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Asymptotic value distributions for certain 2xn games and n-stage games of perfect information

David Reginald Thomas
Iowa State University

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ASYMPTOTIC VALUE DISTRIBUTIONS FOR CERTAIN 2xn GAMES AND n-STAGE GAMES OF PERFECT INFORMATION

by

David Reginald Thomas

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

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Head of Major Department

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I. INTRODUCTION

This thesis is concerned with the distribution of the value of a game with random payoffs. Two types of games are considered: a certain class of games of perfect information with independent and identically distributed terminal payoffs and matrix games with independent and identically distributed matrix elements.

Consider a two-person zero-sum perfect information game, with Player I and Player II alternately choosing one of several alternatives. In the special games of this type considered below Player I and Player II each choose \( n \) times; moreover it is assumed that there are always \( p \) and \( q \) alternatives available respectively to Players I and II. There will be \( (pq)^n \) terminal payoffs (to Player I) \( x(i_1, i_2, \ldots, i_{2n}) \), where the indices \( i_1, i_2, \ldots, i_{2n-1} \), each with range \( (1, 2, \ldots, p) \), indicate the successive alternatives chosen by Player I, and the indices \( i_2, i_4, \ldots, i_{2n} \), each with range \( (1, 2, \ldots, q) \), indicate the successive alternatives chosen by Player II. The value of the game \( v(x(i_1, i_2, \ldots, i_{2n})) \) equals

\[
\max_{i_1} \min_{i_2} \max_{i_3} \min_{i_4} \cdots \max_{i_{2n-1}} \min_{i_{2n}} [x(i_1, i_2, \ldots, i_{2n})] \quad (1.1)
\]

Now replace the \( (pq)^n \) numbers \( x(i_1, i_2, \ldots, i_{2n}) \) by independent random variables \( X(i_1, i_2, \ldots, i_{2n}) \), each with distribution function \( F \).

The asymptotic behavior of the random value \( V_n = \)
\(v([X(i_1, i_2, \ldots, i_{2n})])\) is investigated in Chapters III through VI. Specifically, Chapter III treats the case of uniformly distributed payoffs, and it is shown that the asymptotic distribution of \(V_n\) is everywhere continuous and monotone-increasing and satisfies a certain functional equation. Chapter IV exhibits a functional equation necessarily satisfied by the asymptotic value distributions arising from general payoff distributions; this functional equation leads to a characterization of the set \(\mathcal{L}\) of all possible asymptotic value distributions. It is shown that, after a certain translation, scale normalization alone is sufficient to reach essentially any member of \(\mathcal{L}\); it is also shown to what extent the payoff distribution and the asymptotic value distribution determine this scale normalization. In Chapter V, a certain subset of \(\mathcal{L}\) is shown to attract a large number of payoff distributions and some examples are given. In Chapter VI, it is shown that all the asymptotic value distributions in this same subset possess all moments. It is also shown that moment convergence holds in a very special case.

Chapter VII treats the other general game situation considered here: matrix games with independent and identically distributed matrix elements. Let \(|| x_{ij} ||, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\) be the payoff matrix of a zero-sum two-person game, and let \(v(|| x_{ij} ||)\) be its (possible mixed) value. Consider the random value \(V_{m,n} = v(|| X_{ij} ||)\), where the \(X_{ij}\) are \(mn\) mutually independent random variables, each distributed
according to the probability density $f$. It is established in Chapter VII that the conditional distribution of $V_{m,n}$ given that it is pure, is that of the $n^{th}$ largest of $m + n - 1$ mutually independent random variables, each distributed according to $f$. For uniform $f$, a method is given for determining the conditional distribution of $V_{2,n}$, given that it is mixed. The asymptotic distributions of $V_{2,n}$ are also considered for densities $f$ that are zero for $t < t_0$, and are continuous to the right and discontinuous to the left at $t_0$.

Although the specific sorts of distributional problems considered in this thesis seem not to have been considered before, there is a substantial amount of related work. The distribution of the value of the sort of perfect information game considered here involves a type of functional iteration on the distribution function $F$ analogous to the product involved in the distribution of an extreme. Thus Gnedenko's (9) fundamental treatment of extremes has suggested much to the present treatment of perfect information game value distributions. The problems posed and solved in Chernoff and Teicher's (6) asymptotic treatment of the minimax operator are also related to this thesis, related indeed to both types of game situations treated here. In addition, the topics investigated by Sobel (18) and by Efron (8) relate to the subject matter of Chapter VII.

Although the distributional results obtained may be of
some independent interest, there also is a game-theoretic implication: Suppose two players are going to play a composite game $G$ consisting of the successive playing of $N$ zero-sum games $G_1, G_2, \ldots, G_N$. Then, as often happens also in the case of less trivial composite games (see 16, Appendix 8), $G$ is itself a zero-sum game for which the minimax strategies simply call for minimax strategies in the component games $G_i$. If now $N$ is large and the payoffs in the component games can be thought of as randomly selected from a single distribution, the average per-component game gain of Player I, in a single play of $G$, will be approximated by the expectation $E(V)$ of the value distribution; $E(V)$ thus approximates the per-component game payment of Player I to Player II that makes $G$ fair.

Nearly all of the results of this thesis are asymptotic, in the sense that the component games $G_i$ are large ($n$ large). "Law of large number" results (e.g., Theorem 3.1 and Theorem 4.1) and "central limit" results (e.g., Theorem 3.2 and Theorem 5.1) then provide, respectively, first-order and second-order approximations of $E(V)$.

The present results in Chapters V and VI regarding domains of attraction and moment convergence need, if possible, to be strengthened, and the development of Chapter VII brought

---

1As has been pointed out by Akio Kudo, for games with possibly mixed values (as considered in Chapter VII), there is also a pre-averaging approximation introduced here in the consideration of the distribution of the value, as opposed to the distribution of the random per-component game payoff.
to a level comparable to that of the rest of the work. In addition, recent developments in the theory of extremes (1, 2, 3) point to analogous ramifications here.
II. THE MAXIMIN FUNCTION $\phi$

A. Introductory Remarks

In Section B it will be shown how the distribution of the game value, for the games of perfect information introduced in Chapter I, can be found by iterating a certain function $\phi$. Properties of $\phi$ useful in the study of limiting distributions are given in Section C. Frequent reference will be made to these properties in subsequent chapters.

The behavior of the functional iteration for a large number of iterates (stages) is not easily traceable. However, the asymptotic behavior of the value distribution is obtainable by introducing more tractable bound functions $\mu$ and $\lambda$. This is done in Section E, where $\mu$ and $\lambda$ are shown to bound the portion of $\phi$ of asymptotic interest. In Section D $\mu$ and $\lambda$ are shown to bound $\phi$ everywhere when $p = q = 2$ (i.e., when two alternatives are available to each player at each move). This special property is used in the proof of moment convergence in Chapter VI.

B. Perfect Information Game Value Distributions

Define the "maximin" function $\phi$ as

$$\phi(x) = \left[1 - (1 - x)^q\right]^p \quad \text{for} \quad 0 \leq x \leq 1, \quad (2.1)$$
and the \(k\)th iterate of \(\phi\) as

\[
\phi^{(k)}(x) = \phi^{(k-1)}(\phi(x)) = \phi(\phi^{(k-1)}(x)) \quad \text{for} \quad 0 \leq x \leq 1.
\]

Let \(F\) be the common distribution function for the random terminal payoffs \(X(i_1, i_2, \ldots, i_{2n})\). Then, in view of the alternating minimum and maximum operations performed in the computation of the game value \(V_n\) (Expression 1.1), the distribution function \(F_{V_n}(x)\) for \(V_n\) satisfies the recurrence relation

\[
F_{V_{n+1}}(x) = \phi(\phi_{V_n}(x)),
\]

and hence is given by

\[
F_{V_n}(x) = \phi^{(n)}(\phi(x)) \quad \text{for} \quad 0 \leq x \leq 1.
\]

C. Some Properties of \(\phi\)

Several properties of \(\phi\), illustrated in part by Figure 1, are given in the two lemmas of this section.

**Lemma 2.1:**

(i) \(\phi(x)\) is monotone-increasing for \(0 \leq x \leq 1\).

(ii) There exists a unique point \(a\) (fixed-point) such that:

- \(0 < \phi(x) < x\) for \(0 < x < a\)
- \(x < \phi(x) < 1\) for \(a < x < 1\)
- \(\phi(0) = 0, \phi(a) = a, \text{ and } \phi(1) = 1\).
There exists a point $x_m$ such that the second derivative of $\phi$ satisfies

\[
\phi''(x) > 0 \quad \text{for} \quad 0 < x < x_m
\]

\[
\phi''(x) < 0 \quad \text{for} \quad x_m < x < 1
\]

and $\phi''(x_m) = 0$.

(iv) $x_m < a$ iff $\left[ \frac{q(p-1)}{pq-1} \right]^p < 1 - \left[ \frac{q-1}{pq-1} \right]^{1/q}$.

**Proof:** (1) By examining the form of the derivative

\[
\phi'(x) = pq \left[ 1 - (1-x)^q \right]^{p-1}(1-x)^{q-1},
\]

we see that $\phi'(x)$ is equal to zero at the points $x = 0, 1$ and is positive for $0 < x < 1$. Therefore $\phi(x)$ is monotone-increasing for $0 \leq x \leq 1$.

(iii) From the form of the second derivative,

\[
\phi''(x) = pq \left\{ (p-1)q \left[ 1 - (1-x)^q \right]^{p-2} \left[ (1-x)^{q-1} \right]^2 \\
- (q-1) \left[ 1 - (1-x)^q \right]^{p-1} (1-x)^{q-2} \right\}
\]

\[
= \left\{ pq \left[ 1 - (1-x)^q \right]^{p-2} (1-x)^{q-2} \right\} \left\{ (p-1)q(1-x)^q \\
- (q-1) \left[ 1 - (1-x)^q \right] \right\}
\]

\[
= \left\{ pq \left[ 1 - (1-x)^q \right]^{p-2} (1-x)^{q-2} \right\} \left\{ (pq-1)(1-x)^q - (q-1) \right\},
\]

we see that $\phi''(x) = 0$ iff $x = 0, 1$, or

\[(pq - 1)(1-x)^q - (q - 1) = 0.\]  

Equation 2.5 is satisfied by the unique point
\[ x_m = 1 - \left[ \frac{\frac{q}{p}}{p^2 - 1} \right]^{1/q}. \]  \hfill (2.6)

Since the left side of Equation 2.5 is monotone-decreasing, it follows by 2.6 that

\[ \phi''(x) > 0 \quad \text{for} \quad 0 < x < x_m \]  \hfill (2.7)
\[ \phi''(x) < 0 \quad \text{for} \quad x_m < x < 1. \]

(ii) Since

\[ \phi(0) = 0, \phi(1) = 1 \quad \text{and} \quad \phi'(0) = \phi'(1) = 0, \]  \hfill (2.8)

the continuous function \( \phi(x) \) has at least one fixed-point \( a \) in the interval \( 0 < x < 1 \). Also, since \( \phi''(x) \) changes sign only once in the interval \( 0 < x < 1 \), the fixed-point \( a \) is unique. From the uniqueness of \( a \) and from 2.8 it follows that

\[ 0 < \phi(x) < x \quad \text{for} \quad 0 < x < a \]
\[ x < \phi(x) < 1 \quad \text{for} \quad a < x < 1. \]  \hfill (2.9)

For future reference it will be convenient to have the fixed-point relation

\[ a = \left[ 1 - (1 - a)^q \right]^p \]

expressed in the alternate form

\[ a^{1/p} = 1 - (1 - a)^q. \]  \hfill (2.10)

(iv) In view of 2.9, \( x_m < a \) iff \( \phi(x_m) < x_m \), therefore,
by 2.6,
\[ x_m < a \quad \text{iff} \quad \left[ 1 - \left(1 - \left(1 - \left(\frac{q-1}{pq-1}\right)^{1/q}\right)^q\right)^p \right] < 1 - \left(\frac{q-1}{pq-1}\right)^{1/q}, \]

i.e.,
\[ x_m < a \quad \text{iff} \quad \left[ \frac{q(p-1)}{pq-1} \right]^p < 1 - \left(\frac{q-1}{pq-1}\right)^{1/q}. \quad (2.11) \]

**Lemma 2.2:**

\[ \lim_{n \to \infty} \theta^{(n)}(x) = \begin{cases} 0 & \text{for } 0 \leq x < a \\ a & \text{for } x = a \\ 1 & \text{for } a < x \leq 1. \end{cases} \]

**Proof:** Properties (i) and (ii) of Lemma 2.1 yield

\[ 0 < \theta^{(n)}(x) < \theta(x) < x \quad \text{for } 0 < x < a \quad (2.12) \]
\[ 1 > \theta^{(n)}(x) > \theta(x) > x \quad \text{for } a < x < 1 \]
\[ \theta^{(2)}(0) = 0, \quad \theta^{(2)}(a) = a, \quad \text{and} \quad \theta^{(2)}(1) = 1. \]

Therefore, by induction, it follows from 2.12 that

\[ 0 < \theta^{(n)}(x) < \theta^{(n-1)}(x) < x \quad \text{for } 0 < x < a \quad (2.13) \]
\[ 1 > \theta^{(n)}(x) > \theta^{(n-1)}(x) > x \quad \text{for } a < x \leq 1 \]
\[ \theta^{(n)}(0) = 0, \quad \theta^{(n)}(a) = a, \quad \text{and} \quad \theta^{(n)}(1) = 1 \quad \text{for } n = 1, 2, 3, \ldots \]

Since the sequence \{\theta^{(n)}(x)\} is monotone non-increasing for \(0 \leq x < a\), monotone non-decreasing for \(a < x \leq 1\), and is identically \(a\) for \(x = a\),
Since \( \phi \) is a continuous function, (2.14) yields

\[
H(x) = \lim_{n \to \infty} \phi^{(n+1)}(x)
\]

\[
= \phi \left[ \lim_{n \to \infty} \phi^{(n)}(x) \right] = \phi[H(x)].
\]

\( H(x) \) satisfies the inequality \( 0 \leq H(x) \leq 1 \) for \( 0 \leq x \leq 1 \) since it is the limit of functions which also satisfy this inequality. However, by (ii) of Lemma 2.1, Equation 2.15 is only satisfied when \( H(x) \) is equal to one of the fixed-points, that is,

\[
H(x) = 0, 1, \text{ or } a \quad \text{for } 0 \leq x \leq 1. \tag{2.16}
\]

Hence, (2.13) and (2.16) yield

\[
\lim_{n \to \infty} \phi^{(n)}(x) = 0 \quad \text{for } 0 \leq x < a
\]

\[
= a \quad \text{for } x = a
\]

\[
= 1 \quad \text{for } a < x \leq 1.
\]

D. Bounds for a Special \( \phi \)

In this section bounds are given for \( \phi(x) \) when \( p = q = 2 \). The relationship of the bound functions to the \( \phi \) function is
Figure 1. Maximin functions $\phi^{(n)}(x)$ for $n = 1, 2, 3$

$\phi(a) = a; \phi''(x_m) = 0$ (for $p = q = 2$)
Figure 2. Maximin function $\phi(x)$ and bound functions $\mu(x)$ and $\lambda(x)$ ($p = q = 2$)
Table 1. Fixed-points $a$ and slopes $b = f^*(a)$; $a < 1$; $b > 1$; * denotes $x_m < a$ ($x_m \neq a$ for $2 \leq p, q \leq 10$)

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<td>1.682</td>
<td>1.759</td>
<td>1.806</td>
<td>1.838</td>
<td>1.860</td>
<td>1.877</td>
<td>1.891</td>
</tr>
<tr>
<td>3</td>
<td>* 0.611</td>
<td>1.682</td>
<td>1.951</td>
<td>2.105</td>
<td>2.209</td>
<td>2.284</td>
<td>2.342</td>
<td>2.388</td>
<td>2.426</td>
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<tr>
<td>4</td>
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<td>1.760</td>
<td>2.105</td>
<td>2.314</td>
<td>2.459</td>
<td>2.567</td>
<td>2.653</td>
<td>2.723</td>
<td>2.781</td>
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<tr>
<td>5</td>
<td>* 0.780</td>
<td>* 0.489</td>
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<td>2.209</td>
<td>2.459</td>
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<td>* 0.539</td>
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<td>* 0.653</td>
<td>* 0.487</td>
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<td>2.831</td>
<td>3.108</td>
<td>3.326</td>
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</table>
Illustrated by Figure 2.

**Lemma 2.3:** For the case when $p = q = 2$

$$
\lambda(x) \leq \phi(x) \leq \mu(x) \quad \text{for} \quad 0 \leq x \leq 1,
$$

where $\lambda(x)$, $\phi(x)$, and $\mu(x)$ are defined by

$$
\mu(x) = a \left( \frac{x}{a} \right)^b
$$

$$
\phi(x) = \left[ 1 - (1 - x)^2 \right]^2
$$

$$
\lambda(x) = 1 - (1 - a) \left( \frac{1 - x}{1 - a} \right)^b,
$$

and $a$ and $b$ are defined by

$$
a = \phi(a)
$$

$$
b = \phi'(a).
$$

**Proof:** Let $x_o$ be the point satisfying $x_o = (1 - x_o)^2$. Then

$$
\phi(x_o) = \left[ 1 - (1 - x_o)^2 \right]^2 = (1 - x_o)^2 = x_o,
$$

which implies, by the uniqueness of the "interior" fixed-point $a$, that $x_o = a$. Thus the fixed-point $a$ satisfies the relation

$$
a = (1 - a)^2, \quad (2.17)
$$

and $b$ has the simple form
\[ b = \phi'(a) = 4\left[1 - (1 - a)^2\right](1 - a) = 4(1 - a)(1 - a) = 4a. \] (2.18)

It will first be verified that \( \phi(x) \leq \mu(x) \) for \( 0 \leq x \leq 1 \).

