Analysis and Synthesis Methods for Nonlinear Network Systems

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Analysis and synthesis methods for uncertain nonlinear network systems

by

Amit Vivek Diwadkar

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

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ABSTRACT

Over the past two decades the interactions between systems and their control components have undergone some significant changes. These interactions are no more localized, but usually take place over a network and even the control components may be remotely located, thus involving aspects of communication in control systems. Furthermore, the last decade has also seen a surge in intermingling ideas from control and communication and their application to biological systems, power systems giving rise to new research areas like Networked Control Systems (NCS), Cyber-Physical Systems (CPS), Gene Regulatory Networks (GRN) to name a few. This has led researchers to study control systems with practical constraints imposed on them. One such practical constraint identified as a major challenge, is the fragility of control systems and performance degradation, when the interconnection is not reliable. Design of controllers and estimators for such systems needs to take into account these constraints and mitigate them, to ensure sufficient robustness against unreliability of the interconnection. Considerable research has been done over the past decade in analyzing these new challenges and developing design tools to extract desired performance.

Control over communication channels is one such widely researched area where the effect of unreliable interconnection on the stability performance of the system has been studied. The reliability of communication could manifest in various ways like sensor failure at output measurement, control actuator failure, interconnection links failures in the form of packet erasure channel, fading channel, quantization etc. Significant research progress has been made, in areas of control and estimation over unreliable communication links, consensus over unreliable network interconnections, etc., albeit the work has dealt with linear time invariant (LTI) systems theory. This has led to fruitful results for special cases of packet-drop communication channel modeled as a Bernoulli erasure channel. In the case of linear systems these results have demonstrated a connection between the performance characteristics of the interconnection and the
expansion or destabilizing characteristics of the linear system, in obtaining desired performance of the closed loop system.

Most of the current research for control over communication channels have focused on LTI plant dynamics. Furthermore the results involving nonlinear plant dynamics have reverted to local linearization techniques. It is well-known that for nonlinear systems, results based on local linearization at an equilibrium point will be local in nature and does not account for the global dynamics of the nonlinear system. For the proposed applications of network control systems to electric power grid and biological networks it is essential to develop results for the analysis of nonlinear systems over networks.

In this work, we are primarily interested in the interaction of nonlinear systems and controllers over unreliable interconnections modelled as a stochastic multiplicative uncertainty. We provide analysis and synthesis methods for the control and observation of uncertain nonlinear network controlled systems. Our analysis methods indicate, fundamental limitations arise in the stabilization and observation of nonlinear systems over uncertain channels. Our main result provides the limitation for observation of nonlinear system over erasure channel expressed in terms of the probability of erasure and positive Lyapunov exponents of the open loop nonlinear plant. The positive Lyapunov exponents are measure of dynamical complexity and comparing our results with existing results for LTI systems, we show that Lyapunov exponents emerge as a natural generalization of eigenvalues from linear to nonlinear systems.

Entropy is another measure of dynamical complexity. Using results from ergodic theory of dynamical systems we also relate the limitation for stabilization and observation with the entropy corresponding to the invariant measure capturing the global dynamics of the nonlinear systems. Existing Bode-like fundamental limitation results for nonlinear systems relate limitation for stabilization with the entropy corresponding to the invariant measure at the equilibrium point. Our results are the first to connect the limitation for stabilization with the entropy corresponding to invariant measure other than the one associated with equilibrium point.

Our synthesis methods for the design of robust controller and observer against uncertain channels revolves around special class of nonlinear systems -Lure systems. These systems are
essentially linear systems with sector-bounded nonlinearity in the feedback loop. For this special class of nonlinear systems, we delve into the theoretical tools of absolute stability to obtain some synthesis methods which provide design criteria for nonlinear systems over unreliable interconnections. Stability of Lur’e systems is a special case of the stability of interconnected passive systems. Thus we can characterize the unreliability of the interconnection, that guarantees the desired performance for Lur’e systems, in terms of the passivity of the linear system. Passivity theory is a rich theory with wide spread applications to nonlinear controller design and observation, which extends ideas of system stability to input-output systems using the ideas of dissipativity. Our synthesis methods developed for Lure systems with input and output stochastic channel uncertainties provide natural extension of the powerful passivity based synthesis tools developed for deterministic Lure systems. In particular, our results help understand the trade-off between passivity and stochastic uncertainty in feedback control systems.
CHAPTER 1. Introduction

1.1 Motivation: Interacting Systems and a World of Interconnections

The world of today is a world of interconnected systems. All around us we have various systems and their components, even remotely located, interacting over a web of interconnections. Right from the visible and more obvious network of systems like electric power grids, cellular networks, world wide web and internet to the more subtle interconnections in social networks, biological networks, etc. Thus in the past decade, research directions in a lot of fields have moved beyond the classical study of the area and aligned itself with understanding the effects of such interconnections. This has given rise to new areas of research like Networked Control Systems (NCS), Cyber-Physical Systems (CPS), Smart Grids to cite a few. A key feature of this network of systems is that unreliability in the interconnections of the systems may degrade the performance of the interconnected systems. An even important feature observed in such networked systems is that any degradation of behavior may not remain localized and can have global effects, degrading remote areas of the network. Thus if not designed to subject to certain performance requirements, the interconnection network and its systems may be fragile to deviation of the interconnection from nominal behavior, even causing cascading failures. Such instances have been observed in the North-East Blackout of 2003 in the United States, The Southern Brazil Blackout of 1999, disruption and traffic congestion in cellular and internet networks in Asia after few undersea cables were destroyed by the Taiwan earthquake in 2006, or the congestion of internet traffic and subsequent outage in Europe due to deviant behavior of a router in the Czech Republic. Thus it is important to understand the effects of unreliable interconnections on system performance.

There has been vast technological development over the past two decades in various fields
such as communication, energy systems, biological systems, etc. These developments on the application front aimed at improving the quality of life, have spawned the emergence of new theoretical research areas that hitherto lay unexplored. One such key area of research has been the interaction of control systems with these emerging applications and scientific developments. Control systems are an integral part of a lot of practical world applications. This has led to a merger of control systems with the fields of communication, biology, power systems etc. But all of these applications bring along their own set of challenges. One major challenge is the occurrence of uncertainty in the interconnections between different systems and between the system and its controllers themselves. Such uncertainties in connectivity may be observed in communication systems where wireless channels show fading behavior, erasure of channel information, packet-drop and delays. In other engineering applications like aerospace these uncertainties may occur from the point of view of faults and fault detection which may be modeled as parametric uncertainty Boskovic and Mehra (2003). Uncertainty is also inherent in biological networks due to stochastic nature of these processes. Thus to understand such processes completely we need to understand the interaction of control systems over uncertain interconnections.

Seeing the emergence of these new research directions at the turn of the century, a lot of research has been conducted to understand the effects of practical constraints arising from interconnection of systems and its control components. This has led to the study of control and estimation tools from classical linear control theory being studied with additional constraints on the interconnection. These results which we will survey in some detail later can be summarised as providing the tolerable unreliability in the interconnections, given the expansion rates (instability) of the open-loop linear system subject to the requirement of a predecided performance criterion. Many a times the performance criterion is to optimize a quadratic cost or variance of the closed-loop. This body of work has given us considerable insight into the close relation between reliability characteristics of the interconnection and the characteristic behavior of the open-loop linear system as given by unstable eigenvalues. Building upon this knowledge, we believe that these results need to be studied in the context of nonlinear systems for two reasons. Primarily, all practical systems are nonlinear in nature. Second and most
important, the global behavior of nonlinear systems cannot be judged by observing local eigenvalues at an equilibrium state. Global (non-equilibrium) behavior may be vastly different from local (equilibrium) behavior, creating complex dynamics, not encountered in linear system dynamics. Thus for nonlinear systems inferences drawn on equilibrium eigenvalues may provide to be misleading to the designer trying to achieve robustness of the interconnection. In this body of work we aim to develop analysis and design tools to study the effect of such uncertain interconnections on stability of nonlinear control systems and ways to mitigate them to attain desired performance.

1.2 Network Control Systems: A Guiding Example

Communication network systems are becoming abundant in the world. Purely mechanical systems and mechanical interactions between systems are being replaced with digital communication channels sometimes even wireless communication channels. Furthermore, significant amount of this communication between physical, computational and control elements of these systems takes place through a network of sensors and actuators. Such systems interacting over networks are seen in every walk of life with the advent of the world wide web, power grids, biological networks and social networks to name a few. This blend of control of physical systems taking into account communication strategies over a network framework, has led to the rise of a new research area of Network Controlled Systems (NCS), Antsaklis and Baillieul (2007). NCS can thus be visualized as an interaction of plants and controllers over a network of sensors and actuators as represented in Figure 3.2 Elia (2005). But this interaction of systems through communication networks brings in new challenges like insufficient resources, unreliable communication, quantization and time delay to name a few. Thus a good understanding of interacting nonlinear systems with communication constraints is an important step towards developing NCS in the future. We will now use the the example of a simple NCS model, where the system and its controller are connected over a communication link to put our results into perspective. The question we ask ourselves is the following: What are the connections of such communication imposed constraints with the characteristic properties of a nonlinear system like Lyapunov exponents that characterize global non-equilibrium behavior, system entropy
and passivity and dissipative characteristics? We now give a brief review of such connections established for linear systems.

![Network Control System Schematic Representation](image)

**Figure 1.1** Network Control System Schematic Representation

### 1.3 Literature survey

The problem of state estimation and control of systems over erasure channels has attracted a lot of attention lately given the importance of this problem in the control of systems over network Antsaklis and Baillieul (2007). The problem of state estimation with intermittent observation was first studied in Nahi (1969); Hadidi and Schwartz (1979). In Sinopoli et al. (2003); Epstein et al. (2008), state estimation over erasure channel with different performance metric on the error covariance is studied for linear time invariant (LTI) systems. In Sinopoli et al. (2003), under some assumption on system dynamics, it is proved that there exists a critical erasure probability below which the error covariance is unbounded. The critical probability is further shown to relate to positive eigenvalue of the LTI system. A Markov jump linear system framework is used to model the state estimation problem with intermittent measurement and to provide conditions for the convergence of error covariance Costa (2002). In Huang and Dey (2006); Smith and P.Seiler (2003), state estimation over erasure channel with Markovian packet loss is studied. Under certain assumptions on the system dynamics necessary and sufficient conditions were provided for LTI state estimation in Y.Mo (2008); L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007) and optimal control of LTI systems in Imer et al. (2006a) indicating deep connections between the limitations imposed and defining characteristics of the LTI system. The problem of control over unreliable channels has been looked at in the context of uncertainty threshold principle in Ku and Athans (1977); Koning (1982). The notion of anytime capacity was introduced to study the limitations that
are introduced in control of system over unreliable communication links Sahai and Mitter (2006). Information theoretic results on communication constraints due to packet loss have been addressed in Wong and Brockett (1998); Elia and Mitter (1999); Tatikonda and Mitter (2004). In Garone et al. (2010); Gupta et al. (2007), the problem of optimal control of linear time invariant (LTI) systems over packet-drop links is studied. The combined problem of estimation and control over unreliable links using two different protocols, UDP and TCP, is studied with LTI plant dynamics in Imer et al. (2006b); L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007). The authors also demonstrate the effect of the protocol on the limitations and interdependency of the estimation and control problems based on available information from the protocol. Robust control framework is used in the analysis and synthesis of controllers for Multi Input Multi Output (LTI) systems over unreliable channels in Elia (2005). This critical probability is further linked to the positive eigenvalues of the linear system and in turn can be related to its entropy for a single output system and assumptions of Bernoulli erasure channels. Thus in the linear case a characterization of the limitation is obtained in terms of fundamental system characteristics. In Smith and P.Seiler (2003) the authors consider packet loss between sensor and controller and pose the cotrol design problem as a \( H_\infty \) optimization problem. Similarly Markov jump linear systems results are also used in Mariton (1990); Do Val et al. (1999) to study the network problem over packet-drop links. However all the above results are developed for linear time invariant system and there is no systematic result that addresses the state estimation problem for nonlinear system over erasure channels.

1.4 Theme and Contribution

The popularity of NCS and its foray into various fields of active theoretical research and applications, brings with itself a new set of complexities that need to be tackled. Reliability of channels in terms of packet loss and delays, quantization effects, security of communication, allocation of available communication resources may degrade the performance of the control systems interacting over the communication network. One of the important aspect of NCS, that has been a major research area over the past decade, has been the interaction of controllers
and plants in NCS when the communication network in unreliable. These uncertain NCS with unreliable channels in feedback can be visualized as in Figure 3.3. This interaction of controllers, estimators and plants over the underlying uncertain communication network through feedback imposes fundamental limitations on achieving stability of the system. Characterizing such limitations is an important part of design and analysis of uncertain NCS. The main underlying theme of this body of work and successive research is to understand at a fundamental level, such limitations imposed by communication constraints on performance in stability, observation, synchronization of nonlinear systems interacting through communication networks. We briefly discuss the problems that have been attempted in this doctoral work.

\[ \Delta = \begin{bmatrix} \Delta_1 & \cdots & \Delta_N \end{bmatrix} \]

Figure 1.2 Network Control System with Uncertain Interconnections

1.4.1 Control of LTV systems over uncertain channels

As a first step, we study the problem of stabilization over uncertain interconnection for linear time varying (LTV) systems Diwadkar and Vaidya (2011b). Linear time varying systems though linear in nature do not have the same expansion rates at every instant in time. This is apparent in the fact that stability of eigenvalues of the system matrix at each instant does not guarantee the stability of the system Khalil (1996). Thus LTV systems provide an interesting yet simplified intermediate step before we move from linear systems theory to attacking the problem in nonlinear systems. We solve the problem for LTV systems to gain further insight into the tools required for studying the problem in the context of nonlinear systems.
1.4.2 Fundamental limitation in observation of nonlinear systems over erasure channels

The framework based on the theory of random dynamical system was adapted in Vaidya and Elia (2011), to develop results providing necessary condition for mean square stabilization of nonlinear system over erasure channels. It was shown that global non-equilibrium dynamics of the nonlinear system plays an important role in determining the minimum Quality of Service (QoS) to be delivered by the erasure channel. In Diwadkar and Vaidya (2010) we tackle the problem of observation of a nonlinear system over Bernoulli erasure channels. In Figure 1.3 (Elia (2005)) we give a schematic representation of the problem of observation and control of nonlinear system over unreliable channels at input and output. The main contributions of this work are that, firstly we give a necessary condition on the observer error stability based on the tangent space dynamics of the free dynamics. Secondly under certain constraints this condition is shown to equate to the sum of positive Lyapunov exponents of the free dynamics which captures the nonequilibrium dynamics.

1.4.3 Entropy based fundamental limitation for stabilization of nonlinear systems over uncertain channels

We extend further the results for full state feedback stabilization for general uncertainty and relate the limitations on performance to the QoS for the channel with general multiplicative
uncertainty, and positive Lyapunov exponents of the open loop systems. The limitation is then connected to the dynamical systems entropy using Ruelle's inequality. There has been extensive research in connection of control theoretic limitations with the dynamical system entropy. Bode fundamental limitations for LTI systems Astrom and Murray (2010) is a well known result that connects the unstable eigenvalues of the LTI system with its entropy. The extension of Bode limitations to nonlinear system requires the use of measure theoretic entropy as defined in ergodic theory of dynamical systems Zang and Iglesias (2003). In our results we show that the fundamental limitation associated with stabilization over uncertain channels is connected with the entropy associated with an invariant measure of the system associated with its non-equilibrium dynamics. The main reason for the appearance of entropy associated with non-equilibrium dynamics is the uncertainty present in the feedback loop. This uncertainty is instrumental in driving the system away from its equilibrium dynamics.

1.4.4 Stabilization of Lur’e systems over erasure channels

![Diagram of Nonlinear system observer and control with erasure channel at input and output](image)

Figure 1.4 Nonlinear system observer and control with erasure channel at input and output

In the previous subsections, the problems addressed by us were analytical in nature, giving a limitation on the extracting performance from the control system interacting over a unreliable interconnection. We would now like to provide synthesis techniques for nonlinear systems, where by we may design observers and controllers for nonlinear systems. As such a problem is difficult for general nonlinear systems even in the deterministic scenario, we turn our attention to a special class of systems. Hence in Diwadkar et al. (2012) we focus on a particular class of
nonlinear systems namely nonlinear systems in \textit{Lur’\textprime{e}} form, Figure 1.4. A system in Lure form consist of feedback interconnection of LTI system and static nonlinearity element Khalil (1996). Systematic analysis tools in the form of Positive Real Lemma (PRL) or Kalman-Yakubovich-Popov (KYP) Lemma exist for the synthesis and design of system in Lure form Haddad and Bernstein (1994); Arcak and Kokotovic (2001); Ibrir (2007); Johansson and Robertsson (2002).

The main contributions of this paper are as follows. We discover a stochastic variant of PRL to provide synthesis method for the design of observer based controller for the stabilization of nonlinear systems in Lure form over erasure channels. We provide a sufficient condition for the stabilization of feedback control system in Lure form with general stochastic uncertainty at the input channel. I was the primary researcher and author for the paper, the other author being my advisor and principle investigator.

1.5 The Road Ahead

We now glimpse through some potential research fields where our results may be applied. Our results are a step towards understanding the importance of two features of modern control systems -

1. The effect of uncertainty of the interconnections between systems themselves and their control components.

2. Importance of complex dynamics and global non-equilibrium behavior over unreliable interconnections.

With this we look towards some avenues of research where results may build upon these foundations.

1.5.1 Cyber-Physical Systems

Cyber-Physical Systems (CPS) have become an important area of research over the past couple of years. The amalgamation of physical systems with computational systems and their control, coordination and automation through sensor, actuation networks has given birth to the field of CPS Lee (2008). But given the interaction through sensor networks and feedback
interconnection between the physical and automation components, there is a natural concern of reliability of the communication as unreliability of these may severely limit the performance we can extract from this interconnection of cyber-physical systems. CPS are aimed to be automated and adaptive as much as possible. These pose design challenges as Lee (2008) states - "Cyber physical systems will not be operating in a controlled environment, and must be robust to unexpected conditions and adaptable to subsystem failures" and "... How does a designer avoid brittle designs, where small deviations from expected operating conditions cause catastrophic failures?". There has been some work on understanding the effects of packet-drop, mixed traffic, resource constraints and other constraints of cyber-physical control over wireless networks Xia et al. (2010), event-triggered systems and their stabilization Koutsoukos et al. (2011). Physical systems are inherently nonlinear in nature which makes it imperative that we understand the interaction and effect of uncertainty on nonlinear control systems. Furthermore, cyber-physical systems may also be modeled as passive systems for purpose of controller design Koutsoukos et al. (2008). Thus understanding of interaction and tradeoff between passivity and uncertainty in interconnection of passive nonlinear systems would help us build robust cyber-physical networks.

1.5.2 Biological Networks

Study of biological systems using nonlinear dynamical models can be traced back to Lotka’s model for population dynamics. Today biological networks are important for us to understand various biological processes like gene regulatory networks Danino et al. (2010), neuronal networks in biological applications Brgers and Kopell (2003), and synchronization of fireflies Mirolo and Strogatz (1990); Mohanty (2005) and epidemic spreads Zager and Verghese (2009). There has been considerable work on synthetic gene regulatory networks and their use to obtain tunable oscillators toggle switches Stricker et al. (2008). Synchronization is another important phenomenon in nonlinear chaotic systems observed in biological elements like group of fireflies, neurons Nethoff et al. (2004). In neural activity understanding synchronization is critical in understanding neurological ailments like epilepsy Iasemidis (2003); Chakravarthy et al. (2009); Alamir et al. (2011) and in turn being able to control them. Nonlinear models like Duffing
oscillators, which might be written in the Lur’e form, have also been used to study activity in
neuronal networks, Srebro (1995); Iwasaki and Zheng (2002). Furthermore, in most of the anal-
ysis the synapses connecting neurons are considered to be deterministic. But, these synapses
may be unreliable and show stochastic behavior Guo and Li (2011); Senn (2002); Zador (1998).
Thus trying to understand how uncertainty in interconnections affects synchronization, control
of nonlinear systems with unreliable communication can give us insight into these biological
phenomenon and bring us that closer to unraveling nature. Also, understanding these proper-
ties and their affect of Lur’e class of nonlinear systems might be a fruitful idea to explore.

1.5.3 Passivity and Dissipative Systems

Dissipative systems and passivity theory have been a major research area in the past century.
The characterization of important aspects of passive dissipative systems theory in general can
be found in the seminal papers Willems (1972a,b). Dissipativity theory is generalization of
stability of closed dynamical systems to open systems. It extends ideas of Lyapunov stability
theory to input-output systems. Given a closed dynamical system, a Lyapunov function can
be interpreted as the energy contained in the system. If this system energy decays to zero
asymptotically over time then the system is asymptotically stable. Similarly for input-output
systems, dissipativity theory gives the existence of a storage function which stores the energy
supplied to the system. This supply of energy is a function of the input and output of the
system. If the storage rate of energy is less than the supply rate then the system is said to
be dissipative. This storage function for a closed system acts like a Lyapunov function thus
bringing out the connection between Lyapunov stability theory and dissipativity theory.

Passive systems are a special class of dissipative systems where the supply rate can be
written as a product of the input and output which intuitively signifies the work done on the
system by the input to produce the desired output. Passivity theory and its connection to
absolute stability of nonlinear systems which has given insight into the Lur’e problem. Systems
in Lur’e form are characterized by a linear system with a nonlinearity in a feedback which are
deemed passive. Systems in Lure form are widely studied in control system community because
several systems in engineering application can be modeled as feedback interconnection of LTI
system and static nonlinearity Li et al. (2010); Willems (1971); Cao et al. (2005). The above mentioned examples of power systems or biological systems can be abstracted as a network of oscillators, which may be written as a LTI system with a passive nonlinearity in feedback. Thus at a higher level we aim to understand the interaction of passive systems when connected over unreliable communication networks.

In passivity theory, there is a notion of excess passivity in a system which can be characterized by the so called passivity indices. This excess of passivity maybe exchanged with a non-passive system over a feedback interconnection of the two systems McCourt and Antsaklis (2010); Yu and Antsaklis (2010). An important question we would like to ask is as follows: In case of unreliable feedback connection, what is the price we pay in the exchange of passivity to overcome the uncertainty in communication link? Thus we imagine some amount of passivity being absorbed to overcome the uncertainty and the remaining excess passivity available to make the feedback interconnection passive. Furthermore, given the Lur’e nature of some oscillators we would like to understand the deeper connection between passivity exchange, uncertainty propagation and network properties.

### 1.5.4 Uncertainty propagation in uncertain nonlinear control systems

The field of uncertainty propagation in dynamical system models is an important area of research Konda et al. (2010); Hibbert et al. (2001). All mathematical models used for study of systems do not completely characterize the system, leading to undetermined parameters within the system. A large number of times engineers make use of coarse dynamical models to model the system. The actual dynamics of the system are then considered to depend on the coarse model by varying certain parameters Koslowski and Strachan (2011). As the system dynamics and in turn control of such dynamics depends on the behavior of these parameters it is important to study how the uncertainty in these system parameters affects the output of the system. The current methods for understanding uncertainty propagation make use of numerical methods like monte Carlo simulations or generalized Polynomial Chaos (gPC) Duong and Lee (2012); ngelles Herrador et al. (2005). Using the structure we have designed we could study the extent of the destabilizing effect on the system due to these parametric uncertainties. The
main idea of these results is to bring forth the limitations imposed on systems when we have stochastically switching stable and unstable dynamics, acting on any input given to the system. Thus we could generalize the results obtained in this thesis to a trade-off between stability and instability of two dynamical systems made to interact in a stochastic nature.
CHAPTER 2. Preliminaries

In this chapter we will go over a some mathematical preliminaries which we will be using enroute to deriving our results.

2.1 Stability Definition

In our work we deal with systems that are stochastic in nature. Hence, to study the stability of such systems we require stability definitions which take into account the stochastic processes that drive the system. Here we give some common stability definitions for stochastic systems. Consider the following system

\[ x_{t+1} = S(x_t, \xi_t) \]  

(2.1)

where \( x \in X \subseteq \mathbb{R}^N \) and \( \xi_t \in W \subseteq \mathbb{R} \) are independent identically distributed (i.i.d.) random variables, and \( S(0, \xi_t) = 0 \). We can now define stability for this system as follows

**Definition 1** The system in (2.1) is said to be exponentially \( p \)-th moment stable if for Lebesgue almost all \( x_0 \in X \) there exist positive constants \( M_1 < \infty \) and \( \beta_1 < 1 \) such that,

\[ E_{\xi_0} [ \| x_{t+1} \|^p ] \leq M_1 \beta_1^{t+1} \| x_0 \|^p \]

(2.2)

where \( E_{\xi_0}[\cdot] \) is expectation taken over the sequence \( \{\xi_0, \ldots, \xi_t\} \).

These definitions for continuous-time systems are given in Has’minski˘ı (1980). For our results we will use the specific case of \( p = 2 \) which is more commonly known as exponential mean square stability. Exponential mean square stability is very common stability definition in control literature where it is used to ensure that the second moment or variance of the stochastic dynamical system converges to zero exponentially. Now that we have the stability definitions
which we will use to study the system performance, we need to ensure that the deterministic system is controllable and observable. For this we give some conditions to ascertain the controllability and observability for LTV and nonlinear systems in the next section.

