Existence of almost periodic solutions of functional-differential equations

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EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

by

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I. INTRODUCTION

In this paper we consider sufficient conditions for the existence of almost periodic solutions of functional-differential equations. In this chapter, therefore, we give a brief discussion of the development of almost periodicity and functional-differential equations.

Almost periodicity as a structural property of functions is a generalization of "pure" periodicity. Free oscillations of undamped linear systems with more than one natural frequency are not in general periodic, but are almost periodic. For example \( \sin pt + \sin qt \), where \( p \) and \( q \) are constants and \( p/q \) is irrational, is a function which is almost periodic and is not periodic. We find that the sum, product, and uniform limit of almost periodic functions is again an almost periodic function.

Harald Bohr (6, 7, 8) introduced the theory of almost periodic functions in papers published during the years 1924-1926. Weyl (18), Stepanoff (16), and C. de la Vallee Poussin (17) contributed greatly to the generalizations of his theory and to the establishment of some of Bohr's results in a less complicated fashion. Among those who first applied this theory to such fields as number theory, dynamics, and differential equations were Besicovitch (2), Birkhoff (3), and Bochner (4). Favard (10) gave conditions for the existence of almost periodic solutions of systems of linear differential equations with almost periodic time dependence, and Amerio (1) generalized this result to nonlinear systems. Langenhop and Seifert (15) and Seifert (15) have used Amerio's results to show existence of almost periodic solutions of certain classes of differential equations.
A differential-difference equation can be defined as an equation in an unknown function and certain of its derivatives, evaluated at arguments which differ by any of a fixed number of values. For example,

\[ u'(t) = u(t-1) + 2u'(t-1). \]

The theory of differential-difference equations has been investigated in detail; see, for example, Norlund (14). By generalizing the usual differential-difference equation, Krasovskii (12) has defined functional-differential equations. He used Lyapunov functionals to discuss the problem of stability for these functional-differential equations. Hale (11) and Yoshizawa (19) have also used Lyapunov functions in establishing results pertaining to almost periodicity and functional-differential equations. Bochner (5) gives a condition for almost periodicity which differs from the usual condition and uses this new condition to establish some existence theorems for almost periodic solutions of linear differential-difference equations. G. Seifert\(^1\) has also established a new condition for almost periodicity and has applied it to functional-differential equations.

In Chapter II of this paper we give several definitions and propositions pertaining to almost periodicity. We also define a solution of a functional-differential equation and introduce a definition of separated solutions analogous to that used by Amerio. In Chapter III we extend Amerio's results to the class of functional-differential equations.

\(^1\)G. Seifert, Department of Mathematics, Iowa State University of Science and Technology, Ames, Iowa. A condition for almost periodicity with some applications to functional-differential equations. Private communication, 1964.
II. PRELIMINARIES

A. Almost Periodicity

The following definitions and propositions concerning almost periodic functions are needed. In some cases we have modified the definitions in the classical Bohr-theory of almost periodic functions so that the material in Chapter III may be established. The standard items which are presented here can be found, for example, in Favard's book (10).

**Definition 2.0:** Let \( \mathbb{R}^n \) be a normed n-dimensional space over the complex field. If \( z \in \mathbb{R}^n \), we denote by \( |z| \) the norm of \( z \).

**Definition 2.1:** \( C = C[-h,0] \) is the set of functions \( \phi \) which are continuous on the closed real interval \([-h,0]\) and which have values in \( \mathbb{R}^n \). The topology in \( C \) is in terms of the supremum norm which is denoted by \( \| \phi \| \). Thus, for \( \phi \in C \),

\[
\| \phi \| = \sup_{-h \leq \theta \leq 0} |\phi(\theta)|.
\]

**Definition 2.2:** \( [C] \) is the set of functions \( F \) continuous on \( I \times C \) to \( \mathbb{R}^n \), where \( I = (-\infty, \infty) \).

**Definition 2.3:** \( C_H \) is the subset of \( C \) defined by \( \| \phi \| \leq H \); \( [C]_H \) is the subset of \( [C] \) consisting of functions \( F \) continuous on \( I \times C_H \) to \( \mathbb{R}^n \).

**Definition 2.4:** A set \( E \) of real numbers is called relatively dense if there exists a number \( L > 0 \) such that every interval of length \( L \) contains at least one element of the set \( E \).

**Definition 2.5:** Let \( F(t,\theta) \) be a function defined on \( I \times \Lambda \) to \( \mathbb{R}^n \), where \( \Lambda \) is some subset of \( C \). If a real number \( \tau \) satisfies the inequality
\[ |F(t + \tau, \phi) - F(t, \phi)| \leq \delta \]

for all \( t \in I \), then \( \tau \) is called a \( \delta \)-translation number of \( F(t, \phi) \) belonging to \( \delta \). We denote the set of all \( \delta \)-translation numbers of a function \( F(t, \phi) \) belonging to \( \delta \) by \( \{ \tau_F(\delta, \phi) \} \).

**Definition 2.6:** A function \( F(t, \phi) \) which is continuous in \( t, \phi \) for \( t \in I, \ \phi \in \Lambda \), is said to be almost periodic in \( t \) for fixed \( \phi \in \Lambda \) if for each \( \delta > 0 \) the set \( \{ \tau_F(\delta, \phi) \} \) is relatively dense.

**Definition 2.7:** A function \( F(t, \phi) \) which is continuous in \( t, \phi \) for \( t \in I, \ \phi \in \Gamma \), where \( \Gamma \) is a compact subset of \( C \), is said to be almost periodic in \( t \) uniformly with respect to \( \phi \in \Gamma \) if for each \( \delta > 0 \) the relatively dense set of \( \delta \)-translation numbers is independent of \( \phi \in \Gamma \). We denote this relatively dense set by \( \{ \tau_F(\delta, \Gamma) \} \).

**Definition 2.8:** A function \( F \) in \([C]_g \) is said to be almost periodic in \( t \) uniformly for \( \phi \) in \( C \) if for each compact set \( \Gamma \subset C \) there exists a relatively dense set of \( \delta \)-translation numbers \( \{ \tau_F(\delta, \Gamma) \} \), for each \( \delta > 0 \), which is independent of \( \phi \) in \( \Gamma \).

**Definition 2.9:** If \( F \) is in \([C] \), we define \( H(F) \) to be the set of functions \( G \) on \( I \times C \) to \( R^n \) such that

\[ \lim_{k \to \infty} F(t + \tau_k, \phi) = G(t, \phi) \]

for some sequence \( \{ \tau_k \} \).

**Remark 2.1:** If \( F \) is almost periodic in \( t \) and if \( G \in H(F) \), then without loss of generality, we may assume \( F(t + \alpha_k, \phi) \to G(t, \phi) \) uniformly for \( t \) in \( I \) for some sequence \( \{ \alpha_k \} \). Also, if \( \phi \in \Gamma \) where \( \Gamma \) is a compact subset of \( C \), then for some subsequence \( \{ \beta_k \} \) of the sequence \( \{ \alpha_k \} \), \( F(t + \beta_k, \phi) \to G(t, \phi) \) where the limit is uniform for \( (t, \phi) \) in \( I \times \Gamma \).
Proposition 2.1: If $F(t, \phi)$ is almost periodic in $t$ uniformly for $\phi$ in $C_H$ and if $G \in H(F)$, then $G(t, \phi)$ is almost periodic in $t$ uniformly for $\phi$ in $C_H$.

