A MULTI-MODE SOLUTION FOR ANALYSIS OF THE REFLECTION COEFFICIENT
OF OPEN-ENDED RECTANGULAR WAVEGUIDES RADIATING INTO A
DIELECTRIC INFINITE HALF-SPACE

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INTRODUCTION

The open-ended rectangular waveguide probe is a powerful tool for characterizing
the dielectric and reflection properties of various materials and structures [1-11]. In these
applications the measurable parameter, \( r \) (i.e. the reflection coefficient), is used to
determine the sought-for parameters. The said reflection coefficient is defined as the ratio
of the reflected electric field at the aperture of the waveguide to that of the incident electric
field.

In light of this, many studies have dealt with the radiation properties of the open­
ended waveguide probe radiating into an infinite dielectric half-space. The original studies,
using variational principles, approximate the field distribution at the aperture to that of the
dominant propagating \( TE_{10} \) mode [12-13]. Although these derivations are reasonably
accurate and computationally efficient for most practical cases, they lead to significant
errors when attempting to re-calculate the dielectric properties of the infinite half-space from
the measured \( \Gamma \) for a certain range of dielectric properties and frequency of operation in the
waveguide band [10]. These errors are primarily due to the exclusion of higher-order
modes in the theoretical formulation. To remedy this problem several attempts have been
made which incorporate the effect of higher-order modes in the derivations [1-3, 14-21],
but all except [3] require some form of numerical or theoretical approximation in order to solve for the unknown coefficients that lead to the final solution for a given set of modes and only deal with the particular case of an infinite half-space of free-space. Also none of the above studies discuss the impact of the frequency of operation and of the permittivity of the dielectric half-space on the contribution of the higher-order modes in the final solution.

This paper provides for a brief description of a rigorous and exact formulation in which the dominant mode and the evanescent TE and TM higher-order modes are used as basis functions to obtain the solution for the reflection coefficient, \( \Gamma \), at the waveguide aperture. The analytic formulation uses Fourier analysis similar to that used in [21], in addition to the forcing of the necessary boundary conditions at the waveguide aperture. In this approach only a set of higher-order modes necessary to obtain convergence is included in the formulation whose contribution will be discussed as a function of the dielectric properties of the infinite half-space, and of the frequency of operation.

INTEGRAL SOLUTION

The geometry of the problem is illustrated in Figure 1 in which an open-ended rectangular waveguide with its broad dimension \( 2a \) and narrow dimension \( 2b \) is mounted on an infinite ground plane, and is radiating into an infinite half-space of a dielectric material. The nomenclature for the waveguide dimensions has been chosen for algebraic simplification and to enforce the choice of the origin of the system of axes at the center of the waveguide aperture. The complex relative dielectric constant of the half-space is given by \( \varepsilon_r = \varepsilon_\prime - j\varepsilon''_r \), where \( \varepsilon_\prime \) and \( \varepsilon''_r \) are referred to as the relative permittivity and loss factor of the material, respectively. Loss tangent, \( \tan \delta \), is the ratio of the permittivity to loss factor \( (\varepsilon''_r / \varepsilon_\prime) \). Firstly let us define \( \eta_o = \sqrt{\mu_o / \varepsilon_o} \) as the free-space intrinsic impedance, \( k_o = \omega \sqrt{\varepsilon_o \mu_o} \) as the free-space wave number, \( \varepsilon_o \) and \( \mu_o \) as the permittivity and permeability of free-space, respectively, and \( \omega \) as the radial frequency. Secondly, we express the following intermediate variables: \( a_m = (m\pi / 2a) \), \( b_n = (n\pi / 2b) \), \( k_{mn} = \sqrt{k_o^2 - a_m^2 - b_n^2} \), \( k_1 = k_o \sqrt{\varepsilon_\prime \mu_o} \), and \( \zeta = \sqrt{k_1^2 - \xi^2 - \eta^2} \). Lastly, we define \( A^e_{mn} \) (TM\(_{mn}\) modes) and \( A^h_{mn} \) (TE\(_{mn}\) modes) as the sought for unknown coefficients of the reflected waves at the aperture. Through an exercise of proper boundary matching of the fields at the aperture of the waveguide, two linear sets of equations are obtained that lead to the solution of the unknown coefficients \( A^e_{mn} \) and \( A^h_{mn} \). Their equations are given by
Figure 1. Open-ended rectangular waveguide radiating into infinite half-space.