**2.2 Observability and Controllability Conditions**

We now study the conditions for controllability and observability for LTV and nonlinear systems. Consider an LTV system of the form

\[ x_{t+1} = A_t x_t + B_t u_t \]

\[ y_t = C_t x_t \]  

Firstly we give a condition for uniform controllability of LTV systems from Kwakernaak and Sivan (1972).

**Definition 2 (Uniformly controllable)** The sequence of pairs \((A_t, B_t)\) is said to be uniformly controllable if there exists an integer \(k \geq 1\) and positive constants \(\alpha_0, \alpha_1, \beta_0\) and \(\beta_1\) such that

\[
W(t_0, t_0 + k) > 0 \tag{2.4}
\]

\[
\alpha_0 I \leq W^{-1}(t_0, t_0 + k) \leq \alpha_1 I \tag{2.5}
\]

\[
\beta_0 I \leq \Phi'(t_0 + k, t_0) W^{-1}(t_0, t_0 + k) \Phi(t_0 + k, t_0) \leq \beta_1 I \tag{2.6}
\]

\(\forall t_0\), where \(W(t_0, t_1)\) is a symmetric nonnegative matrix

\[
W(t_0, t_1) = \sum_{t=t_0}^{t_1-1} \Phi(t_1, t + 1) B_t B_t^\prime \Phi'(t_1, t + 1) \tag{2.7}
\]

and \(\Phi(t, t_0) = \prod_{k=t_0}^{t} A_k\) is the transition matrix.

**Remark 3** The condition for uniform reconstructability is the dual of the above definition and can be found in Kwakernaak and Sivan (1972). For the case of nonlinear systems we will employ a similar idea by using the Jacobian matrix for a matrix of observation vectors.
We now provide the following definition of an observability rank condition for nonlinear systems Nijmeijer (1982). Consider a nonlinear system of the form

\[ x_{t+1} = f(x_t) \]
\[ y_t = h(x_t) \]  \hspace{1cm} (2.8)

**Definition 4 (Observability Rank Condition)** Consider the map 
\[ \theta^{N-1}(x) : X \rightarrow Y \times \ldots \times Y \]
\[ \theta^{N-1}(x) := (h(x), h(f(x)), \ldots, h(f^{N-1}(x))'). \]  \hspace{1cm} (2.9)

The system (4.1) is said to satisfy the observability rank condition at \( x \), if

\[ \text{rank} \left( \frac{\partial \theta^{N-1}(x)}{\partial x} \right) = N. \]

**2.3 Ergodic Theory of Dynamical Systems**

We next introduce few preliminary definitions from the ergodic theory of dynamical systems. For more details on this topic refer to Folland (1999); Lasota and Mackey (1994); Froyland (2001).

**Definition 5 (\( \sigma \)-algebra)** A collection \( \mathcal{B}(X) \) of subsets of a set \( X \) is a \( \sigma \)-algebra if

1. When \( B \in \mathcal{B}(X) \) then \( X \setminus B \in \mathcal{B}(X) \);

2. Given a finite or infinite sequence \( \{B_k\} \) of subsets of \( X \), \( B_k \in \mathcal{B}(X) \), then the union \( \bigcup_k B_k \in \mathcal{B}(X) \); and

3. \( X \in \mathcal{B}(X) \).

**Definition 6** A real-valued function \( \mu \) defined on \( \sigma \)-algebra \( \mathcal{B}(X) \) is a measure if

1. \( \mu(\emptyset) = 0 \);

2. \( \mu(B) \geq 0 \) for all \( B \in \mathcal{B}(X) \); and

3. \( \mu(\bigcup_k B_k) = \sum_k \mu(B_k) \) is \( \{B_k\} \) is a finite or infinite sequence of pair wise disjoint sets from \( \mathcal{B}(X) \), that is \( B_i \cap B_j = \emptyset \) for \( i \neq j \).
Definition 7 (Measure Space) If $B(X)$ is a $\sigma$-algebra of subsets of $X$ and if $\mu$ is a measure on $B(X)$, then the triple $(X, B(X), \mu)$ is called a measure space. The sets belonging to $B(X)$ are called measurable sets because, for them, the measure is defined.

Definition 8 (Probability Measure) A measure space $(X, B(X), \mu)$ is called finite if $\mu(X) < \infty$. In particular, if $\mu(X) = 1$, then the measure space is said to be normalized or probabilistic. The measure $\mu$ is called a probability measure.

Definition 9 (Ergodic invariant measure) Let $\mathcal{M}(X)$ be the space of probability measures on $X$. A measure $\mu \in \mathcal{M}(X)$ is said to be invariant for $x_{t+1} = f(x_t)$ if

$$
\mu(f^{-1}(B)) = \mu(B)
$$

for all set $B \in B(X)$ (Borel $\sigma$-algebra generated by $X$). A set $A \subset X$ is said to be an $f$ invariant set if it satisfies $f^{-1}(A) = A$. An invariant measure $\mu$ is said to be ergodic if every $f$ invariant set $A$ has $\mu$ measure equal to zero or one.

Definition 10 (Physical measure) An ergodic invariant probability measure is called a natural or physical measure if

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} g(f^k(x)) \to \int_X g(x) d\mu(x)
$$

(2.10)

for all continuous function $g: X \to \mathbb{R}$ and Lebesgue-almost all $x \in X$.

Typically a chaotic system will have infinitely many invariant measures but only one physical measure. Consider the following example

Example 11 The linear mapping on a torus $F: \mathbb{T}^2 \to \mathbb{T}^2$ defined by $F(x, y) = (2x + y, x + y)$ (mod 1) has infinitely many invariant measures. For example $\delta_{(0,0)}$, and Lebesgue measure. However of all the invariant measures only the Lebesgue measure is a physical measure.

Physical measure are of special interest because in the computer simulation of dynamical systems, the trajectories will distribute themselves according to the physical measure. We next define the Lyapunov exponents for a deterministic dynamical system. We now state the main
result on Oseledec Multiplicative Ergodic theorem from Ruelle (1979). Results of this theorem is used in the proof for the main results of this paper. First we give a definition for which the readers may refer to Arnold (1998) (Chap. 3, Lemma 3.2.6).

**Definition 12** Let \( \wedge \) denote the usual outer product between two quantities and \( V \) be a real \( n \)-dimensional vector space. Let \( \wedge^q V \) denote alternating \( k \)-linear forms. Suppose \( M : V \rightarrow V \) is a linear operator. Then for \( u_1, \ldots, u_q \in V \) the linear extension of

\[
M \wedge^q (u_1 \wedge \cdots \wedge u_q) = Mu_1 \wedge \cdots \wedge Mu_q
\]

defines a linear operator \( M \wedge^q : \wedge^q V \rightarrow \wedge^q V \).

We now state the following theorem from Ruelle (1979), providing condition for the existence of Lyapunov exponents.

**Theorem 13** Consider the dynamical system \( x_{n+1} = f(x_n) \) with \( x_n \in X \subset \mathbb{R}^N \) a compact set. Let

\[
\lim_{t \to \infty} \sup_{t} \frac{1}{t} \log \| D_x f(x_t) \| \leq 0
\]

where \( D_x f(x) = \frac{\partial f}{\partial x} (x) \). Let \( D^t_x f(x_0) := D_x f(x_1) \ldots D_x f(x_0) \). Assume that the following limit exists

\[
\lim_{t \to \infty} \frac{1}{t} \log \| (D^t_x f(x))^\wedge q \|
\]

exists for \( q = 1, \ldots, N \). Then:

\[
\lim_{t \to \infty} \left( (D^t_x f(x))^\prime D^t_x f(x) \right)^{\frac{1}{t}} = \Lambda_x
\]

exists where \( M' \) is transpose of a matrix \( M \). Let \( \lambda^1_{\exp} \geq \ldots \geq \lambda^N_{\exp} \) be the eigenvalues of \( \Lambda_x \), then the Lyapunov exponents \( \Lambda^k_{\exp} \) are given by \( \Lambda^k_{\exp} = \log \lambda^k_{\exp} \) for \( k = 1, \ldots, N \). Also, \( \lambda^1_{\exp} \) is known as the maximum Lyapunov exponent. Furthermore if \( \det \Lambda_x \neq 0 \) then

\[
\lim_{t \to \infty} \frac{1}{t} \log | \det(D^t_x f(x))| = \log \det \Lambda_x = \log \prod_{k=1}^{N} \lambda^k_{\exp} \quad (2.11)
\]

The limiting matrix \( \Lambda_x \) will be independent of the initial condition \( x \) under the assumption of unique ergodicity of the system.
Results of Oseledets Multiplicative Ergodic Theorem are used in the proof for the main results of this paper.

**Remark 14** For the theorem and its proof, refer to Ruelle (1979) Proposition 1.3 and Theorem 1.6. The assumptions of Theorem 13 are satisfied if the system has bounded Jacobian i.e. \( \|D_x f(x_t)\| < K \) for all \( t \geq 0 \) since this will imply that

\[
\lim_{t \to \infty} \sup \frac{1}{t} \log \| D_x f(x_t) \| \leq 0
\]

Furthermore as \( \|D_x f(x_t)\| < K \) we know that \( \|D_x^t f(x_t)\| < K^{t+1} \) and \( |\det(D_x f(x_t))| < \bar{K} \) for some \( \bar{K} < \infty \). Hence the limit

\[
\lim_{t \to \infty} \frac{1}{t} \log \| (D_x^t f(x))^\wedge q \|
\]

exists for \( q = 1 \) and \( q = N \) which ensures that this limit will exist for all \( q = 1, \ldots, N \).

Further information on computation of these exponents from the nonlinear system or trajectory data can be found in Abarbanel (1993); G. Froyland and Mees (1995).

Measure-theoretic entropy, \( H_\mu(f) \), for the dynamical system, \( x_{n+1} = f(x_n) \), is associated with a particular ergodic invariant measure, \( \mu \), and is another measure of dynamical complexity. While the measure-theoretic entropy counts the number of typical trajectories for their growth rate, the positive Lyapunov exponents measure the rate of exponential divergence of nearby system trajectories. For more details on entropy refer to Walters (1982). These two measures of dynamical complexity are related by Ruelle’s inequality.

**Theorem 15 (Ruelle’s Inequality)** (Eckman and Ruelle (1985) Eq. 4.4); (Ruelle (1978) Theorem 2) Let \( x_{n+1} = f(x_n) \) be the dynamical system, \( f : X \to X \) be a \( C^r \) map, with \( r \geq 1 \), of a compact metric space \( X \) and \( \mu \) an ergodic invariant measure. Then,

\[
H_\mu(f) \leq \sum_k (\lambda^k_{\exp})^+,
\]

where \( a^+ = \max\{0, a\} \), \( H_\mu(f) \) is the measure-theoretic entropy corresponding to the ergodic invariant measure \( \mu \), and \( \lambda^k_{\exp} \) are the Lyapunov exponents of the system.
2.4 Lur’e Systems

We now give some basic preliminaries for a special class of nonlinear systems known as the Lur’e systems. Consider a Lur’e system of the form

\[ x_{t+1} = Ax_t + Bu_t \]  \hspace{1cm} (2.13)
\[ y_t = Cx_t \]
\[ u_t = -\phi(y_t) \]  \hspace{1cm} (2.14)

where \( A \) is Hurwitz and \( \phi(y_t) \) is monotonically non-decreasing and satisfies the condition \( \phi(y_t)'(y_t - D\phi(y_t)) \geq 0 \), then we get the following condition for the closed loop system to be stable Haddad and Bernstein (1994), Lancaster and Rodman (1995)

**Lemma 16** Consider a nonlinear system of the form (2.13). Then the system given by (2.13) is stable if either of the following conditions are true

1. There exist matrices \( P = P' > 0 \), \( V \) and \( W \) such that

\[ P \geq A'PA + V'V \]  \hspace{1cm} (2.15)
\[ 0 = B'PA - C + W'V \]  \hspace{1cm} (2.16)
\[ 0 = D + D' - B'PB - W'W \]  \hspace{1cm} (2.17)

2. There exist matrices \( P = P' > 0 \) and \( R = R' > 0 \) such that, \( D + D' - B'PB > 0 \) and,

\[ P = A_1PA_1 + A_1'PB(D + D' - B'PB)^{-1}B'PA_1 + C'(D + D')^{-1}C + R \]  \hspace{1cm} (2.18)

where \( A_1 = A - B(D + D')^{-1}C \).

**Proof.** Consider the system (6.1) given by

\[ x(t+1) = Ax(t) - B\phi(y(t)) \]  \hspace{1cm} (2.19)
\[ y(t) = Cx(t) \]  \hspace{1cm} (2.20)
The first condition is true from the proof given in Haddad and Bernstein (1994). This proof also gives the condition in terms of an equivalent Riccati equation for (2.15), (2.16) and (2.17) given by

\[ P = APA + (A'PB - C') (D + D' - B'PB)^{-1} (B'PA - C) + R \]  

(2.21)

We can obtain (2.21) from the above Riccati equation by a transformation given in Lancaster and Rodman (1995).
CHAPTER 3. Stabilization of LTV Systems Over Erasure Channels

To study the problem of stabilization of nonlinear systems over unreliable actuation, as a first step, we study the problem of control of linear time varying systems over unreliable actuation links. The unreliable communication link is written as a fading channel that consists of a deterministic mean connectivity $\mu$, and a zero mean stochastic uncertainty $\Delta$ with finite variance $\sigma^2$. Exponential mean square stability of the control system is chosen as the desired performance criterion. The main result shows that, fundamental limitation arises when exponential mean square stability of the control system is desired. Main result of this chapter provides a necessary and sufficient condition for the control system to be exponentially mean square stable. Furthermore, offline computable necessary conditions are given for the case of a single input and $N$ input system. This necessary condition for single input case is expressed in terms of the the variance $\sigma^2$, mean connectivity $\mu$ and the product of the positive Lyapunov exponents of the uncontrolled system. In the case of $N$ inputs, this condition is also shown to be sufficient. This result generalizes the existing result known in the case of linear time invariant systems and Lyapunov exponents emerge as the generalization of eigenvalues from linear time invariant systems to linear time varying systems. Simulation result is presented to verify the main result for single input system.

3.1 Introduction

There has been increased research activity in the area of network controlled systems (Antsaklis and Baillieul, 2007). Network controlled systems (NCS) comprise of systems to be controlled with actuators, sensors, and controller communicating over communication channels. One of the important problems that has been addressed in the area of NCS, is that of characterizing the
performance limitations on control and estimation caused by unreliable communication channels. For the problem studied in this chapter, the communication channels we are interested in are described by analog fading channels Elia (2005).

The main contribution of this result is in the development of the results for the control of linear time varying (LTV) systems over fading channels. In this section we study the problem of feedback control of LTV systems in the presence of fading communication links between the plant and the controller (refer to Figure 3.1). The main result provides a necessary and sufficient condition for the control system to be exponentially mean square stable to the origin. Furthermore an offline computable, necessary condition is derived for the case of a single input and $N$-input system, and is also shown to be sufficient for the $N$-input case. This necessary condition is expressed in terms of variance $\sigma^2$ of the stochastic fading link, the mean connectivity $\mu$ and positive Lyapunov exponents of the open loop system. In the special case of binary erasure channel Elia (2005), the result is obtained in terms of channel erasure probability and product of positive Lyapunov exponents. Thus the result generalizes the existing results known in the case of LTI systems where Lyapunov exponent emerge as the natural generalization of eigenvalues from LTI systems to LTV systems. We also provide for the synthesis of the controller that is robust to link failure uncertainty. Simulation result is presented towards the end of the chapter, to verify the main result. Simulation results for the single input case suggest that the proven necessary condition is also sufficient.

### 3.2 Preliminaries

We consider the problem of control of multi-state multi-input LTV systems with a stochastic memoryless multiplicative uncertainty between the plant and the controller (refer to Fig. 3.1a). The LTV system with multiplicative uncertainty channel is described by the following equation:

$$x(t + 1) = A(t)x(t) + \gamma(t)B(t)u(t),$$  \hspace{1cm} (3.1)

where $x(t) \in \mathbb{R}^N$ is the state, $u(t) \in \mathbb{R}^M$ is input with $M \leq N$, and $t \geq 0$. The channel uncertainty, between the plant and the controller, is modeled using the random variable $\gamma(t)$ and is assumed to satisfy following statistics, $E[\gamma(t)] = \mu$ and $E[(\gamma(t) - \mu)^2] = \sigma^2$. By defining
a new random variable, $\Delta(t) := \gamma(t) - \mu$, the feedback control system in Fig. 3.1a can be redrawn as shown in Fig. 3.1b. The random variable $\Delta(t)$ now satisfies

$$E[\Delta] = 0, \quad E[\Delta^2] = \sigma^2, \quad (3.2)$$

The feedback control system inside the dotted line in Fig. 3.1b now represents a nominal system with mean connectivity $\mu$, interacting with zero mean $\Delta$ uncertainty with variance $\sigma^2$. The system Eq. (3.1) can be written as:

$$x(t + 1) = A(t)x(t) + \mu B(t)u(t) + \Delta(t)B(t)u(t). \quad (3.3)$$

Writing the feedback control system with multiplicative channel uncertainty, $\gamma$, as the interconnection of nominal system with mean connectivity, $\mu$, and zero mean random variable, $\Delta$, closely follows Elia (2005).

**Remark 17** The block diagram in Fig. 3.1b, where a nominal system (inside the dotted box) is interconnected with zero mean random variable $\Delta$ allows us to interpret the main results of this paper along the lines of the results known in the robust control literature for LTI systems Dullerud and Paganini (1999); Elia (2005) (refer to Remark 31).

We make the following assumptions about the system dynamics.
**Assumption 18** We assume the system matrix, $A(t)$, is uniformly bounded above and below, and that $B'(t)B(t)$ is uniformly bounded from below. Furthermore, we assume the pair, $(A(t), B(t))$, is uniformly controllable. The definition of uniform controllability is from Kwakernaak and Sivan (1972) and is given as follows.

**Definition 19 (Uniformly controllable)** The sequence of pairs, $(A(t), B(t))$, is said to be uniformly controllable, if there exists an integer $k \geq 1$ and positive constants $\alpha_0, \alpha_1, \beta_0$ and $\beta_1$, such that

$$W(t_0, t_0 + k) > 0 \quad (3.4)$$

$$\alpha_0 I \leq W^{-1}(t_0, t_0 + k) \leq \alpha_1 I \quad (3.5)$$

$$\beta_0 I \leq \Phi'(t_0 + k, t_0)W^{-1}(t_0, t_0 + k)\Phi(t_0 + k, t_0) \leq \beta_1 I, \quad (3.6)$$

\forall t_0, \text{ where } W(t_0, t_1) \text{ is a symmetric nonnegative matrix.}

$$W(t_0, t_1) = \sum_{t = t_0}^{t_1-1} \Phi(t_1, t + 1)B(t)B'(t)\Phi'(t_1, t + 1) \quad (3.7)$$

and $\Phi(t, t_0) = \prod_{k = t_0}^{t} A(k)$ is the transition matrix.

We now provide the following definition of exponential stable and exponentially antistable dynamics for the LTV system. The following two definitions closely follow Iglesias (2001).

**Definition 20 (Exponential stable and antistable Iglesias (2001))** Consider the uncontrollable system in (3.3) given by $x(t + 1) = A(t)x(t)$. Let $k, l$ be positive integers. We say that $\{A(t)\}_{t \geq 0}$ is

1. Uniformly exponentially stable, if there exist positive constants $K_s$ and $\beta_s < 1$, such that

$$\left\| \prod_{t = k}^{k+l-1} A(t) \right\| < K_s\beta_s^l.$$  

2. Uniformly exponentially antistable, if there exist positive constants $K_u$ and $\beta_u > 1$, such that

$$\mu \left( \prod_{t = k}^{k+l-1} A(t) \right) > K_u\beta_u^l,$$  

where for any $N \times N$ matrix $M$, $\mu(M) := \inf \{ \|Mx\| : \|x\| = 1 \}$.

For limitation results involving LTI systems, it is known that the fundamental limitations for stabilization using state feedback controller arise only due to antistable parts of the system.
Elia (2005); L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007). A first step towards proving such results for the LTI system is to perform a change of coordinates that allows one to decompose the system matrix into stable and antistable components. We expect similar conclusions to hold true for the limitations results involving LTV systems. In fact, using results from Ben-Artzi and Gohberg (1991); Iglesias (2001), it can be shown that the LTV system admits decomposition into stable and antistable components under the assumption that system matrices \( \{A(t)\} \) satisfy exponential dichotomy property, defined as follows.

**Definition 21 (Exponential Dichotomy Iglesias (2001))** Let \( \{A(t)\} \) be a sequence of \( N \times N \) matrices and let \( P(t) \) be a bounded sequence of projections in \( \mathbb{R}^N \), such that the rank of \( P(t) \) is constant. The sequence \( \{P(t)\} \) is a dichotomy for \( \{A(t)\} \) if the commutativity condition, \( A(t)P(t) = P(t + 1)A(t) \), is satisfied for all \( t \), and there exists positive constants \( L_d \) and \( \beta_d > 1 \), such that

\[
\left\| \prod_{t=k}^{k+l-1} A(t) P(k) x \right\| > L_d \beta_d \left\| P(k) x \right\|
\]

\[
\left\| \prod_{t=k}^{k+l-1} A(t) (I - P(k)) x \right\| < \frac{1}{L_d \beta_d} \left\| (I - P(k)) x \right\|
\]

for any \( x \in \mathbb{R}^N \).

Under the assumption of exponential dichotomy (Definition 21), it can be proven (Iglesias (2001); Ben-Artzi and Gohberg (1991)) there exists a bounded sequence of matrices \( \{T(t)\} \) with bounded inverses, such that the system matrices pair, \( (A(t), B(t)) \), can be transformed into stable and antistable components, i.e.,

\[
\begin{bmatrix}
T(t+1)A(t)T(t)^{-1} & T(t+1)B(t)
\end{bmatrix} = \begin{bmatrix}
A_u(t) & 0 \\
0 & A_s(t)
\end{bmatrix} \begin{bmatrix}
B_u(t) \\
B_s(t)
\end{bmatrix}, \quad (3.8)
\]

where \( A_s(t) \) is exponentially stable and \( A_u(t) \) is exponentially antistable (Definition 20). We now make the following assumption on system dynamics.

**Assumption 22 (Stable and antistable)** We assume the system matrices, \( \{A(t)\} \), possesses an exponential dichotomy. Hence, there exists a change of coordinates, \( \{T(t)\} \), such that the
system may be transformed into a block diagonal form with stable and antistable components. Henceforth, with no loss of generality, we assume that system pair, \((A(t), B(t))\), is already decomposed into exponentially stable and exponentially antistable components i.e.,

\[
\begin{bmatrix}
  A(t) & B(t)
\end{bmatrix} = \begin{bmatrix}
  A_u(t) & 0 \\
  0 & A_s(t)
\end{bmatrix} \begin{bmatrix}
  B_u(t) \\
  B_s(t)
\end{bmatrix}.
\] (3.9)

Our objective is to design a linear state feedback controller, \(u(t) = K(t)x(t)\), so that the feedback control system (3.10) is mean square exponentially stable (Definition 23).

\[
x(t + 1) = (A(t) + \mu B(t)K(t) + \Delta(t)B(t)K(t))x(t) := A(t, \Delta(t))x(t).
\] (3.10)

**Definition 23 (Mean Square Exponential Stability)** The system (3.10) is said to be mean square exponentially stable, if there exists positive constants \(K < \infty\) and \(\beta < 1\), such that

\[
E_{\Delta_k} \left[ \| x(t + 1) \|^2 \right] \leq K\beta^t \| x(0) \|^2,
\]

for all \(x(0) \in \mathbb{R}^N\), where \(E_{\Delta_k}[:,]\) is the expectation over the sequence \(\{\Delta(k)\}^t_{k=0}\).

It is well known, that the stability information for an LTV system cannot be obtained from the eigenvalues of the time varying matrix computed at each fixed time, \(t\) Khalil (1996). However, stability information for the LTV system can be obtained using *Lyapunov exponents*. The Multiplicative Ergodic Theorem (MET) provides technical conditions for the existence of Lyapunov exponents (Ruelle (1979) Proposition 1.3). Before we proceed with the definition of Lyapunov exponents, we provide a definition for the exterior powers of the matrices Arnold (1998) (Chap. 3, Lemma 3.2.6).

**Definition 24** Let \(\wedge\) denote the usual outer product between two quantities and \(V\) be a real \(n\)-dimensional vector space. Let \(\wedge^q V\) denote alternating \(k\)-linear forms. Suppose \(M: V \rightarrow V\) is a linear operator. Then, for \(u_1, \ldots, u_q \in V\), the linear extension of

\[
M^{\wedge q}(u_1 \wedge \cdots \wedge u_q) = M u_1 \wedge \cdots \wedge M u_q
\]

defines a linear operator, \(M^{\wedge q}: \wedge^q V \rightarrow \wedge^q V\).
Definition 25 (Lyapunov exponents Ruelle (1979) Proposition 1.3) Let \( \{L(t)\}_{t>0} \) be a sequence of real \( m \times m \), matrices such that
\[
\lim_{t \to \infty} \sup \frac{1}{t} \log \| L(t) \| \leq 0.
\]
Define \( \mathcal{L}(t) := L(t)L(t-1)\ldots L(1) \). Furthermore, suppose the following limits exist
\[
\lim_{t \to \infty} \frac{1}{t} \log \| \mathcal{L}(t)^\wedge q \|.
\]
Then, the limit
\[
\Lambda = \lim_{t \to \infty} (\mathcal{L}(t)'\mathcal{L}(t))^{\frac{1}{2}}
\]
exists. Let \( \lambda_{\exp}^i \) for \( i = 1, \ldots, N \) be the eigenvalues of \( \Lambda \), such that \( \lambda_{\exp}^1 \geq \lambda_{\exp}^2 \geq \cdots \geq \lambda_{\exp}^N \).

