Bending of rectangular corrugated sandwich plates

Edgar Oliver Seaquist Jr.

Iowa State University
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BENDING OF RECTANGULAR CORRUGATED SANDWICH PLATES

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Edgar Oliver Seaquist, Jr.

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I. INTRODUCTION

The general problem under consideration is the small displacement theory of bending due to transverse loads of an anisotropic, rectangular plate. The problem is generalized to the extent that the principal axes of elastic symmetry are arbitrarily oriented with respect to the plate boundaries. The analytical and experimental model which will be considered is a corrugated-core sandwich structure which consists of a multiple corrugated sheet rigidly attached to two plane face sheets. Thus, the material is "orthotropic"; that is, it possesses orthorhombic symmetry composed of one plane of symmetry mid-way between the face sheets and two orthogonal planes of symmetry normal to the plate surface. The selection of this particular model introduces a degree of nonhomogeneity; however, in the context of the conventional assumptions regarding nonhomogeneous media it will be assumed that the nonhomogeneity may be neglected, provided the characteristic dimensions of the corrugations are small in comparison to the planar dimensions of the plate. Only a linear theory of bending will be considered in which it is assumed that no local buckling occurs in the face sheets and corrugation and that no in-plane forces exist.

The appropriate boundary value problem is derived for
the special case of a simply supported, rectangular plate. This problem consists of a fourth order, partial differential equation similar to the bi-harmonic together with boundary condition equations involving the sum of second-order derivative terms. A finite difference technique of solution has been applied to this system of equations and the results compared to experimental findings.
II. HISTORICAL REVIEW

The problem of the deformation of an anisotropic plate has apparently not been treated in the most general case for a generalized Hooke's Law and arbitrary orientation of the elastic axes relative to the plate edges. The special case for an orthotropic material for which the principal elastic axes are coincident with rectangular boundaries having simple supports is a logical extension of elementary plate theory as given by Timoshenko (18). Several Russian authors, notably Mauzurkiewicz (12) and Kaczkowski (7), have extended the orthotropic plate problem to more generalized boundary and loading conditions, yet the restriction on the orientation of the elastic axes is still maintained.

Early in the 1940's the development of various types of sandwich panel structures precipitated extensive analysis of anisotropic, nonhomogenous plates; however, most of the analysis was concerned with the honeycomb-core sandwich material. Reissner's analysis (16, 17) of such plates is principally related to the shear deflection problem, and in fact the anisotropic nature of the core is neglected. More recently, Libove and Batdorf (10) have defined the boundary value problem for the small deflection theory of bending of rectangular, corrugated-core, sandwich plates. Although
this development includes both shear and bending deformations, it is limited to the case of coincident elastic and plate axes.

Even though there are no solutions to the orthotropic plate with arbitrary direction of the elastic axes, there are several problems that are very nearly analogous to it. The skew-plate problem is one such analogy. This problem involves a differential equation in the displacement function which has the same derivative terms as the general orthotropic plate differential equation; however, two of the derivative terms have like coefficients which permit a variables separable solution. Krettner (9) obtained a single-series expression for the deflection but he does not discuss the complementary functions required to satisfy actual boundary conditions. Quinlan (15) has presented a more complicated method of solution which includes particular boundary conditions. With some modification the skew-plate problem could be used as an approximation to the general orthotropic plate problem.

Another similar boundary value problem is presented by Muller (13) in connection with the solution of the twisting of a prismatic bar consisting of a general anisotropic material. In this case the homogeneous differential equation is directly analogous to the plate problem but a solution is presented only for the special case in which the
roots of the quartic characteristic equation can be found. In addition, the boundary conditions do not form an analogy to those in the plate problem.
III. THEORETICAL DEVELOPMENT

A. Assumptions

In order to develop the boundary value problem which will define the shape of the deformed surface of a corrugated plate, equilibrium, geometry of distortion, and the elastic characteristics of the material must be considered. The particular configuration of corrugated-core, sandwich plate which will be analyzed consists of two plane sheets separated by a thin plate folded in a corrugated pattern. The corrugated plate is rigidly attached to the face sheets and it is symmetric with respect to a plane lying midway between the face sheets. The cross section normal to the axis of the corrugation is shown in Figure 8. The case of a simply supported, rectangular plate will be considered.

Assumptions consistent with the elementary theory of thin plates regarding the plate dimensions and the geometry of distortion will be made. Also, several additional assumptions specifically concerning the geometry and elastic properties of the core and face materials will be made. The assumptions associated with the elementary plate theory can be summarized as follows:

1. The deflections of the symmetry plane lying in the plane of the plate (middle surface) are sufficient
to define the deflections of the entire plate. Thus, if \( w \) is the normal displacement of this middle surface, then \( w = w(x,y) \) only and \( w \) is not a function of the normal coordinate dimension \( z \).

2. The deflection of the middle surface is small in comparison to the total thickness and the lateral dimensions of the plate, and the slope of the middle surface is small compared to unity.

3. Elements of the plate originally perpendicular to the middle surface remain perpendicular to the middle surface. This assumption, together with the second assumption implies that displacements \( u_1 \) and \( u_2 \) in orthogonal directions \( x_1 \) and \( x_2 \) in the plane of the plate are linearly related to the distance \( z \) measured for the middle surface. In particular, the linear relations will be assumed to be

\[
\begin{align*}
    u_1 &= -z \frac{\partial w}{\partial x_1}, \\
    u_2 &= -z \frac{\partial w}{\partial x_2}.
\end{align*}
\]

4. There are no applied forces or restraints in the plane of the plate.

5. The deformations due to bending are sufficiently large compared to those due to transverse shear that deflections due to shear may be neglected. This assumption was confirmed for the particular
geometry considered in the numerical solution.

6. All loads acting on the plate are normal to the surface, and the normal stresses in a direction perpendicular to the plane of the plate are negligible in comparison to the normal stresses in the plane of the plate.

The additional assumptions concerning the corrugated-core sandwich structure are the following:

7. The total thickness of the plate and the pitch length of the corrugation elements are small in comparison to the lateral dimensions of the plate. Thus, although there is nonhomogeneity throughout the thickness of the plate, it will be assumed that the plate is homogeneous from point to point in its plane.

8. Even though the core and face sheets are relatively thin sections no local buckling occurs at any point in the plate.

9. Both the face sheet and core materials are assumed to be linearly elastic, homogeneous, and isotropic. The bond between the fact sheet and core is assumed to be continuous and of negligible dimensions.

10. The sandwich structure is symmetrical with respect to the middle surface.
B. Development of Equations

1. Equilibrium equations

Consider a rectangular plate with boundaries parallel to $x$ and $y$ axes. Let the $xy$-plane be coincident with the middle plane of the plate and define a $z$-axis normal to this plane. Also, let axes $u$ and $v$ be directed along the principal elastic axes (symmetry axes) for which $u$ is parallel and $v$ perpendicular to the maximum (corrugation axis) and minimum principal elastic axes respectively. Consider an infinitesimal section of the plate cut out by two pairs of planes parallel to the $uz$- and $vz$- planes as shown in Figure 1. Acting on the edges of this element are a system of normal and shear stresses which give rise to resultant bending ($M_u$, $M_v$) and twisting ($M_{uv}$, $M_{vu}$) moments and transverse shear forces ($Q_u$, $Q_v$). These so-called stress resultant moments and forces have units of moment and force per unit length.

