) : q \in L(x_3) - \{\beta, \gamma\}\} \cup \{(\beta, \gamma', q) : q \in L(x_3) - \{\beta, \gamma'\}\} \subseteq S$. Note here that $\gamma$ or $\gamma'$ can be $\alpha$. If $|S'| \geq 3$, the lemma follows. Otherwise, $|L(x_3) - \{\beta, \gamma\}| = 1$ and $|L(x_3) - \{\beta, \gamma'\}| = 1$ implying that $L(x_3) = \{\beta, \gamma, \gamma'\}$. Without loss of generality, assume $\gamma \in \{\gamma, \gamma'\} - \{\alpha\}$. Then $S \supseteq \{(\beta, \gamma, \gamma'), (\beta, \gamma', \gamma), (\alpha, \gamma, \beta)\}$ and the lemma follows. \qed

Lemma 4.12. Let $(u, v, w)$ be a triangle. Assume there are three distinct ordered pairs of colors $(a_i, b_i)$ for $i = 1, 2, 3$ that can be assigned to the vertices $w, u$. If $v$ is assigned a list $L(v)$ of three colors, then there are at least three distinct ordered pairs of colors $(b_i, c_i)$, $i = 1, 2, 3$, that can be assigned to the vertices $u, v$ for which $c_i \in L(v) - \{a_i, b_i\}$ for $i = 1, 2, 3$.

Proof. It is not hard to see that there exists $c_i \in L(v) - \{a_i, b_i\}$ for $i = 1, 2, 3$. Consider the pairs $(b_1, c_1)$, $(b_2, c_2)$, $(b_3, c_3)$. It remains to show that these three pairs are distinct. Assume $|L(v) - \{a_i, b_i\}| = 1$ for all $i = 1, 2, 3$. Otherwise, there are more than three pairs and the result follows more easily. Without loss of generality, assume $b_1 = b_2$ and $c_1 = c_2$. This implies two of the pairs for $w, u$ are actually $(a_1, b_1)$, $(a_2, b_1)$. It then follows that $L(v) - \{a_1, b_1\} = c_1 = L(v) - \{a_2, b_1\}$, hence $a_1 = a_2$. This is a contradiction, as it was assumed that $(a_1, b_1)$ and $(a_2, b_2)$ are distinct pairs. Thus, the pairs $(b_1, c_1)$, $(b_2, c_2)$, $(b_3, c_3)$ are distinct and the lemma follows. \qed

The following lemma will be used to show that outerplane graphs are $\{2, 2\}$-extendable.

Lemma 4.13. Let $G$ be an outerplane near-triangulation with vertex $x$ of degree 2. Let $L$ be an assignment of lists of colors to the vertices of $G$ such that $|L(x)| = 2$ and $|L(w)| = 3$ for all $w \in V(G) - \{x\}$. For any edge $uv$ on the unbounded face of $G$, there are at least three $L$-colorings of $G$ such that the ordered pairs of colors assigned to $u, v$ are distinct.

Proof. If $x \in \{u, v\}$, then there are at least three distinct proper colorings of $G[\{u, v\}]$ that are each extendable to $L$-colorings of $G$ by Theorem 1.13. Thus, assume $x \notin \{u, v\}$. 

The proof is by induction on $|V(G)|$. If $G$ has three vertices, the result follows from Lemma 4.11.

Before proceeding, it can be assumed that $G$ does not contain any non-$\{x, u, v\}$-separating chords, otherwise the lemma follows by induction and Theorem 1.13.

Assume the result holds for all outerplane near-triangulations on less than $n$ vertices with list assignments as described in the hypotheses of the lemma. Now consider an outerplane graph $G$ such that $|V(G)| = n$ and choose an arbitrary edge $uv$ on the unbounded face of $G$ for which $x \not\in \{u, v\}$. Since $G$ is a near-triangulation and there is no non-$\{x, u, v\}$-separating chord in $G$, there is a $w \in V(G)$ such that $(u, v, w)$ is a triangle in $G$ and either $vw$ is an edge on the unbounded face of $G$ or $uw$ is an edge on the unbounded face of $G$. Without loss of generality, assume $vw$ is an edge on the unbounded face of $G$. Consider the graph $G - v$ with lists $L$.

Since $|V(G - v)| = n - 1$ and $uw$ is an edge on the unbounded face of $G - v$, there are at least three $L$-colorings of $G - v$ for which the ordered pairs of colors assigned to $w, u$ are distinct by induction. Let these pairs be $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ where $a_i$ is the color from $L(w)$ assigned to $w$ and $b_i$ is the color from $L(u)$ assigned to $u$ for $i = 1, 2, 3$.

The result then follows by Lemma 4.12. □

In Theorem 4.3, the idea of a reduced graph is used. The following lemma illustrates why this notion is helpful.

**Lemma 4.14.** Let $G$ be a plane graph, $x, y \in V(G)$ be vertices on the unbounded face $C$ of $G$, and $L$ an assignment of lists of colors to the vertices of $G$ such that $|L(x)| = |L(y)| = 2$, $|L(v)| = 3$ for all $v \in V(C) - \{x, y\}$, and $|L(w)| = 5$ for all $w \in V(G) - V(C)$. Let $\tilde{G} = R(G)$ be a Type II reduction of $G$ with respect to $x, y$. If $\tilde{G}$ is $L$-colorable, then $G$ is $L$-colorable.

**Proof.** If $\tilde{G}$ was obtained from $G$ by removing a separating 3-cycle with vertex set $X$ for which $X'$ is the vertex set of the connected component of $G - X$ which contains neither $x$ nor $y$ and $c$ is an $L$-coloring of $G'$, then $c$ may be extended to an $L$-coloring of $G$. Let $L'(w) = L(w)$ for all $w \in X'$ and $L'(z) = \{c(z)\}$ for all $z \in X$, then $G[X \cup X']$ is $L'$-colorable by Theorem 4.7. Thus $G$ is $L$-colorable. If $\tilde{G}$ was obtained from $G$ by letting $\tilde{G} = R(G) = G_A$ where $uv$ is a non-$\{x, y\}$-separating chord that splits $G$ into two graphs $G_A$ and $G_B$ such that $G = G_A \cup G_B,$
Theorem 4.15. Let $G = (V, E)$ be a plane graph and let $C$ be the cycle that corresponds to the boundary of the unbounded face of $G$. Let $x, y \in V(C)$ be two nonadjacent vertices of $C$. Let $L : V \to 2^\mathbb{N}$ be an arbitrary assignment of lists of colors to the vertices of $G$ such that $|L(x)| = |L(y)| = 2$, $|L(v)| = 3$ for all $v \in V(C) - \{x, y\}$, and $|L(w)| = 5$ for all $w \in V - V(C)$. Let $\tilde{G}$ be the Type II reduced graph of $G$ with respect to $x, y$. If $\tilde{G}$ is $L'$-colorable, then $G$ is $L$-colorable.

This theorem follows from Lemma 4.14 because the Type II reduced graph of $G$ is obtained from $G$ via a series of Type II reductions.

4.4 Proof of Theorem 4.3

One of the main tools used in the following proof will be Theorem 4.9 and the notion of 3-extendability. A caterpillar is a tree in which all vertices of the graph are on or incident to a path which contains every vertex of degree at least two.

Proof of Theorem 4.3. Observe first that by Corollary 4.15, if the Type II reduced graph of $G$ is $L$-colorable, then $G$ is $L$-colorable. Thus, assume that $G$ is a Type II reduced graph with respect to $x, y$ for the remainder of the proof.

1. Without loss of generality, assume $G$ is a near-triangulation.

   (a) If $j \in \{2, k\}$, then the result follows from Theorem 1.13.

   (b) If $j \in \{3, k - 1\}$, assume $j = 3$. Add two adjacent vertices $s$ and $t$ with $s \sim \{x, x_2\}$ and $t \sim \{x_2, y\}$ so that $s$ and $t$ now lie on the cycle that corresponds to the unbounded face. Call this new graph $G'$. Let $a$ and $b$ be two colors not in any of the lists $L$. Assign to the vertices of $G'$ the lists $L'$ where $L'(s) = \{a\}$, $L'(t) = \{b\}$, $L'(x) = L(x) \cup \{a\}$, $L'(x_2) = L(x_2) \cup \{a, b\}$, $L'(y) = L(y) \cup \{b\}$, and $L'(w) = L(w)$.
for all \( w \in V(G) - \{x, x_2, y\} \). By Theorem 1.13, \( G' \) is \( L \)-colorable and it follows that \( G \) is \( L \)-colorable.

(c) If \( j \in \{4, k - 2\} \), assume \( j = 4 \). Add a 3-path \( s t u \) with \( s \sim \{x, x_2\} \), \( t \sim \{x_2, x_3\} \) and \( u \sim \{x_3, y\} \) so that \( s t u \) now lies on the cycle that corresponds to the unbounded face, see Figure 4.5. Call this new graph \( G' \) and let \( C' \) be the cycle that corresponds to the unbounded face. Let \( a, b \) and \( c \) be three colors not in any of the lists \( L \). Assign to the vertices of \( G' \) the lists \( L' \), where \( L'(s) = \{a\} \), \( L'(t) = \{b\} \), \( L'(u) = \{c\} \), \( L'(x) = L(x) \cup \{a\} \), \( L'(x_2) = L(x_2) \cup \{a, b\} \), \( L'(x_3) = L(x_3) \cup \{b, c\} \), \( L'(y) = L(y) \cup \{c\} \), and \( L'(w) = L(w) \) for all \( w \in V(G) - \{x, x_2, x_3, y\} \). If \( G' \) is 3-extendable with respect to \( s t u \), then that \( L' \)-coloring of \( G' \) provides an \( L \)-coloring of \( G \). Thus, it remains to verify that \( G' \) does not contain a subgraph \( H \) that is a generalized wheel with principal path \( s t u \) and vertices of outercycle on \( C' \).

Assume such a subgraph \( H \) exists in \( G' \). Observe that \( H \) cannot be a broken wheel because \( t \) does not have any neighbors with lists of size 3. Additionally, \( H \) cannot be a wheel because there is no vertex \( z \) in \( G' \) such that \( z \sim \{s, t, u\} \). Thus, \( H \) must be a generalized wheel formed by identifying principal edges of two wheels as seen in Figure 2.1c. However, this would require \( t \) to have degree 5 in \( H \), a contradiction because \( t \) is of degree 4 in \( G' \). So by Theorem 4.9, \( G' \) is 3-extendable with respect to \( s t u \).

2. Let \( G_T \) be the \( \{x, y\} \)-skeleton of \( G \). Besides the fact that \( G_T \) is a tree, some additional observations may be made. First, \( G_T \) is indeed a caterpillar. There is also an underlying linear ordering of the chords of \( G \). Consider the weak dual of \( G[C] \). This graph is a
path $w_1w_2\ldots w_m$ whose endpoints correspond to the bounded faces of $G$ that contain $x$ and $y$, respectively. Let $G_i$, $i = 1, \ldots, m$ be the subgraph of $G$ whose unbounded face has boundary that is the cycle corresponding to the vertex $w_i$ in the weak dual of $G[C]$. Additionally, each $w_i$ in the vertex set of the weak dual of $G[C]$ corresponds to $P_i$ with vertices $u_{i-1}, u_i, u_{i+1}$ in $G_T$. Let $C_{P_i}$ be the the bad coloring of $u_{i-1}u_iu_{i+1}$ with respect to $(G_i, L)$. As noted earlier, for each $P_i$, there is a $C_{P_i}$ for which the precoloring of $P_i$ does not extend to a proper $L$-coloring of $G_i$. Note that $x \in P_1$ and $y \in P_m$ uniquely. Say an $\{x,y\}$-skeleton has a “good” $L$-coloring if for all $G_i$, $i = 1, \ldots, m$, the corresponding $P_i$ can all be simultaneously $L$-colored so that $P_i$ avoids $C_{P_i}$.

**Claim 4.16.** If $G_T$ has a “good” $L$-coloring, then $G$ is $L$-colorable.

The claim holds because the “good” $L$-coloring of $G_T$ may be extended to an $L$-coloring of $G$ by Theorem 4.9 applied to each $G_i$.

**Claim 4.17.** $G_T$ has a “good” $L$-coloring.

**Proof of Claim 4.17.** By induction on $m$. If $m = 1$, then there are no chords and $x \sim y$, so the results follows by Theorem 1.13.

So assume the result holds for $m - 1$ and consider $G_T$. Without loss of generality, assume $P_m = w_{m-1}w_my$ and $C_{P_m} = (c_{m-1}, c_m, c_y)$. Let $G'_T = G_T - \{y\}$. Let $L'(w_m) = L(w_m) - \{c_m\}$ and $L'(w) = L(w)$ for all $w \in V(G'_T) - \{w_m\}$. By induction, there is a “good” $L'$-coloring $c$ of $G'_T$. This can be extended to a “good” $L$-coloring of $G_T$ by assigning to $y$ a color from $L(y) - \{c(w_m)\}$. □

By the above two claims, $G$ is $L$-colorable. □

### 4.5 Proof of Theorem 4.5

The following lemma will be used in the proof of Theorem 4.5.
Lemma 4.18. Let \( C = x_1x_2 \ldots x_{2k+1}x_1 \) be an odd cycle and \( L : V(C) \rightarrow 2^\mathbb{N} \) be an assignment of lists of colors to the vertices of \( C \). Then the following results are true:

1. If \(|L(x_i)| = 2\) for all \( i = 1, \ldots, 2k + 1\), then \( C \) is \( L \)-colorable unless all of the lists are identical.

2. Let \( x = x_1 \) and \( y = x_j \) for some \( j \in \{3, 2k - 1\} \). If \( L \) is such that \(|L(x)| = |L(y)| = 2\) and \(|L(x_i)| = 3\) for all \( i \neq 1, j \), then \( G \) is \( L \)-colorable.

Proof. 1. Assume all of the 2-lists are not identical. Then there are \( x_i, x_{i+1} \) adjacent with nonidentical lists. If \( L(x_i) \cap L(x_{i+1}) = \emptyset \), then delete \( x_i x_{i+1} \). This leaves a path which is 2-list-colorable. The corresponding coloring will be proper in \( C \) as well. So assume there is \( a \in L(x_i) \cap L(x_{i+1}) \). Without loss of generality, assume \( L(x_i) = \{a, b\} \) and \( L(x_{i+1}) = \{a, c\} \). Assign \( b \) to \( x_i \), then greedily color the other vertices of \( C \) in the following order: \( x_{i-1}x_{i-2} \ldots x_1x_{2k+1}x_2 \ldots x_{i+1} \). Since \( b \not\in L(x_{i+1}) \), \( c(x_{i+1}) \neq b \) and the coloring of \( C \) will be proper. The result follows.

2. This result follows from part (1) because one can always choose two element sublists of the given 3-lists so that they are not all identical.

Proof of Theorem 4.5. Without loss of generality, assume \( G = (V, E) \) is a near-triangulation. Let \( C \) be the cycle that corresponds to the boundary of the unbounded face of \( G \). Let \( x, y \in V(C) \) be two nonadjacent vertices of \( C \). Let \( L : V \rightarrow 2^\mathbb{N} \) be an arbitrary assignment of lists of colors to the vertices of \( G \) such that \(|L(x)| = i, |L(y)| = j, |L(v)| = 3\) for all \( v \in V(C) - \{x, y\} \), and \(|L(w)| = 5\) for all \( w \in V - V(C) \).