Then, the Lyapunov exponents, \( \Lambda_{\exp}^i \) for \( i = 1, \ldots, m \), for the system \( x(t+1) = L(t)x(t) \) are defined as \( \Lambda_{\exp}^i = \log \lambda_{\exp}^i \). Furthermore, if \( \det(\Lambda) \neq 0 \), then
\[
\lim_{t \to \infty} \frac{1}{t} \log |\det(\mathcal{L}(t))| = \log \prod_{k=1}^{N} \lambda_{\exp}^k(x).
\]

Remark 26 The Lyapunov exponents can be used for the stability analysis of the LTV system. In particular, if the maximum Lyapunov exponent of the system \( x(t+1) = L(t)x(t) \) is negative, i.e., \( \Lambda_{\exp}^1 < 0 \), then the system is exponentially stable Arnold (1998).

Assumption 27 We assume the Lyapunov exponents for the uncontrolled system \( x(t+1) = A(t)x(t) \) are well defined, and there are \( 0 < N_1 \leq N \) positive Lyapunov exponents, and \( N_2 := N - N_1 \) negative Lyapunov exponents.

3.3 Main Results

In this section, we prove the main results of this paper for the limitation on control over uncertain channels in actuation. We use mean square exponential stability of the closed-loop systems as the stability metric. Our first theorem provides a Lyapunov function-based necessary condition for the mean square exponential stability of uncertain feedback control system (3.10).
Theorem 28 The feedback control system (3.10) is mean square exponentially stable only if there exists a sequence of positive definite matrices \( \{P(t)\}_{t \geq 0} \) and positive constants \( \alpha_1 \) and \( \alpha_2 \), such that the following conditions are satisfied.

\[
E_{\Delta(t)} \left[ A'(t, \Delta(t)) P(t + 1) A(t, \Delta(t)) \right] < P(t), \quad \alpha_1 I < P(t) < \alpha_2 I, \tag{3.14}
\]

for all \( t \geq 0 \), and \( A(t, \Delta(t)) := A(t) + (\mu + \Delta)B(t)K(t) \) from Eq. (3.10).

Proof. Consider the following construction of \( P(t) \),

\[
P(t) = \sum_{n=t}^{\infty} E_{\Delta(t)} \left[ \left( \prod_{k=t}^{n} A(k, \Delta(k)) \right) \right] \left( \prod_{k=t}^{n} A(k, \Delta(k)) \right),
\]

where \( E_{\Delta(t)}[\cdot] \) means expectation has been taken over \( \Delta(t) \) for \( t = 1, \ldots, n \). Since the closed-loop system, \( x(t + 1) = A(t, \Delta(t)) \), is assumed mean square exponentially stable, the construction for \( P(t) \) is well defined. We can also write the above equation as

\[
E_{\Delta(t)} \left[ A'(t, \Delta(t)) A(t, \Delta(t)) + A'(t, \Delta(t)) P(t + 1) A(t, \Delta(t)) \right] = P(t).
\]

The equation for \( P(t) \) can be rewritten as follows:

\[
E_{\Delta(t)} \left[ A'(t, \Delta(t)) P(t + 1) A(t, \Delta(t)) \right] - P(t) = -E_{\Delta(t)} \left[ A'(t, \Delta(t)) A(t, \Delta(t)) \right].
\]

Since \( A(t, \Delta(t)) \) is invertible for \( \Delta(t) = 0 \) and is continuous with respect to \( \Delta \), it follows that \( E_{\Delta(t)} \left[ A'(t, \Delta(t)) A(t, \Delta(t)) \right] > 0 \). Hence, we obtain

\[
E_{\Delta(t)} \left[ A'(t, \Delta(t)) P(t + 1) A(t, \Delta(t)) \right] < P(t).
\]

We now need to show \( P(t) \) is bounded. The system is assumed mean square exponentially stable as given in Definition 23. There exists \( \beta < 1 \) and \( K < \infty \), such that

\[
E_{\Delta(t)} \left[ \| x(t + 1) \|^2 \right] = E_{\Delta(t)} \left[ \left\| \prod_{k=0}^{t} A(k, \Delta(k)) x(0) \right\|^2 \right] \leq K \beta^t \| x(0) \|^2.
\]

Hence, we have \( \| P(t) \| \leq K \sum_{k=0}^{\infty} \beta^k = \frac{K}{1 - \beta} \). Since matrix \( A(t, \Delta(t)) \) is bounded below in some Lebesgue neighborhood of \( \Delta(t) = 0 \) for all \( t \geq 0 \), we have some constant \( \alpha_1 > 0 \), such that \( \alpha_1 I \leq E_{\Delta(t)} \left[ A'(t, \Delta(t)) A(t, \Delta(t)) \right] \forall t \geq 0 \) which gives

\[
P(t) \geq E_{\Delta(t)} \left[ A'(t, \Delta(t)) A(t, \Delta(t)) \right] \geq \alpha_1 I.
\]
Setting \( \alpha_2 = \frac{K}{1-\beta} \), we get \( \alpha_1 I \leq P(t) \leq \alpha_2 I \). □

We have the following Lemma providing necessary conditions for the mean square exponential stability of (3.10) in terms of the solution of the Riccati equation.

**Lemma 29** The necessary condition for mean square exponentially stability of system (3.10) derived in Theorem 66 is equivalent to

\[
P_0(t) > A'(t)P_0(t+1)A(t)
- \frac{\mu^2}{\mu^2 + \sigma^2} A'(t)P_0(t+1)B(t) \left( B'(t)P_0(t+1)B(t) \right)^{-1} B'(t)P_0(t+1)A(t),
\]

where \( \mu \) is the mean connectivity, \( \sigma^2 \) is the variance of the zero mean uncertainty. \( \{P_0(t)\}_{t \geq 0} \) is the sequence of positive definite symmetric matrices that satisfies the following Riccati equation Kwakernaak and Sivan (1972).

\[
P_0(t) = A'(t)P_0(t+1)A(t)
- A'(t)P_0(t+1)B(t) \left( I_M + B'(t)P_0(t+1)B(t) \right)^{-1} B'(t)P_0(t+1)A(t) + R(t),
\]

where \( R(t) = R'(t) > 0 \) is such that \( P_0(t) \) is uniformly bounded above and below.

**Proof.** From Theorem 66 we know a necessary condition for mean square exponential stability of (3.10) is given by

\[
P(t) > E_{\Delta(t)} \left[ A'(t, \Delta(t))P(t+1)A(t, \Delta(t)) \right],
\]

where \( A(t, \Delta(t)) := A(t) + \mu B(t)K(t) + \Delta(t)B(t)K(t) \) and there exist \( \alpha_1, \alpha_2 > 0 \), such that \( \alpha_1 I < P(t) < \alpha_2 I \) for all \( t > 0 \). Expanding the above equation, we derive the necessary condition

\[
P(t) > A'(t)P(t+1)A(t) - \mu A'(t)P(t+1)B(t)K(t)
- \mu K'(t)B'(t)P(t+1)A(t) + (\mu^2 + \sigma^2) K'(t)B'(t)P(t+1)B(t)K(t).
\]

(3.16)

Taking the trace and minimizing the RHS w.r.t. \( K(t) \), we obtain optimal \( K^*(t) \) Kwakernaak and Sivan (1972) to achieve the mean square exponential stability as

\[
K^*(t) = -\frac{\mu}{\mu^2 + \sigma^2} \left( B'(t)P(t+1)B(t) \right)^{-1} B'(t)P(t+1)A(t).
\]
This provides us the necessary condition for mean square exponential stability of the controlled system (3.10)

\[ P(t) > A'(t)P(t + 1)A(t) \]
\[ - \frac{\mu^2}{\mu^2 + \sigma^2} A'(t)P(t + 1)B(t) \left( B'(t)P(t + 1)B(t) \right)^{-1} B'(t)P(t + 1)A(t). \] (3.17)

Now, using the fact \( P(t) \) is bounded below, there exists \( \Sigma > 0 \), such that \( \frac{\sigma^2}{\mu^2} B'(t)P(t + 1)B(t) \geq \Sigma I_M \) for all \( t \geq 0 \). Substituting this in (3.17) we obtain,

\[ P(t) > A'(t)P(t + 1)A(t) \]
\[ - A'(t)P(t + 1)B(t) \left( \Sigma I_M + B'(t)P(t + 1)B(t) \right)^{-1} B'(t)P(t + 1)A(t). \] (3.18)

Defining \( P_0(t) = \frac{1}{2} P(t) \) we find

\[ P_0(t) > A'(t)P_0(t + 1)A(t) \]
\[ - A'(t)P_0(t + 1)B(t) \left( I_M + B'(t)P_0(t + 1)B(t) \right)^{-1} B'(t)P_0(t + 1)A(t). \] (3.19)

Thus, there exists \( R(t) > 0 \) for all \( t \geq 0 \) as given in Kwakernaak and Sivan (1973), such that

\[ P_0(t) = A'(t)P_0(t + 1)A(t) \]
\[ - A'(t)P_0(t + 1)B(t) \left( I_M + B'(t)P_0(t + 1)B(t) \right)^{-1} B'(t)P_0(t + 1)A(t) + R(t). \] (3.20)

We notice that (3.17) is independent of any constant scaling. Hence, \( P_0(t) \) satisfies

\[ P_0(t) > A'(t)P_0(t + 1)A(t) \]
\[ - \frac{\mu^2}{\mu^2 + \sigma^2} A'(t)P_0(t + 1)B(t) \left( B'(t)P_0(t + 1)B(t) \right)^{-1} B'(t)P_0(t + 1)A(t). \] (3.21)

This gives the required necessary condition.

The first main result of the paper provides a computable necessary condition for stability of (3.10) for \( M < N \) inputs case with \( M < N \).

**Theorem 30** A necessary condition for the mean square exponential stability of system (3.10) for \( M < N \) inputs is given by

\[ \sigma^2 \frac{\left( \prod_{k=1}^{N_1} \lambda_{k_{\text{exp}}}^k \right)^{\frac{2}{\mu^2}} - 1}{\mu^2} < 1, \] (3.22)
where \(\sigma^2 < \infty\) is the variance of uncertainty \(\Delta\) (Eq. (3.2)), and \(\lambda_{exp}^k = e^{\Lambda_{exp}^k}\), and \(\Lambda_{exp}^k\) is the \(k^{th}\) positive Lyapunov exponent of uncontrolled system \(x(t+1) = A(t)x(t)\) for \(k = 1, \ldots, N_1\).

**Proof.** From Theorem 29, we have a necessary condition for a system with \(N\) states and \(M\) inputs given by

\[
P_0(t) > A'(t)P_0(t+1)A(t) - \frac{\mu^2}{\mu^2 + \sigma^2} A'(t)P_0(t+1)B(t) \left( B'(t)P_0(t+1)B(t) \right)^{-1} B'(t)P_0(t+1)A(t).
\]

(3.23)

Let \(P_0(t)\) be given by the blockwise representation,

\[
P_0(t) := \begin{bmatrix}
P_{011}(t) & P_{012}(t) \\
P_{012}^T(t) & P_{022}(t)
\end{bmatrix},
\]

(3.24)

where \(P_{011}\) is an \(N_1 \times N_1\) block, \(P_{022}\) is an \(N_2 \times N_2\) block, and \(P_{012}\) is a \(N_1 \times N_2\) block. We know since the matrix \(P_0(t)\) is positive definite, any \(k \times k\) block for \(k \leq N\) must be positive definite. Hence, from (3.23) and (3.24), the necessary condition for mean square exponential stability provides the positive definiteness of the first \(N_1 \times N_1\) block in (3.23), given by

\[
P_{011}(t) > A_{u}'(t)P_{011}(t+1)A_{u}(t)
- \frac{\mu^2}{\mu^2 + \sigma^2} \left[ A_{u}'(t) \left( P_{011}(t+1)B_{u}(t) + P_{012}(t+1)B_{s}(t) \right) \right] \cdot 
\left[ \left( B'(t)P_0(t+1)B(t) \right)^{-1} \left( P_{011}(t+1)B_{u}(t) + P_{012}(t+1)B_{s}(t) \right)'A_{u}(t) \right].
\]

(3.25)

Taking determinants on both sides and using Sylvester’s determinant theorem, we obtain

\[
\det(P_{011}(t)) > \left[ \det(A_{u}(t))^2 \det(P_{011}(t+1)) \right] \cdot 
\left[ \det \left( I_M - \left( \frac{\mu^2}{\mu^2 + \sigma^2} \right) \left( B'(t)P_0(t+1)B(t) \right)^{-1} \cdot 
\left( P_{011}(t+1)B_{u}(t) + P_{012}(t+1)B_{s}(t) \right)'P_{011}(t+1)^{-1} \left( P_{011}(t+1)B_{u}(t) + P_{012}(t+1)B_{s}(t) \right) \right) \right].
\]

(3.26)
Using the partition for $B(t)$, we write

$$B'(t)P_0(t + 1)B(t)$$

$$= B'_u(t)P_{011}(t + 1)B_u(t) + B_u'(t)P_{012}(t + 1)B_s(t) + B'_s(t)P_{012}(t + 1)B_u(t) + B'_s(t)P_{022}(t + 1)B_s(t)$$

$$> B'_u(t)P_{011}(t + 1)B_u(t) + B_u'(t)P_{012}(t + 1)B_s(t) + B'_s(t)P_{012}(t + 1)B_u(t)$$

$$+ B'_s(t)(P_{012}(t + 1)'P_{011}(t + 1)^{-1}P_{012}(t + 1))B_s(t)$$

$$> \left( P_{011}(t + 1)B_u(t) + P_{012}(t + 1)B_s(t) \right)'P_{011}(t + 1)^{-1}\left( P_{011}(t + 1)B_u(t) + P_{012}(t + 1)B_s(t) \right),$$

(3.27)

since $P_{022}(t + 1) > P_{012}(t + 1)'P_{011}(t + 1)^{-1}P_{012}(t + 1)$ as $P_0(t + 1)$ is positive definite. Hence, from (3.27) we derive

$$I_M > \left[ (B'(t)P_0(t + 1)B(t))^{-1}. \right.$$}

$$\left( P_{011}(t + 1)B_u(t) + P_{012}(t + 1)B_s(t) \right)'P_{011}(t + 1)^{-1}\left( P_{011}(t + 1)B_u(t) + P_{012}(t + 1)B_s(t) \right) \right].$$

(3.28)

Hence, using (3.28) in (3.26), we find

$$1 > \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^M \det (A_u(t))^2 \det (P_{011}(t + 1)) \det (P_{011}(t))^{-1}.$$  

(3.29)

Hence, the necessary condition can be written as

$$1 > \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^{M(t + 1)} \prod_{k=0}^t \left( \det (A_u(k)) \right)^2 \det (P_{011}(t + 1)) \det (P_{011}(0))^{-1}.$$  

Taking the logarithm, averaging over $t + 1$, and taking the limit as $t \to \infty$, we obtain the necessary condition,

$$1 > \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^M \left( \prod_{k=1}^{N_1} \lambda^k_{exp} \right)^2,$$

where we are use the fact $P_0(t)$ (and hence $P_{011}(t)$) is bounded above and below for all $t \geq 0$ and Eq. (4.30) from Definition 53. This condition is rewritten as

$$\sigma^2 \left( \prod_{k=1}^{N_1} \lambda^k_{exp} \right)^{\frac{2}{M}} \frac{1}{\mu^2} < 1.$$  

(3.30)
Remark 31 The necessary conditions derived in Theorem 66, Lemma 29 are equivalent. Lemma 29 implies Theorem 30 though the converse may not be true. Hence Theorem 30 is lower in the hierarchy in comparison with Theorem 66 and Lemma 29. The necessary condition for stability in Eq. (3.22) can be used to provide critical value of variance, \( \sigma^* \), above which the system is guaranteed to be mean square unstable. In particular the critical value of variance using Eq. (3.22) is given by

\[
\sigma^* = \left( \frac{\mu^2}{(\prod_{k=1}^{N_1} \lambda_k^{\exp})^2 - 1} \right)^{\frac{1}{2}}.
\]

The necessary condition for mean square exponential stability derived in the above theorem is tighter for the single input case (i.e., \( M = 1 \)). However, for \( 1 < M < N \), Eq. (3.22) provides a necessary condition for stability and can be made tighter, i.e., improved necessary condition can be obtained that will provide for a smaller value of critical variance \( \sigma^* \) than the one provided by Eq. (3.31). We expect the tighter necessary condition to depend on some combination of Lyapunov exponents and not necessarily on all the Lyapunov exponents as it does in Eq. (3.22).

Borrowing terminology from Elia (2005), the quantity \( \left( \frac{\prod_{k=1}^{N_1} \lambda_k^{\exp}}{\mu^2} \right)^2 - 1 \) in Eq. (3.22) can be viewed as the scaled mean square norm of the nominal system with mean connectivity \( \mu \) as seen by the uncertainty \( \Delta \) for block diagram in Fig. 3.1b. Thus if we consider this as the mean square input-output gain of the nominal system and \( \sigma^2 \) as the mean square gain of the uncertainty, then the necessary condition in Theorem 30 may be interpreted as a necessary small gain condition for mean square stability of nonlinear systems.

The next main result of this paper provides the necessary and sufficient condition for the mean square exponential stability of the feedback system for the \( N \) input case. For this \( N \) input case, we assume the matrix \( B(t) \) is non-singular.

**Theorem 32** A necessary and sufficient condition for the mean square exponential stability of (3.10) with \( N \) inputs is given by

\[
\sigma^2 \left( \frac{\lambda_{\exp}^1}{\mu^2} \right)^2 - 1 < 1,
\]
where $\lambda_{exp}^1 = e^{\Lambda_{exp}^1}$ and $\Lambda_{exp}^1$ is the maximum positive Lyapunov exponent of system $x(t + 1) = A(t)x(t)$.

**Proof.** From Theorem 29, we obtain the following necessary condition for mean square exponential stability

$$P_0(t) > A'(t)P_0(t + 1)A(t) - \frac{\mu^2}{\mu^2 + \sigma^2}A'(t)P_0(t + 1)B(t) (B'(t) P_0(t + 1)B(t))^{-1} B'(t) P_0(t + 1)A(t).$$

Since $B(t)$ is a non-singular $N \times N$ matrix and $P_0(t)$ is invertible, we can write the above Lyapunov function inequality as

$$P_0(t) > \sigma^2 A'(t)P_0(t + 1)A(t).$$

Equation (3.33) implies following inequality to be true

$$P_0(0) > \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^{t + 1} \left( \prod_{k=0}^{t} A(k) \right)' P_0(t + 1) \left( \prod_{k=0}^{t} A(k) \right).$$

Since there exists $\alpha_1 > 0$ and $\alpha_2 > 0$, such that $\alpha_2 I > P_0(t) > \alpha_1 I$ for all $t > 0$, the necessary condition can be written as

$$\frac{\alpha_2}{\alpha_1} I > \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^{t + 1} \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right).$$

Take the logarithm in (3.34), divide by $t + 1$, and take $\lim_{t \to \infty}$, we get the following necessary condition for mean square exponentially stability,

$$\frac{\sigma^2}{\mu^2 + \sigma^2} \Lambda^2 < 1,$$  

which is satisfied only if $\frac{\sigma^2}{\mu^2 + \sigma^2} (\lambda_{exp}^1)^2 < 1$. This can be rewritten as $\sigma^2 \frac{(\lambda_{exp}^1)^2 - 1}{\mu^2} < 1$.

We will now prove the sufficiency part. Consider the controller gain as derived in the necessary condition given by $K = -\frac{\mu}{\mu^2 + \sigma^2} B(t)^{-1} A(t)$. Using this controller gain, the dynamics of the controlled system are given by

$$x(t + 1) = \frac{\sigma^2 - \Delta(t)\mu}{\mu^2 + \sigma^2} A(t)x(t).$$

From (3.36), we obtain

$$E_{\Delta(t)} [||x(t + 1)||^2] = \frac{\sigma^2}{\sigma^2 + \mu^2} x'(t)A'(t)A(t)x(t).$$

(3.37)
Thus, from (3.37) we obtain

\[ E_{\Delta t} \left[ ||x(t+1)||^2 \right] = \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} x'(0) \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) x(0). \]  

(3.38)

Now, we claim there exist positive constants \( K < \infty \) and \( \beta < 1 \), such that

\[ \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2 < K \beta^{t+1} \]  

(3.39)

for all \( t \geq 0 \). We will defer the proof of this claim for later to maintain continuity in the proof of the sufficiency condition. Now, using the claim from (3.39) in (3.37), we derive

\[ E_{\Delta t} \left[ ||x(t+1)||^2 \right] \leq \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2 ||x(0)||^2 \]

\[ < K \beta^{t+1} ||x(0)||^2. \]  

(3.40)

Thus, we have proven the required sufficiency condition. We will now prove the claim made in (3.39). To prove this claim suppose

\[ \sigma^2 \left( \lambda_{exp}^1 \right)^2 - 1 < 1. \]  

(3.41)

Hence, there exists \( \beta < 1 \), such that \( \frac{\sigma^2}{\sigma^2 + \mu^2} \left( \lambda_{exp}^1 \right)^2 = \beta^2 \). Furthermore, from the definition of the Lyapunov exponents (Definition 53), we obtain \( \lambda_{exp}^1 \) for the system \( x(t+1) = A(t)x(t) \) is given by

\[ \lambda_{exp}^1 = ||\Lambda||_2, \]  

(3.42)

where \( ||\Lambda||_2 \) is the matrix 2-norm of the matrix \( \Lambda \) given by

\[ \Lambda := \lim_{t \to \infty} \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \]  

(3.43)

Thus, from the property of the matrix 2-norm and the Lyapunov exponent definition (Proposition 1.3 Ruelle (1979), Eckman and Ruelle (1985)), we have

\[ \lambda_{exp}^1 = \lim_{t \to \infty} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2^{1/(2(t+1))}. \]  

(3.44)
Thus, we can conclude
\[
\lim_{t \to \infty} \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2^{1/(t+1)}
\]
\[
= \frac{\sigma^2}{\sigma^2 + \mu^2} \lim_{t \to \infty} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2^{1/(t+1)}
\]
\[
= \frac{\sigma^2}{\sigma^2 + \mu^2} (\lambda_{\text{exp}}^1)^2
\]
\[
= \beta^2 < \beta < 1. \tag{3.45}
\]

Hence, there exists \( N_\beta < \infty \), such that \( \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2 < \beta^{t+1} \) for all \( t \geq N_\beta \). The existence of \( N_\beta < \infty \) is proved by contradiction as follows. Suppose \( N_\beta < \infty \) does not exist. Thus there exits a subsequence \( \{t_i\}_{i \geq 0} \), such that
\[
\left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t_i+1} \left\| \left( \prod_{k=0}^{t_i} A(k) \right)' \left( \prod_{k=0}^{t_i} A(k) \right) \right\|_2 > \beta^{t_i+1}.
\]

Now, let
\[
s(t) := \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2^{1/(t+1)}.
\]
We have \( s(t_i) > \beta \) for all \( i \geq 0 \). Hence, we have from Rudin (1976) and (3.45)
\[
\beta \leq \limsup_{i \to \infty} s(t_i) \leq \limsup_{t \to \infty} s(t) = \beta^2 < \beta, \tag{3.46}
\]
a contradiction. Thus, we conclude there exists an \( N_\beta < \infty \), such that
\[
\left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2 < \beta^{t+1}, \tag{3.47}
\]
for all \( t \geq N_\beta \). Now, we define
\[
K := \max \left\{ 1, \sup_{0 \leq t \leq N_\beta} \left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2^{\beta^{t+1}} \right\} \tag{3.48}
\]
As the supremum is taken over a finite sequence, it will exist and be finite. Hence, from (3.47) and (3.48), there exist positive constants \( K < \infty \) and \( \beta < 1 \), such that
\[
\left( \frac{\sigma^2}{\sigma^2 + \mu^2} \right)^{t+1} \left\| \left( \prod_{k=0}^{t} A(k) \right)' \left( \prod_{k=0}^{t} A(k) \right) \right\|_2 < K\beta^{t+1}. \tag{3.49}
\]
for all \( t \geq 0 \). This proves the required sufficient condition.
Remark 33 We examine the two different stability conditions derived in Theorems 30 and 32 for single input and \( N \) input case, respectively. We notice that the necessary condition for the single input case is a function of all positive Lyapunov exponents of the system; whereas, the condition for \( N \) input case is a function of only the largest positive Lyapunov exponent. Intuitively, the difference in conditions can be explained as follows. The analysis of an \( N \) state system with \( N \) inputs is similar to that of \( N \) parallel scalar systems with \( N \) parallel input channels. Thus, one derives conditions for stabilization for each individual system. The stabilization condition for each system then depends upon the Lyapunov exponent of individual system and the most restrictive of these \( N \) conditions provides the stability condition for the entire system. On the other hand, for an \( N \)-state single input system, the lone input is responsible for stabilizing all the states. The sum of positive Lyapunov exponents (or the product of exponential of the Lyapunov exponent) is equal to the entropy of a system and is a measure of the rate of expansion of the volume in the state space. For stability in a single input case, we require this expansion of open-loop dynamics be compensated by the controller. Hence, the condition for a single input case turns out to be a function of the sum of all positive Lyapunov exponents of the open-loop system.