Proof. Let $\Gamma$ be a compact subset of $C_H$. By the previous remark there exists a sequence $\{\alpha_n\}$ such that $F(t + \alpha_n, \phi) \rightarrow G(t, \phi)$ uniformly for $(t, \phi)$ in $I \times \Gamma$. Thus for $\delta/3$ there exists $N(\delta/3) > 0$ such that if $n > N$,

$$|G(t, \phi) - F(t + \alpha_n, \phi)| < \delta/3 \quad (2.1)$$

Now let $\tau \in \{\tau_F(\delta/3, \Gamma)\}$ and then consider

$$|G(t + \tau, \phi) - G(t, \phi)| \leq |G(t + \tau, \phi) - F(t + \tau + \alpha_n, \phi)| + |F(t + \tau + \alpha_n, \phi) - F(t + \alpha_n, \phi)| + |F(t + \alpha_n, \phi) - G(t, \phi)|.$$

Using (2.1) and the almost periodicity of $F$ we have

$$|G(t + \tau, \phi) - G(t, \phi)| \leq \delta.$$

Thus $\{\tau_F(\delta/3, \Gamma)\} \subset \{\tau_G(\delta, \Gamma)\}$, and from this we have that $\{\tau_G(\delta, \Gamma)\}$ is relatively dense. Since the set $\{\tau_G(\delta, \Gamma)\}$ is independent of $\phi$ in $\Gamma$, we have that $G(t, \phi)$ is almost periodic in $t$ uniformly for $\phi$ in $C_H$.

Proposition 2.2: If $F(t, \phi)$ is almost periodic in $t$ uniformly with respect to $\phi \in \Gamma$, where $\Gamma$ is a compact subset of $C$, then $F(t, \phi)$ is uniformly continuous in $(t, \phi)$ for $t \in I$, $\phi \in \Gamma$.

Proof. Let $\Gamma$ be as in the hypothesis. For $\delta > 0$, let $L(\delta/4)$ be as in Definition 2.4 and let $\eta (0 < \eta < 1)$ be a number such that

$$|F(t_1, \phi_1) - F(t_2, \phi_2)| < \delta/4$$

for any $t_1, t_2 \in 0 \leq t \leq L(\delta/4)+1$ such that $|t_1 - t_2| < \eta$ and $||\phi_1 - \phi_2|| < \eta$. $\phi_1, \phi_2 \in \Gamma$. Let $t, t'$ be any two numbers in $I$ such that $|t-t'| < \eta$. 


and let $\phi, \phi'$ be any two elements of $\Gamma$ such that $||\phi - \phi'|| < \eta$.

Then there exists $\tau \in \{ \tau_F(\delta/4, \Gamma) \}$ such that $t + \tau, t' + \tau \in (0, L(\delta/4) + 1)$.

Hence,

$$|F(t, \phi) - F(t', \phi')| \leq |F(t, \phi) - F(t + \tau, \phi)|$$

$$+ |F(t + \tau, \phi) - F(t' + \tau, \phi')|$$

$$+ |F(t' + \tau, \phi') - F(t', \phi')|$$

$$< \delta.$$ 

In order to prove the next proposition, we need the following lemma.

**Lemma 2.1:** Let $F(t, \phi)$ be almost periodic in $t$ uniformly with respect to $\phi \in \Gamma$, where $\Gamma$ is a compact subset of $\mathbb{C}$, and let $\{k_i\}$ be any sequence of real numbers. Then, to any $\delta > 0$ there corresponds a subsequence $\{k_{n_i}\}$ such that the norm of the difference of any pair of functions $F(t + k_{n_i}, \phi)$ is less than $\delta$.

**Proof.** Let $\Gamma$ be as in the hypothesis. Observe that $k_i$ can be represented by

$$k_i = \tau_i + \omega_i$$

where $\tau_i \in \{ \tau_F(\delta/4, \Gamma) \}$ and where $\omega_i$ is such that $0 < \omega_i \leq L(\delta/4)$.

We will consider only one representation of each $k_i$ in this form. Let $\omega$ be a limit point of the set of all $\omega_i$. By the uniform continuity of $F(t, \phi)$ we may define a number $\sigma$ such that

$$|F(t'', \phi) - F(t', \phi)| < \delta/2$$

for $|t'' - t'| < 2\sigma, \phi \in \Gamma$.

Let $K = \{ k_i : \omega - \sigma < \omega_i < \omega + \sigma \}$. Then for $k_m, k_n \in K$ we have
\[
\sup_{t \in I} |F(t + k, \phi) - F(t + k, \phi)| = \sup_{t \in I} |F(t + k - k, \phi) - F(t, \phi)|
\]
\[
= \sup_{t \in I} |F(t + \tau_n - \tau_n^* + \omega^* - \omega_n, \phi) - F(t, \phi)|
\]
\[
\leq \sup_{t \in I} |F(t + \tau_n - \tau_n^* + \omega^* - \omega_n, \phi) - F(t + \omega^* - \omega_n, \phi)| + \sup_{t \in I} |F(t + \omega^* - \omega_n, \phi) - F(t, \phi)|.
\]

Since \(\tau_m - \tau_n \in \{\tau_F(\delta/2, \Gamma)\}\), we have
\[
\sup_{t \in I} |F(t + \tau_m - \tau_n + \omega^* - \omega_n, \phi) - F(t + \omega^* - \omega_n, \phi)| < \delta/2.
\]

Also, \(|\omega^* - \omega_n| < 2\delta\), and so by (2.2) we have
\[
\sup_{t \in I} |F(t + \omega^* - \omega_n, \phi) - F(t, \phi)| < \delta/2.
\]

Thus
\[
|F(t + k, \phi) - F(t + k, \phi)| < \delta,
\]
and \(K\) is the set which satisfies the conditions of this lemma.

Proposition 2.3: If \(\Gamma\) is a compact subset of \(C\), then \(F(t, \phi)\) is almost periodic in \(t\) uniformly with respect to \(\phi \in \Gamma\) if and only if for every sequence of real numbers \(\{\tau_i\}\) the sequence \(\{F(t + \tau_i, \phi)\}\) has a subsequence \(\{F(t + \tau_{j_k}, \phi)\}\) which converges uniformly with respect to \(t \in I, \phi \in \Gamma\).

Proof. Let \(\Gamma\) and \(F\) be as in the hypothesis. Let \(\{\tau_i\}\) be a sequence of real numbers. By Lemma 2.1 we can choose a subsequence \(\{\tau_{j_k}\}\) such that for any two positive integers \(m, n\) and for all \(t\)
\[
|F(t + \tau_{j_k}^{(1)} m, \phi) - F(t + \tau_{j_k}^{(1)} n, \phi)| < 1.
\]
Likewise we can choose a subsequence \(\{\tau_{j_k}^{(2)}\}\) of the sequence \(\{\tau_{j_k}^{(1)}\}\) such
that for any positive integers \( m, n \) and for all \( t \) we have
\[
|F(t + \tau^{(2)}_{j_m}, \phi) - F(t + \tau^{(2)}_{j_n}, \phi)| < \frac{1}{2}.
\]
Again choose a subsequence \( \{\tau^{(3)}_{j_k}\} \) of the sequence \( \{\tau^{(2)}_{j_k}\} \) such that
\[
|F(t + \tau^{(3)}_{j_m}, \phi) - F(t + \tau^{(3)}_{j_n}, \phi)| < \frac{1}{3}.
\]
Continue this process and then consider the sequence
\[
\{ F(t + \tau^{(k)}_{j_k}, \phi) \}, \text{ for } k = 1, 2, \ldots \quad (2.3)
\]
Let \( m, n \) \((m < n)\) be two positive integers. From the above we see that
\[
|F(t + \tau^{(m)}_{j_m}, \phi) - F(t + \tau^{(n)}_{j_n}, \phi)| < \frac{1}{m},
\]
and thus we have that the sequence 2.3 is uniformly convergent. By defining \( \tau_k = \tau^{(k)}_{j_k} \) we have the desired result.

