3-DIMENSIONAL FLAW CHARACTERIZATION THROUGH
2-DIMENSIONAL IMAGE RECONSTRUCTIONS

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ABSTRACT

The 3-dimensional Born approximation is a powerful method for characterizing flaws because it can be applied to characterize flaws of any shape. Yet, the method has a number of difficulties, such as taking and processing a large amount of data, and the complications associated with 3-dimensional image reconstructions such as 3-dimensional interpolation, long computing time, etc. For these reasons the method is usually simplified and restricted to characterize symmetrically shaped flaws, which can be characterized by using only a small number of pulse echoes. Though this procedure is simple, it cannot be applied to characterize flaws of more general shape.

In this paper, the 3-dimensional image construction problem is reduced to a series of 2-dimensional image reconstructions, thereby avoiding the difficulties associated with 3-dimensional image reconstructions. The reconstructed 2-dimensional images represent the 2-dimensional projections or shadows of the 3-dimensional flaw characteristic function. Each projection image is reconstructed independently using well developed computerized tomography reconstruction techniques. If the shape of the flaw is not too irregular or if the fine details of the shape are not of interest, only a few of these projection images suffice to characterize the flaw. The magnitude scaling problem and the alignment problem of the echoes at different incidence directions can be handled easily in the algorithm. The results of simulation studies are presented.

INTRODUCTION

In the three-dimensional inverse Born approximation the back-scattered amplitude \( A(\omega, \Omega) \) of a plane acoustic wave incident on an isotropic homogeneous medium can be written in the form:

\[
A(\omega, \Omega) = \omega^2 F(\lambda, \mu, \theta) S(2\omega/v, \Omega)
\]

where \( \omega, \Omega \) are respectively the angular frequency and direction of the incident wave, \( v \) is the velocity of the acoustic wave in the medium, \( F(\lambda, \mu, \theta) \) is a function of the Lame parameters \( \lambda \) and \( \mu \) of the medium and of the scattering angle \( \theta \), and \( S(k, \Omega) \) is the Fourier transform of the characteristic function \( \rho(r) \) of the flaw with \( k \) and \( \Omega \) respectively denoting
the magnitude and the direction of the spatial frequency vector $k$ [1]. Here the characteristic function $\rho(r)$ is defined to be 1 inside the flaw and 0 outside.

With the substitution $k = \omega/v$ and rearranging, equation (1) can be written in the form

$$S(k,\Omega) = 4A(kv/2,\Omega)/\left[k^2v^2F(\lambda,\mu,\theta)\right] \tag{2}$$

Consider the range of the parameters $\omega, k, \Omega$ of the functions $S$ and $A$. The direction $\Omega$ is a constant for an incident wave. If the incident pulse is broad band, it may be considered as to contain frequency components from $\omega = 0$ to $\omega = \omega$. Thus equation (2) shows that from the pulse echo of a broad band pulse incident in the $\Omega$ direction one obtains $S(k,\Omega)$ from $k = 0$ to $k = \infty$ in the same direction $\Omega$. Since $S(k,\Omega)$ is the Fourier transform of the real function $\rho(r)$, $S(k,-\Omega) = S^*(k,\Omega)$, therefore $S(k,-\Omega)$ can also be obtained. In other words the pulse echo measurement yields a line of the Fourier components of the characteristic function $\rho(r)$ of the flaw, with the line running from $\omega$ in the direction $-\Omega$ to $\omega$ in the direction $\Omega$ in Fourier space and passing through the origin. This situation is illustrated in Figure 1. Therefore inspecting the flaw at all angles in a half space will yield all the Fourier components of $\rho(r)$. From these Fourier components $\rho(r)$ can be reconstructed through a 3-dimensional inverse Fourier transformation. The Lamé constants of the flaw can also be obtained in the process.

Thus in order to characterize the flaw one has to inspect it from all $2\pi$ solid angles in 3 dimensions, and to perform an inverse 3-dimensional Fourier transform. Such a procedure involves a number of difficulties: (1) a large amount of data to take and to process; (2) possible inaccessibility of some angles; (3) complications associated with 3-dimensional image reconstructions, such as 3-dimensional interpolation, long computing time, etc. For these reasons the method is usually simplified and restricted to characterize symmetrically shaped flaws, which can be characterized by using only a small number of pulse echoes, such as in the case of the multiviewing inspection system being developed at Ames Laboratory [2]. This simplified procedure is known as the 1-dimensional inverse Born approximation [3]. Though the procedure is simple, it cannot be applied to characterize flaws of a more general shape.

