Baffling of fluid sloshing in cylindrical tanks

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BAFFLING OF FLUID SLOSHING IN CYLINDRICAL TANKS

by

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I. INTRODUCTION

During the past few years a great deal of research has been conducted on the problem of sloshing of liquid propellants in missile and space vehicles. One of the primary objectives of these investigations was to study the effects of this liquid motion on the dynamic stability of the rocket.

It has been observed that, in any rocket flight, the vehicle body is subjected to translatory and oscillatory perturbations from external forces such as guidance and control inputs. Vehicle body motions of this type result in disturbances of the contained liquid. If the perturbations occur at a frequency near that of the control frequency of the space vehicle, the liquid is forced to oscillate at amplitudes sufficiently large to cause severe de-stabilizing forces and moments on the vehicle. Bauer (1) remarks that with the increasing size of space vehicles and their larger tank diameters, which lower the natural frequencies of the propellants, the effects of propellant sloshing upon the stability of the vehicle become extremely critical. Especially since at launch usually more than ninety per cent of the total mass is in the form of liquid propellant. With increasing diameter, the oscillating propellant masses and the corresponding forces increase.

If a space vehicle, due to an atmospheric disturbance, deviates from its original trajectory, it should be returned
quickly to its pre-programmed flight path. This is performed by the control and guidance system and is executed by swiveling the thrust of the space vehicle. A poorly designed control system can therefore continuously excite the motion of the propellant in the fuel tank. For this reason, the forces and moments due to liquid performing harmonic oscillations in a particular container must be determined and their influence upon the stability must be investigated.

In order to minimize the effects of propellant motion, several methods have been suggested. Bauer (2) has discussed the feasibility of dividing the tank into several subtanks by means of separating inner-walls. This could be performed by dividing the tank into sector tanks by means of inner-walls or by a concentric cylinder wall. It has been observed by Bauer that the employment of a cross-device increases the natural frequencies of the propellant; however, the resonant frequencies are the same for all four quarter tanks, causing a superposition of the forces and torques of the propellant in the individual quarter tanks. The configuration which uses a concentric circular cylindrical inner wall, thus dividing the tank into an inner circular cylindrical tank and a circular cylindrical ring tank enables a separation of the propellant resonance frequencies. The relative phase shift between the propellant oscillations in the inner and outer tanks makes it possible to reduce the total propellant forces and torques. A still greater cancellation is possible by
choosing a diameter of the inner wall so that the frequencies are well separated and the propellant masses in the inner and outer tank are approximately equal. On the other hand, the introduction of an annular baffle mounted on the tank wall will break up the flow and create enough turbulence to dampen the oscillations. It is the purpose of this paper to present a theoretical discussion of liquid sloshing in a cylindrical tank containing a baffle.

Miles (3) has studied liquid sloshing in a cylindrical tank containing a baffle and has given some approximate results related to the drag coefficient. Cole and Gambucci (4) and (5) have developed some experimental tests for measuring the effectiveness of baffles in damping the fluid oscillations. The evaluation of force measurements from forced oscillation of twenty-seven baffle configurations in a two-dimensional tank and damping measurements of ring baffles in a cylindrical tank has led them to the following conclusions:

1. For baffle depths greater than twice the width of the baffle, it appears that a flat plate with a sharp edge is the most effective damper.

2. For baffle depths less than twice the width of the baffle, two widely different types of fluid damping occur separated by a critical velocity. Below the critical velocity, a traveling wave forms which breaks and creates turbulent shear layers and very high damping. Above the
critical velocity, a smooth standing wave forms which reduces the damping effectiveness of the baffle.

3. For baffle depths less than the width of the baffle, plates which are angled up are more effective than flat plates.

4. An asymmetrical baffle is substantially more effective a fuel damper than conventional ring-type baffles of equal area.

Eulitz and Glaser (6) have discussed the linear boundary value problem associated with propellant sloshing in a right circular cylinder. Within the framework of linear theory, the free surface of the fluid in a container undergoing transverse harmonic vibrations should exhibit a steady-state, planar, harmonic motion at all frequencies except resonance. They found excellent agreement between experimental results and the linearized theory.

A fairly extensive bibliography of the work done before 1960 on the sloshing problem has been compiled by Cooper (7).
II. DEFINITION OF THE BOUNDARY VALUE PROBLEM

This paper is concerned with the sloshing of an incompressible, inviscid liquid contained in a right circular cylinder which is mounted in a space vehicle which is moving along a prescribed path. Because of perturbations due to the deviation of the vehicle from its path, the system oscillates and produces waves on the surface of the liquid. It is proposed that the introduction of an annular baffle below the surface of the liquid will damp the induced oscillations.

Since the tank is in motion along some path, it appears to be reasonable to refer its motion to an inertial system, for example the earth. However, if any type of measuring device is attached to the tank, then it measures quantities in terms of a tank-fixed reference frame which is moving relative to the inertial system. Thus, it is necessary to be able to express the tank-fixed system in terms of the inertial system and vice versa.

Let $Y_i$, with coordinates $y_i$ ($i = 1, 2, 3$) and origin $0'$, be a fixed Cartesian reference frame and $X_i$ with coordinates $x_i$ and origin $0$ be a Cartesian frame moving relative to $Y_i$. Then, instantaneously, it follows that

$$y_i = \tilde{z}_i + a_{ij} x_j,$$

where the components of $\tilde{z}_i$ are measured in $Y_i$, $\tilde{z}(t)$ gives the instantaneous displacement of $0$ relative to $0'$, and

$$a_{ij}(t) = \cos(x_i, y_j)$$
measures the instantaneous rotation of $X_i$ with respect to $Y_i$.

In the following, a repeated index indicates summation over the range of values of the index. The coordinate transformation simultaneously gives the formulas of transformation for any free vector

$$\mathbf{B}_i(y) = a_{ij} B_j(x)$$

and

$$\mathbf{B}_i(x) = a_{ij} \mathbf{B}_j(y),$$

where $\mathbf{B}_i$ and $\mathbf{B}_i$ are the same vector with components measured respectively in $Y_i$ and $X_i$.

Since the $a_{ij}$ are a set of direction cosines, they satisfy

$$a_{ik}a_{jk} = \delta_{ij}, \quad a_{ki}a_{kj} = \delta_{ij}, \quad (1)$$

where $\delta_{ij}$ is the Kronecker delta, for any $t$. Letting $\dot{a}_{ij}$ denote $da_{ij}/dt$, a simple differentiation of Equation 1 yields

$$a_{ik} \dot{a}_{jk} + \dot{a}_{ik} a_{jk} = 0. \quad (2)$$

Define $w_{ij} = a_{ik} \dot{a}_{jk}$. so that, according to Equation 2

$$w_{ji} = a_{jk} \dot{a}_{ik} = -a_{ik} \dot{a}_{jk} = -w_{ij}.$$

Thus $w_{ij}$ is a skew-symmetric second order quantity (it can be shown to be a tensor). Hence there exists a dual vector $w_i$ defined by

$$w_{ij} = -\varepsilon_{ijk} w_k,$$

where $\varepsilon_{ijk}$ is the third order alternating tensor. Consequently

$$a_{ik} \dot{a}_{jk} = -\varepsilon_{ijk} w_k,$$
where \( w_k \) may be identified as the angular velocity of \( X_1 \) with respect to \( Y_1 \), measured along \( X_1 \).

The absolute velocity of a particle is given by

\[
\frac{dy_1}{dt} = \dot{y}_1 = \dot{z}_1(y) + a_{j1} \dot{x}_j + \dot{a}_{j1} x_j = \ddot{q}_1(y).
\]

However, the control instruments measure \( q_1(x) \), where

\[
q_1(x) = a_{1j} \ddot{q}_j(y).
\]

Therefore

\[
q_1(x) = a_{1j} \dot{z}_j(y) + a_{1j} a_{kj} \ddot{x}_k + a_{1j} \dot{a}_{kj} x_k
\]

\[
= \dot{z}_1(x) + \dot{x}_1 - \epsilon_{jik} w_j x_k
\]

\[
= \ddot{z}_1(x) + \dot{x}_1 + \epsilon_{1jk} w_j x_k. \tag{3}
\]

In vector symbolism, Equation 3 assumes the form

\[
\mathbf{v} = \mathbf{v}_o \omega = \ddot{z} + \dot{\mathbf{r}} + \mathbf{w} \times \mathbf{r},
\]

where \( \mathbf{r} \) is the position vector in \( X_1 \). The absolute acceleration is obtained from

\[
\ddot{a}_1(y) = \frac{d}{dt} \ddot{q}_1(y),
\]

and from the relation

\[
\ddot{q}_1(y) = a_{j1} q_j(x),
\]

which yield

\[
\ddot{a}_1(y) = \frac{d}{dt} \ddot{q}_1(y) = a_{j1} \frac{d}{dt} q_j(x) + \dot{a}_{j1} q_j(x). \tag{4}
\]

But since \( a_1(x) \) is desired and

\[
a_1(x) = a_{1j} \ddot{a}_j(y), \tag{5}
\]
it follows from Equations 4 and 5 that

\[ a_i(x) = a_{ij} a_{kj} \frac{d}{dt} q_k(x) + a_{ij} \dot{a}_{kj} q_k(x) \]

\[ = \frac{d}{dt} q_i(x) - \varepsilon_{ikj} w_j q_k(x) \]

\[ = \frac{d}{dt} q_i(x) + \varepsilon_{ijk} w_j q_k(x). \]

But observing that \( q_i \) is a function of the coordinates \( x_i \) as well as \( t \), it is to be noted that

\[ \frac{d}{dt} q_i(x) = \frac{\partial q_i}{\partial t}(x) + \frac{\partial q_i}{\partial x_k} \frac{dx_k}{dt}. \]  

(6)

It was shown above that

\[ q_i(x) = \dot{z}_i + \dot{x}_i + \varepsilon_{ijk} w_j x_k \]

or

\[ \dot{x}_k = q_k(x) - \dot{z}_k - \varepsilon_{kpq} w_p x_q \]

which is the same as

\[ \ddot{x} = q - z - w \times r. \]  

(7)

Upon insertion of Equation 7 into Equation 6, the result is

\[ \frac{d}{dt} q_i(x) = \frac{\partial q_i}{\partial t}(x) + \frac{\partial q_i}{\partial x_k} [q_k(x) - \dot{z}_k - \varepsilon_{kpq} w_p x_q], \]

and finally the acceleration is found to be

\[ a_i(x) = \frac{\partial q_i}{\partial t}(x) + \varepsilon_{ijk} w_j q_k(x) + \frac{\partial q_i}{\partial x_k} [q_k(x) - \dot{z}_k(x) - \varepsilon_{kpq} w_p x_q] \]

or, in vector symbolism,

\[ a = \frac{\partial q}{\partial t} + w \times q + [(q - \dot{z} - w \times r) \cdot \nabla] q. \]

For an incompressible fluid, the Eulerian equations of
motion are
\[ \ddot{a}_i(y) = F_i(y) + \frac{1}{\rho} \frac{\partial p}{\partial y_i}(y), \]
where \( F_i \) is the specific body force, \( \rho \) is the density, and \( p \) is the pressure. Now
\[ \frac{\partial p}{\partial y_1} = \frac{\partial p}{\partial x_k} \frac{\partial x_k}{\partial y_1} = a_{k1} \frac{\partial p}{\partial x_k}. \]
Transforming to body-fixed axes, it is found that
\[ a_1(x) = a_{ij} \ddot{a}_j(y) = a_{ij} F_j(y) - \frac{1}{\rho} a_{ij} a_{kj} \frac{\partial p}{\partial x_k} = F_1(x) - \frac{1}{\rho} \frac{\partial p}{\partial x_1}. \]
If it is assumed that the fluid motion is irrotational, then there exists a potential function \( \phi \) such that
\[ q = -\nabla \phi \quad \text{or} \quad \ddot{a}_1(y) = -\frac{\partial \phi}{\partial y_1}. \]
It is easily verified that
\[ q_1(x) = -\frac{\partial \phi}{\partial x_1} \]
Then, since the fluid is incompressible \((\nabla \cdot q = 0)\), it follows that
\[ \nabla^2 \phi = 0. \]
Thus the fundamental differential equation to be solved is Laplace's equation for a velocity potential.

To describe the boundary conditions for the problem, consider a tank of arbitrary shape partially filled with liquid. Suppose that a constant acceleration is imposed
along an axis of thrust - call it the $x_3$-axis. Then the liquid assumes a planar surface normal to the thrust axis - herein this surface is called the free surface.

Choose the origin of $X_1$ at the center of gravity of the fluid in this configuration.

The motion of the tank-fixed frame $X_1$ relative to $Y_1$ characterized by $z_1(x)$ and $w_1$ are oscillatory motions superimposed on the constant-acceleration motion. For an observer traveling with the tank, these are, of course, the only forcing motions that he sees.