Now for the "only if" part of this proposition we assume the contrary; that is, we suppose that \( F(t, \phi) \) is not almost periodic in \( t \) uniformly for \( \phi \in \Gamma \). Then there exists a \( \delta_1 > 0 \) such that the set
\[
\{ \tau_F(\delta_1, \Gamma) \}
\]
is not relatively dense. Let \( \Lambda \) denote the set \( \{ \tau_F(\delta_1, \Gamma) \} \).

Pick an arbitrary real number \( \tau_1 \) and let \((a_2, b_2)\) be an interval of length greater than \( 2|\tau_1| \) which does not contain any member of \( \Lambda \). Let \( \tau_2 \) be the mid-point of this interval. Clearly \( \tau_2 - \tau_1 \in (a_2, b_2) \) and therefore \( \tau_2 - \tau_1 \notin \Lambda \). Now define an interval \((a_3, b_3)\) of length greater than \( 2(|\tau_1| + |\tau_2|) \) which does not contain any member of \( \Lambda \).

Let \( \tau_3 \) be the mid-point of \((a_3, b_3)\), and as before the numbers \( \tau_3 - \tau_1, \tau_3 - \tau_2 \) do not belong to \( \Lambda \). We define \( \tau_4, \tau_5, \ldots \) similarly so that none of the numbers \( \tau_i - \tau_j \) belong to \( \Lambda \). Thus for any \( i, j \)
\[
\sup_{t \in \Gamma} |F(t + \tau_i, \phi) - F(t + \tau_j, \phi)| = \sup_{t \in \Gamma} |F(t + \tau_i - \tau_j, \phi) - F(t, \phi)| > \delta_1.
\]
Thus we have that no subsequence of the sequence \( \{ F(t + \tau_k, \phi) \} \) is uniformly convergent, and this contradicts our hypothesis concerning \( F \).

Thus the proposition is proved.

**Proposition 2.4:** If \( F(t, \phi) \) is almost periodic in \( t \) uniformly for \( \phi \in \mathcal{C}_H \), and if \( G \in \mathcal{H}(F) \), \( L \in \mathcal{H}(F) \), then \( G \in \mathcal{H}(L) \).

**Proof.** Let \( \Gamma \) be a compact subset of \( \mathcal{C}_H \). By hypothesis we may assume the existence of sequences \( \{ \tau_k \} \), \( \{ \sigma_k \} \), such that

\[
\lim_{k \to \infty} F(t + \tau_k, \phi) = G(t, \phi)
\]

where this limit is uniform for \( (t, \phi) \in \mathcal{I} \Gamma \), and

\[
\lim_{k \to \infty} F(t + \sigma_k, \phi) = L(t, \phi)
\]

where again this limit is uniform for \( (t, \phi) \in \mathcal{I} \Gamma \).

Thus for \( \delta > 0 \) there exists \( N_1 = N_1(\delta) \) such that for \( k > N_1 \)

\[
|G(t, \phi) - F(t + \tau_k, \phi)| < \delta/2
\]

for \( (t, \phi) \in \mathcal{I} \Gamma \). Also, there exists \( N_2 = N_2(\delta) \) such that for \( k > N_2 \)

\[
|L(t, \phi) - F(t + \sigma_k, \phi)| < \delta/2
\]

for \( (t, \phi) \in \mathcal{I} \Gamma \). So, for \( k > \max(N_1, N_2) \)

\[
|G(t, \phi) - L(t + \tau_k - \sigma_k, \phi)| = |G(t, \phi) - F(t + \tau_k, \phi)| + |F(t + \tau_k - \sigma_k + \sigma_k, \phi) - L(t + \tau_k - \sigma_k, \phi)| < \delta.
\]

Hence, by letting \( \tau_k - \sigma_k = \eta_k \) we have

\[
\lim_{k \to \infty} L(t + \eta_k, \phi) = G(t, \phi).
\]

Thus \( G \in \mathcal{H}(L) \).
B. Functions

In this section we define various types and classes of functions. This will enable us to present the material in the next chapter in a more clear and precise manner. We also give a definition of a solution of a functional-differential equation.

Definition 2.10: Let $S$ be a set of functions defined on a real interval $J$ to $\mathbb{R}^n$. $S$ is said to be uniformly bounded on $J$ if there exists a constant $M > 0$ such that $|s(t)| \leq M$ for all $s \in S$, and for all $t \in J$.

Definition 2.11: Let $S$ be a set of functions defined on a real interval $J$ to $\mathbb{R}^n$. $S$ is said to be equicontinuous on $J$ if, given any $\delta > 0$, there exists a $\eta(\delta) > 0$ independent of $s \in S$ and also $t, t' \in J$ such that

$$|s(t) - s(t')| < \delta$$

whenever $|t - t'| < \eta(\delta)$.

Definition 2.12: A set of functions $S$ is said to be conditionally compact if its closure is compact.

Definition 2.13: A function $f$ satisfies condition $B(H,K)$ if

i) $f : I \rightarrow \mathbb{C}$

ii) for all $t, t' \in I$, $||f(t)|| \leq H$ and $||f(t) - f(t')|| \leq K|t - t'|$.

Definition 2.14: Let $S$ be a class of functions, $I \rightarrow \mathbb{C}$. Consider a function $g \in S$. If there exists $\mu(g) > 0$ such that for every $h \in S$, $h \neq g$, we have

$$||g(t) - h(t)|| \geq \mu$$

for all $t$, then $g$ is said to be separated with respect to $S$.

Definition 2.15: If $\hat{x}$ is a function on $I$ to $\mathbb{C}$, then we denote its functional value by $x_t$, i.e., $\hat{x}(t) = x_t$. Note that $x_t$ is a function
on \([-h, 0]\) to \(\mathbb{R}^n\). We denote the function value of \(x_t\) by \(x_t(\theta)\).

**Definition 2.16:** Let \(x\) be the function whose function value, \(x(t)\), is such that \(x(t) = x_t(0)\), where \(x_t(\theta)\) is as above. Thus \(x\) is a function on \(I\) to \(\mathbb{R}^n\).

**Definition 2.17:** If for a function \(\hat{x}\), \(I \to \mathbb{C}\), and the corresponding \(x\) as defined above, the limit
\[
\lim_{\tau \to 0^+} \frac{x(t + \tau) - x(t)}{\tau}
\]
exists, we denote it by \(\hat{x}(t)\).

**Definition 2.18:** The function \(\hat{x}\), \(I \to \mathbb{C}\), which is such that \(x_t(\theta) = x_{t+\theta}(0)\) for \(-h \leq \theta \leq 0\) and \(t \in I\), is a solution of
\[
x(t) = F(t, x_t)
\]
for \(t\) in \(J\), some real interval, if \(F\) is in \([0]^n\), \(\|x_t\| \leq H\) for \(t \in J\), and (2.4) is true for \(t \in J\), where \(x\) and \(\hat{x}(t)\) are as defined above.

