STRESS EFFECT ON ULTRASONIC WAVE PROPAGATION THROUGH
THE SOLID-SOLID AND LIQUID-SOLID PLANE INTERFACE

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INTRODUCTION

Ultrasonic wave propagation in prestressed materials has been studied extensively
in the last 40 years. Most of this work was concentrated on the effect of stress on the
velocities of different types of ultrasonic waves in homogeneous materials. Actually
stresses affect not only wave velocities but also the boundary conditions at the inter­
face. Many practical applications of ultrasonic stress characterization involve wave
propagation through the interface between fluid and solid or two solids. In immer­
sion measurements one needs to consider the effect of stress on wave propagation from
fluid to solid. This leads to change in propagation direction and energy redistribution.
Also additional modes could be excited leading to stress-induced birefringence. These
are all important phenomena which require rigorous quantitative description since the
stress effect in general is very small. Another important problem is ultrasonic charac­
terization of residual stresses in composite materials [1]. It involves wave propagation
through an interface between layers with different properties and stress levels.

Boundary conditions on a free surface for small perturbations in deformed materi­
als were first formulated by Hayes and Rivlin [3], when they considered Rayleigh wave
propagation in a stressed medium. Generalized relations on a free boundary with an
arbitrary orientation were presented by Iwashimizu and Kobori [4].

In this paper a unified approach for numerical solution of the reflection-transmission
problem on the plane boundary of separation between two generally anisotropic stressed
solids is described. Modifications when one of the solids is replaced by fluid is also
discussed. The approach is based on the computational procedure for wave propagation
through the plane interface between two anisotropic media proposed by Rokhlin et
al. [5]. Stresses are assumed to be locally homogeneous and satisfy static boundary
conditions. No assumption is made on the nature of the stresses which can be both ap­
plied and residual resulting from elastic or plastic deformations. Calculation examples
are presented for a Ti/SiC, water/plexiglass and water/aluminum interfaces. Stress­
induced and texture-induced birefringence and the resulting shear wave interference
are discussed. Acoustoelastic constants are calculated assuming hyperelasticity using
second and third order elastic constants and stresses [2].
To describe wave propagation in a prestressed medium the approach proposed by Man and Lu [6] is used. The prestressed configuration is the only reference configuration in this approach and the initial stress is included in the constitutive equation:

\[ \sigma_{ij} = \sigma_{ij}^0 + C_{ijkl} \varepsilon_{kl} + u_{i,k} \sigma_{kj}^0. \]  

where \( \sigma_{ij} \) is the first Piola-Kirchhoff stress tensor, \( \sigma_{ij}^0 \) is the initial static stress, \( \varepsilon_{ij} \) is the elastic strain due to wave propagation, \( u_{i,k} \) is the displacement gradient and \( C_{ijkl} \) is the fourth rank tensor of stress-dependent elastic constants. Eq. (1) gives the relation between stresses and displacements which arise due to wave propagation in the prestressed medium. It is an analog of Hooke’s law for the unstressed case. In general the stress \( \sigma_{ij} \) can be both applied and residual since there is no restriction that the resulting deformation be elastic.

The equation of motion has the following form:

\[ \sigma_{ij,j} = \rho \ddot{u}_i. \]  

Using (1), (2) can be rewritten as

\[ (C_{ijkl} + \sigma_{ii}^0 \delta_{jk}) u_{k,lt} = \rho \ddot{u}_i. \]  

Now assuming that the material and local (over the size of the transducer) stresses are homogeneous and using a plane wave solution for \( u \)

\[ u_k = AP_k e^{iK(n \cdot x - Vt)} \]

where \( A \) is the amplitude of the wave, \( P_k \) is the unit displacement vector, \( K = Kn = (\omega / V)n \) is the wave number, \( V = Vn \) is the wave velocity, \( n \) is the wave normal, and \( x \) is the position vector, one has the Christoffel equation for an anisotropic material under stress:

\[ [C_{ijkl} n_i n_l + (\sigma_{ii}^0 n_i n_l - \rho V^2) \delta_{jk}] P_k = 0. \]  

Eq. (5) has nontrivial solutions when the determinant is equal to zero

\[ |G_{jk} - \rho V^2 \delta_{jk}| = 0 \]

where \( G \) is the generalized Christoffel tensor with components

\[ G_{jk} = (C_{ijkl} + \sigma_{ii}^0 \delta_{jk}) n_i n_l. \]

It can be shown that \( G \) is symmetric (\( G_{ij} = G_{ji} \)) and the eigenvalue problem has three real solutions as for an unstressed medium.