Directly from the definitions of \( \mu \) and \( \phi \) we have

\[ \mu(x) - \phi(x) = a\left(\frac{x}{a}\right)^b - x^2(2 - x)^2 
= \left[\sqrt{a}\left(\frac{x}{a}\right)^{\frac{b}{2}} - x(2 - x)\right]\left[\sqrt{a}\left(\frac{x}{a}\right)^{\frac{b}{2}} + x(2 - x)\right]. \] (2.19)

Since

\[ \sqrt{a}\left(\frac{x}{a}\right)^{\frac{b}{2}} + x(2 - x) \geq 0 \quad \text{for} \quad 0 \leq x \leq 1, \] (2.20)

with equality holding iff \( x = 0 \), 2.19 yields \( \mu(x) \geq \phi(x) \) iff

\[ \sqrt{a}\left(\frac{x}{a}\right)^{\frac{b}{2}} - x(2 - x) \geq 0. \] (2.21)

(Note that equality also holds in 2.21 iff \( x = 0 \).) Define

\[ \gamma(x) = a^{\frac{1-b}{2}}x^{\frac{b}{2}-1} - 2 + x \quad \text{for} \quad 0 < x \leq 1. \] (2.22)

Thus, 2.21 holds iff \( x = 0 \) or

\[ \gamma(x) \geq 0 \quad \text{for} \quad 0 < x \leq 1. \] (2.23)

But the derivative of \( \gamma(x) \) is

\[ \gamma'(x) = -\left[(1 - \frac{b}{2})a^\frac{1-b}{2}\right] / x^2 - \frac{b}{2} + 1, \] (2.24)
and Table 1 shows that $1 < b < 2$; therefore, $\gamma''(x)$ is monotone-increasing for $0 < x < 1$. Hence, if there is an $x_0$, $0 < x_0 < 1$, with $\gamma'(x_0) = 0$, then $\gamma(x_0) \leq \gamma(x)$ for $0 < x < 1$.

The function $\gamma(x)$ evaluated at $x = 0$ is, by 2.17, 2.18 and 2.22,

$$
\gamma(a) = a^{-b} a^{-2} - 2 + a
$$

$$
= 1/a^{3/2} - 1 - (1 - a)
$$

$$
= 1/(1 - a) - 1 - (1 - a)
$$

$$
= \left[1 - (1 - a) - (1 - a)^2 \right]/(1 - a)
$$

$$
= \left[1 - (1 - a) - a \right]/(1 - a) = 0, \quad (2.25)
$$

and $\gamma'(x)$ evaluated at $x = a$ is

$$
\gamma'(a) = - \left[ (1 - \frac{b}{2})a^{-\frac{b}{2}} \right]/a^{2 - \frac{b}{2}} + 1
$$

$$
= -(1 - 2a)/a^{3/2} + 1
$$

$$
= -(1 - 2a)/\left[ a(1 - a) \right] + 1
$$

$$
= \left[-1 + 2a + (1 - a) \right]/\left[ a(1 - a) \right]
$$

$$
= \left[-(1 - a)^2 + a \right]/\left[ a(1 - a) \right]
$$

$$
= \left[-a + a \right]/\left[ a(1 - a) \right] = 0. \quad (2.26)
$$

Hence, 2.26 implies by the monotonicity of $\gamma'(x)$ that $\gamma(x)$ is minimum at $x = a$ for $0 < x < 1$; this fact, together with 2.25, verifies Expression 2.23 and, consequently, Expression 2.20.

The next step is to verify that $\lambda(x) \leq \mu(x)$ for $0 \leq x \leq 1$. Directly from the definitions of $\lambda$ and $\mu$,
\[ \phi(x) - \lambda(x) = 1 - (1 - a) \left( \frac{1 - x}{1 - a} \right)^b - x^2(2 - x)^2 \]
\[ = \left[ \frac{1 - x(x(2 - x))}{1 + x(2 - x)} \right] - (1 - a) \left( \frac{1 - x}{1 - a} \right)^b \]
\[ = (1 - x)^2 \left[ 1 + x(2 - x) \right] - (1 - a)^l x^b(1 - x)^b \]
\[ = (1 - x)^2 \left[ 1 + 2x - x^2 - \frac{1}{[1 - (1 - a)^b](1 - x)^{2 - b}} \right] \]
\[ = (1 - x)^2 \delta(x). \quad (2.27) \]

Hence, 2.27 yields

\[ \phi(x) > \lambda(x) \quad \text{for} \ 0 \leq x \leq 1 \quad \text{iff} \ x = 1 \ \text{or} \]
\[ \delta(x) = 1 + 2x - x^2 - \frac{1}{[1 - (1 - a)^b](1 - x)^{2 - b}} \geq 0 \]
\[ \quad \text{for} \ 0 \leq x < 1. \quad (2.28) \]

The derivative of \( \delta(x) \) is

\[ \delta'(x) = 2(1 - x) - (2 - b)\left[ 1 - \frac{1}{(1 - a)^b(1 - x)^{2 - b}} \right] \]
\[ = (1 - x)\left[ 2 - (2 - b)\left[ 1 - \frac{1}{(1 - a)^b(1 - x)^{2 - b}} \right] \right]. \]

Hence, \( \delta'(x) = 0 \) for \( 0 \leq x < 1 \) iff

\[ 2 - (2 - b)\left[ 1 - \frac{1}{(1 - a)^b(1 - x)^{2 - b}} \right] = 0. \quad (2.29) \]

Since the left side of 2.29 is monotone-decreasing for \( 0 \leq x < 1 \), \( \delta'(x) = 0 \) for at most one point and \( \delta(x) \) is a minimum at that point. By using 2.17 and 2.18, Expression 2.28 evaluated at \( x = a \) is
\[ \delta(a) = 1 + 2a - a^2 - \frac{1}{(1 - a)^{b-1}(1 - a)^{2-b}} \]
\[ = 1 + 2a - a^2 - \frac{1}{1-a} \]
\[ = 2 - (1-a)^2 - \frac{1}{1-a} \]
\[ = 2 - a - \frac{1}{1-a} \]
\[ = \left\{1 + (1-a)\right\}(1-a) - 1)/\left(1-a\right) \]
\[ = \left\{1 - a + (1-a)^2 - 1\right\}/\left(1-a\right) \]
\[ = \left\{-a + a\right\}/\left(1-a\right) = 0, \quad (2.30) \]

and 2.29 evaluated at \( x = a \) is

\[ \frac{\delta'(a)}{1-a} = 2 - (2-b)/\left\{1-a\right\}^{b-1}(1-a)^{4-b} \]
\[ = 2 - (2-4a)/(1-a)^3 \]
\[ = 2\left\{1 - (1-2a)/\left\{a(1-a)\right\}\right\} \]
\[ = 2\left\{a - a^2 - 1 + 2a\right\}/\left\{a(1-a)\right\} \]
\[ = 2\left\{a - (1-a)^2\right\}/\left\{a(1-a)\right\} \]
\[ = 2(a-a)/\left\{a(1-a)\right\} = 0. \quad (2.31) \]

Hence, 2.30 and 2.31 establish that \( \delta(x) \geq 0 \) for \( 0 \leq x < 1 \);
therefore, Inequality 2.28 has been verified.

**Lemma 2.4:** For the case when \( p = q = 2 \)

\[ \lambda^{(n)}(x) \leq \phi^{(n)}(x) \leq \mu^{(n)}(x) \quad \text{for } 0 \leq x \leq 1; \quad n = 1,2,3,\ldots, \]

where \( \lambda(x) \), \( \phi(x) \), and \( \mu(x) \) are defined as in Lemma 2.3, and the
iterates \( \lambda^{(n)} \), \( \mu^{(n)} \) of \( \lambda \) and \( \mu \) are analogous to the iterates
\( \phi^{(n)} \) of \( \phi \).
proof: directly from the definitions of $\mu(x)$ and $\lambda(x)$ we see that $\lambda(x)$ and $\mu(x)$ are monotone-increasing for $0 \leq x \leq 1$. hence,

$$\lambda^{(2)}(x) = \lambda[\lambda(x)] \leq \lambda[\theta(x)] \leq \theta[\theta(x)] = \theta^{(2)}(x)$$

for $0 \leq x \leq 1$ \hspace{1cm} (2.32)

$$\theta^{(2)}(x) = \theta[\theta(x)] \leq \mu[\theta(x)] \leq \mu[\mu(x)] = \mu^{(2)}(x)$$

for $0 \leq x \leq 1$,

and the lemma follows from Lemma 2.3 and 2.32 by induction.

the iterates $\lambda^{(n)}$ and $\mu^{(n)}$ have simple form:

for $n = 2$

$$\lambda[\lambda(x)] = 1 - (1 - a)\left[\frac{1 - \lambda(x)}{1 - a}\right]^b$$

$$= 1 - (1 - a)\left\{1 - \frac{1 - (1 - a)(1 - x)^b}{1 - a}\right\}^b$$

$$= 1 - (1 - a)\left[\frac{(1 - x)^b}{1 - a}\right]^b$$

$$= 1 - (1 - a)(1 - x)^b^2, \hspace{1cm} (2.33)$$

and

$$\mu[\mu(x)] = a\left(\frac{\mu(x)}{a}\right)^b$$

$$= a\left(\frac{a(x)^b}{a}\right)^b$$

$$= a\left[(x)^b\right]^b$$
and, by induction,

\[\lambda^{(n)}(x) = 1 - (1 - a) \left( \frac{1 - x}{1 - a} \right)^n \quad \text{for } 0 \leq x \leq 1\]

\[\mu^{(n)}(x) = a \left( \frac{x}{a} \right)^n \quad \text{for } 0 \leq x \leq 1.

E. Fixed-Point Neighborhood Bounds for \( \phi \)

Numerical computations support the conjecture that for \( p \geq 3 \) and \( q \geq 3 \), \( \lambda \) and \( \mu \) bound \( \phi \) for \( 0 \leq x \leq 1 \); but proving this appears to be difficult. Therefore, since bounds in a neighborhood of the fixed point are sufficient for the development of Chapters III through V, it will be shown that \( \lambda \) and \( \mu \) bound \( \phi \) in some neighborhood of \( a \).

**Lemma 2.5:** There exists some neighborhood \( N(a) \) of the fixed-point \( a \) such that

\[\lambda(x) \leq \phi(x) \leq \mu(x) \quad \text{for } x \in N(a),\]

where

\[\lambda(x) = 1 - (1 - a) \left( \frac{1 - x}{1 - a} \right)^b\]

\[\mu(x) = a \left( \frac{x}{a} \right)^b, \quad \text{and} \quad b = \phi'(a).\]
Proof: The first and second derivatives of $\lambda$ and $\mu$ are

\[
\lambda'(x) = b \left( \frac{1 - x}{a} \right)^{b-1}
\]

\[
\lambda''(x) = -\frac{b(b - 1)}{1 - a} \left( \frac{1 - x}{a} \right)^{b-2}
\]

and

\[
\mu'(x) = b \left( \frac{x}{a} \right)^{b-1}
\]

\[
\mu''(x) = \frac{b(b - 1)}{a} \left( \frac{x}{a} \right)^{b-2}
\]

By the definition of $b$ and the evaluation of $\lambda'(a)$ and $\mu'(a)$,

\[
\lambda'(a) = \mu'(a) = \phi'(a) = b.
\]

Also, directly from the definitions,

\[
\lambda(a) = \mu(a) = \phi(a) = a.
\]

Equalities 2.37 and 2.38 constitute, by a simple application of the mean value theorem (see, for example, Kaplan (14, p. 120)), a proof for the lemma if it can be shown that

\[
\mu''(a) > \phi''(a),
\]

and

\[
\phi''(a) < \lambda''(a),
\]

both of which are now verified.

By using identity 2.10, evaluation of $\phi'(x)$ (Expression
2.3) at \( x = a \) yields

\[
\frac{b}{a} = \phi'(a) = pq\left[1 - (1 - a)^q\right]^{p-1}(1 - a)^{q-1} = pq(a^{1/p})^{p-1}\left[1 - a^{1/p}\right]/(1 - a) = \frac{pqa(a^{-1/p} - 1)}{(1 - a)}.
\]

(2.41)

Evaluation of \( \mu''(x) \) and \( \lambda''(x) \) at \( x = a \) gives

\[
\lambda''(a) = -\frac{b(b - 1)}{(1 - a)}, \quad \mu''(a) = \frac{b(b - 1)}{a}.
\]

(2.42)

By repeated use of identity 2.10, Evaluation of \( \phi''(x) \)

(Expression 2.4) at \( x = a \) yields, by 2.41,

\[
\phi''(a) = \left\{pq\left[1 - (1 - a)^q\right]^{p-2}(1 - a)^{q-2}\right\}\left\{(pq - 1)(1 - a)^q - (q - 1)\right\}
\]

\[
= \left\{pq\left[a^{1/p}\right]^{p-2}\frac{(1 - a^{1/p})}{(1 - a)^2}\right\}\left\{(pq - 1)(1 - a^{1/p}) - q + 1\right\}
\]

\[
= \frac{1}{1 - a}\left\{\frac{pqa(a^{-1/p} - 1)}{1 - a}\right\}\left\{q(p - 1)a^{-1/p} - pq + 1\right\}
\]

\[
= \frac{1}{1 - a}\left[\frac{b(p - 1)a^{-1/p}}{1 - a}\right] - pq + 1\right]\]

(2.43)

It now remains to compare the quantities \( \lambda''(a), \mu''(a), \) and \( \phi''(a) \). By dividing \( \mu''(a) \) and \( \phi''(a) \) by \( b \), it follows from 2.42 and 2.43 that 2.39 holds iff

\[
\frac{b}{a} - 1 > \frac{1}{1 - a}\left[q(p - 1)a^{-1/p} - pq + 1\right],
\]

or, equivalently, by substituting Expression 2.41 for \( b \), iff
\[ \frac{pq(a^{-1/p} - 1)/(1 - a) - 1}{a} > \frac{1}{1 - a} [ q(p - 1)a^{-1/p} - pq + 1 ], \]
i.e., iff
\[ \frac{pq(a^{-1/p} - 1)}{1 - a} - \frac{1}{a} > \frac{1}{1 - a} [ pq a^{-1/p} - qa^{-1/p} + 1 ], \]
which reduces after cancellation to
\[ -\frac{1}{a} > \frac{(-qa^{-1/p} + 1)}{1 - a}, \]
i.e., to
\[ a > \left( \frac{1}{a} \right)^\frac{p}{p-1}. \quad (2.44) \]

Thus \( \mu^n(a) > \emptyset^n(a) \) is equivalent to 2.44. The validity of 2.44 is now demonstrated. By (ii) of Lemma 2.1, 2.44 holds iff
\[ \emptyset \left[ \left( \frac{1}{q} \right)^{p-1} \right] < \left( \frac{1}{q} \right)^{p-1}, \quad \text{i.e.,} \quad \left[ 1 - \left( 1 - \left( \frac{1}{q} \right)^{p-1} \right)^q \right]^p < \left( \frac{1}{q} \right)^{p-1}, \]
and taking the pth root of each side yields
\[ 1 - \left( 1 - \left( \frac{1}{q} \right)^{p-1} \right)^q < \left( \frac{1}{q} \right)^{p-1} \quad \text{iff} \quad 1 - \left( \frac{1}{q} \right)^{p-1} < \left( 1 - \left( \frac{1}{q} \right)^{p-1} \right)^q, \]
and taking the qth root of each side gives
\[ \left( 1 - \left( \frac{1}{q} \right)^{p-1} \right)^{\frac{1}{q}} < 1 - \left( \frac{1}{q} \right)^{p-1} \quad \text{,} \quad (2.45) \]
Expanding the natural logarithm of each side of inequality 2.45 yields

\[ - \frac{1}{q} \left[ \frac{1}{p-1} \left( \frac{1}{q} \right)^{p-1} + \frac{1}{2} \left( \frac{1}{q} \right)^{p-1} + \frac{1}{3} \left( \frac{1}{q} \right)^{p-1} + \cdots \right] \]

or

\[ < - \left[ \left( \frac{1}{q} \right)^{p-1} + \frac{2}{p-1} \left( \frac{1}{q} \right)^{p-1} + \frac{3}{p-1} \left( \frac{1}{q} \right)^{p-1} + \cdots \right], \]

i.e.,

\[
\left[ \left( \frac{1}{q} \right)^{p-1} + \frac{1}{2} \left( \frac{1}{q} \right)^{p-1} + \frac{1}{3} \left( \frac{1}{q} \right)^{p-1} + \cdots \right] > \left[ \left( \frac{1}{q} \right)^{p-1} + \frac{2}{p-1} \left( \frac{1}{q} \right)^{p-1} + \frac{3}{p-1} \left( \frac{1}{q} \right)^{p-1} + \cdots \right],
\]

and inequality 2.46 holds since each term in the series on the left is larger than the corresponding term in the series on the right, with the exception of the first term in each series, these being equal. Therefore inequality 2.45 and consequently inequality 2.44 hold, and 2.39 has been verified.

Inequality 2.40 is now demonstrated in analogous fashion.
By multiplying \( \varphi''(a) \) and \( \lambda''(a) \) by \((1 - a)/b\) it follows from 2.42 and 2.43 that \( \varphi''(a) > \lambda''(a) \) iff

\[
q(p - 1)a^{-1/p} - pq + 1 > -b + 1,
\]

or, equivalently, by cancelling the 1 and substituting expression 2.41 for \( b \),

\[
q(p - 1)a^{-1/p} - pq > -pqa^{-1/p} - 1/(1 - a),
\]

or equivalently by multiplying by \((1 - a)/q\) and cancelling common terms,

\[
(1 - a)(p - 1)a^{-1/p} - (1 - a)p > -pa(a^{-1/p} - 1)
\]

iff

\[
(p - ap + a)a^{-1/p} - p + ap > -paa^{-1/p} + ap
\]

iff

\[
(p - l + a)a^{-1/p} < 0 \quad \text{iff} \quad p(a^{-1/p} - 1) > (1 - a)a^{-1/p},
\]

or equivalently by multiplying by \( a^{1/p} \), \( p(l - a^{1/p}) > l - a \),

and by using identity 2.10, \( p(1 - a)^q > 1 - a \), iff

\[
(1 - a)^{q-1} > \frac{1}{p}, \quad \text{iff} \quad 1 - a > \left(\frac{1}{p}\right)^{q-1} \quad \text{iff}
\]

\[
a < 1 - \left(\frac{1}{p}\right)^{q-1}. \quad (2.47)
\]

By (ii) of Lemma 2.1, 2.47 holds iff
\[ 1 - \left( \frac{1}{p} \right)^{q-1} < \phi \left[ 1 - \left( \frac{1}{p} \right)^{q-1} \right] \]

iff

\[ 1 - \left( \frac{1}{p} \right)^{q-1} < \left[ 1 - \left( 1 - \left( \frac{1}{p} \right)^{q-1} \right)^{q} \right]^{p} \]

iff

\[ 1 - \left( \frac{1}{p} \right)^{q-1} < \left[ 1 - \left( \frac{1}{p} \right)^{q-1} \right]^{p} \]

or, equivalently, by taking the \( p^{th} \) root of each side,

\[ \left[ 1 - \left( \frac{1}{p} \right)^{q-1} \right]^{\frac{1}{p}} < 1 - \left( \frac{1}{p} \right)^{q-1} \quad \text{(2.48)} \]

Notice that 2.48 is 2.45 with \( p \) and \( q \) interchanged. Therefore, the argument following 2.45 substantiates inequality 2.48. Consequently, 2.47 must also hold, which establishes 2.40.
III. THE LIMIT DISTRIBUTION FOR UNIFORM TERMINAL PAYOFFS

A. The Limit Distribution $L_u$

A proof is given in this chapter of the existence of a non-degenerate limiting distribution for $V_n$ when $F$ is uniform.