![Diagram](https://via.placeholder.com/150)

Fig. 1. A pulse echo and the line of Fourier components calculated from it.

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1-DIMENSIONAL AND 2-DIMENSIONAL PROJECTED IMAGES

In this paper we present a method which simplifies the 3-dimensional characterization process for flaws of general shape by reducing the 3-dimensional image reconstruction problem to a series of 2-dimensional image reconstructions, thereby avoiding the difficulties associated with direct 3-dimensional image reconstructions.

First of all let us find out how much we know about the 3-dimensional flaw shape \( \rho(r) \) from only a line of its Fourier transform. We will choose to work in rectangular co-ordinate systems rather than the spherical co-ordinate system employed in equation (2). \((x,y,z)\) denotes spatial co-ordinates in object space, and \((k_x,k_y,k_z)\) denotes spatial frequency co-ordinates in Fourier space. Let \( S(k_x,k_y,k_z) \) be the Fourier transform of an arbitrary 3-dimensional function \( \rho(x,y,z) \), and consider its Fourier components located on the \( k_x \) axis. These components are characterized by \( k_y = k_z = 0 \), and can be written as

\[
S(k_x,k_y,k_z) |_{k_y=k_z=0} = \iint dx \ e^{2\pi i k_x x} \iint dy \ dz \rho(x,y,z)
\]

where

\[
p_{yz}(x) = \iint dy \ dz \rho(x,y,z)
\]

Hence the line of Fourier components is the 1-dimensional Fourier transform of the function \( p_{yz}(x) \).

The physical meaning of the function \( p_{yz}(x) \) is that it is the 1-dimensional projection or projected image of \( \rho(x,y,z) \) onto the \( x \) axis, as shown in Figure 2. In computerized tomography the relationship expressed in equations (3) and (4) is known as the central slice theorem or projection theorem. If the function \( \rho(r) \) is the characteristic function of a flaw, then equation (4) reduces to

\[
p_{yz}(x') = \text{cross-sectional area of the flaw at the plane } x=x'
\]

Thus \( p_{yz}(x) \) represents the cross-sectional area profile of the flaw. Recall that this area profile is proportional to the inverse Fourier transform of the pulse echo frequency components after dividing by \( \omega^2 \) in the frequency

![Diagram of 3-Dimensional and 2-Dimensional Projected Images](image)

Fig. 2. Two-dimensional and 1-dimensional projected images.
domain is equivalent to integrating twice in the time domain. In other words, up to a multiplicative constant the area profile can be obtained directly by integrating the pulse echo waveform twice.

\[
\text{area profile} \propto \text{I.F.T. } \frac{\text{pulse echo spectrum}}{\omega^2} 
= \iint \text{[pulse echo]} 
\]

Equation (7) is the basis of the work on ramp response by B. D. Cook et al. [4]. On the other hand, by differentiating equation (7) twice we get the result that the pulse echo is proportional to the 2nd derivative of the area profile of the flaw

\[
\text{pulse echo} \propto \frac{d^2}{dt^2} \text{[area profile]} 
\]

The same result is arrived at in a different way by J. H. Rose and J. M. Richardson [5] in their work on the time-domain inverse Born approximation.

Equation (5) exhibits the relationship between the measured quantity \( p_{yz}(x) \) and the 3-dimensional flaw shape. In this paper we propose to relate \( p_{yz}(x) \) to a 2-dimensional quantity. Equation (4) can be written in the form

\[
p_{yz}(x) = \int dy p_z(x,y) 
\]

where

\[
p_z(x,y) = \int dz \rho(x,y,z) 
\]

Thus \( p_{yz}(x) \) can be considered as the 1-dimensional projection of the 2-dimensional function \( p_z(x,y) \) projected along the \( y \) direction onto the \( x \) axis. Given enough of these projects of \( p_z(x,y) \) at all angles from 0 to \( \pi \) in the \( x-y \) plane, the function \( p_z(x,y) \) itself can be reconstructed uniquely. This is a well-known result in computerized tomography. That only projections from 0 to \( \pi \) instead of from 0 to \( 2\pi \) are needed is due to the inversion symmetry of the projection operation, i.e. the projection at angle \( \theta \) is the same as the projection at angle \( \theta + \pi \). The reconstruction of \( p_z(x,y) \) from its projections can be accomplished using object-space reconstruction algorithms such as filtered back projection, or it can proceed in Fourier space after taking the transform of the projections [6]. In the rest of the paper we will employ the more standard notation \( p_z(s;\theta) \) used in computerized tomography to denote the 1-dimensional projection of \( p_z(x,y) \) at angle \( \theta \), where \( s \) represents the coordinate on the \( x \) axis after the \( x \) axis is rotated by the angle \( \theta \). By way of example, \( p_{yz}(x) \) becomes \( p_z(s;\pi/2) \) in this notation.