Stress dependent elastic constants can be determined from second and third order elastic constants and stresses assuming that the deformation is hyperelastic. Formulas for stress dependent elastic constants are presented in [2].
BOUNDARY CONDITIONS ON A PLANE INTERFACE

In order to describe wave propagation through the interface between stressed media, we need to formulate boundary conditions on it. Let us consider a plane interface between two generally anisotropic stressed media (Fig. 1a). \( \mathbf{\nu} \) is the vector normal to the interface. Index I refers to the upper medium and II to the lower medium. The initial stresses are \((\sigma_{ij}^0)_I\) and \((\sigma_{ij}^0)_{II}\) for upper and lower media respectively.

For the initial static stressed state the boundary conditions represent the continuity of the traction forces:

\[
(\sigma_{ij}^0)_I = (\sigma_{ij}^0)_{II}. \quad (8)
\]

Consider a monochromatic plane wave (4) propagating from the upper to the lower medium. The boundary conditions at the interface represent continuity of displacements and traction forces. In the general case, for a wave incident from the upper medium there are three reflected (in the upper medium) and three transmitted (in the lower medium) waves. The boundary conditions are:

\[
\begin{align*}
\sigma_{ik}^{inc} \nu_k + \sum_{\alpha=1}^{3} (\sigma_{\alpha ij}^0)_{I} & = \sum_{\alpha=1}^{3} (\sigma_{\alpha ij}^0)_{II} \\
\mathbf{u}^{inc} + \sum_{\alpha=1}^{3} \mathbf{u}_{I}^{\alpha} & = \sum_{\alpha=1}^{3} \mathbf{u}_{II}^{\alpha} \\
\end{align*}
\]

Each of the terms \(\sigma_{ij} \nu_j\) can be written

\[
\sigma_{ij} \nu_j = C_{ijkl} u_{k,l} \nu_j + \sigma_{ij}^0 u_{ij} \nu_j \quad (10)
\]

Note that, in comparison with the boundary conditions for traction forces in the case of wave propagation through a plane boundary between two unstressed solids [5], the elastic constants are replaced with stress dependent elastic constants and an additional term \(\sigma_{ij}^0 u_{ij} \nu_j\) appears. Boundary conditions for displacements are the same as in the unstressed state.

In the case when the upper medium (I) is fluid (Fig. 1b), static boundary conditions represent the lack of traction force in the lower solid medium (II):

\[
(\sigma_{ij}^0)_{II} = 0. \quad (11)
\]

Dynamic boundary conditions represent the continuity of normal traction forces \(\sigma_n = \sigma_{ij} \nu_j \nu_i\) and normal displacements \(u_n = u_i \nu_i\):

\[
\begin{align*}
\mathbf{u}^{inc} + \mathbf{u}^{refl} & = \sum_{\alpha=1}^{3} \mathbf{u}_{II}^{\alpha} \\
\sigma^{inc} + \sigma^{refl} & = \sum_{\alpha=1}^{3} (\sigma_{n}^{\alpha})_{II} \\
\end{align*}
\]

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Also shear traction forces are equal to zero in the solid which can be expressed by the condition that traction vector is parallel to the interface normal $\mathbf{\nu}$:

$$\sigma_n \times \mathbf{\nu} = 0.$$  \hspace{1cm} (13)

**COMPUTATIONAL PROCEDURE**

Here we will discuss the changes which must be introduced into the computational procedure for reflection-refraction on the boundary between anisotropic unstressed solids described in [5] to take the stresses into account. These changes result from the new equation for wave propagation inside the stressed solid (Eq. 5) and the modified boundary conditions (Eq. 10).