When $F$ is uniform $F_{V_n}(x)$ (Expression 2.2) takes the simple form

$$F_{V_n}(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\phi^{(n)}[F(x)] = \phi^{(n)}(x) & \text{for } 0 \leq x \leq 1 \\
1 & \text{for } x > 0.
\end{cases} \quad (3.1)$$

It will not be necessary to give explicit consideration to the two extreme portions of $\phi^{(n)}[F(x)]$; all subsequent arguments will be given in terms of its central part $\phi^{(n)}(x)$, $0 \leq x \leq 1$.

**Theorem 3.1**: When the terminal payoffs have a common uniform distribution, $V_n$ converges in probability to the constant $a$.

**Proof**: The proof follows directly from Lemma 2.2 and 3.1.

**Theorem 3.2**: When the terminal payoffs have a common uniform distribution, the sequence of random variables $\{b^n(V_n - a)\}$ converges in distribution as $n \to \infty$ to a non-
degenerate limit distribution $L_u(y)$. Furthermore, $L_u(y)$ is continuous, monotone-increasing and satisfies the functional equation

$$g^{(k)}\left[ L_u\left(\frac{y}{b^k}\right) \right] = L_u(y) \quad \text{for } -\infty < y < \infty; \ k = 1, 2, 3, \ldots.$$

The proof of Theorem 3.2 follows from Lemmas 3.1, 3.2, and 3.4 of Sections B and C.

The explicit functional form of $L_u(y)$ is not known. However, if solutions to the functional equation

$$g^{(k)}\left[ L_u\left(\frac{y}{b^k}\right) \right] = L_u(y) \quad \text{for } -\infty < y < \infty; \ k = 1, 2, 3, \ldots,$$

could be found, the solution $L_u$ could perhaps be characterized by use of the bound functions defined in Chapter II.

Frequent use will be made in this chapter of the fact that

$$P\left[ b^n(v_n - a) \leq y \right] = P\left[ v_n \leq \frac{y}{b^n} + a \right]$$

$$= g^{(n)}\left(\frac{y}{b^n} + a\right) \quad (3.2)$$

for $-\infty < y < \infty; \ n = 1, 2, 3, \ldots$. 
B. Functional Equation for $L_u$

**Lemma 3.1:**

$$\phi^{(k)} \left[ L_u \left( \frac{y}{b^k} \right) \right] = L_u(y) \quad \text{for } -\infty < y < \infty; \ k = 1, 2, 3, \ldots,$$

where

$$L_u(y) = \lim_{n \to \infty} \phi^{(n)} \left( \frac{y}{b^n} + a \right) \quad \text{for } -\infty < y < \infty.$$

**Proof:** The existence of $L_u$ will be shown in Lemma 3.2. Let $k$ be any positive integer. Then

$$L_u(y) = \lim_{n \to \infty} \phi^{(k+n)} \left( \frac{y}{b^{k+n}} + a \right)$$

$$= \phi^{(k)} \left[ \lim_{n \to \infty} \phi^{(n)} \left( \frac{y}{b^n} + a \right) \right]$$

$$= \phi^{(k)} \left[ L_u \left( \frac{y}{b^k} \right) \right]. \quad (3.3)$$

Note that the limit operation and the $k$-fold iteration can be interchanged, since the function $\phi^{(k)}$ is continuous.

For future reference, replacing $y$ by $b^k y$ in 3.3 yields

$$\phi^{(k)} \left[ L_u(y) \right] = L_u(b^k y). \quad (3.4)$$

For the case when $p = q = 2$, 3.4 has, for $k = 1$, the form
for $-\infty < y < \infty$ and $b = \phi'(a) = 0.382$. The author has been unable to find a treatment of functional equations of this form.

C. The Existence, Continuity, and Monotonicity of $L_u$

Lemma 3.2:

(i) $\lim_{n \to \infty} \phi(n) \left( \frac{x}{b^n} + a \right) = L_u(y)$ exists for $-\infty < y < \infty$.

(ii) $0 < L_u(y) < a$ for $y < 0$

$L_u(0) = a$

$a < L_u(y) < 1$ for $y > 0$,

and $L_u(-\infty) = 0$; $L_u(\infty) = 1$.

Proof: Let $z = x - a$ for $0 \leq x \leq 1$. Then the tangent line to the function $\phi(z + a)$ at the point $z = 0$ has the simple form

$bz + a$ for $-a \leq z \leq 1 - a$.

In view of (iii) of Lemma 2.1 consider the following three cases.
(a) If \( \varphi''(a) > 0 \), there is exactly one point \( z_o \), 
\[ 0 < z_o < 1 - a \], satisfying
\[ \varphi(z + a) \geq bz + a \quad \text{for} \quad -a \leq z \leq z_o. \]

(b) If \( \varphi''(a) < 0 \), there is exactly one point \( z_o \), 
\[ -a < z_o < 0 \], satisfying
\[ \varphi(z + a) \leq bz + a \quad \text{for} \quad z_o \leq z < 1 - a. \]

(c) If \( \varphi''(a) = 0 \), then
\[ \varphi(z + a) \geq bz + a \quad \text{for} \quad -a \leq z \leq 0 \]
\[ \varphi(z + a) \leq bz + a \quad \text{for} \quad 0 < z < 1 - a. \]

The function \( \varphi^{(n)} \) is monotone-increasing since it is the iterative composition of the monotone-increasing function \( \varphi \) ((i) of Lemma 2.1). Therefore, the relations
\[ \varphi^{(n+1)}\left(\frac{y}{b^{n+1}} + a\right) = \varphi^{(n)}\left[\varphi\left(\frac{y}{b^{n+1}} + a\right)\right] \leq \varphi^{(n)}\left(\frac{y}{b^n} + a\right) \]  
(3.5)
hold iff
\[ \varphi\left(\frac{y}{b^{n+1}} + a\right) \leq \frac{y}{b^n} + a. \]  
(3.6)

When \( z \) is replaced by \( \frac{y}{b^{n+1}} \) in (a), (b), and (c), the equivalence of 3.5 and 3.6 yields

(a') if \( \varphi''(a) > 0 \), then
\[ \varphi^{(n+1)}\left(\frac{y}{b^{n+1}} + a\right) \geq \varphi^{(n)}\left(\frac{y}{b^n} + a\right) \quad \text{for} \quad -b^{n+1}a \leq y \leq b^{n+1}z_o, \]
where $0 < z_0 < 1 - a$,

(b') if $\theta''(a) < 0$, then

$$\varrho^{(n+1)}(\frac{y}{b^{n+1}} + a) \leq \varrho^{(n)}(\frac{y}{b^n} + a)$$

for $b^{n+1}z_0 \leq y < b^{n+1}(1-a)$,

where $-a < z_0 < 0$, and

(c') if $\theta''(a) = 0$, then

$$\varrho^{(n+1)}(\frac{y}{b^{n+1}} + a) \leq \varrho^{(n)}(\frac{y}{b^n} + a)$$

for $-b^{n+1}a \leq y \leq 0$

$$\varrho^{(n+1)}(\frac{y}{b^{n+1}} + a) \geq \varrho^{(n)}(\frac{y}{b^n} + a)$$

for $0 < y \leq b^{n+1}(1-a)$.

Since $-b^{n+1}a \to -\infty$, $b^{n+1}(1-a) \to \infty$, and $b^{n+1}z_0 \to \pm \infty$

for $z_0 \geq 0$, as $n \to \infty$, relations (a'), (b'), and (c') yield

(a'') if $\theta''(a) > 0$, then the sequence $\{\varrho^{(n)}(\frac{y}{b^n} + a)\}$ is

eventually monotone non-decreasing for each $y$,

(b'') if $\theta''(a) < 0$, then the sequence $\{\varrho^{(n)}(\frac{y}{b^n} + a)\}$ is

eventually monotone non-increasing for each $y$,

(c'') if $\theta''(a) = 0$, then $\{\varrho^{(n)}(\frac{y}{b^n} + a)\}$ is eventually monotone

non-increasing for each $y \leq 0$ and eventually monotone

non-decreasing for each $y > 0$.

Thus it follows, since the sequence $\{\varrho^{(n)}(\frac{y}{b^n} + a)\}$ is bounded

between zero and one for all $y$, that

$$\lim_{n \to \infty} \varrho^{(n)}(\frac{y}{b^n} + a) = L_u(y)$$

exists and $0 \leq L_u(y) \leq 1$ for all $y$.

(ii) By Lemma 2.5, there exists some neighborhood $N(a)$

of the point $z + a = a (z = 0)$ such that
\( \lambda(z + a) \leq \phi(z + a) \leq \mu(z + a) \quad \text{for} \quad z + a \in N(a) \), \hspace{1cm} (3.7)

where

\[
\lambda(z + a) = 1 - (1 - a)(1 - \frac{z}{1 - a})^b \\
\mu(z + a) = a(1 + \frac{z}{a})^b.
\] \hspace{1cm} (3.8)

Since \( \lambda \) and \( \mu \) are monotone-increasing functions with common fixed-point \( a \), a number \( \tau > 0 \) can be chosen small enough so that

\[
a - \tau \in N(a) \quad \text{and} \quad a + \tau \in N(a) \quad (3.9)
\]

and

\[
a - \tau \leq \lambda(z + a) \quad \text{and} \quad \mu(z + a) \leq a + \tau \quad \text{imply} \quad z + a \in N(a) \quad (3.10)
\]

Define the sets

\[
Z_{k, \tau} = \{ z : a - \tau \leq \lambda^{(k)}(z + a) \quad \text{and} \quad \mu^{(k)}(z + a) \leq a + \tau \}
\] \hspace{1cm} (3.11)

for \( k = 1, 2, \cdots \);

then by 2.35 and 3.8

\[
\lambda^{(k)}(z + a) = \left( 1 - (1 - a)\left( \frac{1 - (z + a)}{1 - a} \right)^b \right)^k < 1 - (1 - a)\left( \frac{1 - (z + a)}{1 - a} \right)^{b^{k-1}} = \lambda^{(k-1)}(z + a)
\] \hspace{1cm} (3.12)

since \( \frac{1 - (z + a)}{1 - a} > 1 \), and similarly,
\[ \mu^k(z + a) = a(1 + \frac{z}{a})^k \quad (3.13) \]

\[ > a(1 + \frac{z}{a})^{k-1} = \mu^{k-1}(z + a) \quad \text{for } 0 < z \leq 1 - a, \]

since \(1 + \frac{z}{a} > 1\). Hence, by induction

\[ \cdots \cdots < Z_{k, \tau} < Z_{k-1, \tau} < \cdots < Z_{1, \tau} < N(a). \quad (3.14) \]

Now define

\[ Y_{\infty, \tau} = \left\{ y : a - \tau \leq 1 - (1 - a)e^{-\frac{y}{1-a}} \leq ae^{y/a} \leq a + \tau \right\}, \quad (3.15) \]

and let \( y \) and \( \bar{y} \) be the lower endpoint and upper endpoint, respectively, of \( Y_{\infty, \tau} \). Then for any interior point \( y \in Y_{\infty, \tau} \)

\[ \lim_{n \to \infty} \lambda^{(n)}(\frac{y}{b^n} + a) = 1 - (1 - a)e^{-\frac{y}{1-a}} > 1 - (1 - a)e^{-\frac{y}{1-a}} = a - \tau, \]

and

\[ \lim_{n \to \infty} \mu^{(n)}(\frac{y}{b^n} + a) = ae^{y/a} < ae^{\bar{y}/a} = a + \tau; \]

therefore, there exists an integer \( N \) such that

\[ a - \tau \leq \lambda^{(n)}(\frac{y}{b^n} + a) \leq \mu^{(n)}(\frac{y}{b^n} + a) \leq a + \tau \quad \text{for } n \geq N, \]

i.e.,

\[ \frac{y}{b^n} \in Z_{n, \tau} \quad \text{for } n \geq N. \quad (3.16) \]
Hence 3.14 and 3.15 imply that for any \( y \) in the open interval \((y, \bar{y})\)

\[
\frac{y}{b^n} < Z_k \tau < Z_{k-1} \tau < \cdots < Z_1 \tau < N(a) \quad \text{for } n \geq N;
\]

\[k = 3, 4, \ldots.
\]

(3.17)

Since \( \mu \) and \( \lambda \) are monotone-increasing 3.7, 3.9, 3.10 and 3.17 (with \( z \) replaced by \( y/b^n \) in 3.7, 3.9 and 3.10) yield, for \( y \in Y_\infty, \tau \) and \( n \geq N \),

\[
\lambda^{(2)}\left(\frac{y}{b^n} + a\right) = \lambda \left[ \lambda \left(\frac{y}{b^n} + a\right) \right] \leq \lambda \left[ \lambda \left(\frac{y}{b^n} + a\right) \right] \leq \lambda \left(\frac{y}{b^n} + a\right) \\
= \mu^{(2)}\left(\frac{y}{b^n} + a\right) = \mu \left[ \mu \left(\frac{y}{b^n} + a\right) \right] \leq \mu \left(\frac{y}{b^n} + a\right),
\]

i.e.,

\[
\lambda^{(2)}\left(\frac{y}{b^n} + a\right) \leq \mu^{(2)}\left(\frac{y}{b^n} + a\right) \leq \mu^{(2)}\left(\frac{y}{b^n} + a\right). 
\]

(3.18)

And applying 3.7 to 3.18 gives

\[
\lambda^{(3)}\left(\frac{y}{b^n} + a\right) = \lambda \left[ \lambda^{(2)}\left(\frac{y}{b^n} + a\right) \right] \leq \lambda \left[ \lambda^{(2)}\left(\frac{y}{b^n} + a\right) \right] \leq \lambda \left(\frac{y}{b^n} + a\right) \\
\leq \lambda \left(\frac{y}{b^n} + a\right) \leq \lambda \left(\frac{y}{b^n} + a\right) \leq \mu \left(\frac{y}{b^n} + a\right) \\
\leq \mu \left(\frac{y}{b^n} + a\right) \leq \mu^{(3)}\left(\frac{y}{b^n} + a\right),
\]

and in general, by induction,

\[
\lambda^{(k)}\left(\frac{y}{b^n} + a\right) \leq \mu^{(k)}\left(\frac{y}{b^n} + a\right) \leq \mu^{(k)}\left(\frac{y}{b^n} + a\right) \text{ for } k = 1, 2, \ldots, n.
\]
Hence, \( y \in (\overline{y}, \bar{y}) \) implies
\[
\lambda^{(n)}(\frac{Y}{b^n} + a) \leq \phi^{(n)}(\frac{Y}{b^n} + a) \leq \mu^{(n)}(\frac{Y}{b^n} + a)
\] for \( n \geq N \),

which leads to
\[
1 - (1 - a)e^{-\frac{Y}{1-a}} = \lim_{n\to\infty} \lambda^{(n)}(\frac{Y}{b^n} + a) \leq \lim_{n\to\infty} \phi^{(n)}(\frac{Y}{b^n} + a) = L_u(y)
\]
\[
\leq \lim_{n\to\infty} \mu^{(n)}(\frac{Y}{b^n} + a) = ae^{Y/a}.
\]

Then it follows from 3.19 that
\[
0 < L_u(y) < a \quad \text{for } y < y < 0
\]
\[
L_u(0) = a \quad \text{(3.20)}
\]
\[
a < L_u(y) < 1 \quad \text{for } 0 < y < \overline{y}.
\]

The functional equation (Lemma 3.1)
\[
\lim_{k\to\infty} \phi^{(k)}[L_u(\frac{Y}{2})] = \lim_{k\to\infty} L_u(b^k \frac{Y}{2}) = L_u(-\infty)
\]
then yields
\[
L_u(-\infty) = 0
\]

by Lemma 2.2 since \( L_u(\overline{y}/2) < a \), and, similarly, \( L_u(\overline{y}/2) > a \),
gives
\[
L_u(\infty) = 1.
\]
The alternate form of the functional equation, that is,
\[ \varphi(k)[L_u(y)] = L_u(b^k y) \quad \text{for } k = 1, 2, 3, \ldots, \]
implies by the fixed-point property of zero and one that
\[ 0 < L_u(y) < 1 \quad \text{for } -\infty < y < \infty, \quad (3.23) \]
for, if not, \( k \) could be chosen large enough, for any \( y \), so that the inequalities in 3.20 would be violated. (ii) now follows from 3.20, 3.21, 3.22, and 3.23.

**Lemma 3.3:** Let \( f, g, \) and \( f_n, n = 1, 2, 3, \ldots \) represent uniformly bounded functions defined over some interval \( I \).

(i) If \( f(x) \) is convex (concave) and bounded for \( x \in I \), then \( f(x) \) is continuous for \( x \in I \).

(ii) If \( f(x) \) and \( g(x) \) are convex (concave) and \( f(x) \) is monotone-increasing for \( x \in I \), then \( f \left[ g(x) \right] \) is convex (concave) for \( x \in I \).

(iii) If \( \{f_n(x)\} \) is a sequence of convex (concave) functions for \( x \in I \), then \( \lim_{n \to \infty} f_n(x) = f(x) \) is convex (concave) for \( x \in I \).

(iv) If \( f''(x) \geq 0 \) (\( \leq 0 \)) for \( x \in I \), then \( f(x) \) is convex (concave) for \( x \in I \).

For proofs of (i) through (iv), except the easily verified (iii), see Hardy (12, Chapter 3).
Lemma 3.4:

(i) \( L_u(y) \) is continuous for \(-\infty < y < \infty\).

(ii) \( L_u(y) \) is monotone-increasing for \(-\infty < y < \infty\).