1. Let \( G \) be an outerplane graph, it contains two vertices of degree 2. It can be assumed that these two vertices are \( x \) and \( y \). If \( z \not\in \{x, y\} \) is a vertex of degree 2 in \( G \) then the graph \( G' = G - z \) is \( L \)-colorable by induction on the number of vertices in \( G \) and the coloring extends to \( z \) because \(|L(z)| \) is greater than the degree of \( z \). So, \( x \) and \( y \) are the only two vertices of degree 2.
It can also be assumed that \( x \sim y \), otherwise the result follows by Theorem 1.13. So \( y \) is a vertex of degree 2 with neighbors \( u \) and \( v \) which are adjacent.

Apply Lemma 4.13 to the graph \( G' = G - y \) with respect to the edge \( uv \). This yields three proper \( L \)-colorings of \( G' \) and three distinct pairs \( (a_1, b_1), (a_2, b_2), (a_3, b_3) \) such that \( c(u) = a_i \) and \( c(v) = b_i \), for \( i = 1, 2, 3 \), corresponding to each coloring. As these pairs are distinct, at least one of them is such that \( L(y) - \{a_i, b_i\} \neq \emptyset \). Without loss of generality, assume this pair is \( (a_1, b_1) \). Thus, the coloring \( c \) for which \( c(u) = a_1 \) and \( c(v) = b_1 \) can be extended to \( y \) by assigning to it the available color in \( L(y) - \{a_1, b_1\} \). The theorem follows since this will hold for all pairs of vertices \( x, y \in V(C) \) and all such list assignments \( L \).

2. Let \( G \) be a wheel and let \( u \) be the center of \( G \). Since \( |L(u)| = 5 \), there is \( a \in L(u) - \{L(x) \cup L(y)\} \). Let \( c(u) = a \). Delete \( u \) and remove \( a \) from the lists of adjacent vertices to create lists \( L' \) for \( G - \{u\} \).

Case 1: \( G - \{u\} \) is an even cycle.

Its lists are of size at least 2. Even cycles are 2-list-colorable by the classification of all 2-list-colorable graphs by Erdős et al. [28], so the claim follows.

Case 2: \( G - \{u\} \) is an odd cycle.

It is list-colorable, unless \( |L'(w)| = 2 \) for all \( w \in G - \{u\} \) and \( L'(w) = L'(w') \) for all \( w, w' \in G - \{u\} \). If this is the case, without loss of generality, assume \( L(x) = \{b, c\} = L(y) \), \( L(u) = \{a, b, c, d, e\} \) and \( L(v) = \{a, b, c\} \) for all \( v \in V(G) - \{u, x, y\} \). Thus, instead let \( c(u) = d \) and proceed as before by deleting \( u \) and removing \( d \) from the lists of adjacent vertices to create lists \( L' \) for \( G - \{u\} \). This will yield an odd cycle in which all but two vertices have lists of size at least 3. It is not hard to see that such a graph is list-colorable, see Lemma 4.18.

Thus, wheels are \( \{2, 2\} \)-extendable.
4.6 Properties of a minimal counterexample to Conjecture 4.2

If Conjecture 4.2 is false, then there is a graph $G$ for which there is at least one pair of nonadjacent vertices $x$ and $y$ on the unbounded face of $G$ for which $G$ is not $(2, 2)$-extendable with respect to $(x, y)$. This section explores the structure of such a graph that is a minimal counterexample and shows that if such a minimal counterexample exists, it falls into one of three specific cases.

**Theorem 4.19.** Let $G = (V, E)$ be a plane graph that is not $(2, 2)$-extendable and $G$ has the least number of vertices. Let $C = x_1x_2 \ldots x_kx_1$ be the cycle that corresponds to the boundary of the unbounded face of $G$. Then, without loss of generality, $G$ is not $(2, 2)$-extendable with respect to $(x, y)$ and a certain list assignment $L : V \to 2^N$, such that $|L(x)| = |L(y)| = 2$, where $x = x_1$ and $y = x_j$ for some $j \in \{3, \ldots, k - 1\}$; $|L(x_i)| = 3$ for $i \neq 1, j$; $|L(v)| = 5$ for all $v \in V - V(C)$, and one of the following is true:

1. $\deg(x) = 2$, $L(x) \subseteq L(x_2) = L(x_3) = L(x_{k-1}) = L(x_k)$, or

2. $\deg(x) = 3$, $L(x) \subseteq L(x_2) = L(x_3)$, $L(x) \subseteq L(x_{k-1}) = L(x_k)$, there is a vertex $v \in V(G) - F$ such that $v \sim \{x, x_2, x_k\}$, there is a path $x_{k-1}v_1 \ldots v_mv$ with internal vertices from $V(G) - F$ such that $G[x_k, x_{k-1}, v_1, \ldots, v_m, v]$ is a broken wheel with principal path $vx_kx_{k-1}$, and there is a path $x_3v'_1 \ldots v'_mv$ with internal vertices from $V(G) - F$ such that $G[x_2, x_3, v'_1, \ldots, v'_m, v]$ is a broken wheel with principal path $vx_2x_3$, or

3. $\deg(x) > 3$, $L(x) \subseteq L(x_2) = L(x_3)$, $L(x) \subseteq L(x_{k-1}) = L(x_k)$, there are two vertices $u, v \in V(G) - F$ such that $u \sim \{x, x_2\}$ and $v \sim \{x, x_k\}$, there is a path $x_{k-1}v_1 \ldots v_mv$ with internal vertices from $V(G) - F$ such that $G[x_k, x_{k-1}, v_1, \ldots, v_m, v]$ is a broken wheel with principal path $vx_kx_{k-1}$, and there is a path $x_3u_1 \ldots u_mv$ with internal vertices from $V(G) - F$ such that $G[x_2, x_3, u_1, \ldots, u_m, u]$ is a broken wheel with principal path $ux_kx_{k-1}$.

**Proof.** Without loss of generality, assume $G$ is a near-triangulation. The proof of this theorem will be a series of claims showing that $G$ is not a minimal counterexample unless conditions (1), (2), or (3) from above hold.
Claim 1. $G = R(G)$.

If $R(G) \neq G$, it follows that $R(G)$ is $(2, 2)$-extendable with respect to $(x, y)$. However, Corollary 4.15 implies that $G$ will also be $(2, 2)$-extendable with respect to $(x, y)$, a contradiction.

So $G$ does not contain any separating 3-cycles or any non-$\{x, y\}$-separating chords.

Claim 2. If the distance between $x$ and $y$ in $G[V(C)]$ is at most 3, then $G$ is $(2, 2)$-extendable with respect to $(x, y)$.

This follows from Theorem 4.3(1).

Claim 3. If $G$ contains a chord $x_2x_l$ for some $l \in \{j + 1, \ldots, k - 1\}$ and $G[x, x_2, x_l, \ldots, x_k]$ is not a broken wheel, then $G$ is $(2, 2)$-extendable with respect to $(x, y)$.

Note that $x_2x_l$ is an $\{x, y\}$-separating chord. The chord $x_2x_l$ splits $G$ into two graphs, say $G_1$ and $G_2$, for which $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{x_2, x_l\}$. Let $G_1$ be the graph containing $y$. Let $L'$ be a list assignment for $G_1$ such that $L'(u) = L(u)$ for all $u \in V(G_1) - \{x_2\}$ and $L'(x_2) = L(x_2) - \{a\}$, where $a$ is the bad color of $x_2$ in $C_{x_2x_l}(G_2, L)$. By the minimality of $G$, it follows that $G_1$ is $(2, 2)$-extendable with respect to $(x_2, y)$ using lists $L'$. This coloring must be extended to $G_2$. Let $L''$ be a list assignment for $G_2$ such that $L''(x_2)$ and $L''(x_l)$ consist of the colors assigned to them in the $L'$-coloring of $G_1$, respectively, $L''(x) = L(x) - L''(x_2)$, and all other lists agree with $L$. By Theorem 4.9, this is 3-extendable with respect to $x_2x_l$.

Note that 3-extendability can be used here because $x \sim x_l$. If $x \sim x_l$, then $G$ would contain a non-$\{x, y\}$-separating chord $xx_l$. Also, the assumption that $G[x, x_2, x_l, \ldots, x_k]$ is not a broken wheel ensures that $G_2$ does not contain a subgraph whose vertices lie on the unbounded face of $G_2$ that is a broken wheel with principal path $xx_2x_l$. For this reason, the bad color $a$ of $x_2$ in $C_{x_2x_l}(G_2, L)$ is unique.

Observe that Claim 3 implies the analogous result for such chords $x_kx_l$, $x_{j-1}x_l$ and $x_{j+1}x_l$. 
Claim 5. If there is a vertex \( v \) in \( G \) that is a vertex of degree 3 in \( G \), then it is not hard to see that \( \deg(v) = 2 \). This is because \( x_2x_k \) must be a chord and if \( x \) had any additional neighbors, then \( \{x, x_2, x_k\} \) forms a separating 3-cycle, a contradiction. Thus, if \( \deg(x) > 2 \), then Claim 3 will apply to any chord \( x_2x_i \). This implies the analogous result for \( y \).

Claim 4. If \( G \) contains a separating 3-path \( x_ix_l \) that does not separate \( \{x, y\} \) and the graph \( G[v, x_i, \ldots, x_l] \) is not a broken wheel, then \( G \) is \((2, 2)\)-extendable with respect to \( (x, y) \).

First note that \( v = x_m \) for any \( m \in \{1, \ldots, k\} \), otherwise at least one of \( x_v, x_vx_l \) would be a non-\( \{x, y\} \)-separating chord. The 3-path \( x_vx_l \) splits \( G \) into two graphs \( G_1 \) and \( G_2 \) for which \( G = G_1 \cup G_2 \) and \( V(G_1) \cap V(G_2) = \{x_v, x_l\} \). Assume \( x, y \in V(G_1) \). Let \( L' \) be a list assignment for \( G_1 \) such that \( L'(u) = L(u) \) for all \( u \in V(G_1) \setminus \{v\} \) and \( L'(v) = L(v) - \{a\} \), where \( a \) is the bad color of \( v \) in \( C_{x_vx_l}(G_2, L) \). By the minimality of \( G, G_1 \) is \((2, 2)\)-extendable with respect to \( (x, y) \) using lists \( L' \). This coloring must be extended to \( G_2 \). Let \( L'' \) be a list assignment for \( G_2 \) such that \( L''(x_v), L''(v), L''(x_l) \) consist of the colors assigned to them in the \( L' \)-coloring of \( G_1 \), respectively, and all other lists agree with \( L \). This is \( 3 \)-extendable with respect to \( x_vx_l \) by Theorem 4.9. It is important to note here that the assumption that \( G[v, x_i, \ldots, x_l] \) is not a broken wheel ensures that \( G_2 \) does not contain a subgraph whose vertices lie on the unbounded face of \( G_2 \) that is a broken wheel with principal path \( x_vx_l \). For this reason, the bad color \( a \) of \( v \) in \( C_{x_vx_l}(G_2, L) \) is unique.

Claim 5. If there is a vertex \( w \in V(G) - F \) such that \( w \sim \{x_i, x_{i+1}, x_{i+2}\}, i + 1 \notin \{1, j\}, \) and \( L(x_i) \subseteq L(x_{i+2}) \) or \( L(x_i) \supseteq L(x_{i+2}) \), then \( G \) is \((2, 2)\)-extendable with respect to \( (x, y) \).

Without loss of generality, assume there is a \( w \sim \{x, x_2, x_3\} \). Let \( G' \) be the graph obtained by contracting the path \( xx_2x_3 \) into a new vertex \( x' \). Let \( L' \) be a list assignment for \( G' \) such that \( L'(x') = L(x) \) and \( L'(v) = L(v) \) for \( v \neq x' \). By the minimality of \( G \), the graph \( G' \) is \((2, 2)\)-extendable with respect to \( (x', y) \) from lists \( L' \). Let \( c \) be such a coloring of \( G' \) and consider this coloring in \( G \), where \( c(x) = c(x_3) = c(x') \). Since \( x_2 \) is a vertex of degree 3 in \( G \) with a list of size 3 and two of its neighbors are assigned the same color, the coloring \( c \) may be extended to \( x_2 \).
Claim 6. If \( L(x) \not\subseteq L(x_2) \) or \( L(x) \not\subseteq L(x_k) \), then \( G \) is \((2,2)\)-extendable with respect to \((x,y)\).

Without loss of generality, assume \( L(x) \not\subseteq L(x_2) \) and there exists \( a \in L(x) - L(x_2) \). Assign \( a \) to \( x \), remove \( a \) from all lists of vertices adjacent to \( x \), and delete \( x \) from \( G \). Call these new lists \( L' \) and the new graph \( G' \). Observe that \( |L'(x_2)| = 3 \), \( |L'(x_k)| \geq 2 \), and other modified lists on the unbounded face of \( G' \) have size at least 3. By the minimality of \( G \), the graph \( G' \) is \((2,2)\)-extendable with respect to \((x_k,y)\).

Thus, it may be assumed that \( L(x) \subseteq L(x_2) \) and \( L(x) \subseteq L(x_k) \). Similarly, it may be assumed that \( L(y) \subseteq L(x_{j-1}) \) and \( L(y) \subseteq L(x_{j+1}) \).

Claim 7. If \( L(x_2) \neq L(x_3) \) or \( L(x_k) \neq L(x_{k-1}) \), then \( G \) is \((2,2)\)-extendable with respect to \((x,y)\).

Without loss of generality, assume \( L(x_2) \neq L(x_3) \), so there is \( a \in L(x_2) - L(x_3) \). Assign color \( a \) to vertex \( x_2 \), then assign to \( x \) a color \( b \) from \( L(x) - \{a\} \) and to \( x_k \) a color from \( L(x_k) - \{a,b\} \). Remove the colors assigned to vertices \( x_2, x, x_k \) from the lists of vertices adjacent to them and delete \( x_2, x, x_k \) from \( G \). Call these new lists \( L' \) and the new graph \( G' \). Observe that \( L'(x_3) = L(x_3), \) so \( |L'(x_3)| = 3, \) \( |L'(x_{k-1})| \geq 2 \) and modified lists of vertices on the unbounded face of \( G' \) have size at least 3. Thus, \( G' \) is \((2,2)\)-extendable with respect to \((x_{k-1},y)\) from lists \( L' \).

Therefore, it can be assumed that \( L(x_2) = L(x_3) \) and \( L(x_k) = L(x_{k-1}) \). By symmetry, it can also be assumed that \( L(x_{j-1}) = L(x_{j-2}) \) and \( L(x_{j+1}) = L(x_{j+2}) \).

What follows is a case analysis that looks at, without loss of generality, \( \deg(x) \).

Case 1: \( \deg(x) = 3 \).

If \( \deg(x) = 3 \), then there is a vertex \( v \in V(G) - F \) such that \( v \sim \{x,x_2,x_k\} \). This follows because \( G \) is a near-triangulation and \( \deg(x) = 3 \), otherwise one of the earlier claims would
be contradicted. In addition, without loss of generality, one of the following two subcases will occur:

1. \( x_{k-1} \sim v \), or
2. there is a path \( x_{k-1}v_1 \ldots v_mv \) with internal vertices from \( V(G) - F \) such that \( G[x_k, x_{k-1}, v_1, \ldots, v_m, v] \) is a broken wheel with principal path \( vx_kx_{k-1} \).

*Case 2:* \( \deg(x) > 3 \).