Let us introduce a system of coordinates such that the 1-3 plane is the incident plane and the 2-3 plane is the plane of the interface (Fig. 1a). In the coordinate system selected the slowness vectors $\mathbf{m} = \mathbf{n}/V = \mathbf{K}/\omega$ for incident, reflected and transmitted waves will have only two nonzero components $m_1$ and $m_3$. It follows from Snell's law that all projections of the slowness vectors on the interface ($m_1^\text{inc}$) are equal and thus they are known since $m_1^\text{inc}$ is known. The unknown $m_3^a$ components can be determined from Eq. (6), rewritten in terms of the slowness vector:

$$\frac{1}{\rho} \left( \sigma_{ij}^{(0)} + \sigma_{ji}^{(0)} \delta_{jk} \right) m_i m_j - \rho V^2 \delta_{jk} \right| = 0$$  \hspace{1cm} (14)

This procedure should be carried out separately for upper and lower media. There will be six solutions for $m_3$ for each half space. To choose the three physical solutions for reflected and three for transmitted waves one must require that the energy flow be directed into the appropriate half space.

Wave velocities and propagation directions are determined from the slowness vectors for all existing waves. Then the wave polarizations are determined as the eigenvectors corresponding to the eigenvalues (velocities).

The unknown amplitudes $A^a$ of the displacements for reflected and transmitted waves can be found from boundary conditions (9) using slowness and polarization vec-
tors found. The boundary conditions can be written in the form of six linear algebraic equations:

\[ A^0 P^0_i + \sum_{\alpha=1}^{6} A^\alpha P^\alpha_i = 0, \]  
\[ A^0 C_{j\delta k} m^0_k P^0_i + \sum_{\alpha=1}^{6} A^\alpha C_{j\delta k} m^\alpha_k P^\alpha_i = 0. \]

The components of the ray velocity vector can be found if the displacement vector is known:

\[ V_{g_i} = \frac{1}{\rho} (C_{ijkl} + \sigma_{il}\delta_{jk}) m_i P_j P_k \]  

Coefficients of reflection and refraction can be found as

\[ R^\alpha = A^\alpha / A^{inc}, \]  
where \( A^\alpha \) is the amplitude of the reflected or refracted wave and \( A^{inc} \) is the amplitude of the incident wave.

It is more useful to discuss energy flow ratios than amplitude ratios. These transformation factors are defined in terms of the energy fluxes normal to the interfaces:

\[ r^\alpha = U^\alpha_3 / U^{inc}_3 \]  

The \( r^\alpha \) characterizes the redistribution of the incident energy flux between different reflected and transmitted modes. The energy flux density vector is determined as
\[ U_i = \bar{\sigma}_{ik} \bar{u}_k, \]  

(19)

where the bar indicates time averaging. It can be calculated as

\[ U_3 = \frac{1}{2} A^2 \omega^2 (C_{ijjk} + \sigma_{ij} \delta_{jk}) m_i P_j P_k \]  

(20)

The direction of the vector \( \mathbf{U} \) coincides with the direction of the ray velocity \( \mathbf{V}_g \).

For a fluid/solid interface Eq. (14) is to be solved for only the solid half space and the system of linear equations (15) representing boundary conditions reduces from 6x6 to 4x4. The four unknowns are the amplitudes of the three waves transmitted into the solid and the one wave reflected into the fluid.

**CALCULATION RESULTS AND DISCUSSION**

In this section we present computational examples based on the procedure described above to illustrate the effect of stress on wave propagation through solid/solid and liquid/solid interfaces. Also we discuss how stress influences shear wave interference.