**Proof:** (i) Let \( x_m \) be defined as in Lemma 2.1, that is, \( \varphi^n(x_m) = 0 \). From (iii) of Lemma 2.1 and (iv) of Lemma 3.3 \( \varphi(x) \) is convex for \( 0 \leq x \leq x_m \) and concave for \( x_m \leq x \leq 1 \).

Then (ii) of Lemma 2.1 and (ii) of Lemma 3.3 yield:

(a) when \( x_m > a \), \( \varphi^{(2)}(x) \) is convex for all \( x \) satisfying 
\[ \varphi^{(2)}(x) \leq x_m, \]

(b) when \( x_m < a \), \( \varphi^{(2)}(x) \) is concave for all \( x \) satisfying 
\[ \varphi^{(2)}(x) \geq x_m, \]

(c) when \( x_m = a \), \( \varphi^{(2)}(x) \) is convex (concave) for \( x \leq a \) (\( x \geq a \)).

Thus, by induction, for any positive integer \( n \),

(a') when \( x_m > a \), \( \varphi^{(n)}(x) \) is convex for all \( x \) satisfying 
\[ \varphi^{(n)}(x) \leq x_m, \]

(b') when \( x_m > a \), \( \varphi^{(n)}(x) \) is concave for all \( x \) satisfying 
\[ \varphi^{(n)}(x) \geq x_m, \]

(c') when \( x_m = a \), \( \varphi^{(n)}(x) \) is convex (concave) for \( x \leq a \) (\( x \geq a \)).

Let \( y = b^n(x - a) \). The convexity (concavity) of a function is preserved under a linear transformation of its argument; therefore (a'), (b'), and (c') imply

(a'') when \( x_m > a \), \( \varphi^{(n)}(\frac{y}{b^n} + a) \) is convex for all \( y \) satisfying 
\[ \varphi^{(n)}(\frac{y}{b^n} + a) \leq x_m, \]

(b'') when \( x_m > a \), \( \varphi^{(n)}(\frac{y}{b^n} + a) \) is concave for all \( y \) satisfying 
\[ \varphi^{(n)}(\frac{y}{b^n} + a) \geq x_m, \]

(c'') when \( x_m = a \), \( \varphi^{(n)}(\frac{y}{b^n} + a) \) is convex (concave) for \( y \leq a \) (\( y \geq a \)).
(b") when $x_m < a$, $\varphi(n)\left(\frac{y}{b^n} + a\right)$ is concave for all $y$ satisfying $\varphi(n)\left(\frac{y}{b^n} + a\right) \geq x_m$, and

(c") when $x_m = a$, $\varphi(n)\left(\frac{y}{b^n} + a\right)$ is convex (concave) for $y \leq 0$ ($y \geq 0$).

Then, from (iii) of Lemma 3.3 the corresponding limit statements for (a"), (b"), and (c") are

(a""") when $x_m > a$, $L_u(y)$ is convex for all $y$ satisfying $L_u(y) < x_m$,

(b""") when $x_m < a$, $L_u(y)$ is concave for all $y$ satisfying $L_u(y) > x_m$, and

(c""") when $x_m = a$, $L_u(y)$ is convex (concave) for $y \leq 0$ ($y \geq 0$).

Thus, continuity of $L_u(y)$ in a neighborhood of $y = 0$ follows from (i) of Lemma 3.3. Since $|y/b^k|$ can be made arbitrarily small by choosing $k$ sufficiently large, and $\varphi(k)\left[L_u\left(\frac{y}{b^k}\right)\right]$ is continuous iff $L_u\left(\frac{y}{b^k}\right)$ is continuous, it follows from the functional equation (Expression 3.3) that $L_u(y)$ is continuous for $-\infty < y < \infty$.

(ii) Suppose that $L_u(y)$ is not monotone-increasing; then there are points $y_1 < y_2$ satisfying $L_u(y_1) = L_u(y_2)$.

($L_u(y)$ is non-decreasing since it is the limit of a sequence of non-decreasing functions.) By 3.3,

$$\varphi(n)\left[L_u\left(\frac{y_1}{b^n}\right)\right] = L_u(y_1) = L_u(y_2) = \varphi(n)\left[L_u\left(\frac{y_2}{b^n}\right)\right]$$

for $n = 1, 2, 3, \ldots$. 
which implies, by the monotonicity of $\phi^{(n)}$,

$$L_u\left(\frac{y_1}{b^n}\right) = L_u\left(\frac{y_2}{b^n}\right) \quad \text{for } n = 1, 2, 3, \ldots \quad (3.25)$$

From 3.20 it follows that either $y_1/b^n < y_2/b^n < a$ or $a < y_1/b^n < y_2/b^n$ for $n = 1, 2, 3, \ldots$. From statements $(a''''), (b''')$, and $(c''')$ it follows that there exists some $\delta > 0$ such that $L_u(y)$ is either convex or concave for $-\delta \leq y \leq 0$ and either convex or concave for $0 \leq y \leq \delta$. Consider now the case where $a < y_1 < y_2$, and let $N$ be chosen large enough so that $y_2/b^N < \delta$. Then, in view of 3.20,

$$a < L_u\left(\frac{y_1}{b^{n+1}}\right) = L_u\left(\frac{y_2}{b^{n+1}}\right) \leq L_u\left(\frac{y_1}{b^n}\right) = L_u\left(\frac{y_2}{b^n}\right) \quad \text{for all } n \geq N. \quad (3.26)$$

However, $L_u(y)$ convex or concave for $a \leq L_u(y) \leq \delta$ implies that $L_u(y_2/b^{n+1}) = L_u(y_1/b^n)$ for all $n \geq N$ (i.e., a bounded convex (or concave) function can have at most one horizontal line segment), which implies $L_u(+0) > a$. But this contradicts, by the continuity of $L_u(y)$, that $L_u(0) = a$. A similar argument holds when $y_1 < y_2 < a$; hence, $L_u(y)$ is monotone-increasing.
IV. LIMIT DISTRIBUTIONS

A. Introductory Remarks

In this chapter limit distributions will be considered for game value distributions arising from arbitrary $F$, where, as before, $F$ represents the common distribution of the random terminal payoffs. Definition 4.1 and Lemmas 4.1, 4.2, and 4.3 are taken from the references cited.

For distribution functions $H$, $\{H_n\}$ the phrase "$H_n \to H$ as $n \to \infty$" will indicate convergence in distribution, i.e.,

\[
\lim_{n \to \infty} H_n = H \text{ for all continuity points of } H.
\]

**Definition 4.1** (Gnedenko and Kolmogorov, 10, p. 41): The distribution functions $H_1(x)$ and $H_2(x)$ are said to be of the same type if, for some constants $\alpha > 0$ and $\beta$, $H_1(x) = H_2(\alpha x + \beta)$ for $-\infty < x < \infty$.

**Lemma 4.1** (Gnedenko and Kolmogorov, 10, pp. 40-42): If $H_n(x) \to H(x)$ as $n \to \infty$, $H(x)$ non-degenerate, then for any choice of constants $a_n > 0$ and $b_n$ the sequence $\{H_n(a_n x + b_n)\}$ can converge only to a non-degenerate distribution of the same type as $H(x)$.

**Lemma 4.2** (Gnedenko, 9, pp. 435-437): Let $H(x)$, $H_n(x)$ be distribution functions, $H(x)$ non-degenerate. If for constants $a_n > 0$, $b_n$, $a'_n > 0$, $b'_n$
Lemma 4.3 (Gnedenko, 9, pp. 437-438): Let $H(z)$, $H'(x)$ be distribution functions, $H(x)$ non-degenerate. If for constants $a_n > 0$, $b_n$, $a'_n > 0$, $b'_n$

\[ H_n(a_n x + b_n) \rightarrow H(x), \]

and

\[ H_n(a'_n x + b'_n) \rightarrow H(x) \]

as $n \rightarrow \infty$, then

\[ \frac{a_n}{a'_n} \rightarrow 1 \quad \text{and} \quad \frac{b_n - b'_n}{a_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \]

Directly from Lemma 2.2 one has:

Theorem 4.1: The sequence of distribution functions $\{g^{(n)}[F(x)]\}$ converges as $n \rightarrow \infty$ to the distribution function $G(x)$, defined at its continuity points by

\[ G(x) = \begin{cases} 
0 & \text{for } F(x) < a \\
1 & \text{for } F(x) > a \\
a & \text{for } F(x) = a
\end{cases} \quad (4.1) \]
If $G(x) \neq a$ for any $x$, then $V_n$ converges in probability to the constant $x_a$ defined by $F(x_a - 0) \leq a \leq F(x_a)$, that is, $V_n$ converges to the degenerate distribution function with saltus unity at $x_a$. Thus, by Lemma 4.1, if $G(x) = a$ on some interval (which implies $F(x) = a$ for some $x$), all the non-degenerate limit distributions for $V_n$ are of the two-step form defined by 4.1. This type of limit distribution for $V_n$ belongs to the $4^{th}$ class of limit distributions defined in Theorem 4.2.

B. Functional Equation for Limit Distributions

Gnedenko (9), in his work on extreme values, shows that the limiting distributions $G$ (Gnedenko's $\phi$) must satisfy a functional equation of the form

$$\left[G(\alpha x + \beta)\right]^t = G(x) \quad \text{for} \quad -\infty < x < \infty; \quad t = 1, 2, 3, \ldots. \quad (4.2)$$

He was able to show that $\alpha_1 \leq 1$ for some $t$ implies $\alpha_1 \leq 1$ for all $t$.

Define the functional operator $\psi(x) = x^t$, in terms of which 4.2 may suggestively be written

$$\psi \left[G(\alpha x + \beta)\right] = G(x) \quad (4.3)$$

and, indeed, Gnedenko's relation

$$G\left[\alpha^k x + \beta (1 + \alpha_1 + \cdots + \alpha_{k-1})\right]^t = G(x)$$
may be written

\[ \psi^{(k)} [ G(a^k x + \beta (l + a + \cdots + a^{k-1})] = G(x). \]

This last relation differs from the relation of Lemma 4.4 only in the particular form of the operator \( \psi (\beta) \) involved, and explains the similarities between this section and the corresponding portion of Gnedenko's work.

Lemma 4.4: If

\[ \phi^{(n)} [ F(a_n x + b_n) ] \longrightarrow L^*(x) \quad \text{as} \quad n \rightarrow \infty, \]

where \( a_n > 0, -\infty < b_n < \infty \), and \( L^*(x) \) is a non-degenerate distribution function, then \( L^*(x) \) satisfies the functional equations

\[ \phi^{(k)} [ L^*(a^k x + \beta (l + a + a^2 + \cdots + a^{k-1}))] = L^*(x), \]

for \( -\infty < x < \infty ; k = 1, 2, 3, \ldots \), where \( a > 0; -\infty < \beta < \infty \).

Proof: Define the function \( G(x) \) by

\[ L^*(x) = \phi [ G(x) ] \quad \text{for} \quad -\infty < x < \infty. \quad (4.4) \]

Then, since \( \phi^{-1} \) is monotone-increasing and continuous
\( G(x) = \phi^{-1} [ L^*(x) ] \) is a non-degenerate distribution function.
Now
\[ \phi [ G(x) ] = L^*(x) = \lim_{n \rightarrow \infty} \phi^{(n+1)} [ F(a_{n+1} x + b_{n+1}) ] = \phi \left( \lim_{n \rightarrow \infty} \phi^{(n)} [ F(a_{n+1} x + b_{n+1}) ] \right), \]
where the limits are defined for all continuity points of $L'(x)$ (of $G(x)$). But

$$\varphi(n) [F(a_n x + b_n)] \rightarrow L'(x),$$

and

$$\varphi(n) [F(a_{n+1} x + b_{n+1})] \rightarrow G(x)$$

as $n \rightarrow \infty$ implies by Lemma 4.1 that $G(x)$ is of the same type as $L'(x)$, that is, that there exist constants $a > 0$, $-\infty < \beta < \infty$ such that

$$G(x) = L'(ax + \beta) \quad \text{for} \quad -\infty < x < \infty,$$

i.e., by 4.4,

$$\varphi[L'(ax + \beta)] = L'(x) \quad \text{for} \quad -\infty < x < \infty. \quad (4.5)$$

In terms of the $2^{nd}$ iterate $\varphi^{(2)}$ 4.5 gives

$$\varphi^{(2)}[L'(a^2 x + \beta(1 + a))] = \varphi(\varphi[L'(a(ax + \beta) + \beta)])$$

$$= \varphi(L'(ax + \beta))$$

$$= L'(x) \quad \text{for} \quad -\infty < x < \infty,$$

and by induction

$$\varphi^{(k)}[L'(a^k x + \beta(1 + a + a^2 + \cdots + a^{k-1}))] = L'(x)$$

$$\quad \text{for} \quad -\infty < x < \infty. \quad (4.6)$$

For $a \neq 1$, note that by letting
\[ x_a' = \frac{\beta}{1 - \alpha}, \quad (4.7) \]

4.6 reduces to

\[ \psi^{(k)} [L' (a^k (x - x_a') + x_a')] = L'(x), \quad (4.8) \]

and, when \( \alpha = 1 \), to

\[ \psi^{(k)} [L' (x + k\beta)] = L'(x) \quad (4.9) \]

for \(- \infty < x < \infty \); \( k = 1, 2, 3, \ldots \). For notational convenience, when \( \alpha \neq 1 \), let \( y = x - x_a' \), and define

\[ L(y) = L'(y + x_a'). \quad (4.10) \]

Note that \( L \) is of the same type as \( L' \) and from 4.8

\[ \psi^{(k)} [L' (a^k y + x_a')] = L'(y + x_a'), \]

i.e.,

\[ \psi^{(k)} [L(a^k y)] = L(y) \quad \text{for } - \infty < y < \infty ; \quad k = 1, 2, 3, \ldots. \quad (4.11) \]

Note that any distributions \( L \) satisfying the functional equations in 4.6 are limit distributions of \( V_n \) for some \( F \), namely \( L \) itself, i.e.,

\[ \lim_{n \to \infty} \psi^{(n)} [L(a_n x + b_n)] = L(x) \]

with \( a_n = \alpha^n \) and \( b_n = \beta(1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}) \).
C. Classes of Limit Distributions

As pointed out in the beginning of Section B, Gnedenko's $\alpha_\psi$ corresponds to $\alpha$, and it is fruitful, analogously to Gnedenko's distinguishing between the three cases $0 < \alpha_\psi < 1$, $\alpha_\psi = 1$ and $\alpha_\psi > 1$, to distinguish also here between $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$. The conclusions, detailed in Theorem 4.2 differ from those of Gnedenko in view of the interior fixed-point $a$ of the functional operator $\phi$ of which $\psi$ does not possess an analogue.

(i) The non-degenerate limit distributions for $V_n$ that are solutions of 4.6 (i.e. of 4.8 or 4.11) for $\alpha < 1$ fall into three easily distinguishable classes defined below.

(ii) There are no limit distributions for $V_n$ that are solutions of 4.6 for $\alpha > 1$.

(iii) As already indicated in Section A, the limit distributions for $V_n$ that are solutions of 4.6 (i.e. of 4.9) for $\alpha = 1$ are precisely the two-step distribution functions taking values 0, $a$, and 1; these solutions are designated as belonging to Class IV.

Theorem 4.2:

(i) When $\alpha < 1$, there are three easily distinguishable classes (Class I, II, and III) of non-degenerate distribution function types satisfying 4.6 (i.e., 4.11). The distribution types $L_I$, $L_{II}$, $L_{III}$ belong, respectively,
to Class I, II, or III if they satisfy 4.11 and if

\[ 0 < L_I(y) < a \quad \text{for} \quad -\infty < y < 0 \]
\[ L_I(-0) = L_I(0) = a \]
\[ a < L_I(y) < 1 \quad \text{for} \quad 0 < y < \infty \]
\[ 0 < L_{II}(y) < a \quad \text{for} \quad -\infty < y < 0 \]
\[ L_{II}(-0) = a \]
\[ L_{II}(y) = 1 \quad \text{for} \quad 0 \leq y < \infty \]
\[ L_{III}(y) = 0 \quad \text{for} \quad -\infty < y < 0 \]
\[ L_{III}(0) = a \]
\[ a < L_{III}(y) < 1 \quad \text{for} \quad 0 < y < \infty . \]

(ii) When \( a > 1 \), there are no distribution functions satisfying 4.6 (i.e., 4.11).

(iii) When \( a = 1 \), any non-degenerate distribution function \( L' \) satisfying 4.6 (i.e., satisfying 4.9) must in fact satisfy

\[ g^{(k)}[L'(x)] = L'(x) \quad \text{for} \quad -\infty < x < \infty ; \]
\[ k = 1, 2, 3, \ldots , \]

and must have form:

\[ 0 \quad \text{for} \quad -\infty < x < x_L \]
\[ L_{IV}(x) = a \quad \text{for} \quad x_L \leq x < x_U \]
\[ 1 \quad \text{for} \quad x_U \leq x < \infty , \]

for some \( x_L < x_U \).
Proof: (i) \( \alpha < 1 \): Let \( L(y) \) be any non-degenerate distribution satisfying 4.11 and let \( \alpha \) and \( \beta \) be defined as in Lemma 4.4. Then, for \( y < 0 \), it follows that \( y < ay \). Since \( L(y) \) is non-decreasing

\[
L(y) \leq L(ay) \quad \text{for } -\infty < y < 0. \tag{4.12}
\]

Also, \( L(y) = \mathcal{O}[L(ay)] \) implies by (ii) of Lemma 2.1 and 4.12 that

\[
L(y) \leq a \quad \text{for } -\infty < y < 0. \tag{4.13}
\]

Similarly, \( y > 0 \) implies \( ay < y \), which further implies

\[
L(ay) \leq L(y) \quad \text{for } 0 < y < \infty. \tag{4.14}
\]

Then it also follows, since \( L(y) \) satisfies \( \mathcal{O}[L(ay)] = L(y) \), that

\[
a \leq L(y) \quad \text{for } 0 < y < \infty. \tag{4.15}
\]

For \( y = 0 \), \( L(ay) = L(y) \); therefore \( \mathcal{O}[L(0)] = L(0) \).

Hence, \( L(0) \) must equal one of the fixed-points 0, \( a \), or 1.