If \( \deg(x) > 3 \), then there are two vertices \( u, v \in V(G) - F \) such that \( u \sim \{x, x_2\} \) and \( v \sim \{x, x_k\} \). This follows because \( G \) is a near-triangulation and all of the previous claims must be satisfied. Additionally, there may be vertices \( w_1, \ldots, w_i \) from \( V(G) - F \) that are also adjacent to \( x \). These vertices will also be the internal vertices on a path from \( u \) to \( v \). As in Case 1, without loss of generality, one of the following two subcases will occur:

1. \( x_{k-1} \sim v \), or
2. there is a path \( x_{k-1}v_1 \ldots v_mv \) with internal vertices from \( V(G) - F \) such that \( G[x_k, x_{k-1}, v_1, \ldots, v_m, v] \) is a broken wheel with principal path \( vx_kx_{k-1} \).

If Case 1.2 or Case 2.2 occurs, then by Claims 6 and 7, it follows that \( L(x) \subset L(x_k) \) and \( L(x_k) = L(x_{k-1}) \). Hence, \( L(x) \subset L(x_{k-1}) \) and \( G \) is \((2,2)\)-extendable with respect to \((x,y)\) by Claim 5 because \( v \sim \{x,x_k,x_{k-1}\} \).

*Case 3:* \( \deg(x) = 2 \).

Since \( G \) is a near-triangulation, it follows that \( x_2 \sim x_k \). Additionally, the edge \( x_2x_k \) does not lie on the unbounded face of \( G \) so it must be part of two triangles. These triangles will be \((x, x_2, x_k)\) and \((v, x_2, x_k)\) for some vertex \( v \in V(G) \). In fact, \( v \in V(G) - F \), otherwise one of \( vx_2 \) and \( vx_k \) is a non-\( \{x,y\} \)-separating chord, contradicting Claim 1.

Note that \( x_2, x_k \) are both already adjacent to \( x \in F \). It is also the case that \( x_2 \) and \( x_k \) have no additional neighbors in \( F \) besides \( x_3 \) and \( x_{k-1} \), respectively. Assume, without loss of generality, that \( x_2 \sim x_l \). By Claims 1 and 3, it follows that \( l \in \{j + 1, \ldots, k - 1\} \) and \( G[x, x_2, x_3, \ldots, x_k] \) is a broken wheel. However, this would mean that \( \{x_2, x_{k-1}, x_k\} \) form a separating 3-cycle that separates \( v \) from other vertices of the graph. This contradicts the
fact that $G$ is a Type II reduced graph. Note here that this separating 3-cycle must indeed exist, otherwise \( \{x,x_2,x_k\} \) is a separating 3-cycle that separates $v$ from other vertices of the graph. This would also contradict the fact that $G$ is a Type II reduced graph. Thus, let $N(x_2) - \{x,x_3,x_k,v\} = \{u_1,\ldots,u_m\}$ and $N(x_k) - \{x,x_{k-1},x_2,v\} = \{w_1,\ldots,w_l\}$.

Assume $L(x) = \{a,b\}$, $L(x_2) = L(x_3) = \{a,b,c\}$ and $L(x_k) = L(x_{k-1}) = \{a,b,d\}$. If $c \neq d$, then $G$ is (2,2)-extendable with respect to $(x,y)$.

Proceed as follows: delete $x$ and $x_k$ from $G$. Remove $d$ from $L(x_{k-1})$, $L(v)$ and $L(w_i)$ for $i = 1,\ldots,l$. The resulting graph is (2,2)-extendable with respect to $(x_{k-1},y)$. Such a coloring may be extended to the vertices $x$ and $x_k$ by assigning color $d$ to $x_k$ and the color $a$ or $b$ to $x$, so that $x_2$ and $x$ are not assigned the same color.

Note the above argument can be analogously applied to show a similar result for when $\deg(y) = 2$.

The preceding results and case analysis shows that $G$ must be one of the three graphs described in the statement of the theorem. □
CHAPTER 5. CATALOG

5.1 Introduction and preliminaries

This chapter is a catalog of small configurations, graphs with size functions associated with them. In this chapter, we determine which of these size functions are choice functions. These configurations with choice functions can then be used in reductions, as seen in Chapter 3 with \(D\) and \(W\). In Section 5.2 we look at broken wheels and the “nice” property. In Sections 5.3 and 5.4 we look at diamonds and wheels and the “good” property. The “nice” property deals with nicely coloring subgraphs of a small graph, while the “good” property deals with determining whether a graph is \(f\)-choosable for a given size function \(f\). We will determine whether or not a given graph with a size function has the desired property by either providing a list assignment for which a graph cannot be colored in the desired way, or by explicitly describing a coloring \(c\) given an arbitrary \(f\)-assignment \(L\). We review some definitions here for convenience and to describe some notation. Throughout, we will fix a linear ordering of the vertices and consider size functions with respect to these orderings. It shall be noted here that in the figures provided in this catalog, when a vertex \(v\) is labeled with a number, that number indicates \(f(v)\) for a given size function \(f\).

5.2 Broken wheels

Let \(BW_k = (V, E)\) be the broken wheel with vertex set \(V = (u, x_1, x_2, \ldots, x_k)\) and edge set \(E = \{ux_1, ux_2, \ldots, ux_k\}\). Let \(f\) be a size function for \(BW_k\). See Figure 5.1 for examples of \(BW_3\) and \(BW_4\). We say that \((BW_k; f(u), f(x_1), \ldots, f(x_k))\) is nice if \(S = BW_k[x_1, x_2, \ldots, x_k]\) can be nicely colored with respect to \(L\) for any \(f\)-assignment \(L\). Otherwise, we say that \((BW_k; f(u), f(x_1), \ldots, f(x_k))\) is not nice. In this section, we determine all nice and not nice
broken wheels for $k = 3$ and $k = 4$ given size functions $f$ for which $f(u) = 5$ and $1 \leq f(x_i) \leq 5$ for all $i = 1, \ldots, k$.

**Lemma 5.1.** If $f(x_i) + f(x_{i+2}) \leq 5$ for some $i = 1, \ldots, k - 2$, then $(BW_k; 5, f(x_1), \ldots, f(x_k))$ is not nice.

**Proof.** We will consider two cases: (1) $f(x_i) = 1$, $f(x_{i+2}) = 4$ and (2) $f(x_i) = 2$, $f(x_{i+2}) = 3$ and provide $f$-assignments $L$ that show $(BW_k; 5, f(x_1), \ldots, f(x_k))$ is not nice. Let $L$ be an $f$-assignment such that $L(u) = \{a, b, c, d, e\}$, $L(x_{i+1}) \subseteq L(u)$, and (1) $L(x_i) = \{a\}$, $L(x_{i+2}) = \{b, c, d, e, 1\}$ or (2) $L(x_i) = \{a, b\}$, $L(x_{i+2}) = \{c, d, e\}$. In either case, for any $L$-coloring $c$ of $BW_k[x_1, \ldots, x_k]$, it must be that $c(x_i) \neq c(x_{i+2})$. Thus, three distinct colors from $L(u)$ will be used to properly color $BW_k[x_i, x_{i+1}, x_{i+2}]$. \hfill \qed

Assume $k < m$. Let $f$ be a size function for $BW_k$ and let $g$ be a size function for $BW_m$ such that $f(u) = g(u)$ and $f(x_i) = g(x_i)$ for all $i = 1, \ldots, k$. If $(BW_k; f(u), f(x_1), \ldots, f(x_k))$ is not nice, then $(BW_m; g(u), g(x_1), \ldots, g(x_m))$ is not nice.

**5.2.1 Broken 3-wheels**

The only two broken 3-wheels that are not nice and not classified by Lemma 5.1 are $(BW_3; 5, 2, 1, 4)$ and $(BW_3; 5, 3, 1, 3)$. Figures 5.2a and 5.2b provide specific $f$-assignments that show these graphs are not nice.

All $(BW_3; 5, f(x_1), f(x_2), f(x_3))$ that are nice can be determined by showing that the following are nice: $(BW_3; 5, 1, 2, 5)$, $(BW_3; 5, 2, 2, 4)$, $(BW_3; 5, 3, 1, 4)$, and $(BW_3; 5, 3, 2, 3)$. 

Figure 5.1: Examples of $BW_k$. 

\[ \text{(a) } BW_3 \quad \text{(b) } BW_4 \]
Assume \( L(x_1) \subset L(x_2) \subset L(u) \), otherwise \( L(u) - \{L(x_1), L(x_2)\} \) contains at least four elements and the desired result follows. So let \( L(x_1) = \{a\} \) and \( L(x_2) = \{a, b\} \), which gives \( c(x_1) = a \) and \( c(x_2) = b \). If \( a \in L(x_3) \), let \( c(x_3) = a \) and \( |L_c(u, S)| \geq 3 \). Otherwise, there is \( c \in L(x_3) - L(u) - \{b\} \), so let \( c(x_3) = c \) and \( |L_c(u, S)| \geq 3 \).

\((BW_3; 5, 1, 2, 5)\) is nice.

Assume \( L(x_1), L(x_2), L(x_3) \subset L(u) \), otherwise \( S \) can be nicely colored. If there is \( a \in L(x_1) \cap L(x_3) \), let \( c(x_1) = c(x_3) = a \) and take \( c(x_2) \in L(x_2) - \{a\} \). Otherwise, let \( c(x_2) = b \in L(x_2) \) and take \( c(x_1) = a \in L(x_1) - \{b\} \). Then there is \( c \in L(x_3) - L(u) - \{b\} \), so let \( c(x_3) = c \). In either case, \( |L_c(u, S)| \geq 3 \).

\((BW_3; 5, 2, 2, 4)\) is nice.

Assume \( L(x_2) = \{b\} \subset L(u) \), otherwise \( S \) can be nicely colored. Let \( c(x_2) = b \). If there is \( a \in L(x_1) \cap L(x_3) \), let \( c(x_1) = c(x_3) = a \) and the result follows. Otherwise, there is \( c \in L(x_1) \cup L(x_3) - L(u) - \{b\} \). Assume \( c \in L(x_1) \), so let \( c(x_1) = c \) and take \( c(x_3) \in L(x_3) - \{b\} \). In either case, \( |L_c(u, S)| \geq 3 \).

\((BW_3; 5, 3, 1, 4)\) is nice.

Assume \( L(x_2) \subset L(u) \), otherwise \( S \) can be nicely colored. If there is \( a \in L(x_1) \cap L(x_3) \), let \( c(x_1) = c(x_3) = a \) and choose \( c(x_2) \in L(x_2) - \{a\} \). Otherwise, let \( c(x_2) = b \in L(x_2) \) and there is \( c \in L(x_1) \cup L(x_3) - L(u) - \{b\} \). Assume \( c \in L(x_1) \), so let \( c(x_1) = c \) and take \( c(x_3) \in L(x_3) - \{b\} \). In either case, \( |L_c(u, S)| \geq 3 \).

\((BW_3; 5, 3, 2, 3)\) is nice.

Assume \( L(x_2) \subset L(u) \), otherwise \( S \) can be nicely colored. If there is \( a \in L(x_1) \cap L(x_3) \), let \( c(x_1) = c(x_3) = a \) and choose \( c(x_2) \in L(x_2) - \{a\} \). Otherwise, let \( c(x_2) = b \in L(x_2) \) and there is \( c \in L(x_1) \cup L(x_3) - L(u) - \{b\} \). Assume \( c \in L(x_1) \), so let \( c(x_1) = c \) and take \( c(x_3) \in L(x_3) - \{b\} \). In either case, \( |L_c(u, S)| \geq 3 \).
Lemma 5.2. If $L$ is an assignment of lists of colors to the vertices of $BW_3$ such that $|L(u)| = 4$, $|L(x_2)| = 5$, (1) $|L(x_1)| = |L(x_3)| = 3$, or (2) $|L(x_1)| = 2$, $|L(x_3)| = 4$, and $L(x_1) \cap L(x_3) = \emptyset$, then $BW_3[x_1, x_2, x_3]$ can be nicely colored with respect to $L$.

Proof. There is $a, b \in L(x_1) \cup L(x_3) - L(u)$. If $a \in L(x_1)$, $b \in L(x_3)$, then the result follows. Otherwise, without loss of generality, $a, b \in L(x_1)$ and $c \in L(x_2)$, so $x_1$ and $x_2$ can both be assigned distinct colors not in $L(u)$. \qed

5.2.2 Broken 4-wheels

Lemma 5.1 determines many of the $(BW_4; 5, f(x_1), f(x_2), f(x_3), f(x_4))$ that are not nice, but not all. This classification is completed with the following that are not nice:

$(BW_4; 5, 1, 1, 5, 5)$, $(BW_4; 5, 1, 2, 5, 4)$, $(BW_4; 5, 1, 3, 5, 3)$, $(BW_4; 5, 1, 4, 5, 2)$, $(BW_4; 5, 1, 5, 5, 1)$, $(BW_4; 5, 2, 2, 4, 4)$, $(BW_4; 5, 2, 3, 4, 3)$, $(BW_4; 5, 2, 4, 4, 2)$, $(BW_4; 5, 3, 2, 3, 4)$, and $(BW_4; 5, 3, 3, 3, 3)$. See Figure 5.3 for examples of lists that illustrate this. To complete the classification of all $(BW_4; 5, f(x_1), f(x_2), f(x_3), f(x_4))$, it remains to show that all of the other possible $(BW_4; 5, f(x_1), f(x_2), f(x_3), f(x_4))$ are nice.
(BW\(_4\); 5, 1, 2, 5, 5) is nice.

Without loss of generality, \(L(x_1) = \{a\} \subseteq L(u)\), otherwise the result follows by letting \(c(x_1) = a\), then consider \(L(x_2) - \{a\}\) and apply the fact that \((BW_3; 5, 1, 5, 5)\) is nice. Thus, let \(c(x_1) = a\) and choose \(c(x_2) = b \in L(x_2) - \{a\}\). If \(b \notin L(u)\), then consider \(L(x_3) - \{b\}\) and apply the fact that \((BW_3; 5, 1, 4, 5)\) is nice. So assume that \(b \in L(u)\).

**Case 1:** \(a \in L(x_3)\).

Let \(c(x_3) = a\). If \(b \in L(x_4) - \{a\}\), let \(c(x_4) = b\). If \(b \notin L(x_4) - \{a\}\), there is \(c \in L(x_4) - L(u)\) so let \(c(x_4) = c\).

**Case 2:** \(a \notin L(x_3)\).

There is \(c \in L(x_3) - L(u)\), so let \(c(x_3) = c\). If \(b \in L(x_4) - \{c\}\), let \(c(x_4) = b\). If \(a \in L(x_4) - \{c\}\), let \(c(x_4) = a\). Otherwise, there is \(d \in L(x_4) - L(u)\), so let \(c(x_4) = d\).
$(BW_4; 5, 1, 3, 5, 4)$ is nice.

Without loss of generality, $L(x_1) = \{a\} \subseteq L(u)$, otherwise let $c(x_1) = a$, then consider $L(x_2) - \{a\}$ and use the fact that $(BW_3; 5, 2, 5, 4)$ is nice. So let $c(x_1) = a$.

**Case 1:** $a \in L(x_3)$.

Let $c(x_3) = a$. If there is $b \in L(x_2) \cap L(x_4) - \{a\}$, let $c(x_2) = c(x_4) = b$. Otherwise, $L(x_2) \cap L(x_4) - \{a\} = \emptyset$, so $|L(x_2) \cup L(x_4) - \{a\}| = 5$ and at least one of those five colors is not an element of $L(u)$. Without loss of generality, assume this color is $c \in L(x_2)$. So let $c(x_2) = c(x_4) = b$.