3.4 Simulations

In this section, we present simulation results for the controller design for LTV systems in the presence of the stochastic uncertain channel for a single input system. The uncertain channel considered in the simulations, is an erasure channel modeled as a Bernoulli random variable. Although the main results of this paper provide only necessary conditions for the mean square exponential stability, the simulation results show the derived necessary condition is close to be sufficient.

3.4.1 Example 1

We consider the continuous time LTV system as described in Khalil (1996) by
\[
\dot{x}(t) = A(t)x(t) + \gamma(t)Bu(t)
\]
with \( B = [1 \ 1]' \). The eigenvalues of \( A(t) \) are located in the left-half plane at \(-0.25 \pm j0.25\sqrt{7}\) and; hence, independent of \( t \). However, the origin is exponentially
unstable. This can be verified from the state transition matrix for $A(t)$ written as follows Khalil (1996):

$$\Phi(t, 0) = \begin{pmatrix} e^{0.5t} \cos(t) & e^{-t} \sin(t) \\ -e^{0.5t} \sin(t) & e^{-t} \cos(t) \end{pmatrix}.$$ 

The state transition matrix can be used to construct a discrete time system as follows:

$$x(\Delta(t + 1)) = \Phi(\Delta(t + 1), 0)\Phi^{-1}(\Delta t, 0)\Phi(\Delta t, 0)x(0) =: A(\Delta t)x(\Delta t), \quad (3.50)$$

where $A(\Delta t) = \Phi(\Delta(t + 1), 0)\Phi^{-1}(\Delta t, 0)$. For $\Delta = 0.1$, the Lyapunov exponents of the system are computed equal to $\lambda_1 = 0.05$ and $\lambda_2 = -0.1$. The critical probability, $p^*$, is the function of the positive Lyapunov exponent and computed equal to $p^* = 1 - \frac{1}{e^{2\lambda_1}} = 0.0952$. In Figure (3.2) and Figure (3.3), we show the plots for the state norm for non-erasure probability above and below the critical value of $p^*$, respectively. The plots are obtained by averaging the state norm over 1000 different realizations of the Bernoulli random variable. A zero mean white Gaussian noise with unit variance is added to the system to visualize the mean square unstable dynamics. We see for $p = 0.09 < p^*$, the state norm fluctuates to substantially high values, while for $p = 0.11 > p^*$, the state norm stabilizes to a small band an order of magnitude smaller than the values at $p = 0.09$. The small asymptotic variance for the case of $p = 0.11 > p^*$ is due to the addition of the Gaussian noise vector, and will decrease as the noise variance is decreased. Thus, we may conclude for the controlled system to be robust to the actuation link failure uncertainty in the exponential mean square sense, the probability of non-erasure must be at least given by $p^* = 0.095$. Furthermore, the condition given by the positive Lyapunov exponent seems sufficient, as a small increase in non-erasure probability above $p^*$ shows mean square stable behavior.
Figure 3.2  State norm for non-erasure probability $p^* < p = 0.11$.

Figure 3.3  State norm for non-erasure probability $p^* > p = 0.09$. 
3.4.2 Example 2

In the next example, we choose a linear time periodic system with all Lyapunov exponents positive. The system is given by the following sets of periodic $A$ and $B$ matrices

$$A_1 = \begin{pmatrix} -0.4 & 0.8 & 1.2 \\ 1 & 0.8 & -0.4 \\ 0.6 & -0.8 & 0.4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1.6 & -1.4 & 1.2 \\ 0.8 & -1.6 & 2.8 \\ 1.6 & -2.2 & 1.2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -0.8 & 1.6 & 1.2 \\ 1.6 & -1.2 & -1.2 \\ 1.6 & -2.4 & 1.2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

This system has all Lyapunov exponents positive given by $\lambda_1 = 0.4578$, $\lambda_2 = 0.1191$, and $\lambda_3 = 0.0544$. Then, from Theorem 29 we have the critical probability as

$$p^* = 1 - \frac{1}{e^{2(\sum_i \lambda_i)}} = 0.7170.$$

![Figure 3.4 State norm for non-erasure probability above $p^* < p = 0.8170$.](image)

We add to the system some white zero mean Gaussian noise with variance $\sigma_G^2 = 0.01$. Now, we plot the norm of the state for the case with uncertainty in control for two values
of the non-erasure probability, $p = 0.8170 > p^*$ and $p = 0.6170 < p^*$. Furthermore, in case of uncertainty in control the norm has been averaged over 1000 realizations of the actuation uncertainty sequence. We clearly see the norm of the state above the critical probability, $p = 0.8170 > p^*$ in Figure 3.4, stabilizes close to zero, due to the addition of the Gaussian noise vector. In the case of the probability of non-erasure $p = 0.6170 < p^*$ Figure 3.5 less than the critical probability, the state norm fluctuates significantly as compared to the case for $p = 0.8170 > p^*$. This indicates the system is fragile to the sequence of uncertainties below the critical probability.

### 3.5 Conclusions

In this paper, we have studied the problem of control over uncertain channels between the plant and the controller for an LTV system. The results provided necessary and sufficient conditions for the feedback control system to be mean square exponentially stable. We provide computable necessary condition for the $M < N$ input case, where $N$ is the dimension of the state space. For the $N$-input case, we give a computable necessary condition, that is also shown to be sufficient. The necessary conditions are expressed in terms of the mean and variance of the stochastic channel uncertainty and the instability of the open-loop dynamics, as captured by the positive Lyapunov exponents of the open-loop system. The results in this paper generalize the existing results known in the case of LTI systems and Lyapunov exponents emerge as the
natural generalization of eigenvalues from LTI systems to LTV systems. Simulation results verify the main conclusion for the single input case for a special case of an erasure channel. While the result provides a necessary condition, our simulation results indicate this condition may also be sufficient. The proof technique presented in this paper can be extended to prove limitation results for the estimation of LTV systems over erasure channels Diwadkar and Vaidya (2011a).
CHAPTER 4. Observer Design for Nonlinear Systems Over Erasure Channels

In this chapter, we study the problem of state observation of nonlinear systems over an erasure channel. The notion of mean square exponential stability is used to analyze the stability property of observer error dynamics. The main result proves, fundamental limitation arises for mean square exponential stabilization of the observer error dynamics, expressed in terms of probability of erasure, and positive Lyapunov exponents of the system. Positive Lyapunov exponents are a measure of average expansion of nearby trajectories on an attractor set for nonlinear systems. Hence, the dependence of limitation results on the Lyapunov exponents highlights the important role played by non-equilibrium dynamics in observation over an erasure channel. The limitation on observation is also related to measure-theoretic entropy of the system, which is another measure of dynamical complexity. The limitation result for the observation of linear systems is obtained as a special case, where Lyapunov exponents are shown to emerge as the natural generalization of eigenvalues from linear systems to nonlinear systems. Finally, we prove a separation theorem for nonlinear systems under certain assumptions.

4.1 Introduction

The problem of state estimation of systems over erasure channels has attracted a lot of attention lately, given the importance of this problem in the control of systems over a network Antsaklis and Baillieul (2007). The problem of state estimation with intermittent observation was first studied in Nahi (1969); Hadidi and Schwartz (1979). In Sinopoli et al. (2003); Epstein et al. (2008), state estimation over an erasure channel with different performance metrics on the error covariance is studied. In Sinopoli et al. (2003), under some assumptions on system
dynamics, it is proved that there exists a critical non-erasure probability below which the error covariance is unbounded. A Markov jump linear system framework is used to model the state estimation problem with intermittent measurement and to provide conditions for the convergence of error covariance in Costa (2002). In Smith and P.Seiler (2003), state estimation over erasure channel with Markovian packet loss is studied. However, all the above results are developed for linear time invariant (LTI) systems. There is no systematic result that addresses the state estimation problem for nonlinear systems over erasure channels. Thus there is a need for extension and development of such results for nonlinear systems, with regard to their applications in network systems consisting of nonlinear components, such as power system networks, biological networks, and Internet communication networks. The two important reasons for developing fundamental limitation results for control systems with unreliable interconnection are

1. All systems in nature evolve with nonlinear dynamics and it is not always easy to approximate them as linear systems.

2. The global characteristics of nonlinear systems cannot be predicted by the study of local equilibrium eigenvalues.

We will mainly focus on the second reason and aim to discover the relationship between the fundamental limitations and indicators of global characteristics of a nonlinear system.

We study the problem of state observation of nonlinear systems over an erasure channel, with the objective to develop limitation results for state observation. We expect the limitation results for the state observation problem, to provide useful insight into the more challenging problem of state estimation over an erasure channel. The erasure channel is modeled as an on/off Bernoulli switch. We use mean square exponential (MSE) stability to study the state observation problem over an erasure channel. The main result shows, that a fundamental limitation arises in MSE stabilization of the observer error dynamics. This limitation is expressed in terms of erasure probability and global instability of the nonlinear system. In particular, under a certain ergodicty assumption, we show the instability of a nonlinear system can be expressed in terms of the sum of positive Lyapunov exponents of the system. Using Ruelle's
inequality from ergodic theory of a dynamical system Walters (1982), the sum of the positive Lyapunov exponents can be related to the entropy of a nonlinear system. Hence, the limitation result can be interpreted in terms of the entropy of a nonlinear system. Our result involving Lyapunov exponents of a non-trivial (other than equilibrium point) invariant measure is also the first to highlight the important role played by the non-equilibrium dynamics in the limitations on nonlinear observation.

There are two main contributions of this result. First, it adopts and extends the formalism from ergodic theory of random dynamical systems to study the problem of nonlinear observation over an erasure channel. Second, the result provides an analytical relationship between the maximum tolerable channel uncertainty (i.e., the maximum erasure probability) and the inability of the system to maintain mean square exponential stability of the observer error dynamics.

4.2 Preliminaries

4.2.1 Preliminaries for observation problem

The set-up for nonlinear observations with a unique erasure channel at the output is described by the following equations:

\[ x_{t+1} = f(x_t), \quad y_t = \xi_t h(x_t), \]  

where \( x_t \in X \subseteq \mathbb{R}^N \) is the state, \( y_t \in Y \subseteq \mathbb{R}^M \) is the output, and \( \xi_t \in \{0, 1\} \) is a Bernoulli random variable with probability distribution \( \text{Prob}(\xi_t = 1) = p \) for all \( t \geq 0 \), with \( 0 < p < 1 \), and independent of \( \xi_\tau \) for \( \tau \neq t \). The IID (independent identically distributed) random variable, \( \xi_t \), models the erasure channel between the plant and the observer through which all the outputs are sent to the observer simultaneously.

Remark 34 To make the problem interesting, we assume that \( M < N \) and \( 0 < p < 1 \). The \( 0 < p \) assumption implies that the system dynamics, \( x_{t+1} = f(x_t) \), is unstable and hence requires some non-zero probability of erasure for the observer to work.
We now provide the following definition of an observability rank condition for nonlinear systems Nijmeijer (1982).

**Definition 35 (Observability Rank Condition)** Consider the map \( \theta^{N-1}(x) : X \to Y \times \ldots \times Y \)

\[
\theta^{N-1}(x) := (h(x), h(f(x)), \ldots, h(f^{N-1}(x))').
\]  

(4.2)

The system (4.1) is said to satisfy the observability rank condition at \( x \), if

\[
\text{rank} \left( \frac{\partial \theta^{N-1}(x)}{\partial x} \right) = N.
\]

We make following assumption on the system dynamics.

**Assumption 36** The system mapping, \( f \), and output function, \( h \), are \( C^r \) functions of \( x \), for \( r \geq 1 \), with \( f(0) = 0 \), \( h(0) = 0 \), and the Jacobian \( \frac{\partial f}{\partial x}(x) \) is uniformly bounded above and below for all \( x \in X \). Furthermore, the system satisfies the observability rank condition (Definition 35) and there exist \( \alpha_\theta > 0 \) and \( \beta_\theta > 0 \), such that

\[
\alpha_\theta I_N < \frac{\partial \theta^{N-1}(x)}{\partial x} \frac{\partial \theta^{N-1}}{\partial x}(x) < \beta_\theta I_N \]

(4.3)

for all \( x \in X \) and, \( I_N \) is the \( N \times N \) Identity matrix.

**Remark 37** Assumption 36 and in particular the observability rank condition are essential for the observer design for the system with no erasure at the output.

The stochastic notion of stability we use to analyze the observer error dynamics is defined in the context of a general random dynamical system (RDS) of the form \( x_{t+1} = S(x_t, \zeta_t) \), where \( x_t \in X \subseteq \mathbb{R}^N \), \( \zeta_t \in W = \{0, 1\} \) for \( t \geq 0 \), are IID random variables with probability distribution \( \text{Prob}(\zeta_t = 1) = p \). The system mapping \( S : X \times W \to X \) is assumed to be at least \( C^1 \) with respect to \( x_t \in X \) and measurable w.r.t \( \zeta_t \). We assume \( x = 0 \) is an equilibrium point, i.e., \( S(0, \zeta_t) = 0 \). The following notion of stability can be defined for RDS Has’minskiı (1980); Applebaum and Siakalli (2009).
Definition 38 (Mean Square Exponential (MSE) Stable) The solution, $x = 0$, is said to be MSE stable for $x_{t+1} = S(x_t, \zeta_t)$, if there exist positive constants $L < \infty$ and $\beta < 1$, such that
\[
E_{\zeta_0} [\| x_{t+1} \|^2] \leq L \beta^t \| x_0 \|^2, \quad \forall t \geq 0
\]
for Lebesgue almost all initial condition, $x_0 \in X$, where $E_{\zeta} [\cdot]$ is the expectation taken over the sequence $\{\zeta_0, \ldots, \zeta_t\}$.

4.3 Main results

We will first give the results for the individual problems of observer design and full state feedback stabilization. Later we will give a separation theorem for nonlinear systems under certain assumptions which will help us derive fundamental limitations for the observer based controller problem.

4.3.1 Observation over erasure channels

The main results of this section are derived under the following assumption on the observer dynamics.

Assumption 39 The observer gain, $K$, is assumed deterministic and not an explicit function of the channel erasure state $\xi_t$ nor its history (i.e., $\xi_t^{t-1}$). The observer dynamics is assumed to be of the form:
\[
\hat{x}_{t+1} = f(\hat{x}_t) + K(y_t) - K(\hat{y}_t), \quad \hat{y}_t = \xi_t h(\hat{x}_t),
\]
where $\hat{x} \in X$ is the observer state, $\hat{y} \in Y$ is the observer output, and $K: Y \rightarrow X$ is the observer gain and assumed to be a $C^r$ function of $y$, for $r \geq 1$, and satisfies $K(0) = 0$. Thus the property $K(0) = 0$ and $\xi_t \in \{0, 1\}$, allows us to rewrite the observer dynamics (6.14) as follows:
\[
\hat{x}_{t+1} = f(\hat{x}_t) + \xi_t K(h(x_t)) - \xi_t K(h(\hat{x}_t)).
\]
We assume that the observer output $\hat{y}_t$ is an explicit function of channel state, $\xi_t$. This assumption is justified by assuming a TCP-like protocol, where the observer receives an immediate acknowledgement of the channel erasure state Sinopoli et al. (2003).
Remark 40 In Sinopoli et al. (2003), the problem of state estimation for an LTI system over an erasure channel is studied. The optimal estimator gain that minimizes the error covariance is shown to be a function of the channel erasure state history. With the estimator gain, a function of the channel erasure state history, the results in Sinopoli et al. (2003) only prove the error covariance will remain bounded and not converge to a steady state value, unlike the regular Kalman filtering problem for an LTI system with no loss of measurement. Hence, we conjecture (Assumption 39) on the observer gain, not being a function of the channel erasure state history, the results in Sinopoli et al. (2003) only prove the error covariance will remain bounded and not converge to a steady state value, unlike the regular Kalman filtering problem for an LTI system with no loss of measurement. Hence, we conjecture (Assumption 39) on the observer gain, not being a function of the channel erasure state or its history, is necessary for the error dynamics to be MSE stable.

We first prove Lemma 41 that provides a necessary condition for MSE stability of the error dynamics $e_t = x_t - \hat{x}_t$ in terms of MSE stability of the linearized error dynamics.

Lemma 41 Consider the observer dynamics in Eq. (4.5) and let the error dynamics (i.e., $e_t = x_t - \hat{x}_t$) be MSE stable (Definition 38). Then, the following linearized error dynamics,

$$
\eta_{t+1} = \left( \frac{\partial f}{\partial x}(x_t) - \xi_t \frac{\partial K \circ h}{\partial x}(x_t) \right) \eta_t, \quad x_{t+1} = f(x_t)
$$

(4.6)

is also MSE stable, i.e., there exist positive constants $L < \infty$ and $\beta < 1$, such that $E_{t_0} \left[ \|\eta_{t+1}\|^2 \right] \leq L \beta^t \|\eta_0\|^2 \quad \forall t \geq 0$. The functions $K$ and $h$ in (4.6) are the observer gain and output function, respectively, from Eq. (6.14).

Proof. Define $g(x_t, \xi_t) := f(x_t) - \xi_t K(h(x_t))$ and $A(x_t, \xi_t) := \frac{\partial g}{\partial x}(x_t, \xi_t)$. Then using Mean Value Theorem for the vector valued function, the error dynamics, can be written as

$$
e_{t+1} = g(x_t, \xi_t) - g(x_t - e_t, \xi_t) = \left( \int_0^1 \frac{\partial g}{\partial x}(x_t - s e_t, \xi_t) ds \right) e_t = \prod_{k=0}^{t} \left( \int_0^1 A(x_k - s e_k, \xi_k) ds \right) e_0,
$$

Here $e_t$ is an implicit function of the initial error $e_0$, initial state $x_0$, and the sequence of uncertainties $\xi_{0}^{t-1}$. We define $B_k(x_0, \xi_{0}^t, e_0) := \int_0^1 A(x_k - s e_k, \xi_k) ds$ and $B'_0(x_0, \xi_{0}^t, e_0) := \prod_{k=0}^{t} B_k(x_0, \xi_{0}^t, e_0)$. This gives

$$
E_{t_0} \left[ \|e_{t+1}\|^2 \right] = E \left[ e'_{t+1} e_{t+1} \right] = e'_0 E_{t_0} \left[ B'_0(x_0, \xi_{0}^t, e_0) B'_0(x_0, \xi_{0}^t, e_0) \right] e_0.
$$

Using Assumption 36, we know there exists a positive constant $\bar{L} < \infty$, such that $\|B_k(x_0, \xi_{0}^k, \alpha e_0)\| < \bar{L}$ for Lebesgue almost all $x_0 \in X$ and for some scalar, $\alpha > 0$. Let $B_k(x_0, \xi_{0}^k, \alpha e_0)_{ij}$ denote the $i^{th}$ row $j^{th}$ column entry in $B_k(x_0, \xi_{0}^k, \alpha e_0)$. Now consider a sequence, $\{\alpha_l\}_{l=1}^{\infty}$, such
that $\lim_{l \to \infty} \alpha_l = 0$. Then, we have by Dominated Convergence Theorem Folland (1999) and continuity of $A(x_k - se_k, \xi_k)$, $\lim_{l \to \infty} B_k(x_0, \xi_0^k, \alpha_l e_0)_{ij} = B_k(x_0, \xi_0^k, 0)_{ij}$ which implies $\lim_{l \to \infty} B_k(x_0, \xi_0^k, \alpha_l e_0) = B_k(x_0, \xi_0^k, 0)$. Hence, we have

$$\lim_{l \to \infty} B_0^k(x_0, \xi_0^k, \alpha_l e_0) = B_0^k(x_0, \xi_0^k, 0). \quad (4.7)$$

From MSE stability of the error, we obtain $e_0' E_{\xi_0} [B_0^t(x_0, \xi_0^t, e_0)' B_0^t(x_0, \xi_0^t, 0)] e_0 \leq L \beta' e_0' e_0$, for some positive constants $L < \infty$ and $\beta < 1$. Since the above inequality is true for any initial error, this will be true if the initial error vector used to compute the product of matrices is scaled by $\alpha_l$, where $\lim_{l \to \infty} \alpha_l = 0$. Substituting $\alpha_l e_0$ for $e_0$, we can write

$$e_0' E_{\xi_0} [B_0^t(x_0, \xi_0^t, \alpha_l e_0)' B_0^t(x_0, \xi_0^t, \alpha_l e_0)] e_0 \leq L \beta' e_0' e_0. \quad (4.8)$$

Now, letting $l \to \infty$ and by Fatou’s Lemma, we have

$$e_0' E_{\xi_0} \left[ \lim_{l \to \infty} B_0^t(x_0, \xi_0^t, \alpha_l e_0)' B_0^t(x_0, \xi_0^t, \alpha_l e_0) \right] e_0 \leq \lim_{l \to \infty} e_0' E_{\xi_0} [B_0^t(x_0, \xi_0^t, \alpha_l e_0)' B_0^t(x_0, \xi_0^t, \alpha_l e_0)] e_0 \leq L \beta' e_0' e_0. \quad (4.8)$$

Thus, using (4.7) and (4.8), we obtain $e_0' E_{\xi_0} [B_0^t(x_0, \xi_0^t, 0)' B_0^t(x_0, \xi_0^t, 0)] e_0 \leq L \beta' e_0' e_0$, where $B_0^t(x_0, \xi_0^t, 0)$ is the product of the Jacobian matrices $A(x_k, \xi_k)$, with zero initial error and computed along the nominal trajectory, $x_{t+1} = f(x_t)$. Hence,

$$E_{\xi_0} \left[ e_0' \left( \prod_{k=0}^t A(x_k, \xi_k) \right)' \left( \prod_{k=0}^t A(x_k, \xi_k) \right) e_0 \right] \leq L \beta' e_0' e_0.$$

Since the matrices in the above equation are independent of $e_0$, we can substitute $\eta_0$ for $e_0$. Now, using the evolution of $\eta_t$ from Eq. (4.6), we obtain the desired result. □

Our next theorem provides the necessary condition for MSE stability of the linearized error dynamics.

**Theorem 42** Let the $\eta_t$ dynamics for the system (4.6) be MSE stable (Definition 38). Then, there exists a matrix function of $x_t$, $P(x_t)$, such that $\gamma_1 I \leq P(x_t) \leq \gamma_2 I$ and

$$E_{\xi_t} [A'(x_t, \xi_t) P(x_{t+1}) A(x_t, \xi_t)] < P(x_t), \quad (4.9)$$

for some positive constants $\gamma_1, \gamma_2$, where $x_{t+1} = f(x_t)$ and $A(x_t, \xi_t) = \frac{\partial f}{\partial x}(x_t) - \xi_t \frac{\partial K}{\partial y}(h(x_t)) \frac{\partial h}{\partial x}(x_t)$ from (4.6).
Proof. To prove the necessary part, assume the system is MSE stable and consider the following construction of $P(x_t)$.

$$P(x_t) = \sum_{k=t}^{\infty} E_{\xi_t} \left[ \left( \prod_{j=t}^{k} A(x_j, \xi_j) \right) \left( \prod_{j=t}^{k} A(x_j, \xi_j) \right)' \right],$$

where $E_{\xi_t} [\cdot]$ is the expectation over the random sequence $\{\xi_t, \ldots, \xi_j\}$. The existence of positive constants $\gamma_1, \gamma_2$ follows from the fact that $\eta_t$ dynamics is MSE stable and the Jacobian $\frac{\partial f}{\partial x}$ is bounded from above and below. The inequality (4.9) follows from construction of $P(x_t)$. 

We have Corollary 43 to the Theorem 42.

Corollary 43 Let the RDS (4.6) be MSE stable. Then, there exists a matrix function of $x_t$, $Q(x_t)$ and positive constants $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$, such that

$$\tilde{\gamma}_1 I \leq Q(x_t) \leq \tilde{\gamma}_2 I, \quad E_{\xi_t} [A(x_t, \xi_t) Q(x_t) A'(x_t, \xi_t)] < Q(x_{t+1}). \quad (4.10)$$

Proof. The proof follows from Theorem 42 and by constructing $Q(x_t) = P(x_t)^{-1}$. 

Remark 44 We will refer to matrix $Q(x_t)$, satisfying the conditions (4.10) of Corollary 43 as the matrix Lyapunov function.

Our goal is to derive a necessary condition for the MSE stability of the linearized error dynamics; thereby, providing a necessary condition for MSE stability of the true error dynamics.