When the principles of equilibrium are applied to the elementary block, three equilibrium equations in terms of the various stress resultants are obtained.
\[
\Sigma F_z = \frac{\partial Q_u}{\partial u} + \frac{\partial Q_v}{\partial v} + q = 0
\]

\[
\Sigma M_u = \frac{\partial M_{uv}}{\partial u} - \frac{\partial M_u}{\partial v} + Q_v = 0
\]  
*(Eq. 1)*

\[
\Sigma M_v = \frac{\partial M_{vu}}{\partial v} + \frac{\partial M_u}{\partial u} - Q_u = 0
\]

for which \( q \) is the uniformly distributed load acting on the surface of the plate.

2. **Stress-strain relations**

Several alternate methods are available for introducing state relations into the problem. One possible approach is to recognize that for the complete plate orthorhombic symmetry exists so that there are two perpendicular axes of elastic symmetry in the plane of the plate. For this class of symmetry Hearmon (4) shows the stress-strain relationship to be of the form

\[
\sigma_u = C_u \varepsilon_u + C_{uv} \varepsilon_v
\]

\[
\sigma_v = C_{uv} \varepsilon_u + C_v \varepsilon_v
\]  
*(Eq. 2)*

\[
\tau_{uv} = C_{uv} \gamma_{uv}
\]
for which $\sigma$ and $\tau$ are the normal and shearing components of stress, $\varepsilon$ and $\gamma$ are the normal components of strain, and $C_u, C_v, C, \text{ and } C_{uv}$ the elastic stiffnesses. With the use of Equations 2 the same procedure as in elementary plate theory could be followed in developing relations between the stress resultants and strain components; however, the elastic stiffnesses must still be determined in terms of the known geometry and elastic characteristics of the face sheets and core materials.

A more direct approach is to examine the manner in which the individual elements of face sheet and core deform and thus define appropriate stress-strain relations for these elements. Thus, consistent with the ninth assumption, the planar face sheets adhere to an isotropic, linearly elastic law for which the stress-strain relationship is given by

\[
\begin{align*}
\sigma_u' &= \frac{E}{1 - \mu^2} \left[ \varepsilon_u' + \mu \varepsilon_v' \right] \\
\sigma_v' &= \frac{E}{1 - \mu^2} \left[ \varepsilon_v' + \mu \varepsilon_u' \right] \\
\tau_{uv}' &= G \gamma_{uv}'
\end{align*}
\]

(Eq. 3)

for which $E$ is Young's modulus, $G$ the shearing modulus, and $\mu$ Poisson's ratio. The primes refer to the face sheet
material, while double primes will be used in reference to the core material.

Although the core is assumed to be composed of the same material as the face sheets, when it is formed into a corrugated section and normal, in-plane forces are applied to it in the direction of elastic symmetry, it is much more flexible in a direction normal to the corrugation than in a direction parallel to the corrugation. Specifically, it will be assumed that for strains of the same order of magnitude in the \( u \) and \( v \) directions \( \sigma''_u \ll \sigma''_v \). Thus, the normal stress-strain relation for the core is

\[
\begin{align*}
\sigma''_u &= E\varepsilon''_u \\
\sigma''_v &= 0
\end{align*}
\]  
(Eq. 4)

Similarly, for any system of orthogonal shear stresses parallel to the plane of the plate, the corrugated core will simply deform in accordion fashion and thus, it can be assumed that the core can support no in-plane shearing stresses, i.e. \( \tau''_{uv} = \tau''_{vu} = 0 \).

With the conclusion that the in-plane shearing stresses in the core do not exist and considering the third of Equations 3, it can be concluded that all the in-plane shearing stresses are carried in the face sheets. If
moment equilibrium of an element cut from the face sheet is to exist, the conventional relation $\tau'_{uv} = \tau'_{vu}$ must be satisfied. Since the twisting moment stress resultants can be expressed in the form

$$M_{uv} = -\int_A z\tau_{uv} dA, \quad M_{vu} = \int_A z\tau_{vu} dA,$$

a direct substitution gives $M_{uv} = -M_{vu}$. Substitute this relation into the last two equilibrium equations and differentiate; the equilibrium equations reduce to

$$\frac{\partial^2 M_u}{\partial u^2} - 2 \frac{\partial^2 M_{uv}}{\partial u \partial v} + \frac{\partial^2 M_v}{\partial v^2} = -q \quad \text{(Eq. 5)}$$

3. Moment-displacement relations

The strains in the direction of the elastic symmetry axes $(u,v)$ are defined in the conventional manner to be

$$\epsilon_u = \partial u_u / \partial u; \quad \epsilon_v = \partial u_v / \partial v, \quad \text{and} \quad \gamma_{uv} = \partial u_u / \partial v + \partial u_v / \partial u.$$

The third assumption requires that the displacements $u_u$ and $u_v$ in the $u$ and $v$ directions, respectively, are linear functions of the distance from the middle surface. In particular, the law is assumed to be
\[ u_u = -z \frac{\partial w}{\partial u} \quad \text{and} \quad u_v = -z \frac{\partial w}{\partial v}, \quad \text{for which} \quad \bar{w} = \bar{w}(u,v). \]

When these relations are differentiated and substituted into the strain equations, the strain-curvature equations are found to be

\[ \varepsilon_u = -z \frac{\partial^2 \bar{w}}{\partial u^2}, \quad \varepsilon_v = -z \frac{\partial^2 \bar{w}}{\partial v^2}, \quad \text{and} \quad \gamma_{uv} = -2z \frac{\partial^2 \bar{w}}{\partial u \partial v}. \]

(Eq. 6)

It has been assumed that the core and face sheet act together in a continuous fashion; thus Equations 6 is valid for appropriate values of \( z \) in both the core and face sheet.

Using the definition of the moment stress resultant, \( M_u \), the appropriate stress-strain relation (Equations 3 and 4), and the strain-curvature equations (Equation 6), the moment can be expressed as

\[ M_u = \int_{A_c} \sigma_u z dA_c + \int_{A_s} \sigma_u' z dA_s \]

\[ = \int_{A_c} E \varepsilon_u'' z dA_c + \int_{A_s} \left( \frac{E}{1-\mu^2} \right) (\varepsilon_u' + \mu \varepsilon_v') z dA_s \]

\[ = \int_{A_c} E(-z^2 \frac{\partial^2 \bar{w}}{\partial u^2}) dA_c + \int_{A_s} \left( \frac{E}{1-\mu^2} \right) (-z^2) \left( \frac{\partial^2 \bar{w}}{\partial u^2} + \mu \frac{\partial^2 \bar{w}}{\partial v^2} \right) dA_s \]

for which \( A_c \) and \( A_s \) are the areas per unit length of the
core and face sheet respectively. Now, recalling that \( \bar{w} = \bar{w}(u,v) \) alone, the curvature terms and the elastic constants can be factored out of the integral. Also, by definition, the second moments of area of the core and face sheet are

\[
I_c = \int_{A_c} z^2 dA_c \quad \text{and} \quad I_s = \int_{A_s} z^2 dA_s . \quad \text{(Eq. 7)}
\]