**Case 2:** $a \notin L(x_3)$.

So there is $b \in L(x_3) - L(u)$. If $b \in L(x_2) \cap L(x_4)$, let $c(x_2) = c(x_4) = b$ and choose $c(x_3) \in L(x_3) - \{b\}$. Otherwise, let $c(x_3) = b$ and since $b \notin L(x_2) \cap L(x_4)$, at least one of $L(x_2)$ and $L(x_4)$ does not contain $b$. If $a \in L(x_4)$, let $c(x_4) = a$ and choose $c(x_2) \in L(x_2) - \{a, b\}$. If there is $c \in L(x_2) \cap L(x_4) - \{a, b\}$, let $c(x_2) = c(x_4) = c$. If $L(x_2) \cap L(x_4) - \{a, b\} = \emptyset$, there are at least five distinct colors in $L(x_2) \cap L(x_4) - \{a, b\}$. So at least one of them, call it $d$, is not in $L(u)$. Without loss of generality, assume $d \in L(x_2)$. So let $c(x_2) = d$ and choose $c(x_4) \in L(x_4) - \{b\}$. 
(BW$_4$; 5, 1, 4, 5, 3) is nice.

Without loss of generality, $L(x_1) = \{a\} \subseteq L(u)$, otherwise the result follows: let $c(x_1) = a$ and, with $L(x_2) - \{a\}$, use the fact that (BW$_3$; 5, 3, 5, 3) is nice. So let $c(x_1) = a$.

Case 1: $a \in L(x_3)$.

Let $c(x_3) = a$. If there is $b \in L(x_2) \cap L(x_4) - \{a\}$, let $c(x_2) = c(x_4) = b$. Otherwise, $L(x_2) \cup L(x_4) - \{a\} = \emptyset$, so $L(x_2) \cup L(x_4) - \{a\}$ contains five distinct colors and at least one of them, say $c \in L(x_2)$, is not in $L(u)$. So let $c(x_2) = c$ and choose $c(x_4) \in L(x_4) - \{a\}$.

Case 2: $a \notin L(x_3)$.

If $a \in L(x_4)$, let $c(x_4) = a$ and there is $b \in L(x_3) - L(u)$, so let $c(x_3) = b$ and choose $c(x_2) \in L(x_2) - \{a, b\}$.

If there is $b \in L(x_2) \cap L(x_4) - \{a\}$, let $c(x_2) = c(x_4) = b$. Without loss of generality, $b \in L(u)$, otherwise the result follows for all elements of $L(x_3) - \{a, b\}$. Therefore, there is $c \in L(x_3) - L(u) - \{a, b\}$, so let $c(x_3) = c$. If $L(x_2) \cap L(x_4) - \{a\} = \emptyset$, then apply Lemma 5.2.

(BW$_4$; 5, 1, 5, 5, 2) is nice.

Without loss of generality, $L(x_1) = \{a\} \subseteq L(u)$, otherwise let $c(x_1) = a$ and, with $L(x_2) - \{a\}$, apply the fact that (BW$_3$; 5, 2, 5, 4) is nice. Let $c(x_1) = a$.

Case 1: $a \in L(x_3)$.

Let $c(x_3) = a$ and $c(x_4) = b \in L(x_4) - \{a\}$. Assume $b \in L(u)$, otherwise the result follows. If $b \in L(x_2)$, let $c(x_2) = b$. Otherwise, there is $c \in L(x_2) - L(u)$, so let $c(x_2) = c$.

Case 2: $a \notin L(x_3)$.

If $a \in L(x_4)$, let $c(x_4) = a$. Then there is $b \in L(x_3) - L(u)$, so let $c(x_3) = b$ and choose $c(x_2) \in L(x_2) - \{a, b\}$. If there is $b \in L(x_2) \cap L(x_4) - \{a\}$, let $c(x_2) = c(x_4) = b$. Then there is $c \in L(x_3) - L(u) - \{b\}$, so let $c(x_3) = c$. If $L(x_2) \cap L(x_4) - \{a\} = \emptyset$, then apply Lemma 5.2.
(BW\textsubscript{4}; 5, 2, 1, 5, 5) is nice.

Without loss of generality, assume \( L(x_2) = \{a\} \subseteq L(u) \), otherwise let \( c(x_2) = a \), then consider \( L(x_1) - \{a\} \) and apply the fact that \( (BW\textsubscript{3}; 5, 1, 4, 5) \) is nice.

Without loss of generality, assume \( b \in L(x_1) - \{a\} \) is in \( L(u) \), else let \( c(x_1) = b \) and use the fact that \( (BW\textsubscript{3}; 5, 1, 5, 5) \) is nice. So let \( c(x_2) = a \) and \( c(x_1) = b \).

**Case 1:** \( a \in L(x_4) \).

Let \( c(x_4) = a \). If \( b \in L(x_3) - \{a\} \), let \( c(x_3) = b \). Otherwise, there is \( c \in L(x_3) - L(u) \) so let \( c(x_3) = c \).

**Case 2:** \( a \notin L(x_4) \).

If \( b \in L(x_3) \), let \( c(x_3) = b \) and there is \( c \in L(x_4) - L(u) \), so let \( c(x_4) = c \).

If \( b \in L(x_4) \), let \( c(x_4) = b \) and choose \( c(x_3) \in L(x_3) - L(u) \neq \emptyset \). Otherwise, \( b \notin L(x_3) \cup L(x_4) \), so \( |L(x_3) - L(u)| \geq 1 \) and \( |L(x_4) - L(u)| \geq 2 \). Then choose \( c(x_3) = d \in L(x_3) - L(u) \) and \( c(x_4) \in L(x_4) - L(u) - \{d\} \).
is nice.

Without loss of generality, $L(x_1) \subseteq L(u)$, otherwise let $c(x_1) = a \in L(x_1) - L(u)$ and, with $L(x_2) - \{a\}$, apply the fact that $(BW_3; 5, 1, 4, 5)$ is nice. Without loss of generality, $L(x_2) \subseteq L(u)$, otherwise let $c(x_2) = b \in L(x_2) - L(u)$ and, with $L(x_1) - \{b\}$ and $L(x_3) - \{b\}$, apply the fact that $(BW_3; 5, 1, 3, 5)$ is nice.

**Case 1:** There is $a \in L(x_1) \cap L(x_3)$.

Let $c(x_1) = c(x_3) = a$, then let $c(x_2) = b \in L(x_2) - \{a\}$. If $b \in L(x_4)$, let $c(x_4) = b$. Otherwise, there is $c \in L(x_4) - L(u)$, so let $c(x_4) = c$.

**Case 2:** $L(x_1) \cap L(x_3) = \emptyset$.

This implies there is $a \in L(x_3) - L(u)$.

- If there is $b \in L(x_2) \cap L(x_4)$, let $c(x_2) = c(x_4) = b$. Now, $b \notin L(x_1)$ or $b \notin L(x_3)$, so $|L(x_1) \cup L(x_3) - \{b\}| \geq 5$ and one of those colors, without loss of generality say $c \in L(x_3)$ as $L(x_1) \subset L(u)$, is not in $L(u)$. So let $c(x_3) = c$ and choose $c(x_1) \in L(x_1) - \{b\}$.

- If $L(x_2) \cap L(x_4) = \emptyset$, there is $b, c \in L(x_4) - L(u)$. Let $c(x_3) = a, c(x_4) = d \in \{b, c\} - \{a\}$ then choose $c(x_2) \in L(x_2)$ and $c(x_1) \in L(x_1) - c(x_2)$. 
(BW₄; 5, 2, 2, 5, 4) is nice.

**Case 1:** There is \( a \in L(x_2) \cap L(x_4) \).
Let \( c(x_2) = c(x_4) = a \) and choose \( c(x_1) = b \in L(x_1) - \{a\} \). Assume \( a \in L(u) \), otherwise the result follows. If \( b \in L(x_3) \), let \( c(x_3) = b \). Otherwise, there is \( c \in L(x_3) - L(u) \), so let \( c(x_3) = c \).

**Case 2:** \( L(x_2) \cap L(x_4) = \emptyset \).
- If there is \( a \in L(x_1) \cap L(x_3) \), let \( c(x_1) = c(x_3) = a \). Assume \( a \in L(u) \), otherwise the result follows. Note that \( a \not\in L(x_2) \) or \( a \not\in L(x_4) \), otherwise Case 1 applies. So \( |L(x_2) \cup L(x_4) - \{a\}| \geq 5 \) and at least one color, say \( c \in L(x_2) \), is not in \( L(u) \). Hence let \( c(x_2) = c \) and choose \( c(x_4) \in L(x_4) - \{a\} \).
- If there is \( a \in L(x_1) \cap L(x_4) \), let \( c(x_1) = c(x_4) = a \). Since \( a \not\in L(x_2) \cup L(x_3) \), \( |L(x_2) \cup L(x_3)| \geq 5 \), so at least one color, say \( c \in L(x_2) \), is not in \( L(u) \). Then let \( c(x_2) = c \) and choose \( c(x_3) \in L(x_3) - \{a, c\} \).
- If none of the above occur, then \( L(x_1) \cap L(x_3) = \emptyset = L(x_1) \cap L(x_4) \). If there is \( a \in L(x_1) - L(u) \), let \( c(x_1) = a \) and choose \( c(x_2) = b \in L(x_2) - \{a\} \). If \( b \in L(u) \), there is \( c \in L(x_3) - L(u) \), so let \( c(x_3) = c \) and choose \( c(x_4) \in L(x_4) - \{c\} \).
  If \( b \not\in L(u) \), choose \( c(x_3) \in L(x_3) - \{b\} \) and choose \( c(x_4) \in L(x_4) - \{c(x_3)\} \).
Otherwise, \( L(x_1) \subseteq L(u) \) and there is \( a \in L(x_4) - L(u) \), so let \( c(x_4) = a \) and, with \( L(x_3) - \{a\} \), use the fact that \( (BW_3; 5, 2, 2, 4) \) is nice.
(BW₄; 5, 2, 3, 4, 4) is nice.

Assume \( L(x_1) \subseteq L(u) \), otherwise there is \( a \in L(x_1) - L(u) \) so let \( c(x_1) = a \) and, with \( L(x_2) - \{a\} \), apply the fact that (BW₃; 5, 2, 4, 4) is nice.

**Case 1:** There is \( a \in L(x_1) \cap L(x_3) \).

Let \( c(x_1) = c(x_3) = a \). If there is \( b \in L(x_2) \cap L(x_4) - \{a\} \), let \( c(x_2) = c(x_4) = b \). Otherwise, \( |L(x_2) \cup L(x_4) - \{a\}| \geq 5 \), so there is \( b \in L(x_2) \cup L(x_4) - L(u) \). Without loss of generality, assume \( b \in L(x_2) \), so let \( c(x_2) = b \) and choose \( c(x_4) \in L(x_4) - \{a\} \).

**Case 2:** \( L(x_1) \cap L(x_3) = \emptyset \).

This implies there is \( a \in L(x_3) - L(u) \).

- If there is \( b \in L(x_2) \cap L(x_4) \), let \( c(x_2) = c(x_4) = b \). Assume \( b \in L(u) \), else the result follows. Also, \( b \notin L(x_1) \) or \( b \notin L(x_3) \), otherwise Case 1 applies. Now let \( c(x_3) = a \) and choose \( c(x_1) \in L(x_1) - \{b\} \).

- If \( L(x_2) \cap L(x_4) = \emptyset \), then there is \( b, c \in L(x_2) \cup L(x_4) - L(u) \). If \( b \) and \( c \) lie in the same list, without loss of generality assume \( b, c \in L(x_2) \), let \( c(x_3) = a \) and \( c(x_2) = d \in \{b, c\} - \{a\} \). Then \( x_1 \) and \( x_4 \) can be colored properly. Otherwise, \( b \) and \( c \) are in different lists, so, without loss of generality, let \( c(x_2) = b \) and \( c(x_4) = c \) and \( x_1 \) and \( x_4 \) can be colored properly.
\((BW_4; 5,2,3,5,3)\) is nice.

Assume \(L(x_4) \subseteq L(u)\), otherwise there is \(a \in L(x_4) - L(u)\) so let \(c(x_4) = a\) and, with \(L(x_3) - \{a\}\), apply the fact that \((BW_3; 5,2,3,4)\) is nice.

**Case 1:** There is \(a \in L(x_2) \cap L(x_4)\).

Let \(c(x_2) = c(x_4) = a\). Assume \(a \in L(u)\), otherwise choose \(c(x_1) \in L(x_1) - \{a\}\) and \(c(x_3) \in L(x_3) - \{a\}\) and the result holds. If there is \(b \in L(x_1) \cap L(x_3) - \{a\}\), let \(c(x_1) = c(x_3) = b\). Otherwise, \(|L(x_1) \cup L(x_3) - \{a\}| = 5\) and there is \(b \in L(x_1) \cup L(x_3) - L(u)\). Without loss of generality, assume \(b \in L(x_1)\) and let \(c(x_1) = b\) then choose \(c(x_3) \in L(x_3) - \{a\}\).

**Case 2:** \(L(x_2) \cap L(x_4) = \emptyset\).

This implies there is \(a \in L(x_2) - L(u)\).

- If there is \(b \in L(x_1) \cap L(x_3)\), let \(c(x_1) = c(x_3) = b\). Note that \(b \not\in L(x_2)\) or \(b \not\in L(x_4)\), otherwise Case 1 applies. Then let \(c(x_2) = a\) and choose \(c(x_4) \in L(x_4) - \{b\}\).
- Otherwise, \(L(x_1) \cap L(x_3) = \emptyset\) and there is \(b, c \in L(x_1) \cup L(x_3) - L(u)\). If \(b\) and \(c\) are in distinct lists, then, without loss of generality, let \(c(x_1) = b\), \(c(x_3) = c\) and \(x_2, x_4\) can be colored properly to obtain the desired result. If \(b\) and \(c\) are both in the same list, then let \(c(x_2) = a\) and without loss of generality \(c(x_1) = d \in \{b, c\} - \{a\}\) and \(x_3, x_4\) can be colored properly to obtain the desired result.
(BW\(_4\); 5, 2, 4, 4, 3) is nice.

Assume \(L(x_1) \subseteq L(u)\), otherwise choose \(c(x_1) \in L(x_1) - L(u)\) and with \(L(x_2) - \{c(x_1)\}\) apply the fact that \((BW_3; 5, 3, 4, 3)\) is nice.

**Case 1:** There is \(a \in L(x_1) \cap L(x_3)\).

Let \(c(x_1) = c(x_3) = a\). If there is \(b \in L(x_2) \cap L(x_4) - \{a\}\), let \(c(x_2) = c(x_4) = b\).

Otherwise, \(|L(x_2) \cup L(x_4) - \{a\}| \geq 5\) so there is \(b \in L(x_2) \cup L(x_4) - L(u)\).

Without loss of generality, assume \(b \in L(x_2)\), so let \(c(x_2) = b\) and choose \(c(x_4) \in L(x_4) - \{a\}\).

**Case 2:** \(L(x_1) \cap L(x_3) = \emptyset\).

So there is \(a \in L(x_3) - L(u)\).

- If there is \(b \in L(x_2) \cap L(x_4)\), let \(c(x_2) = c(x_4) = b\).

If \(b = a\), then choose \(c(x_1) \in L(x_1) - \{a\}\), \(c(x_3) \in L(x_3) - \{a\}\) and the result follows as \(a \not\in L(u)\).

If \(b \neq a\), let \(c(x_3) = a\) and choose \(c(x_1) \in L(x_1) - \{b\}\).

- Otherwise, \(L(x_2) \cap L(x_4) = \emptyset\), so there is \(b, c \in L(x_2) \cup L(x_4) - L(u)\).

If \(b, c\) are both only contained in one of the two lists \(L(x_2), L(x_4)\), without loss of generality say \(L(x_2)\), let \(c(x_2) = d \in \{b, c\} - \{a\}\), \(c(x_3) = a\), and choose \(c(x_1) \in L(x_1) - \{d\}\), \(c(x_4) - \{a\}\).