$$
\sum_{m,n=1}^{\infty} \left[ a_{m} I_1(m,n,p,q) + b_{n} I_2(m,n,p,q) \right] \cdot k_{mn} A_{mn}^e \\
+ \sum_{m,n=0, m=n \neq 0}^{\infty} \left[ b_{n} I_1(m,n,p,q) - a_{m} I_2(m,n,p,q) \right] \cdot k_{mn} A_{mn}^h \\
+ k^2_{o} a b \left[ \frac{A_{pq}^e b_{k}}{k_{o}} - \frac{A_{pq}^h a_{k_{pq}} (1 + \delta_{p,q})}{k^2_{o}} \right] \cdot \frac{\eta_{i}}{\eta_{o}} \\
= \left[ k_{o} a I_2(1,0,p,q) - 2k_{o} a_{10} \frac{\eta_{i}}{\eta_{o}} a b \delta_{p} \delta_{q} \right] \cdot A^{i} \\
$$

where for $p = 1, 2, 3, \ldots, \infty, \quad q = 0, 1, 2, \ldots, \infty,$ and

$$
\sum_{m,n=1}^{\infty} \left[ a_{m} I_3(m,n,p,q) + b_{n} I_4(m,n,p,q) \right] \cdot k_{mn} A_{mn}^e \\
+ \sum_{m,n=0, m=n \neq 0}^{\infty} \left[ b_{n} I_3(m,n,p,q) - a_{m} I_4(m,n,p,q) \right] \cdot k_{mn} A_{mn}^h \\
+ k^2_{o} a b \left[ \frac{A_{pq}^e a_{k}}{k_{o}} + \frac{A_{pq}^h b_{k_{pq}} (1 + \delta_{p,q})}{k^2_{o}} \right] \cdot \frac{\eta_{i}}{\eta_{o}} \\
= \left[ k_{o} a I_4(1,0,p,q) \right] \cdot A^{i} \\
$$

(1)
for \( q = 0,1,2,\ldots,\infty \), \( p = 1,2,3,\ldots,\infty \) and where the \( I_{m=1,2,3,4}(m,n,p,q) \) integrals are defined as

\[
I_1(m,n,p,q) = \frac{1}{4\pi^2} \int \int \frac{\xi \eta}{\xi_k} C_m^a(-\xi) S_p^b(-\eta) S_q^b(\xi) C_q^b(\eta) d\eta d\xi
\]

and the remaining intermediate variables are defined as

\[
\delta_{pq} = \begin{cases} 
1 & \text{if } p = q \\
0 & \text{otherwise} 
\end{cases} 
\]

\[
S_p^b(\xi) = \int_{-a}^{a} \sin a_p(x+a) e^{-j\xi x} \, dx 
\]

\[
C_q^b(\eta) = \int_{-b}^{b} \cos b_q(y+b) e^{-j\eta y} \, dy 
\]

\[
C_p^a(\xi) = \int_{-a}^{a} \cos a_p(x+a) e^{-j\xi x} \, dx 
\]

\[
S_q^b(\eta) = \int_{-b}^{b} \sin b_q(y+b) e^{-j\eta y} \, dy. 
\]

The expressions for (3)-(6) can be modified, which results in a double integration over the aperture of the waveguide (bounded integrals) containing simple trigonometric functions without singularities in the integrand. The solution for \( A_{m,n}^+ \) and \( A_{m,n}^- \) is then simply dependent on solving \( I_{m=1,2,3,4}(m,n,p,q) \) integrals and on the truncated linear set of equations. Because of the even geometry of the problem, only modes possessing odd \( m \) and even \( n \) indices will be coupled with the incident TE_{10} mode.
The study of the reflection properties of the open-ended rectangular waveguide was performed theoretically at J-band (5.85-8.2 GHz). To this end, the convergence of the measurable parameter which is the reflected portion of the dominant TE$_{10}$ mode (\(A_{10}^r\)), was calculated via (1) and (2) for \(A^i = 1\) throughout the waveguide frequency band for the case of 1, 6 and 15 modes. The computation time for this procedure was 0.38, 8.35 and 47.84 seconds on a Pentium 133 MHz platform. It can be observed in Figure 2 and 3 that when using 6 or more modes the solution converges quickly to its final solution. It should also be noticed that the single mode and higher-order mode solutions diverge as the frequency of operation increases. Because of this, the computation of \(\Gamma\) vs. the permittivity of the infinite half-space was performed at the end frequency of the waveguide band where the influence of the higher-order modes is more pronounced. Figure 4 presents both the single mode solution and the multi-mode solution of \(|\Gamma|\) for a set of 6 modes (\(m = 1, 3\) and \(n = 0, 2\)) for different values of \(\varepsilon_r\) (3, 5 and 7) and \(\tan \delta = \varepsilon''/\varepsilon_r\) (0.0, 0.1, 0.2, 0.3, 0.4 and 0.5). It can be observed that there does exist an appreciable difference between the approximate single mode and higher-order mode solutions. This difference decreases as the loss of the infinite half-space material decreases. Therefore, for dielectric property measurements of high loss materials, omission of the higher-order modes in the final solution will not lead to significant errors in the extraction of \(\varepsilon_r\) from \(\Gamma\).
CONCLUSION

A rigorous multi-mode solution to the reflection properties of the open-ended waveguide probe radiating into a dielectric infinite half-space was presented. It was observed that the influence of the higher-order modes was more pronounced at the higher end of the waveguide frequency band and for a relatively low-loss infinite half-space. The results of the forward problem, that is calculating $\Gamma$ vs. $\varepsilon_r$, can therefore be used for the precise nondestructive determination of the permittivity of the infinite half-space.

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