On the wetted surface of the tank, the boundary condition must be that the velocity of the liquid normal to the tank wall must equal the normal component of velocity of the tank itself (it is assumed here that the tank is rigid). Thus, if $v$ denotes the unit exterior normal to the tank, then

$$q \cdot v = v \cdot [\ddot{z} + w \times r],$$

since $\ddot{r} = 0$ for a rigid tank, or

$$- v \cdot \nabla \phi = v \cdot [\ddot{z} + w \times r].$$

There are two conditions at the free surface. If the disturbed free surface is denoted by $\eta(x_1,x_2,t)$ and the unit normal to the quiescent free surface is taken to be $n = (0,0,1)$, then there exists a kinematic condition such that a particle of fluid which travels with the free surface as it moves must have the same velocity as the free surface itself,
i.e., if \( x_3 \) is the displacement of a particle in the \( x_3 \)-direction, then
\[
\frac{d}{dt} (x_3 - \eta) \bigg|_{x_3=\eta} = 0. \tag{8}
\]
Expanding the left-hand side of Equation 8, it is found that
\[
\left\{ \frac{dx_3}{dt} - \frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x_1} \dot{x}_1 - \frac{\partial \eta}{\partial x_2} \dot{x}_2 \right\} \bigg|_{x_3=\eta} = 0,
\]
and since
\[
\dot{x} = g - \ddot{x} - \omega \times r
\]
if follows that
\[
\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_1} \dot{x}_1 + \frac{\partial \eta}{\partial x_2} \dot{x}_2 = (g - \ddot{x} - \omega \times r) \cdot \eta \bigg|_{x_3=\eta}.
\]

The second condition is a dynamic one which states that the pressure at the free surface of the liquid must equal the ambient pressure. The form of this condition can be obtained from an integration of the equation of motion. Suppose that the only specific body force is that due to the gravitational field in which the liquid-tank system is operating. The equation of motion is
\[
\frac{\partial \eta}{\partial t} + wxg + \left[ (g - \ddot{x} - \omega \times r) \cdot \nabla \right] \eta = F - \frac{1}{\rho} \nabla p.
\]
Now
\[
\eta = -\nabla \varphi, \quad F = -\nabla \Omega, \quad \frac{1}{\rho} \nabla p = \nabla \left( \frac{1}{\rho} p \right),
\]
so that the equation of motion assumes the form
\[
-\nabla \left( \frac{\partial \varphi}{\partial t} \right) + wxg + \left[ (g - \ddot{x} - \omega \times r) \cdot \nabla \right] \eta = -\nabla (\Omega + p/\rho).
\]
Recall the vector identity
\[
\nabla \cdot \left( s \nabla \right) \phi = \nabla \cdot s \nabla \phi + s \nabla \cdot \nabla \phi
\]
(g - \ddot{z}) \cdot \nabla g = (g - \ddot{z}) \cdot \nabla (g - \ddot{z}) = \frac{1}{2} \nabla (g - \ddot{z})^2

and substitute into the differential equation of motion; the result is

\[ w \times g - [(w \times r) \cdot \nabla]g = \nabla \left( \frac{\partial \omega}{\partial t} - \frac{1}{2}(g - \ddot{z})^2 - \Omega \cdot \frac{p}{\rho} \right). \tag{9} \]

Consider next

\[ \nabla [g \cdot (w \times r)] = [(w \times r) \cdot \nabla]g + (g \cdot \nabla)(w \times r) \]

\[ + (w \times r) \times (\nabla \times g) + g \times [\nabla \times (w \times r)] \]

\[ = [(w \times r) \cdot \nabla]g + (g \cdot \nabla)(w \times r) + g \times [\nabla \times (w \times r)] \]

since \( \nabla \times g = 0 \). It is to be noted also that

\[ \nabla \times (w \times r) = 2w \text{ and } (g \cdot \nabla)(w \times r) = w \times g, \]

so that

\[ \nabla [g \cdot (wxr)] = [(w \times r) \cdot \nabla]g + w \times g + 2g \times w \]

\[ = [(w \times r) \cdot \nabla]g + g \times w. \tag{10} \]

When Equation 10 is inserted into Equation 9, the equation of motion, the result is

\[ \nabla \left( \frac{\partial \omega}{\partial t} - \frac{1}{2}(g - \ddot{z})^2 - \Omega \cdot (w \times r) - \frac{p}{\rho} \right) = 0, \]

or upon integration and replacement of \( \Omega \) by \(-ax_3\), this becomes

\[ \frac{p - P_o}{\rho} = ax_3 - \frac{1}{2}(g - \ddot{z}) - g \cdot (w \times r) + \frac{\partial \omega}{\partial t}, \tag{11} \]

where \( P_o \) is the constant ambient pressure, \( \alpha \) is the magnitude of the acceleration of the liquid-tank system. Thus at \( x_3 = \eta \),
\[ \frac{\partial \varphi}{\partial t} = \alpha \eta + \frac{1}{2} \left( \nabla \varphi + \dot{\varphi} \right)^2 + (w \times \varphi) \cdot \nabla \varphi. \]

For small free surface oscillations, the problem may be linearized. Thus second order terms in velocities can be neglected and it is assumed that not only \( \eta \) is small, but also \( \partial \eta / \partial x_1 \).

Under these conditions, the boundary conditions on the free surface become
\[ \frac{\partial \eta}{\partial t} = - n \cdot \nabla \varphi - n \cdot \left( \dot{\varphi} + w \times \varphi \right), \]
\[ \frac{\partial \varphi}{\partial t} = \alpha \eta, \quad (12) \]
which are usually combined in the form
\[ - n \cdot \nabla \varphi - \frac{1}{\alpha} \frac{\partial^2 \varphi}{\partial t^2} = n \cdot \left( \dot{\varphi} + w \times \varphi \right) \bigg|_{x_3 = \eta}. \]

If the assumption is made that \( \dot{z} \) and \( w \) can be represented as harmonic oscillations,
\[ \dot{z} = u e^{i \beta t} \quad \text{and} \quad w = w_0 e^{i \beta t}, \]
then it is usually assumed that \( \varphi(x_1, x_2, x_3, t) = \psi(x_1, x_2, x_3) e^{i \beta t} \).

The problem then reduces to solving
\[ \nabla^2 \psi = 0 \]
subject to
\[ - n \cdot \nabla \psi = n \cdot \left( u + w_0 \times \varphi \right) \]
(13)
on the wetted surface and
\[ - n \cdot \nabla \psi + \frac{\beta^2}{\alpha} \psi = n \cdot \left( u + w_0 \times \varphi \right) \]
(14)on the free surface.
III. PROPELLANT SLOSHING IN A CIRCULAR CYLINDRICAL TANK CONTAINING AN ANNULAR BAFFLE

In an attempt to minimize the effects of propellant sloshing, an annular baffle of infinitesimal thickness and width \((1 - \gamma)a\), \((0 < \gamma < 1)\), is mounted on the wall of a circular cylindrical tank of radius \(a\). It is convenient to set up a coordinate system centered along the axis of symmetry of the cylinder with the origin located in the plane of the baffle. Accordingly, it is expedient to use cylindrical polar coordinates \((r, \theta, z)\), so that \(z = -h\) denotes the bottom of the tank, \(z = 0\) is the plane of the baffle, and \(z = z_o\) is the quiescent free surface, whereas \(r = a\) refers to the wall of the tank.

Therefore, in terms of cylindrical polar coordinates, the problem under consideration here is the solution of Laplace's equation

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0
\]  

subject to the condition prescribed in Equation 13 for the wetted surface of the tank and to the condition prescribed in Equation 14 for the free surface.

In order to be more specific about these boundary conditions for the geometry of the tank involved, consider, first of all, the vertical wall of the tank \(r = a\). The unit vector normal to the wall of the tank is
\[ v = \cos \theta \hat{i} + \sin \theta \hat{j} = e_r. \] (16)

The components of the vectors \( u \) and \( w \) which appear in Equations 13 and 14 may be expressed as

\[ u = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \quad \text{and} \quad w = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}, \]

where \( u_i \) and \( w_i \) \((i = 1, 2, 3)\) are constants. Consequently, it is evident that

\[ -v \cdot \nabla \psi = -\frac{\partial \psi}{\partial r} \] (17)

and

\[ v \cdot (u + w \times r) = v \cdot u + v \cdot (w \times r) \]

\[ = u_1 \cos \theta + u_2 \sin \theta + \begin{vmatrix} \cos \theta & \sin \theta & 0 \\ w_1 & w_2 & w_3 \\ a \cos \theta & a \sin \theta & z \end{vmatrix} \]

\[ = u_1 \cos \theta + u_2 \sin \theta + z(w_2 \cos \theta - w_1 \sin \theta). \] (18)

Hence, in view of Equations 13, 17, and 18, on the wall of the tank \( r = a \), the boundary condition is

\[ \left( \frac{\partial \psi}{\partial r} \right)_{r=a} = -u_1 \cos \theta - u_2 \sin \theta + z(w_1 \sin \theta - w_2 \cos \theta). \] (19)

Secondly, on the bottom of the tank, where \( z = -h \) and \( v = (0,0,-1) \), it follows again from Equation 13 that

\[ -v \cdot \nabla \psi = \frac{\partial \psi}{\partial z} = -u_3 + \begin{vmatrix} 0 & 0 & -1 \\ w_1 & w_2 & w_3 \\ r \cos \theta & r \sin \theta & -h \end{vmatrix}, \]

i.e.,
Thirdly, consider Equation 14 and observe that \( \mathbf{n} = (0,0,1) \) is the unit normal vector to the quiescent free surface \( z = z_0 \). Thus, this condition becomes

\[
\psi_z z = z_0 = u_3 + \begin{bmatrix} 0 & 0 & 1 \\ w_1 & w_2 & w_3 \\ r \cos \theta & r \sin \theta & z_0 \end{bmatrix}.
\]

or

\[
\left( \frac{\partial^2}{\partial z^2} \right) \psi z = z_0 = u_3 + r(w_1 \sin \theta - w_2 \cos \theta). \tag{21}
\]

Finally, because of Equation 13, the condition

\[
\left( \frac{\partial \psi}{\partial z} \right) z = 0 = -u_3 + r(w_2 \cos \theta - w_1 \sin \theta), \tag{22}
\]

holds on the baffle, i.e., for \( \gamma a < r < a \), where \( \gamma \) has been taken to be \((0,0,\pm 1)\).

Furthermore, it seems to be reasonable that the \( z \)-component of the fluid particle velocity vector should be continuous in the region \( z = 0, 0 \leq r < \gamma a \) which corresponds to the opening in the baffle; and \( \psi \) itself should be continuous there. In other words, for \( z = 0 \) and \( 0 \leq r < \gamma a \), the conditions

\[
\frac{\partial \psi}{\partial z} (r, \theta, 0+) = \frac{\partial \psi}{\partial z} (r, \theta, 0-)
\]

\[
\psi (r, \theta, 0+) = \psi (r, \theta, 0-) \tag{23}
\]

must hold.

Therefore, the linearized sloshing problem for a
cylindrical tank containing an annular baffle is now completely specified in Equations 15 to 23, inclusive.