Thus,

\[
M_u = - \frac{\partial^2 \bar{w}}{\partial u^2} \int_{A_c} z^2 dA_c - \frac{E}{1-\mu^2} \left( \frac{\partial^2 \bar{w}}{\partial u^2} + \mu \frac{\partial^2 \bar{w}}{\partial v^2} \right) \int_{A_s} z^2 dA_s
\]

\[
M_v = - \frac{[EI_c + \frac{EI_s}{1-\mu^2}] \frac{\partial^2 \bar{w}}{\partial u^2} - \frac{\mu EI_s}{1-\mu^2} \frac{\partial^2 \bar{w}}{\partial v^2}}{\partial u} . \quad \text{(Eq. 8)}
\]

In a similar manner the moment stress resultant in the \( v \)-direction becomes

\[
M_v = \int_A \sigma_v zdA = \int_{A_c} \sigma''_v zdA + \int_{A_s} \sigma'_v zdA
\]

\[
M_v = \frac{E}{1-\mu^2} \left( \frac{\partial^2 \bar{w}}{\partial v^2} \right) \int_{A_s} z^2 dA_s + \mu \frac{\partial^2 \bar{w}}{\partial u^2} \int_{A_s} z^2 dA
\]

\[
M_v = - \frac{\mu EI_s}{1-\mu^2} \frac{\partial^2 \bar{w}}{\partial u^2} - \frac{EI_s}{1-\mu^2} \frac{\partial^2 \bar{w}}{\partial v^2} \quad \text{ (Eq. 9)}
\]

and the twisting moment stress resultant is
\[ M_{uv} = - M_{vu} = - \int_A \tau_{uv} z dA = - G \int_{A_s} \gamma_{uv} z dA_s = 2G \frac{\partial^2 w}{\partial u \partial v} \int_{A_s} z^2 dA_s \]

\[ M_{uv} = 2GI \frac{\partial^2 w}{\partial u \partial v} . \]

Since \( G = \frac{E}{2(1+\mu)} \),

\[ M_{uv} = \frac{EI_s}{1+\mu} \frac{\partial^2 w}{\partial u \partial v} \quad (\text{Eq. 10}) \]

Equations 8, 9, and 10 can be written in the form

\[ M_u = - D_{uu} \frac{\partial^2 w}{\partial u^2} - D \frac{\partial^2 w}{\partial v^2} \]

\[ M_v = - D \frac{\partial^2 w}{\partial u^2} - D_{vv} \frac{\partial^2 w}{\partial v^2} \quad (\text{Eq. 11}) \]

\[ M_{uv} = D_{uv} \frac{\partial^2 w}{\partial u \partial v} \]

for which the elastic stiffnesses are

\[ D_{uu} = EI_c + \frac{EI_s}{1-\mu^2} \]

\[ D = \frac{\mu EI_s}{1-\mu^2} \]

\[ D_{vv} = \frac{EI_s}{1-\mu^2} \]
\[ D_{uv} = \frac{EI_s}{1+\mu} \]  

(Eq. 12)

By comparing Equations 11 and 2 it can be seen that the moment-curvature relations for the corrugated plate are of the same form as the stress-strain relations for an orthotropic material.

4. Transformation equations

At this point Equations 5 and 11, together with appropriate boundary conditions, are sufficient to define the deformed surface; however, any boundary conditions can most easily be expressed in terms of the plate edge coordinate system x and y. Thus, the various moments and curvatures must be transformed by a rotation of axes from the uv-coordinate system to the xy-coordinate system. If \( \alpha \) is the angle measured between the positive x and u axes, the equations of transformation of the coordinate axes are

\[
\begin{align*}
x &= u\cos \alpha - v\sin \alpha = u_m - v_t \\
y &= u\sin \alpha + v\cos \alpha = u_l + v_m
\end{align*}
\]

(Eq. 13)

for which \( \cos \alpha = m \) and \( \sin \alpha = t \).
The displacement function of the middle surface, \( \overline{w} = \overline{w}(u,v) \) goes over to the function \( w = w(x,y) \) under the transformation defined by Equations 13. By successive differentiation the transformation equations for the curvature components become

\[
\frac{\partial^2 w}{\partial u^2} = \frac{\partial^2 w}{\partial x^2} m^2 + \frac{\partial^2 w}{\partial y^2} t^2 + 2 \frac{\partial^2 w}{\partial x \partial y} l_m
\]

\[
\frac{\partial^2 w}{\partial v^2} = \frac{\partial^2 w}{\partial x^2} t^2 + \frac{\partial^2 w}{\partial y^2} m^2 - 2 \frac{\partial^2 w}{\partial x \partial y} l_m \quad \text{(Eq. 14)}
\]

\[
\frac{\partial^2 w}{\partial u \partial v} = \left( \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) l_m + \frac{\partial^2 w}{\partial x \partial y} \left( m^2 - t^2 \right)
\]

Next, consider a triangular wedge element cut from the plate so that the normals to the three edges are in the \( x, y, \) and \( u \)-directions as shown in Figure 2. From

Fig. 2. Wedge element.
equilibrium it follows directly that

\[ M_u = M_x m^2 + M_y l^2 - 2M_{xy} l \mu \]  

(Eq. 15)

\[ M_{uv} = M_{xy} (m^2 - l^2) + (M_x - M_y) l \mu . \]

Applying the equilibrium equations to a similar wedge with normals in the \(x\), \(y\), and \(v\) directions, the transformation equation for the moment in the \(v\)-direction becomes

\[ M_v = M_x l^2 + M_y m^2 + 2M_{xy} l \mu . \]  

(Eq. 16)

Substituting Equations 14, 15, and 16 into 11, the transformed moment-curvature equations become

\[ M_x = - A_1 \frac{\partial^2 w}{\partial x^2} - A_3 \frac{\partial^2 w}{\partial y^2} - 2B_1 \frac{\partial^2 w}{\partial x \partial y} \]

\[ M_y = - A_3 \frac{\partial^2 w}{\partial x^2} - A_2 \frac{\partial^2 w}{\partial y^2} - 2B_2 \frac{\partial^2 w}{\partial x \partial y} \]  

(Eq. 17)

\[ M_{xy} = B_1 \frac{\partial^2 w}{\partial x^2} + B_2 \frac{\partial^2 w}{\partial y^2} + B_3 \frac{\partial^2 w}{\partial x \partial y} \]

for which
\[ A_1 = D_{uu} t^4 + D_{vv} t^4 + 2t^2 m^2 (D + D_{uv}) \]
\[ A_2 = D_{uu} t^4 + D_{vv} m^4 + 2t^2 m^2 (D + D_{uv}) \]
\[ A_3 = m^2 t^2 (D_{uu} - 2D_{uv} + D_{vv}) + D(t^4 + m^4) \]  
(Eq. 18)
\[ B_1 = m t [D_{uu} m^2 - D_{vv} t^2 + (t^2 - m^2) (D + D_{uv})] \]
\[ B_2 = m t [D_{uu} t^2 - D_{vv} m^2 - (t^2 - m^2) (D + D_{uv})] \]
\[ B_3 = 2t^2 m^2 (D_{uu} - 2D + D_{vv}) + D_{uv} (t^2 - m^2)^2. \]

