APPLICATION OF FINITE ELEMENT METHODS TO STUDY TRANSIENT WAVE PROPAGATION IN ELASTIC WAVE GUIDES

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INTRODUCTION

A thorough understanding of the mechanics of wave propagation is critical for the quantitative nondestructive evaluation of structural components. This is of special importance when considering guided waves and how they are influenced by geometric features. Analytical solutions are available for relatively simple geometries, like plates or perfect annular structures. However, there are no closed form solutions available for more complicated geometries that are present when interrogating actual structural components. As a result, other, more robust solution methodologies are needed.

The finite element (FE) method is one potential solution technique. This method is used in this research because of its ability to accurately model complicated geometries. First, the FE method is used to reproduce known analytical results (guided waves in a plate) and thus confirm the FE method's accuracy for this problem type. Next, the FE method is applied to a more complicated geometry, an annular structure that contains a simplified crack; these results are used to show the robustness of the FE method. Finally, the FE results are briefly compared to an analytical solution and to experimentally obtained data.

THEORY OF FINITE ELEMENT METHOD

This research is based on the assumptions of linear elasticity. The general
equations of motion in matrix form are given as:

\[ M \ddot{u} + C \dot{u} + Ku = F_a, \]  

(1)

where \( M \) is the structural mass matrix; \( C \) is the structural damping matrix; \( K \) is the structural stiffness matrix; \( F_a \) is the vector of applied loads; and, \( u, \dot{u}, \) and \( \ddot{u} \) is the displacement vector and its time derivatives, respectively. Damping is not considered in this study. Equation (1) is solved using the Newmark Time Integration method.

4-node quadrilateral elements are used in this research. These elements provide a uniform mass distribution over the entire meshed geometry. A non-uniform mass distribution would cause spurious results for wave propagation problems. This research is restricted to 2D plane strain material behavior.

Temporal and spatial resolution of the finite element model is critical for the convergence of these numerical results [1]. The integration time step, \( \Delta t \), is the step size for which equation (1) is solved. This step is

\[ \Delta t = \frac{1}{20 f_{\text{max}}}, \]  

(2)

where \( f_{\text{max}} \) is the highest frequency of interest. The waveform must be accurately modeled by the element size; this leads to a required element length, \( l_e \), of

\[ l_e = \frac{\lambda_{\text{min}}}{20}, \]  

(3)

where \( \lambda_{\text{min}} \) is the shortest wavelength of interest.

NUMERICAL RESULTS

A 2 mm thick and 100 mm long steel plate is investigated first. The upper left corner of this plate, which is modeled with square shaped elements (\( l_e = 0.1 \) mm), is loaded with a displacement boundary condition in the \( x \) and \( y \) directions. Due to equation (2), and the time dependence of the applied load (the load is linearly ramped up and down within 2 \( \mu s \)), a time step of \( \Delta t = 10 \) ns is required. The in-plane displacement solution at the upper surface is spatially (\( \Delta x = 0.5 \) mm) and temporally (\( \Delta T = 0.1 \) \( \mu s \)) sampled. A 2D-FFT is applied to the numerically calculated time domain signals [2]. From the transformed data, the frequency spectrum of the steel plate is extracted. Figure 1 shows the numerically predicted dispersion curves (the numerical values are noted by “x”), compared to the exact solution (solid lines) of the Rayleigh-Lamb equation. Clearly, the FE method does an outstanding job in modeling dispersive wave propagation in a flat plate.

A steel annular structure is studied next. The inner radius is \( r_i = 50.8 \) mm; and the outer radius is \( r_o = 62.8 \) mm. The inner surface is traction free. A time dependent point load, \( f(t) \), is applied normal to the outer surface. The time function, \( f(t) \), of
Figure 1. Exact solution of Rayleigh-Lamb equation and numerically obtained values.

this load, as well as the frequency content is given in Figure 2. The frequencies above
1 MHz are expected to have little influence on the results. For this load, the
parameters \( l_e \) and \( \Delta t \) are varied to study convergence. According to Equation (2) and
\( f_{\text{max}} = 1 \text{ MHz} \) (see Figure 2), the results should converge for \( \Delta t = 0.05 \mu s \). This is
clearly shown in Figure 3, where the radial surface displacements, for an 90° angle
between source and receiver are shown. In addition, the element length \( l_e \) is varied;
Figure 4 shows that the results converge for \( l_e = 0.3 \text{ mm} \). Smaller time steps or
smaller elements are not necessary.
The FE method is very robust in modeling complicated geometries. Such complexities can consist of cracks or mounting fillets in a real part. To demonstrate the robustness of the method, a simplified radial crack in the (previously described) annular structure is modeled. This is accomplished by removing a 0.1 mm wide rectangular area from the ring cross section at a 45° angle away from the point of application of the load. Two crack lengths are investigated: 3 and 6 mm. The same point load, with time dependence shown in Figure 2, is used. Figure 5 compares the ideal ring with the "cracked" rings.

Figure 3. Convergence results for different integration time steps, $\Delta t$.

Figure 4. Convergence results for different element lengths, $l_e$. 
The results of the FE method are now compared to an analytical mode superposition solution presented in [3]. Again, the point load, \( f(t) \), of Figure 2 is applied in both calculations. The FE solution with optimal parameters (\( \Delta t = 0.05 \mu s \), \( l_e = 0.3 \)) is plotted in Figure 6, as well as the analytical solution. The almost perfect match of these two approaches confirms the applicability of the FE method to wave propagation. It also shows that the FE solution converges toward an analytical result for parameters chosen according to equation (2) and (3).

As a benchmark, the FE results are compared with experimentally obtained waveforms. Optical generation and detection of ultrasonic waves is used in these experiments; this comes very close to the ideal case of a point source and receiver (modeled with FE method). The frequency content of the experimental source is broad band, while the numerical source has a limited frequency content. This is the main difference between the two models. Both sources excite mainly the first mode, which is the surface wave propagating along the outer surface. This mode propagates in almost a non-dispersive fashion. Therefore, its arrival time is nearly frequency independent and a valid comparison between the experimental and FE results can be made. The arrival time of the main peak of the first mode is measured (numerically and experimentally) for different travel distances between source and receiver. In Figure 7, results for different travel distances (indicated by angle \( \beta \)) are shown. The numerical peaks arrive 1.7 to 4.2 percent earlier than the experimentally measured peaks. This can be explained by the slightly decreasing phase velocity of the first
mode with increasing frequency. The early arrival makes therefore sense, thus providing another demonstration of robustness of the FE method.

CONCLUSION

This work shows the effectiveness of using the FE method to solve 2D wave propagation problems. The FE method is a tool that can be used when an analytical

Figure 6. Comparison of FE solution and mode superposition result.

Figure 7. Experimental (left) and numerical (right) waveforms.
solution is not possible because "complicated" component geometry. The FE method can also be used to predict or interpret experimental results. Another big advantage is the powerful postprocessing capabilities commercial FE codes offer. These features allow very detailed insight into wave propagation phenomena. So far, somewhat easy geometries are investigated. This is done in order to establish confidence in this method by comparing to analytical or experimental results. For NDT purposes, further work is needed into more detailed modeling of cracks and their influence on propagating modes.

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