Note that (2.4) is a differential equation of the type mentioned in the introduction, that is, a functional-differential equation.

**Remark 2.2:** If \(\hat{x}\) is a solution of (2.4), the existence and continuity on \(I\) of \(\hat{x}(t)\) implies the existence and continuity of the ordinary derivative. Thus (2.4) can be written in the standard integral equation form.

The following definition is very important since the set which is defined is one that is utilized in the hypothesis of many of the lemmas and theorems of the next chapter.

**Definition 2.19:** Let \(X\) denote the set of all functions \(\hat{x}\) which are such that

1) \(\hat{x}\) satisfies condition \(B(H,K)\),
ii) \( x_{t+\theta} (0) = x_t (\theta) \) for \(-h \leq \theta \leq 0\) and \( t \in I \), where \( \hat{x}(t) = x_t \) as previously defined.

**Remark 2.3:** Let \( \hat{x} : I \to C \), be such that \( x_{t+\theta}(0) = x_t(\theta) \) for \(-h \leq \theta \leq 0\) and for \( t \in I \), and let \( x \) be related to \( \hat{x} \) as in Definition 2.16. Then it follows easily that \( \hat{x} \) is almost periodic if and only if \( x \) is almost periodic.

**Remark 2.4:** Amerio's definition of separation (1) applies to functions on \( I \to \mathbb{R}^n \). If we apply the above separation condition (Definition 2.14) to functions defined on \( I \to \mathbb{R}^n \) which arise from functions on \( I \to C \) as in Definition 2.16, it follows that Amerio's condition is stronger than the one in Definition 2.14. For \( \hat{x}, \hat{y} \in X \), \( \| x_t - y_t \| \geq \rho \) which is equivalent to \( \sup_{\theta \in [-h,0]} |x(t + \theta) - y(t + \theta)| \geq \rho \) is obviously weaker than \( |x(t) - y(t)| \geq \rho \).
III. ALMOST PERIODIC SOLUTIONS 
OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

In this chapter we present an extension of Amerio's results (1) to functional-differential equations. The material in the previous chapter as well as several lemmas in this chapter are utilized in obtaining the main result, Theorem 3.2. We also define a module for an almost periodic function and present a theorem concerning modules of solutions of functional-differential equations. A proof of Lemma 3.1 (Ascoli) can be found, for example, in Coddington and Levinson's book (9).

Lemma 3.1 (Ascoli): Let \( S = \{s\} \) be an infinite, uniformly bounded, equicontinuous set of functions defined on a compact subset \( J \) of \( I \). Then \( S \) contains a sequence \( \{s_n\} \), \( n = 1, 2, \ldots \), which is uniformly convergent on \( J \).

Lemma 3.2: Let \( \hat{x} \in X \), then the range of \( \hat{x} \) is conditionally compact.

Proof. By hypothesis we have

\[
\| x_t - x_{t'} \| = \sup_{\theta \in [-h,0]} |x(t+\theta) - x(t'+\theta)| \leq K|t-t'|.
\]

for all \( t, t' \) in \( I \), where \( x \) is the function described in Definition 2.16. Thus we have

\[
|x(t) - x(t')| \leq K|t - t'|
\]

and from this we obtain

\[
|x(t + \theta') - x(t + \theta)| \leq K|\theta - \theta'|.
\]

Now for \( \delta > 0 \) let \( \gamma = \delta/K \), then if \( |\theta - \theta'| < \gamma \) we have

\[
|x(t + \theta') - x(t + \theta)| < \delta
\]

for \( \theta, \theta' \in [-h,0] \). Thus we have shown that the set \( \{x_t\} \), for all \( t \in I \).
is an equicontinuous set of functions in $C$. We also have from the hypothesis that $\|x_t\| \leq H$, and from this we see that $\{x_t\}$ is also a uniformly bounded set for all $t \in I$. Hence by Lemma 3.1, every sequence in the set $\{x_t\}$ contains a uniformly convergent subsequence. We conclude by noting that since the set $\{x_t\}$ is just the range of $\hat{x}$, we have established the lemma.

Definition 3.1: If $\hat{x} \in X$, let $\Gamma$ denote the closure of the range of $\hat{x}$. By the previous lemma we note that $\Gamma$ is a compact subset of $C$.

Definition 3.2: Let $\{\tau_n\}$ be a sequence of real numbers. If $\{k_n\}$ is a subsequence of $\{\tau_n\}$, we denote this by $\{k_n\} \subset \{\tau_n\}$.

Lemma 3.3: Let $\hat{x} \in X$ and let $\{\tau_k\}$ be any sequence of real numbers. Then there exists a subsequence $\{\tau_{k_n}\}$ of the sequence $\{\tau_k\}$ such that

$$\lim_{n \to \infty} x_{t + \tau_{k_n}} = y_t$$

uniformly on each compact subset of $I$, where $\hat{y} \in X$.

Proof. Let $m$ be a positive integer and let $x$ be as in Definition 2.16. Fix $\tau \in I$ and consider the function $x^m$ whose functional values are $x^m(u + \tau)$, for $-m - m \leq u \leq m$. Now by letting $\tau$ vary over $I$ we obtain a set of functions defined on $[-m, m]$, a compact subset of $I$. This set of functions can be shown to be equicontinuous and uniformly bounded in the same manner as the set $\{x_t\}$ in Lemma 3.2. Thus by Lemma 3.1 there exists a sequence $\{\tau_{k}^m\} \subset \{\tau_k\}$ such that

$$\lim_{k \to \infty} x(u + \tau_{k}^m) = y^m(u),$$

where this limit exists uniformly for $u \in [-m, m]$. Now if we replace $u$ by $t + \theta$ we have
\[ \lim_{k \to \infty} x(t + \theta + \tau^m_k) = y^m(t + \theta) \]
uniformly for \( t \in [-m, m] \), \( \theta \in [-h, 0] \). And since \( y^m(t + \theta) \) is the functional value of \( y^m_t \), for \( -h < \theta < 0 \), we have
\[ x_{t+\tau^m_k} \to y^m_t \]
as \( k \to \infty \), uniformly for \( t \in [-m, m] \).

We now continue the above argument for \( m = 1, 2, \ldots \), such that
\[ \tau^{m+1}_k \subset \tau^m_k, \quad m = 1, 2, \ldots \] Thus, if \( j > m \)
y^{j}_t = y^{m}_t
for \( t \in [-m, m] \). For any \( t \in I \) we define \( y^m_t = y^m_t \), where \( m \) is the least integer such that \( t \) is in \([-m, m]\), and define
\[ \tau^n_k = \tau^n_n, \quad n = 1, 2, \ldots \]
Therefore for any integer \( N > 0 \) we have
\[ \lim_{n \to \infty} x_{t+\tau^n_k} = y^m_t \]
uniformly for \( t \in [-N, N] \).

Now we must show \( y \in X \). The fact that \( \|y_t\| \leq H \) follows easily from \( \|x_t\| \leq H \). Also, we have
\[ \|y_t - y_{t'}\| \leq \|y_t - x_{t+\tau^m_k} - x_{t'+\tau^m_k} - y_{t'}\| + \|x_{t+\tau^m_k} - x_{t'+\tau^m_k}\| \] (3.1)
For \( t, t' \) arbitrary there exists a compact subset of \( I \) containing \( t, t' \) for which the first and third terms on the right side of (3.1) approach zero as \( n \to \infty \) from the first part of this lemma. Since \( x \in X \), we know that for any \( n, \)
\[ \|x_{t+\tau^m_k} - x_{t'+\tau^m_k}\| \leq K|t - t'|. \]
Thus,

$$\| y_t - y_{t'} \| \leq K |t - t'|,$$

and we have shown $\hat{x} \in X$.