From continuity to the right it is seen by 4.15 that the zero fixed-point is impossible; thus

\[
L(0) = a \text{ or } 1. \tag{4.16}
\]

Now consider 4.11, or its equivalent

\[
\mathcal{O}^{(k)}[L(y)] = L(V_{\alpha^k}) \quad \text{for } -\infty < y < \infty; \quad k = 1, 2, 3, \ldots, \tag{4.17}
\]
and suppose for some $y' \neq 0$ that $L(y') = a$. Then since $L(y')$ is a fixed-point of $\varnothing$, 4.17 implies

$$L\left(\frac{y'}{\alpha^k}\right) = a \quad \text{for } k = 1, 2, 3, \ldots. \quad (4.18)$$

Therefore, since $0 < a < 1$,

$$L(-\infty) = a \quad \text{if } y' < 0$$
$$L(\infty) = a \quad \text{if } y' > 0;$$

however, this contradicts the assumption that $L(y)$ is a distribution function. Thus, it has been established that

$$L(y) < a \quad \text{for } -\infty < y < 0 \quad (4.19)$$
$$L(y) > a \quad \text{for } 0 < y < \infty.$$

Suppose for some $y' < 0$ that $L(y') = 0$. Then since $L(0)$ is a fixed-point of $\varnothing^{(k)}$, it follows from 4.11 that

$$L(-0) = 0. \quad (4.20)$$

Now suppose for some $y'' > 0$ that $L(y'') = 1$. Then since $L(y'')$ is also a fixed-point of $\varnothing^{(k)}$, 4.11 and 4.16 imply

$$L(+0) = L(0) = 1. \quad (4.21)$$

From 4.20 and 4.21 it can be seen that if there were points $y', y''$ satisfying $L(y') = 0$ and $L(y'') = 1$ then $L(y)$ would be degenerate. Therefore by 4.19

$$0 < L(y) < a \quad \text{for } -\infty < y < 0, \quad (4.22)$$
and/or

\[ a < L(y) < 1 \quad \text{for} \quad 0 < y < 0. \quad (4.23) \]

From 4.22 and the functional equation \( \phi(k)[L(a^ky)] = L(y) \)

it follows that

\[ L(-0) = a. \quad (4.24) \]

Hence, 4.16 and 4.19 through 4.24 imply that \( L \) must belong to
Class I, II, or III.

(ii) \( a > 1 \): Let \( L(y) \) be any non-degenerate distribution
satisfying 4.11 and let \( a \) and \( \beta \) be defined as in Lemma 4.4.
Then for \( y < 0 \), \( ay < y \), which implies, since \( L(y) \) is non-
decreasing,

\[ L(ay) \leq L(y). \]

However, since \( \phi[L(ay)] = L(y) \) it follows from (ii) of
Lemma 2.1 that

\[ a \leq L(y) \quad \text{for} \quad -\infty < y < 0; \]

this contradicts that \( L(y) \) is a distribution. Hence, there
are no distribution functions satisfying 4.11 with \( a > 1 \).

(iii) \( a = 1 \): For this case \( L'(x) \) must satisfy 4.6,
i.e.,

\[ \phi(k)[L'(x + k\beta)] = L'(x) \quad \text{for} \quad -\infty < x < \infty; \ k = 1, 2, 3, \ldots. \]

From (ii) of Lemma 2.1 and the fact that \( L' \) is non-decreasing
it follows that for $\beta < 0$,

$$a < L'(x) \quad \text{for} \quad -\infty < x < \infty,$$

and for $\beta > 0$,

$$a > L'(x) \quad \text{for} \quad -\infty < x < \infty;$$

however, this contradicts that $L'(x)$ is a distribution function. Hence, $\beta = 0$ in 4.9, i.e.,

$$\hat{\theta}^{(k)}[L'(x)] = L'(x) \quad \text{for} \quad -\infty < x < \infty;$$

and the form of $L_T(y)$ follows from the type of argument given at the end of Section C, Chapter II.

D. Norming Sequences

The following lemma establishes that only a "scale sequence" \{a_n\} of norming constants is required for convergence to a limit distribution $L(y)$ belonging to Classes I, II, or III. In terms of a normed sequence of game values the following lemma shows that $(V_n - b_n)/a_n$ converging in distribution to $L(y)$ implies $(V_n - x_a)/a_n$ converges in distribution to $L(y)$ where $x_a$ is defined by $F(x_a - 0) < a \leq F(x_a)$. Note that the analysis will correspond to Gnedenko's treatment of the case $\alpha_\xi < 1$, except that the role of the "extreme" fixed-point 1 of Gnedenko's functional operator $\hat{\psi}$ is here assumed, essentially, by the "interior" fixed-point $a$ of $\theta$. 

**Lemma 4.5**: If \( \{a_n, b_n\} \ (a_n > 0) \) is a sequence of constants such that
\[
\phi^{(n)}[F(a_n y + b_n)] \to L(y) \quad \text{for} \quad -\infty < y < \infty
\]
as \( n \to \infty \), where \( L(y) \) belongs to Class I, II, or III, then
\[
\phi^{(n)}[F(a_n y + x_a)] \to L(y) \quad \text{for} \quad -\infty < y < \infty
\]
as \( n \to \infty \), where \( x_a \) is the point defined by
\[
F(x_a - 0) \leq a \leq F(x_a).
\]

**Proof**: Note that the point \( x_a \) is well-defined by 4.25, for suppose it was not \( F(x) = a \) for more than one point), then, by Theorem 4.1, \( \phi^{(n)}[F(x)] \to L_T(x) \) as \( n \to \infty \); however, this contradicts, by Lemma 4.1, that \( L(y) \) belongs to Class I, II, or III, since the distributions belonging to different classes must be of different types.

Let \( L(y) \) be any distribution belonging to Class I, II, or III and \( \{a_n, b_n\} \) any sequence of constants \( (a_n > 0) \) satisfying
\[
\phi^{(n)}[F(a_n y + b_n)] \to L(y) \quad \text{for} \quad -\infty < y < \infty
\]
as \( n \to \infty \). Define the sequence \( \{y_n\} \) by
\[
y_n = \frac{x_a - b_n}{a_n}, \ \text{i.e.,} \quad a_n y_n + b_n = x_a \quad \text{for} \quad n = 1, 2, 3, \ldots,
\]
and let $a_n' = a_n$, $b_n' = x_a$ for $n = 1, 2, 3, \ldots$. Then, by Lemma 4.3, all that needs to be shown to establish

$$
\varphi^{(n)} \left[ F(a_n y + x_a) \right] \to L(y) \quad \text{for } -\infty < y < \infty,
$$

as $n \to \infty$, is

$$
\lim_{n \to \infty} \left( \frac{b_n - b_n'}{a_n} \right) = \lim_{n \to \infty} \left( \frac{x_a - b_n'}{a_n} \right) = 0,
$$

i.e., by 4.27,

$$
\lim_{n \to \infty} y_n = \lim_{n \to \infty} \left( \frac{x_a - b_n}{a_n} \right) = 0. \tag{4.29}
$$

From Theorem 4.2,

$$
L(-0) \leq a \quad \text{and} \quad L(0) \geq a. \tag{4.30}
$$

Suppose that for some $\varepsilon > 0$

$$
\liminf_{n \to \infty} y_n = -\varepsilon. \tag{4.31}
$$

Let $\tau$ be chosen such that $0 < \tau < \varepsilon$ and $-\tau$ is a continuity point of $L$; then from 4.25, 4.26, 4.30, and 4.31:

$$
a \leq \liminf_{n \to \infty} \varphi^{(n)} \left[ F(x_a) \right] = \liminf_{n \to \infty} \varphi^{(n)} \left[ F(a_n y_n + b_n) \right] \leq \liminf_{n \to \infty} \varphi^{(n)} \left[ F(a_n(-\tau) + b_n) \right] = L(-\tau) < a;
$$

thus establishing that
\[ \liminf_{n \to \infty} (y_n) \geq 0. \quad (4.32) \]

Suppose now that for some \( \epsilon > 0 \)
\[ \limsup_{n \to \infty} (y_n) = \epsilon. \quad (4.33) \]

Choose \( \tau \) such that \( 0 < \tau < \epsilon/2 \) and \( \tau \) is a continuity point of \( L \); then from 4.25, 4.26, 4.30, and 4.33

\[ a \geq \limsup_{n \to \infty} \phi^{(n)} \left[ F(x_n - \frac{\epsilon a_n}{2}) \right] \]
\[ = \limsup_{n \to \infty} \phi^{(n)} \left[ F(a_n(y_n - \frac{\epsilon}{2}) + b_n) \right] \]
\[ \geq \limsup_{n \to \infty} \phi^{(n)} \left[ F(a_n \tau + b_n) \right] \]
\[ = L(\tau) > a; \]

thus establishing that
\[ \limsup_{n \to \infty} (y_n) \leq 0. \quad (4.34) \]

Expressions 4.32 and 4.34 imply 4.29, and therefore 4.28.

Q.E.D.

For future reference, it is shown that
\[ \phi^{(n)} \left[ F(a_n y + x_a) \right] \to L(y + x_a) \quad \text{for } -\infty < y < \infty, \]
as \( n \to \infty \), implies
when \( L \) belongs to Class I, II, or III. By the uniqueness of the point \( x_a \) in 4.25

\[
F(z + x_a) < a \quad \text{for } z < 0 \tag{4.36}
\]

\[
F(z + x_a) > a \quad \text{for } z > 0.
\]

Now, suppose that \( \limsup_{n \to \infty} (a_n) = \epsilon \) for some \( 0 < \epsilon < 1 \). Choose \( \tau > 0 \) so that \( 0 < \tau < \epsilon \) and \( -\tau, \tau \) are continuity points of \( L \); then 4.36 implies, by Lemma 2.2,

\[
L(1) = \limsup_{n \to \infty} \left\{ \phi(n) \left[ F(a_n(1) + x_a) \right] \right\} \geq \limsup_{n \to \infty} \left\{ \phi(n) \left[ F(\tau + x_a) \right] \right\} = 1, \tag{4.37}
\]

and since \( -\liminf_{n \to \infty} (-a_n) = \limsup_{n \to \infty} (a_n) = \epsilon \),

\[
L(-1) = \liminf_{n \to \infty} \left\{ \phi(n) \left[ F(a_n(-1) + x_a) \right] \right\} \leq \liminf_{n \to \infty} \left\{ \phi(n) \left[ F(-\tau + x_a) \right] \right\} = 0. \tag{4.38}
\]

However, 4.37 and 4.38 contradict that \( L \) belongs to Class I, II, or III; therefore 4.35 has been established since \( a_n > 0 \) for all \( n \).

Also, for future reference, it is shown that

\[
\frac{a_{n+1}}{a_n} \to a \quad \text{as } n \to \infty \tag{4.39}
\]
in Lemma 4.5. Since
\[ g^{(n+1)}[F(a_{n+1}y + x_a)] \rightarrow L(y) \]
and
\[ g\{g^{(n)}[F(a_n(\alpha y) + x_a)]\} \rightarrow g\{L(\alpha y)\} = L(y), \]
it follows by Lemma 4.2 that
\[ \frac{a_{n+1}}{an} \rightarrow 1 \quad \text{as } n \rightarrow \infty, \]
which implies 4.39.

E. General Limit Distributions in the Form \( L_u[C(y)] \)

Underlying the development of this section is the fact that, in order that a distribution \( F \), continuous and monotone-increasing in some "neighborhood" of \( x_a \), lead to a non-degenerate limit distribution, there must exist a scaling \( C(y) \) in terms of which normed arguments of the uniform distribution \( U \) and of the distribution \( F \) are in a sense equivalent for large \( n \):

\[ F(a_n y + x_a) = U\left(\frac{C(y)}{b^n} + a\right) = \frac{C(y)}{b^n} + a. \]

In terms of \( C(y) \), necessary and sufficient conditions are given for \( \{a_n\} \) to be a norming sequence such that \( (V_n - x_a)/a_n \) converges in distribution to some \( L_u[C(y)] \) belonging to
Class I, II, or III. These results pertain to distributions $F(x)$ which are continuous and monotone-increasing for regions defined as follows:

Class I: $N(x_a, \delta) = \{x: x_a - \delta \leq x \leq x_a + \delta\}$ (4.40)
Class II: $N(x_a, \delta) = \{x: x_a - \delta \leq x < x_a\}$ (4.41)
Class III: $\bar{N}(x_a, \delta) = \{x: x_a \leq x \leq x_a + \delta\}$ (4.42)

where $\delta > 0$, and $x_a$ is defined as in Lemma 4.5, i.e.,

$$F(x_a - \delta) \leq a \leq F(x_a). \quad (4.43)$$

It seems not entirely unnatural that a certain amount of regularity is introduced at this point, in contrast to the complete generality maintained by Gnedenko. This is because the analogue of this section in (9) is the relatively straightforward Gnedenko Lemma 4.4, based on the equivalence for small $x$ of $\ln(1+x)$ and $x$, which shows $C(x)$ to be, simply, the negative of the ln of the limit distribution in that case, i.e.,

$$\left[F(a_n y + b_n)\right]^n \equiv G(y)$$

iff $-n \ln[F(a_n y + b_n)] \equiv -\ln[G(y)] \ (\equiv C(y))$

iff $n[1 - F(a_n y + b_n)] \equiv C(y)$

iff $F(a_n y + b_n) \equiv -\frac{C(y)}{n} + 1.$

It may be of interest to point out in this connection
that Gnedenko, using the Functional Equation 4.3, was able to narrow down the possible scalings \( C(y) \) (and hence limit distributions \( G_n = e^{-C(y)} \)) to three one-parameter families in a manner not possible here. This is because \( G \) must in fact satisfy not one but a sequence of functional equations, \( t = 1, 2, \ldots \).

It will be convenient to represent the domain of attraction of a limit distribution \( L \) by \( \Theta(L) \). Thus, in view of Lemma 4.5, \( P \in \Theta(L) \) means that there exists a sequence \( \{a_n\} \) satisfying \( \phi^{(n)}[F(a_n y + x_a)] \to L(y) \) as \( n \to \infty \).

**Lemma 4.6:** Let \( F(x) \) be a distribution which is continuous and monotone-increasing in \( N(x, \delta) \) for some \( \delta > 0 \), and let \( L_I(y) \) be a distribution function belonging to Class I. Then \( P \in \Theta[I^I(y)] \) iff there exists a sequence of positive constants \( \{a_n\} \) and a function \( C(y) \) defined for \(-\infty < y < \infty \) such that

(i) \( C(y) \) is continuous to the right
(ii) \( C(-\infty) = -\infty, C(y) < 0 \) for \( y < 0, C(-0) = C(0) = 0, C(y) > 0 \) for \( y > 0, C(\infty) = \infty \)
(iii) \( b^n[F(a_n y + x_a) - a] \to C(y) \) at all continuity points \( C(y) \),

where \( L_I(y) = I_n[C(y)] \).

**Proof:** First some results are derived from the fact that \( F(x) \) is continuous and monotone-increasing in \( N(x_a, \delta) \); these results will be used in both the necessary and
the sufficient part of the proof. From the continuity and
monotonicity of $F$ in $N(x, \delta)$, it follows that $F^{-1}(\frac{w}{b^n} + a)$ is
defined for all $w$ satisfying $F^{-1}(\frac{w}{b^n} + a) = z + x_a =
x \in N(x_a, \delta)$. Let $w$ be an arbitrary number and let $M(w)$ be a
positive integer chosen large enough so that

$$F^{-1}(\frac{w}{b^n} + a) \in N(x_a, \delta) \quad \text{for } n \geq M(w). \quad (4.44)$$

Then 4.44 yields

$$P\{b^n[F(V_n) - a] \leq w\} = P\{F(V_n) \leq \frac{w}{b^n} + a\}$$

$$= P\{V_n \leq F^{-1}(\frac{w}{b^n} + a)\}$$

$$= \phi(n) [F[F^{-1}(\frac{w}{b^n} + a)]\}$$

$$= \phi(n)(\frac{w}{b^n} + a) \quad (4.45)$$

for $n \geq M(w)$.

Therefore, in view of the results of Chapter II,

$$\lim_{n \to \infty} P\{b^n[F(V_n) - a] \leq w\} = L_u(w). \quad (4.46)$$

Let $y$ be any number; then, since $a_n \to 0$ (Expression 4.35) and
$F(x_a - 0) = a = F(x_a)$, a positive integer $M'$ can be chosen
large enough so that $a_n y + x_a \in N(x_a, \delta)$ for $n \geq M'$. Thus, for
$n \geq M'$

$$\phi(n)[F(a_n y + x_a)] = P\{V_n \leq a_n y + x_a\}$$
\[ P[F(V_n) \leq F(a_n y + x_a)] = P[b^n \left[ F(V_n) - a \right] \leq b^n \left[ F(a_n y + x_a) - a \right]]. \] (4.47)

Now, assuming there exist a sequence \( \{a^n\} \) and a function \( C(y) \) defined for \( -\infty < y < \infty \) such that (i), (ii), and (iii) hold, we show that \( F \in \mathcal{D} \big[ L_1(y) = L_u(C(y)) \big] \). \( C(y) \) must be non-decreasing since it is defined to be continuous to the right and the limit function (on a dense set of the real line) of a sequence of non-decreasing functions. This fact, together with (i) and (ii), insure that \( L_u(C(y)) \) is a distribution function belonging to Class I (see (ii) of Lemma 3.2).

Let \( y \) be any continuity point of \( C(y) \) (i.e., continuity point of \( L_u(C(y)) \)). Since \( L_u \) is continuous (Lemma 3.4), for a given \( \epsilon > 0 \) there exists a \( \delta' > 0 \) satisfying

\[ L_u\left[ C(y) - \delta' \right] > L_u\left[ C(y) \right] - \epsilon \] (4.48)

and

\[ L_u\left[ C(y) + \delta' \right] < L_u\left[ C(y) \right] + \epsilon. \]

Condition (iii) implies that there exists an integer \( M'' \) such that for all \( n \geq M'' \)

\[ b^n \left[ F(a_n y + x_a) - a \right] > C(y) - \delta', \] (4.49)

\[ b^n \left[ F(a_n y + x_a) - a \right] < C(y) + \delta'. \] (4.50)

Set \( w = C(y) - \delta' \) in 4.45; then from 4.45, 4.47, and 4.49
\begin{align*}
P\{V_n \leq a_n y + x_a\} &= P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq b^n \left\lfloor F(a_n y + x_a) - a \right\rfloor\} \\
&\geq P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq C(y) - \delta'\}
\end{align*}
for \(n \geq \max \left\{M(C(y) - \delta'), M', M''\right\} \).