The following discussion treats only the case of pure translational oscillations \( \omega = 0 \). The details of the solution of the problem for the case of pure rotational oscillations \( \omega = 0 \) are essentially the same, and by the superposition principle the solutions for the two cases may be added to obtain the complete solution of the sloshing problem involving both translational and rotational oscillations of the tank. Therefore the problem is reduced to the solution of Laplace's equation

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0,
\]

subject to the prescribed conditions

\[
\left. \left( \frac{\partial \psi}{\partial r} \right) \right|_{r=a} = -u_1 \cos \theta - u_2 \sin \theta, \quad 0 \leq \theta \leq 2\pi, \quad -h \leq z < 0, \quad 0 < z < z_0 \]

\[
\left. \frac{\partial \psi}{\partial z} \right|_{z=0} = -u_3, \quad r < a, \quad 0 \leq \theta \leq 2\pi,
\]

\[
\left. \frac{\partial \psi}{\partial z} \right|_{z=-h} = -u_3, \quad 0 \leq r < a, \quad 0 \leq \theta \leq 2\pi,
\]

\[
\left. \left( \frac{\partial^2 \psi}{\partial a^2} - \frac{\partial \psi}{\partial z} \right) \right|_{z=0} = u_3, \quad 0 \leq r < a, \quad 0 \leq \theta \leq 2\pi,
\]

\[
\psi(r, \theta, 0-) = \psi(r, \theta, 0+), \quad 0 \leq r < \gamma a, \quad 0 \leq \theta \leq 2\pi,
\]

\[
\left. \frac{\partial \psi}{\partial z} \right|_{z=0} = \left. \frac{\partial \psi}{\partial z} \right|_{z=0}, \quad 0 \leq r < \gamma a, \quad 0 \leq \theta \leq 2\pi.
\]

From a mathematical point of view, it is necessary to treat the tank as though it were composed of two regions, one
above the baffle labeled I and the second below the baffle labeled II, which leads to two "potential functions" \( \psi_1(r, \theta, z) \) and \( \psi_2(r, \theta, z) \) for regions I and II, respectively, defined such that

\[
\psi(r, \theta, z) = \begin{cases} 
\psi_1(r, \theta, z) & \text{for } 0 \leq z \leq z_0, \\
\psi_2(r, \theta, z) & \text{for } -h \leq z \leq 0,
\end{cases}
\]

where \( \psi_1 \) satisfies Laplace's equation

\[
\frac{\partial^2 \psi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi_1}{\partial \theta^2} + \frac{\partial^2 \psi_1}{\partial z^2} = 0
\]

and the boundary

\[
\left( \frac{\partial \psi_1}{\partial r} \right)_{r=a} = -u_1 \cos \theta - u_2 \sin \theta
\]

\[
\left( \frac{\partial \psi_1}{\partial z} \right)_{z=0} = -u_3, \quad \forall a < r < a,
\]

\[
\left( \frac{\partial^2 \psi_1}{\partial \theta^2} - \frac{\partial \psi_1}{\partial z} \right)_{z=z_0} = u_3,
\]

and \( \psi_2 \) also satisfies Laplace's equation

\[
\frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi_2}{\partial \theta^2} + \frac{\partial^2 \psi_2}{\partial z^2} = 0
\]

plus the boundary conditions

\[
\left( \frac{\partial \psi_2}{\partial r} \right)_{r=a} = -u_1 \cos \theta - u_2 \sin \theta
\]

\[
\left( \frac{\partial \psi_2}{\partial z} \right)_{z=-h} = -u_3,
\]

\[
\left( \frac{\partial \psi_2}{\partial z} \right)_{z=0} = -u_3, \quad \forall a < r < a.
\]

In addition to these conditions, the functions \( \psi_1 \) and \( \psi_2 \) must also satisfy the "continuity conditions"
\[ \psi_1(r, \theta, 0) = \psi_2(r, \theta, 0) \quad \text{for } 0 \leq r < \gamma a. \]

\[ \frac{\partial \psi_1}{\partial z}(r, \theta, 0) = \frac{\partial \psi_2}{\partial z}(r, \theta, 0) \]

Since none of the boundary conditions embodied in Equations 24 and 25 is homogeneous, it is convenient to make a change of dependent variable such that certain transformed boundary conditions are homogeneous. This is accomplished by defining a function \( \tilde{\psi}_1(r, \theta, z) \), so that

\[ \tilde{\psi}_1(r, \theta, z) = \psi_1(r, \theta, z) + r(u_1 \cos \theta + u_2 \sin \theta) + u_3 z. \]  \( \tag{26} \)

A few simple calculations lead to the new boundary value problem

\[ \frac{\partial^2 \tilde{\psi}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{\psi}_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{\psi}_1}{\partial \theta^2} + \frac{\partial^2 \tilde{\psi}_1}{\partial z^2} = 0, \]  \( \tag{27} \)

\[ \left( -\frac{\partial \tilde{\psi}_1}{\partial r} \right)_{r=a} = 0, \]

\[ \left( -\frac{\partial \tilde{\psi}_1}{\partial z} \right)_{x=0} = 0, \quad \text{for } \gamma a < r < a \]  \( \tag{28} \)

\[ \left( \frac{\partial}{\partial z} \tilde{\psi}_1 - \frac{\partial^2 \tilde{\psi}_1}{\partial z^2} \right)_{z=z_0} = \frac{\beta^2}{\alpha} [u_3 z_0 + r(u_1 \cos \theta + u_2 \sin \theta)] \]  \( \tag{29} \)

In a similar fashion, defining for region II the function \( \tilde{\psi}_2(r, \theta, z) \) by

\[ \tilde{\psi}_2(r, \theta, z) = \psi_2(r, \theta, z) + r(u_1 \cos \theta + u_2 \sin \theta) + u_3 z, \]  \( \tag{30} \)

a new problem is obtained:

\[ \frac{\partial^2 \tilde{\psi}_2}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{\psi}_2}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{\psi}_2}{\partial \theta^2} + \frac{\partial^2 \tilde{\psi}_2}{\partial z^2} = 0, \]  \( \tag{31} \)
\[
\frac{\partial \phi_2}{\partial r} \bigg|_{r=a} = 0,
\]
\[
\frac{\partial \phi_2}{\partial z} \bigg|_{z=-h} = 0,
\]
\[
\frac{\partial \phi_2}{\partial z} \bigg|_{z=0} = 0, \quad \text{for } \gamma a < r < a.
\]

(32)

(33)

From the conditions of continuity, it follows that

\[
\phi_1(r,\theta,0) = \phi_2(r,\theta,0)
\]

for \(0 < r < \gamma a\).  \(34\)

Separating the variables in Equation 27 in the classical fashion, it is possible to show, after some labor, that the solution of Laplace's equation assumes the form

\[
\phi_1(r,\theta,z) = A_0 + \sum_{m=1}^{\infty} \left[ A_m \cosh \left( \frac{\xi_m z}{a} \right) + B_m \sinh \left( \frac{\xi_m z}{a} \right) \cos \theta J_1(\xi_m r/a) \right] + \sum_{m=1}^{\infty} \left[ C_m \cosh \left( \frac{\xi_m z}{a} \right) \sin \theta J_1(\xi_m r/a) \right] + \sum_{m=1}^{\infty} \left[ D_m \sinh \left( \frac{\xi_m z}{a} \right) \right]
\]

(35)

where \(A_0, A_m, B_m, C_m, D_m\) are constants, \(J_1(\xi_m r/a)\) is the Bessel function of the first kind of order one, and the \(\xi_m (m = 1,2,3\ldots)\) are the positive roots of the transcendental equation

\[
J_1(x) = 0.
\]

In view of the boundary condition in Equation 28, the function
\( \tilde{\varphi}_1 \) must satisfy the condition
\[
\left( \frac{\partial}{\partial z} \right)_{z=0} = \sum_{m=1}^{\infty} \frac{1}{a} \xi_m P_m \cos \theta J_1(\xi_m r/a) + \sum_{m=1}^{\infty} \frac{1}{a} \xi_m D_m \sin \theta J_1(\xi_m r/a) = 0, \tag{36}
\]
for \( \gamma a < r < a \). Multiply Equation 36 by \( \cos \theta \) and integrate with respect to \( \theta \) from 0 to \( 2\pi \) to obtain
\[
\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = 0, \text{ for } \gamma a < r < a, \tag{37}
\]
and a similar statement results for the term containing \( \sin \theta \), i.e.,
\[
\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = 0, \text{ for } \gamma a < r < a. \tag{38}
\]

From the free surface condition given by Equation 29, it follows that
\[
\left( \frac{\partial}{\partial z} \right) \tilde{\varphi}_1 = \frac{\partial^2}{\partial z^2} \tilde{\varphi}_1 = \frac{\partial^2}{\partial z^2} \tilde{\varphi}_1 = \sum_{m=1}^{\infty} \left\{ A_m \text{chsh}(m; z_o) + B_m \text{shch}(m; z_o) \right\} \left[ \cos \theta J_1(\xi_m r/a) \right] + \sum_{m=1}^{\infty} \left\{ C_m \text{chsh}(m; z_o) + D_m \text{shch}(m; z_o) \right\} \sin \theta J_1(\xi_m r/a) = \frac{\partial^2}{\partial z^2} \left[ u_2 z_o + r(u_1 \cos \theta + u_2 \sin \theta) \right], \tag{39}
\]
where
\[
\text{chsh} (m; z_o) = \frac{\beta^2}{a} \cosh (\xi_m z_o/a) - \frac{1}{a} \xi_m \sinh (\xi_m z_o/a)
\]
and
\[
\text{shch} (m; z_o) = \frac{\beta^2}{a} \sinh (\xi_m z_o/a) - \frac{1}{a} \xi_m \cosh (\xi_m z_o/a). \tag{40}
\]
Integrating Equation 39 with respect to \( \theta \) from 0 to \( 2\pi \), it is
easily found that

A_o = u_o^3 Z_o . \tag{41}

If Equation 39 is now multiplied by \cos \theta and integrated with respect to \theta from 0 to 2\pi, it becomes

\sum_{m=1}^{\infty} \left[ A_m \cosh(m; z_o) + B_m \sinh(m; z_o) \right] j_1(\xi_m r/a) = \frac{\beta^2}{\alpha} u_1 r; \tag{42}

and, in turn, if Equation 42 is multiplied by \rho j_1(\xi_k r/a) and integrated with respect to r from 0 to a, then

\left[ A_m \cosh(m; z_o) + B_m \sinh(m; z_o) \right] \int_0^a r^2 j_1^2(\xi_m r/a) dr

= \frac{\beta^2}{\alpha} u_1 \int_0^a r^2 j_1(\xi_m r/a) dr,

making use of the fact that the set \{j_1(\xi_m r/a)\} is a complete orthogonal set on the interval (0,a). Using the well-known integrals

\int_0^a r^2 j_1(\xi_m r/a) dr = \frac{\pi^2}{2} \frac{1}{\xi_m^2} j_1(\xi_m) \tag{43}

and

\int_0^a r^2 j_1(\xi_m r/a) dr = \frac{a^3}{\xi_m^2} j_1(\xi_m) \tag{44}

it follows immediately that

A_m \cosh(m; z_o) + B_m \sinh(m; z_o) = \frac{2a^2 \beta^2}{\alpha(\xi_m^2 - 1) j_1(\xi_m)} u_1. \tag{45}

Proceeding in an analogous fashion with the term in Equation 39 containing \sin \theta, a similar relation is obtained:

C_m \cosh(m; z_o) + D_m \sinh(m; z_o) = \frac{2a^2 \beta^2}{\alpha(\xi_m^2 - 1) j_1(\xi_m)} u_2, \tag{46}
where again Equations 43 and 44 have been employed. Inserting Equations 41, 45, and 46 into Equation 35 and rearranging terms, the expression for the function \( \tilde{\varphi}_1(r,\theta,z) \) now assumes the form

\[
\tilde{\varphi}_1(r,\theta,z) = u_3z_0 + \frac{2a\beta^2}{a} (u_1 \cos \theta + u_2 \sin \theta) \sum_{m=1}^{\infty} \left( \begin{array}{c}
\cosh(\xi_m z/a)J_1(\xi_m r/a) \\
J_1(\xi_m) (\xi_m^2 - 1) \text{chsh}(m; z_0) - \sum_{m=1}^{\infty} \frac{(B_m \cos \theta + D_m \sin \theta)}{\text{chsh}(m; z_0)}
\end{array} \right)
\]

(47)

At this point, no specific statements can yet be made regarding the coefficients \( B_m \) and \( D_m \) which appear in Equation 47, despite the fact that these coefficients satisfy Equations 37 and 38, respectively. It is necessary now to investigate the function \( \tilde{\varphi}_2(r,\theta,z) \) as well as the continuity conditions given in Equation 34 in order to determine the coefficients \( B_m \) and \( D_m \).