The reconstruction can be visualized easily in Fourier space. Equations (3) and (8) show that the Fourier transform of the projection \( p_z(s;\pi/2) \) of \( p_z(x,y) \) onto the \( x \) axis is the line of Fourier components of \( \rho(x,y,z) \) located on the \( k_x \) axis, as illustrated in Figure 3. Therefore the Fourier components on the entire \( k_x-k_y \) plane in Fourier space can be filled up by taking the projections of \( p_z(x,y) \) at all angles from 0 to \( \pi \) in the \( x-y \) plane, since each projection gives rise to a line of Fourier components in the \( k_x-k_y \) plane. Now the Fourier components in the \( k_x-k_y \) plane can be viewed as the 2-dimensional Fourier transform of \( p_z(x,y) \):

\[
S(k_x,k_y) \mid_{k_z=0} = \iint \rho(x,y,z) e^{i(k_x x + k_y y + k_z z)} \ dx dy dz \mid_{k_z=0} 
= \iint dx e^{ik_x x} \int dy \ i e^{ik_y y} p_z(x,y) 
\]

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Hence from this plane of Fourier components the function $p_z(x,y)$ can be reconstructed through 2-dimensional inverse Fourier transformation. The reconstruction process is sketched in Figure 4.

**RECONSTRUCTING 2-D PROJECTIONS FROM PULSE ECHOES**

Because of the one to one correspondence between a pulse echo and a line of Fourier components as expressed in equation (2), pulse echo inspection of a flaw at all angles from 0 to $\pi$ in the x-y plane will yield all the Fourier components of the flaw on the $k_x-k_y$ plane. This plane of Fourier components in turn will enable $p_z(x,y)$ to be reconstructed. The complete procedure for reconstructing $p_z(x,y)$ from the pulse echoes is graphically illustrated in Figure 5. Pulse echo measurements using broad band incident pulses are taken at each incident angle $\theta$ in the x-y plane. Fourier transforming these waveforms yields the frequency components $A(k,\theta)$, from which the spatial Fourier components $S(k,\theta)$ of the flaw can be obtained by dividing $k^2$ (equation (2)). Inverse transforming $S(k,\theta)$ gives the quantities $p_z(s;\theta)$ (equation (3)), which are the 1-dimensional projections $p_z(x,y)$ at angle $\theta$, $\theta \in [0,\pi/2]$. From these 1-dimensional projections the quantity $p_z(x,y)$ can be reconstructed, either by using object-space reconstruction algorithms or by using Fourier-space algorithms.
Now we note some technical aspects of the reconstruction problem. First, $S(k,\theta)$ can be obtained from the pulse echoes for all $k$ except at $k = 0$, which corresponds to the D.C. level. The D.C. level can be determined from the value of $p_z(s;\theta)$ for large $|s|$, since $p(x,y,z) = 0$ for $x$, $y$, or $z$ outside the interval $[-a,a]$ means $p_z(s;\theta) = 0$ for $s$ in the same range, where $a$ is the radius of a sphere circumscribing the flaw and thus is also the radius of a circle circumscribing $p_z(x,y)$, and $s$ is measured from the projected location of the center of the circumscribing circle on the $s$ axis. The geometry is shown in Figure 6.
Second, since equation (2) is only approximately true (the inverse Born approximation), the quantities \( p_z(s; \theta) \) calculated from the pulse echoes contain errors. In frequency space the most serious errors are found in the high frequency range \( k a > 1 \) where the Born approximation becomes invalid, and in the low frequency range \( k a < 0 \) because the errors there are magnified by the factor \( 1/k^2 \). The frequency components in the intermediate frequency can be considered to be relatively reliable, and therefore \( p_z(s; \theta) \) can be treated as band-limited in the spatial frequency space. On the other hand \( p_z(s; \theta) \) is space-limited in the object space, i.e.

\[
p_z(s; \theta) = 0 \quad s \notin [-a, a]
\]

These two conditions make it possible to use the Gerchberg-Papoulis algorithm [7,8] to further improve the quantities \( p_z(s; \theta) \)'s calculated from the pulse echoes. The algorithm works on the basis that the Fourier transform of any space-limited function is an entire function, and for an entire function it is possible to estimate its value anywhere from its values within a finite interval. The algorithm is sketched in Figure 7. \( p_z(s; \theta) \) is transformed back and forth between the spatial frequency space and object space, being corrected in the spatial frequency space by the intermediate frequency components, and in object space by the fact that \( p_z(s; \theta) = 0 \) for \( s = a \) and \( a = s \). The additional a priori information that \( p_z(s; \theta) \geq 0 \) for all \( s \) is also incorporated into the iterative scheme. The interval \( [-a, a] \) can be estimated from the first estimate of \( p_z(s; \theta) \).