Now, set \(w = C(y) + \delta'\) in 4.45, similarly, it follows that
\begin{align*}
P\{V_n \leq a_n y + x_a\} &= P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq b^n \left\lfloor F(a_n y + x_a) - a \right\rfloor\} \\
&\leq P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq C(y) + \delta'\}
\end{align*}
for \(n \geq \max \left\{M(C(y) + \delta'), M'(y), M''\right\} \).

Combining 4.51 and 4.52 gives
\begin{align*}
P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq C(y) - \delta'\} \\
&\leq P\{V_n \leq a_n y + x_a\} \\
&\leq P\{b^n \left\lfloor F(V_n) - a \right\rfloor \leq C(y) + \delta'\}
\end{align*}
for \(n \geq \max \left\{M(C(y) - \delta'), M(C(y) + \delta'), M''(y), M''\right\} \).

By letting \(n\) tend to infinity in 4.53, Expressions 4.45, 4.46, and 4.48 give
\begin{align*}
L_u\left[C(y)\right] - \epsilon &\leq L_u\left[C(y) - \delta'\right] \leq \lim_{n \to \infty} \inf P\{V_n \leq a_n y + x_a\} \\
&\leq \limsup_{n \to \infty} P\{V_n \leq a_n y + x_a\} \\
&\leq L_u\left[C(y) + \delta'\right] \leq L_u\left[C(y)\right] + \epsilon.
\end{align*}
Hence,
\[ \lim_{n \to \infty} \phi(n) \mathbb{P}(a_n y + x_a) = \lim_{n \to \infty} \mathbb{P}(V_n \leq a_n y + x_a) \]
\[ = L_u \mathbb{C}(y) = L_I(y), \]
which completes the sufficiency part of the proof.

\( F \in \mathcal{O}[L_I(y)] \) is now shown to imply Conditions (i), (ii) and (iii). \( C(y) \) is well-defined by \( L_u \mathbb{C}(y) = L_I(y) \) since the continuity and monotonicity of \( L_u \) imply the existence of \( C(y) = L_u^{-1}[L_I(y)] \). Since the range for both \( L_u \) and \( L_I \) is the open interval \((0,1)\) it follows that
\[ C(-\infty) = \lim_{y \to -\infty} C(y) = \lim_{y \to -\infty} L_u^{-1}[L_I(y)] = -\infty \]
\[ C(\infty) = \lim_{y \to \infty} C(y) = \lim_{y \to \infty} L_u^{-1}[L_I(y)] = \infty. \]

Also,
\[ C(-0) = L_u^{-1}\left[ \lim_{y \to 0^-} L_I(y) \right] = L_u^{-1}[L_I(0)] = L_u^{-1}[a] = 0; \]
the remaining properties in (ii) are easily seen to hold. \( C(y) \) is continuous to the right since \( L_u^{-1} \) and \( L_I \) are both continuous to the right. Thus only (iii) remains to be verified. \( F \in \mathcal{O}[L_u(C(y))] \) means that there exists a sequence \( \{a_n\} \) of positive constants satisfying
\[ \lim_{n \to \infty} \phi(n) \mathbb{P}(a_n y + x_a) = L_u \mathbb{C}(y) \quad (4.54) \]
at all continuity points of $L_u$ and hence of $C(y)$. Let $y$ be any continuity point of $C(y)$, and set $w = C(y)$ in 4.44. Then 4.46 yields

$$\lim_{n \to \infty} P\{b^n[F(V_n) - a] \leq C(y)\} = L_u[C(y)]. \quad (4.55)$$

$L_u$ monotone-increasing implies

$$L_u[C(y) - \delta'] < L_u[C(y)] < L_u[C(y) + \delta'] \quad (4.56)$$

for all $\delta' > 0$.

Expressions 4.47 and 4.54 yield

$$\lim_{n \to \infty} \rho^{(n)}[F(a_ny + x_a)] = \lim_{n \to \infty} P\{b^n[F(V_n) - a] \leq b^n[F(a_ny + x_a) - a]\} = L_u[C(y)]. \quad (4.57)$$

Suppose that

$$\lim_{n \to \infty} b^n[F(a_ny + x_a) - a] \neq C(y) \quad (4.58)$$

Then there exists a $\delta' > 0$ such that an infinite number of terms of the sequence $\{b^n[F(a_ny + x_a) - a]\}$ are either less than $C(y) - \delta'$ or greater than $C(y) + \delta'$. Therefore, by 4.55 and 4.57, either

$$L_u[C(y)] = \lim_{n \to \infty} P\{b^n[F(V_n) - a] \leq b^n[F(a_ny + x_a) - a]\}$$
\begin{align*}
\lim_{n \to \infty} P\{b^n [F(n) - a] \leq C(y) - \delta'\} \\
= L(C(y) - \delta'), \quad (4.59)
\end{align*}

or,

\begin{align*}
\lim_{n \to \infty} P\{b^n [F(n) - a] \leq \infty\} \\
= L(\infty); \quad (4.60)
\end{align*}

however, 4.59 or 4.60 constitute, by 4.56, a contradiction of 4.58.

**Lemma 4.7:** Let \(P(x)\) be a distribution function which is continuous and monotone-increasing in \(N(x, \delta)\) for some \(\delta > 0\) \((4.41, 4.42)\). Then \(P \in L_{II}(y)\) \((P \in \overline{L}_{III}(y))\) iff there exists a sequence of positive constants \(\{a_n\}\) and a function \(C(y)\) defined for \(-\infty < y < 0\) \((0 < y < \infty)\) such that

(i) \(C(y)\) is continuous to the right

(ii) \(C(-\infty) = -\infty, C(0) = 0, C(\infty) = \infty\)

(iii) \(b^n [F(a_n y + x_a) - a] \to C(y)\) at all continuity points of \(C(y)\); \(b^n [F(a_n y + x_a) - a] \to \infty (-\infty)\) for \(0 < y < \infty\) \((-\infty < y < 0)\),
$L_{II}(y) = L_u[C(y)]$ for $-\infty < y < 0$

$1$ for $0 \leq y < \infty$

and

$L_{III}(y) = 0$ for $-\infty < y < 0$

$L_u[C(y)]$ for $0 \leq y < \infty$.

Proof: By replacing $N(x, \delta)$ by $N(x, \delta)$ and $\bar{N}(x, \delta)$, the proof of Lemma 4.6 applies for the region where $C(y)$ is defined, i.e. $-\infty < y < 0$ for Class II and $0 \leq y < \infty$ for Class III.

Assume that there exists a sequence $\{a_n\}$ and a function $C(y)$ defined for $-\infty < y < 0$ ($0 \leq y < \infty$) such that (i), (ii), and (iii) hold and show that $\lim_{n \to \infty} \phi(n) \left[ F(a_n y + b_n) \right] = 1$ for $y > 0$ ($0$ for $y < 0$).

For the case when $b^n \left[ F(a_n y + x_a) - a \right] \to \infty$ for $0 < y < \infty$, define

$C_n(y) = b^n \left[ F(a_n y + x_a) - a \right]$ for $0 < y < \infty$; $n = 1, 2, 3, \ldots$.  \hfill (4.61)

Then

$\phi(n) \left[ F(a_n y + x_a) \right] = \phi(n) \left[ \frac{C_n(y)}{b^n} + a \right]$ for $0 < y < \infty$; $n = 1, 2, 3, \ldots$.  \hfill (4.62)

From the uniform results (Chapter III), we have

$\lim_{n \to \infty} \phi(n) \left( \frac{w}{b^n} + a \right) = L_u(w)$.  \hfill (4.63)
Then for any $\epsilon > 0$ there is a number $w_U(\epsilon)$ such that

$$L_u \left[ w_U(\epsilon) \right] > 1 - \epsilon. \quad (4.64)$$

By the definition of $C_n(y)$, (iii) gives

$$\lim_{n \to \infty} C_n(y) = \infty \quad \text{for} \quad 0 < y < \infty.$$ Hence, for any $y > 0$ there exists an integer $M(y)$ such that

$$C_n(y) > w_U(\epsilon) \quad \text{for all} \quad n \geq M(y), \quad (4.65)$$

which leads to, by 4.62,

$$\phi(n) \left[ F(a_n y + x_a) \right] = \phi(n) \left[ \frac{C_n(y)}{b^n} + a \right]$$

$$\geq \phi(n) \left[ \frac{w_U(\epsilon)}{b^n} + a \right] \quad \text{for all} \quad n \geq M(y).$$

Thus,

$$\lim_{n \to \infty} \phi(n) \left[ F(a_n y + x_a) \right] \geq \lim_{n \to \infty} \phi(n) \left[ \frac{w_U(\epsilon)}{b^n} + a \right]$$

$$= L_u \left[ w_U(\epsilon) \right] > 1 - \epsilon,$$

which implies

$$\lim_{n \to \infty} \phi(n) \left[ F(a_n y + x_a) \right] = 1$$

since the $\epsilon$ was chosen arbitrarily.

Similarly, for the case when $b^n \left[ F(a_n y + x_a) - a \right] \to \infty$
for $-\infty < y < 0$, as $n \to \infty$, define
\[ C_n(y) = b^n \left[ F(a_n y + x_a) - a \right] \quad \text{for} \quad -\infty < y < 0; \quad n = 1, 2, 3, \ldots \] \hspace{1cm} (4.66)

For any $\varepsilon > 0$, there is a number $w_L(\varepsilon)$ such that
\[ L_u \left[ w_L(\varepsilon) \right] < \varepsilon. \] \hspace{1cm} (4.67)

By the definition of $C_n(y)$, (iii) yields the existence of a number $M(y)$ such that
\[ C_n(y) < w_L(\varepsilon) \quad \text{for all} \quad n \geq M(y), \] \hspace{1cm} (4.68)

which leads to, by 4.66
\[
\phi(n) \left[ F(a_n y + x_a) \right] = \phi(n) \left[ \frac{C_n(y)}{b^n} + a \right] \\
\leq \phi(n) \left[ \frac{w_L(\varepsilon)}{b^n} + a \right] \quad \text{for all} \quad n \geq M(y).
\]

Thus,
\[
\limsup_{n \to \infty} \phi(n) \left[ F(a_n y + x_a) \right] \leq \lim_{n \to \infty} \phi(n) \left[ \frac{w_L(\varepsilon)}{b^n} + a \right] \\
= L_u \left[ w_L(\varepsilon) \right] < \varepsilon,
\]

which yields,
\[
\lim_{n \to \infty} \phi(n) \left[ F(a_n y + x_a) \right] = 0.
\]

Now, we shall assume that $\lim_{n \to \infty} \phi(n) \left[ F(a_n y + x_a) \right] = 1$. 


for \( y > 0 \) (0 for \( y < 0 \)), and show that \( b^n \left[ F(a_n y + x_a) - a \right] \to \infty \)
for \( y > 0 \) (-\( \infty \) for \( y < 0 \)). For the case when
\[
\lim_{n \to \infty} \varphi(n) \left[ F(a_n y + x_a) \right] = 1 \text{ for } y > 0,
\]
define \( C_n(y) \) as in 4.61, and suppose that for some \( y > 0 \)
\[
C_n(y) \to \infty \quad \text{as } n \to \infty. \tag{4.69}
\]
Then, \( \lim \inf \ C_n(y) = C_0(y) < \infty \), and
\[
\lim \inf_{n \to \infty} \varphi(n) \left[ F(a_n y + x_a) \right] = \lim \inf_{n \to \infty} \varphi(n) \left[ \frac{C_n(y)}{b^n} + a \right]
\]
\[
\leq \lim \inf_{n \to \infty} \varphi(n) \left[ \frac{C_0(y)}{b^n} + a \right] = L_u \left[ C_0(y) \right] < 1,
\]
by the monotonicity of \( L_u \). However, this is a contradiction
that \( \lim_{n \to \infty} \varphi(n) \left[ F(a_n y + x_a) \right] = 1 \); therefore, \( C_n(y) \to \infty \)
for \( 0 < y < \infty \), as \( n \to \infty \).

Similarly, for the case when \( \lim_{n \to \infty} \varphi(n) \left[ F(a_n y + x_a) \right] = 0 \)
for \( y < 0 \), define \( C_n(y) \) as in 4.66, and suppose that some \( y < 0 \),
\[
C_n(y) \to -\infty \quad \text{as } n \to \infty. \tag{4.70}
\]
Then, \( \lim \sup_{n \to \infty} C_n(y) = C_0(y) > -\infty \), and
\[
\limsup_{n \to \infty} \varrho(n) \left( F(a_n y + x_a) \right) = \limsup_{n \to \infty} \varrho(n) \left( \frac{c_n(y)}{b^n} + a \right) \\
\geq \limsup_{n \to \infty} \varrho(n) \left( \frac{c_0(y)}{b^n} + a \right) = L_u \left[ \frac{c_0(y)}{b^n} \right] > 0,
\]

by the monotonicity of \( L_u \). This is a contradiction that
\[
\lim_{n \to \infty} \varrho(n) \left( F(a_n y + x_a) \right) = 0; \text{ hence, } c_n(y) \to -\infty \text{ for } -\infty < y < 0, \text{ as } n \to \infty.
\]
V. DOMAINS OF ATTRACTION

A. Theory

The domains of attraction for three particular families of limit distributions $L_u^{[C(y)]}$, one in each of the Classes I, II, and III, are studied in this chapter. Sufficient conditions are given for distributions $F$, satisfying the continuity and monotonicity conditions introduced in Section E of Chapter IV, to belong to the domains of attraction of the distributions $L_{I,\gamma,\tau}$, $L_{II,\gamma}$, and $L_{III,\gamma}$ defined as follows:

**Class I**

$$L_{I,\gamma,\tau}(y) = \begin{cases} L_u(-|y|^{\gamma}) & \text{for } -\infty < y < 0 \\ L_u(\tau y^{\gamma}) & \text{for } 0 \leq y < \infty \end{cases}$$

**Class II**

$$L_{II,\gamma}(y) = \begin{cases} L_u(-|y|^{\gamma}) & \text{for } -\infty < y < 0 \\ 1 & \text{for } 0 \leq y < \infty \end{cases}$$

**Class III**

$$L_{III,\gamma}(y) = \begin{cases} 0 & \text{for } -\infty < y < 0 \\ L_u(y^{\gamma}) & \text{for } 0 \leq y < \infty \end{cases}$$

where $\tau > 0$ and $\gamma > 0$.

**Lemma 5.1:** The distributions $L_{I,\gamma,\tau}$, $L_{II,\gamma}$, and $L_{III,\gamma}$ belong respectively to Classes I, II, and III.
Proof: To begin with, it is clear that the distributions are of the required form. It remains to show that $L$ satisfies

$$g^{(k)} \left[ L(\alpha^k y) \right] = L(y) \quad \text{for} \quad -\infty < y < \infty ; \quad k = 1, 2, 3, \ldots ,$$

and for some $0 < \alpha < 1$.

Define

$$\alpha = \left( \frac{1}{b} \right)^{1/\gamma} ; \quad (5.2)$$

then $0 < \alpha < 1$ since $b > 1$ and $\gamma > 0$; also recall (Lemma 3.1) that

$$g^{(k)} \left[ L_u \left( \frac{z}{b^k} \right) \right] = L_u(z) \quad \text{for} \quad -\infty < z < \infty ; \quad k = 1, 2, 3, \ldots . \quad (5.3)$$

Let

$$z = -|y|^\gamma \quad \text{for} \quad -\infty < y < 0 \quad (5.4)$$

then, by 5.2,

$$L_{I, \gamma, \tau}(\alpha^k y) = L_u(-|\alpha^k y|^\gamma)$$

$$= L_u\left( -\left| \frac{|y|^\gamma}{b^k} \right| \right)$$

$$= L_u\left( \frac{z}{b^k} \right) \quad \text{for} \quad -\infty < z < 0 (-\infty < y < 0),$$

and
Therefore, by 5.3,

\[ \phi^{(k)} \left[ L_{I, \gamma, \tau}(a^k y) \right] = \phi^{(k)} \left[ L_u \left( \frac{x}{b^k} \right) \right] \]

\[ = L_u(z) \]

\[ = L_{I, \gamma, \tau}(y) \text{ for } -\infty < y < \infty; \quad k = 1, 2, 3, \ldots. \]

Since the fixed-points 0 and 1 trivially satisfy 5.1, the proof given above for \( L_{I, \gamma, \tau} \) also holds for \( L_{II, \gamma} \) and \( L_{III, \gamma} \) if now \( z \) (defined in 5.4) is used for the ranges

\[ -\infty < z < 0 \text{ and } 0 \leq z < \infty, \text{ respectively.} \]

For easy reference, the results of Lemma 4.5 and 4.6, when applied to \( L_{I, \gamma, \tau}, L_{II, \gamma}, \text{ and } L_{III, \gamma} \), are summarized in the following lemma.

**Lemma 5.2:**

(i) If \( F(x) \) is continuous and monotone-increasing in \( N(x_0, \delta) \) for some \( \delta > 0 \), then \( F \in \mathcal{O}(L_{I, \gamma, \tau}) \) iff there exists a sequence of positive constants \( \{a_n\} \) satisfying
\[ b^n \left[ F(a_n y + x_a) - a \right] \to -|y|^\gamma \quad \text{for } -\infty < y < 0, \quad (5.5) \]
\[ b^n \left[ F(a_n y + x_a) - a \right] \to \gamma y \quad \text{for } 0 < y < \infty, \]

as \( n \to \infty \).

(iii) If \( F(x) \) is continuous and monotone-increasing in \( N(x_a, \delta) \) for some \( \delta > 0 \), then \( F \in \mathcal{O}(L_{III}, \gamma) \) iff there exists a sequence of positive constants \( \{ a_n \} \) satisfying
\[ \begin{align*}
    b^n \left[ F(a_n y + x_a) - a \right] & \to -|y|^\gamma \quad \text{for } -\infty < y < 0, \\
    b^n \left[ F(a_n y + x_a) - a \right] & \to \infty \quad \text{for } 0 < y < \infty,
\end{align*} \]

as \( n \to \infty \).