The equilibrium equation in terms of the moments (Equation 5) is valid in any coordinate system, and thus
\[ \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q. \]  
(Eq. 5a)

Finally, differentiating Equations 17 and substituting the results into Equation 5a, a single differential equation in the displacement function is obtained.
\[ A_1 \frac{\partial^4 w}{\partial x^4} + 4B_1 \frac{\partial^4 w}{\partial x^3 \partial y} + 2(A_3 + B_3) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4B_2 \frac{\partial^4 w}{\partial x \partial y^3} + A_2 \frac{\partial^4 w}{\partial y^4} = q. \]  
(Eq. 19)
5. Boundary conditions

The special case will be considered for which the edges of the plate are simply supported. In this case the boundary conditions become

(a) On a boundary \( y = y_0 \) the conditions which must be satisfied are \( w(x, y_0) = 0 \) and \( M_y(x, y_0) = 0 \). From the second of Equations 17

\[
M_y(x, y_0) = -A_3 \frac{\partial^2 w}{\partial x^2} - A_2 \frac{\partial^2 w}{\partial y^2} - 2B_2 \frac{\partial^2 w}{\partial x \partial y}.
\]

Since \( w(x, y_0) \) and \( \frac{\partial w}{\partial x} \bigg|_{y=y_0} \) are continuous functions in \( x \), then \( w(x, y_0) = 0 \) implies that \( \frac{\partial^2 w(x, y_0)}{\partial x^2} = 0 \). Thus, the boundary conditions on \( y = y_0 \) reduce to

\[
w(x, y_0) = 0 \tag{Eq. 20}
\]

\[
A_2 \frac{\partial^2 w}{\partial y^2} + 2B_2 \frac{\partial^2 w}{\partial x \partial y} = 0 \quad 0 < x < a \text{ and } y = y_0.
\]

(b) On \( x = x_0 \) the boundary conditions are \( w(x_0, y) = 0 \) which implies that \( \frac{\partial^2 w(x_0, y)}{\partial y^2} = 0 \) and

\[
M_x(x_0, y) = 0, \text{ or}
\]
\[ w(x_0,y) = 0 \quad (\text{Eq. 21}) \]

\[ A_1 \frac{\partial^2 w}{\partial x^2} + 2B_1 \frac{\partial^2 w}{\partial x \partial y} = 0 \quad 0<y<b \text{ and } x = x_0. \]

Equations 19, 20, and 21 constitute the complete boundary value problem required to define the deformed surface of the plate.

Hormander (5) quotes a proof by B. Malgrange that any partial differential equation with constant coefficients has a fundamental solution. Hormander (6) later presents theorems which show that for the special case where the coefficients of the differential operators in the boundary conditions are constant and the boundary is rectilinear, the boundary problem is correctly posed and the solution is regular in the interior and at the boundary of the region. Since the boundary value problem defined by Equations 19, 20, and 21 meet these criteria, the existence of a solution is assured.

Finally, it can be noted that there exists a two-fold axis of symmetry for the deformed surface at the center of the plate. This can be shown to be true since, if \( w(x,y) \) is a solution, then by direct substitution into Equations 19, 20, and 21 it is also true that \( w(a-x,b-y) \) is also a solution.
IV. ANALYSIS

A. Series Solution

The boundary value problem for the deformed surface of a sandwich plate defined in the preceding section can be solved by the Navier method used in isotropic plates for the special case in which the elastic axes of symmetry are coincident with the plate edge axes \(18\). In this case either \(\alpha = 0^\circ\) or \(\alpha = 90^\circ\) and the direction cosines, \(m\) and \(l\), which appear in the stiffness coefficient expressions are either zero or unity. Thus, the differential equation (Equation 19) reduces to

\[
A_1 \frac{\partial^4 w}{\partial x^4} + 2(A_3 + B_3) \frac{\partial^4 w}{\partial x^2 \partial y^2} + A_2 \frac{\partial^4 w}{\partial y^4} = q(x, y)
\]

Similarly, the boundary conditions given by Equations 20 and 21 become

\[
\frac{\partial^2 w}{\partial y^2} = 0 \text{ on a } y = \text{ constant edge},
\]

\[
\frac{\partial^2 w}{\partial x^2} = 0 \text{ on a } x = \text{ constant edge}, \text{ and}
\]

\[w = 0 \text{ on the entire boundary}.
\]
The special case will be considered in which the load $q(x,y)$ is a uniform load, $q_o$, over the entire surface of the plate. Thus, if $a$ and $b$ are the plate dimensions in the $x$ and $y$ directions, respectively, the load can be represented in the form of a double Fourier series:

$$q(x,y) = \frac{16q_o}{\pi^2} \sum_{m=1,3,5} \sum_{n=1,3,5} \frac{1}{mn} \sin \frac{mnx}{a} \sin \frac{nny}{b} .$$

A solution for the deformed surface which satisfies the boundary conditions can be taken as a double sine series. When this series is substituted into the differential equation the complete solution for the special cases $\alpha = 0^\circ$ and $\alpha = 90^\circ$ becomes

$$w(x,y) = \frac{16q_o}{\pi^2} \sum_{m=1,3,5} \sum_{n=1,3,5} \sin \frac{mnx}{a} \sin \frac{nny}{b}$$

$$\sin \frac{mnx}{a} \sin \frac{nny}{b}$$

$$\sum mn \{A_1 \frac{n^4}{a^4} + 2(A_3 + B_3) \frac{m^2 n^2}{a^2 b^2} + A_2 \frac{n^4}{b^4}\} .$$ (Eq. 22)

This series was summed on the IBM 7074 digital computer for values of $m$ and $n = 1,3,5,...,15$ and the results tabulated in Table 1 for the case $\alpha = 0^\circ$. The values of the elastic constants selected for this example and all the following numerical analysis correspond to those of the test panels.
described in Section V. It is convenient to dimensionalize the displacements by multiplying them by a parameter $K_\sigma = \frac{\overline{D}\pi^6}{16q_0 a^4}$, for which $\overline{D} = EI$.

B. Numerical Solution

In the general case for arbitrary values of $\alpha$ no closed form analytical solution was found, and therefore only a numerical solution was obtained. The particular numerical method used in solving the boundary value problem was a finite difference technique. The domain of the independent variables was replaced by a finite set of points (mesh points) and each differential term in the differential equation and boundary condition expressions was replaced by a linear algebraic equation (difference equation) with the values of the displacement at each mesh point as the unknowns.