In analogy to Equation 35, the solution of Equation 31 may be expressed in the form

\[
\tilde{\varphi}_2(r,\theta,z) = E_0 + \sum_{m=1}^{\infty} \left[ F_m \cosh(\xi_m z/a) + G_m \sinh(\xi_m z/a) \right] \cos \theta J_1(\xi_m r/a)
\]

\[
+ \sum_{m=1}^{\infty} \left[ H_m \cosh(\xi_m z/a) + K_m \sinh(\xi_m z/a) \right] \sin \theta J_1(\xi_m r/a)
\]

(48)

According to Equation 32, it follows that
\[
\sum_{m=1}^{\infty} \xi_m \left\{ -F_m \sinh(\xi_m h/a) + G_m \cosh(\xi_m h/a) \right\} \cos \theta J_1(\xi_m r/a)
\]

\[
+ \sum_{m=1}^{\infty} \xi_m \left\{ -H_m \sinh(\xi_m h/a) + K_m \cosh(\xi_m h/a) \right\} \sin \theta J_1(\xi_m r/a) = 0,
\]

which leads immediately to

\[ F_m \sinh(\xi_m h/a) = G_m \cosh(\xi_m h/a) \]

\[ H_m \sinh(\xi_m h/a) = K_m \cosh(\xi_m h/a) \]

because of the orthogonality property of the trigonometric and Bessel functions. Consequently Equation 48 may be put in the somewhat more compact form

\[
\varphi_2(r, \theta, z) = E_0 + \sum_{m=1}^{\infty} \frac{(G_m \cos \theta + K_m \sin \theta) \cosh(\xi_m h/a) \, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)} \]

For \( z = 0 \) and \( \rho_0 < r < a \), Equation 33 comes into play and yields

\[
\left( \frac{\partial \varphi_2}{\partial z} \right)_{z=0} = \sum_{m=1}^{\infty} \frac{a}{a} \xi_m (G_m \cos \theta + K_m \sin \theta) J_1(\xi_m r/a) = 0,
\]

and therefore

\[
\sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a) = 0, \quad (49)
\]

for \( \rho_0 < r < a \). The so-called continuity conditions given in Equation 34 may now be applied. Setting \( \varphi_1 = \varphi_2 \) at \( z = 0 \) for \( 0 < r < \rho_0 \), it follows that

\[ u_3 z_0 + \frac{2a^2}{a} \left( u_1 \cos \theta + u_2 \sin \theta \right) \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2-1) \cosh(\xi_m z_0)} \]
\[
- \sum_{m=1}^{\infty} \frac{B_m \cos \theta + D_m \sin \theta}{\operatorname{chsh}(m; z_0)} \operatorname{shch}(m; z_0) J_1(\xi_m r/a) = \\
\sum_{m=1}^{\infty} \frac{G_m \cos \theta + K_m \sin \theta}{\sinh (5_m h/a)} \cosh (5_m h/a) J_1(\xi_m r/a).
\]

The orthogonality properties of the trigonometric functions may next be exploited to obtain the result

\[
E_0 = u_3 z_0
\]
as well as

\[
\sum_{m=1}^{\infty} \frac{B_m \operatorname{chsh}(m; z_0)}{\operatorname{chsh}(m; z_0)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} G_m \coth(\xi_m h/a) J_1(\xi_m r/a) = \frac{2a^2}{\alpha} u_1 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(5^2_m - 1) \operatorname{chsh}(m; z_0)},
\]

(50)

\[
\sum_{m=1}^{\infty} \frac{D_m \operatorname{chsh}(m; z_0)}{\operatorname{chsh}(m; z_0)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} K_m \coth(\xi_m h/a) J_1(\xi_m r/a) = \frac{2a^2}{\alpha} u_2 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(5^2_m - 1) \operatorname{chsh}(m; z_0)},
\]

(51)

which are valid for \(0 < r < \gamma a\). Matching the derivatives according to Equation 34 and again observing the orthogonality of the trigonometric functions on the interval \((0, 2\pi)\), it can be shown that

\[
\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a), \quad 0 \leq r < \gamma a,
\]

and

\[
\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a), \quad 0 \leq r < \gamma a.
\]
In summary, then, the pertinent results are
\[
\sum_{m=1}^{\infty} \frac{\mathrm{shch}(m;z_0)}{\mathrm{chsh}(m;z_0)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} \xi_m \coth(\xi_m h/a) J_1(\xi_m r/a) = \frac{2a^2}{\alpha} \left( \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2-1) \mathrm{chsh}(m;z_0)} \right), \quad 0 \leq r < \gamma a \quad (52)
\]

\[
\sum_{m=1}^{\infty} \frac{\mathrm{shch}(m;z_0)}{\mathrm{chsh}(m;z_0)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} \xi_m \coth(\xi_m h/a) J_1(\xi_m r/a) = \frac{2a^2}{\alpha} \left( \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2-1) \mathrm{chsh}(m;z_0)} \right), \quad 0 \leq r < \gamma a \quad (53)
\]

\[
\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a), \quad 0 \leq r < \gamma a, \quad (54)
\]

\[
\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a), \quad 0 \leq r < \gamma a, \quad (55)
\]

\[
\sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a, \quad (37)
\]

\[
\sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a, \quad (38)
\]

\[
\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a, \quad (39)
\]

Suppose now that the infinite series in Equation 54 converge to a function \( F(r) \), i.e.,
But, according to Equations 37, 49, and 55, it is observed that

\[\sum_{m=1}^{\infty} B_m J_m(\frac{r}{a}) = F(r), \quad 0 \leq r < \gamma a,\]

and

\[\sum_{m=1}^{\infty} G_m J_m(\frac{r}{a}) = F(r), \quad 0 \leq r < \gamma a,\]

From Equations 56 and 57, it is apparent that \(G_m = B_m\) since the two infinite series converge to the same function in the fundamental interval \((0, a)\). In an analogous fashion, it can also be shown that \(D_m = K_m\) \((m = 1, 2, 3, \ldots)\). Because of these last two statements Equations 50, 51, 37, and 38 can be collected in the forms

\[\sum_{m=1}^{\infty} B_m \text{chsh}(m; z) J_1(\frac{m r}{a}) = \frac{2a \Delta_{2}}{\alpha} \sum_{n=1}^{\infty} \frac{J_1(\frac{n r}{a})}{J_1(\frac{m}{a})(\frac{n^2-1}{m^2-1})\text{chsh}(m; z)}\]

for \(0 \leq r < \gamma a,\)

\[\sum_{m=1}^{\infty} B_m J_1(\frac{m r}{a}) = 0,\]

for \(\gamma a < r < a,\)

and

\[\sum_{m=1}^{\infty} D_m \text{chsh}(m; z) J_1(\frac{m r}{a}) = \frac{2a \Delta_{2}}{\alpha} \sum_{n=1}^{\infty} \frac{J_1(\frac{n r}{a})}{J_1(\frac{m}{a})(\frac{n^2-1}{m^2-1})\text{chsh}(m; z)}\]

for \(0 \leq r < \gamma a,\)
for \( \gamma a < r < a \). Therefore, the coefficients \( B_m \) and \( D_m \) must be determined from these last four equations, which reduce to the single pair

\[
\sum_{m=1}^{\infty} B_m \text{chsh}(m; z) J_1(\xi_m x) = \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2 - 1) \text{chsh}(m; z)}
\]

for \( 0 < x < \gamma \),

\[
\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m x) = 0, \quad \gamma < x < 1,
\]

upon setting \( x = r/a \) and defining \( B_m = 2a^2 u_1 B^*/a \) and \( D_m = 2a^2 u_2 B^*/a \). Equations 58 and 59 may be called a dual series pair, and, since the \( \xi_m \) are the zeros of \( J_1(x) = 0 \), the pair may be termed a dual Dini series. The solution of the dual series for the coefficients \( B^*_m \) presents a real problem in itself and will be considered in some detail in Chapter V.

Thus, once the values of \( B^*_m \) (\( m = 1, 2, 3, \ldots \)) have been determined, all the required coefficients in the expressions for the functions \( \tilde{\varphi}_1(r, \theta, z) \) and \( \tilde{\varphi}_2(r, \theta, z) \) will be known.
IV. FREE SURFACE DISPLACEMENTS, PRESSURE, FORCES, AND MOMENTS

Supposing for the moment that the dual Dini series embodied in Equations 58 and 59 have been solved for the coefficients $B^*_m$, the functions $\tilde{\varphi}_1(r,\theta,z)$ and $\tilde{\varphi}_2(r,\theta,z)$ may now be expressed in the form

$$\tilde{\varphi}_1(r,\theta,z) = u_3z_o + \frac{2aS}{a}(u_1 \cos \theta + u_2 \sin \theta)$$

$$\sum_{m=1}^{\infty} \frac{\cosh(\xi_mz/a)}{J_1(\xi_m)(\xi_m^2-1)\cosh(m;z_o)} - \sum_{m=1}^{\infty} \frac{B^*_m \sinh(m;z_o-z)J_1(\xi_mz/a)}{\cosh(m;z_o)},$$

and

$$\tilde{\varphi}_2(r,\theta,z) = u_3z_o + \frac{2aS}{a}(u_1 \cos \theta + u_2 \sin \theta)$$

$$\sum_{m=1}^{\infty} \frac{B^*_m \cosh(\xi_mh+z)/aJ_1(\xi_mz/a)}{\sinh(\xi_mh/a)},$$

and, by Equations 26 and 30, the functions $\psi_1(r,\theta,z)$ and $\psi_2(r,\theta,z)$ are defined by

$$\psi_1(r,\theta,z) = \tilde{\varphi}_1(r,\theta,z) - r(u_1 \cos \theta + u_2 \sin \theta) - u_3z,$$  \hspace{1cm} (60)

for $i = 1, 2$. Thus

$$\psi_1(r,\theta,z) = u_3(z_o-z) + \frac{2aS}{a}(u_1 \cos \theta + u_2 \sin \theta)$$

$$\sum_{m=1}^{\infty} \frac{\cosh(\xi_mz/a)J_1(\xi_mz/a)}{J_1(\xi_m)(\xi_m^2-1)\cosh(m;z_o)} - \sum_{m=1}^{\infty} \frac{B^*_m \sinh(m;z_o-z)J_1(\xi_mz/a)}{\cosh(m;z_o)}$$

$$- \frac{ar}{2aS^2}.$$  \hspace{1cm} (61)
\[ \psi_2(r, \theta, z) = u_3(z_0 - z) + \frac{2a^2}{\alpha} (u_1 \cos \theta + u_2 \sin \theta) \]

\[ \sum_{m=1}^{\infty} \frac{B^* \cosh \xi_m(h+z)/aJ_1(\xi_m r/a)}{\sinh (\xi_m h/a)} - \frac{\alpha r}{2a^2} \]  

Next define

\[ \varphi_1(r, \theta, z, t), \quad 0 < z < z_0 \]

\[ \varphi(r, \theta, z, t) = \varphi_2(r, \theta, z, t), \quad -h < z < 0, \]

so that \( \varphi_j(r, \theta, z, t) = e^{i\beta t} \psi_j(r, \theta, z), \quad (j = 1, 2 \text{ and } i = \sqrt{-1}). \)

Then from Equation 12 the (assumed small) free surface displacement of the fluid in the baffled tank may be computed from

\[ \eta(r, \theta, t) = \frac{1}{a} \left( \frac{\partial \varphi_1}{\partial t} \right)_{z=z_0} = \frac{i\beta}{\alpha} e^{i\beta t} \psi_1(r, \theta, z_0) \]

\[ = \frac{2ia^{3} (u_1 \cos \theta + u_2 \sin \theta)}{\alpha^2} \sum_{m=1}^{\infty} \frac{\cosh(\xi_m z_0/a) J_1(\xi_m r/a)}{J_1(\xi_m) (\xi_m^2 - 1) \cosh(\xi_m z_0)} \]

\[ + \frac{1}{a} \sum_{m=1}^{\infty} \frac{B^* \xi_m J_1(\xi_m r/a)}{\cosh(\xi_m z_0)} - \frac{\alpha r}{2a^2} \]

where \( \psi_1 \) as given in Equation 61 has been utilized.

The linearized form of Equation 11 is

\[ p(r, \theta, z, t) = p_0 + \rho (az - \frac{\partial \varphi}{\partial t}), \quad (63) \]

and hence the expression for the pressure may more conveniently be written in the form

\[ p(r, \theta, z, t) = p_0 + \rho [az - i\beta e^{i\beta t} \psi_j(r, \theta, z)], \quad (j = 1, 2), \quad (64) \]

in virtue of Equations 60 and 63. Thus, for the region above
the baffle, the appropriate expression for the pressure in
the fluid is

\[ p_1(r, \theta, z, t) = p_0 + \rho \left[ \alpha z - i \beta e^{i \beta t} u_3(z-z_0) \right] \]

\[ - \frac{2 \alpha a \beta^3}{\alpha} e^{i \beta t} \left( u_1 \cos \theta + u_2 \sin \theta \right) \left\{ \sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a) J_1(\xi_m r/a)}{J_1(\xi_m) (\xi_m^2 - 1) \text{chsh}(m; z_0)} \right\} \]

\[ - \frac{\alpha}{2 \beta^2} . \] (65)

For \( r = a \), the pressure on the wall of the tank above the
baffle is

\[ p_1(a, \theta, z, t) = p_0 + \rho \left[ \alpha z - i \beta e^{i \beta t} u_3(z-z_0) \right] - \frac{2 \alpha a \beta^3}{\alpha} e^{i \beta t} \]

\[ \left( u_1 \cos \theta + u_2 \sin \theta \right) \left\{ \sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)}{\xi_m^2 - 1 \text{chsh}(m; z_0)} \right\} - \frac{\alpha}{2 \beta^2} . \] (65)

From Equations 62 and 64, the expression for the pressure
in the fluid below the baffle is found to be

\[ p_2(r, \theta, z, t) = p_0 + \rho \left[ \alpha z - i \beta e^{i \beta t} u_3(z-z_0) \right] \]

\[ - \frac{2 \alpha a \beta^3}{\alpha} \left( u_1 \cos \theta + \sin \theta \right) \left\{ \sum_{m=1}^{\infty} \frac{\text{B*shch}(m; z_0 - z) J_1(\xi_m r/a)}{\text{chsh}(m; z_0) (\xi_m^2 - 1) \text{chsh}(m; z_0)} \right\} \]

\[ - \frac{\alpha}{2 \beta^2} ; \] (66)

and consequently, on the wall \( r = a \), the pressure distribution
is
p_2(a, \theta, z, t) = p_0 + \rho [az - i\beta e^{i\omega t} u_3(z-z_0)]

\frac{2ia\beta^3}{\alpha} e^{i\omega t} (u_1 \cos \theta + u_2 \sin \theta) \left\{ \sum_{m=1}^{\infty} \frac{B_m \cosh \frac{m(z+h)}{a} J_m(\xi_m)}{\sinh(\xi_m/2a)} - \frac{\alpha r}{2a\beta^2} \right\}. \quad (67)

Setting z = -h in Equation 66, the pressure on the bottom of the tank is

p(r, \theta, -h, t) = p_0 + \rho [-ch + i\beta e^{i\omega t} u_3(h + z_0)]

\frac{2ia\beta^3}{\alpha} e^{i\omega t} (u_1 \cos \theta + u_2 \sin \theta) \left\{ \sum_{m=1}^{\infty} \frac{B_m J_m(\xi_m r/a)}{\sinh(\xi_m/2a)} - \frac{\alpha r}{2a\beta^2} \right\}. \quad (68)

The net force \( F \) acting on an area \( S \) can be computed from the surface integral

\[ F = - \int_S (p - p_0) \mathbf{n} \, dS, \quad (69) \]

where \( p - p_0 \) denotes the net pressure at a point and \( \mathbf{n} \) is the unit exterior normal to the surface \( S \).