Third, the magnitude of the pulse echoes depends on such variables as the distance of the transducer from the flaw and the inspection angle \( \theta \) through the factor \( F(\lambda, \mu, \theta) \) in equation (1). Hence the quantities \( p_z(s; \theta) \)'s calculated from these pulse echoes are not on the same scale. They should be normalized relative to each other before input to the reconstruction of \( p_z(x, y) \). Normalization can be achieved easily by noting that for a properly normalized \( p_z(s; \theta) \) the following reaction holds

\[
\int p_z(s; \theta) ds = \int \rho(x, y, z) dx dy dz = \text{volume of flaw}
\]

![Fig. 7. Gerchberg-Papoulis iterations to improve \( p_z(s; \theta) \).](image-url)
where we have used equations (8) and (9). Thus \( p_z(s; \theta) \) can be normalized by dividing by its own integral value:

\[
p_z(s; \theta) \rightarrow \frac{p_z(s; \theta)}{\int p_z(s; \theta) \, ds}
\]

Fourth, the \( p_z(s; \theta) \)'s at different \( \theta \) should also be properly aligned for the reconstruction of \( p_z(x,y) \). Let point \( C \) be the centroid of \( p_z(x,y) \). By definition any line passing through \( C \) divides \( p_z(x,y) \) into 2 parts of equal weight, therefore the projections of these 2 parts along the direction of the dividing line also have equal weight. In other words the centroid \( C \) of \( p_z(x,y) \) projects onto the centroid of \( p_z(s; \theta) \) for all \( \theta \). Hence the calculated \( p_z(s; \theta) \)'s at different angles can be aligned by adjusting their positions so that the perpendicular lines through their centroids converge at a common point.

What use can we make of the reconstructed \( p_z(x,y) \)? Equation (9) indicates that \( p_z(x,y) \) is the 2-dimensional image of the flaw shape projected along the z direction onto the x-y plane. The numerical value of the function is a measure of the thickness of the 3-dimensional object at each lateral position \((x,y)\), and its shape represents the shadow or silhouette of the object viewed in the z direction. By rotating the plane on which the pulse echo measurements are made (the x-y plane in Figure 5), one can obtain the 2-dimensional projection of \( \rho(r) \) in other directions. If the shape of the flaw is not too irregular or if the fine details of the shape are not of interest, only a few of these projection images suffice to characterize the flaw. In general this condition is satisfied in the ultrasonic characterization of flaws whose size is small compared to the characteristic wavelength of the sound wave: the high spatial frequency variation in the flaw shape cannot be detected by the sound beam, and as a result the flaw appears to be smooth. Consequently only a few views of the flaw are needed to give an overall idea of its shape and size, and one needs only to inspect the flaw in the planes perpendicular to the views of interest. Of course, if enough of these 2-dimensional projections of \( \rho(r) \) in other directions can be obtained by pulse echo measurements, then \( \rho(r) \) itself can be reconstructed completely.

There are several advantages in reducing the flaw characterization problem from reconstructing the 3-dimensional flaw shape to reconstructing the individual 2-dimensional projections of the flaw. First of all, it saves a lot of measurement time and computing time. As mentioned before, a small number of 2-dimensional projections usually suffice to give a fairly good estimate of the 3-dimensional flaw shape. Therefore only measurements in those planes are needed. In contrast, with 3-dimensional image reconstruction, inspections at all \( 4\pi \) steradians are always needed if only a few views are actually of interest.