Lemma 3.4: If each $x \in X$ is separated with respect to $X$, then $X$ is finite.

Proof. Let $\hat{x}$ be as in the hypothesis and $x$ as in Definition 2.16. We assume the contrary, i.e. we assume $X$ is infinite. Then the method of proof is as follows: we show that the associated set $\{x\}$ satisfies the hypothesis of Lemma 3.1; we then consider the infinite sequence $\{x^n\}$, $x^n \neq x^m$, $m \neq n$, which exists by Lemma 3.1, and in the same manner as in the previous lemma we pick a "diagonal" subsequence $\{x^k\}$ which converges uniformly to $\hat{x}$; finally we show that $x^k$ and $\hat{x}$ are members of $X$ and thus have a contradiction.

We have that the set $\{x\}$ is uniformly bounded, for by hypothesis

$$h \geq \| x_t \| = \sup_{\theta \in [-h,0]} |x_t(\theta)| = |x(t)|.$$

Again from the hypothesis we have

$$\| x_t - x_{t'} \| = \sup_{\theta \in [-h,0]} |x(t + \theta) - x(t' + \theta)| \leq K |t - t'|.$$

From this we see that

$$|x(t) - x(t')| \leq K|t - t'|$$

for all $t, t' \in I$. Thus for $\delta > 0$ let $\gamma = \delta/K$. Then if $|t-t'| < \gamma$ we have

$$|x(t) - x(t')| < \delta$$

and $\{x\}$ is also an equicontinuous set of functions.

Let $\{x^n\}$ be the infinite sequence which exists by Lemma 3.1, and let $\{x^k\}, \{x^k\} \subset \{x^n\}$, be the "diagonal" subsequence chosen as in Lemma 3.2 from subsequences of $\{x^n\}$. Thus we assert that $x^k(t) \rightarrow \hat{x}(t)$ uniformly on any compact subset of $I$, and by our previous definitions we
have that $x^k_t \to \tilde{x}_t$ uniformly for $t$ on any compact subset of $I$. Now if we can show that $\tilde{x}^k$ and $\tilde{x}$ are members of $X$ the proof will be complete, for then we will have an infinite number of functions in $X$ and $x^k_t \to \tilde{x}_t$ shows that $\tilde{x}$ is not separated with respect to $X$.

For $y \in \{x\}$ we know that for all $t, t' \in I$
$$|y(t) - y(t')| \leq K |t - t'| .$$
Now since $x^k \in \{x\}$ for all $k$, we have for all $t \in I$
$$\|x^k_t - \tilde{x}^k_t\| = \sup_{\theta \in [-h, 0]} |x^k(t+\theta) - x^k(t'+\theta)| \leq K |t - t'| .$$
Also, since $|y(t)| \leq H$ for $y \in \{x\}$ we have for all $t \in I$
$$\|x^k_t\| = \sup_{\theta \in [-h, 0]} |x^k(t+\theta)| \leq H .$$
Thus, $\tilde{x}^k \in X$.

Pick $t, t'$ in $I$. Let $J$ be any compact subset of $I$ containing $t, t'$.
Observe that
$$\|\tilde{x}_t - \tilde{x}_{t'}\| \leq \|\tilde{x}_t - x^k_t\| + \|x^k_t - \tilde{x}_{t'}\| + \|\tilde{x}_{t'} - \tilde{x}_t\| . \quad (3.2)$$
Since $x^k_t \to \tilde{x}_t$ and $\tilde{x}^k \in X$ we have from (3.2),
$$\|\tilde{x}_t - \tilde{x}_{t'}\| \leq K |t - t'| .$$
We also observe that $\|\tilde{x}_t\| \leq H$, and so we have that $\tilde{x} \in X$. This concludes the proof.

**Lemma 3.5:** Let $F$ be almost periodic in $t$ uniformly for $\phi$ in $C_H$, let 
$G \in H(F)$, and consider the following functional-differential equations

$$\dot{x}(t) = F(t, x_t) \quad (3.3)$$
$$\dot{x}(t) = G(t, x_t) \quad (3.4)$$
If for some Equation 3.4 there exists a solution $\tilde{x} \in X$, then for every
Equation 3.4 there exists a solution which is a member of X.

Also, given $\{t_n\}$ there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that

i) $F(t + t_{n_k}) \to \tilde{G}(t, \phi)$ uniformly on $I \times \Gamma$ where $\tilde{G}$ is some element of $H(F)$ and

\[ \lim_{k \to \infty} x_{t + t_{n_k}} = y_t \] uniformly on $I$, where $y_t \in X$ is a solution of $\dot{y}(t) = G(t, y_t)$.

Proof. Let $G, \tilde{G} \in H(F)$. By Proposition 2.4, $\tilde{G} \in H(G)$. Since $G \in H(F)$, $G$ has the same almost periodicity and continuity properties as $F$, thus the propositions in Chapter II hold when $F$ is replaced by $G$.

Let $\tilde{x} \in X$ be a solution of (3.4). Then given $\tilde{G}(t, \phi)$ of (3.4), since $\tilde{G} \in H(G)$ there exists a sequence $\{\tau_k\}$ such that

\[ G(t + \tau_k, \phi) \to \tilde{G}(t, \phi) \] uniformly for $(t, \phi)$ in $I \times \Gamma$. By Lemma 3.3 we know that there exists a subsequence $\{\tau_{n_k}\}$ of the sequence $\{\tau_k\}$ such that $x_{t + \tau_{n_k}} \to y_t$ uniformly on any compact subset of $I$, and such that $y_t \in X$. To conclude the first part of this lemma we must show that $\tilde{y}$ is a solution of (3.4).

Fix $t, t_o \in I$ and let $x$ and $y$ be as in Definition 2.16. Then

\[ |y(t) - \int_{t_o}^{t} \tilde{G}(s, y_s)ds - y(t_o)| \leq |y(t) - x(t + \tau_{n_k})| + |x(t_o + \tau_{n_k}) - y(t_o)| 
+ |x(t_o + \tau_{n_k}) - \int_{t_o}^{t} G(s + \tau_{n_k}, x_{s + \tau_{n_k}})ds - x(t_o + \tau_{n_k})| \]
Now we note that the first two terms on the right side of (3.6) approach zero as \( n \to \infty \). The third term is zero since \( x_{t+\tau_{k_n}} \) is the functional value of a solution of

\[
\dot{x}(t) = \overline{G}(t + \tau_{k_n}, x_t)
\]

The fourth term approaches zero by (3.5), and by Proposition 2.2 we see that the last term on the right side of (3.6) approaches zero as \( n \to \infty \). Thus

\[
\hat{y}(t) = \overline{G}(t, y_t)
\]

and we have that \( \hat{y} \) is a solution of (3.4) for \( \overline{G} \).

Part (i) of the latter part of this lemma is clear since \( \overline{G} \in H(F) \), where \( F \) is almost periodic, implies the existence of the sequence \( \{\tau_{k_n}\} \). The convergence in (ii) can be shown as in Lemma 3.3, and the fact that \( \hat{y} \) is a solution follows in the same manner as in the first part of this lemma.