For simplicity the plate was divided into a square array of mesh points of size $\lambda$ by $\lambda$. In addition a set of off-boundary points, or virtual points, was defined at a distance $\lambda$ from the edge. The independent variables, $x$ and $y$, were replaced by the discrete variables $(i-2)\lambda$ and $(j-2)\lambda$ for which $i = 1, 2, 3 \ldots N$ and $j = 1, 2, 3 \ldots M$. Thus, $\lambda = b/(M-3) = a/(N-3)$. When each of the differential terms is replaced with a symmetric difference equation, then for
all values of $i$ and $j$ in the interior of the plate, the
difference equation representing the differential equation
at a generic point $(i,j)$ becomes

$$2 \sum_{k=-2}^{2} \sum_{l=-2}^{2} A_{k,l} w(i+k, j+l) = q_0 \lambda^4,$$  \hspace{1cm} (Eq. 23)

for which the values of $A_{k,l}$ are

$$A_{0,0} = 2(3A_1 + 4(A_3 + B_3) + 3A_3).$$

$$A_{1,0} = A_{-1,0} = -4(A_1 + A_3 + B_3)$$

$$A_{0,1} = A_{0,-1} = -4(A_2 + A_3 + B_3)$$

$$A_{2,0} = A_{-2,0} = A_1$$

$$A_{0,2} = A_{0,-2} = A_2$$

$$A_{1,1} = A_{-1,-1} = 2(A_3 + B_3 - B_1 - B_2)$$

$$A_{-1,1} = A_{1,-1} = 2(A_3 + B_3 + B_1 + B_2)$$

$$A_{2,1} = A_{-2,-1} = -A_{-2,1} = -A_{2,-1} = B_1$$

$$A_{1,2} = A_{1,-2} = -A_{-1,2} = -A_{1,-2} = B_2$$

$$A_{\pm 2, \pm 2} = 0.$$

Similarly, the boundary condition equations at a generic
point $(i,j)$ on the plate edge can be represented in finite
difference form as follows:

1. On a $x = \text{constant}$ boundary:

$$w(2,j) = w(N-1,j) = 0 \text{ for } j = 2,3,\ldots,M-1 \text{ and}$$

$$\sum_{k=-1}^{1} \sum_{l=-1}^{1} c_{k,l} w(i+k,j+l) = 0 \text{ for } j = 2,3,\ldots,M-1, \quad \begin{align*}
&i = 2 \text{ and} \\
&i = N-1.
\end{align*}$$

(Eq. 24)

2. On a $y = \text{constant}$ boundary:

$$w(i,2) = w(i,M-1) = 0 \text{ for } i = 2,3,\ldots,N-1 \text{ and}$$

$$\sum_{k=-1}^{1} \sum_{l=-1}^{1} b_{k,l} w(i+k,j+l) = 0 \text{ for } i = 2,3,\ldots,N-1, \quad \begin{align*}
&j = 2 \text{ and} \\
&j = M-1.
\end{align*}$$

(Eq. 25)

where

$$b_{0,1} = b_{0,-1} = 2A_2$$

$$b_{-1,1} = b_{-1,-1} = -b_{1,-1} = -b_{-1,1} = B_2$$

and

$$c_{0,1} = c_{0,-1} = 2A_1$$

$$c_{-1,1} = c_{-1,-1} = -c_{1,-1} = -c_{-1,1} = B_1.$$

For an adequately determined system four additional equations are required. Since the corner virtual points
are the only points in the array which only appear once in Equations 23 through 25, it seems reasonable to make some additional specification concerning these four points, although there is no a priori reason why these particular points were selected. The first difference approximations in the x and y directions at each corner were averaged in order to obtain the four additional equations. For example, the difference equation for the (1,1) corner is

\[ w_{1,1} = \frac{1}{2}[w_{2,1} - \lambda \left( \frac{w_{3,1} - w_{2,1}}{\lambda} \right)] + [w_{1,2} - \lambda \left( \frac{w_{1,3} - w_{1,2}}{\lambda} \right)] \]

\[ w_{1,1} = w_{2,1} + w_{1,2} - \frac{1}{2}(w_{3,1} + w_{1,3}). \]  

(Eq. 26)

Nominally, Equations 23 through 26 represent MxN equations in MxN unknowns; however, the conditions of zero deflection on the plate edges can be applied directly, thus leaving \([MN - 2(N+M) + 12]\) unknowns. Finally, if the symmetry condition, \(w(x,y) = w(x-a,y-b)\) is applied, the total number of unknowns when M and N are both odd integers reduces to \(\frac{1}{2}[MN - 2(M+N) + 13]\). M and N were always taken as odd numbers so that one set of mesh points was along the plate center-line.

A relaxation technique and a direct matrix inversion approach were used to obtain numerical values for the unknowns in the system of difference equations. The
relaxation technique involves making an initial guess for the values of the unknowns and successively iterating in order to improve the initial guess. Equations 23 through 26 can be solved in terms of the displacement at the point at which they are applied, or in general,

\[ w^{(p)}(i,j) = f[w(i+k, j+l)] \text{ for } k \text{ and } l \text{ not both zero.} \]  
(Eq. 27)

for which the superscript \((p)\) represents the iterative step.

The first point-iterative method attempted was the Gauss-Seidel method in which, starting with a set of initial values, \(w^0(i,j)\), the new values computed were used as they were available. Symbolically this operation can be represented as

\[ w^{(p+1)}(i,j) = f_1[w^{(p+1)}(i+k, j+l)] + f_2[w^{(p)}(i+k, j+l)] \]

where \(t = -2,-1,0,1,2\) when \(k = -1,-2\) and \(t = -1,-2\) when \(k = 0\) for \((p+1)\), and \(t = -2,-1,0,1,2\) when \(k = 1,2\) and \(t = 1,2\) when \(k = 0\) for \((p)\).

(Eq. 28)

Although the corrections after successive iterations were everywhere less than 0.10% after 500 iterations, the results lacked a symmetry element which should have existed. This was due to the non-symmetric nature of the order of
the operation in Equation 28. In the Jacobi iterative method the improved values are not used until after a complete iteration over all values of i and j, and thus the iterative operation has the same symmetry as the original finite difference expressions. A Jacobi method was attempted, but unfortunately after 80 iterations the computation became unstable and the values of \( w(i,j) \) cycled between extremely large positive and negative values. Finally, a modified Gauss-Seidel method, or over-relaxation method, was attempted. For this method the following formula was used.

\[
\tilde{w}(p+1)(i,j) = \varphi_b \tilde{w}(p+1)(i,j) - (\varphi_b - 1)w(p)(i,j),
\]

(Eq. 29)

for which \( \varphi_b \) is a parameter known as a relaxation factor, the choice of which determines the rate of convergence of the method. Young (21) suggests an approximation to the best value of \( \varphi_b \) based on the results from a simple Gauss-Seidel method. Several values of the relaxation factor were tried. A value of \( \varphi_b = 1.35 \) produced the most rapid convergence. For the case \( \alpha = 0^\circ \) after 500 iterations the values of the displacements were within 2% of the values obtained from the series solution; however, the results still lacked the required symmetry. Since the differences
in displacements between the cases for various values of $\alpha$ are in the order of 2%, it was concluded that the relaxation methods would never generate sufficiently accurate results in a reasonable number of iteration steps. For this reason the direct matrix inversion method was used to solve the system of finite difference equations.

When the solution of the finite difference equations requires a matrix inversion procedure the number of equations, and therefore the grid mesh density, is limited by computational errors and computer storage capacity. The computing machines available could reasonably handle approximately 100 equations. Thus, $M$ and $N$ were chosen to be 19 and 11 respectively. The total number of unknowns was 81 with 153 mesh points on the plate. A program was written for the IBM 7074 computer which generated the 81 equations, and the Cyclone computer was used to solve the system of equations. The displacements were computed for values of $\alpha$ of $0^\circ$, $22.5^\circ$, $45^\circ$, $67.5^\circ$, and $90^\circ$.