Lemma 45 The necessary condition for exponential mean square stability of the linearized error dynamics (4.6) is given by

$$(1 - p)^M (\det(A(x_t)))^2 \frac{\det(Q_0(x_t))}{\det(Q_0(x_{t+1}))} < 1, \quad (4.11)$$

for Lebesgue almost all $x_t \in X$. In (4.11) $Q_0(x_t)$ is a solution of the following Riccati equation,

$$Q_0(x_{t+1}) = R(x_t) + A(x_t) Q_0(x_t) A'(x_t)$$

$$- A(x_t) Q_0(x_t) C'(x_t) \left( I_M + C(x_t) Q_0(x_t) C'(x_t) \right)^{-1} C(x_t) Q_0(x_t) A'(x_t), \quad (4.12)$$

where $R(x_t) \geq 0$ is some symmetric positive semi-definite matrix. Furthermore, $Q_0(x_t)$ is uniformly bounded above and below with $A(x_t) := \frac{\partial f}{\partial x}(x_t)$, $C(x_t) := \frac{\partial h}{\partial x}(x_t)$, $x_{t+1} = f(x_t)$, $I_M$ is $M \times M$ identity matrix, and $(1 - p)$ is the probability of erasure.
Proof. Using the result of Corollary 43, the necessary condition for MSE stability of (4.6) can be expressed in terms of the existence of \( \tilde{\gamma}_1 I \leq Q(x_t) \leq \tilde{\gamma}_2 I \), such that \( \tilde{\gamma}_1, \tilde{\gamma}_2 > 0 \) and,

\[
E_{\xi_t} \left[ A(x_t, \xi_t) Q(x_t) A'(x_t, \xi_t) \right] < Q(x_{t+1}),
\]

where \( A(x_t, \xi_t) = A(x_t) - \xi_t \tilde{K}(x_t) C(x_t) \) and \( \tilde{K}(x_t) := \frac{\partial K}{\partial y}(h(x_t)) \). Minimizing trace of the left-hand side of (4.13) with respect to \( \tilde{K}(x_t) \), we obtain \( \tilde{K}^*(x_t) = A(x_t)Q(x_t)C'(x_t) (C(x_t)Q(x_t)C'(x_t))^{-1} \)

and \( Q(x_t) \) to satisfy

\[
A(x_t)Q(x_t)A'(x_t) - pA(x_t)Q(x_t)C'(x_t) (C(x_t)Q(x_t)C'(x_t))^{-1} C(x_t)Q(x_t)A'(x_t) < Q(x_{t+1}).
\]

(4.14)

It is important to notice that the inequality (4.14) is independent of any positive scaling i.e., if \( Q(x_t) \) satisfies the above inequality then \( cQ(x_t) \) also satisfies the above inequality for any positive constant \( c \). Since \( Q(x_t) \) is a matrix Lyapunov function and hence lower bounded, it follows from Remark 34, that there exists a positive constant \( \Delta > 0 \) such that \( C(x_t)Q(x_t)C'(x_t) (\frac{1-p}{p}) \geq \Delta I_M \). Hence (4.14) implies following inequality to be true

\[
A(x_t)Q(x_t)A'(x_t) - A(x_t)Q(x_t)C'(x_t) (\Delta I_M + C(x_t)Q(x_t)C'(x_t))^{-1} C(x_t)Q(x_t)A'(x_t) < Q(x_{t+1}).
\]

(4.15)

Now define \( Q_0(x_t) := \frac{1}{\tilde{\gamma}^2} Q(x_t) \), then using the fact that (4.15) is independent of positive scaling, we obtain following inequality for \( Q_0(x_t) \)

\[
A(x_t)Q_0(x_t)A'(x_t) - A(x_t)Q_0(x_t)C'(x_t) (I_M + C(x_t)Q_0(x_t)C'(x_t))^{-1} C(x_t)Q_0(x_t)A'(x_t) < Q_0(x_{t+1}).
\]

(4.16)

Inequality (4.16) implies there exists \( R(x_t) \geq 0 \), such that the following equality is true.

\[
Q_0(x_{t+1}) = R(x_t) + A(x_t)Q_0(x_t)A'(x_t)
\]

\[
- A(x_t)Q_0(x_t)C'(x_t) (I_M + C(x_t)Q_0(x_t)C'(x_t))^{-1} C(x_t)Q_0(x_t)A'(x_t).
\]

(4.17)

For any fixed trajectory \( \{x_t\} \) generated by the system, \( x_{t+1} = f(x_t) \), the above equality resembles the Riccati equation obtained for the minimum covariance estimator design problem
for the linear time varying system, where the matrices $Q_0(x_t)$ and $R(x_t)$ can be identified with the error and input noise covariance matrices, respectively Kwakernaak and Sivan (1972) with output noise variance matrix equal to identity matrix. The difference between the regular Riccati equation obtained from the minimum variance estimator problem for the linear time varying system and Eq. (4.17) is that, the various matrices appearing in (4.17) are parameterized by $x_t$ instead of time. Furthermore $Q_0(x_t)$ as the solution of Riccati-like equation (4.17) is both bounded above and below and is proved as follows. The system matrices $A(x_t)$ and $C(x_t)$ satisfy Assumption 36 along any given trajectory. Hence, the linearized system, \[ \eta_{t+1} = A(x_t)\eta_t, \zeta_t = C(x_t)\eta_t, \] along any fixed trajectory is uniformly completely reconstructible as defined in Kwakernaak and Sivan (1972) (Definition 6.6). It then follows from Jazwinski (2007) (Lemmas 7.1 and 7.2) that the covariance matrix $Q_0(x_t)$ is uniformly bounded above and below for all $x \in X$. The matrix $Q_0(x_t)$ satisfies (4.14) follows from the definition of $Q_0(x_t)$ (i.e., $Q_0(x_t) := \frac{1}{\Delta} Q(x_t)$) and the fact that (4.14) is independent of positive scaling. We obtain,

\[
A(x_t)Q_0(x_t)A'(x_t) - pA(x_t)Q_0(x_t)C'(x_t)(C(x_t)Q_0(x_t)C'(x_t))^{-1}C(x_t)Q_0(x_t)A'(x_t)
\]

\[< Q_0(x_{t+1}).\]

This proves that $Q_0(x_t)$ obtained as a solution of Riccati-like equation is a valid matrix Lyapunov function. To derive the required necessary condition (4.11), we take determinants on both sides of (4.18) to obtain

\[
\left(\det(A(x_t))\right)^2 \frac{\det(Q_0(x_t))}{\det(Q_0(x_{t+1}))} \det \left(I_N - pC'(x_t)(C(x_t)Q_0(x_t)C'(x_t))^{-1}C(x_t)Q_0(x_t)\right) < 1.\]  

(4.19)

By Sylvester’s determinant Theorem (i.e., $\det(I_N + GJ) = \det(I_M + JG)$, $G \in \mathbb{R}^{N \times M}, J \in \mathbb{R}^{M \times N}$), we obtain

\[
\det \left(I_N - pC'(x_t)(C(x_t)Q_0(x_t)C'(x_t))^{-1}C(x_t)Q(x_t)\right) = (1 - p)^M.
\]  

(4.20)

We obtain the required inequality (4.11) by combining Eqs. (4.19) and (4.20).

Before we proceed further we would like to remark on the nature of the Riccati equation obtained in Lemma 45 and its connection with the Riccati equation for the minimum variance estimator from Linear Time Varying system theory.
Remark 46 In the case of nonlinear systems, let us assume that the system (4.1) is such that it satisfies the observability rank condition, and it satisfies Assumption 36. Furthermore, consider the tangent space dynamics for the open loop system (4.1) given by

\[ \eta_{t+1} = \frac{\partial f}{\partial x} \eta_t : = A(x_t)\eta_t \]  
\[ \chi_t = \frac{\partial h}{\partial x} \eta_t : = C(x_t)\eta_t \]  

(4.21) (4.22)

Then we will obtain that the system evolving on the tangent space (4.21), for the nonlinear system (4.1), satisfies the uniform reconstructability condition Kwakernaak and Sivan (1972) along Lebesgue almost all trajectories of the free dynamics.

In that case we can define a linear time varying system along each trajectory of the nonlinear system using the Jacobian matrices, for which we may now define a minimum variance estimator as follows,

**Definition 47 (Minimum variance estimator for tangent space)** Consider the system

\[ x_{t+1} = f(x_t), \ y_t = h(x_t) \]
\[ \eta_{t+1} = A(x_t)\eta_t + w_t \]
\[ \chi_t = C(x_t)\eta_t + v_t \]  

(4.23)

where \( A(x_t) := \frac{\partial f}{\partial x}(x_t), C(x_t) := \frac{\partial h}{\partial x}(x_t) \), \( w_t \) and \( v_t \) are the white Gaussian noise for the states and measurement process respectively. Let \( Q_t = Q'_t \geq 0 \) and \( R_t = R'_t \) be the noise covariance matrices for the state and measurement noise respectively. Consider an estimator of the form

\[ \hat{\eta}_{t+1} = A(x_t)\hat{\eta}_t + K(x_t)(\chi_t - C(x_t)\hat{\eta}_t) \]

Design the optimal gain matrices \( K(x_t) \), that minimizes the error variance

\[ E[(\eta_t - \hat{\eta}_t)'(\eta_t - \hat{\eta}_t)]. \]  

(4.24)

**Lemma 48** Consider the problem of minimum variance estimator problem for the tangent space dynamics (Definition 47). The optimal gain matrix \( K(x_t) \) that minimizes the error
variance given in (4.24) is given by
\[ K(x_t) = \frac{A(x_t)P(x_t)C'(x_t)}{C(x_t)P(x_t)C'(x_t) + R_t} \] (4.25)
\[ P(x_{t+1}) = A(x_t)P(x_t)A'(x_t) + Q_t - \frac{A(x_t)P(x_t)C'(x_t)C(x_t)P(x_t)A'(x_t)}{C(x_t)P(x_t)C'(x_t) + R_t} \] (4.26)
for all \( t > 0 \), where \( P(x_t) = E[(\eta_t - \hat{\eta}_t)(\eta_t - \hat{\eta}_t)^\top] \) is the linearized error covariance matrix with initial condition \( P(x_0) = I \) the \( N \times N \) identity matrix.

**Proof.** Consider a trajectory of the nonlinear system \( x_{t+1} = f(x_t) \) given by \( \{x_t\}_{t=0}^\infty \). Along this trajectory the linear system
\[ \eta_{t+1} = A_t \eta_t + w_t \]
\[ \chi_t = C_t \eta_t + v_t \]
where \( A_t := A(x_t) \) and \( C_t = C(x_t) \), is exactly like an linear time varying (LTV) system. Results for minimum variance estimator for a an LTV system with additive white gaussian noise are well known in literature Kwakernaak and Sivan (1972). Hence the proof for this theorem follows along the lines of LTV systems proof as given in Kwakernaak and Sivan (1972). We see that the Riccati equation obtained in Lemma 45 can be viewed as a minimum variance estimator equation for the LTV system along any trajectory. The results of Lemma 45 will now be used to prove the main results under various assumptions on the system dynamics.

**Theorem 49 (Linear Systems)** Let \( f(x) = Ax \) with \( x \in \mathbb{R}^N \) and \( h(x) = Cx \in \mathbb{R}^M \). Assume that all eigenvalues \( \lambda_k \) for \( k = 1, \ldots, N \) of \( A \) have absolute value greater than one. The necessary condition for the observer error dynamics to be MSE stable is given by
\[ (1 - p)^M \left( \prod_{k=1}^N |\lambda_k| \right)^2 < 1. \] (4.27)

**Proof.** For the linear system, the solution of Riccati-like equation (4.12) from Lemma 45 leads to a constant matrix \( Q_0 \) independent of \( x_t \). Hence the necessary condition (4.11) for the stability will reduce to
\[ (1 - p)^M \det(A^2) < 1. \]
The required necessary condition (4.27) then follows by substituting \( \det(A^2) = \left( \prod_{k=1}^N |\lambda_k| \right)^2 \).
Remark 50 A careful examination of the proofs for Lemma 41 and 45, and Theorem 42 for the special case of linear systems with single output, reveals the necessary condition (4.27) is also sufficient for MSE stability of the linear system.

Theorem 51 (Nonlinear systems on unbounded space) Consider system (4.1) with system mapping $f$ and output $h$ satisfying Assumption 36 and state space $X$ possibly unbounded. The necessary condition for MSE stability of the observer error dynamics (6.14) is given by

$$(1 - p)^M \left( \frac{\det(Q_0(x_t))}{\det(Q_0(x_{t+1}))} \right)^2 < 1,$$

for Lebesgue almost all $x \in X$, \hspace{1cm} (4.28)

where $A(x) = \frac{\partial f}{\partial x}(x)$ and $Q_0(x)$ satisfy the Riccati-like Eq. (4.12).

Proof. The proof follows by combining results from Lemmas 41 and 45, and Theorem 42. \hspace{1cm} □

In Theorem 56, we show, for a nonlinear system evolving on a compact state space, the term $(\det(A(x)))^2 \frac{\det(Q(x_t))}{\det(Q(x_{t+1}))}$ from (4.28) relates to the sum of postive Lyapunov exponents of the system. For Theorem 56 we provide the following definitions Mane (1987).

Definition 52 (Physical measure) Let $\mathcal{M}(X)$ be the space of probability measures on $X$. A measure $\mu \in \mathcal{M}(X)$ is said to be invariant for $x_{t+1} = f(x_t)$ if $\mu(f^{-1}(B)) = \mu(B)$ for all sets $B \in \mathcal{B}(X)$ (Borel $\sigma$-algebra generated by $X$). An invariant probability measure, $\mu$, is said to be ergodic if any continuous bounded function $\varphi$ that is invariant under $f$, i.e., $\varphi(f(x)) = \varphi(x)$, is $\mu$ almost everywhere constant. Ergodic invariant measure, $\mu$, is said to be physical if $\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \varphi(f^k(x)) = \int_X \varphi(x) d\mu(x)$ for positive Lebesgue measure of the initial condition $x \in X$ and all continuous function $\varphi : X \to \mathbb{R}$.

Definition 53 (Lyapunov exponents) For a deterministic system $x_{t+1} = f(x_t)$, let

$$\Lambda(x_0) = \lim_{t \to \infty} \left( \frac{D_x f(x_0)^t D_x^t f(x_0)}{t!} \right),$$

(4.29)

where $D_x f(x) = \frac{\partial f}{\partial x}(x)$ and $D_x f(x_0) := D_x f(x_t) \cdots D_x f(x_0)$. Let $\lambda^i_{\text{exp}}$ for $i = 1, \ldots, N$ be the eigenvalues of $\Lambda(x_0)$, such that $\lambda^1_{\text{exp}} \geq \lambda^2_{\text{exp}} \geq \cdots \geq \lambda^N_{\text{exp}}$. Then, the Lyapunov exponents $\Lambda^i_{\text{exp}}$ are defined as $\Lambda^i_{\text{exp}} = \log \lambda^i_{\text{exp}}$ for $i = 1, \ldots, N$. Furthermore, if $\det(\Lambda(x_0)) \neq 0$, then

$$\lim_{t \to \infty} \frac{1}{t} \log |\det(D_x^t f(x_0))| = \log \prod_{k=1}^{N} \lambda^k_{\text{exp}}(x).$$

(4.30)
Remark 54 The technical conditions for the existence of limits in (4.29) and (4.30) are provided by the Multiplicative Ergodic Theorem Ruelle (1979) (Theorem 1.6), Walters (1982) (Theorem 10.4), Eckman and Ruelle (1985) (Section D). The limits in (4.29) and (4.30) are known to be independent of the initial condition and are unique under the assumption of unique ergodic invariant measure for system dynamics. For a compact state space, the existence of an invariant measure is always guaranteed Walters (1982) (Corollary 6.9.1). Furthermore, every invariant measure admits ergodic decomposition Walters (1982) (Remarks pp. 153), Mane (1987) (Theorem 6.4). We now make Assumption 55 on the system dynamics.

Assumption 55 We assume the nonlinear system, \( x_{t+1} = f(x_t) \), has a unique physical measure with all Lyapunov exponents positive.

The assumption of a unique physical measure is not restrictive and it allows us to prove the main result in Theorem 56, that is independent of initial conditions. With ergodic invariant measures that are guaranteed to exist (Remark 54), the main result in Theorem 56 will be a function of a particular ergodic measure under consideration. The assumption of all Lyapunov exponent being positive is analogous to the assumption made in the LTI case that all eigenvalues are positive. We verify through simulation results in section 4.4 that the result of Theorem 56 also applies to the case where one of the Lyapunov exponent is negative.

Theorem 56 (Nonlinear systems on compact space) Consider the system (4.1) with system mapping \( f \) and output \( h \) satisfying Assumptions 36 and 55 and state space \( X \) compact. The necessary condition for MSE stability of the observer error dynamics (6.14) is given by

\[
(1 - p)^M \left( \prod_{k=1}^{N} \lambda_{exp}^k \right)^2 < 1, \tag{4.31}
\]

where \( \lambda_{exp}^k = e^{\Lambda_{exp}^k} \), and \( \Lambda_{exp}^k \) is the \( k \)th positive Lyapunov exponent of \( x_{t+1} = f(x_t) \).

Proof. We follow the notations from Lemma 45. The necessary condition for MSE stability (Eq. 4.11) is true for almost all points \( x \in X \), and, hence in particular for \( x_t \) evaluated along the system trajectory \( x_{t+1} = f(x_t) \). Evaluating (4.11) along the system trajectory and taking
the product, we write the necessary condition as
\[
((1 - p)^M)^n \det(Q_0(x_0)Q_0^{-1}(x_{n+1})) \prod_{t=1}^{n} \det(A(x_t))^2 < 1.
\]

Taking time average for the log of the expression and in the limit as \( n \to \infty \), we obtain the following necessary condition for MSE stability,
\[
\lim_{n \to \infty} \frac{1}{n} \log \left( ((1 - p)^M)^n \det(Q_0(x_0)Q_0^{-1}(x_{n+1})) \prod_{t=1}^{n} \det(A(x_t))^2 \right) < 0. \quad (4.32)
\]

Using the fact that both \( Q_0(x_t) \) and \( Q_0^{-1}(x_t) \) are almost always uniformly bounded and using (4.30) from Definition 53, (4.32) gives the required necessary condition (4.31) for MSE stability.

\[ \blacksquare \]

**Remark 57** The necessary condition for MSE stability in Theorems 49, 51, and 56 for single input case is tighter however for \( 1 < M < N \), we expect the condition to be improved further.

The necessary condition for MSE stability from our main results provides a critical dropout rate, i.e., the erasure probability, \( q^* = 1 - p^* \), above which the system is guaranteed MSE unstable.

In particular, the critical dropout rate for a nonlinear system with single output, evolving on compact space from Theorem 56 is given by \( q^* = \left( \prod_{k=1}^{N} \lambda_k \right)^{-2} \).

### 4.3.2 Entropy and limitation for observation

Measure-theoretic entropy, \( H_\mu(f) \), for the dynamical system, \( x_{n+1} = f(x_n) \), is associated with a particular ergodic invariant measure, \( \mu \), and is another measure of dynamical complexity. While the measure-theoretic entropy counts the number of typical trajectories for their growth rate, the positive Lyapunov exponents measure the rate of exponential divergence of nearby system trajectories. For more details on entropy refer to Walters (1982). These two measures of dynamical complexity are related by Ruelle’s inequality.

**Theorem 58 (Ruelle’s Inequality)** (Eckman and Ruelle (1985) Eq. 4.4); (Ruelle (1978) Theorem 2) Let \( x_{n+1} = f(x_n) \) be the dynamical system, \( f : X \to X \) be a \( C^r \) map, with \( r \geq 1 \), of a compact metric space \( X \) and \( \mu \) an ergodic invariant measure. Then,
\[
H_\mu(f) \leq \sum_{k} (\Lambda_k^{\exp})^+, \quad (4.33)
\]
where $a^+ = \max\{0, a\}$, $H_\mu(f)$ is the measure-theoretic entropy corresponding to the ergodic invariant measure $\mu$, and $\Lambda_{\text{exp}}^k$ are the Lyapunov exponents of the system.

The Ruelle inequality (4.33) can be used to relate the limitation for observation with system entropy.

**Theorem 59** Consider the system (4.1) with system mapping $f$ and output $h$ satisfying Assumptions 36 and 55 and state space $X$ compact. The necessary condition for MSE stability of the observer error dynamics (6.14) is given by

$$M \log(1 - p) + 2H_\mu(f) < 0$$ (4.34)

where $\mu$ is the physical invariant measure of $f$ (Definition 52 and Assumption 55) and $H_\mu(f)$ is the measure-theoretic entropy corresponding to measure $\mu$.

**Proof.** The proof follows by applying the results of Theorems 56 and 58.

### 4.4 Simulation

**Henon map** is one of the widely studied examples of two-dimensional chaotic maps. The small random perturbation of a two-dimensional Henon map is described by following equations:

$$x_{1t+1} = 1 - ax_{1t}^2 + x_{2t} + r_{1t}, \quad x_{2t+1} = bx_{1t} + r_{2t}, \quad y_t = \xi_t x_{1t},$$ (4.35)

where $a = 1.4$, $b = 0.3$ are constant parameters, and $r_{it} \in [0, 1E-6]$, $i \in \{1, 2\}$, are uniform random variables. The small amount of external noise, $r_{it}$, is essential to see the effect of mean square instability. The system has Lyapunov exponents given by $\lambda_1 = 0.426$ and $\lambda_2 = -1.63$. Although the main results are proved under the assumption that all Lyapunov exponents are positive, the simulation results verify that the results hold true even for this example with one Lyapunov exponent negative. The critical probability $p^*$ is computed, based on the positive exponent and is equal to $p^* = 1 - \frac{1}{\exp^{2\lambda_1}} = 0.5734$. The observer is designed such that error dynamics with no erasure is asymptotically stable. In Figure (4.1a) and Figure (4.1b), we plot the error norm for the observer dynamics, averaged over 50 realizations of the erasure sequence, at probabilities below and above the critical probability $p^*$, respectively. We clearly
see the average error norm for non-erasure probability, $p = 0.7 > p^*$, is negligible compared to fluctuations in the average error norm for $p = 0.55 < p^*$, which are four orders of magnitude higher than the uniform noise in the system. In Figure (4.1c), we plot the peak error variance for linearized error dynamics vs. non-erasure probability. The dashed line indicates the critical probability, $p^* = 0.5734$. We observe the peak linearized error variance is unbounded below critical probability.

### 4.5 Conclusions

In this work, the problem of state observation for a nonlinear system over erasure channel is studied. We prove that fundamental limitation arises for MSE stabilization of observer error dynamics. We show that instability of the non-equilibrium dynamics of the nonlinear system, as captured by positive Lyapunov exponents, plays an important role in obtaining the limitation result for nonlinear observation. The limitation result for LTI systems is obtained as a special case, where Lyapunov exponents emerge as the natural generalization of eigenvalues from linear systems to nonlinear systems. The proof technique presented in section can be easily extended to prove results for the estimation of linear time varying systems over erasure channels.
CHAPTER 5. Entropy based Fundamental Limitations in Nonlinear Control with General Uncertainty

5.1 Introduction

In recent literature, network control systems has been a topic of extensive research. The majority of this literature has focus on linear time invariant (LTI) systems L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007); V. Gupta and N. Martins and J. Baras (2007); Epstein et al. (2008); Imer et al. (2006a); Elia (2004); N. Elia and J. N. Eisenbeis (2011); Martins and Dahleh (2004); N.C. Martins, M.A. Dahleh, and N.Elia (2006); Costa (2002); Sahai (2001). One of the important problems in the area of network control system is to characterize the limitations imposed on the performance of the closed loop system due to channel uncertainty. For LTI systems the performance limitation problem has been addressed in various setting. The performance limitations for stabilization and estimation are shown to arise due to unstable eigenvalues of the open loop plant dynamics. Very few results have focused on addressing the performance limitation problem with nonlinear system dynamics. In Nair et al. (2004); Mehta et al. (2008) Bode-like fundamental limitation results are derived for stabilization of nonlinear systems. The limitations for stabilization are expressed in terms of topological and measure theoretic entropy corresponding to the equilibrium point of open loop unstable dynamics. In Vaidya and Elia (2011, 2012); Diwadkar and Vaidya (2012) performance limitations for stabilization and observation with erasure channel in feedback loop is addressed. The limitations for stabilization and observation are expressed in terms of the positive Lyapunov exponents of the open loop unstable dynamics. The positive Lyapunov exponents are the measure of dynamical complexity and measure the exponential rate of divergence of nearby trajectories. We continue this investigation further and connect
the performance limitation with one another notion of dynamical complexity i.e., entropy. In particular, the performance limitation for stabilization with channel uncertainty is expressed in terms of the measure theoretic entropy of open loop unstable dynamics. Following are the main contributions of this section. The performance limitation results are derived for nonlinear stabilization with general channel uncertainty. The limitation for stabilization are expressed in terms of entropy corresponding to nonequilibrium dynamics of the open loop nonlinear plant.

5.2 Preliminaries

5.2.1 Preliminaries for stabilization problem

We consider the problem of stabilization of a multi-state single-input system

\[ x_{n+1} = f(x_n) + bw_n + bw(n, \gamma_n) \] (5.1)

where \( x_n \in X \subset \mathbb{R}^N \) is the state, \( v_n \in U \subset \mathbb{R} \) is the plant control input. The objective is to design a stabilizing feedback control input \( u_n = k(x_n) \) where \( u_n \) is the controller output and is assumed to reach the plant over a fading communication link. In this paper we assume the following multiplicative channel model

\[ v_n = \xi_n u_n \] (5.2)

where \( \xi_n \in \mathbb{R} \) is a random variable with density function \( \psi(\xi) \) and mean \( \mu := E[\xi_n] \) and variance \( \sigma^2 := E[(\xi_n - \mu)^2] \). To make the problem interesting we assume the random variable to have non-zero variance i.e. \( \sigma^2 \neq 0 \). Thus in this case we assume that at a given discrete time step \( n \) the control signal fed to the system, is amplified by the fading channel according to the variable \( \xi_n \). The external time-dependent forcing is modeled by \( w \in W \subset \mathbb{R} \) and \( \gamma_n \in \mathbb{R} \) is assumed to be IID random variable. The external forcing signal \( w \) can be used to model either the disturbance input or tracking signal. The random variable \( \gamma_n \) is not necessarily independent of \( \xi_n \). This gives us the following model of the controlled system over a general fading channel

\[ x_{n+1} = f(x_n) + \xi_n bk(x_n) + bw(n, \gamma_n) \] (5.3)

We make the following assumptions on the system mapping
Assumption 60 The system mapping \( f : X \rightarrow X \) is assumed to be \( C^r \) with \( r \geq 1 \) and the Jacobian \( \frac{\partial f}{\partial x}(x) \) is assumed to be invertible and uniformly bounded for almost all (w.r.t. Lebesgue measure) \( x \in X \) and for \( x = 0 \). Furthermore, \( x = 0 \) is an unstable equilibrium point of the system with eigenvalues \( |\lambda_i| > 1 \) for \( i = 1, \ldots, N \) of the Jacobian \( \frac{\partial f}{\partial x}(0) \). The control matrix \( b \) satisfies \( b' b > 0 \).