**Definition 3.2**: A function \( f \) satisfies condition \( B'(H,K,\bar{t}) \) if

1) \( f: I \to \mathbb{C} \)

ii) for all \( t, t' \in I \) such that \( t, t' \geq \bar{t} \),

\[
|| f(t) || \leq H \quad \text{and} \quad || f(t) - f(t') || \leq K |t - t'| .
\]

**Definition 3.3**: If \( \hat{x} \) satisfies condition \( B'(H,K,\bar{t}) \), let \( \Lambda \) be the closure of the set \( \{x_0\}, t \geq \bar{t} + h \), where \( \hat{x}(t) = x_t \).
Remark 3.1: By using an argument similar to the one used in Lemma 3.2 we see that the set \( \{x_t \} , t \geq t+h \), is conditionally compact, and hence that \( A \) is a compact subset of \( C \).

Theorem 3.1: If for some Equation 3.3 there exists a solution \( \hat{x} \) which satisfies condition \( B'(H,K,t) \), then for every Equation 3.4 there exists a solution which is a member of \( X \).

Proof. Let \( \hat{x} \), a solution of (3.3), be as in the hypothesis. Consider the set \( \{x_{t+n} \} n > h, t \geq \hat{t} \). Each member of this set satisfies
\[
\dot{x} (t) = F(t + n, x_t) .
\]
It can be shown, as in Lemma 3.3, that there exists a subsequence \( \{n_r \} \) of the sequence of integers such that as \( r \to \infty \)
\[
x_{t+n_r} \to y_t
\]
uniformly on every compact subset of \( I \). We also may assume as \( r \to \infty \)
\[
F(t + n_r, \phi) \to \bar{F} (t, \phi)
\]
where this convergence is uniform for \( (t,\phi) \) in \( I \times \Lambda \). So by using techniques similar to those used in establishing Lemma 3.5 we have
\[
\dot{y} (t) = \bar{F} (t, y_t)
\]
where \( y \in X \).

From Proposition 2.1 we have that \( \bar{F} (t,\phi) \) has the same properties as \( F(t,\phi) \). So, if we apply the above result to the equation
\[
\dot{x}(t) = \bar{F} (t, x_t)
\]
we have that for each \( G \in H(\bar{F}) \), the equation
\[
\dot{x} (t) = \bar{G} (t, x_t)
\]
has a solution which is a member of \( X \). But by Proposition 2.4, each \( G \in H(F) \) is such that \( G \in H(\bar{F}) \). Hence for each \( G \in H(F) \), (3.4) has a solution
which is a member of $X$.

**Remark 3.2:** For a fixed $G \in \mathcal{H}(F)$ we have from Lemma 3.4 that if each $\hat{x} \in X$ is separated with respect to $X$, then $X$ is finite. Thus there exists a positive 'separation constant' for each fixed $G$.

One of the results in this next lemma is that if each $\hat{x} \in X$, a solution of (3.4), is separated with respect to $X$, then the finite number of these solutions is independent of $G \in \mathcal{H}(F)$.

**Lemma 3.6:** There exists $\sigma > 0$ independent of $G \in \mathcal{H}(F)$ such that if $\hat{x}^1$ and $\hat{x}^2$ are solutions of (3.4) which are members of $X$ and which are separated with respect to $X$, then for all $t \in I$

$$\| x^1_t - x^2_t \| > \sigma .$$

**Proof.** For $\tilde{G}, \tilde{G} \in \mathcal{H}(F)$, let $\tilde{\omega}$ be the separation constant for $\tilde{G}$ and let $\tilde{\omega}$ be the corresponding one for $\tilde{G}$. By Proposition 2.4 $\tilde{G} \in \mathcal{H}(\tilde{G})$ and so there exists $\{\tau_k \}$ such that

$$\lim_{k \to \infty} \tilde{G}(t + \tau_k, \phi) = \tilde{G}(t, \phi)$$

uniformly for $(t, \phi) \in \text{int} I$. Let $\hat{x}^1, \hat{x}^2$ in the hypothesis be the solutions of

$$\hat{x}(t) = \tilde{G}(t, x_t) . \quad (3.7)$$

By Lemma 3.3 there exists $\{\tau_k'\}$ a subsequence of $\{\tau_k\}$ such that

$$x^1_{t + \tau_k'} \to y^1_t .$$

Again, there exists $\{\tau_k''\} \subseteq \{\tau_k'\}$ such that

$$x^2_{t + \tau_k''} \to y^2_t .$$

Hence for $\{\tau_k''\}$,
where these limits are uniform on any compact subset of $I$. As in Lemma 3.5, we have that $\tilde{y}^1$ and $\tilde{y}^2$ are solutions of

$$\dot{y}(t) = \bar{F}(t, y_t)$$

(3.9)

where $\tilde{y}^1$ and $\tilde{y}^2$ are elements of $X$.

Now since $\tilde{y}^1$ and $\tilde{y}^2$ are separated with respect to $\bar{A}$, we have

$$\inf_{t \in I} \| x^1_{t+\tau_R^k} - x^2_{t+\tau_R^k} \| = \inf_{t \in I} \sup_{\theta \in [-h,0]} |x^1(t+\tau_R^k+\theta) - x^2(t+\tau_R^k + \theta)|$$

$$= \inf_{t \in I} \sup_{\theta \in [-h,0]} |x^1(t + \theta) - x^2(t + \theta)|$$

$$= \alpha_{12} > 0.$$  (3.10)

Also,

$$\inf_{t \in I} \| y^1_t - y^2_t \| = \beta_{12} \geq \alpha_{12}.$$  (3.11)

The above inequality follows by assuming the contrary, i.e.

$$\inf_{t \in I} \sup_{\theta \in [-h,0]} |y^1(t+\theta) - y^2(t+\theta)| = \beta_{12} < \alpha_{12}.$$  

This implies that there exists $t^* \in I$ such that

$$\sup_{\theta \in [-h,0]} |y^1(t^*+\theta) - y^2(t^*+\theta)| < \alpha_{12},$$

or

$$\| y^1_{t^*} - y^2_{t^*} \| < \alpha_{12}.$$  

Now by using properties of the norm, (3.8), and (3.10), we have

$$\| y^1_{t^*} - y^2_{t^*} \| = \lim_{r \to \infty} x^1_{t^*+\tau_R^k} - \lim_{r \to \infty} x^2_{t^*+\tau_R^k} \|$$
which is a contradiction.

Let \( m \) be the number of solutions of (3.7) which are members of \( X \) and which are such that each solution is separated with respect to \( X \), and let \( n \) be the corresponding number for (3.9). For each sequence \( \{\tau_k^i\} \subset \{\tau_k\} \) it follows by the separation hypothesis that distinct solutions \( \hat{x} \) of (3.7) go into distinct solutions \( \hat{y} \) of (3.9), hence we have \( m \leq n \). By interchanging \( \tilde{G} \) and \( \bar{G} \) we can show that \( n \leq m \); hence \( m = n \) and we have that the number of solutions of (3.4) which are elements of \( X \) and are separated with respect to \( X \) is independent of \( G(t, \phi) \).

If

\[
K = \min_{i, k = 1, \ldots, m} \alpha_{ik}
\]

\[
L = \min_{j, s = 1, \ldots, m^2} \beta_{js}
\]

we have by (3.11) that \( L \geq K \). Again by interchanging \( \tilde{G} \) and \( \bar{G} \) we have \( K \geq L \). Thus,

\[
K = L = \sigma
\]

and we have completed the proof.