The unit normal to the vertical wall \( (r = a) \) of the tank is

\[ \mathbf{n} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}, \quad (70) \]

so that the force acting on the wall above the baffle can be found from Equation 69 upon replacement of \( p - p_0 \) from Equation 65; this leads to

\[ F = \frac{2ia\beta^3}{\alpha} e^{i\omega t} \int_0^{2\pi} (u_1 \cos \theta + u_2 \sin \theta)(\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \, d\theta \]
\[ \int_{z_0}^{0} \sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)}{(\xi_m^2 - 1) \text{chsh}(m; z_o)} = - \sum_{m=1}^{\infty} \frac{B_m \text{shch}(m; z_0 - z) J_1(\xi_m)}{\text{chsh}(m; z_0)} - \frac{a}{2\beta^2} \text{d}z, \]

where \( dS = ad\theta dz \), \( 0 \leq \theta \leq 2\pi \), and \( 0 \leq z \leq z_o \) have been used.

If \( F = F_1 + F_2 \) and \( u = u_1 + u_2 \), then

\[ F_j = \frac{2na^2 \beta^3}{a} u_j e^{i\beta t} \sum_{m=1}^{\infty} \frac{\sinh(\xi_m z_0/a)}{\xi_m (\xi_m^2 - 1) \text{chsh}(m; z_0)} \]

\[ = \frac{B_m [a^2 \xi_m^2 (\cosh(\xi_m z_0/a) - 1) - \sinh(\xi_m z_0/a)] J_1(\xi_m)}{\text{chsh}(m; z_0)} - \frac{a}{2\beta^2}, \]

where in view of Equation 40

\[ \int_{z_0}^{0} \text{shch}(m; z_0 - z) dz = \frac{a^2 \xi_m^2}{a^2} [\cosh(\xi_m z_0/a) - 1] - \sinh(\xi_m z_0/a) \]

In the region \(-h \leq z < 0\) on the wall \( r = a\), the net force is found to be

\[ F_j = \frac{2na^2 \beta^3}{a} u_j e^{i\beta t} \sum_{m=1}^{\infty} \frac{B_m \xi_m J_1(\xi_m)}{\xi_m^2} - \frac{a}{2\beta^2}, \] \( j = 1, 2, \)

using Equations 69, 70, and 67.

Finally, on the bottom of the tank, the unit exterior normal is \(-k\) and the pressure difference is given by Equation 68, so that the net force acting on that portion of the tank is, since \( dS = rdrd\theta \),

\[ F = \rho k \int_{0}^{2\pi} d\theta \int_{0}^{a} [-ah + i\beta e^{i\beta t} u_3(h + z_0)] rdr \]

\[ = \rho a^2 [i\beta e^{i\beta t} u_3(h + z_0) - ah]k. \]
Letting \( n \) denote the unit normal vector to a surface \( S \), \( r \) the position vector of a point, and \( p - p_o \) the pressure difference at a point on the surface, the moment, \( M \) of force taken with respect to the origin of coordinates may be computed from the surface integral

\[
M = - \int_S (p - p_o) \, r \times n \, dS. \quad (71)
\]

For the portion of the wall of the tank above the baffle, the appropriate pressure distribution is given by Equation 65 and \( n \) by Equation 70, so that

\[
M = \frac{2ia^2 \rho n^3}{\alpha} e^{i\theta} (u_{21} - u_{11}) \left\{ \sum_{m=1}^{\infty} \frac{z_0}{\chi(m; z_0)} \right. \\
- \sum_{m=1}^{\infty} \frac{B^* J_1(\xi_m)}{\chi(m; z_0)} \right\}
\]

\[
\int_0^{z_0} z \cosh(\xi_m z/a) \, dz = \frac{a^2}{\xi_m} \left[ 1 - \cosh(\xi_m z_o/a) \right]
\]

\[
\int_0^{z_o} z \sinh(\xi_m z_0/a) \, dz = \frac{a^2}{\xi_m} \left[ \sinh(\xi_m z_o/a) - \frac{a^2}{\xi_m} \cosh(\xi_m z_o/a) \right]
\]

\[
\int_0^{z_o} z \cosh(\xi_m z/a) \, dz = \frac{a^2}{\xi_m} \left[ \sinh(\xi_m z_o/a) - \frac{a^2}{\xi_m} \cosh(\xi_m z_o/a) \right]
\]
have been used. Using Equation 67, \( r \times n = -z(\sin \theta - \cos \theta) \), and Equation 71, it is found that

\[
M = \frac{2\pi m a^2 \rho b^3}{\alpha} e^{i \beta t} (u_2 - u_1) \left\{ \sum_{m=1}^{\infty} \frac{\rho* J_1(\xi_m)}{\xi_m^2 \sinh(\xi_m h/a)} \right\} - \frac{\alpha^2 h^2}{4b^2},
\]

gives the moment of force acting on the portion of the wall of the tank below the baffle. Lastly, the moment of force on the bottom of the tank is evaluated from the integral

\[
M = \int_0^{2\pi} d\theta \int_0^a (p - p_o) \mid_{z=-h} r(\sin \theta - \cos \theta) \, rdr
\]

which, in view of Equation 68, leads to the expression

\[
M = -\frac{2\pi m a^2 \rho b^3}{\alpha} e^{i \beta t} (u_2 - u_1) \left\{ \sum_{m=1}^{\infty} \frac{\rho* J_1(\xi_m)}{\xi_m^2 \sinh(\xi_m h/a)} \right\} - \frac{\alpha^2 h^2}{8b^2}.
\]
The solution of the dual Dini Series

\[ \sum_{m=1}^{\infty} \frac{B_m \text{chsh}(m; z_0 + h) J_1(\xi_m x)}{\sinh(\xi_m h/a) \text{chsh}(m; z_0)} = \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m) (\xi_m^2 - 1) \text{chsh}(m; z_0)}, \]  

(58)

\[ 0 < x < Y \]

\[ \sum_{m=1}^{\infty} B_m^* J_1(\xi_m x) = 0, \quad Y < x < 1, \]  

(59)

may be expressed in a variety of ways, and in the following pages several methods of determining the coefficients \(B_m^*\) will be given.

Being somewhat more general than above, consider the dual series

\[ \sum_{m=1}^{\infty} \xi_m G(m) a_m J_v(\xi_m x) = F(x), \quad 0 < x < Y \]  

(72)

\[ \sum_{m=1}^{\infty} a_m J_v(\xi_m x) = 0, \quad Y < x < 1, \]  

(73)

where \(J_v(\xi_m x)\) is the Bessel function of the first kind of order \(v\), \(\xi_m (m = 1, 2, 3, \ldots)\) are the positive roots of \(J_v(x) = 0\), \(F(x)\) is a known function, \(-1 \leq p \leq 1\), \(G(m)\) is known, and the coefficients \(a_m\) are to be determined. The technique of determining the \(a_m\) is based on a method due to Tranter and Cooke (8).

The following theorem, which is analogous to a theorem stated in (8), is necessary for the ensuing discussions:
Theorem. If \( n \) is zero or a positive integer, \( v > -1 \), \( k > 0 \), and \( \xi_m \) are the positive roots of the transcendental equation \( J^v(x) = 0 \), then

\[
\sum_{m=1}^{\infty} \frac{J_{v+2n+k}(\gamma \xi_m)}{\xi_m^k - 2} \frac{J_v(\xi_m x)}{\xi_m^{v^2 - (\xi_m^2)^2}} = 0 < x < Y, \ 0 < Y < 1
\]

\[
\frac{\Gamma(v+n+1)}{2^{k-1} \Gamma(v+1) \Gamma(n+k)} x^v (1 - x^2 / \gamma^2)^{k-1} J_n(k+v+1, x^2 / \gamma^2),
\]

\( 0 \leq x < \gamma, \ 0, \ Y < x < 1 \)

\[
\frac{1}{2} \int_0^{\infty} r^{1-k} J_{v+2n+k}(\gamma r) J_v(xr) \, dr,
\]

where

\[
J_n(k+v+1, x^2 / \gamma^2) = \frac{1}{2} F_1(-n, k+v+n, v+1, x^2 / \gamma^2)
\]

is Jacobi's polynomial, as given by Magnus and Oberhettinger (9).

Proof. Consider the Weber-Schafheitlin integral

\[
\int_0^{\infty} r^{1-k} J_{v+2n+k}(\gamma r) J_v(xr) \, dr =
\]

\[
\frac{\Gamma(v+n+1)}{2^{k-1} \Gamma(v+1) \Gamma(n+k)} x^v (1 - x^2 / \gamma^2)^{k-1} J_n(k+v+1, x^2 / \gamma^2),
\]

\( 0 \leq x < \gamma, \ 0, \ Y < x \)

which is given in Watson (10), and apply the Hankel inversion formula to obtain
\[ r^{-k} J_{v+2n+k}(y r) \]

\[ = \frac{\Gamma(v+n+1)}{2^{k-1} \gamma^{v-k+2} \Gamma(v+1) \Gamma(n+k)} \int_0^y x^{v+1} (1-x^2/\gamma^2)^{k-1} J_{n}(k+v, v+1, x^2/\gamma^2) \]

\[ J_{v}(y r) dx. \]  

(72)

Now the Dini expansion of the function \( f(x) \) defined by

\[ f(x) = \begin{cases} 
0 & 0 \leq x < \gamma \\
\gamma & \gamma < x < 1,
\end{cases} \]

is

\[ f(x) = \sum_{m=1}^{\infty} A_m J_v(\xi_m x), \]  

(75)

the coefficients \( A_m \) being determined from

\[ A_m = \frac{2 \xi_m^2}{(\xi_m^2 - \gamma^2) J_v(\xi_m)} \int_0^1 x J_v(\xi_m x) f(x) dx. \]  

(76)

Therefore, using Equation 76, the \( A_m \) are found to be

\[ A_m = \frac{2 \xi_m^2}{(\xi_m^2 - \gamma^2) J_v(\xi_m)} \frac{\Gamma(v+n+1)}{2^{k-1} \gamma^{v-k+2} \Gamma(v+1) \Gamma(n+k)} \]

\[ \int_0^y x^{v+1} (1-x^2/\gamma^2)^{k-1} J_{n}(k+v, v+1, x^2/\gamma^2) J_v(\xi_m x) dx \]

\[ = 2^{k-1} \xi_m^{k-2} \frac{J_{v+2n+k}(\gamma \xi_m)}{(\xi_m^2 - \gamma^2) J_v(\xi_m)} \]  

(77)
and hence, upon substitution of the result of Equation 77 into Equation 75, it is evident that

\[
\sum_{m=1}^{\infty} \frac{J_v(\xi_m^2 x) J_v(\xi_m^2 + 2n + k)}{\xi_m^2 \xi_m^2 - (\xi_m^2 - \gamma^2) J_v(\xi_m^2)} = \frac{\Gamma(v + 1)}{2^k \Gamma(v - k + 1)} \tau^v \left(1 - \frac{x^2}{\gamma^2}\right)^{k-1} J_n(k,v+1,x^2/\gamma^2),
\]

for \(0 < \tau < \gamma\), 0, \(\gamma < \tau < 1\),

which is the desired result.