If \(b, c\) are in more than one list, without loss of generality, let \(c(x_2) = b\), \(c(x_4) = c\) and choose \(c(x_1) \in L(x_1) - \{b\}\), \(c(x_3) \in L(x_3) - \{b, c\}\).
(BW; 5, 2, 4, 5, 2) is nice.

Assume \( L(x_4) \subseteq L(u) \), otherwise let \( c(x_4) \in L(x_4) - L(u) \) and with \( L(x_3) - \{c(x_4)\} \) apply the fact that \( (BW; 5, 2, 4, 4) \) is nice.

**Case 1:** There is \( a \in L(x_2) \cap L(x_4) \).

Let \( c(x_2) = c(x_4) = a \). If there is \( b \in L(x_1) \cap L(x_3) - \{a\} \), let \( c(x_1) = c(x_3) = b \). Otherwise, \( |L(x_1) \cup L(x_3) - \{a\}| \geq 5 \) so there is \( b \in L(x_1) \cup L(x_3) - L(u) \).

Without loss of generality, assume \( b \in L(x_1) \), so let \( c(x_1) = b \) and choose \( c(x_3) \in L(x_3) - \{a\} \).

**Case 2:** \( L(x_2) \cap L(x_4) = \emptyset \).

There is \( a \in L(x_2) - L(u) \). If there is \( b \in L(x_1) \cap L(x_3) - \{a\} \), then let \( c(x_1) = c(x_3) = b \), \( c(x_2) = a \) and choose \( c(x_4) \in L(x_4) - \{b\} \). If \( L(x_1) \cap L(x_3) - \{a\} = \emptyset \), then there is \( b \in L(x_1) \cup L(x_3) - L(u) - \{a\} \). Without loss of generality, assume \( b \in L(x_3) \), so let \( c(x_2) = a \), \( c(x_3) = b \) then choose \( c(x_1) \in L(x_1) - \{a\} \) and \( c(x_4) \in L(x_4) - \{b\} \).

(BW; 5, 3, 1, 4, 5) is nice.

Assume \( L(x_2) = \{a\} \), so \( c(x_2) = a \). Assume \( a \in L(u) \), otherwise apply the fact that \( (BW; 5, 2, 3, 5) \) is nice. Also assume \( L(x_1) \subseteq L(u) \). Otherwise, choose \( c(x_1) \in L(x_1) - L(u) \) and apply the fact that \( (BW; 5, 1, 4, 5) \) is nice.

**Case 1:** \( a \in L(x_4) \).

Let \( c(x_4) = a \). If there is \( b \in L(x_1) \cap L(x_3) - \{a\} \), let \( c(x_1) = c(x_3) = b \). Otherwise, \( |L(x_1) \cup L(x_3) - \{a\}| \geq 5 \) so there is \( c \in L(x_3) - L(u) \), so let \( c(x_3) = c \) and choose \( c(x_1) \in L(x_1) - \{a\} \).

**Case 2:** \( a \notin L(x_4) \).

So there is \( b \in L(x_4) - L(u) \). If there is \( c \in L(x_1) \cap L(x_3) - \{a\} \), let \( c(x_1) = c(x_3) = c \) and \( c(x_4) = b \). Otherwise \( |L(x_1) \cup L(x_3) - \{a\}| = 5 \), so there is \( d \in L(x_3) - L(u) \), so let \( c(x_3) = d \). If there is \( b \in (L(x_1) - \{a\}) \cap (L(x_4) - \{d\}) \), let \( c(x_1) = c(x_4) = b \). Else, choose \( c(x_1) \in L(x_1) - \{a\} \). Thus there is \( e \in L(x_4) - L(u) - \{c\} \), so let \( c(x_4) = e \).
(BW_4; 5, 3, 2, 3, 5) is nice.

Assume L(x_1) \subseteq L(u), otherwise let c(x_1) = a \in L(x_1) - L(u) and apply the fact that (BW_3; 5, 1, 3, 5) is nice. Also assume that L(x_2) \subseteq L(u), otherwise let c(x_2) = b \in L(x_2) - L(u) and apply the fact that (BW_3; 5, 2, 2, 5) is nice with L(x_1) - \{b\} and L(x_3) - \{b\}.

Case 1: There is a \in L(x_1) \cap L(x_3).

Let c(x_1) = c(x_3) = a and c(x_2) = b \in L(x_2) - \{a\}. If b \in L(x_4), let c(x_4) = b. Else there is c \in L(x_4) - L(u), so let c(x_4) = c.

Case 2: L(x_1) \cap L(x_3) = \emptyset.

There is a \in L(x_3) - L(u), so let c(x_3) = a. Then, delete x_3, add the edge x_2x_4 and, with lists L(x_2) - \{a\}, L(x_4) - \{a\}, apply the fact that (BW_3; 5, 3, 1, 4) is nice.

(BW_4; 5, 3, 2, 4, 4) is nice.

Assume L(x_4) \subseteq L(u), otherwise let c(x_4) = a \in L(x_4) - L(u), then, with L(x_3) - \{a\}, apply the fact that (BW_3; 5, 3, 2, 3) is nice.

Case 1: There is a \in L(x_2) \cap L(x_4).

Let c(x_2) = c(x_4) = a. If there is b \in L(x_1) \cap L(x_3) - \{a\}, let c(x_1) = c(x_3) = b. Else, |L(x_1) \cup L(x_3) - \{a\}| \geq 5, so there is b \in L(x_1) \cup L(x_3) - L(u). Without loss of generality, assume b \in L(x_3), so let c(x_3) = b and choose c(x_1) \in L(x_1) - \{a\}.

Case 2: L(x_2) \cap L(x_4) = \emptyset.

There is a \in L(x_2) - L(u), so let c(x_2) = a. Now, delete x_2, add the edge x_1x_3 and with lists L(x_1) - \{a\}, L(x_3) - \{a\} apply the fact that (BW_3; 5, 2, 3, 4) is nice.
\((BW_4; 5,3,3,3,4)\) is nice.

Assume \(L(x_1) \subseteq L(u)\), otherwise let \(c(x_1) = a \in L(x_1) - L(u)\). Then, with 
\(L(x_2) - \{a\}\), apply the fact that \((BW_3; 5,2,3,4)\) is nice. Assume \(L(x_2) \subseteq L(u)\), 
otherwise let \(c(x_2) = b \in L(x_2) - L(u)\). Then, with 
\(L(x_1) - \{b\}\) and \(L(x_3) - \{b\}\), 
apply the fact that \((BW_3; 5,2,2,4)\) is nice. Also assume \(L(x_3) \subseteq L(u)\), otherwise 
let \(c(x_3) = c \in L(x_3) - L(u)\). Then, with 
\(L(x_2) - \{c\}\) and \(L(x_4) - \{c\}\), apply 
the fact that \((BW_3; 5,3,2,3)\) is nice. It follows that there is 
\(a \in L(x_1) \cap L(x_3)\).

Let \(c(x_1) = c(x_3) = a\). If there is \(b \in L(x_2) \cap L(x_4) - \{a\}\), let \(c(x_2) = c(x_4) = b\).

Else, \(|L(x_2) \cup L(x_4) - \{a\}| \geq 5\), so there is \(b \in L(x_2) \cup L(x_4) - L(u)\). Without loss 
of generality, assume \(b \in L(x_2)\), so let \(c(x_2) = b\) and choose \(c(x_4) \in L(x_4) - \{a\}\).

\((BW_4; 5,3,3,4,3)\) is nice.

Without loss of generality, \(L(x_4) \subseteq L(u)\), else let \(c(x_4) = a \in L(x_4) - L(u)\) and 
with \(L(x_3) - \{a\}\) apply the fact that \((BW_3; 5,3,3,3)\) is nice.

**Case 1:** There is \(a \in L(x_2) \cap L(x_3)\).

Let \(c(x_2) = c(x_4) = a\). If there is \(b \in L(x_1) \cap L(x_3) - \{a\}\), let \(c(x_1) = c(x_3) = b\).

Otherwise, \(|L(x_1) \cup L(x_3) - \{a\}| \geq 5\), so there is \(b \in L(x_1) \cup L(x_3) - L(u)\).

Without loss of generality, assume \(b \in L(x_3)\), so let \(c(x_3) = b\) and choose 
\(c(x_1) \in L(x_1) - \{a\}\).

**Case 2:** \(L(x_2) \cap L(x_4) = \emptyset\).

So there is \(a \in L(x_2) - L(u)\). If there is \(b \in L(x_1) \cap L(x_3) - \{a\}\), let \(c(x_1) = c(x_3) = b\), 
\(c(x_2) = a\) and choose \(c(x_4) \in L(x_4) - \{b\}\). If \(L(x_1) \cap L(x_3) - \{a\} = \emptyset\), then there is 
\(b \in L(x_1) \cup L(x_3) - L(u) - \{a\}\). Without loss of generality, 
\(c(x_3) = b\), \(c(x_2) = a\) then choose \(c(x_1) \in L(x_1) - \{a\}\) and 
\(c(x_4) \in L(x_4) - \{b\}\).
\((BW_4; 5, 4, 1, 3, 5)\) is nice.

Assume \(L(x_2) = \{a\}\), so let \(c(x_2) = a\). Without loss of generality, \(a \in L(u)\), otherwise, delete \(x_2\), add the edge \(x_1x_3\) and with lists \(L(x_1) - \{a\}\), \(L(x_3) - \{a\}\) apply the fact that \((BW_3; 5, 3, 2, 5)\) is nice. Assume \(L(x_1) \subseteq L(u)\), else choose \(c(x_1) \in L(x_1) - L(u)\) and apply the fact that \((BW_3; 5, 1, 3, 5)\) is nice.

**Case 1**: There is \(b \in L(x_1) \cap L(x_3) - \{a\}\).

Let \(c(x_1) = c(x_3) = b\). If \(a \in L(x_4)\), let \(c(x_4) = a\). Otherwise, there is \(c \in L(x_4) - L(u)\), so let \(c(x_4) = c\).

**Case 2**: \(L(x_1) \cap L(x_3) - \{a\} = \emptyset\).

This implies there is \(b \in L(x_3) - L(u)\), so let \(c(x_3) = b\). If there is \(c \in L(x_1) \cap L(x_4) - \{a, b\}\), then let \(c(x_1) = c(x_4) = c\). Otherwise, \(L(x_1) \cap L(x_4) = \emptyset\) and there is \(d \in L(x_4) - L(u) - \{b\}\), so let \(c(x_4) = d\) and choose \(c(x_1) \in L(x_1) - \{a\}\).

\((BW_4; 5, 4, 2, 2, 4)\) is nice.

Assume \(L(x_2) \subseteq L(u)\), else let \(c(x_2) = a \in L(x_2) - L(u)\), then delete \(x_2\), add the edge \(x_1x_3\) and with lists \(L(x_1) - \{a\}\), \(L(x_3) - \{a\}\) apply the fact that \((BW_3; 5, 3, 1, 4)\) is nice. By symmetry, this implies that \(L(x_3) \subseteq L(u)\).

**Case 1**: There is \(a \in L(x_1) \cap L(x_3)\).

Let \(c(x_1) = c(x_3) = a\), then \(c(x_2) = b \in L(x_2) - \{a\}\). If \(b \in L(x_4)\), let \(c(x_4) = b\).

Otherwise, there is \(c \in L(x_4) - l(u)\) so let \(c(x_4) = c\).

**Case 2**: By symmetry of Case 1, \(L(x_1) \cap L(x_3) = \emptyset = L(x_2) \cap L(x_4)\).

Choose \(c(x_2) \in L(x_2)\) and \(c(x_3) \in L(x_3) - \{c(x_2)\}\). There is \(a \in L(x_1) - L(u)\), so let \(c(x_1) = a\) and there is \(b \in L(x_4) - L(u)\), so let \(c(x_4) = b\).

\((BW_4; 5, 5, 1, 2, 5)\) is nice.

Assume \(L(x_2) = \{a\}\) and \(L(x_3) = \{a, b\}\), so let \(c(x_2) = a\) and \(c(x_3) = b\).

Assume \(\{a, b\} \subseteq L(u)\), otherwise the result follows by an application of the fact that \((BW_3; 5, 4, 1, 5)\) is nice. If \(a \in L(x_4)\), let \(c(x_4) = a\). Otherwise there is \(c \in L(x_4) - L(u)\) so let \(c(x_4) = c\). If \(b \in L(x_1)\), let \(c(x_1) = b\). Otherwise, there is \(d \in L(x_1) - L(u)\), so let \(c(x_1) = d\). In each case, \(|L_c(u, S)| \geq 3\).
5.3 Diamonds

Let $D = (V, E)$ be the graph with $V = (x_1, y_1, x_2, y_2)$ and $E = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2, y_1y_2\}$. See Figure 5.4. We call $D$ a diamond. Note that $D$ is isomorphic to $BW_3$, but we use a different name here in order to create a distinction between the broken wheels which we classified as nice or not nice and these diamonds which we will classify as good or bad with respect to whether or not $D$ is $f$-choosable for a given size function $f$.

We will consider size functions $f : V \to \mathbb{N}$ for $D$ such that $f(v) \leq 5$ for all $v \in V$ and $f(v) = 1$ for at most one $v \in V$. It will be determined which of these size functions are choice functions for $D$.

The diamonds $(D; 1, 2, 5, 2)$, $(D; 1, 3, 2, 3)$, $(D; 2, 1, 2, 3)$, and $(D; 2, 2, 5, 2)$ are bad, as illustrated by the examples of list assignments in Figure 5.5.

We complete the classification by showing that the following six diamonds are good. This is done by explicitly constructing a proper coloring $c$ given an arbitrary $f$-assignment $L$. 

![Figure 5.4: D.](image)

![Figure 5.5: Diamonds that are not reducible.](image)
(D; 1, 2, 2, 4) is good.

Without loss of generality, assume \( L(x_1) = \{a\} \), so \( c(x_1) = a \). Choose \( c(y_1) \in L(y_1) - \{a\} \), then \( c(x_2) \in L(x_2) - \{c(y_1)\} \). There is at least one color in \( L(y_2) \) that can be chosen for \( c(y_2) \).

(D; 1, 2, 3, 3) is good.

Without loss of generality, assume \( L(x_1) = \{a\} \), so \( c(x_1) = a \). Choose \( c(y_1) \in L(y_1) - \{a\} \), then \( c(y_2) \in L(y_2) - \{a, c(y_1)\} \). There is at least one color in \( L(x_2) \) that can be chosen for \( c(x_2) \).

(D; 2, 1, 4) is good.

Without loss of generality, assume \( L(y_1) = \{a\} \), so \( c(y_1) = a \). Then choose \( c(x_1) \in L(x_1) - \{a\} \) and \( c(x_2) \in L(x_2) - \{a\} \). There is at least one color in \( L(y_2) \) that can be chosen for \( c(y_2) \).

(D; 2, 1, 3, 3) is good.

Without loss of generality, assume \( L(y_1) = \{a\} \), so \( c(y_1) = a \). Then let \( c(x_1) = b \in L(y_1) - \{a\} \) and choose \( c(y_2) \in L(y_2) - \{a, b\} \). There is at least one color in \( L(x_2) \) that can be chosen for \( c(x_2) \).

(D; 2, 2, 2, 3) is good.

Without loss of generality, assume \( L(x_1), L(x_2), L(y_1) \subseteq L(y_2) \), otherwise \( x_1, x_2, y_1 \) can be colored using at most two colors in \( L(y_2) \). If \( L(x_1) \cap L(x_2) \neq \emptyset \), then assign the same color to \( x_1 \) and \( x_2 \). This implies there is a color available to choose for \( c(y_1) \). At most two colors in \( L(y_2) \) have been used, so \( y_2 \) can also be colored. Otherwise, \( L(x_1) \cap L(x_2) = \emptyset \), a contradiction as one of those lists would have a color not in \( L(y_2) \).
(D; 3, 1, 3, 2) is good.