C. Results of Numerical Solution

The results for $\alpha = 0^\circ$ are tabulated in Table 1 for comparison with the results from the series solution. The difference between these two solutions is less than 1% for all values of $i$ and $j$. 
Table 1. Displacement coefficients for the case $\alpha = 0^\circ$

<table>
<thead>
<tr>
<th>(I,J)</th>
<th>$w(I,J)K_b$ from solution by</th>
<th>(I,J)</th>
<th>$w(I,J)K_b$ from solution by</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Series</td>
<td>Finite diff.</td>
<td></td>
</tr>
<tr>
<td>2,2</td>
<td>0.0501</td>
<td>0.0506</td>
<td>4,2</td>
</tr>
<tr>
<td>2,3</td>
<td>0.0941</td>
<td>0.0949</td>
<td>4,3</td>
</tr>
<tr>
<td>2,4</td>
<td>0.1302</td>
<td>0.1312</td>
<td>4,4</td>
</tr>
<tr>
<td>2,5</td>
<td>0.1582</td>
<td>0.1594</td>
<td>4,5</td>
</tr>
<tr>
<td>2,6</td>
<td>0.1788</td>
<td>0.1801</td>
<td>4,6</td>
</tr>
<tr>
<td>2,7</td>
<td>0.1928</td>
<td>0.1943</td>
<td>4,7</td>
</tr>
<tr>
<td>2,8</td>
<td>0.2009</td>
<td>0.2024</td>
<td>4,8</td>
</tr>
<tr>
<td>2,9</td>
<td>0.2035</td>
<td>0.2051</td>
<td>4,9</td>
</tr>
<tr>
<td>2,10</td>
<td>0.2009</td>
<td>0.2024</td>
<td>4,10</td>
</tr>
<tr>
<td>2,11</td>
<td>0.1928</td>
<td>0.1943</td>
<td>4,11</td>
</tr>
<tr>
<td>2,12</td>
<td>0.1788</td>
<td>0.1801</td>
<td>4,12</td>
</tr>
<tr>
<td>2,13</td>
<td>0.1582</td>
<td>0.1594</td>
<td>4,13</td>
</tr>
<tr>
<td>2,14</td>
<td>0.1302</td>
<td>0.1312</td>
<td>4,14</td>
</tr>
<tr>
<td>2,15</td>
<td>0.0941</td>
<td>0.0949</td>
<td>4,15</td>
</tr>
<tr>
<td>2,16</td>
<td>0.0501</td>
<td>0.0506</td>
<td>4,16</td>
</tr>
<tr>
<td>3,2</td>
<td>0.0908</td>
<td>0.0915</td>
<td>5,2</td>
</tr>
<tr>
<td>3,3</td>
<td>0.1712</td>
<td>0.1724</td>
<td>5,3</td>
</tr>
<tr>
<td>3,4</td>
<td>0.2375</td>
<td>0.2390</td>
<td>5,4</td>
</tr>
<tr>
<td>3,5</td>
<td>0.2891</td>
<td>0.2909</td>
<td>5,5</td>
</tr>
<tr>
<td>3,6</td>
<td>0.3272</td>
<td>0.3292</td>
<td>5,6</td>
</tr>
<tr>
<td>3,7</td>
<td>0.3530</td>
<td>0.3553</td>
<td>5,7</td>
</tr>
<tr>
<td>3,8</td>
<td>0.3679</td>
<td>0.3703</td>
<td>5,8</td>
</tr>
<tr>
<td>3,9</td>
<td>0.3728</td>
<td>0.3753</td>
<td>5,9</td>
</tr>
<tr>
<td>3,10</td>
<td>0.3679</td>
<td>0.3703</td>
<td>5,10</td>
</tr>
<tr>
<td>3,11</td>
<td>0.3530</td>
<td>0.3553</td>
<td>5,11</td>
</tr>
<tr>
<td>3,12</td>
<td>0.3272</td>
<td>0.3292</td>
<td>5,12</td>
</tr>
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<td>3,13</td>
<td>0.2891</td>
<td>0.2909</td>
<td>5,13</td>
</tr>
<tr>
<td>3,14</td>
<td>0.2375</td>
<td>0.2390</td>
<td>5,14</td>
</tr>
<tr>
<td>3,15</td>
<td>0.1712</td>
<td>0.1724</td>
<td>5,15</td>
</tr>
<tr>
<td>3,16</td>
<td>0.0908</td>
<td>0.0915</td>
<td>5,16</td>
</tr>
</tbody>
</table>

The results for different values of $\alpha$ are presented graphically in Figures 3 through 7. Since the virtual points have no direct physical meaning and are not tabulated, the designation of the grid mesh points was
Fig. 3. Matrix solution of plate displacements.
Fig. 4. Matrix solution of plate displacements for $\alpha = 22.5^\circ$. 
Fig. 5. Matrix solution of plate displacements for $\alpha=45^\circ$. 
Fig. 6. Matrix solution of plate displacement for $\alpha = 67.5^\circ$. 

- Stiffness coefficient difference $\sim (w_{67.5} - w_0) K_b x 10^2$
- $y$-direction increment $\sim J$
Fig. 7. Matrix solution of plate displacements for $\alpha=90^\circ$. 
redefined such that \( I = 1, J = 1 \) corresponds to the x-y origin at the corner of the plate. The dimensionless displacement coefficient, \( w(I,J)K_\beta \), is quoted only for \( \alpha = 0^\circ \). For other values of \( \alpha \) the displacement coefficient curves are very similar; however, due to the degree of anisotropy of the plate the skewed nature of these curves is difficult to detect visually, and therefore the difference between the displacement coefficient for the given angle and for zero degrees is shown. In a histrionic sense the interpretation of these curves is that positive displacement differences represent larger deflection, and therefore a more "flexible" plate. Moreover, the skew between points corresponding to symmetry elements on the plate with \( \alpha = 0^\circ \), i.e. \( w(x,y) = w(x-a, y-b) \), can be thought of as points which are relatively more flexible. Thus, all the curves indicate that the lower left quadrant is relatively more flexible than the upper left quadrant. By analogy with elementary beam theory this situation is consistent, for in the upper left corner the corrugation elements span across the corner, while in the lower left corner the corrugations are directed into the corner. It follows that the relatively stiffer "load path" is in the upper left quadrant.
V. EXPERIMENTAL INVESTIGATION

A. Objective

A series of tests was undertaken in order to experimentally determine the shape of the deformed surface. It was proposed to measure the deformed surface of several orthotropic, rectangular plates for which the elastic axes of symmetry were oriented at various angles with respect to the plate boundaries. For simplicity it was decided to have simply supported edge conditions and a uniform load on the plate surface; furthermore, edge length ratios of 2 to 1 were selected.

B. Plate Geometry and Material Properties

Through the courtesy of Mr. William R. Roote, the Space and Information Systems Division of North American Aviation, Inc. contributed four specimens of "Spacemetal Sandwich." This sandwich material consists of a 0.002 in. thick (Type 301) stainless steel, corrugated core attached (by a continuous seam spot weld) between two, 0.010 in. nominal thickness, full-hard, stainless steel face sheets. The overall height of the panels varied between 0.167 in. and 0.172 in. and the core pitch was 6.35 corrugations.
per inch.