Theorem 5.1: Let \( F(x) \) be a distribution function which is continuous and monotone-increasing in \( N(x_a, \delta) \) for some \( \delta > 0 \). Then \( F \in \mathcal{O}(L_{I}, \gamma, \tau) \) if

(i) \[ \frac{F(Kx + x_a) - a}{F(z + x_a) - a} \to K^\gamma, \quad \text{as } z \to 0, \quad \text{for all } K > 0 \]
Proof: Define the sequence \( \{a_n\} \) by

\[
a_n = \min \left[ z : \frac{F(z + x_a) - a}{a - F(- z + x_a)} \geq \frac{r}{b^n} \right]. \tag{5.8}
\]

Then \( F(x) \) continuous and monotone-increasing in \( N(x_a) \) implies there is an integer \( N \) large enough so that equality holds in 5.8; thus

\[
F(a_n + x_a) - a = \frac{r}{b^n} \quad \text{for } n \geq N. \tag{5.9}
\]

Also, \( F(x_a) = a \) implies that

\[
a_n > 0 \quad \text{for } n = 1, 2, 3, \ldots \tag{5.10}
\]

and

\[
a_n \to 0 \quad \text{as } n \to \infty. \tag{5.11}
\]

Condition (1) and 5.11 imply

\[
\frac{F(a_n y + x_a) - a}{F(a_n + x_a) - a} \to y^\gamma \quad \text{for } y > 0 \tag{5.12}
\]

as \( n \to \infty \). Hence, 5.9 and 5.12 yield

\[
b^n [F(a_n y + x_a) - a] \to \tau y^\gamma \quad \text{for } y > 0 \tag{5.13}
\]

as \( n \to \infty \). On the other hand,
as \( n \to \infty \), and Condition (ii) and 5.11 imply

\[
\lim_{n \to \infty} \frac{F(a_n y + x_a) - a}{F(-a_n y + x_a) - a} = -\frac{1}{\tau} \quad \text{for } y < 0;
\]

thus 5.14 gives

\[
\lim_{n \to \infty} \{b_n[F(a_n y + x_a) - a]\} = \left(-\frac{1}{\tau}\right) \lim_{n \to \infty} \{F(a_n(-y) + x_a) - a\}
\]

\[= \left(-\frac{1}{\tau}\right) \left[\tau(-y)^\gamma\right]
\]

\[= -|y|^\gamma \quad \text{for } y < 0. \tag{5.15}
\]

Hence 5.13 and 5.15 imply by 5.5 that \( F \in \mathcal{D}(L_1,\gamma,\tau) \).

**Corollary 5.1:** If the density \( f(x) \) exists and is larger than zero in \( N(x_a,\delta) \) for some \( \delta > 0 \), then \( F \in \mathcal{D}(L_1) \).

**Proof:** Since \( F(Ky + x_a) \to a \), as \( y \to 0 \), for all \( K > 0 \), and the derivative of \( F \) exists in some neighborhood of \( x_a \), L'Hospital Rule may be used, which yields

\[
\lim_{y \to 0} \frac{F(Ky + x_a) - a}{F(y + x_a) - a} = \lim_{y \to 0} \frac{K f(Ky + x_a)}{f(y + x_a)} = K,
\]

and
\[
\lim_{y \to 0} \frac{F(y + x_a) - a}{a - F(-y + x_a)} = \lim_{y \to 0} \frac{f(y + x_a)}{f(-y + x_a)} = 1.
\]

Hence, \( F \in \bigcap \mathbb{L}_u(y) = \mathbb{L}_{I,1,1}(y) \) by Theorem 5.1.

**Theorem 5.2:** Let \( F(x) \) be a distribution function which is continuous and monotone-increasing in \( N(x_a, \delta) \) for some \( \delta > 0 \) and \( F(x_a - 0) = a \). Then \( F \in \bigcap \mathbb{L}_{II,\gamma}(y) \) if

\[
(1) \quad \frac{F(Ky + x_a) - a}{F(z + x_a) - a} \to K^\gamma, \text{ as } z \to -0, \text{ for } K > 0
\]

\[
(ii) \quad \frac{F(z + x_a) - a}{a - F(-z + x_a)} \to \infty, \text{ as } z \to +0.
\]

**Proof:** Define the sequence \( \{a_n\} \) by

\[
a_n = \inf \left\{ z : a - F(-z + x_a) \geq \frac{1}{b^n} \right\}.
\]

Then since \( F(x) \) is continuous and monotone-increasing in \( N(x_a, \delta) \) and \( F(x_a - 0) = a \) it follows that there exist an integer \( N \) such that

\[
a - F(-a_n + x_a) = \frac{1}{b^n} \quad \text{for } n \geq N, \quad (5.16)
\]

where

\[
a_n > 0 \quad \text{for } n = 1, 2, 3, \ldots \quad (5.17)
\]

and

\[
a_n \to 0 \quad \text{as } n \to \infty. \quad (5.18)
\]
From Condition (1) and 5.18 it follows that

\[ \frac{F(-a_n(-y) + x_a) - a}{F(-a_n + x_a) - a} \rightarrow (-y)^\gamma \quad \text{for } y < 0 \quad (5.19) \]

as \( n \rightarrow \infty \), and from 5.16 and 5.19 that

\[ b^n \left[ \frac{F(a_n y + x_a) - a}{\left( a_n (-y) + x_a \right) - a} \right] \rightarrow -|y|^{\gamma} \quad \text{for } y < 0 \quad (5.20) \]

as \( n \rightarrow \infty \). In addition, Condition (ii) and 5.17 imply

\[ \frac{F(a_n y + x_a) - a}{F(a_n (-y) + x_a) - a} \rightarrow -\infty \quad \text{for } y > 0 \]

as \( n \rightarrow \infty \), and therefore, by 5.20,

\[ b^n \left[ \frac{F(a_n y + x_a) - a}{\left( a_n (-y) + x_a \right) - a} \right] \rightarrow +\infty \quad \text{for } y > 0. \quad (5.21) \]

Thus from 5.6, 5.20, and 5.21 it follows that \( F \in \mathcal{C}(L_{III, \gamma}) \).

**Theorem 5.3:** Let \( F(x) \) be a distribution function which is continuous and monotone-increasing in \( N(x_0, \delta) \) for some \( \delta > 0 \). Then \( F \in \mathcal{C}(L_{III, \gamma}) \) if

1. \( \frac{F(Kz + x_a) - a}{F(z + x_a) - a} \rightarrow K^\gamma \), as \( z \rightarrow +0 \), for all \( K > 0 \),

2. \( \frac{F(z + x_a) - a}{a - F(-z + x_a)} \rightarrow +0 \), as \( z \rightarrow +0 \).

**Proof:** By setting \( r = 0 \) the first part, 5.8 through 5.12, of the proof of Theorem 5.1 is applicable. Condition
(ii) and 5.11 imply

\[
\frac{F(a_n y + x_a) - a}{F(a_n (-y) + x_a) - a} \to -\infty \quad \text{for } y < 0
\]
as \(n \to \infty\), and therefore by 5.13

\[
b^n [F(a_n y + x_a) - a] \to -\infty \quad \text{for } y < 0;
\]
thus \(F \in \mathcal{S}(L_{III}, y)\) by 5.7.

B. Examples

From Corollary 5.1 it follows that all of the common distributions with density functions, e.g., normal, beta, gamma, Student's t, Snedecor's F, belong to \(\mathcal{S}[L_u = L_{I, 1, 1}]\). The following example exhibits a distribution \(F\) which belongs to \(\mathcal{S}(L_{III}, 2)\).

**Example 1:** Define \(F(x)\) as follows:

\[
\begin{align*}
0 & \quad \text{for } x < 0 \\
\frac{a - (x - \sqrt{a})^2}{a} & \quad \text{for } 0 \leq x \leq \sqrt{a} \\
\frac{\sqrt{a}x}{a} & \quad \text{for } \sqrt{a} \leq x \leq 1/a \\
1 & \quad \text{for } x > 1,
\end{align*}
\]

where \(a\) is the fixed-point of \(\phi\). Thus, \(F(x)\) is continuous for all \(x\), monotone-increasing for \(0 \leq x \leq 1/a\), and \(F(\sqrt{a}) = a\), i.e. \(x_a = \sqrt{a}\). We now show that Conditions (1) and (ii) of
Theorem 5.2 are satisfied with $\gamma = 2$, and therefore $F \in \mathcal{O}(L_{II}, 2)$. Verification of (i) and (ii) gives

$$\lim_{z \to 0} \frac{F(Kz + \sqrt{a}) - a}{F(z + \sqrt{a}) - a} = \lim_{z \to 0} \frac{a - \left[ \frac{(Kz + \sqrt{a}) - \sqrt{a}}{z} \right]^2}{a - \left[ \frac{(z + \sqrt{a}) - \sqrt{a}}{z} \right]^2} = \lim_{z \to 0} \frac{k^2 z^2}{z^2} = k^2 \quad \text{for all} \quad K > 0,$$

and

$$\lim_{z \to +0} \frac{F(z + \sqrt{a}) - a}{a - F(-z + x_a)} = \lim_{z \to +0} \frac{a - \left[ \frac{(z + \sqrt{a}) - \sqrt{a}}{a - \sqrt{a}(z + \sqrt{a})} \right]^2}{a - \sqrt{a}(z + \sqrt{a})} = \lim_{z \to +0} \frac{\frac{z^2}{\sqrt{a} z}}{\sqrt{a} z} = 0.$$

Furthermore, the norming sequence $\{a_n\}$ is easily gotten from 5.28 (b is defined as in Chapter II)

$$a - F(-a_n + \sqrt{a}) = \frac{1}{b^n},$$

i.e.,

$$a - \left[ a - ((-a_n + \sqrt{a}) - \sqrt{a})^2 \right] = \frac{1}{b^n},$$

i.e.,

$$a_n = \frac{1}{b^{n/2}} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Thus, $b^{n/2}(v_n - \sqrt{a})$ converges in distribution to
\[ L_{I,2}(y) = L_u(-|y|^2) \quad \text{for} \quad -\infty < y < 0 \]
\[ L_{I,2}(y) = 1 \quad \text{for} \quad 0 \leq y < \infty. \]

The following example exhibits a distribution \( F \) belonging to \( \mathcal{D}(L_{I,1,2}) \).

**Example 2:** Define \( F(x) \) as follows:

\[
F(x) =
\begin{align*}
0 & \quad \text{for} \quad x < 0 \\
x & \quad \text{for} \quad 0 \leq x < a \\
 a + 2(x - a) & \quad \text{for} \quad a \leq x < \frac{1-a}{2} \\
1 & \quad \text{for} \quad x > \frac{1-a}{2}
\end{align*}
\]

where \( a \) is the fixed-point of \( \phi \). For this example, \( x_a = a \) and \( F(x) \) is continuous and monotone-increasing for \( 0 \leq x \leq 1 \).

Therefore, by showing Conditions (i) and (ii) are satisfied with \( \gamma = 1, \tau = 2 \), we establish that \( F \in \mathcal{D}(L_{I,1,2}) \). Verification of (i) and (ii) gives

\[
\lim_{z \to -0} \frac{F(Kz + a) - a}{F(z + a) - a} = \lim_{z \to -0} \frac{(Kz + a) - a}{(z + a) - a} = K,
\]
\[
\lim_{z \to +0} \frac{F(Kz + a) - a}{F(z + a) - a} = \lim_{z \to +0} \frac{a + 2(Kz + a - a) - a}{a + 2(z + a - a) - a} = K,
\]

and

\[
\tau = \lim_{z \to +0} \frac{F(z + a) - a}{a - F(-z + a)} = \lim_{z \to +0} \frac{a + 2(z + a - a) - a}{a - \left[-z + a\right]} = 2.
\]
VI. MOMENTS

A. Existence of Moments of Certain Limit Distributions

When \( p = q = 2 \) the existence of all moments about the origin (and consequently about any point) is established for the distributions \( L_{I,V,T}, L_{II,V}, \) and \( L_{III,V} \) defined in Chapter V.

The restriction to the case \( p = q = 2 \) is made since the proof of Theorem 6.1 is dependent on \( \psi(x) \) being bounded by \( \mu(x) \) and \( \lambda(x) \) for all \( x \) in the interval \([0,1]\). However, as stated in Section D of Chapter II, it is conjectured by the author that \( \mu(x) \) and \( \lambda(x) \) bound \( \psi(x) \) for \( 0 \leq x \leq 1 \) and all values of \( p \) and \( q \). If this were the case, then the results of this section would hold without restriction.

**Lemma 6.1** (Cramer, 7, p. 71): If, for some positive integer \( k > 0 \), the distribution function \( L(x) \) satisfies

\[
L(y) = O(|y|^{-(k+1)}) \quad \text{as} \quad y \to -\infty
\]

\[
1 - L(y) = O(y^{-(k+1)}) \quad \text{as} \quad y \to +\infty
\]

then

\[
\int_{-\infty}^{\infty} |y|^{1-i} dL(y) \quad \text{exists} \quad \text{for} \quad i = 1, 2, \ldots, k.
\]
Theorem 6.1: For the case when \( p = q = 2 \),

\[
\int_{-\infty}^{\infty} |y|^k \, dL_{I,Y,Y}(y), \quad \int_{-\infty}^{\infty} |y|^k \, dL_{II,Y,Y}(y),
\]

and

\[
\int_{-\infty}^{\infty} |y|^k \, dL_{III,Y,Y}(y) \quad \text{exist for } k = 1, 2, 3, \ldots.
\]

Proof: From Lemma 2.4 and 2.35 we have

\[
L_u(-|y|^y) = \lim_{n \to \infty} \varrho^{(n)}(-|y|^y + a) \leq \lim_{n \to \infty} \mu^{(n)}(-|y|^y + a) = \frac{|y|^y}{a}
\]

for \( -\infty < y \leq 0 \),

and

\[
L_u(\gamma y^y) = \lim_{n \to \infty} \varrho^{(n)} \left( \frac{\gamma y^y}{b^n} + a \right) \geq \lim_{n \to \infty} \lambda^{(n)} \left( \frac{\gamma y^y}{b^n} + a \right) = 1 - \frac{\gamma y^y}{1 - a}
\]

for \( 0 \leq y < \infty \).

Hence,

\[
\lim_{y \to -\infty} \left[ |y|^k L_u(-|y|^y) \right] \leq \lim_{y \to -\infty} \left( a|y|^k \right) = 0
\]

for \( k = 1, 2, 3, \ldots \),

and
\[
\lim_{y \to \infty} \frac{(1 - a)|y|^k}{e^{\frac{\tau y^*}{a}}} = 0
\]

for \( k = 1, 2, 3, \ldots \).

The conclusions of the lemma then follow directly from Lemma 6.1 and the definitions, in terms of \( L_u \), for \( L_{I, y^*}, L_{II, y^*} \) and \( L_{III, y^*} \).

B. Moment Convergence for Uniform Terminal Payoffs

It will be shown in Lemma 6.2 that, for \( p = q = 2 \) and \( F \) uniform, all moments of \( U_n = b^n \left( \sum_{i=1}^{n} V_i - a \right) \) converge to the corresponding moments of \( L_u \). As in Section A the restriction to \( p = q = 2 \) is due to the fact that the question of bounding other \( \phi \)'s over their entire domain is at the present unresolved. The restriction to uniform \( F \) can be somewhat relaxed in certain straightforward ways. But it has not been significantly lightened because of the difficulties in establishing requisite forms of uniformity.

The following identities will be useful.

\[
k \int_{-\infty}^{0} |y|^{k-1} F(y) dy = k \int_{-\infty}^{0} |y|^{k-1} dy \int_{-\infty}^{y} dF(x)
= \int_{-\infty}^{0} dF(x) \int_{x}^{0} x|y|^{k-1} dy
= \int_{-\infty}^{0} dF(x) \int_{x}^{0} (-y)^{k-1} dy
\]
where \( k \) is any positive integer and \( F \) is any distribution for which the above integrals exist.

**Lemma 6.2:** For \( p = q = 2 \) and \( F \) uniform all moments of \( U_n = b^n(V_n - a) \) converge to the corresponding moments of \( L_u \).

**Proof:** The relations

\[
\mu^{(n)}(\frac{y}{b^n} + a) = a(\frac{y}{b^n} + 1)b^n < ae^{y/a} \quad \text{for} \quad y < 0; \quad n = 1,2,3,\ldots \tag{6.4}
\]

\[
1 - \lambda^{(n)}(\frac{y}{b^n} + a) = (1-a)\left(1 - \frac{y}{b^n}\right)b^n < (1-a)e^{\frac{y}{1-a}} \tag{6.5}
\]

are first verified:

\[
\left(\frac{y}{b^n} + 1\right)b^n < e^y \quad \text{iff} \quad b^n \ln(1 + \frac{y}{b^n}) < y,
\]

i.e.,

\[
= \int_{-\infty}^{0} (-x)^kdF(x)
= \int_{-\infty}^{0} |x|^kdF(x), \quad (6.2)
\]

and similarly

\[
k \int_{0}^{\infty} (y)^{k-1}[1 - F(y)]dy = \int_{0}^{\infty} (x)^{k}dF(x), \quad (6.3)
\]
iff \( b^n \left( \frac{z}{b^n} - \frac{1}{2} \left( \frac{z}{b^n} \right)^2 + \frac{1}{3} \left( \frac{z}{b^n} \right)^3 - \cdots \right) < z \),

i.e., for negative \( z \),

\[
\text{iff } - b^n \left( \frac{1}{2} \left| \frac{z}{b^n} \right|^2 + \frac{1}{3} \left| \frac{z}{b^n} \right|^3 + \cdots \right) < 0.
\]

Thus 6.4 and 6.5 have been verified. Then from the fact (Lemma 2.4) that \( \mu^{(n)} \) and \( \lambda^{(n)} \) bound \( \phi^{(n)} \) respectively from above and below it follows that

\[
|y|^{k_{F_U_n}}(y) < a |y|^k e^{y/a} \quad \text{for } y < 0; \ n = 1, 2, 3, \ldots \tag{6.6}
\]

\[
y^k \left[ 1 - F_{U_n}(y) \right] < (1 - a)y^k e^{-\frac{y}{1 - a}} \quad \text{for } y > 0; \ n = 1, 2, 3, \ldots
\]