Without loss of generality, assume \( L(y_1) = \{a\} \), so \( c(y_1) = a \). Then choose \( c(y_2) \in L(y_2) - \{a\} \). There is at least one color in each of \( L(x_1) \) and \( L(x_2) \) that can be chosen for \( c(x_1) \) and \( c(x_2) \), respectively.

### 5.4 Wheels

Let \( W_k = (V, E) \) be the wheel with center \( u \) and outercycle \( x_1x_2 \ldots x_kx_1 \). In particular, \( V = (u, x_1, x_2, \ldots, x_k) \) and \( E = \{x_1x_2, x_2x_3, \ldots, x_{k-1}x_k, x_kx_1, ux_1, ux_2, \ldots, ux_k\} \). See Figure 5.6 for examples of \( W_k \) for \( k = 3, 4, 5 \). We will consider size functions \( f : V \to \mathbb{N} \) for \( W_k \) such that \( f(x_i) \leq 5 \) for all \( i = 1, \ldots, k \), \( f(u) \in \{1, 5\} \), and if \( f(x_i) = 1 \), then \( f(x_{i-1}), f(x_{i+1}) \neq 1 \). It will be determined which of these size functions are choice functions for \( W_k \).

![Figure 5.6: Examples of \( W_k \) for \( k = 3, 4, 5 \).](image)

We begin by making some general observations. It is known that \( \chi_l(C_{2m}) = 2 \) and \( \chi_l(C_{2m+1}) = 3 \), so even cycles are 2-list-colorable and odd cycles are 3-list-colorable. Furthermore, when \( f(u) = 1 \), we may employ known results about list-coloring the vertices of paths and cycles. These imply the following:

1. If \( k = 2m \) and \( f(x_i) \geq 3 \) for all \( i = 1, \ldots, k \), then \( (W_k; 1, f(x_1), f(x_2), \ldots, f(x_k)) \) is good.

2. If \( k = 2m + 1 \) and \( f(x_i) \geq 3 \) for all \( i = 1, \ldots, k \), and there is a \( j \in \{1, \ldots, k\} \) for which \( f(x_j) \geq 4 \), then \( (W_k; 1, f(x_1), f(x_2), \ldots, f(x_k)) \) is good.

3. If \( k = 2m + 1 \) and \( f(x_i) \leq 3 \) for all \( i = 1, \ldots, k \), then \( (W_k; 1, f(x_1), f(x_2), \ldots, f(x_k)) \) is bad.
4. If \( f(x_i) = f(x_{i+j}) = 2 \) and \( f(x_{i+1}) = \ldots = f(x_{i+j-1}) = 3 \) for some \( i, j \in \{1, \ldots, k\} \), then \((W_k; 1, f(x_1), f(x_2), \ldots, f(x_k))\) is bad.

5. If \( f(x_i) = f(x_{i+j}) = 1 \) and \( f(x_{i+1}) = \ldots = f(x_{i+j-1}) = 2 \) for some \( i, j \in \{1, \ldots, k\} \), then \((W_k; 5, f(x_1), f(x_2), \ldots, f(x_k))\) is bad.

6. If \((W_k; 5, f(x_1), f(x_2), \ldots, f(x_k))\) is good, then \((W_k; 1, f(x_1) + 1, f(x_2) + 1, \ldots, f(x_k) + 1)\) is good.

### 5.4.1 3-wheels

The observations at the beginning of Section 5.4 and the following complete the classification of all good and bad \((W_3; 1, f(x_1), f(x_2), f(x_3))\).

\[(W_3; 1, 2, 3, 4)\] is good.

Without loss of generality, assume \(L(u) = \{a\}\), so \(c(u) = a\) and remove \(a\) from \(L(x_i), i = 1, 2, 3\) which leaves \(|L(x_1)| \geq 1, |L(x_2)| \geq 2\) and \(|L(x_3)| \geq 3\). The remaining vertices can be colored in the following order: \(x_1\), then there is an available color for \(x_2\) and then an available color for \(x_3\).

### 5.4.2 4-wheels

The observations at the beginning of Section 5.4 determine whether or not many of the possible \((W_4; 1, f(x_1), f(x_2), f(x_3), f(x_4))\) are good or bad. The classification is completed via some examples of list assignments that show \((W_4; 1, 2, 3, 2, 5)\) and \((W_4; 1, 2, 3, 3, 3)\) are bad, see Figure 5.7, and explanations that \((W_4; 1, 2, 3, 3, 4)\), \((W_4; 1, 2, 3, 4, 3)\), and \((W_4; 1, 2, 4, 2, 4)\) are good.

\[(W_4; 1, 2, 3, 3, 4)\] is good.

Remove \(L(u)\) from \(L(x_i), i = 1, 2, 3, 4\). This leaves a 4-cycle with lists of sizes at least 1, 2, 2, 3 for each of the \(x_i, i = 1, 2, 3, 4\). This determines the color of \(x_1\), then the colors of \(x_2\) and \(x_3\). There remains at least one color available to assign to \(x_4\).
Figure 5.7: 4-wheels that are bad.

(W_4; 1, 2, 3, 2, 5) **is good.**

Remove $L(u)$ from $L(x_i), i = 1, 2, 3, 4$. This leaves a 4-cycle with lists of sizes at least 1, 2, 3, 2 for each of the $x_i, i = 1, 2, 3, 4$. This determine the color of $x_1$, then the colors of $x_2$ and $x_4$. There remains at least one color available to assign to $x_3$.

(W_4; 1, 2, 4, 2, 4) **is good.**

Remove $L(u)$ from $L(x_i), i = 1, 2, 3, 4$. This leaves a 4-cycle with lists of sizes at least 1, 3, 1, 3 for each of the $x_i, i = 1, 2, 3, 4$. This determine the colors of $x_1$ and $x_3$. There remains at least one color available to assign to each of $x_2$ and $x_4$.

**5.4.3 5-wheels with $f(u) = 1$**

The observations at the beginning of Section 5.4 determine whether or not many of the possible $(W_5; 1, f(x_1), f(x_2), f(x_3), f(x_4), f(x_5))$ are good or bad.

The remaining bad $W_5$ are determined by showing $(W_5; 1, 2, 3, 2, 5, 5)$ and $(W_5; 1, 2, 3, 3, 2, 5)$ are bad. See Figure 5.8 for examples of list assignments.

Combined with the above results, all good $W_5$ are classified by showing that the following are good: $(W_5; 1, 2, 3, 3, 4), (W_5; 1, 2, 3, 3, 4, 3), (W_5; 1, 2, 3, 4, 2, 4),$ and $(W_5; 1, 2, 4, 2, 4, 4)$. 
(W₅; 1, 2, 3, 3, 3, 4) is good.

Remove L(u) from L(xᵢ) for i = 1, ..., 5. What remains is a 5-cycle with lists of sizes at least 1, 2, 2, 2, 3 for each of the xᵢ, i = 1, ..., 5. This forces a color for x₁, then x₂, x₃ and x₄. There is at least one color available to choose for x₅.

(W₅; 1, 2, 3, 3, 4, 3) is good.

Remove L(u) from L(xᵢ) for i = 1, ..., 5. What remains is a 5-cycle with lists of sizes at least 1, 2, 2, 3, 2 for each of the xᵢ, i = 1, ..., 5. This forces a color for x₁, then x₂ and x₅, and then x₃. There is at least one color available to choose for x₄.

(W₅; 1, 2, 3, 4, 2) is good.

Remove L(u) from L(xᵢ) for i = 1, ..., 5. What remains is a 5-cycle with lists of sizes at least 1, 2, 3, 1, 3 for each of the xᵢ, i = 1, ..., 5. This forces a color for x₁ and x₄, then x₂ and x₃. There is at least one color available to choose for x₅.

(W₅; 1, 2, 4, 2, 4) is good.

Remove L(u) from L(xᵢ) for i = 1, ..., 5. What remains is a 5-cycle with lists of sizes at least 1, 3, 1, 3, 3 for each of the xᵢ, i = 1, ..., 5. This forces a color for x₁ and x₃, then x₂. There are at least two colors available to choose for x₄ and x₅, so they can both be colored.
5.4.4 5-wheels with $f(u) = 5$

The observations at the beginning of Section 5.4 determine whether or not many of the possible $(W_5; 5, f(x_1), f(x_2), f(x_3), f(x_4), f(x_5))$ are good or bad. As shown in Figure 5.9. Additionally, $(W_5; 5, 1, 2, 2, 3, 3)$, $(W_5; 5, 1, 2, 3, 1, 3)$, $(W_5; 5, 1, 2, 3, 3, 2)$, $(W_5; 5, 1, 3, 1, 3, 3)$, and $(W_5; 5, 1, 3, 2, 2, 3)$ are bad.

![5-wheels with $f(u) = 5$]

Figure 5.9: 5-wheels with $f(u) = 5$ that are bad.

The classification is completed by showing that the following are good:

$(W_5; 5, 1, 2, 2, 4)$ is good. The color for $x_1$ is forced, then there is at least one color to be chosen for $x_2$, followed by $x_3$, $x_4$ and $u$. Finally, there is at least one color that can be chosen for $x_5$.

$(W_5; 5, 1, 2, 2, 4, 2)$ is good. The color for $x_1$ is forced, then there is at least one color to be chosen for $x_2$ and $x_5$, followed by $x_3$ and $u$. Finally, there is at least one color that can be chosen for $x_4$.

$(W_5; 5, 1, 2, 3, 1, 4)$ is good. The colors for $x_1$ and $x_4$ are forced, then there is at least one color to be chosen for $x_2$, followed by $x_3$ and $u$. Finally, there is at least one color that can be chosen for $x_5$. 
(W₅; 5, 1, 2, 3, 2, 3) is good. The color for x₁ is forced, then there is at least one color to be chosen for x₂. What remains to be colored is a diamond formed by the vertices x₃, x₄, x₅, u with lists of size 2, 2, 2, 3, respectively. As (D; 2, 2, 2, 3) is a reducible diamond, the remaining vertices can be colored from their lists.

(W₅; 5, 1, 2, 4, 1, 3) is good. The colors for x₁ and x₄ are forced, then there is at least one color to be chosen for x₂ and x₅. There is then at least one color that can be chosen for u, followed by x₃.

(W₅; 5, 1, 3, 1, 3, 4) is good. The colors for x₁ and x₃ are forced, then there is at least one color to be chosen for x₂. Next, choose a color for u from L(u) that will extend the proper coloring. Then there is at least one color that can be chosen for x₄, followed by x₅.

(W₅; 5, 1, 3, 2, 3, 3) is good. The color for x₁ is forced, then properly color x₅ from L(x₅). What remains to be colored is a diamond formed by the vertices x₂, x₃, x₄, u with lists of size 2, 2, 2, 3, respectively. Since (D; 2, 2, 2, 3) is a reducible diamond, the remaining vertices can be colored from their lists.

(W₅; 5, 2, 2, 2, 2, 3) is good. Color x₁ from L(x₁) with a color that is not in L(x₅), at least one such color exists. Then properly color x₂ from L(x₂). What remains to be colored is a diamond formed by the vertices x₃, x₄, x₅, u whose lists now have sizes 1, 2, 3, 3, respectively. Since (D; 1, 2, 3, 3) is a reducible diamond, the remaining vertices can be colored from their lists.
CHAPTER 6. SUM-LIST-COLORING

6.1 Introduction

In this chapter we will explore the notion of sum-list-coloring. In particular, we will show that trees of cycles are sc-greedy and determine information about the sum choice number of other graphs, including graphs on at most five vertices. Many of the important terms for this chapter have been defined in Chapter 1. Here, we begin with a brief review and provide some background and preliminary results.

To show that $\chi_{SC}(G) = m$, one must provide a choice function $f$ of size $m$ for $G$ and show that for each size function $g$ of size $m - 1$, there is a $g$-assignment $L$ for which $G$ is not $L$-colorable. In other words, if $\text{size}(g) = m - 1$, then $g$ is not a choice function for $G$. In the next section, some known results about sum-list-coloring will be presented.

6.2 Background and preliminaries

A survey by Heinold [36] compiles results about sum-list-coloring, the sum choice number, and states open problems in the area. Specifically, it contains a summary of graphs that are known to be sc-greedy as well as other graphs whose sum choice number is known. Recall that a graph $G$ is sc-greedy if $\chi_{SC}(G) = GB(G)$. These graphs as well as previous results included in the survey that will be used in this paper are stated here for convenience. Note that the number of graphs known to be sc-greedy is not large, nor is the number of graphs whose sum choice number is known. Recall that many of the graphs referred to below were defined in Section 2.1.

The following graphs are known to be sc-greedy: complete graphs, paths, cycles, trees, and cycles with pendant paths. Also, the Petersen graph is sc-greedy, as well as $P_2 \times P_n$ and the
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\chi_{SC}(G)$</th>
<th>$GB(G)$</th>
<th>sc-greedy?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_3 \times P_n$</td>
<td>$8n - 3 - \left\lfloor \frac{n}{3} \right\rfloor$</td>
<td>$8n - 3$</td>
<td>$n = 1, 2$</td>
</tr>
<tr>
<td>$K_{2,n}$</td>
<td>$2n + 1 + \left\lfloor \sqrt{4n + 1} \right\rfloor$</td>
<td>$3n + 2$</td>
<td>$n = 1, 2$</td>
</tr>
<tr>
<td>$K_{3,n}$</td>
<td>$2n + 1 + \left\lfloor \sqrt{12n + 4} \right\rfloor$</td>
<td>$4n + 3$</td>
<td>$n = 1$</td>
</tr>
<tr>
<td>$K_2 \times K_n$</td>
<td>$n^2 + \left\lfloor \frac{2n}{3} \right\rfloor$</td>
<td>$n^2 + 2n$</td>
<td>$n = 1, 2$</td>
</tr>
<tr>
<td>$K_3 \times K_3$</td>
<td>25</td>
<td>27</td>
<td>no</td>
</tr>
<tr>
<td>$\Theta_{1,1,2k+1}$</td>
<td>$4k + 10$</td>
<td>$4k + 11$</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 6.1: Graphs that are not generally sc-greedy whose sum choice number is known.

theta graph $\Theta_{k_1,k_2,k_3}$ unless $k_1 = k_2 = 1$ and $k_3$ is odd. Paths of cycles are sc-greedy. While this result is stated in [36] and elsewhere, this chapter contains a proof of the more general result that certain trees of cycles are sc-greedy. The graph $P_2^2$ is also sc-greedy. Additional graphs whose sum choice number is known are $P_3 \times P_n$, $K_{2,n}$, $K_{3,n}$, $K_2 \times K_n$ and $K_3 \times K_3$. See Table 6.1 for more information. In most cases, these graphs are not sc-greedy.

It can be noted that the graph $K_{2,3}$ is the smallest graph which is not sc-greedy. Note that $K_{2,3}$ is a graph on five vertices and all graphs on at most four vertices are sc-greedy. There do exist other graphs on five vertices that are not sc-greedy. These will be presented in Subsection 6.4.1.

A **non-simple size function** is a function $f$ for which $2 \leq f(v) \leq \deg(v)$ for all $v \in V(G)$. Otherwise, $f$ is a **simple size function**, i.e. $f(v) = 1$ or $f(v) > \deg(v)$ for some $v \in V(G)$. Sum-choosability questions for simple functions can be determined by answering sum-choosability questions about $G-v$. This section contains some known results that illustrate this idea.