The original panels of Spacemetal were 15 in. wide and
17, 18, 19 and 60 in. long, respectively. From these four
panels five test plates were required such that the corruga-
tion axis was at an angle of $0^\circ$, $22.5^\circ$, $45^\circ$, $67.5^\circ$, and $90^\circ$
with the plate edges. Thus, the maximum size plate with an
edge length ratio of 2 to 1 which could be cut from the
original panels was $7 \times 14$ in. Allowing $1/8$ in. on a side
for the bearing supports, the final test plate dimensions
were $6.75 \times 13.5$ in. between support points.

The geometry of the Spacemetal panels was not com-
pletely uniform. The face sheet thickness varied between
0.0096 in. and 0.0098 in. and the flats at the seam weld
on the corrugated sheet varied between 0.0170 in. and
0.0330 in. Although the face sheets were continuous, the
core was spliced at irregular intervals varying between
6 in. and 21 in. At these splices the core was overlapped
one or two corrugations, and in some cases the spot weld
was completely omitted. After inspecting all the test
plates an average cross sectional geometry as shown in
Figure 8 was determined. For this cross section the planar
moments of interia normal and parallel to the corrugation
axis were computed to be $139.2 \times 10^{-6}$ in$^4$/in and $126.6 \times$
$10^{-6}$ in$^4$/in, respectively. Since the ratio of these
moments of inertia is 1.10:1, it will be convenient to
Fig. 8. Typical Sandwich Plate Cross-section.
refer to the "amount of anisotropy" as being 10 percent.

Two methods of determining the Young's Modulus of the stainless steel specimens were used. First, the panels were disassembled and a simple stress-strain test was performed on the face sheet material. Unfortunately, the process of disassembling the panels involved literally ripping the seam welds which left small pits along the weld line. Moreover, the sheet was apparently locally annealed along the weld line. In the 300 series of stainless steels the Young's Modulus differs as much as 15% between the annealed and full-hard conditions. Thus, the combination of these thickness and modulus discontinuities suggests that the Young's modulus determined from a simple tension test of the sheet may not be completely valid.

The alternate method of determining the modulus involved loading narrow strips of the panel for which simple beam bending theory was applicable and computing the modulus from measured deflections. Another advantage of this approach was that by using several combinations of loading, different ratios of bending stress to transverse shearing stress could be produced. By comparing the computed Young's modulus from these tests a qualitative determination of the importance of deformation due to transverse shearing stress could be made. Two 13 in. long, 1.50 in. wide beam elements were cut from the panels such that the corrugation axis was
normal and parallel to the beam axes. With the beam elements supported along knife edges, three loading conditions were used: (a) pure bending, (b) concentrated loads at 1.5 in. from the ends (partial shear), and (c) a single concentrated load at mid span (maximum shear). The stress situation produced in the second loading condition roughly corresponds to the equivalent bending and shearing stress ratio which would exist in an isotropic, rectangular plate, while the first and third cases represent the extremes in the relative proportions of bending and shearing stress. Thus a comparison of the computed modulus from these three cases should indicate the relative importance of shear deformation. The Young's modulus for the three loading conditions is tabulated in Table 2.

Table 2. Computed Young's modulus from simple beam tests

<table>
<thead>
<tr>
<th>Loading condition</th>
<th>Geometry</th>
<th>Corrugation axis</th>
<th>E ( \times 10^6 ) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Pure bending</td>
<td></td>
<td>Axial</td>
<td>25.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>25.1</td>
</tr>
<tr>
<td>2. Partial shear</td>
<td></td>
<td>Axial</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>24.9</td>
</tr>
<tr>
<td>3. Maximum shear</td>
<td></td>
<td>Axial</td>
<td>25.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Transverse</td>
<td>24.8</td>
</tr>
</tbody>
</table>
If the moduli for the various loading conditions are compared, it can be seen that there is no orderly trend as the shear to bending stress ratio is increased, and therefore it is reasonable to conclude that the principal mode of deformation is that due to bending stress.

The measured values of moduli are slightly lower than the values commonly quoted for full-hard, cold rolled sheet; a typical value being $26 \times 10^6$ psi. Furthermore, it is generally found that the modulus in the roll direction (longitudinal) is 5 to 6 percent lower than in the direction normal to the roll axis (transverse). The orientation of the roll axis was not specifically known; however, since the Spacemetal panels are produced in long sheets in the direction of the corrugation, it is reasonable to assume that the roll axis is in this direction. Although this implies another slight inconsistency in the results of the simple beam modulus determination, the flexural rigidity, the significant variable in plate deformation theory, can still be determined from the beam tests. Average values of the flexural rigidity were determined to be:

$$(EI)_{axial} = 3508 \text{ lb-in}^2/\text{in} \quad \text{and} \quad (EI)_{trans.} = 3188 \text{ lb-in}^2/\text{in}.$$  

Several additional beam elements were loaded in order to determine a failure criterion. Local buckling occurred in the corrugation at loads corresponding to 28,000 psi fiber stress in the face sheets and nonlinear deformation
was observed for higher loads. This established the maximum allowable total load on the test plates to be 900 lb, or roughly 10 psi distributed load. The distributed loading condition was simulated by applying local loads at 32 points on the plate surface. Holes 0.040 in. in diameter were drilled in the plate, piano wire was inserted in these holes and attached to a one inch square loading block which in turn rested on a foam rubber cushion. The load was distributed to the individual piano wires by a whiffle-tree arrangement as shown in Figure 9a.

C. Test Apparatus

Two welded steel frame structures were used as the plate supports. The plates rested on one frame while the other frame was bolted above to prevent the corners of the plate lifting off the lower supports. The surfaces of the frames were milled in order to provide a planar support. Sharp knife edge bearings could not be used against the relative flexible sandwich plates, and so 1/4 in. diameter drill rods were bolted to the milled surfaces to serve as supports. A series of stand-off bolts was located between the upper and lower frames so that a uniform clamping pressure could be maintained against the plate surfaces. Finally, an aluminum rack was fitted above the frames which
Fig. 9a. Load distribution whiffle-tree.  
Fig. 9b. General view of test apparatus.
Fig. 10a. Dial indicator support bracket.

Fig. 10b. Support structure with upper frame disassembled.
provided support for 1/10,000 dial indicator gages.

A scale poise lever with a multiplication ratio of 20:1 was fabricated and used to apply the loads to the whiffle tree. The poise was calibrated using a strain gage load cell and it was determined that the loads at the power fulcrum were within ± 1% for loads between zero and 1000 lb.

Several photographs of the test apparatus are shown in Figures 9 and 10.