With the results of the above theorem now available, it is observed that

\[
\sum_{m=1}^{\infty} \frac{J_v(\xi_m^2 x) J_v(\xi_m^2 + 2n + k)}{\xi_m^2 \xi_m^2 - (\xi_m^2 - \gamma^2) J_v(\xi_m^2)} = 0.
\]

(78)

for \(\gamma < \tau < 1\), where \(k\) has been replaced by \(1 + \frac{1}{2}p\). Define

\[
a_n = \frac{1}{\xi_m^2 \xi_m^2 - (\xi_m^2 - \gamma^2) J_v(\xi_m^2)} \sum_{n=0}^{\infty} b_n J_v(2n + p + 1)(\xi_m^2) - J_v(\xi_m^2)
\]

(79)

so that at Equation 73, i.e., the second equation of the pair forming the dual series, is satisfied identically. Thus

\[
\sum_{m=1}^{\infty} a_m J_v(\xi_m^2 x) = \sum_{m=1}^{\infty} a_m J_v(\xi_m^2) \sum_{n=0}^{\infty} b_n J_v(2n + p + 1)(\xi_m^2)
\]

\[
= \sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} a_m J_v(\xi_m^2) J_v(\xi_m^2) = 0,
\]

for \(\gamma < \tau < 1\), by Equation 78, assuming that the order of the
summations may be interchanged.

Next substitute Equation 79 into Equation 72, the first equation in the dual series, to obtain
\[
\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{G(m) J_{v+2n+\frac{1}{2}p+1}(\gamma x_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)} = F(x),
\] (80)
for 0 \leq x < \gamma, after interchanging the order of summation.

But Equation 74 can be put in the form
\[
\int_0^{\infty} \rho^v (1-x^2/\gamma^2)^{\frac{3}{2}p} J_k^2(1+\frac{1}{2}p+v, v+1, x^2/\gamma^2) J_v(\xi_m x) dx.
\]
Then if Equation 80 is multiplied by
\[
x^{v+1}(1-x^2/\gamma^2)^{\frac{3}{2}p} J_k(1+\frac{1}{2}p+v, v+1, x^2/\gamma^2)
\]
and integrated with respect to x from 0 to \gamma, the result is
\[
\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{G(m) J_{v+2n+\frac{1}{2}p+1}(\gamma x_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)}
\]
= \frac{\Gamma(v+k+1)}{2^{3/2} \pi v^{-\frac{3}{2}p+1} \Gamma(v+1) \Gamma(k+1+\frac{1}{2}p)}
\int_0^{\gamma} x^{v+1}(1-x^2/\gamma^2)^{\frac{3}{2}p} J_k(1+\frac{1}{2}p+v, v+1, x^2/\gamma^2) F(x) dx.
\] (81)

For the sake of notational brevity, define
\[
S(k,n;v,p) = \sum_{m=1}^{\infty} \frac{G(m) J_{v+2k+\frac{1}{2}p+1}(\gamma x_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)}.
\] (82)
and

\[
E(k;v,p) = \frac{\Gamma(v+k+1)}{2^{3p} \Gamma\left\{\frac{v}{2}+p+1\right\} \Gamma(v+1) \Gamma(k+\frac{1}{2}p+1)}.
\]

\[
\gamma \int_0^\infty x^{v+1}(1-x^2/\gamma^2)^{3p/2} J_k(1+\frac{1}{2}p+v,v+1,x^2/\gamma^2) F(x) dx,
\]

so that the form of Equation 81 becomes

\[
\sum_{n=0}^\infty b_n S(k,n;v,p) = E(k;v,p),
\]

from which the coefficients \(b_n\) may be calculated by considering the result as an infinite system of linear equations in \(b_n\), for each choice of \(v\) and \(p\).

For the dual series given by Equations 58 and 59, it is convenient to define

\[
C_m^* = \xi_m B_m^*.
\]

so that the dual series

\[
\sum_{m=1}^\xi \frac{C_m^* \text{chsh}(m;Z_0+h)J_1(\xi_m x)}{\sinh(\xi_m h/a) \text{chsh}(m;Z_0)} = \sum_{m=1}^\xi \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1) \text{chsh}(m;Z_0)},
\]

\(0 \leq x < \gamma\),

\[
\sum_{m=1}^\xi C_m^* J_1(\xi_m x) = 0, \quad \gamma < x < 1,
\]

is of the form expressed in Equations 72 and 73, where

\(p = -1, v = 1,\)

\[
G(m) = \frac{\text{chsh}(m;Z_0+h)}{\sinh(\xi_m h/a) \text{chsh}(m;Z_0)},
\]

(85)
and $C^*_m$ has replaced $a^*_m$. Therefore Equation 79 is now written

$$
C^*_m = \frac{\xi_m^{3/2}}{(\xi_m^2 - 1)J_1^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{2n+3/2}(Y_m^\xi),
$$

so that, in view of Equation 84,

$$
B^*_m = \frac{\xi_m^{1/2}}{(\xi_m^2 - 1)J_1^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{2n+3/2}(Y_m^\xi).
$$

According to Equation 82, it follows that

$$
S(k,n;1,-1) = S(k,n) = \sum_{m=1}^{\infty} \frac{\cosh(mz_0+h) J_{2n+3/2}(Y_m^{\xi})}{\sinh(\xi_m h/a) J_1^2(\xi_m)(\xi_m^2 - 1) \cosh(mz_0)},
$$

and from Equation 83

$$
E(k;1,-1) = E(k) = \sum_{m=1}^{\infty} \frac{1}{J_1(\xi_m)(\xi_m^2 - 1) \cosh(mz_0)} \left[ \int_0^\infty x^{2(1-x^2/\gamma^2)^{-\frac{1}{2}}} J_{k,1/2}(x^2/\gamma^2) J_1(\xi_m x) dx \right]
$$

$$
= \sum_{m=1}^{\infty} \frac{J_{2k+3/2}(Y_m^{\xi})}{J_1(\xi_m) \xi_m^{2/3}(\xi_m^2 - 1) \cosh(mz_0)},
$$

by Equation 74.

To summarize, then, it is desired to obtain the coefficients $b_n$ from the following equation

$$
\sum_{n=0}^{\infty} b_n S(k,n) = E(k),
$$

where $S(k,n)$ and $E(k)$ are given by Equations 87 and 88, respectively. Then once the values of $b_n (n = 0,1,2,\ldots)$ are
known, the values of the coefficients $B_m^* (m=1,2,3,...)$ can be computed from Equation 86.

Making use of a certain contour integral in the complex plane, Tranter and Cooke (8) were able to sum the infinite series which corresponds to $S(k,n,1,-1)$ in this paper, and this sum was expressible in the form of an improper integral of the first kind involving the modified Bessel functions of both the first and second kinds. Because the form of $S(k,n,1,-1)$, as it appears here, is much more complicated than the corresponding expression appearing in (8), the contour integral approach yielded for the sum of $S(k,n,1,-1)$ an improper integral, as expected, and, in addition, another infinite series. In other words, it was found that the infinite series in Equation 85 could be replaced by another infinite series plus an improper integral.

At this point it appeared futile to pursue the Tranter-Cooke (8) method further, and it was decided to attempt to determine the $b_n$'s numerically using the IBM 7074 computer, given values of the various parameters appearing in the problem.

Now Equation 89 represents an infinite system of linear algebraic equations in the $b_n$'s, and a set of numerical values of the quantities $b_0, b_1, b_2, ...$ is called a solution of the system if on substituting these values in the left member of Equation 89 the infinite series converges and all the equations
are satisfied for \( k = 0, 1, 2, \ldots \). According to Kantorovich and Krylov (11), approximate solutions of Equation 87 may, under certain conditions, be obtained by terminating the infinite series at, say, \( n = N \) and by then assigning to \( k \) the values, 0, 1, 2, \ldots \( N \), in such a way that an \( N+1 \) by \( N+1 \) system of linear algebraic equations is obtained, i.e.,

\[
\begin{align*}
S(0,0)b_0 + S(0,1)b_1 + \ldots + S(0,N)b_N &= E(0) \\
S(1,0)b_0 + S(1,1)b_1 + \ldots + S(1,N)b_N &= E(1) \\
\vdots & \\
S(N,0)b_0 + S(N,1)b_1 + \ldots + S(N,N)b_N &= E(N).
\end{align*}
\]

A somewhat different system of equations can be obtained by making use of a special case of the integral (12)

\[
\int_0^a x^{v+1} (a^2-x^2)^u J_v(yx)dx = \frac{2^u \Gamma(u+1)}{\sqrt{\pi}} \frac{a^{v+u+1}}{y^{u+1}} J_{v+u+1}(ay),
\]

providing that \( y > 0, a > 0, \text{Re} \ u > -1 \), and \( \text{Re} \ v > -1 \). Set \( v = 1, a = Y, u = k - \frac{1}{2}, \) and \( y = \xi_m \); the form of Equation 90 now becomes

\[
\int_0^Y x^2 (\gamma^2-x^2)^{k-\frac{1}{2}} J_1(\xi_m x)dx = \frac{2^{k-\frac{1}{2}} \Gamma(k+\frac{1}{2}) Y^{k+3/2}}{\xi_m^{k+\frac{1}{2}}} J_{k+3/2}(\gamma \xi_m).
\]

Multiply Equation 80 by \( x^2(\gamma^2-x^2)^{k-\frac{1}{2}} \), and integrate with respect to \( x \) from 0 to \( Y \) to obtain

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\text{chsh}(m; z_0+h) J_{2n+3/2}(\gamma \xi_m) J_{k+3/2}(\gamma \xi_m)}{\sinh(\xi_m h/\alpha) J_1(\xi_m) \xi_m^{k+\frac{1}{2}} (\xi_m^2 - 1) \text{chsh}(m; z_0)}
\]
\[
= \sum_{m=1}^{\infty} \frac{J_{k+3/2}(\gamma_m^\gamma)}{J_1(\xi_m^\gamma)\xi_m^{k+\frac{1}{2}}(\xi_m^2-1)\text{chsh}(m;z_o)}
\]

where the integral of Equation 91 has been used, assuming that the interchange of the order of integration and summation is permissible.

Again define
\[
S(k,n) = \sum_{m=1}^{\infty} \frac{\text{chsh}(m;z_o+h) J_{2n+3/2}(\gamma_m^\gamma) J_{k+3/2}(\gamma_m^\gamma)}{\sinh(\xi_m h/a) J_1(\xi_m^\gamma)\xi_m^{k+\frac{1}{2}}(\xi_m^2-1)\text{chsh}(m;z_o)}
\]

and
\[
E(k) = \sum_{m=1}^{\infty} \frac{J_{k+3/2}(\gamma_m^\gamma)}{J_1(\xi_m^\gamma)\xi_m^{k+\frac{1}{2}}(\xi_m^2-1)\text{chsh}(m;z_o)}
\]

where \( k = 1,2,3,\ldots \), so that an infinite system of equations
\[
\sum_{n=0}^{\infty} b_n S(k,n) = E(k)
\]

is obtained, and the values of \( B_m \) can be determined from Equation 86 as soon as the values of the \( b_n \) are known.

As a matter of fact, it is possible to obtain the formal solution of the dual Dini series given by Equations 58 and 59 in a variety of forms depending upon one's ability to find a Dini expansion of a function which converges to zero for \( \gamma < x < 1 \). This ability seems to be dictated by the availability of integrals involving the Bessel function \( J_1(\xi_m x) \) and two free parameters, e.g., consider Equations 74 and 90. As might be expected one method of solving the dual series may
be more amenable to numerical computation than another method. To be more specific, it has been observed that the solutions represented by Equations 87, 88, and 89 and by Equations 92, 93, and 94 are not well suited to machine computation since the matrices represented here by $S(k,n)$ become ill-conditioned as the dimensions of the matrices exceed 15 x 15 in one case and 20 x 20 in the other case. Therefore, in the subsequent pages, other formulas for the solution of the dual series are given. Basically the method of solution is the same as that given above; however different integrals involving $J_1(\xi_m x)$ are employed to give various forms for $S(k,n)$ and $E(k)$. From a numerical standpoint, it is desirable to obtain infinite series for $S(k,n)$ which converge fairly rapidly and which are such that the elements of the $S(k,n)$ matrix do not become too small or lead to an ill-conditioned system.