Theorem 3.2: If for each Equation 3.4, every solution which is a member of \( X \) is separated with respect to \( X \), then every such solution is almost
periodic.

Proof. Let \( \hat{x} \in X \), a solution of (3.3), be separated with respect to \( X \). \( \hat{x} \) is almost periodic in \( t \) if for an arbitrary sequence \( \{ \tau_n \} \) there exists a subsequence \( \{ \tau_{n'} \} \) such that \( \{ x_{t+\tau_{n'}} \} \) converges uniformly for \( t \in \mathbb{I} \).

Assume \( \hat{x} \) is not almost periodic in \( t \); i.e., for every subsequence \( \{ \tau_n'' \} \) of some sequence \( \{ \tau_n' \} \), \( \{ x_{t+\tau_{n''}} \} \) does not converge uniformly for \( t \in \mathbb{I} \). However, for \( \{ \tau_n' \} \subset \{ \tau_n \} \) we know by Lemma 3.5 that

\[
F(t + \tau_{n'}, \phi) \to G(t, \phi)
\]

uniformly for \( (t, \phi) \) in \( I \times \Gamma \) for \( G \in H(F) \), and

\[
x_{t+\tau_{n'}} \to y_t
\]

uniformly for \( t \) on each compact subset of \( I \) where \( \hat{y} \) is a solution of

\[
\dot{y}(t) = G(t, y_t)
\]

Here we are assuming that \( \{ x_{t+\tau_{n'}} \} \) does not converge uniformly for \( t \in \mathbb{I} \).

Now we show that for each \( \mu > 0 \) there exists an \( \alpha \) such that \( 0 < \alpha < \mu \), and three sequences \( \{ m_i \} \), \( \{ n_i \} \), and \( \{ t_i \} \), \( i = 1, 2, \ldots \) where \( m_i \) and \( n_i \) are positive integers with \( m_i < m_{i+1} \) and \( n_i < n_{i+1} \) for which

\[
\alpha \leq \| x_{t_i+\tau_{m_i}} - x_{t_i+\tau_{n_i}} \| \leq \mu.
\]

To establish (3.12) we proceed in the same manner as Amerio (1). For \( m < n \) we define a new function \( \psi_{m,n} \) which is such that

\[
\psi_{m,n}(t) = \| x_{t+\tau_m} - x_{t+\tau_n} \|. \tag{3.13}
\]

Then let

\[
J_{m,n} = \{ t | \psi_{m,n}(t) \leq \mu, \, t \in \mathbb{I} \}. \tag{3.14}
\]

Since we have \( x_{t+\tau_n} \to y_t \), we see that
\[
\psi_{m,n}(0) \leq \mu
\]

for \( m \) sufficiently large, and thus \( J_{m,n} \) is non-void for sufficiently large \( m \). Without loss of generality we may assume \( J_{m,n} \) non-void for \( 1 \leq m < n \) and define

\[
\xi_{m,n} = \sup_{t \in J_{m,n}} \psi_{m,n}(t)
\]  
(3.15)

From (3.14) we see that

\[
\xi_{m,n} \leq \mu.
\]  
(3.16)

Now suppose

\[
\lim_{m,n \to \infty} \xi_{m,n} = 0.
\]

Thus for each \( \varepsilon > 0 \), \( 0 < \delta < \mu \), there exists \( m_0 \) such that for \( m_0 \leq m < n \),

\[
\xi_{m,n} < \delta.
\]

This implies that for \( t \in J_{m,n} \) we have

\[
\psi_{m,n} < \delta.
\]

By (3.14), if \( t \in I_{m,n} \), we must have

\[
\psi_{m,n} \geq \mu,
\]

and since \( \psi_{m,n} \) is continuous for \( t \in I \) it follows that \( I = J_{m,n} \) for \( m_0 \leq m < n \).

Thus (3.17) implies uniform convergence of \( \{x_{t+\tau_n}\} \) for \( t \in I \) and this is a contradiction. So, instead of (3.17) we now suppose

\[
\lim_{m,n \to \infty} \xi_{m,n} = 2\alpha
\]  
(3.18)

for \( \alpha > 0 \). Then from (3.16) we have \( 2\alpha \leq \mu \), or \( \alpha < \mu \). And from (3.18) there exist sequences \( \{m_i\}, \{n_i\} \) such that

\[
\frac{m_i}{n_i} > \sqrt{\frac{3}{\alpha}}.
\]
Thus there exists a sequence \( \{t_{i}\} \), \( t_{i} \in [m_{i}, n_{i}] \), such that
\[
\delta_{m_{i}, n_{i}}(t_{i}) \geq \alpha,
\]
and we have therefore established (3.12).

From Lemma 3.3 we know that for the sequences \( \{x_{t_{i}+\tau_{m_{i}}}n_{i}\} \) and \( \{x_{t_{i}+\tau_{n_{i}}}m_{i}\} \), there exists a sequence \( \{\delta_{i}\} \) such that
\[
x_{t_{0_{i}}} + \tau_{m_{\delta_{i}}} \rightarrow z,
\]
\[
x_{t_{0_{i}}} + \tau_{n_{\delta_{i}}} \rightarrow w,
\]
where \( z, w, \in C_{H} \).

Letting \( t_{\delta_{i}} = t_{i}, \tau_{m_{\delta_{i}}} = \gamma_{i}, \) and \( \gamma_{n_{\delta_{i}}} = \beta_{i} \) we have
\[
\lim_{i \rightarrow \infty} x_{t_{i}+\gamma_{i}} = z,
\]
\[
\lim_{i \rightarrow \infty} x_{t_{i}+\beta_{i}} = w.
\]

Then by (3.12) we have
\[
\alpha \leq || z - w || = \mu. \tag{3.19}
\]

Now we show that there exists a sequence of integers \( \{k_{i}\} \) such that
\[
x_{t_{i}+\gamma_{k_{i}}} \rightarrow y_{i}^{1}
\]
\[
x_{t_{i}+\beta_{k_{i}}} \rightarrow y_{i}^{2} \tag{3.20}
\]
uniformly for \( t \) in any compact subset of \( I \), and such that
\[
F(t + t_{k_{i}}^{i} + \gamma_{k_{i}}^{i}, \phi) \rightarrow G_{1}(t, \phi)
\]
\[
F(t + t_{k_{i}}^{i} + \beta_{k_{i}}^{i}, \phi) \rightarrow G_{2}(t, \phi) \tag{3.21}
\]
where these limits are uniform for \((t, \phi)\) in \(\mathbb{I} \times \Gamma\), where \(G_1 \in \mathcal{E}(F)\) for 
\(i = 1, 2\), and where 
\[ y_i^1(t) = G_1(t, y_i^u), \quad i = 1, 2. \]
We proceed by considering the sequences 
\[
\{x_{t+t_i^1} + r_i^1\}
\]
\[
\{x_{t+t_i^1} + \beta_i^1\}
\]
From Lemma 3.5 we know that for the sequence \(\{t_i^1 + r_i^1\}\) there exists a subsequence \(\{r_i^1\}\) of the sequence of integers such that 
\[ F(t + t_i^1 + r_i^1, \phi) \rightarrow G_1(t, \phi) \]
where \(G_1 \in \mathcal{E}(F)\), uniformly for \((t, \phi) \in \mathbb{I} \times \Gamma\). Also from the same lemma we can show that there exists \(\{j_i^1\} \subseteq \{r_i^1\}\) such that 
\[ x_{t+t_j^1} + r_j^1 \rightarrow y_j^1 \]
uniformly for \(t\) in any compact subset of \(\mathbb{I}\), where 
\[ y_j^1(t) = G_1(t, y_j^1) . \]
By using the above arguments again we know that there exists \(\{s_i^1\} \subseteq \{j_i^1\}\) such that 
\[ F(t + t_s^1 + \beta_s^1, \phi) \rightarrow G_2(t, \phi) \]
for some \(G_2 \in \mathcal{E}(F)\), uniformly for \((t, \phi) \in \mathbb{I} \times \Gamma\). And there exists \(\{k_i^1\} \subseteq \{s_i^1\}\) such that 
\[ x_{t+t_k^1} + \beta_k^1 \rightarrow y_k^2 \]
uniformly for \(t\) in any compact subset of \(\mathbb{I}\), where 
\[ y_k^2(t) = G_2(t, y_k^2) . \]
Thus we have obtained the desired sequence \( \{ k_i \} \).