(recall from 3.2 that \( F_{U_n}(y) = \phi^{(n)}(\frac{y}{b^n} + a) \)), where the two right sides are integrable over their respective domains of definition; therefore by Lebesgue's Theorem on dominated convergence (see 7, p. 47), 6.2, 6.3, and Theorem 6.1

\[
\lim_{n \to \infty} \int_{-\infty}^{0} |x|^{k_{F_{U_n}}}(x) = \lim_{n \to \infty} \left\{ k \int_{-\infty}^{0} |y|^{k-1} F_{U_n}(y) dy \right\} = k \int_{-\infty}^{0} |y|^{k-1} \left[ \lim_{n \to \infty} F_{U_n}(y) \right] dy = k \int_{-\infty}^{0} |y|^{k-1} L_n(y) dy
\]
\begin{align*}
(x)^n_{T_{\mathcal{I}}x} \int_0^\infty x &= \\
\mathcal{A}_{n-0} \left[ (\mathcal{A})^n_{T_{\mathcal{I}}x} \right] \int_0^\infty x &= \\
\mathcal{A}_{n-0} \left[ (\mathcal{A})^n_{x} \right] \mathcal{M}^{\mathcal{I}} - \mathcal{M}^{\mathcal{I}} \int_0^\infty x &= \\
\{ \mathcal{A}_{n-0} \left[ (\mathcal{A})^n_{x} \right] \mathcal{M}^{\mathcal{I}} - \mathcal{M}^{\mathcal{I}} \int_0^\infty x \} \mathcal{M}^{\mathcal{I}} &= (x)^n_{\mathcal{A}_{n-0}x} \int_0^\infty \mathcal{M}^{\mathcal{I}}
end{align*}

Then:

\begin{align*}
(x)^n_{T_{\mathcal{I}}x} \int_0^\infty |x|
\end{align*}
VII. MATRIX GAMES WITH INDEPENDENT IDENTICALLY DISTRIBUTED MATRIX ELEMENTS

A. Conditional Density of a Pure Value

Let $||x_{ij}||$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ be the payoff matrix of a zero-sum two-person game, and let $v(||x_{ij}||)$ be its value. In this section the conditional distribution of $V_{n,m} = v(||x_{ij}||)$ given that $V_{n,m}$ is a pure value (corresponding to the existence of a saddle point) is considered, where the $x_{ij}$ are $mn$ mutually independent random variables, each distributed according to the probability density $f$. The probability that the random matrix $||x_{ij}||$ possesses a pure value (possesses a saddle point) does not depend on $f$ and is given (16, p. 79) by

$$P(V_{m,n} \text{ is a pure value}) = \frac{m! \ n!}{(m + n - 1)!}. \quad (7.1)$$

This is exploited in the proof of

**Lemma 7.1:** The conditional density $f_{V_m,n}$ given that it is pure, is that of the $n^{th}$ largest of $m + n - 1$ mutually independent random variables, each distributed according to $f$, i.e.,

$$f_{V_m,n}(t) = \frac{(m + n - 1)!}{(m - 1)! (n - 1)!} \left[ F(t) \right]^{m-1} \left[ 1 - F(t) \right]^{n-1} f(t). \quad (7.2)$$

**Proof:** If an element, $x_{11}$ say, is a saddle point
(minimum in its row and maximum in its column) it is the pure value of the game, i.e., $V_{m,n} = x_{11}$. Since each of the independent random variables $X_{ij}$ possess a density $f$ the probability of a unique saddle point is one. Thus

$$P[V_{m,n} \leq t \text{ and is pure}]$$

$$= mn P[V_{m,n} \leq t \text{ and } X_{11} \text{ is a saddle point}]$$

$$= mn P[X_{11} \leq t; X_{12}, \ldots, X_{1n} \geq X_{11}; X_{21}, \ldots, X_{mn} \leq X_{11}]$$

$$= mn \left( \frac{1}{(m+n-1)(m+n-2)} \right) \frac{1}{(n-1)!} \frac{1}{(m+n-1)!} P[X_{(n)}^{(n)} \text{ is } n^{th} \text{ largest of independent random variables } \leq t].$$

Therefore by 7.2

$$F_{V|\mathcal{G}}(t) = P[V_{m,n} \leq t \mid V_{m,n} \text{ is pure}]$$

$$= \frac{1}{\binom{m+n-1}{n-1}} \frac{1}{(m+n-1)!} P[X_{m+n-1}^{(n)} \leq t]$$

$$= P[X_{m+n-1}^{(n)} \leq t].$$

(7.3)

The density corresponding to 7.3 is that given by 7.2 (see 7, p. 370).
B. Conditional Density of the Mixed Value for Uniform $f$

In this section the conditional density $f_{V|\mathcal{M}}(n)$ of $V_n$, given that it is mixed, is considered for $f$ uniform and $m = 2$. The method described below may be used for any density $f$ which has no lower tail (i.e., there is a point $x_o$ such that $f(x) = 0$ for $x \leq x_o$); however the computations required, although elementary in nature, would be considerably more complex than for the uniform $f$. Due to the geometric formulation of the problem the extension of the solution to $m \geq 3$ is computationally (but not conceptually) very difficult. The results concerning the density $f_{V|\mathcal{M}}$ were also obtained by a more direct procedure; however the ease with which asymptotic (as $n \to \infty$) results are gotten by the method to be described warrants its use. A detailed derivation of the results of this section would be lengthy and quite elementary; therefore the complete derivation will not be carried through in detail.

The geometric construction (see 16, p. 405) associated with expressing the mixed value $v_n = v(\| x_{1j} \|)$ as the function $v_n = \frac{ab}{a + b}$ of the intercepts $a, b$ of a certain "separating line" $L_s$ is used. Let $(x_j) = (x_{1j})$ for $j = 1, 2, \ldots, n$ and let $M = \begin{bmatrix} x_j \\ y_j \end{bmatrix}$. The separating line $L_s$ is constructed as follows (see Figure 3): (a) form the convex hull $\text{CH}(M)$ of the $n$ points $P_1 = (x_1, y_1)$, (b) define

$$W_{v}(M) = \{(x, y): 0 \leq x \leq v, 0 \leq y \leq v\}$$
\[ v^* = \min \{ v: \mathcal{W}_V(M) \cap \mathcal{CH}(M) \neq \emptyset \} \]

(c) "separate" the convex sets \( \mathcal{W}_V(M) \) and \( \mathcal{CH}(M) \) by any line \( L_s \) lying outside of the interiors of both \( \mathcal{W}_V(M) \) and \( \mathcal{CH}(M) \). Let \( v_n \) be the value of the game corresponding to \( M \). It is easily verified (16, p. 450) that \( v_n = v^* = \frac{ab}{a + b} \).

Also, conditional on mixing, the probability is one that (i) \( L_s \) is unique, (ii) the slope of \( L_s \) satisfies \( -\infty < \sigma < 0 \), and (iii) exactly two points \( P_j \) lie on \( L_s \), with one on either side of the equiangular line \( y = x \). Since \( v_n = \frac{ab}{a + b} \) it suffices to compute the bivariate density \( g_n(a, b) \) of \( a \) and \( b \). Let \( P_1 \) and \( P_2 \) be the points which lie on the line \( L_s \) and define \( h(P_1, P_2) \) and \( v(P_1, P_2) \) as the horizontal and vertical intercepts of \( L_s \). Let \( \Delta \) be any small positive number and define

\[ P_{a, b, \Delta} = \mathbb{P}\left\{ a \leq h(P_1, P_2) \leq a + \frac{a}{5} \Delta; \ b \leq v(P_1, P_2) \leq b + \Delta; \ P_1 \text{ above line } y = x; \ P_2 \text{ below line } y = x \right\} \]

and

\[ g_{12}(a, b) = \lim_{\Delta \to 0} \left( \frac{P_{a, b, \Delta}}{\Delta^2 a} \right). \]

In view of (i), (ii), and (iii) it is convenient to write the density \( g_n(a, b) \) in the form

\[ \frac{g_n(a, b)}{P\{ \text{mixed value} \}} = n(n - 1) \left[ g_{12}(a, b) \right] \mathbb{P}\left\{ P_3, \ldots, P_n \text{ are above } L_s \right\}. \]
Figure 3. Geometric construction of the game value $V_n$ resulting from the $2 \times n$ payoff matrix $M$
Figure 4. Regions $R_1, R_{II}, R_{III}$ in relation to $v = \frac{ab}{a+b}$
Figure 5. Regions $R_{IL}$, $R_{IM}$, $R_{IU}$ for $P_1$ such that
$a \leq h(P_1;P_2) \leq a + \frac{a}{b} \Delta$, $b \leq v(P_1;P_2) \leq b + \Delta$, $0 < b \leq a \leq 1$

$L_1: y = \frac{b(a - x)}{a}$
$L_2: y = b - \frac{b^2 x}{a(b + \Delta)}$
$L_3: y = (b + \Delta)(\frac{a - x}{a})$
$L_4: y = b + \Delta - \frac{b}{a} x$
Figure 6. Region $A_{IM}(P_1)$ for $P_2$ such that

- $a \leq h(P_1, P_2) \leq a + \frac{a}{b} \Delta$, $b \leq v(P_1, P_2) \leq b + \Delta$,
- $a < b < a < 1$ when $P_1 \in R_{IL}$

$L_5: \quad y = b - \frac{b - y_1}{x_1} x$

$L_6: \quad y = \frac{a - x}{a - x_1} y_1$
Figure 7. Region $A_{IM}(P_1)$ for $P_2$ such that

$$a \leq h(P_1, P_2) \leq a + \frac{a}{b} \Delta, \quad b \leq v(P_1, P_2) \leq b + \Delta,$$

$0 < b \leq a \leq 1$ when $P_1 \in R_{IM}$

$L_7: \quad y = \frac{a(b + \Delta) - bx}{a(b + \Delta) - bx_1} y_1$

$L_8: \quad y = \frac{a - x}{a - x_1} y_1$
Figure 8. Region $A_{IU}(P_1)$ for $P_2$ such that

- $a \leq h(P_1, P_2) \leq a + \frac{a}{b} \Delta$, $b \leq v(P_1, P_2) \leq b + \Delta$,
- $0 < b \leq a \leq 1$ when $P_1 \in B_{IU}$

$L_9: \quad y = \frac{a(b + \Delta)}{a(b + \Delta) - bx_1} y_1$

$L_{10}: \quad y = b + \Delta - \frac{b + \Delta - y_1}{x_1} x$
The evaluation of $g_n(a,b)$ is made for the three regions $\mathcal{R}_I$, $\mathcal{R}_{II}$, $\mathcal{R}_{III}$ defined as follows (see Figure 4):

$$\mathcal{R}_I = \{(a,b): 0 < b \leq a \leq 1\}$$

$$\mathcal{R}_{II} = \{(a,b): 0 < b < 1 < a\}$$  \hspace{1cm} (7.7)

$$\mathcal{R}_{III} = \{(a,b): 1 < b < a\};$$

these regions are sufficient to define $g_n(a,b)$ since $g_n(b,a) = g_n(a,b)$ and $g_n(a,b)$ is non-zero only in the positive quadrant (for $f$ uniform).

The method used for evaluating $P_{a,b,\Delta}$ (see Equation 7.4) for region $\mathcal{R}_I$ will now be described with the aid of Figures 5 through 8. The points $P_1$ are weighted by the area of the region $A_1(P_1)$ (to order $O(\Delta^2)$) that the point $P_2$ must occupy in order to satisfy $a \leq h(P_1,P_2) \leq a + \frac{c}{b} \Delta$ and $b \leq v(P_1,P_2) \leq b + \Delta$. The area function of $A_1(P_1)$ is then integrated over the appropriate region for $P_1$. It is convenient to divide the region for $P_1$ (to order $O(\Delta^2)$) into three regions $\mathcal{R}_{II}$, $\mathcal{R}_{IM}$, $\mathcal{R}_{IU}$ (see Figures 5 through 8) corresponding to the regions $A_{II}(P_1)$, $A_{IM}(P_1)$, and $A_{IU}(P_1)$ being bounded from above and below by three pairs of lines of different form. Let $A_{II}(P_1)$ also represent the area of region $A_{II}(P_1)$, and similarly for $A_{IM}(P_1)$, $A_{IU}(P_1)$; then the calculation
\[ s_{12}(a, b) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \int_{p_1 \in R_{IL}} A_{IL}(p_1) + \int_{p_1 \in R_{IM}} A_{IM}(p_1) \right\} \]

\[ + \int_{p_1 \in R_{IU}} A_{IU}(p_1) \]

leads to the simple expression

\[ s_{12}(a, b) = \frac{1}{2} \left( \frac{ab}{a + b} \right)^2 \quad \text{for} \quad (a, b) \in \mathcal{R}_1. \] (7.9)

An analogous sequence of computations is made for \( s_{12}(a, b) \) for regions \( \mathcal{R}_{II} \) and \( \mathcal{R}_{III} \). The regions \( R_{IL}, R_{IM}, R_{IU} \) are the same as the corresponding regions in Figure 5; however the regions \( A_{II}(p_1), A_{IM}(p_1), \) and \( A_{IU}(p_1) \) (see Figure 6, 7, and 8) are bounded on the right by the line \( x = 1 \).

The regions \( R_{III}, R_{IM}, R_{IU} \) differ from the corresponding regions in Figure 5 in that they are partially bounded from above by the line \( y = 1 \). The regions \( A_{III}(p_1), A_{IIM}(p_1), A_{III}(p_1) \) are the same as the corresponding regions \( A_{III}(p_1), A_{IIM}(p_1), A_{III}(p_1) \).

Computations analogous to those in 7.8 yield

\[ s_{12}(a, b) = \frac{b}{2a} \left( 1 - \frac{ab}{a + b} \right) \frac{ab}{a + b} \quad \text{for} \quad (a, b) \in \mathcal{R}_{II} \]

\[ s_{12}(a, b) = \frac{(1 - \frac{a}{a + b})^3}{\left( \frac{ab}{a + b} \right)} \quad \text{for} \quad (a, b) \in \mathcal{R}_{III} \] (7.10)

The probabilities \( P\{p_3, \ldots, p_n \text{ lie above } L_b\} \) are
The distribution function $F_{V_n \mid \mathcal{M}}(t)$ of the conditional distribution of $V_n$, given $V_n$ is a mixed value, is found by substituting Expressions 7.9, 7.10, and 7.11 in 7.6 and performing the integration

$$F_{V_n \mid \mathcal{M}}(t) = \frac{1}{P[\text{mixed value}]} \int \int_{a+b \leq t} g_n(a,b) \, da \, db. \quad (7.12)$$

For $n = 2$ the conditional density $f_{V_2 \mid \mathcal{M}}(t)$ has the form

$$(P[\text{mixed value}])f_{V_2 \mid \mathcal{M}}(t) =$$

$$4t^2 - \frac{t^3}{1-t} + 4t^3 \ln(\frac{1-t}{t}) \quad \text{for } 0 < t \leq \frac{1}{2} \quad (7.13)$$

$$4(1-t)^2 - \frac{(1-t)^3}{t} + 4(1-t)\ln(\frac{t}{1-t}) \quad \text{for } \frac{1}{2} < t < 1,$$

where (from 7.1) $P[\text{mixed value}] = \frac{1}{3}$. Also, from 7.2 it is seen that the conditional density $f_{V_2 \mid \mathcal{M}}(t)$ has the form
(P[pure value]) \( f_{V_2}(t) = 4t(1 - t) \) for \( 0 \leq t \leq 1 \). 

Hence, 7.13 and 7.14 yield 

\[
f_{V_2}(t) = 4t - \frac{t^3}{1 - t} + 4t^3 \ln(\frac{1 - t}{t}) \quad \text{for} \quad 0 < t \leq \frac{1}{2}
\]

\[
f_{V_2}(t) = 4(1 - t) - \frac{(1 - t)^3}{t} + 4(1 - t)^3 \ln(\frac{t}{1 - t}) \quad \text{for} \quad \frac{1}{2} < t < 1,
\]

where \( f_{V_2}(t) \) is the unconditional density of the game value for uniform \( f \) and \( m = n = 2 \).

C. Limit Distributions

The limiting distributions of \( V_n = V_{2,n} \) as \( n \to \infty \), are easily obtained from 7.6. Write \( g_n ; (A,B)(a,b) = g_n(a,b) \), where \( A \) and \( B \) are the random intercepts with density \( g_n(a,b) \), and consider the sequence of random variables \( (\sqrt{n} A, \sqrt{n} B) \), \( n = 1,2,3, \ldots \). Transforming 7.6 (see 7.9 and 7.11) gives

\[
g_n ; (\sqrt{n} A, \sqrt{n} B)(a,b) = (P[mixed value]) (1 - \frac{1}{n}) \left( \frac{ab}{a + b} \right)^2 (1 - \frac{ab}{2n})^{n-2}
\]

for \( 0 < a \leq \sqrt{n}, 0 < b \leq \sqrt{n} \), which is the region of asymptotic interest.

Hence,

\[
g(a,b) = \lim_{n \to \infty} g_n ; (\sqrt{n} A, \sqrt{n} B)(a,b)
\]

\[
= \frac{1}{2} \left( \frac{ab}{a + b} \right)^2 e^{-ab/2} \quad \text{for} \quad 0 < a < \infty; 0 < b < \infty
\]

\[
= 0 \quad \text{otherwise.}
\]
The function $g(a,b)$ is a probability density; therefore, by Scheffe's Theorem on convergence of densities (see 18, p. 436), 7.15 implies that

$$L(t) = \lim_{n \to \infty} \frac{F_{\sqrt{n}V_n}(t)}{n} = \lim_{n \to \infty} \int \int \frac{g_n((\sqrt{n} A, \sqrt{n} B)(a,b))db}{ab} \leq t$$

$$= \int \int \frac{g(a,b)db}{ab} \leq t$$

(7.16)

where $L(t)$ is the limiting distribution of $\sqrt{n} V_n$ and $g(a,b)$ is defined as in 7.15.

As a simple generalization, it was established that for densities $f$ satisfying

(i) $f(t) = 0$ for $t < t^o$

(ii) $f(t)$ continuous to the right and discontinuous to the left at $t^o$,

that $U_n = \sqrt{n} f(t^o)(V_n - t^o)$ converges in distribution to $L(t)$, defined in 7.16.

Moment convergence, $\int_{-\infty}^{\infty} t^k dP(U_n(t)) \to \int_{-\infty}^{\infty} t^k dL(t)$, as $n \to \infty$, for $k = 1, 2, 3, \ldots$, was also established for uniform densities. Briefly, the moment convergence is established by observing that $g_n((\sqrt{n} A, \sqrt{n} B)(a,b)) \to g(a,b)$, as $n \to \infty$, from below (see Lemma 6.2) and that

$$\int \int \frac{(ab)^k}{a+b} g(a,b)db$$
exists for $k = 1, 2, 3, \cdots$; application of Lebesgue's Theorem on dominated convergence then yields

$$\lim_{n \to \infty} \int \int (\frac{ab}{a+b})^k g_n; (\sqrt{n} A, \sqrt{n} B)(a, b) \, da \, db$$

and hence the moment convergence for $V_n$ follows since

$v_n = \frac{ab}{a+b}$.
VIII. REFERENCES


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