What follows are some previous results that can be used to show a graph is sc-greedy. Let $f$ be a size function, then $f^v$ is the size function assigned to $G - v$ where $f^v(w) = f(w) - 1$ if $vw \in E(G)$ and $f^v(w) = f(w)$ otherwise. For a subgraph $H$ of $G$, let $f_H$ be the size function restricted to $H$.

**Lemma 6.1** (Isaak [41]). Let $G$ be a graph and $f$ a size function for $G$.

1. If $f(v) = 1$ for some $v \in V(G)$, then $G$ is $f$-choosable if and only if $G-v$ is $f^v$-choosable.
2. If \( f(v) > \deg(v) \) for some \( v \in V(G) \), then \( G \) is \( f \)-choosable if and only if \( G - v \) is \( f_{G-v} \)-choosable.

Let \( \tau(G) \) and \( \rho(G) \) be defined as follows:

\[
\tau(G) = \min \{ \text{size}(f) : G \text{ is } f\text{-choosable and } 2 \leq f(v) \leq \deg(v) \forall v \in V(G) \},
\]

\[
\rho(G) = \min \{ \chi_{SC}(G - v) + \deg(v) + 1 : v \in V(G) \}.
\]

**Lemma 6.2** (Heinold [35, 36]). For all graphs \( G \), \( \chi_{SC}(G) = \min \{ \rho(G), \tau(G) \} \). In particular, if \( G - v \) is sc-greedy for all \( v \in V(G) \), then \( \chi_{SC}(G) = \min \{ GB(G), \tau(G) \} \).

To show that a graph is sc-greedy, one can show that \( G - v \) is sc-greedy for all \( v \in V(G) \) and that there does not exist a non-simple choice function of size one less than the greedy bound. By recursively applying this idea, eventually a known sc-greedy graph can be obtained and it suffices to show that at each step, there does not exist a non-simple choice function of size one less than the greedy bound.

**Lemma 6.3** (Berliner et al. [13]). Let \( G \) and \( G' \) be such that \( V(G) \cap V(G') = \{v_0\} \). Then

\[
\chi_{SC}(G \cup G') = \chi_{SC}(G) + \chi_{SC}(G') - 1.
\]

Lemmas 6.2 and 6.3 imply the following.

**Corollary 6.4.** Let \( G \) be a graph.

1. If \( \tau(G) \geq GB(G) \) and \( G - v \) is sc-greedy for all \( v \in V \), then \( G \) is sc-greedy.

2. If \( G = G_1 \cup G_2 \) with \( V(G_1) \cap V(G_2) = \{v_0\} \) and \( G_1 \) and \( G_2 \) are both sc-greedy, then \( G \) is sc-greedy.

In fact, even more can be said.

**Lemma 6.5** (Heinold [36]). Let \( G \) be a graph with blocks \( G_1, \ldots, G_k \). Then

\[
\chi_{SC}(G) = \sum_{j=1}^{k} \chi_{SC}(G_j) - k + 1.
\]

In particular, a graph whose blocks are sc-greedy, is itself sc-greedy.
The following special case will be useful later on.

**Corollary 6.6.** If $G'$ is obtained by appending a vertex of degree one to a graph $G$, then $\chi_{SC}(G') = \chi_{SC}(G) + 2$. In particular, if $G$ is sc-greedy, then so is $G'$.

The following lemma is especially helpful in the case $r = 0$ because it can be used to force a color on a specific vertex.

**Lemma 6.7** (Berliner et al. [13]). Let $G$ be a graph and $f$ a choice function for $G$. If $\text{size}(f) = \chi_{SC}(G) + r$ for some $r \geq 0$, then for any $v \in V(G)$ and any set $S$ of $r + 1$ colors, there is an $f$-assignment $L$ for which every proper $L$-coloring of $G$ uses a color from $S$ on $v$.

### 6.3 General results and some examples

**Claim 6.8.** Let $G = (V, E)$ be a graph. If there is a $v \in V(G)$ for which $G - v$ is not sc-greedy, then $G$ is not sc-greedy.

*Proof.* Assume $|V| = n$ and $|E| = e$. Let $f'$ be a choice function of minimum size for the graph $G - v$ which is not sc-greedy. Then $\text{size}(f') < (n - 1) + (e - \deg(v))$. Now let $f$ be a size function for $G$ such that $f(u) = f'(u)$ for all $u \neq v$ and $f(v) = \deg(v) + 1$. Then $f$ is a choice function for $G$ and $\text{size}(f) = \text{size}(f') + 1 + \deg(v) < n + e$. Since $f$ is clearly a choice function for $G$, it follows that $G$ is not sc-greedy.

This claim generalizes in the form of the following corollary.

**Corollary 6.9.** If a graph $G$ contains an induced subgraph that is not sc-greedy, then $G$ is not sc-greedy.

*Proof.* Let $H$ be an induced subgraph of $G$ that is not sc-greedy. The proof is by induction on $|V(G)| - |V(H)|$. If $|V(G)| - |V(H)| = 1$, then the result follows from Claim 6.8. Now assume the result holds for all induced subgraphs $H'$ for which $|V(G)| - |V(H')| < |V(G)| - |V(H)|$. Specifically, it holds for $H \cup v$ for any $v \in N(H)$. By Claim 6.8, $H \cup v$ is not sc-greedy. Since $|V(G)| - |V(H \cup v)| < |V(G)| - |V(H)|$, it follows by induction that $G$ is not sc-greedy.
Corollary 6.9 implies that not all triangulations are sc-greedy because it is possible to add three vertices of degree 4 to $K_{2,3}$ to obtain a triangulation, see Figure 6.1. This graph is a triangulation with induced subgraph $K_{2,3}$, which is not sc-greedy.

![Figure 6.1: An example that illustrates not all triangulations are sc-greedy.](image)

We may also look at what can happen to the sum choice number of a graph upon addition of an edge. One might predict that adding an edge to a graph would increase the sum choice number by at most one. However, this is not the case. The following two facts can be observed:

1. There exist graphs that differ by an edge that have the same sum choice number.
2. There exist graphs that differ by an edge, but whose sum choice numbers differ by two.

The first fact can be observed in Figure 6.2 which displays two graphs that differ by an edge. The graph in Figure 6.2a is a 4-cycle with a pendant path, hence sc-greedy, and the graph in Figure 6.2b is $K_{2,3}$. Both of these graphs have sum choice number 10. The second fact can be observed in Figure 6.3 which displays two graphs that differ by an edge. The graph in Figure 6.3a is $K_{2,3}$, whose sum choice number is 10, while the graph in Figure 6.3b is $G5.5$ whose sum choice number is 12, see Section 6.4.1.

6.3.1 **Edge subdivision and minors**

Some things can be said about edge subdivision and when graphs are sc-greedy.
Claim 6.10. There exist graphs that are sc-greedy for which it is possible to subdivide an edge and obtain a graph that is not sc-greedy.

See Figure 6.4 for an example. An edge of $BW_3$, which is sc-greedy, can be subdivided to obtain $K_{2,3}$, which is not sc-greedy.

Claim 6.11. There exists graphs that are not sc-greedy for which it is possible to subdivide an edge and obtain a graph that is sc-greedy.

See Figure 6.5 for an example. An edge of $K_{2,3}$, which is not sc-greedy, can be subdivided
to obtain $\Theta_{1,1,2}$, which is sc-greedy.

![Figure 6.5: $\Theta_{1,1,2}$ can be obtained by subdividing an edge of $K_{2,3}$.](image)

The above two claims imply the following:

1. There exist graphs that are not sc-greedy that have minors that are sc-greedy.

2. There exist graphs that are sc-greedy that have minors that are not sc-greedy.

### 6.3.2 Minimally not sc-greedy graphs

**Definition 6.12.** A graph is **minimally not sc-greedy** if for all $S \subseteq V(G)$, $G[S]$ is sc-greedy, but $G$ is not sc-greedy.

Two examples of minimally not sc-greedy graphs are $K_{2,3}$ and $P_3 \times P_3$. Two more examples are the graphs $G5.4$ (see Figure 6.6d) and $G5.8$ (see Figure 6.6h), which will be presented in Subsection 6.4.1.

**Question 6.13.** If $G$ is minimally not sc-greedy, does it follow that $\chi_{SC}(G)$ is equal to $GB(G) - 1$?

Note that for the examples of minimally not sc-greedy graphs mentioned above, the answer to this question is yes.

Assume $G$ is a minimally not sc-greedy graph for which $\chi_{SC}(G) = GB(G) - 2$. Then, there is a non-simple choice function $g$ for $G$ of size $GB(G) - 2$ by Lemma 6.2. Additionally,
\( \chi_{SC}(G - v) = GB(G - v) = GB(G) - \deg(v) - 1 \) for all \( v \in V(G) \). This implies the following:

\[
\text{size}(g_{G-v}) = GB(G) - 2 - g(v) \\
\geq \chi_{SC}(G - v) \\
= GB(G) - \deg(v) - 1 \\
\Rightarrow GB(G) - 2 - g(v) \geq GB(G) - \deg(v) - 1 \\
\Rightarrow g(v) \leq \deg(v) - 1
\]

This must be true for all \( v \in V(G) \), which implies that \( \delta(G) \geq 3 \). However, the graphs \( K_{2,3}, P_3 \times P_3, \) and \( G_{5,4} \) are all minimally not sc-greedy graphs of minimum degree 2. Thus, if \( G \) is a minimally not sc-greedy graph for which \( \delta(G) = 2 \), then \( \chi_{SC}(G) = GB(G) - 1 \). Note that \( G_{5,8} \) is a minimally not sc-greedy graph of minimum degree 3.

Lemma 6.2 implies that if a graph \( G \) is minimally not sc-greedy, then (1) \( G - v \) is sc-greedy for all \( v \in V(G) \) and (2) there is a non-simple choice function \( f \) for \( G \) such that \( \text{size}(f) = GB(G) - 1 \). Hence, the question remains if there is a non-simple choice function \( f \) for \( G \) such that \( \text{size}(f) < GB(G) - 1 \). Does such a graph \( G \) exist? This would provide an answer to Question 6.13.

### 6.4 Graphs on a small number of vertices

In this section we explore graphs on at most six vertices and determine their sum choice number. As noted earlier, all graphs on at most four vertices are sc-greedy. This is not hard to check. Since \( K_{2,3} \) a graph on five vertices that is not sc-greedy, we look at all other graphs on five vertices.

#### 6.4.1 Graphs on five vertices

Let \( G \) be a connected graph on five vertices. If \( G \) has a cut-vertex, then \( G \) is sc-greedy by Lemma 6.5. Thus, it remains to consider all graphs on five vertices that do not have a cut-vertex, of which there are nine nonisomorphic such graphs. These graphs are depicted in Figure 6.6.
Figure 6.6: All graphs on five vertices without a cut-vertex.

Table 6.2 summarizes the greedy bound, sum choice number, and whether or not the result was known for each of the graphs on five vertices in consideration.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\chi_{SC}(G)$</th>
<th>$GB(G)$</th>
<th>sc-greedy?</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G5.1$</td>
<td>10</td>
<td>10</td>
<td>yes</td>
<td>$C_5$</td>
</tr>
<tr>
<td>$G5.2$</td>
<td>10</td>
<td>11</td>
<td>no</td>
<td>$K_{2,3}$</td>
</tr>
<tr>
<td>$G5.3$</td>
<td>11</td>
<td>11</td>
<td>yes</td>
<td>$\Theta_{0,1,2}$</td>
</tr>
<tr>
<td>$G5.4$</td>
<td>11</td>
<td>12</td>
<td>no</td>
<td></td>
</tr>
<tr>
<td>$G5.5$</td>
<td>12</td>
<td>12</td>
<td>yes</td>
<td></td>
</tr>
<tr>
<td>$G5.6$</td>
<td>12</td>
<td>12</td>
<td>yes</td>
<td>$BW_4, P_5^2$</td>
</tr>
<tr>
<td>$G5.7$</td>
<td>13</td>
<td>13</td>
<td>yes</td>
<td></td>
</tr>
<tr>
<td>$G5.8$</td>
<td>12</td>
<td>13</td>
<td>no</td>
<td>$W_4$</td>
</tr>
<tr>
<td>$G5.9$</td>
<td>14</td>
<td>14</td>
<td>yes</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2: Sum choice number of graphs on five vertices.

We note here that in many of the case analyses that follow, specific size functions of size $GB - 1$ will be considered. These size functions were determined by Steve Butler using Sage and are the only such such functions that need to be considered. This is because for all other
size functions $g$ of size $GB - 1$, Steve was able to use Sage to produce $g$-assignments that are not list-colorable. Also, any vertex labeling is as in Figure 6.6. In the cases for which we show a graph is sc-greedy, it can be observed that $G - v$ is a graph on four vertices for all $v \in V(G)$ and hence sc-greedy. Thus, it suffices to show that there does not exist a non-simple choice function of size $GB - 1$.

**The graph $G_{5.4}$ is not sc-greedy.** This graph has greedy bound 12, but $\chi_{SC}(G_{5.4}) = 11$. It will be shown that the size function $f$ as shown in Figure 6.7a is a choice function of size 11 for $G_{5.4}$. Fix an arbitrary $f$-assignment $L$. We determine information about $L$. If $L(v_1) \cap L(v_2) = \emptyset$ or $L(v_1) \cap L(v_5) = \emptyset$, then $G_{5.4}$ will be $L$-colorable. This is because, without loss of generality, $G_{5.4} - v_1v_2$ has a cut-vertex so it is sc-greedy. So assume $L(v_1) \cap L(v_2) \neq \emptyset$ and $L(v_1) \cap L(v_5) \neq \emptyset$.

Case 1: There is an element $a \in L(v_1) \cap L(v_2)$ such that $a \notin L(v_5)$. Assign color $a$ to $v_1$, remove it from $L(v_2)$, then delete $v_1$. Figure 6.7b illustrates the list sizes on the remaining vertices. This graph can be colored from lists of these sizes unless the lists are as illustrated in Figure 6.7c. This implies $v_1$ should not be assigned $a$. It also provides information about all of the lists. There are two possible list assignments obtained from this situation, see Figures 6.7d and 6.7e. The graph $G_{5.4}$ can be colored from both of these list assignments.

Case 2: There is an element $a \in L(v_1) \cap L(v_2) \cap L(v_5)$. Assign color $a$ to $v_2$ and $v_5$, remove it from $L(v_1)$, $L(v_3)$ and $L(v_4)$, then delete $v_2$ and $v_5$. Figure 6.7f illustrates the list sizes on the remaining vertices. This graph can be colored from lists of these sizes. This shows that $f$ is a choice function of size 11 for $G_{5.4}$.

**The graph $G_{5.5}$ is sc-greedy.** This graph has greedy bound 12. It is not hard to see that the list sizes as indicated in Figure 6.8a is the only non-trivial size function of size 11 to check. Figure 6.8b provides an $f$-assignment for which there does not exist a list-coloring. Thus, $\chi_{SC}(G_{5.5}) = 12$.

**The graph $G_{5.7}$ is sc-greedy.** This graph has greedy bound 13. The list sizes as indicated
Figure 6.7: $G_{5.4}$ is not sc-greedy.

in Figures 6.9a, 6.9b, 6.9c are the only non-trivial size functions of size 12 to check. Figures 6.9d, 6.9e, 6.9f provide $f$-assignments for which there do not exist a list-colorings for each of the non-simple size functions in question. Thus, $\chi_{SC}(G_{5.7}) = 13$.