D. Test Procedure

Perhaps the most important parameter which could alter the experimental results is associated with the edge conditions. Theoretically, the conditions which are required are zero deflection and zero normal bending moment about an axis parallel to the edge. Although these boundary conditions do not imply that the twisting moment, viz. $M_{xy}$, is zero along an edge, it is not valid to apply a twisting moment through the supports. If the bearing surfaces located above and below the plate were tightly clamped against the plate, a twisting moment produced from friction forces could conceivably be transmitted into the plate edge. Thus, the bearing surfaces must be close enough to the plate surface to prevent significant translation and
yet not so tightly clamped that a couple is induced. The procedure used to establish the correct edge conditions was to first lubricate the bearing surfaces with a silicon lubricant. The stand-off bolts were removed and the clamping bolts were tightened to a specified torque against the plate. Next, the stand-off bolts were tightened against the milled surface on the support frame and the clamping bolts released. The stand-off bolts were then advanced 1/8 turn, or 0.005 in., and the clamping bolts were re-torqued. Thus, the bearing was taken between the stand-off bolts and the milled surface of the support frame and not against the plate. In addition, the initial clamping bolt torque was sufficient to compress the edge of the plate 0.004 in., and thus there was a gap of zero to 0.001 in. between the bearing supports and the plate surface. Several initial tests were performed on the same plate in which this positioning operation was repeated in order to determine the consistency of this procedure. Not only was the plate stiffness unchanged after each positioning operation, but dial indicator gages located within 1/16 in. of the supports indicated that the deflection at the supports never exceeded 0.0003 in. A qualitative indication that the normal bending moment was zero can be obtained from the shape of the displaced surface curves shown in Figures 12 through 14. On these curves there is no reverse curvature
near the edges, a situation which would be true if a normal bending moment existed.

Figure 11 shows typical test data for several locations on the test plate for which \( \alpha = 45^\circ \). Not only was the data linear, but also all displacement measurements were repeatable within 0.0005 in. Each plate was loaded at least twice in load increments of roughly 100 to 200 lb.

The displacements of the plate surface were measured at the 45 positions shown in Figure 12. It was convenient to locate the plate with respect to the center lines rather than relative to any edge, and therefore an XY-experimental coordinate system was defined at the center of the plate. Thus, displacements were measured at the locations \( X = \pm 1.50N, Y = \pm 1.50M \), for which \( N = 0,1,2 \) and \( M = 0,1,2,3,4 \). Both the load points and displacement points were carefully layed off on the plate surface to within \( \pm 0.050 \) in.

E. Experimental Results and Comparison to Numerical Results

The values of the measured displacements are quoted as influence coefficients (in./lb total load) in Table 3 and presented graphically in Figures 12 through 14. Advantage is taken of the symmetry pole at the XY-origin,
Fig. 11. Typical load-deflection data for $\alpha = 45^\circ$. 

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<td>(0,4)</td>
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Table 3. Experimental influence coefficients--in./lb x 10^{-4}

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Test plate coordinates measured from center such that X = N 1.5; Y = M 1.5.

*Point of corrugation weld discontinuity.
Fig. 12. Experimental displacements and coordinate system.
Fig. 13. Experimental displacements.
Fig. 14. Experimental displacements.
i.e. \( w(X,Y) = w(-X,-Y) \), and the influence coefficients are quoted for only one-half the plate. On the plate for which \( \alpha = 67.5^\circ \) the displacement measurement locations \((1,-1), (1,1), \) and \((1,0)\) were at a splice in the corrugation where the seam weld was omitted and a noticeable discontinuity existed.

It is easy to see that all the influence coefficients have the same characteristic skew which was observed in the numerical results; however, in a purely visual sense the curves of numerical and experimental results differ since the numerical results are essentially the differences in displacements, while the experimental results are quoted as coefficients directly proportional to absolute displacements.

There are two useful methods of comparing the numerical and experimental results. The first method is to compare the general functional relation of the two deformed surfaces. The second comparison is with respect to the absolute values of the displacements. Figure 15 shows the normalized displacements for the plate with \( \alpha = 0^\circ \) for which the displacements at any point, \( w(X,Y) \), are divided by the displacement at the center of the plate, \( w(0,0) \). Along the X centerline the agreement is excellent; however, near the edges of the plate the experimental displacement ratios are slightly larger than the numerical analysis
Fig. 15. Experimental and numerical normalized displacements for $\alpha = 0^\circ$. 

Normalized displacement $\frac{w(x,y)}{w(0,0)}$.

- Experimental
- Numerical

Y - Distance from plate center ~ in.
indicates. For other values of the angle $\alpha$ similar curves indicate that the functional relation of the deformed surface is very similar between the two methods of solution; however, a slightly higher degree of anisotropy appears to exist in the experimental plates.

Since the "shape" of the deformed surface is similar for both the experimental and numerical approaches, an absolute comparison of the displacements may be made by considering the center deflections. Figure 16 shows the variation in the center displacement for different values of the angle $\alpha$. In this figure the experimental influence coefficients are converted to stiffness coefficients by multiplying by the term $Dm^6/8a^2$. From this curve it is clear that the absolute displacements are not in good agreement. The numerical results indicate that for the "10% anisotropy" the displacements for the zero and 90° cases differ by 6.5%. The experimental results show a variation of 80% between the zero and 90° cases. This would imply that the experimental plates had a much higher degree of anisotropy.

F. Discussion of Results

The numerical and experimental results indicate the same general shape of the deformed surface; however, there
Fig. 16. Comparison of experimental and numerical stiffness coefficients at center.
is a significant difference between the two prediction methods in the absolute magnitude of displacements. In particular, the experimental plate was found to be 23% more flexible for the $\alpha = 0^\circ$ case and 122% more flexible for the $\alpha = 90^\circ$ case than the theory predicted.

Thus, in addition to being more flexible on the average, the experimental plates appeared to have a much higher degree of anisotropy. It would be superficial to conclude that the two methods of determining the displacements simply differ by measured elastic constants, since the beam element tests gave consistent verification of the flexural rigidities in the principal elastic directions. Furthermore, the assumption for the analytical model that deformation due to shearing stresses are negligible seems to have been validated by the beam tests. It is of course possible that shearing stress deformation is negligible in a beam element yet significant in a plate; however, recognizing that the plate problem is the two dimensional generalization of the beam problem, it is unlikely that a basic mechanism which would affect one would not affect the other.
G. Conclusions

On the basis of the preceding discussion, several conclusions can be drawn.

1. The experimental plates were more flexible and possessed a higher degree of anisotropy than the theory predicted.

2. Although it is possible to suppose that some additional mechanism was overlooked in the experimental investigation which would contribute to increasing the overall flexibility, it is difficult to envision any phenomenon which would generate a higher degree of anisotropy. Therefore, the experimental results represent a valid description of the deformed surface of the plates.

3. The analytical model may have been idealized to such an extent that it is not a valid representation of the experimental plates.

4. The numerical solution agrees with the conventional plate theory at the extreme cases for which the elastic axes are coincident with the edge axes. Moreover, the numerical solution correctly defines the deformed surface in a qualitative sense. Thus, the theoretical development for the general orthotropic plate is applicable to the same degree and for the same class of problems as conventional, isotropic plate theory.
VI. REFERENCES


VII. ACKNOWLEDGMENTS

In retrospect I realize how many individuals I owe a debt of gratitude for their help and encouragement in the preparation and completion of my research. In particular, I thank Dr. Glenn Murphy for his direction and continual encouragement of my entire research program. Also, I must express my appreciation to Dr. H. J. Weiss for his helpful suggestions in the preparation of the analysis. Finally, I would like to thank Mrs. J. F. Smith for her skillful aid in the computer programming.