Assumption 61 We assume that the pair \( (f(x), b) \) satisfies following assumption. There exists positive constants \( k_1 \) and \( k_2 \) and integer \( k \geq 0 \) such that

\[
k_1 I \leq \sum_{l=0}^{k} \Phi(x_k, x_l) b' \Phi'(x_k, x_l) \leq k_2 I
\]

for almost all with respect to Lebesgue measure initial conditions \( x_0 \in X \) and for \( x_0 = 0 \), where \( x_{n+1} = f(x_n) \) and \( \Phi(x_n, x_0) := \frac{\partial f}{\partial x}(x_n) \cdots \frac{\partial f}{\partial x}(x_0) = \prod_{k=0}^{n} \frac{\partial f}{\partial x}(x_k) \).

The objective is to design a state feedback control input, \( u_n = k(x_n) \) with \( k : X \rightarrow U \) and \( k(0) = 0 \), such that the system (5.1) is mean square exponentially incrementally stable i.e., any two trajectories of the system (5.1) will exponentially converge to each other as given in Definition 62. Under the assumption that the trajectories starting from two different initial conditions \( x_0 \) and \( y_0 \) are subjected to the same external forcing i.e., \( w(n, \xi_n) \), we define mean square exponential incremental stability of (5.3) as follows

Definition 62 (Mean Square Exponential Incrementally Stable) The system (5.3) is said to be mean square exponential incrementally stable if there exists a positive constants \( K < \infty \) and \( \beta < 1 \) such that

\[
E_{\xi_0, \gamma_0}^{n-1} \left[ \| x_{n+1} - y_{n+1} \|_2^2 \right] \leq K^\beta \| x_0 - y_0 \|_2^2, \quad \forall t \geq 0
\]

for almost all w.r.t. Lebesgue measure initial condition \( x_0 \in X \) and \( x_0 = 0 \), where \( E_{\xi_0, \gamma_0}^{n-1}[\cdot] \) is the expectation taken over the sequence \( \{\xi_0, \ldots, \xi_n\} \) and \( \{\gamma_0, \ldots, \gamma_{n-1}\} \).

Having defined the performance criterion we desire the system to achieve, we now define the Quality of Service (QoS) delivered by our stochastic input and output channels as defined in Xiao and Xie (2010). The QoS is the performance offered by the stochastic channel and will be defined for a stochastic channel with uncertainty \( \varsigma_t \in W \subseteq \mathbb{R} \), with mean \( \mu > 0 \) and variance \( \sigma^2 > 0 \).
Definition 63 (Quality of Service) For the channel with multiplicative stochastic uncertainty \( \varsigma \in W \subset \mathbb{R} \) with finite mean \( \mu > 0 \) and finite variance \( \sigma^2 > 0 \), the quality of service of the channel is defined as

\[ Q_S := \frac{\mu^2}{\sigma^2}. \]

Remark 64 The choice as QoS as \( \frac{\mu^2}{\sigma^2} \) is not ad hoc. This definition of QoS is related to other performance measures in statistics as coefficient of variation or in signal processing to the popular signal to noise ratio (SNR). The signal to noise ratio is defined as \( SNR = \frac{\mu}{\sigma} \) (sometimes defined as \( \frac{\mu^2}{\sigma^2} \) also to ensure positivity). Thus the QoS as defined in Definition 82 is a practically useful measure of performance. For Bernoulli channel uncertainty with probability of erasure \( p \), the QoS is given by

\[ Q_S = \frac{p}{1-p} \]  

(5.5)

5.3 Main results

The main result of this paper proves that, to achieve mean square exponential incremental stability of the networked system one requires a certain minimal QoS from the network. We now outline the key steps involved in proving the main result of this paper.

1. We first show that a necessary condition for the mean square exponential incremental stability of (5.3) is given by the mean square exponential stability of its linearization along a system trajectory Vaidya and Elia (2011).

2. We then use Lyapunov analysis to obtain a necessary condition for mean square exponential stability of the linearized system.

3. In the main theorem, the optimal control derived from the Lyapunov analysis is used to obtain the main result on mean square exponential incremental stability.

We now state the theorem which gives a necessary condition for mean square exponential incremental stability of (5.3) in terms of the mean square exponential stability of its linearization. This theorem is proved in Vaidya and Elia (2011) and will be simply stated here for convenience.
Theorem 65  Consider the following system linearized dynamics along the system trajectory
\[
\eta_{n+1} = \left( \frac{\partial f}{\partial x}(x_n) + \xi_n b \frac{\partial k}{\partial x}(x_n) \right) \eta_n
\](5.6)
\[
x_{n+1} = f(x_n)
\](5.7)

Let the system (5.3) be mean square exponential incremental stable, then the linearized dynamics (5.6) is mean square exponential stable, i.e., there exists positive constants \(K < \infty\) and \(\beta < 1\) such that
\[
E_{\xi_n} \left[ ||\eta_{n+1}||^2 \right] \leq K\beta^n ||\eta_0||^2
\]
for Lebesgue almost all initial conditions \(x_0 \in X\) and \(x_0 = 0\).

Proof. For proof please refer to Vaidya and Elia (2011)  

We now provide the Lyapunov based necessary condition for the linearized system
\[
\eta_{n+1} = \left( \frac{\partial f}{\partial x}(x_n) + \xi_n b \frac{\partial k}{\partial x}(x_n) \right) \eta_n
\](5.8)

Theorem 66  The necessary condition for the linearized system (5.8) to be mean square exponential stable is that there exist positive constants \(\alpha_1\) and \(\alpha_2\) and the matrix function of \(x\) given by \(P(x) = P(x)' \geq 0\) such that \(\alpha_1I \leq P(x) \leq \alpha_2I\) and
\[
E_{\xi_n} \left[ A'(x_n, \xi_n) P(x_{n+1}) A(x_n, \xi_n) \right] < P(x_n)
\](5.9)
for almost all with respect to Lebesgue measure \(x_0 \in X\) and \(x_0 = 0\), where \(A(x_n, \xi_n) = \frac{\partial f}{\partial x}(x_n) + \xi_n b \frac{\partial k}{\partial x}(x_n)\) and \(x_{n+1} = f(x_n)\).

Proof. Consider the following construction of \(P(x)\)
\[
P(x_n) := \sum_{k=n}^{\infty} E_{\xi_n} \left[ \left( \prod_{j=n}^{k} A(x_j, \xi_j) \right)' \left( \prod_{j=n}^{k} A(x_j, \xi_j) \right) \right]
\](5.10)
With the above construction the required inequality (5.9) follows using the fact that
\[
E_{\xi_n} \left[ A(x_n, \xi_n)' A(x_n, \xi_n) \right] > 0\] since \(A(x, \xi = 0)\) is assumed to be invertible and by continuity \(A(x, \xi)\) is invertible for \(\xi\) in some \(\delta > 0\) neighborhood of zero. Furthermore, since (5.8) is
assumed to be mean square exponentially stable, we know there exist positive constants $K < \infty$ and $\beta < 1$ such that

$$E_{\xi_0} \left[ ||A(x_n, \xi_n) \cdots A(x_0, \xi_0)||^2 \right] \leq K \beta^n$$

Hence from the construction of $P(x)$ that there exists a positive constant $\alpha_2$ such that $P(x) \leq \alpha_2 I$. The lower bound on $P(x)$ exists from the construction of $P(x)$ and the condition that $E_{\xi_n} [A(x_n, \xi_n)'A(x_n, \xi_n)] > 0$. Hence there exists a positive $\alpha_1$ such that $\alpha_1 I \leq P(x)$.

**Definition 67 (Matrix Lyapunov Function)** We refer to the matrix function $P(x)$ satisfying the necessary condition (5.9) of Theorem 66 as a matrix Lyapunov function.

We now use the matrix Lyapunov function to derive the necessary condition for mean square exponential stability of the linearized dynamics.

**Theorem 68** Consider the problem of mean square exponential stabilization of (5.8). The necessary condition for mean square exponential stability of the linearized system (5.8) is given by

$$\left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right) \det(A(x_n))^2 \frac{\det(Q(x_n))}{\det(Q(x_{n+1}))} < 1$$

(5.11)

where $A(x_n) := \frac{\partial f}{\partial x}(x_n)$ and $x_{n+1} = f(x_n)$. The matrix function $Q(x)$ can be obtained as the solution to the following Riccati-like equation corresponding to an optimal control problem.

$$Q(x_n) = A(x_n)'Q(x_{n+1})A(x_n)$$

$$= - \frac{A(x_n)'Q(x_{n+1})b'b'Q(x_{n+1})A(x_n)}{1 + b'Q(x_{n+1})b} + R(x_n)$$

(5.12)

where $R(x_n) \geq 0$ is some positive semidefinite matrix.

**Proof.** Let $P(x_n)$ be the matrix Lyapunov function satisfying the condition of the Theorem 66. To derive the optimal control for the derivative map dynamics (5.8), we write the control Lyapunov inequality as follows:

$$E_{\xi_n} \left[ (A(x_n) + \xi_n b v(x_n))'P(x_{n+1})(A(x_n) + \xi_n b v(x_n)) \right] < P(x_n)$$

(5.13)
where \( v(x) := \frac{\partial k}{\partial x}(x) \). Taking expectation w.r.t. \( \xi_n \) and using the fact that \( x_n \) is independent of \( \xi_n \) we get

\[
A'(x_n)P(x_{n+1})A(x_n) - \mu A'(x_n)P(x_{n+1})bv(x_n) \\
- \mu v(x_n)'b'P(x_{n+1})A(x_n) + (\mu^2 + \sigma^2)v(x_n)'b'P(x_{n+1})bv(x_n) \\
< P(x_n)
\]  

(5.14)

minimizing w.r.t. \( v \), we get the following expression for the optimal control \( v(x_n) \)

\[
v(x_n) = - \left( \frac{\mu}{\mu^2 + \sigma^2} \right) \frac{b'P(x_{n+1})A(x_n)}{b'P(x_{n+1})b}
\]

(5.15)

Substituting (5.15) in (5.14), we get

\[
A'(x_n)P(x_{n+1})A(x_n) \\
- \left( \frac{\mu^2}{\mu^2 + \sigma^2} \right) \frac{A'(x_n)P(x_{n+1})bb'P(x_{n+1})A(x_n)}{b'P(x_{n+1})b} < P(x_n)
\]

(5.16)

This can be written as

\[
A'(x_n)P(x_{n+1})A(x_n) - \frac{A'(x_n)P(x_{n+1})bb'P(x_{n+1})A(x_n)}{\Delta_p + b'P(x_{n+1})b} < P(x_n)
\]

(5.17)

where \( \Delta_p = b'P(x_{n+1})b\sigma^2 \). Since \( P(x_{n+1}) \) is the matrix Lyapunov function and hence bounded from below and \( b'b > 0 \), we know that there exists some constant \( \Delta > 0 \) such that \( \Delta_p \geq \Delta \).

The above inequality necessarily implies

\[
A'(x_n)P(x_{n+1})A(x_n) - \frac{A'(x_n)P(x_{n+1})bb'P(x_{n+1})A(x_n)}{\Delta_p + b'P(x_{n+1})b} < P(x_n)
\]

(5.18)

Scaling with the constant \( \Delta \) gives us the equation

\[
A'(x_n)Q(x_{n+1})A(x_n) - \frac{A'(x_n)Q(x_{n+1})bb'Q(x_{n+1})A(x_n)}{1 + b'Q(x_{n+1})b} < Q(x_n)
\]

(5.19)
where \( Q(x_n) = \frac{1}{A}P(x_n) \). Hence there exists matrix \( R(x) \geq 0 \) such that

\[
Q(x_n) = A'(x_n)Q(x_{n+1})A(x_n) - \frac{A'(x_n)Q(x_{n+1})b'b'Q(x_{n+1})A(x_n)}{1 + b'Q(x_{n+1})b} + R(x_n)
\]

(5.20)

The above equation resembles the Riccati like equation satisfied by the Hessian of the optimal cost in the optimal control problem. To obtain the necessary condition, we take determinants on both sides of (5.16) to get

\[
\left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^2 \det(A(x_n))^2 \frac{\det(Q(x_n))}{\det(Q(x_{n+1}))} < 1
\]

(5.21)

We will now prove the main theorem in this section. This theorem provides stability condition for a nonlinear system evolving on a compact space. Given the compact nature of the state space, we can use results from ergodic theory of dynamical systems to provide a computable necessary condition for mean square exponential incremental stability of the nonlinear system (5.3). We now give some definitions and results from ergodic theory of dynamical systems which will be useful in proving the results.

**Definition 69 (Physical Invariant Measure)** A probability measure \( \nu \in \mathcal{M}(X) \) is said to be invariant for \( x_{n+1} = f(x_n) \) if \( \nu(B) = \nu(f^{-1}(B)) \) for all sets \( B \in \mathcal{B}(X) \) (where \( f^{-1}(B) \) is the inverse image of the set \( B \) and \( \mathcal{B}(X) \) is the Borel-\( \sigma \) algebra on \( X \)). An invariant probability measure is said to be ergodic if any \( f \)-invariant set \( A \) i.e., \( f^{-1}(A) = A \) has \( \nu \) measure equal to one or zero. The ergodic invariant measure is said to be physical if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \phi(x_k) = \int_X \phi(x) d\nu(x)
\]

for Lebesgue almost all initial conditions \( x_0 \in X \) and for all continuous \( \phi : X \to \mathbb{R} \).

**Definition 70 (Lyapunov Exponents)** For \( x_{n+1} = f(x_n) \), let

\[
L(x) = \lim_{n \to \infty} \left[ \left( \prod_{k=0}^{n} \frac{\partial f}{\partial x}(x_k) \right)' \left( \prod_{k=0}^{n} \frac{\partial f}{\partial x}(x_k) \right) \right]^{\frac{1}{n}}, \quad x_0 = x
\]
If $\lambda_{\text{exp}}^i$ are the eigenvalues of $L(x_0)$ then the Lyapunov exponents $\Lambda_{\text{exp}}^i$ are given by $\Lambda_{\text{exp}}^i = \log \lambda_{\text{exp}}^i$ for $i = 1, \ldots, N$. Furthermore, if $\det(L(x)) \neq 0$ then

$$
\lim_{n \to \infty} \frac{1}{n} \log \left| \det \left( \prod_{k=0}^{n} \frac{\partial f}{\partial x}(x_k) \right) \right| = \log \prod_{k=1}^{N} \lambda_{\text{exp}}^k(x). \quad (5.22)
$$

**Remark 71** The technical conditions for the existence of Lyapunov exponents and the limit (5.22) are given by the Oseledet Multiplicative Ergodic Theorem. Lyapunov exponents for nonlinear systems are defined with respect to any invariant measure and hence may be a function of the initial condition $x_0$. Under the assumption that there exists a unique ergodic physical measure the Lyapunov exponents are independent of the initial condition. The assumptions on the system in Assumption 60 guarantee that the system satisfies all the technical conditions of the Oseledet Multiplicative Ergodic Theorem.

**Assumption 72** We assume that the system, $x_{n+1} = f(x_n)$ has a unique physical invariant measure with all its Lyapunov exponents positive.

**Theorem 73** Consider the system (5.3) with mapping $f$ satisfying Assumptions 60, 61 and 72 with the state space $X$ compact. Then the necessary condition for the mean square exponential incremental stability is given by

$$
\frac{\sigma^2}{\mu^2 + \sigma^2} \left( \prod_{k=1}^{N} \lambda^k_{\text{exp}} \right)^2 < 1 \quad (5.23)
$$

where $\lambda^k_{\text{exp}} = \exp \Lambda^k_{\text{exp}}$ and $\Lambda^k_{\text{exp}} > 0$ is the $k^{th}$ positive Lyapunov exponent of the system $x_{n+1} = f(x_n)$ and $\lambda_0^k$ are the unstable eigenvalues of the Jacobian $\frac{\partial f}{\partial x}(0)$ at the origin.

**Proof.** Using the results of Theorem 68, the necessary condition for mean square exponential incremental stability is given by

$$
\frac{\sigma^2}{\mu^2 + \sigma^2} \left( \prod_{k=0}^{n} \det(A(x_k)) \right)^2 \frac{\det(Q(x_n))}{\det(Q(x_{n+1}))} < 1 \quad (5.24)
$$

where $x_{n+1} = f(x_n)$ and $A(x_n) = \frac{\partial f}{\partial x}(x_n)$. As this equation holds for all $x_n$ we evaluate the above condition on a trajectory $x_{n+1} = f(x_n)$ to obtain

$$
\left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^n \left( \prod_{k=0}^{n} \det(A(x_k)) \right)^2 \frac{\det(Q(x_0))}{\det(Q(x_{n+1}))} < 1 \quad (5.25)
$$
Taking logarithm and average with respect to \( n \) and studying the limit as \( n \to \infty \) for the above expression we get
\[
\lim_{n \to \infty} \frac{1}{n} \left( \prod_{k=0}^{n} \left( \frac{\det(A(x_k))}{\det(A(x_0))} \right)^2 \right)^{\det(Q(x_0))} \det(Q(x_{n+1})) \right) + \log \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right) < 1 \quad (5.26)
\]
Using the fact that \( Q(x_n) \) is bounded above and below for all \( n \) and using the equality from (5.22) from Definition 70 we get the necessary condition for stability as
\[
\left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right)^N \prod_{k=1}^{\lambda_{exp}^k} < 1 \quad (5.27)
\]

5.3.1 Entropy and limitations on stabilization

Characterizing limitations on stabilization and estimation of systems has been a problem of interest to the control community for a long time. Especially, providing these limitations in terms of open loop characteristics of the dynamical system has been a topic of extensive research. In the main result of this paper we provide the limitations on stabilization of nonlinear systems based on a measure of dynamical complexity of the system given by entropy of the invariant measure that arises mainly from non-equilibrium dynamics. The emergence of non-equilibrium dynamics and the limitations thereof mainly arise due to the presence of uncertainty in the feedback loop. We have already proved the necessary condition for the mean square exponential incremental stability of the system (5.3) based on the positive Lyapunov exponents of the open loop system in Theorem 73. Lyapunov exponents measure the average exponential rate at which nearby trajectories diverge. We now establish a connection between entropy and Lyapunov exponents with the following Ruelle inequality (Eckman and Ruelle (1985) Eq. 4.4 and Ruelle (1978) Theorem 2).

**Theorem 74 (Ruelle’s Inequality)** Let \( F : X \to X \) be a \( C^r \) map with \( r \geq 1 \), of a compact metric space \( X \) and \( \nu \) be an ergodic invariant measure. Then we have
\[
h_\nu(F) \leq \sum_{k} \left( \lambda_{exp}^k \right)^+ \quad (5.28)
\]
where \( a^+ = \max\{0, a\} \) and \( \Lambda^k_{\exp} \) are the Lyapunov exponents of the system.

**Remark 75** For the Ruelle inequality to become an equality we require \( f \) to be a \( C^2 \)-diffeomorphism. When \( f \) is at least a \( C^2 \)-diffeomorphism, the inequality becomes an equality if and only if \( \nu \) is so called Sinai-Ruelle-Bowen (SRB) measure. The equality in this case is known as Pesin’s formula. Me now make use of the Ruelle inequality to prove the main result of this paper.

**Theorem 76** Consider the system (5.3) with the system mapping \( f \) satisfying Assumptions 60, 61 and 72 on a compact state space \( X \). The necessary condition for mean square exponential incremental stability for the single input system is given by

\[
\log \left( \frac{\sigma^2}{\mu^2 + \sigma^2} \right) + 2 \max (h_\nu(f), h_{\nu_0}(f)) < 0 
\]

where \( \nu \) is the physical invariant measure of \( f \) (Assumption 72 and Definition 70 and \( \nu_0 = \delta_0 \) is the Dirac-delta ergodic invariant measure associated with the equilibrium point of \( f \) at \( x = 0 \).

**Proof.** The proof follows from combining the results of Theorem 73 and 74. 

**5.3.2 Sufficiency condition for nonlinear stabilization**

**Theorem 77** The discrete time random dynamical system \( x_{n+1} = F(x_n, \xi_n, \gamma_n) \) is mean square exponential incremental stable if, there exists \( \beta < 1 \), a symmetric, positive definite \( P(x_n) \) and positive constants \( \kappa_1, \kappa_2 \) such that \( \kappa_1 I \leq P(x_n) \leq \kappa_2 I \) and,

\[
E_{\xi_n, \gamma_n} \left[ \left( \frac{\partial F}{\partial x}(x_n, \xi_n, \gamma_n) \right)' P(x_{n+1}) \left( \frac{\partial F}{\partial x}(x_n, \xi_n, \gamma_n) \right) \right] < \beta P(x_n) \]  

**Proof.** We assume the existence of the symmetric positive definite matrix function \( P(x_n) \), satisfying (5.30) and positive constants \( \kappa_1, \kappa_2 \) such that \( \kappa_1 I \leq P(x_n) \leq \kappa_2 I \). Let us begin by defining \( z_0(\theta) = \theta x_0 + (1 - \theta)y_0 \), where \( \theta \in [0, 1] \). The dynamics of \( z \) is given by \( z_{n+1}(\theta) = F(z_n(\theta), \xi_n, \gamma_n) \). We define a new variable \( v_n = \frac{\partial z_n}{\partial \theta} \).

\[
v_{n+1} = \frac{\partial z_{n+1}}{\partial \theta} = \frac{\partial F}{\partial x}(z_n, \xi_n, \gamma_n)v_n
\]
We define a change of coordinates which gives \( w_n = P_1(z_n)v_n \). This implies, \( w_{n+1} = P_1(z_{n+1})v_{n+1} \).

\[
E_{\xi_0, \gamma_0}^n [w_{n+1}w_{n+1}] = E_{\xi_0, \gamma_0}^n [v_n' \frac{\partial F}{\partial x}(z_n, \xi_n, \gamma_n) P(z_{n+1}) \frac{\partial F}{\partial x}(z_n, \xi_n, \gamma_n)v_n]
\]

\[
\leq \beta E_{\xi_0, \gamma_0}^{n-1} [w_n' P(z_n)v_n]
\]

\[
= \beta E_{\xi_0, \gamma_0}^{n-1} [w_n'w_n]
\]  

(5.31)

Iterating the above inequality we get,

\[
E_{\xi_0, \gamma_0}^n [w_{n+1}w_{k+1}] < \beta^{n+1}w_0'w_0
\]

Now, using the fact \( w_{n+1} = P_1(z_{n+1})v_{n+1} \) and \( \kappa_1 I \leq P(z_{n+1}) \leq \kappa_2 I \) we get,

\[
E_{\xi_0, \gamma_0}^n [v_{n+1}^2] = E_{\xi_0, \gamma_0}^n [w_{n+1}' P(z_{n+1})^{-1} w_{n+1}]
\]

\[
< \frac{1}{\kappa_1} E_{\xi_0, \gamma_0}^n [w_{n+1}' w_{n+1}]
\]

\[
< \frac{1}{\kappa_1} \beta^{n+1}w_0'w_0 = \frac{1}{\kappa_1} \beta^{n+1}v_0'P(z_0)v_0
\]

\[
< \left( \frac{\kappa_2}{\kappa_1} \right) \beta^{n+1} \| v_0 \|^2
\]  

(5.32)

Using this we get

\[
E_{\xi_0, \gamma_0}^n [\| x_{n+1} - y_{n+1} \|^2] = E_{\xi_0, \gamma_0}^n \left[ \left( \int_0^1 \| v_{n+1} \| d\theta \right)^2 \right]
\]

\[
\leq E_{\xi_0, \gamma_0}^n \left[ \int_0^1 \| v_{n+1} \|^2 d\theta \right]
\]

\[
\leq \int_0^1 E_{\xi_0, \gamma_0}^n [\| v_{n+1} \|^2] d\theta
\]

\[
< \int_0^1 \left( \frac{\kappa_2}{\kappa_1} \right) \beta^{n+1} \| v_0 \|^2 d\theta
\]

\[
= \left( \frac{\kappa_2}{\kappa_1} \right) \beta^{n+1} \| x_0 - y_0 \|^2
\]  

(5.33)

This gives us the required proof.

\[\blacksquare\]

5.4 Simulation

We will now verify the results proved in this chapter with simulations. The system we choose for simulation is a system of the Lur’e form which shows complex dynamics (refer Figure (6.2a))
given by

\[
x_{t+1} = A_d x_t - B_d \phi(y_t) + B_d u_t
\]

(5.34)

\[
y_t = C_d x_t
\]

(5.35)

where \(A_d, B_d\) and \(C_d\) are given by

\[
A_d = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0.5 & 0.5 & 0
\end{bmatrix};
B_d = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix};
C_d = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix}
\]

(5.36)

The nonlinearity \(\phi_I(y)\) is given by

\[
\phi_I(y) = \begin{cases}
-k y, & ||y|| < 1 \\
2k y - 3 k \text{sgn}(y), & 1 < ||y|| \leq 3 \\
3 k \text{sgn}(y), & 3 < ||y||
\end{cases}
\]

(5.37)

where \(k = 0.8\). We perform a loop transformation to bring the nonlinearity in a passive form to get a system

\[
\dot{x} = A x - B \phi(y) + B u
\]

(5.38)

\[
y = C x
\]

(5.39)

where \(A = A_d + (k + \epsilon) B_d C_d\), \(B = B_d\), \(C = C_d\) and \(\phi(y) = (k + \epsilon) y + \phi_I(y)\), where \(\epsilon = 0.05\).