The second advantage of reducing the 3-dimensional image reconstruction to 2-dimensional image reconstruction is that in most cases better image quality can be achieved. It often happens that the flaw cannot be inspected in some angular range, and therefore the corresponding Fourier components at those angles are not available in reconstructing the flaw shape. The quality of the reconstructed image will be degraded if the reconstruction is performed in a 3-dimensional manner involving all the \( 4\pi \) Fourier components in the 3-dimensional Fourier space. In the reconstruction of a 2-dimensional projection of the flaw shape, however, only the Fourier components on the corresponding plane in Fourier space are needed. Therefore it is possible to reconstruct some 2-dimensional projections of the flaw without loss of information if, for these projection images, inspection is accessible at all the angles from 0 to \( \pi \) in the corresponding inspection plane.
A number of computer simulation studies have been undertaken to show the feasibility of the 2-dimensional reconstruction algorithm. The pulse echoes of a broad band incident pulse from spherical flaws were calculated using the Ying and Truel algorithm [9]. The 2-dimensional projections \( p_z(x,y) \)'s of the flaws were reconstructed from these simulated pulse echoes using the schemes illustrated in Figures 5 and 7. Because of the spherical geometry of the flaws, for each flaw only 1 pulse echo was calculated and used for all the incident directions. All the images reported in this paper were reconstructed using the filtered back projection algorithm. For a void of radius 100 \( \mu \text{m} \) in quartz, Figure 8 A shows the comparison between the intermediate result \( p_z(s;\theta) \) obtained in this way and that calculated from an ideal sphere using equation (4). Figure 8 B shows the comparison in the spatial frequency space. The discrepancies in the low and high frequency range confirm our discussion (Reconstructing 2-D Projections from Pulse Echoes) on the frequency distribution of errors. Figures 9 A and 9 B show the improved \( p_z(s;\theta) \) after 40 iterations using the algorithm in Figure 7. The improvement is more significant in the low frequency region than in the high frequency region.

The ideal reconstructed 2-dimensional projection \( p_z(x,y) \) of a sphere is shown in Figure 10. The display area contains 64 x 64 pixels (picture elements), and each pixel is 7.46 \( \mu \text{m} \times 7.46 \mu \text{m} \) in dimension. The image is reconstructed from 80 mathematically calculated projections equally spaced from 0 to \( \pi \). It is circular in shape and decreases in magnitude

Fig. 8. \( p_z(s;\theta) \) and its spectrum for a void in quartz, radius = 100 \( \mu \text{m} \), calculated from pulse echo.
Fig. 9. The quantities in Figure 8 after 40 Gerchberg-Papoulis iterations.

Fig. 10. $p_2(x,y)$ of a sphere, radius = 100 µm, reconstructed from 80 mathematically calculated projections equally spaced from 0 to $\pi$. The pixel size is 7.46 µm x 7.46 µm.
radially from the centre. The \( p_z(x,y) \) reconstructed from \( p_z(s;\theta) \)'s calculated from 5 pulse echoes equally spaced from 0 to \( \pi \) is shown in Figure 11. The pulse echo waveforms were sampled at 10 ns intervals, and the \( p_z(s;\theta) \)'s calculated from them were interpolated 4 times. The reconstructed image appears to be half-circular in shape rather than circular. The problem can be traced to the shape of \( p_z(s;\theta) \) in Figure 8A calculated from pulse echo. Since the ideal \( p_z(x,y) \) of a sphere is radially symmetrical, its one-dimensional projection \( p_z(s;\theta) \) should be a symmetrical function in \( s \). Yet the shape of \( p_z(s;\theta) \) in Figure 8A is asymmetrical. The implication is that \( p_z(s;\theta) \neq p_z(-s;\theta+\pi) \), and the consequence of this inequality is that all the pulse echoes from 0 to 2\( \pi \) on the x-y plane are needed to reconstruct \( p_z(x,y) \).

The \( p_z(x,y) \) reconstructed from 10 pulse echoes equally spaced from 0 to 2\( \pi \) is shown in Figure 12. It compares favorably to the ideal reconstruction in Figure 10. To demonstrate the effect of the number of pulse echoes used on the quality of the reconstructed image, the number of pulse echoes

Fig. 11. \( p_z(x,y) \) of the void in quartz, radius = 100 \( \mu \)m, reconstructed from 5 pulse echoes equally spaced from 0 to \( \pi \).

Fig. 12. \( p_z(x,y) \) of the void reconstructed from 10 pulse echoes equally spaced from 0 to 2\( \pi \).
Fig. 13. $p(x,y)$ of the void reconstructed from 40 pulse echoes equally spaced from 0 to $2\pi$.

is increased to 40 equally spaced from 0 to $2\pi$, and the resulting reconstruction is shown in Figure 13. It is almost identical to the ideal reconstruction. Simulation studies on a $\text{Al}_2\text{O}_3$ inclusion of radius 100 $\mu$m in quartz gave similar results.

CONCLUSION

A 2-dimensional approach to the ultrasound flaw characterization problem is developed to reconstruct the 2-dimensional projected images of the flaw shape on the inspection planes. The algorithm can be applied to the characterization of flaws of general shape, and does not suffer from many of the difficulties associated with the direct 3-dimensional reconstruction of the flaw shape. Preliminary simulation studies produced very encouraging results.

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REFERENCES