Consider a function $f(x)$ defined as follows:

$$ f(x) = \begin{cases} 
  x^{-1}(\gamma^2-x^2)^{\frac{1}{2}} \cos(n\sqrt{\gamma^2-x^2}), & 0 \leq x < \gamma \\
  0, & \gamma < x < 1.
\end{cases} $$

If the $\xi_m (m = 1, 2, 3, \ldots)$ are the positive roots of the transcendental equation $J_1(x) = 0$, then $f(x)$ may be expanded in a Dini series of the form

$$ f(x) = \sum_{m=1}^{\infty} A_m J_1(\xi_m x), \quad (95) $$

where
According to Erdelyi (12), page 39, formula 47, it is known that

\[
\int_0^\gamma (x^2 - \xi^2)^{-\frac{1}{2}} \cos (n \sqrt{\gamma^2 - x^2}) J_1(\xi x) \, dx = \frac{\pi}{2} J_1\left(\frac{1}{2} n \sqrt{\gamma^2 + \xi^2}\right) J_1\left(\frac{1}{2} n \sqrt{\gamma^2 + \xi^2}\right)
\]

\[
= \frac{1}{\gamma \xi} \left[ \cos(\gamma n) - \cos(\xi n + \xi^2) \right],
\]

and therefore Equation 97 assumes the form

\[
A_m = \frac{2\xi^2}{(\xi^2 - 1) J_1^2(\xi)} \int_0^1 x f(x) J_1(\xi x) \, dx
\]

\[
= \frac{2\xi^2}{(\xi^2 - 1) J_1^2(\xi)} \int_0^\gamma (x^2 - \xi^2)^{-\frac{1}{2}} \cos (n \sqrt{\gamma^2 - x^2}) J_1(\xi x) \, dx.
\]

Hence it has been established that

\[
\sum_{m=1}^\infty \frac{\xi_m [\cos(\gamma n) - \cos(\gamma \sqrt{n^2 + \xi_m^2})]}{J_1^2(\xi_m) (\xi_m^2 - 1)} \approx \frac{1}{2} \gamma x^{-\frac{1}{2}} (\gamma^2 - x^2)^{-\frac{1}{2}} \cos (n \sqrt{\gamma^2 - x^2}), \quad 0 \leq x < \gamma,
\]

\[
= 0, \quad \gamma < x < 1.
\]

Now define

\[
G^*_m = \frac{\xi_m}{J_1^2(\xi_m) (\xi_m^2 - 1)} \sum_{n=1}^\infty b_n [\cos(\gamma n) - \cos(\gamma \sqrt{n^2 + \xi_m^2})],
\]
where the $C^*_m (m = 1, 2, 3, \ldots)$ are the unknown coefficients in the dual series

$$
\sum_{m=1}^{\infty} C^*_m J_1(\xi_m x) = F(x), \quad 0 \leq x \gamma,
$$

(100)

$$
\sum_{m=1}^{\infty} C^*_m J_1(\xi_m x) = 0, \quad \gamma < x < 1,
$$

(101)

where $G(m)$ is given by Equation 85 and

$$
F(x) = \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1) \text{chsh} (m; z)}.
$$

(102)

Equation 101 is formally satisfied identically for the choice of $C^*_m$ as made in Equation 99 as is seen by direct substitution

$$
\sum_{m=1}^{\infty} \frac{\xi_m J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)} \sum_{n=1}^{\infty} b_n [\cos (\gamma n) - \cos (\gamma \sqrt{n^2+m^2})]
$$

$$
= \sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m [\cos (\gamma n) - \cos (\gamma \sqrt{n^2+m^2})]J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)} = 0
$$

for $\gamma < x < 1$, because of Equation 98, assuming that the interchange of the order of summation is permissible.

Returning to Equation 100 and again replacing the $C^*_m$ by Equation 99,

$$
\sum_{m=1}^{\infty} \frac{G(m) J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)} \sum_{n=1}^{\infty} b_n [\cos (\gamma n) - \cos (\gamma \sqrt{n^2+m^2})] = F(x)
$$

for $0 \leq x \gamma$, or upon interchanging the order of summation,

$$
\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{G(m) [\cos (\gamma n) - \cos (\gamma \sqrt{n^2+m^2})] J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)} = F(x). \quad (103)
$$
Multiply Equation 103 by \((\gamma^2 - x^2)^{-\frac{1}{2}} \cos k \sqrt{\gamma^2 - x^2}\) and integrate with respect to \(x\) from 0 to \(y\) to obtain

\[
\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{G(m)[\cos (yn) - \cos (\gamma \sqrt{n^2 + \xi_m^2})][\cos (yk) - \cos (\gamma \sqrt{k^2 + \xi_m^2})]}{J_1^2(\xi_m) \xi_m (\xi_m^2 - 1)}
\]

\[
= \sum_{m=1}^{\infty} \frac{[\cos (yk) - \cos (\gamma \sqrt{k^2 + \xi_m^2})]}{J_1(\xi_m) \xi_m (\xi_m^2 - 1) \cosh (m; z_0)}
\]

where Equations 97 and 102 have been employed.

Therefore, the infinite system

\[
\sum_{n=1}^{\infty} b_n S(k,n) = E(k),
\]

where

\[
S(k,n) = \sum_{m=1}^{\infty} \frac{G(m)[\cos (yn) - \cos (\gamma \sqrt{n^2 + \xi_m^2})][\cos (yk) - \cos (\gamma \sqrt{k^2 + \xi_m^2})]}{J_1^2(\xi_m) \xi_m (\xi_m^2 - 1)},
\]

and

\[
E(k) = \sum_{m=1}^{\infty} \frac{[\cos (yk) - \cos (\gamma \sqrt{k^2 + \xi_m^2})]}{J_1(\xi_m) \xi_m (\xi_m^2 - 1) \cosh (m; z_0)},
\]

for \(n,k = 1, 2, 3, \ldots\), must now be solved for the coefficients \(b_n\).

Alternatively, one might make the change of variable \(x = \sin y\) in Equation 103 and then multiply by \(\sin^2 y \cos(k \cos y)\) and integrate with respect to \(y\) from 0 to \(\pi/2\) to obtain

\[
\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m G(m)[\cos (yn) - \cos (\gamma \sqrt{n^2 + \xi_m^2})] J_{3/2}(\sqrt{k^2 + \gamma^2 \xi_m^2})}{J_1^2(\xi_m) (\xi_m^2 - 1)(k^2 + \gamma^2 \xi_m^2)^{3/4}}
\]
\[ \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\sqrt{k^2 + \gamma^2 \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(k^2 + \gamma^2 \xi_m^2)^{3/4}} \]

using the integral

\[ \int_{0}^{\pi/2} \sin^2 y \cos(k \cos y) J_1(\gamma \xi_m \sin y) dy = \left( \frac{\pi}{2} \right)^{1/2} \frac{\gamma \xi_m}{(k^2 + \gamma^2 \xi_m^2)^{3/4}} J_{3/2}(\sqrt{k^2 + \gamma^2 \xi_m^2}) \]

which is a special case of a more general integral given by Erdelyi (12), page 361, formula 19. Now define

\[ S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_m g(m)[\cos(\gamma n) - \cos(\sqrt{n^2 + \xi_m^2})]}{J_1(\xi_m)(\xi_m^2 - 1)(k^2 + \gamma^2 \xi_m^2)^{3/4}} J_{3/2}(\sqrt{k^2 + \gamma^2 \xi_m^2}) \]

and

\[ E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\sqrt{k^2 + \gamma^2 \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(k^2 + \gamma^2 \xi_m^2)^{3/4}} \]

for \( n, k = 1, 2, 3, \ldots \)

In practice, it was observed that the values of the \( b_n \) obtained by solving the system in Equation 104 and the systems in Equation 105 were roughly of the same order of magnitude, and were such as to cast serious doubt on the convergence of the series for \( C_m^* \), Equation 99. Consequently it would appear that the infinite series in Equation 99 converges too slowly, if at all, to be of any practical value.

Consider next a function \( f(x) \) defined as follows:

\[ f(x) = \begin{cases} x \sin(n \sqrt{\gamma^2 - x^2}), & 0 \leq x < \gamma, \\ 0, & \gamma < x < 1. \end{cases} \]
With reference to Equations 95 and 96, the coefficients $A_m$ in the Dini expansion of $f(x)$ are obtained from the integral

$$A_m = \frac{2\xi_m^2}{J_1(\xi_m)(\xi_m^2-1)} \int_0^\gamma x^2 \sin (n\sqrt{\gamma^2-x^2})J_1(\xi_m x) \, dx$$

$$= \frac{n\gamma^2 \sqrt{2\pi \xi_m^3}}{J_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}} J_{5/2}(n\sqrt{\gamma^2+\xi_m^2})$$

where the integral

$$\int_0^\gamma x^2 \sin (n\sqrt{\gamma^2-x^2})J_1(\xi_m x) \, dx$$

$$= \left(\frac{n}{2}\right)^{1/2} \frac{\xi_m^{5/2}}{(n^2+\xi_m^2)^{5/4}} J_{5/2}(n\sqrt{\gamma^2+\xi_m^2})$$

as given by Erdélyi (12), page 335, formula 19, has been used.

Therefore, it has been established that

$$\sum_{m=1}^\infty \frac{\xi_m^3 J_{5/2}(n\sqrt{\gamma^2+\xi_m^2})J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}}$$

$$= \frac{x}{n\gamma^2 \sqrt{2\pi \gamma^2}} \sin (n\sqrt{\gamma^2-x^2}), \quad 0 \leq x < \gamma,$$

$$= 0,$$

Define

$$C_m = \frac{\xi_m^3}{J_1(\xi_m)(\xi_m^2-1)} \sum_{n=1}^\infty \frac{b_n J_{5/2}(n\sqrt{\gamma^2+\xi_m^2})}{(n^2+\xi_m^2)^{5/4}}$$

so that Equation 101 is satisfied identically. The details of verifying this last statement are essentially the same as in the previous cases outlined a few pages earlier, and therefore they are omitted here. However, Equation 100
assumes the form

\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_n \frac{\xi_m^2 G(m) J_{5/2}(\sqrt{n^2 + \xi_m^2}) J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2 - 1)(n^2 + \xi_m^2)^{5/4}} \]

\[ = \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2 - 1) \text{chsh}(m; z_o)}, \]

for \( 0 \leq x < \gamma \). Proceeding as before and using Equation 106, it is found that

\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_n \frac{\xi_m^3 G(m) J_{5/2}(\sqrt{n^2 + \xi_m^2}) J_{5/2}(\sqrt{k^2 + \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(n^2 + \xi_m^2)^{5/4}(k^2 + \xi_m^2)^{5/4}} \]

\[ = \sum_{m=1}^{\infty} \frac{\xi_m J_{5/2}(\sqrt{\xi_m^2 + k^2})}{J_1(\xi_m)(\xi_m^2 - 1)(k^2 + \xi_m^2)^{5/4} \text{chsh}(m; z_o)}, \]

which leads to the infinite system of equations

\[ \sum_{n=1}^{\infty} b_n S(k, n) = E(k), \]

for \( k = 1, 2, 3, \ldots \), where

\[ S(k, n) = \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_{5/2}(\sqrt{n^2 + \xi_m^2}) J_{5/2}(\sqrt{k^2 + \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(n^2 + \xi_m^2)^{5/4}(k^2 + \xi_m^2)^{5/4}}, \quad (107) \]

and

\[ E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{5/2}(\sqrt{k^2 + \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(k^2 + \xi_m^2)^{5/4} \text{chsh}(m; z_o)}. \]

While the infinite series in Equation 107 converges fairly rapidly, it was found that the matrix \( S(k, n) \) was ill-conditioned for \( k = n = 15 \).