This discrete time system also demonstrates chaotic dynamics as shown in Figure (6.2b), Figure (6.2c) and Figure (6.2d).

We wish to design a full state feedback controller with uncertain actuation links \(u_t = \gamma_t v_t\).

We choose the uncertainty of the actuation links to be modelled by a Bernoulli random variable \(\text{Prob}\{\gamma_t = 1\} = p\). The maximum Lyapunov exponent of this system is positive \(\lambda^{1}_{\exp} = 1.1442\) while the other exponents are negative. This gives us the minimum allowable probability of non-erasure below which it is not possible to guarantee mean square exponential incremental stabilization is given by \(p_c := 1 - \left(\frac{1}{\lambda^{1}_{\exp}}\right)^2 = 0.2506\), as given by Theorem 65. We now use the sufficiency condition derived for stabilization of nonlinear systems. To study the sufficiency condition we consider the following matrix based on the linearization at the origin \(A_0 := \)
Figure 5.1 (a) Schematic of continuous time system, (b) State dynamics in 3 dimensions, (c) State dynamics in X-Y plane, (d) X-state dynamics as a function of time.

\[ A_d - \frac{\partial \phi}{\partial y} B_d C_d = A_d - \epsilon B_d C_d. \]  

We now construct \( P^* \) such that

\[
P^* = A'_0 P^* A_0 - \frac{A'_0 P^* B_d B'_d P^* A_0}{1 + B'_d P^* B_d} + I
\]

(5.40)

Thus \( P^* \) is the matrix that computes the optimal cost \( J^*(x, u) = x'P^*x \), for an optimal control problem for the system linearized at the origin, for the cost function \( J(x, u) = x'x + u'u \). We design the control as \( u_t = -\frac{B'_d P^*}{B'_d P^* B_d} f(x_t) \) and use it to stabilize the system. We plot the results in Figure (5.2a) which shows the incremental error for two systems with almost identical noise, at a probability of non-erasure \( p = 0.5006 > p_c \), and Figure (5.2b) which plots the incremental
error for \( p = 0.2006 < p_c \). We observe that, for a value of non-erasure probability of the actuator higher than the critical probability \( (p > p_c) \) obtained from Theorem 65 as in Figure (5.2a), the incremental error dynamics shows some large fluctuations but eventually converges to zero. When we choose the probability below the critical probability \( (p < p_c) \) derived from Theorem 65, the incremental error between two trajectories grows unbounded with time. The error values in Figure (5.2) have been averaged over 50 iterations of the uncertainty sequence of \( \gamma_t \).

We now compute the sufficiency condition with Lyapunov matrix \( P^* \) given in Theorem 77.

\[
P_{suff} := P^* - \frac{\partial f}{\partial x}(x_t)' \left( P^* - p \frac{P^*B_dB_d'P^*}{B_d'P^*B_d} \right) \frac{\partial f}{\partial x}(x_t)
\]

Figure 5.2 (a) Incremental error dynamics at \( p = 0.38 > p_c \), (b) Incremental error dynamics at \( p = 0.24 < p_c \).

Let us define

According to the condition given in Theorem 77, if \( P_{suff} > 0 \) it guarantees mean square exponential incremental stability. By numerical simulations we can see that this quantity is positive for \( p \geq 0.95 \). It can be seen that this condition is conservative as the simulation results for incremental error do converge for \( p > 0.5 \) as seen in Figure (5.2). The result from the sufficiency condition can be improved by a better choice of the matrix \( P^* \).
5.5 Conclusion

We have generalized the stabilization problem over unreliable actuation channels with a general multiplicative uncertainty, with the performance criterion of mean square exponential incremental stability. We have given a sufficiency condition for mean square incremental stabilization for a general nonlinear system. Though the sufficiency condition does not have a closed form expression in terms of offline computable quantities, we can obtain a slightly conservative sufficiency condition from this, based on the linearization at the origin as shown in the simulation results. Thus the sufficiency condition could be a useful tool to analyze mean square exponential incremental stability of the nonlinear system. In the next chapter we will further pursue the search for a sufficiency condition. Narrowing the field to a special class of nonlinear systems of the Lur’e form we will obtain a sufficiency condition in terms of system matrices.
CHAPTER 6. Stabilization of Lur’e Systems Over Erasure Channels

In this chapter we desire to obtain a synthesis method for observer based controller for nonlinear systems with uncertain measurement and actuation links. For this purpose we look into the class of nonlinear systems known as Lur’e systems, the stability of which is obtained from absolute stability theory and passivity theory. We thus study the problem of stabilization of nonlinear system in Lure form with uncertainty at the input and output channels. The channel uncertainty is modeled using Bernoulli random variable. The main result provides a sufficient condition for the maximum allowable erasure in the input and output channels to maintain mean square stability of the closed loop system. We generalize this result to provide sufficient condition for stabilization over general uncertain channel at the input and perfect measurement channel at the output. The results provide synthesis method for the design of controller and observer that are robust to channel uncertainty. Due to nonlinear plant dynamics, the controller and observer design problem are coupled, however we provide explicit relation between the erasure probability of the input and output channels to mainatin stability of the feedback control system.

6.1 Introduction

The problem of control and estimation over stochastic input and output channels is of importance in control of systems over networks Antsaklis and Baillieul (2007). Specifically systems with packet drop channels at input and output, modeled as Bernoulli erasure channels has garnered much attention N. Elia (2005). The problems of state estimation and optimal control over such Bernoulli erasure channels have been solved for linear time invariant (LTI) systems for various stability criteria like mean square stability, stable in probability Sinopoli
et al. (2003); Epstein et al. (2008); Huang and Dey (2006). Under certain assumptions on the system dynamics necessary and sufficient conditions were provided for LTI state estimation in Y.Mo (2008); L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007) and optimal control of LTI systems in Imer et al. (2006a).

The framework based on the theory of random dynamical system was adapted in Vaidya and Elia (2011) Diwadkar and Vaidya (2010), to develop results providing necessary condition for mean square stabilization and observation of general nonlinear system over erasure channels. It is shown that global nonequilibrium dynamics of the nonlinear system play an important role in determining the minimum Quality of Service (QoS) of the erasure channel. The necessary condition for stabilization and observation of nonlinear systems were expressed in terms of the positive Lyapunov exponents of the nonlinear systems capturing the nonequilibrium dynamics. We continue this line of research to provide sufficient condition for mean square stabilization of nonlinear systems over uncertain channels.

Deriving non-trivial sufficient condition for the stabilization of general nonlinear system over uncertain channels is a challenging problem. Hence we focus on a particular class of nonlinear systems namely nonlinear systems in Lure form. A system in Lure form consist of feedback interconnection of LTI system and static nonlinearity element. Systems in Lure form are widely studied in control system community because several systems in engineering application can be modeled as feedback interconnection of LTI system and static nonlinearity. Systematic analysis tools in the form of Positive Real Lemma (PRL) and Kalman-Yakubovich-Popov (KYP) Lemma exist for the synthesis and design of system in Lure form Haddad and Bernstein (1994); Arcak and Kokotovic (2001); Ibrir (2007); Johansson and Robertsson (2002). We make use of these powerful analysis methods in the development of the main results of this chapter. The main contributions of the results derived in this chapter are as follows,

1. We discover a stochastic variant of PRL to provide synthesis method for the design of observer based controller for the stabilization of nonlinear systems in Lure form over erasure channels.

2. We provide sufficient condition for the stabilization of feedback control system in Lure
form with general stochastic uncertainty at the input channel.

The results developed in this chapter, are also a step forward in answering an important question related to tradeoff between the passivity property of the systems and uncertainty in feedback loop. In particular, we know that feedback interconnection of two passive system is passive and with uncertainty in feedback loop it would be of interest to determine how the uncertainty can be traded off for passivity in the loop.

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \]

\( \xi \)
\( \phi \)
\( \text{Delay} \)
\( \text{Observer based Controller} \)
\( \text{1-Step} \)
\( \tilde{u} \)
\( \tilde{y} \)

Figure 6.1 Observer based controller for Lur’e system over uncertain channels

6.2 Preliminaries

Consider the nonlinear system in Lur’e form with channel uncertainty at the inputs and outputs (refer to Figure 6.1) and described by following equations.

\[ x_{t+1} = Ax_t - B\phi(y_t) + B\tilde{u}_t, \quad y_t = Cx_t \]

\[ \tilde{u}_t = \gamma_t u_t, \quad \tilde{y}_t = \xi_t y_t \]

where \( x_t \in \mathbb{R}^N \), \( \tilde{u}_t \in \mathbb{R}^M \), and \( y_t \in \mathbb{R}^M \) are the state, control input, and output vector respectively. The random variables \( \gamma_t \) and \( \xi_t \) model the uncertainty at the input and output channels respectively.

We make the following assumptions on the system dynamics, channels uncertainties, and information structure between the input and output channels.
**Assumption 78** The nonlinearity \( \phi(y) \) is globally Lipschitz, at least \( C^1 \) and monotonic non-decreasing function of \( y \in \mathbb{R}^M \). Furthermore this nonlinearity satisfies the following sector conditions

\[
\phi(y)' (y - D_1 \phi(y)) > 0
\]

\[
(\phi(y_1) - \phi(y_2))' ((y_1 - y_2) - D_2 (\phi(y_1) - \phi(y_2))) > 0
\]

where \( D_1 + D_1' > 0 \) and \( D_2 + D_2' > 0 \).

**Remark 79** With no stability assumption on the system matrix \( A \) it is possible to transform a nonlinearity in the feedback loop to satisfy the above conditions after appropriate loop transformation Khalil (1996) provided the nonlinearity is globally Lipschitz.

**Assumption 80** We assume that the input channel uncertainty \( \gamma_t \in W \subseteq \mathbb{R} \) and output channel uncertainty \( \xi_t \in \{0,1\} \). Both the channel uncertainties are assumed to be independent identically distributed (i.i.d) random variables with following statistics

\[
E[\gamma_t] = \mu, \quad E[(\gamma_t - \mu)^2] = \sigma^2 < \infty
\]

\[
\text{Prob}\{\xi_t = 1\} = q, \quad \forall t.
\]

To make the problem interesting, we assume that \( 0 < q < 1 \).

**Assumption 81** We assume that the observer receive the acknowledgment about the input channel state with one-step delay (refer to Figure 6.1). This acknowledgement structure is called as Transmission Control Protocol (TCP) L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007).

We next provide the definition of Quality of Service (QoS) for a channel.

**Definition 82** [QoS] For the channel with multiplicative stochastic uncertainty \( \gamma \in W \subseteq \mathbb{R} \) with finite mean \( \mu \) and finite variance \( \sigma^2 \), the quality of service of the channel is defined as

\[
Q = \frac{\mu^2}{\sigma^2}
\]
Remark 83 The definition of QoS is related to other performance measure in statistics as coefficient of variation or in signal processing to popular signal to noise ratio (SNR). The signal to noise ratio is defined as \( \text{SNR} = \frac{\mu}{\sigma} \) (sometimes defined as \( \text{SNR} = \frac{\mu^2}{\sigma^2} \) to ensure positivity). Thus the QoS as defined above is a practical useful measure of performance. For an erasure channel with non-erasure probability \( p \), the QoS \( Q = \frac{P}{1-p} \).

The notion of stability that we adapt to analyze the feedback control system (6.1) is the mean square exponential (MSE) stability. We define this stability in the context of random dynamical system of the form

\[
x_{t+1} = S(x_t, \varsigma_t),
\]

(6.7)

where, \( x_t \in X \subseteq \mathbb{R}^N \), \( \varsigma_t \in W \subseteq \mathbb{R} \) for \( t \geq 0 \), are i.i.d random variables with finite mean and second moment. The system mapping \( S : X \times W \rightarrow X \) is assumed to be at least \( C^1 \) with respect to \( x_t \in X \) and measurable w.r.t \( \varsigma_t \). We assume that \( x = 0 \) is an equilibrium point i.e., \( S(0, \varsigma_t) = 0 \). The following notion of stability can be defined for RDS Has’minski˘ı (1980); Applebaum and Siakalli (2009).

Definition 84 (Mean Square Exponential (MSE) Stable) The solution \( x = 0 \) is said to be MSE stable for \( x_{t+1} = S(x_t, \varsigma_t) \) if there exists a positive constants \( M < \infty \) and \( \beta < 1 \) such that

\[
E_{\varsigma_t} [\| x_{t+1} \|^2] \leq M \beta^t \| x_0 \|^2, \quad \forall t \geq 0
\]

for almost all w.r.t. Lebesgue measure initial condition \( x_0 \in X \) where \( E_{\varsigma_t}[\cdot] \) is the expectation taken over the sequence \( \{\varsigma(0), \ldots, \varsigma_t\} \).

6.3 Main Results

The first main result of this paper provides sufficient condition for the stabilization of Lure system expressed in terms of statistics of channel uncertainties and the solution of Riccati equation. The result also provides synthesis method for the design of observer-based controller robust to channels uncertainties.
Theorem 85 Consider the observer-based controller design problem for the system in Lure form (6.1) and as shown in Figure 6.1. The feedback control system is mean square exponentially stable if there exists \( P^* > 0, \ Q^* > 0 \) such that following conditions are satisfied

\[
\Sigma_1 - B'P^*B > \left(1 - \frac{Q}{1+Q}\right)(\Sigma_1 + B'P^*B) \tag{6.8}
\]

\[
\Sigma_2 - CQ^*C' > (1 - q)(\Sigma_2 + CQ^*C') \tag{6.9}
\]

where \( p \) and \( q \) are the non-erasure probability of the input and output channel respectively and the matrix \( P^* \) and \( Q^* \) satisfy following Riccati equations

\[
P^* = A_1P^*A_1 - A_1P^*B(\Sigma_1 + B'P^*B)^{-1}B'PA_1 + C\Sigma_1^{-1}C
\]

\[
Q^* = A_2Q^*A_2' - A_2Q^*C'(\Sigma_2 + CQ^*C')^{-1}CQ^*A_2' + B\Sigma_2^{-1}B'
\]

where \( \Sigma_1 = D_1 + D_1', \Sigma_2 = D_2 + D_2', A_1 = A - B\Sigma_1^{-1}C, A_2 = A - B\Sigma_2^{-1}C \). Furthermore the controller gain \( K^* \) and observer gain \( L^* \) are given by following expressions

\[
K^* = -(B'P^*B)^{-1}B'P^*A_1
\]

\[
L^* = A_2Q^*C'(CQ^*C')^{-1}
\]

We postpone the proof of this theorem till the end of this section and now provide some intuition behind the sufficiency condition of this theorem.

The conditions (6.8) and (6.9) can be interpreted as generalizations of the positivity conditions from deterministic PRL (i.e., \( \Sigma_1 - B'P^*B > 0 \) and \( \Sigma_2 - CQ^*C' > 0 \)). Thus for the uncertain system we require \( \Sigma_1 - B'P^*B \) and \( \Sigma_2 - CQ^*C' \) to be strictly bounded below by a function of channel uncertainty characteristics. The closer these values are to zero the the amount of tolerable uncertainty decreases. We notice from Eqs. (6.8) and (6.9) that the sufficient conditions involving input and output channels uncertainty are decoupled. This implies that the separation principle applies for the design of observer-based controller for the system in Lure form with input and output channels uncertainties. The observer-based controller problem can be decomposed into two separate problems of design of full state feedback controller and observer design problems. The separation property is in fact the consequence of assumed
TCP like acknowledgment structure (Assumption 81) Imer et al. (2006a); L. Schenato and B. Sinopoli and M. Franceschetti and K. Poolla and S. Sastry (2007). Equations (6.8) and (6.9) then provides sufficient conditions for the mean square stabilization of Lure system with full state feedback control and for the observer error dynamics respectively. We now outline the various steps involved in the proof of Theorem 85.

1. In Theorem 88, we prove results for the design of observer for system in Lure form. Theorem 88 provides bound on the minimum allowable erasure probability of the output channel to maintain mean square exponential stability of the observer error dynamics.

2. We provide the solution to the full state feedback stabilization problem with channel uncertainty at the input in Theorem 89.

3. Finally using Assumption 81, we prove in Theorem 94 that the observer-based controller design problem for the Lure system over uncertain channel enjoys the separation property.

Lemma 86 provides sufficiency condition for the mean square exponential stability of general stochastic dynamical systems.

**Lemma 86** The stochastic dynamical system (6.7) is exponentially mean square stable as given in Definition 84 if there exists a Lyapunov function $V_S(x_t) = x_t^TPx_t$ for some matrix $P = P' > 0$ and positive constants $c_1, c_2$ and $c_3$ such that

\[ c_1 \| x_t \|^2 \leq V_S(x_t) \leq c_2 \| x_t \|^2 \] (6.10)

\[ E_{\zeta_t} [V_S(x_{t+1})] - V_S(x_t) < -c_3 \| x_t \|^2; \forall t \geq 0 \] (6.11)

**Proof.** From (6.11) we get

\[ E_{\zeta_t} [V_S(x_{t+1})] < V_S(x_t) - c_3 \| x_t \|^2, \forall t \geq 0 \] (6.12)

Substituting $V_S(x_t)$ for $\| x_t \|^2$ from (6.10) into (6.12) we get

\[ E_{\zeta_t} [V_S(x_{t+1})] < \left( 1 - \frac{c_3}{c_2} \right) V_S(x_t). \] (6.13)
Let \( 1 - \frac{c_3}{c_2} := \beta_1 < 1 \), since \( c_2 \) can be chosen greater than \( c_3 \). Taking expectation over \( \zeta_t^0 \) in the above equation we get

\[
E_{\zeta_t^0} [V_S(x_{t+1})] < \beta_1 E_{\zeta_t^0} [V_S(x_t)] < \beta_1^2 E_{\zeta_t^0} [V_S(x_{t-1})] < \beta_1^{t+1} V_S(x_0).
\]

Finally using the bounds on \( V_S(x) \) from (6.10) in the above equation we get the desired result.

We propose a observer design with linear gain and is similar to the circle criteria-based observer design proposed in Ibrir (2007); Arcak and Kokotovic (2001). The observer dynamics is assumed to be of the form:

\[
\dot{x}_{t+1} = A\hat{x}_t - B\phi(\hat{y}_t) + \gamma_t Bu_t + L(\hat{y}_t - \tilde{\hat{y}}_t)
\]

(6.14)

\[
\hat{y}_t = C\hat{x}_t, \quad \tilde{\hat{y}}_t = \xi_t\hat{y}_t
\]

(6.15)

This gives the error dynamics, \( e_t := x_t - \hat{x}_t \), to be

\[
e_{t+1} = (A - \xi_t LC) e_t - B(\phi(y_t) - \phi(\hat{y}_t)), \quad w_t = Ce_t
\]

(6.16)

where \( w_t := y_t - \hat{y}_t \).

**Remark 87** It is important to notice that because of the erasure channel uncertainty at the output channel it is possible to assume that the observer has access to channel erasure state, \( \xi_t \). In particular, whenever the system output, \( \tilde{\hat{y}}_t \), is zero (non-zero) the channel erasure state can be assumed to be equal to zero (one).

Writing \( \psi(t,w_t) := (\phi(y_t) - \phi(\hat{y}_t)) \) we can write the error dynamics as

\[
e_{t+1} = (A - \xi_t LC) e_t - B\psi(t,w_t), \quad w_t = Ce_t
\]

(6.17)

where it is clear that \( \psi(t,w_t) \) satisfies the sector condition \( \psi(t,w_t)'(w_t - D_2\psi(t,w_t)) > 0 \) as given by (6.4). Theorem 88 is the main result on observer design for system in Lure form (6.1).

**Theorem 88** Consider a nonlinear system in Lure form (6.1) satisfying Assumptions 78, 80, and 81 and the observer dynamics as given in (6.14). Then the error dynamics (6.17) is mean square exponentially stable if

\[
(\Sigma_2 - CQ^*C') > (1 - q)(\Sigma_2 + CQ^*C')
\]

(6.18)
where $\Sigma_2 = D_2 + D_2' > 0$ and $Q^* = (Q^*)' > 0$ satisfies the Riccati equation

$$Q^* = A_2Q^*A_2' - A_2Q^*C' (\Sigma_2 + CQ^*C')^{-1} CQ^*A_2' + B\Sigma_2^{-1} B'$$

with $A_2 := A - B\Sigma_2^{-1} C$. Furthermore the observer gain, $L$, is given by

$$L = A_2Q^* C'(CQ^*C')^{-1}$$

**Proof.** Consider the candidate Lyapunov function $V_t = e_t^t P_o e_t$, where $P_o$ satisfies following equation.

$$P_o = R_o + E_\xi \left[ A_o(\xi_t)'P_o A_o(\xi_t) + \left( A_o(\xi_t)' P_o B - C' \right) (\Sigma_2 - B'P_o B)^{-1} \left( B'P_o A_o(\xi_t) - C \right) \right]$$

(6.19)

where $A_o(\xi_t) := A - \xi_t LC$ and $R_o > 0$. Equation 6.19 can be viewed as a stochastic variant of positive real Lemma Riccati equation Haddad and Bernstein (1994). Using (6.19) and writing $\Delta V_t := E_\xi [V_{t+1} - V_t]$ we get

$$\Delta V_t = -e_t^t R_o e_t - E_\xi \left[ e_t^t (A_o(\xi)' P_o B - C') (\Sigma_2 - B'P_o B)^{-1} (B'P_o A_o(\xi) - C) e_t \right]$$

$$- E_\xi \left[ e_t^t A_o(\xi)' P_o B \psi(t, w_t) \right] - E_\xi \left[ \psi(t, w_t)' B'P_o A_o(\xi) e_t \right] + E_\xi \left[ \psi(t, w_t)' B'P_o B \psi(t, w_t) \right]$$

(6.20)

Add and subtract $\psi(t, w_t)' \Sigma_2 \psi(t, w_t)$ and $2\psi(t, w_t)' w_t$ to (6.20) to get

$$\Delta V_t = -e_t^t R_o e_t - E_\xi \left[ e_t^t (A_o(\xi)' P_o B - C') (\Sigma_2 - B'P_o B)^{-1} (B'P_o A_o(\xi) - C) e_t \right]$$

$$- E_\xi \left[ e_t^t (A_o(\xi)' P_o B - C') \psi(t, w_t) \right] - E_\xi \left[ \psi(t, w_t)' (C - B'P_o A_o(\xi) e_t \right]$$

$$- E_\xi \left[ \psi(t, w_t)' (\Sigma_2 - B'P_o B) \psi(t, w_t) \right] - 2\psi(t, w_t)' (w_t - D_2 \psi(t, w_t))$$

(6.21)

Using the argument given in Haddad and Bernstein (1994) we get

$$\Delta V_t = -e_t^t R_o e_t - E_\xi \left[ \nu(\xi)' \nu(\xi) \right] - 2\psi(t, w_t)' (w_t - D_2 \psi(t, w_t))$$

(6.22)

where $\nu(\xi) = (\Sigma_2 - B'P_o B)^{-1/2} (B'P_o A_o(\xi) - C) e_t + (\Sigma_2 - B'P_o B)^{1/2} \psi(t, w_t)$. Thus using the fact that $\psi(t, w_t)$ satisfies the sector condition we get

$$E_\xi [V_{t+1} - V_t] < -e_t^t R_o e_t$$
Hence the asymptotic observer with erasure in sensor measurement is mean square exponentially stable. From Lancaster and Rodman (1995) and the transformation \( A_{2o} := A_o - B\Sigma_2^{-1}C \), the equation (6.19) can be written as

\[
P_o > E_\xi [A_{2o}(\xi)'P_oA_{2o}(\xi) + A_{2o}(\xi)'P_o(B(\Sigma_2 - B'P_oB)^{-1}B'P_oA_{2o}(\xi)] + C'S_2^{-1}C
\]

This may then be written as

\[
I > E_\xi [\tilde{A}_2(\xi)'\tilde{A}_2(\xi)]
\]  
(6.23)

where \( \tilde{A}_2(\xi) = (P_o^{-1} - B\Sigma_2^{-1}B')^{-\frac{1}{2}}(A_{2o}(\xi) - B\Sigma_2^{-1}C)(P_o - C'S_2^{-1}C)^{-\frac{1}{2}} \). We know that (6.23) is true if and only if

\[
I > E_\xi [\tilde{A}_2(\xi)\tilde{A}_2(\xi)']
\]  
(6.24)

Now defining \( Q_o := P_o^{-1} \) and expanding (6.24) we get

\[
Q_o - B\Sigma_2^{-1}B' > E_\xi [A_{2o}(\xi)(Q_o^{-1} - C'S_2^{-1}C)^{-1}A_{2o}(\xi)]
\]  
(6.25)

Minimizing R.H.S. in above equation, with respect to \( L \) we get

\[
L = A_2(Q_o^{-1} - C'S_2^{-1}C)^{-1}C'(C(Q_o^{-1} - C'S_2^{-1}C)^{-1}C')^{-1}
\]

where \( A_2 = A - B\Sigma_2^{-1}C \). Simple matrix computation gives us

\[
A_2Q_oC'(\Sigma_2 - CQ_oC')^{-1}\Sigma_2 \quad \text{and} \quad C(Q_o^{-1} - C'S_2^{-1}C)^{-1}C' = CQ_oC'(\Sigma_2 - CQ_oC')^{-1}\Sigma_2
\]

Applying these matrix simplifications to the gain \( L \) we get

\[
L = A_2Q_oC'(CQ_oC')^{-1}
\]

Substituting this structure of \( L \) in (6.25) we get

\[
Q_o > A_2Q_oA_2' - A_2Q_oC'[q(CQ_oC')^{-1} - (1 - q)(\Sigma_2 - CQ_oC')^{-1}]CQ_oA_2' + B\Sigma_2^{-1}B'
\]  
(6.26)

We now wish to give design a \( Q^* \) that will satisfy the above equation. Now suppose \( Q^* \) satisfies the minimum covariance like Riccati equation given by

\[
Q^* = A_2Q^*A_2' - A_2Q^*C'(\Sigma_2 + CQ^*C')^{-1}CQ^*A_2' + B\Sigma_2^{-1}B'
\]
then $Q^*$ satisfies (6.26) if

$$q(CQ^*C')^{-1} - (1 - q)(\Sigma_2 - CQ^*C')^{-1} < (\Sigma_2 + CQ^*C')^{-1}$$

Thus the observer error dynamics (6.17) is exponentially mean square stable if

$$\Sigma_2 - CQ^*C' > (1 - q)(\Sigma_2 + CQ^*C')$$

(6.27)

This proves the result. □

Theorem 89, provide results for the design of full state feedback controller, $u_t = Kx_t$ for system (6.1) in Lure form.