**Wave Propagation through Ti/SiC Interface**

First we consider a Ti/SiC interface. Stresses are chosen to be similar to the residual stresses in fiber (SiC) and matrix (Ti alloy) in Ti/SiC metal matrix composites. They arise due to the mismatch in coefficients of thermal expansion during processing at high temperatures and subsequent cooling. These stresses can be estimated using a concentric cylinders model [7]. They vary through the thickness of the composite. In our calculations we take the average stress values for each phase. Thus, we assume a compressional normal stress of 483 MPa perpendicular to the interface in both materials, tensional normal stress of 400 MPa along the interface direction in Ti and compressional normal stress of 970 MPa along the interface in SiC (Fig. 2). Shear stresses are assumed to be zero. Both substrates are considered to be isotropic and also it is assumed that these stresses result from elastic deformations. In this case stress dependent elastic constants can be determined using formulas from [2], provided third order elastic constants are known. Unfortunately, third order elastic constants for SiC are not available in the literature. In these calculations they were replaced by those of Si. Third order elastic constants for Ti and Si are taken from [8]. Figure 2a shows the energy conversion coefficient from the quasilongitudinal wave in Ti to the quasishear wave in SiC. The dashed line represents the results for the unstressed state and the solid line for the stressed state. Figure 2b displays the refracted angle versus incident...
Figure 4. (a) energy transmission coefficient through water/plexiglas interface for shear waves in unstressed and stressed states; (b) shear wave velocities in plexiglas.

angle for the same mode transformation. One can see that the largest deviation is observed close to the critical angles and overall, despite the high stress level, the stress effect on energy redistribution is small.

Wave Propagation through Water/Solid Interface. Shear Wave Interference

In the next example we consider fluid/stressed solid interface. Ultrasonic wave is incident from water to plexiglass. The plexiglass is considered to be isotropic and under uniaxial tensile stress of 20 MPa. Calculations were made for the incident plane tilted by the angle $\phi = 45^\circ$ with respect to the stress direction (3). Energy transmission coefficients for the shear wave in unstressed and stressed states are shown at Fig. 4a. In the absence of stress there is only one shear wave in plexiglass (solid line), since the material is isotropic. The presence of stress causes birefringence and the appearance of two shear waves slow (short dashed line) and fast (long dashed line).

Stress-induced birefringence described above is very difficult to detect by trying to measure shear wave velocities. These waves have very similar velocities (Fig. 4b) and they need to travel a significant distance to produce a measurable time delay between each other. It is more feasible to consider the interference of these waves. Another problem is that birefringence could also be caused by material texture or other sources of anisotropy.

The next example is for a water/textured aluminum system. The material is considered to be orthotropic with degree of anisotropy of 1%. The angle between the incident plane and the axis of material symmetry is called the azimuthal angle (Fig.3b). There exists shear wave interference even in the absence of stresses due to anisotropy. The amplitudes of shear waves for this case are shown at Figure 5a as a function of azimuthal angle $\phi$ for a fixed incident angle $\theta_i = 18^\circ$. Figure 5b shows the amplitudes of the same waves in the presence of the uniaxial stress of 100 MPa which is applied along the material axis. One can see that in the unstressed state there exists a propagation direction at which the amplitudes of the transmitted shear waves are equal. There will be destructive interference if the wave is incident in this direction. It is shown at Figure 5c. In the presence of stress there is no direction in the chosen incident plane where the amplitudes are equal. Thus, in this case the interference is much less pronounced. The interference pattern is affected significantly by stresses.

SUMMARY

A rigorous procedure to describe ultrasonic wave propagation through the plane interface between two anisotropic stressed solids and fluid and anisotropic stressed solid was presented in this paper. The reflection/refraction problem is solved for arbi-
Figure 5. Amplitudes of shear waves transmitted from water to textured aluminum (a) without and (b) with stress; (c) Shear wave interference in unstressed and stressed textured aluminum.

It was also shown that shear wave interference is influenced by stresses. This effect is mostly due to changes in wave velocities. Destructive interference occurs only in directions where the shear wave amplitudes are equal or close to each other. The minimum position is sensitive to stresses. The main obstacle here is to distinguish the interference of shear waves induced by stresses and texture.

REFERENCES