The graph $G_{5.8}$ is not sc-greedy. This graph is $W_4$ and has greedy bound 13, but $\chi_{SC}(G_{5.8}) = \chi_{SC}(W_4) = 12$. It will be shown that the size function $f$, as illustrated in Figure 6.10a, is a choice function for $G_{5.8}$. If there is an element $a \in L(v_2) \cap L(v_4)$, then assign $a$ to vertices $v_2$ and $v_4$. The remaining vertices can then be colored. Thus, assume $L(v_2) \cap L(v_4) = \emptyset$. Without loss of generality, let $L(v_2) = \{a, b\}$, $L(v_4) = \{c, d\}$, and assume $d \in L(v_4) - L(v_5)$. Assign $d$ to $v_4$, then remove $d$ from the lists of adjacent vertices, and delete $v_4$ from the graph. Figure 6.10b illustrates the list sizes on the remaining vertices. This graph can be colored from lists of the indicated sizes, unless the lists are as illustrated in Figure 6.10c. This implies $v_4$ should not be assigned $d$. However, this indicates what all of the lists will be, and Figure 6.10d shows these lists. This graph can be colored from the provided $L$-assignment, as shown in Figure 6.10e. Thus, $\chi_{SC}(G_{5.8}) = 12$. 
The graph $G_{5.9}$ is sc-greedy. This graph has greedy bound 14. The list sizes as indicated in Figures 6.11a, 6.11b, 6.11c, 6.11d are the only non-trivial size functions of size 13 to check. Figures 6.11e, 6.11f, 6.11g, 6.11h provide $f$-assignments for which there do not exist a list-colorings for each of the non-simple size functions in question. Thus, $\chi_{SC}(G_{5.9}) = 14$.

6.5 Wheels and broken wheels

In his PhD dissertation, Heinold [35] explored the broken wheel $BW_k$ with respect to sum-choosability. In particular, he showed that $BW_{10}$ is not sc-greedy. This provided some very useful information. First, it implies that not all outerplanar graphs are sc-greedy. It also implies that $BW_k$ will not be sc-greedy for all $k \geq 10$ since such graphs will have $BW_{10}$ as an induced subgraph. It was also shown that $BW_k$ is sc-greedy for all $k \leq 9$. It may be observed here that $BW_{10}$ is minimally not sc-greedy. Heinold also showed that there exist $k$ for which the gap between $GB(BW_k)$ and $\chi_{SC}(BW_k)$ is arbitrarily large. While it is known which broken wheels are sc-greedy, it is not known what $\chi_{SC}(BW_k)$ is in general. Heinold guessed that $\chi_{SC}(BW_k) = GB(BW_k) - \left\lfloor \frac{k+1}{11} \right\rfloor$. See [35] for more details. His dissertation also establishes many techniques that could be used to determine $\chi_{SC}(BW_k)$.

As far as we know, the determination of $\chi_{SC}(BW_k)$ remains an open problem. However, the results mentioned above can be used to obtain information about the sum-choosability of wheels $W_k$. Since $BW_{10}$ is not sc-greedy, it follows that $W_k$ is not sc-greedy for all $k \geq 11$ since all of these graphs will have $BW_{10}$ as an induced subgraph. Thus, with respect to being
sc-greedy, it remains to examine $W_k$ for $k \leq 10$. For small values of $k$, the result follows quickly. The graph $W_3$ is isomorphic to $K_4$, and thus is sc-greedy. It was shown in Subsection 6.4.1 that $W_4$ is not sc-greedy. The classification would be completed by looking at $W_k$ for $k = 5, \ldots, 10$.

We summarize this information in the following theorem.

**Theorem 6.14.** Let $BW_k$ be a broken wheel and $W_k$ be a wheel. The following is known:

1. if $k \leq 9$, then $BW_k$ is sc-greedy [35],

2. if $k \geq 10$, then $BW_k$ is not sc-greedy [35],

3. if $k \leq 3$, then $W_k$ is sc-greedy, and

4. if $k = 4$ or $k \geq 11$, then $W_k$ is not sc-greedy.

Some general observations may be made about what remains to be done. First, for any $k$, removing an arbitrary vertex of $W_k$ will yield either a broken wheel or a cycle. Thus, for $k = 5, \ldots, 10$, the graph $W_k - v$ is sc-greedy for all $v \in V(W_k)$. To determine whether or not $W_k$ is sc-greedy, it must be determined whether or not there exists a non-simple choice function for $W_k$ of size $GB(W_k) - 1 = 3k$.
In this section we show that trees of cycles, as defined in Section 2.1 are sc-greedy. More specifically, we prove that paths of cycles are sc-greedy and the result for trees of cycles follows as a corollary.

Let $G$ be a path of $k$ cycles as defined in Section 2.1. Recall that $G$ can be embedded in the plane so that the weak dual of $G$ is a path of length $k$. For $i = 2, \ldots, k$, let $L_i$ be the subgraph of $G$ induced by the vertices of $G_1, \ldots, G_{i-1}$, let $R_i$ be the subgraph of $G$ induced by the vertices of $G_i, \ldots, G_k$, and let $I_i = G[t_i, b_i]$. Note that the greedy bound for $G$ is equal
Theorem 6.15. Paths of cycles are sc-greedy.

Proof. The proof is by induction on $k$. The result clearly holds for $k = 1$, as cycles are known to be sc-greedy. Now assume paths of at most $m$ cycles are sc-greedy for all $m < k$. It will be shown that the result holds for $k$. Recall that the greedy bound for $G$ is $2\sum_{i=1}^{k} a_i - 3(k - 1)$.

Assume $f$ is a minimal choice function for a graph $G$ which is a path of $k$ cycles. The proof requires a case analysis on $f(t_i) + f(b_i)$ for $2 \leq i \leq k$. Observe here that $f(t_i) + f(b_i) \geq 3$ as $I_i$ is isomorphic to $P_2$ and $\chi_{SC}(P_2) = 3$.

Assume first that $f(t_i) = 1$. Now, in $G - t_i$ the vertex $b_i$ will be a cut-vertex that splits $G - t_i$ into two shorter paths of cycles, perhaps with pendant paths attached to them. By induction, shorter paths of cycles are sc-greedy, hence attaching any pendant paths will also yield an sc-greedy graph. Thus, $G - t_i$ is sc-greedy.

Furthermore, assume next there is a $j$ for which there exists $w_j \in V(G_j)$ such that $f(w_j) = 1$ then in $G - w_j$ there is a cut-vertex that splits $G - w_j$ into two shorter paths of cycles, perhaps with pendant paths attached to them. By induction, shorter paths of cycles are sc-greedy, hence attaching any pendant paths will also yield an sc-greedy graph. Thus, $G - w_j$ is sc-greedy.

It follows that $G - v$ is sc-greedy for all $v \in V(G)$. Thus, to show that $G$ is sc-greedy, it remains to show there does not exist a non-simple size function of size $GB - 1$ for $G$. So we assume that $f(v) > 1$ for all $v \in V(G)$ for the remainder of the proof.

Case 1: $f(t_i) + f(b_i) \geq 5$ for all $i = 2, \ldots, k$.

Recall that $f(w_i) \geq 2$ for all $w_i \neq t_i, b_i$. So

$$\text{size}(f) \geq 2(a_1 - 2) + 2 \sum_{i=2}^{k-1} (a_i - 4) + 2(a_k - 2) + 5(k - 1)$$

$$= 2 \sum_{i=1}^{k} a_i - 3(k - 1)$$

$$= GB(G)$$
and the result follows.

Case 2: There is a $j$ such that $f(t_j) = f(b_j) = 2$.

Assume $\text{size}(f) = GB(G) - 1 = 2 \sum_{i=1}^{k} a_i - 3(k - 1) - 1$. By induction,

$$\chi_{SC}(L_j) = 2 \sum_{i=1}^{j-1} a_i - 3(j - 2),$$

$$\chi_{SC}(R_j) = 2 \sum_{i=j}^{k} a_i - 3(k - j),$$

so it follows that

$$\text{size}(f_{L_j - \{t_j, b_j\}}) \geq 2 \sum_{i=1}^{j-1} a_i - 3(j - 2) - 4,$$

$$\text{size}(f_{R_j - \{t_j, b_j\}}) \geq 2 \sum_{i=j}^{k} a_i - 3(k - j) - 4.$$

In fact, the above inequalities must be equalities so that $\text{size}(f_{L_j - \{t_j, b_j\}}) + 4 + \text{size}(f_{R_j - \{t_j, b_j\}}) = \text{size}(f)$ holds. This allows for an $f$-assignment $L$ for which $G$ is not $L$-colorable to be defined as follows:

Let $g_1$ be a size function for $L_j - \{t_j, b_j\}$ such that

$$g_1(w) = \begin{cases} 
  f(w) - 1 & \text{if } w \sim t_j \text{ or } b_j \\
  f(w) & \text{else}
\end{cases}$$

and let $g_2$ be a size function for $R_j - \{t_j, b_j\}$ such that

$$g_2(w) = \begin{cases} 
  f(w) - 1 & \text{if } w \sim t_j \text{ or } b_j \\
  f(w) & \text{else}
\end{cases}.$$

Thus $\text{size}(g_1) < \chi_{SC}(L_j - \{t_j, b_j\}) \leq \text{size}(f_{L_j - \{t_j, b_j\}})$ and $\text{size}(g_2) < \chi_{SC}(R_j - \{t_j, b_j\}) \leq \text{size}(f_{R_j - \{t_j, b_j\}})$, implying $L_j - \{t_j, b_j\}$ and $R_j - \{t_j, b_j\}$ are not $g_1$- and $g_2$-choosable, respectively. There are $g_1$- and $g_2$-assignments $L_L$ and $L_R$, respectively, for which $L_j - \{t_j, b_j\}$ and $R_j - \{t_j, b_j\}$ cannot be list-colored.

Let $L$ be an $f$-assignment for $G$ defined as follows: $L(b_j) = L(t_j) = \{c_1, c_2\}$ where $c_1$ and $c_2$ do not appear in any of the lists $L_L$ and $L_R$, $L = L_L$ on $L_j - \{t_j, b_j\}$ and $L = L_R$ on
$R_j - \{t_j, b_j\}$, except append $c_1$, $c_2$, $c_2$ and $c_1$ to the lists of neighbors of $t_j$ and $b_j$ in $L_j$ and $R_j$, respectively. If $c$ is a proper $L$-coloring of $G$, then either $c(t_j) = c_1$ and $c(b_j) = c_2$ or $c(t_j) = c_2$ and $c(b_j) = c_1$. By the construction of $L$, this coloring will not provide a proper coloring of either $L_j$ or $R_j$, a contradiction.

Therefore, paths of cycles are sc-greedy.

We now observe that Theorem 6.15 extends to trees of cycles. Let $G$ be a tree of cycles as defined in Section 2.1. Recall that $G$ can be embedded in the plane so that the weak dual, call it $G'$, of $G$ is a tree on $k$ vertices.

Let $\mathcal{I} = \{\{i, j\} : V(G_i) \cap V(G_j) \neq \emptyset\}$. Then $|\mathcal{I}| = |E(G)|$. In particular, the number of pairs of cycles in $G$ that share vertices is equal to the number of edges in the weak dual of $G$. This allows us to compute the greedy bound of $G$ as

$$GB(G) = 2 \sum_{i=1}^{k} a_i - 3|\mathcal{I}| = 2 \sum_{i=1}^{k} a_i - 3|E(G')|.$$ 

It then follows as a corollary to Theorem 6.15 that trees of cycles are sc-greedy:

**Corollary 6.16.** Trees of cycles are sc-greedy.

This result follows from Theorem 6.15 and its proof because the argument and case analysis is applied to the intersection of $G_i$ with $G_j$ and the same properties will hold.
CHAPTER 7. GENERAL CONCLUSIONS AND FUTURE WORK

In this chapter we make some general conclusions and discuss plans for future work on some of the topics of this thesis.

7.1 List precoloring extensions

Since the publishing of [11], there has been some progress on this problem and related questions.

The following result is a generalization of Thomassen’s 5-list-coloring theorem, in particular the 5-list-colorability of planar graphs, that applies to graphs that are almost planar and allows for some vertices to be assigned lists of size 4.

**Theorem 7.1** (Dvořák, Lidický, Mohar). Let $G = (V,E)$ be a graph drawn in the plane with some crossings. Let $P \subseteq V$ be a set of vertices such that the distance between any pair of crossed edges is at least 15, the distance between any pair of crossed edges and a vertex of $P$ is at least 13, and dist$(P) \geq 11$. Let $L : V \to 2^\mathbb{N}$ be an assignment of lists of colors to the vertices of $G$ such that $|L(v)| = 4$ for all $v \in P$ and $|L(w)| = 5$ for all $w \in V - P$. Then $G$ is $L$-colorable.

Additionally, the following group claims to have found a $d$ such that the following is true:

**Theorem 7.2** (Dvořák, Lidický, Mohar, Postle). If $G$ is a planar graph with list assignment $L$ that gives a list of size 1 or 5 to each vertex and the distance between any pair of vertices with list size 1 is at least $d$, then $G$ is $L$-colorable.

Thus answering Albertson’s question in the positive. What does this mean for us? There are certainly many other modifications to this question that can be considered. For example,
what if there are far apart vertices that are assigned lists of size 2 or 3? What if instead of precolored vertices, we consider precolored edges or triangles? This answer to Albertson’s question only leads to even more questions to explore in the future.

7.2 \{2,2\}-extendability

The conjecture of Hutchinson \cite{38} remains open. Thus, the first goal is to finish examining the remaining cases. Some related questions can also be asked. For example, what if \( G \) is 3-connected? Can we prove the conjecture in that case? Additionally, in what other ways can the hypotheses of Thomassen’s 5-list-coloring theorem be modified so that we obtain a list-colorable graph? Can we determine what planar graphs are \{1,2\}-extendable?

7.3 Catalog

The catalog allows for much expansion. In particular, a classification of all good 6-wheels with \( f(u) = 1 \) and \( f(u) = 5 \) is currently near completion. What other small graphs can be determined to be good? How can we incorporate information obtained about graphs with respect to sum-list-coloring to simplify some of the existing results and make it easier to find new results? For example, knowing the sum choice number of a graph will provide a lower bound for the sum of list sizes needed for a graph to be good. Consider the 4-wheel. It was shown in Chapter 6 that \( \chi_{SC}(W_4) = 12 \). Thus, the sum of list sizes assigned to the vertices of \( W_4 \) would need to be at least 12 in order for it to possibly be good.

7.4 Sum-list-coloring

The notion of sum-list-coloring is still a fairly new area and there are not a lot of results known. This means that there are many open questions in the area. Some goals for the near future are to complete a characterization of the sum choice number of all graphs on six vertices and other graphs on a small number of vertices. In particular determining whether the wheels \( W_5, \ldots, W_{10} \) would complete a classification of all sc-greedy wheels and broken wheels begun by Heinold \cite{35}. While in the midst of working on this characterization, we are exploring the
question of how many colors are needed to create an $f$-assignment for a graph $G$ that shows $f$ is not a choice function for $G$. How does this number relate to certain parameters of the graph?

As we showed that paths of cycles and certain trees of cycles are sc-greedy, it is a natural progression to explore showing that cycles of cycles are sc-greedy and that paths of cliques are sc-greedy. In what other ways can we ‘glue’ sc-greedy graphs together to obtain graphs that are sc-greedy? In particular, what if two sc-greedy graphs are joined by a cut edge? These are things we are currently working on.

While it is known which theta graphs are sc-greedy, we seek to determine the sum choice number of generalized theta graphs with an arbitrary number of disjoint paths between two vertices. We note that there are some partial results on this question that I have completed with Michael Young.
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