Now by (3.19) and (3.20) we have
\[
\alpha \leq \| y_{r_0}^1 - y_{r_0}^2 \| \leq \mu .
\]

Next we will show that
\[
G_1(t, \phi) = G_2(t, \phi) .
\]

Define \( t_i'' = t_i' + \lambda_i = \gamma_{k_i} \), and \( \eta_i = \beta_{k_i} \). Recalling that \( \{ \lambda_i \} \) and \( \{ \eta_i \} \) are subsequences of \( \{ \tau_n \} \), it follows that
\[
F(t + \lambda_i, \phi) \to G(t, \phi)
\]
\[
F(t + \eta_i, \phi) \to G(t, \phi)
\]
uniformly for \((t, \phi)\) in \( I \times \Gamma \). Now let \( \delta > 0 \) be arbitrary. Then there exists an integer \( \omega_0 \) such that for \( i \geq \omega_0 \) and \((t, \phi)\in I \times \Gamma \) one has
\[
|F(t + \lambda_i, \phi) - F(t + \eta_i, \phi)| < \delta/3 .
\]

From (3.21) there exists \( \omega_0 \) such that for \( i \geq \omega_0 \) and for \((t, \phi)\in I \times \Gamma \),
\[
|G_1(t, \phi) - F(t + t_i'' + \lambda_i, \phi)| < \delta/3
\]
\[
|F(t + t_i'' + \eta_i, \phi) - G_2(t, \phi)| < \delta/3 .
\]

Letting \( t = t_i' + t \) in (3.23), we have for \( i \geq \max(\omega_i', \omega_0) \) and for \((t, \phi)\in I \times \Gamma \),
\[
|G_1(t, \phi) - G_2(t, \phi)| < \delta .
\]

That is,
\[
G_1(t, \phi) = G_2(t, \phi) .
\]

We have previously shown that \( \hat{y}^1 \) and \( \hat{y}^2 \) are distinct solutions of
\[
\hat{y}(t) = G(t, y_t)
\]
where \( \hat{y}^1, \hat{y}^2 \in X \), and where
\[
\| y_{r_0}^1 - y_{r_0}^2 \| \leq \mu .
\]
for arbitrary $\mu$. Thus these two solutions are members of $X$ but are not separated with respect to $X$. Therefore the proof is complete.

To any function $F$ which is almost periodic in $t$ with function values in $\mathbb{R}^n$ there corresponds a Fourier series

$$
\sum_{n=1}^{\infty} a_n e^{i\lambda_n t},
$$

where the $a_n$ are elements of $\mathbb{R}^n$ and the $\lambda_n$ are real numbers, the so called Fourier exponents of the functions. In case $F = 0$, define $a_n = 0$ for all $n$, and $\lambda_n = 0$ for all $n$. If $F \in C^\infty$ is almost periodic in $t$ uniformly for $\phi \in \mathbb{C}_H$, then its Fourier series (3.24) is such that $\lambda_n = \lambda_n(\phi)$; and if $\Gamma \subset \mathbb{C}_H$ is compact then $\{\lambda_n(\phi)\}$, $\phi \in \Gamma$, is countable.

Definition 3.4: If $F \in C^\infty$ is almost periodic in $t$ uniformly for $\phi \in \mathbb{C}_H$, then the set $\left\{\sum_{j=1}^{N} n_j \lambda_j\right\}$ for all integers $N$, where the $n_j$ are integers and $\lambda_j$ are the Fourier exponents of $F$, is called the module of $F$.

Theorem 3.3: If each system (3.4) has exactly one solution belonging to $X$, then the solution of (3.3) is almost periodic and its module is contained in the module of $F$.

Proof: Let $\hat{x}$ be the solution of (3.3). The almost periodicity follows as a special case of the previous theorem, hence we have left to show that the module of $\hat{x}$ is contained in the module of $F$.

Let $\Gamma$ be a compact subset of $\mathbb{C}_H$ and let $\{t_n\}$ be any sequence such that $\{F(t + t_n, \phi)\}$ converges uniformly for $(t, \phi)$ in $\mathbb{I} \times \Gamma$. If we can show that $\{x_{t+t_n}\}$ converges uniformly for $t \in \mathbb{I}$ we will have completed the proof. For it follows in a manner analogous to Favard (10, page 80)
that if \( \{ F(t + t_n, \phi) \} \) converges uniformly and if \( \{ x_{t+t_n} \} \) converges uniformly, then the module of \( \hat{x} \) is contained in the module of \( F(t, \phi) \).

First we show \( \{ x_{t+t_n} \} \) converges for each \( t \in I \). Since \( \hat{x} \) is almost periodic, by Proposition 2.3 there exists \( \{ t_{n_j} \} \subset \{ t_n \} \) such that \( x_{t+t_{n_j}} \to v_t \) uniformly for \( t \in I \). Now for \( t = t_0 \), assume \( x_{t_0+t_n} \) does not converge to \( v_{t_0} \). Thus there exists \( \{ t_n \} \subset \{ t_n \} \) and a \( K > 0 \) such that

\[
\| x_{t_0+t_n} - v_{t_0} \| \geq K,
\]

for \( n = 1, 2, \ldots \). Again by Proposition 2.3 there exists \( \{ t_{n_j} \} \subset \{ t_n \} \) such that \( x_{t+t_{n_j}} \to z_t \) uniformly for \( t \in I \). From (3.25) \( z \neq \hat{y} \), where \( z \) and \( \hat{y} \) belong to \( X \) by Lemma 3.5. Then since they are both solutions of (3.4) for \( G = \hat{G} \), where

\[
\lim_{n \to \infty} F(t + t_n, \phi) = \hat{G}(t, \phi),
\]

we have a contradiction. Thus \( \{ x_{t+t_n} \} \) converges for each \( t \in I \).

We can show that \( \{ x_{t+t_n} \} \) converges uniformly by assuming the contrary and proceeding as in the proof of Theorem 3.2 to again show that we would have two solutions belonging to \( X \) for some Equation 3.4. Thus we have that the module of \( \hat{x} \) is contained in the module of \( F(t, \phi) \).

**Corollary 3.1:** If each system (3.4) has exactly one solution belonging to \( X \), and if \( G \) is periodic with period \( \omega \) for all \( \phi \in C_H \), then the almost periodic solution that exists by Theorem 3.2 is periodic with period \( \omega \).

The proof of this follows as in Yoshizawa (19, Theorem 5).
IV. LITERATURE CITED


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