**Theorem 89** Consider the system (6.1) in Lure form satisfying Assumptions 78, 80, and 81. Let $u_t = Kx_t$ be the linear full state feedback controller, then the state dynamics is mean square exponentially stable if

$$(\Sigma_1 - B'P^*B) > \left(1 - \frac{Q}{1 + Q}\right)(\Sigma_1 + B'P^*B)$$

(6.28)

where $\Sigma_1 = D_1 + D_1' > 0$ and $P^* = (P^*)' > 0$ satisfies the Riccati equation

$$P^* = A_1'P^*A_1 - A_1'P^*B(\Sigma_1 + B'P^*B)^{-1}B'P^*A_1 + C'\Sigma_1^{-1}C$$

where $A_1 := A - B\Sigma_1^{-1}C$. Furthermore the observer gain is given by

$$K = -(B'P^*B)^{-1}B'P^*A_1$$

**Proof.** Consider the candidate Lyapunov function $V_t = x'_tP_cv_t$, where $P_c$ satisfies following equation.

$$P_c = R_c + E_{\gamma_t}\left[ A_c(\gamma_t)'P_cA_c(\gamma_t) + (A_c(\gamma_t)'P_cB - C') (\Sigma_1 - B'P_cB)^{-1} (B'P_cA_c(\gamma_t) - C) \right]$$

(6.29)

where $A_c(\gamma_t) := A + \gamma_tBK$ and $R_c > 0$. Equation 6.19 can be viewed as a stochastic variant of positive real Lemma Riccati equation Haddad and Bernstein (1994). Using (6.19) and writing $\Delta V_t := E_{\gamma_t}[V_{t+1}] - V_t$ we get

$$\Delta V_t = -x'_tR_cx_t - E_{\gamma_t}\left[ x'_t (A_c(\gamma_t)'P_cB - C') (\Sigma_1 - B'P_cB)^{-1} (B'P_cA_c(\gamma_t) - C) x_t \right]$$

$$- E_{\gamma_t}\left[ x'_tA_c(\gamma_t)'P_cB(y_t) \right] - E_{\gamma_t}\left[ \phi(y_t)'B'P_cA_c(\gamma) x_t \right] + E_{\gamma_t}\left[ \phi(y_t)'B'P_cB\phi(y_t) \right]$$

(6.30)
Add and subtract $\phi(y_t)'\Sigma_1\phi(y_t)$ and $2\phi(y_t)'y_t$ to (6.20) to get

$$
\Delta V_t = -x'_tR_c x_t - E_\gamma \left[ x'_t \left( A_c(\gamma)'P_cB - C' \right) \left( \Sigma_1 - B'P_cB \right)^{-1} \left( B'P_cA_c(\gamma) - C \right) x_t \right]
$$

$$
- E_\gamma \left[ x'_t \left( A_c(\gamma)'P_cB - C' \right) \phi(y_t) \right] - E_\gamma \left[ \phi(y_t)' \left( C - B'P_cA_c(\gamma) \right) x_t \right]
$$

$$
- E_\gamma \left[ \phi(y_t)' \left( \Sigma_1 - B'P_cB \right) \phi(y_t) \right] - 2\phi(y_t)'(y_t - D_1\phi(y_t))
$$

(6.31)

Using the argument given in Haddad and Bernstein (1994) we get

$$
\Delta V_t = -x'_tR_c x_t - E_\gamma \left[ \eta(\gamma)'\eta(\gamma) \right] - 2\phi(y_t)'(y_t - D_1\phi(y_t))
$$

(6.32)

where $\eta(\gamma) = (\Sigma_1 - B'P_cB)^{-1/2} \left( B'P_cA_c(\gamma) - C \right) x_t + (\Sigma_1 - B'P_cB)^{1/2}\phi(y_t)$. Thus using the fact that $\phi(y_t)$ satisfies the sector condition we get

$$
E_\gamma [V_{t+1}] - V_t < -x'_tR_c x_t
$$

Hence the asymptotic observer with erasure in sensor measurement is mean square exponentially stable. From Lancaster and Rodman (1995) and the transformation $A_{1c} := A_c - B\Sigma_1^{-1}C$, the equation (6.19) can be written as

$$
P_c > E_\gamma \left[ A_{1c}(\gamma)'P_cA_{1c}(\gamma) + A_{1c}(\gamma)'P_cB(\Sigma_1 - B'P_cB)^{-1}B'P_cA_{1c}(\gamma) \right] + C'\Sigma_1^{-1}C
$$

(6.33)

Minimizing R.H.S. in above equation, with respect to $K$ we get

$$
K = -\frac{\mu}{\mu^2 + \sigma^2} \left( B'(P_c^{-1} - B\Sigma_1^{-1}B')^{-1}B \right)^{-1} B'(P_c^{-1} - B\Sigma_1^{-1}B')^{-1} A_1
$$

where $A_1 = A - B\Sigma_1^{-1}C$. Simple matrix computation gives us $A_1(P_c^{-1} - B\Sigma_1^{-1}B')^{-1}B = A_1P_cB(\Sigma_1 - B'P_cB)^{-1}\Sigma_1$ and $B'(P_c^{-1} - B\Sigma_1^{-1}B')^{-1}B = B'P_cB(\Sigma_1 - B'P_cB)^{-1}\Sigma_1$. Applying these matrix simplifications to the gain $K$ we get

$$
K = -\frac{\mu}{\mu^2 + \sigma^2} (B'P_cB)^{-1}B'P_cA_1
$$

Substituting this structure of $K$ in (6.33) we get

$$
P_c > A_1'P_cA_1 - A_1'P_cB \left[ \frac{\mu^2}{\mu^2 + \sigma^2} (B'P_cB)^{-1} - \frac{\sigma^2}{\mu^2 + \sigma^2} (\Sigma_1 - B'P_cB)^{-1} \right] B'P_cA_1 + C'\Sigma_1^{-1}C
$$

(6.34)
We now wish to give design a $P^*$ that will satisfy the above equation. Now suppose $P^*$ satisfies the minimum covariance like Riccati equation given by

$$P^* = A_1 P^* A_1' - A_1 P^* B (\Sigma_1 + B' P^* B)^{-1} B' P^* A_1 + C' \Sigma_1^{-1} C$$

then $P^*$ satisfies (6.34) if

$$\frac{\mu^2}{\mu^2 + \sigma^2} (B' P^* B)^{-1} - \frac{\sigma^2}{\mu^2 + \sigma^2} (\Sigma_1 - B' P^* B)^{-1} > (\Sigma_1 + B' P^* B)^{-1}$$

Thus the controller dynamics (6.1) is exponentially mean square stable if

$$\Sigma_1 - B' P^* B > \left( 1 - \frac{Q}{1 + Q} \right) (\Sigma_1 + B' P^* B)$$

(6.35)

This proves the result.

Our next theorem in on separation principle. We prove that the problem of observer-based controller design for nonlinear systems with uncertainties at the input and output channels can be decomposed into two separate problems of designing a full state feedback controller and a observer design problem. We prove the theorem on separation principle for more general nonlinear systems with channel uncertainties at the input and output.

$$x_{t+1} = f(x_t, \gamma_t, u_t), \quad y_t = h(x_t), \quad \tilde{y}_t = \xi_t y_t$$

$$\hat{x}_{t+1} = g(\hat{x}_t, \gamma_t, u_t, \xi_t, h(x_t))$$

(6.36)

where $x_t \in X \subseteq \mathbb{R}^N$, $y_t \in Y \subseteq \mathbb{R}^M$ and $u_t \in U \subseteq \mathbb{R}^M$ are the state, output and input respectively. $\hat{x}_t \in \mathbb{R}^N$ is the observer state. $\gamma_t \in W \subseteq \mathbb{R}$ and $\xi_t \in \{0, 1\}$ are assumed to be i.i.d random variables modeling the uncertainty at the input and output channels respectively. We notice that the observer dynamics $g$ in (6.36) is assumed to be the function of function of input channel erasure state $\gamma_t$. This is because of the assumed acknowledgement structure in the form of TCP (Assumption 81). At any given instant we have the observed state for $x_t$ given by $\hat{x}_t$. Using $\hat{x}_t$ the controller generates the input to be used $u_t = K \hat{x}_t$, which is further multiplied by $\gamma_t$ as the control is applied over the uncertain communication channel. As the system generates $x_{t+1}$ using $\gamma_t$ and $u_t$, we obtain the output $y_t$. The output is communicated to the observer through an uncertain channel with multiplicative uncertainty $\xi_t$. Using the
output $y_t$, uncertainty $\xi_t$, the control $u_t$ and the control channel uncertainty value $\gamma_t$ from the previous step, the observer generates the observation for $x_{t+1}$ given by the observed state $\hat{x}_{t+1}$. This is then used by the controller to generate the next control output $u_{t+1}$ which will be used to generate the next state $x_{t+2}$. Thus the TCP acknowledgement structure allows us to use the control uncertainty to generate the observed state in the following step. We now make following assumption on system (6.36).

**Assumption 90** For system (6.36), let $u_t = k(x_t)$ be the full state feedback control input, we assume that there exist Lyapunov functions $V_1(x_t)$ and $V_2(e_t)$, with $e_t := x_t - \hat{x}_t$, and positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, d_1, d_2$ and $d_3$ such that following conditions are satisfied.

\[
\bar{c}_1 ||x_t||^2 \leq V_1(x_t) \leq \bar{c}_2 ||x_t||^2, \quad E_{\gamma_t}[V_1(x_{t+1})] - V_1(x_t) \leq -\bar{c}_3 ||x_t||^2 \quad (6.37)
\]

\[
d_1 ||e_t||^2 \leq V_2(e_t) \leq d_2 ||e_t||^2, \quad E_{\gamma_t, \xi_t}[V_2(e_{t+1})] - V_2(e_t) \leq -d_3 ||e_t||^2 \quad (6.38)
\]

**Assumption 91** We assume that there exists positive constants $L_3, L_4, L_5, L_6, L_7$ such that $||\frac{\partial f}{\partial x}|| \leq L_3, ||\frac{\partial f}{\partial u}|| \leq L_4, ||k(x_t) - k(\hat{x}_t)|| \leq L_5 ||e_t||, ||\frac{\partial V_1}{\partial x}(x)|| \leq L_6 ||x||$, and $||\frac{\partial V_2}{\partial e}(e)|| \leq L_7 ||e||$.

**Remark 92** The results on separation principle for deterministic nonlinear systems exist. Our results in Theorem 94 extends these results for nonlinear systems with input and output channel uncertainty. The results in Theorem 94 can be considered as one of the contribution of this paper. Furthermore, separation theorem is proved for more general uncertainty and this will allow us to use the results of Theorem 94 in the proof of second main result of this paper (Theorem ??).

**Remark 93** The existence of Lyapunov functions satisfying conditions (6.37) and (6.38) in Assumption 90 combined with the results from Lyapunov based Theorem 86 ensure that state dynamics and observer error dynamics for system (6.36) is mean square exponentially stable. In particular, it follows that state dynamics, with full state feedback, and observer error dynamics for (6.36) satisfies following stability conditions.

\[
E_{\gamma_0}[||x_{t+1}||^2] \leq M_1 \beta_1^{t+1} ||x_0||^2, \quad E_{\gamma_0, \xi_0}[||e_{t+1}||^2] \leq M_2 \beta_2^{t+1} ||e_0||^2
\]
Theorem 94 is our main result on principle of separation.

**Theorem 94** Consider the observer-based controller design problem for system \((6.36)\) satisfying Assumptions 90 and 91. Then the state dynamics of system \((6.36)\) using estimated state, \(\hat{x}_t\), for the feedback control input (i.e., \(u_t = k(\hat{x}_t)\)) is mean square exponentially stable.

**Proof.** For \(u_t = k(z_t)\) we get

\[
E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] - V_1(x_t) = E_{\gamma_t} [V_1(f(x_t, \gamma_t, r(x_t)))] - V_1(x_t) + E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] - E_{\gamma_t} [V_1(f(x_t, \gamma_t, r(x_t)))]
\]

\[
\leq -c_3 ||x_t||^2 + E_{\gamma_t, \xi_t} \left[ ||\frac{\partial V_1}{\partial x}(f(x_t, \gamma_t, u))|| ||\frac{\partial f}{\partial u}(x_t, \gamma_t, u)|| ||r(z_t) - r(x_t)|| \right]
\]

\[
\leq -c_3 ||x_t||^2 + E_{\gamma_t, \xi_t} \left[ L_4 L_5 L_6 ||f(x_t, \gamma_t, u)|| ||z_t - x_t|| \right]
\]

where \(u = sr(x_t) + (1 - s)r(z_t)\) for some \(0 \leq s \leq 1\). Thus writing \(c_4 := L_4 L_5 L_6\) we get

\[
E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] \leq \beta_1 V_1(x_t) + c_4 E_{\gamma_t, \xi_t} ||f(x_t, \gamma_t, u) - f(x_t, \gamma_t, r(x_t))|| ||x_t - z_t||
\]

\[
+ c_4 E_{\gamma_t, \xi_t} ||f(x_t, \gamma_t, r(x_t))|| ||x_t - z_t||
\]

\[
\leq \beta_1 V_1(x_t) + c_4 L_4 L_5 ||x_t - z_t||^2
\]

\[
+ c_4 E_{\gamma_t, \xi_t} ||f(x_t, \gamma_t, r(x_t))||^2 \frac{1}{2} ||x_t - z_t||
\]

\[
\leq \beta_1 V_1(x_t) + c_4 L_4 L_5 ||x_t - z_t||^2
\]

\[
+ \frac{c_4}{\sqrt{c_1}} (\beta_1 V_1(x_t))^{\frac{1}{2}} ||x_t - z_t||
\]

Define \(c_5 := C_4 L_4 L_5\) and \(c_6 := \frac{c_4}{\sqrt{c_1}}\). Since \(\beta_1 < 1\) there exists \(\delta_1 > 0\) such that \(\beta_3 := (1 + \delta_1)\beta_1 < 1\). Then if \((\beta_1 V_1(x_t))^{\frac{1}{2}} \geq \frac{c_6}{\delta_1} ||x_t - z_t||\) we get

\[
E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] \leq \beta_3 V_1(x_t) + c_5 ||x_t - z_t||^2. \tag{6.48}
\]
In case \((\beta_1 V_1(x_t))^\frac{1}{2} \leq \frac{c_6}{\delta_1} ||x_t - z_t||\) we get
\[
E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] \leq \beta_1 V_1(x_t) + c_7 ||x_t - z_t||^2,
\]
where \(c_7 = c_5 + \frac{c_2}{\delta_1}\). Thus taking the supremum over both conditions (6.48) (6.49) we get
\[
E_{\gamma_t, \xi_t} [V_1(f(x_t, \gamma_t, r(z_t)))] \leq \beta_3 V_1(x_t) + c_7 ||x_t - z_t||^2.
\]
Thus we get
\[
E_{\gamma_0^{\ell_t}, \xi_0^{\ell_t}} [V_1(f(x_t, \gamma_t, r(z_t)))] \leq \beta_3^{\ell_t+1} V_1(x_0) + c_7 \sum_{i=0}^{\ell_t} E_{\xi_0^{\ell_t-1}} [||x_t - z_t||^2] \beta_3^{\ell_t-i}
\]
\[
\leq \beta_3^{\ell_t+1} V_1(x_0) + c_8 \beta_4^{\ell_t+1} ||x_0 - z_0||^2
\]
where \(\beta_4 = \max(\beta_3, \beta_2), c_8 := \frac{c_7 \beta_4}{\beta_4 - \min(\beta_2, \beta_3)}\). We can now use the inequality \(||x_0 - z_0||^2 \leq 2(||x_0||^2 + ||z_0||^2\) to get the desired result that the coupled system (6.36) is mean square exponentially stable.

We are now ready to provide the proof of Theorem 85.

**Proof of Theorem 85.** It is easy to verify that the system in Lure form (6.1) satisfies the Assumption 91. Furthermore using the results from Theorems 88 and 89 it follows that there exist quadratic Lyapunov functions satisfying Assumption 90. Hence results of Theorem 94 applies. The proof then follows by combining the results of Theorems 88, 89, and 94.

**6.4 Simulation**

In the simulation section we will look at a system in the Lur’e form that demonstrates chaotic behavior, as given in Brockett (1982). These systems are interesting as varying a parameter which determines the slope of the nonlinearity at the origin, displays varied behavior. As we vary the parameter the system goes through a bifurcation from a stable origin to an unstable equilibrium at the origin and demonstrates stable limit cycle oscillations. Further increase in the parameter causes the two stable equilibria to undergo a bifurcation and the system demonstrates chaotic behavior. A very similar behavior is observed in the dynamical model for flutter dynamics of an aeroelastic wing. Hence we feel that it is important to study these type
of systems. We consider the continuous time system (refer Figure (6.2a)) given by

\[
\begin{align*}
\dot{x} &= A_I x - B \phi_I(y) + Bu \\
y &= C x
\end{align*}
\]

where \( A, B \) and \( C \) are given by

\[
A_I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3.5 & -1 \end{bmatrix}; \\
B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \\
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

The nonlinearity \( \phi_I(y) \) is given by

\[
\phi_I(y) = \begin{cases} 
-k y, & ||y|| < 1 \\
2k y - 3 k \text{sgn}(y), & 1 < ||y|| \leq 3 \\
3 k \text{sgn}(y), & 3 < ||y||
\end{cases}
\]

where \( k = 4.5 \). We perform a loop transformation to bring the nonlinearity in a passive form to get a system

\[
\begin{align*}
\dot{x} &= A x - B \phi(y) + Bu \\
y &= C x
\end{align*}
\]

where \( A = A_I + (k + \epsilon) BC \) and \( \phi(y) = (k + \epsilon) y + \phi_I(y) \), where \( \epsilon = 0.1 \). We then transform this system to a discrete time system with a zero order hold to get the system

\[
\begin{align*}
x_{t+1} &= A_d x_t - B_d \phi(y_t) + B_d u_t \\
y_t &= C_d x_t
\end{align*}
\]

This discrete time system also demonstrates chaotic dynamics as shown in Figure (6.2b). We now implement our observer and controller design on the above discrete time system. From our sufficiency condition we get \( p_c = 0.3441 \) and \( q_c = 0.3441 \). We choose the probability of erasure for our simulation to be \( p = 0.35 \) and \( q = 0.35 \). In Figure (6.2c) and Figure (6.2d) we plot the observer based controlled state and observer error dynamics. We see that they both decay to zero as expected.
The system state and observer error decay to zero for almost all initial conditions and almost all sequences of the uncertainties $\gamma$ and $\xi$. In Figure (6.3a) we plot in blue the region of control and measurement non-erasure probabilities that guarantee mean square stable controlled system and observer dynamics. To be able to observe the mean square stable behavior we would need to simulate the system for all possible uncertainty sequences which is not practically possible. To observe the effect of high erasure probability on the system we plot the average time required by the system to converge to zero over 100 realizations of the uncertainty sequence in Figure (6.3b). Shown in red is average time to decay for the observer error and in blue is shown the average time to decay for the controlled state. We clearly see a sharp drop in the average decay time as the non-erasure probability is increased. We observe that the sharp drop in time of decay occurs below the critical non-erasure probability indicating that

Figure 6.2  (a) Schematic of continuous time system, (b) State dynamics in 3 dimensions, (c) Dynamics of controlled state, (d) Dynamics of observer error
the system spends significant time away from the origin for probabilities less than the critical probability that guarantees mean square stability. Thus during the time the system is away from the origin, roughly speaking the trajectories are move along the chaotic attractor of the uncontrolled system. Hence to be able to capture the mean square stable behavior we would like to displace the system from the origin and observe the effects of the attractor dynamics when the control is uncertain. In particular, a small amount of additive noise will make the system sensitive to the uncertainty in the control, preventing the system from converging to zero and we will be able to observe the effects of the attractor dynamics in presence of controller uncertainty. Similar effects will be observed from the observer error dynamics. We thus insert small additive gaussian noise with zero mean and covariance $R_v = 0.1$. In Figure (6.4a) and Figure (6.4b) we plot the system at probabilities above and below the critical probability at $p = 0.35$ and $p = 0.15$ respectively for the state dynamics. In Figure (6.4c) and Figure (6.4d) we plot the system at probabilities above and below the critical probability at $p = 0.35$ and $p = 0.15$ respectively for the observer error dynamics. We can see that for probability values where the system is mean square stable the system state and error have larger deviation from the origin.
6.5 Conclusion

We study the problem of observer based controller design for Lur’e systems over erasure channels. The erasure channels are modeled as on/off Bernoulli switches. We obtain a condition on the minimum required QoS which will guarantee exponential mean square stability of the observer error dynamics and the controlled state dynamics. This minimum QoS essentially gives us the minimum probability of non-erasure of the measurement and actuation channels. We see that the critical probability above which the observer is mean square stable needs to be increased if we wish to stabilize the state dynamics with a static gain controller which is based on the observed states.
CHAPTER 7. Conclusion

The importance of studying problems of control with practical constraints has been established over the past decade. Most of the research in this area has been done with a LTI structure of the plant dynamics. This study though insightful is inadequate to understand the true nature of the destabilizing effects of constraints in controller and plant interaction. We have tried to take this research to a new level by studying the problem for nonlinear systems which is more relevant in the practical world. We started out with the simple case of LTV plants, which simplified the analysis of the fundamental limitations and was a building block in deriving results for nonlinear systems. The performance criterion chosen was mean square exponential stability, which aims to minimize the variance of the closed loop system. This performance criterion is a common performance metric in the study of control systems and has been used to study the performance of linear systems over erasure channels. The fundamental limitations for LTV systems were obtained in terms of positive Lyapunov exponents for the LTV system. We then studied the problem of observation over erasure uncertainties in output measurement, for nonlinear systems. The analysis was carried out over the linear time varying state dependent system given by the tangent space dynamics. Thus borrowing from the ideas of the LTV results, under certain conditions on the system dynamics, we obtained the fundamental limitations for nonlinear systems in terms of the positive Lyapunov exponents of the system. Lyapunov exponents characterize the global expansion characteristics of nonlinear systems and were thus obtained as a generalization of eigenvalues in moving from linear to nonlinear systems. The results obtained for nonlinear systems were able to generate the results for linear systems under the assumption of linearity and time invariance of the plant dynamics. We then obtained similar results for stabilization problem of nonlinear systems over actuation channels with general stochastic multiplicative uncertainty. We were also able to connect the
fundamental limitations obtained for nonlinear systems to the entropy of the plant dynamics through the use of Ruelle’s inequality. Entropy is an important measure of complexity of dynamical systems that has been widely connected to fundamental limitations results in linear and nonlinear systems.

After obtaining analytical results for the fundamental limitations for nonlinear systems, we turned our attention to the problem of controller and observer synthesis for nonlinear systems with channel uncertainties, and the limitations imposed thereby. For this problem we chose the special class of nonlinear systems of the Lur’e form with sector bounded nonlinearity. We expanded the deterministic synthesis tools of KYP Lemma and Positive Real Lemma for absolute stability of Lur’e systems to the case of control and observation over uncertain channels, deriving a stochastic version of the Positive Real Lemma. We used this condition to design controller and observer gains for Lur’e systems. We finally proved a separation theorem for observer and controller of Lur’e systems thus obtaining the fundamental limitations on stabilization using state detection as the decoupled limitations on observation and control. We finally gave an interesting example of Lur’e system that demonstrates complex dynamics and can be used to model flutter dynamics. We showed through simulation results the effectiveness of the synthesis conditions in achieving the desired performance.
BIBLIOGRAPHY