The integral
\[
\int_0^{\pi/2} J_u(z \sin t) J_v(s \cos t) \sin^{u+1} t \cos^{v+1} t \, dt
= \frac{1}{y} \left( \frac{z}{y} \right)^u \left( \frac{s}{y} \right)^v J_{u+v+1}(y),
\]

(108)

where \( y^2 = z^2 + s^2 \), \( \text{Re}(u) > -1 \), and \( \text{Re}(v) > -1 \), is given in Luke (13), page 299, formula 26. If the substitutions \( z = \gamma \xi_m \) and \( s = \gamma w \) and the change of variable \( x = \gamma \sin t \) are made, then the form of the integral in Equation 108 becomes

\[
\int_0^\gamma J_u(\xi_m x) J_v(\sqrt{\gamma^2-x^2}) x^{u+1} (\gamma^2-x^2)^{v/2} \, dx
= \frac{\gamma^u}{(\xi_m^2+w^2)^{u+v+1}} \frac{\gamma^2}{(\xi_m^2+n^2)^{u+v+1}} J_{u+v+1}(\gamma \sqrt{n^2+\xi_m^2}).
\]

Consider the special case of this last integral for which \( u = 1 \), \( v = 0 \), and \( w = n \), a positive integer:

\[
\int_0^\gamma x^2 J_0(\sqrt{\gamma^2-x^2}) J_1(\xi_m x) \, dx = \frac{\gamma^2 \xi_m}{(\xi_m^2+n^2)^2} J_2(\gamma \sqrt{n^2+\xi_m^2}).
\]

(109)

Recalling the integrand of Equation 109, define a function \( f(x) \) such that

\[
f(x) = \begin{cases} 
\xi_m J_1(n \sqrt{\gamma^2-x^2}), & 0 \leq x < \gamma \\
0, & \gamma < x < 1.
\end{cases}
\]

Now the coefficients \( A_m \) which appear in the Dini expansion of \( f(x) \) can be found by carrying out the integration

\[
A_m = \frac{2 \xi_m^2}{J_1^2(\xi_m) (\xi_m^2-1)} \int_0^1 x f(x) J_1(\xi_m x) \, dx.
\]
\[
\frac{2\xi^2_m}{J_1(\xi_m)(\xi_m^2-1)} \int_0^\gamma x^2 J_o(n\sqrt{\gamma^2-x^2})J_1(\xi_m x) \, dx
= \frac{2\gamma^2 \xi^2_m J_2(\gamma\sqrt{\xi_m^2+\xi^2_m})}{J_1(\xi_m)(\xi_m^2-1)(\xi_m^2+\xi^2_m)},
\]

according to Equation 109. Therefore, it has been shown that

\[
\sum_{m=1}^\infty \frac{\xi^3_m J_2(\gamma\sqrt{\xi_m^2+\xi^2_m})}{J_1(\xi_m)(\xi_m^2-1)(\xi_m^2+\xi^2_m)} = \frac{x}{2\gamma^2} J_o(n\sqrt{\gamma^2-x^2}), \quad 0 \leq x < \gamma,
0, \quad \gamma < x < 1.
\]

Defining

\[
C^*_m = \frac{\xi^3_m}{J_1(\xi_m)(\xi_m^2-1)} \sum_{n=1}^\infty \frac{b_n J_2(\gamma\sqrt{n^2+\xi^2_m})}{(\xi_m^2+n^2)},
\]

(110)

it is evident that the second equation which appears in the dual series is satisfied identically. If Equation 110 is inserted into Equation 100, it is found that the following equation arises:

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\xi^2_m G(m)}{J_1(\xi_m)(\xi_m^2-1) \cosh(mz_0)} J_2(\gamma\sqrt{n^2+\xi^2_m}) J_1(\xi_m x)
= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\xi^3_m G(m)}{J_1(\xi_m)(\xi_m^2-1)(\xi_m^2+n^2)(\xi_m^2+k^2)} J_2(\gamma\sqrt{n^2+\xi^2_m}) J_2(\gamma\sqrt{k^2+\xi^2_m})
\]

for \(0 \leq x < \gamma\). Multiply Equation 111 by \(x^2 J_o(k\sqrt{\gamma^2-x^2})\) and integrate with respect to \(x\) from 0 to \(\gamma\) in order to obtain

\[
\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{\xi^3_m G(m)}{J_1(\xi_m)(\xi_m^2-\xi_m^2-n^2)(\xi_m^2+k^2)} J_2(\gamma\sqrt{n^2+\xi^2_m}) J_1(\xi_m x)
\]

which may be abbreviated as
\[ S(k, n) = \sum_{n=1}^{\infty} S_n \cdot S(k, n) = E(k), \]

where

\[ S(k, n) = \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_2(\gamma \sqrt{n^2 + \xi_m^2}) J_2(\gamma \sqrt{k^2 + \xi_m^2})}{J_1(\xi_m)(\xi_m^2 - 1)(\xi_m^2 + n^2)(\xi_m^2 + k^2)}, \quad (112) \]

\[ E(k) = \sum_{m=1}^{\infty} \frac{\xi_m^3 J_2(\gamma \sqrt{\xi_m^2 + k^2})}{J_1(\xi_m)(\xi_m^2 - 1) \text{chsh}(m; z_0)(\xi_m^2 + k^2)}, \]

and \( n, k = 1, 2, 3, \ldots \). It is not difficult to show that the infinite series of Equation 112 converges absolutely, and, as a matter of fact, the asymptotic form of the terms in the series for large \( m \) is

\[ \frac{1}{\xi_m^3 \sin^2(\xi_m - \pi/4)}, \]

neglecting constant multiplicative factors. Nonetheless, the matrix \( S(k, n) \) proved to be too ill-conditioned for \( k = n = 15 \) to be handled even by a double precision matrix inversion routine on the computer.

Finally, define

\[ x(\gamma^2 - x^2)^{-\frac{1}{2}} \cos (n \sqrt{\gamma^2 - x^2}), \quad 0 \leq x < \gamma, \]

\[ f(x) = \begin{cases} 0, & \gamma < x < 1, \end{cases} \]

and determine its Dini expansion for the interval \((0, 1)\).

The coefficients \( A_m \) which appear in the expansion are to be computed from the integral
Thus

\[ A_m = \frac{2\xi_m^2}{J^2_1(\xi_m)(\xi_m^2-1)} \int_0^1 x f(x) J_1(\xi_m x) \, dx \]

Thus

\[ A_m = \frac{2\xi_m^2}{J^2_1(\xi_m)(\xi_m^2-1)} \int_0^\gamma x^2 (\gamma^2-x^2)^{-\frac{1}{2}} \cos(n \sqrt{\gamma^2-x^2}) J_1(\xi_m x) \, dx \]

\[ = \frac{\gamma(2\pi)^{3/2} \xi_m^3}{J^2_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}} J_{3/2}(\gamma \sqrt{n^2+\xi_m^2}), \]

where the integral

\[ \int_0^\gamma x^2 (\gamma^2-x^2)^{-\frac{1}{2}} \cos(n \sqrt{\gamma^2-x^2}) J_1(\xi_m x) \, dx \]

\[ = \frac{\gamma(2\pi)^{3/2} \xi_m^3}{(n^2+\xi_m^2)^{3/4}} J_{3/2}(\gamma \sqrt{n^2+\xi_m^2}), \]

given by Erdelyi (12), page 361, formula 19, has been employed. Hence it has been established that

\[ \sum_{m=1}^\infty \frac{\xi_m^3}{J^2_1(\xi_m)(\xi_m^2-1)} J_{3/2}(\gamma \sqrt{n^2+\xi_m^2}) J_1(\xi_m x) \]

\[ = \frac{x}{\gamma(2\pi)^{3/2} (\gamma^2-x^2)^{-\frac{1}{2}} \cos(n \sqrt{\gamma^2-x^2}), \quad 0 \leq x < \gamma, \]

\[ 0, \quad \gamma < x < 1. \]

Continuing in the usual fashion, define now

\[ C^*_m = \frac{\xi_m^3}{J^2_1(\xi_m)(\xi_m^2-1)} \sum_{n=1}^\infty \frac{b_n J_{3/2}(\gamma \sqrt{n^2+\xi_m^2})}{(n^2+\xi_m^2)^{3/4}}. \quad (113) \]

This choice of the form of \( C_m^* \) is appropriate because Equation 101 is satisfied identically as is easily shown by direct substitution.
Inserting the expression for $C^*_m$ given in Equation 113 into Equation 100, it follows that

\[
\sum_{n=1}^{\infty} b_n \frac{\xi_k^2 G(m) J_{3/2}(\gamma \sqrt{n^2 + \xi_k^2}) J_1(\xi_k x)}{J_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}}
\]

\[
= \sum_{n=1}^{\infty} \frac{J_1(\xi_k x)}{J_1(\xi_m)(\xi_m^2-1)\cosh(m;\zeta_0)}, \quad 0 \leq x < \gamma.
\]

Multiply this last equation by $x^2(\gamma^2-x^2)^{-\frac{1}{2}} \cos(\sqrt{\gamma^2-x^2})$, and integrate with respect to $x$ from 0 to $\gamma$ to obtain

\[
\sum_{n=1}^{\infty} b_n \frac{\xi_k^3 G(m) J_{3/2}(\gamma \sqrt{n^2 + \xi_k^2}) J_{3/2}(\gamma \sqrt{k^2 + \xi_k^2})}{J_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}(k^2+\xi_m^2)^{3/4}}
\]

\[
= \sum_{n=1}^{\infty} \frac{\xi_m J_{3/2}(\gamma \sqrt{k^2+\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{3/4} \cosh(m;\zeta_0)},
\]

which leads to the infinite system of linear equations in $b_n$

\[
\sum_{n=1}^{\infty} b_n S(k,n) = E(k),
\]

where

\[
S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_k^3 G(m) J_{3/2}(\gamma \sqrt{n^2 + \xi_k^2}) J_{3/2}(\gamma \sqrt{k^2 + \xi_k^2})}{J_1(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}(k^2+\xi_m^2)^{3/4}}
\]  \hspace{1cm} (114)

and

\[
E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\gamma \sqrt{k^2+\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{3/4} \cosh(m;\zeta_0)}.
\]

For large $m$, the terms in the infinite series in Equation 114 behave as
\[
\frac{1}{\xi_m^2 \sin^2(\xi_m - \pi/4)},
\]
so that, it would appear that the series should converge. It should be pointed out that \( \sin (\xi_m - \pi/4) \neq 0 \) for any \( m = 1, 2, 3, \ldots \).
VI. SOME REMARKS ON THE NUMERICAL ASPECTS OF THE PROBLEM

Because the infinite series in Equation 114 converges slowly, methods of speeding convergence have been investigated. In particular, Lubkin (14) has discussed the transformation of a given infinite series

\[ S = a_0 + a_1 + a_2 + a_3 + \ldots + a_n + \ldots \]  

into a new series

\[ T = b_0 + b_1 + b_2 + b_3 + \ldots + b_n + \ldots \]

Define the partial sums of the series by

\[ S_n = a_0 + a_1 + a_2 + \ldots + a_n \]  

and

\[ T_n = b_0 + b_1 + b_2 + \ldots + b_n \]

and define the T series by the relation

\[ T_{n+1} = S_n + \frac{a_{n+1}}{1 - (a_{n+1}/a_n)}, \quad n \geq 0, \]

which, after minor manipulations, can be expressed as

\[ T_n = \frac{S_n^2 - S_{n-1}S_{n+1}}{2S_n - S_{n-1} - S_{n+1}}, \quad n > 0, \quad T_0 = \frac{S_0^2}{2S_0 - S_1}. \]

Going one step further, it can be shown that the individual terms of the T series may be computed from

\[ b_0 = \frac{a_0 R_0}{R_0 - 1}, \quad \text{and} \quad b_n = a_n \left[ \frac{R_{n-1} - R_n}{(R_{n-1})(R_{n-1}-1)} \right], \quad n \geq 1, \]

where \( R_n = a_n/a_{n+1} \).
In accord with usual terminology, a series \( C = \sum_{n=0}^{\infty} c_n \) with partial sums \( C_n = \sum_{m=0}^{n} c_m \) (1) is said to be more rapidly convergent than \( S \) (see Equation 115) if both \( S \) and \( C \) converge, and \( (C - C_n)/(S - S_n) \), the ratio of corresponding remainders, tends to zero as \( n \) tends to infinity; (2) is of the same order of rapidity of convergence as \( S \) if both series converge, and \( |(C - C_n)/(S - S_n)| \) remains in value between two finite positive constants for all sufficiently large \( n \); and (3) is no less rapidly convergent than \( S \) if both series converge and the ratio of corresponding remainders is bounded as \( n \) tends to infinity.

In particular, the following theorem has proved useful:

**Theorem.** Let

\[ Q_n = n(R_n - 1) \]

and

\[ Q = \lim_{n \to \infty} Q_n. \]

If \( S \) converges, \( Q \) exists, \( \neq 1 \), and \( n(Q_n - Q_{n-1}) \to 0 \) as \( n \to \infty \), then the series \( U = \sum_{n=0}^{\infty} u_n \) with

\[ u_n = \frac{(Qb_n - a_n)}{Q - 1} \]

converges more rapidly than \( S \) and has the same sum.

However, since \( T \) converges and \( T = S \), Lubkin (14) shows that

\[ U_n = \sum_{m=0}^{n} u_m \frac{Q T_m - S_m}{Q - 1} \to \frac{Q S - S}{Q - 1} = S, \]
and thus the approximation

$$U_n = \frac{Q^n - S_n}{Q - 1} \quad (117)$$

to the value of $S$ is more accurate than is $S_n$ itself.

Another non-linear transformation, called the $W$ transformation, is useful, especially so since the value of $Q$ is not required. Lubkin (14) points out that there are also peculiar cases where the $T$ transformation is usable but not the $W$.

Define

$$P_o = \frac{b_0}{a_0} = \frac{R_0}{R_0 - 1}$$

and

$$P_n = \frac{b_n}{a_n} = \frac{R_{n-1} - R_n}{(R_{n-1})(R_{n-1} - 1)}, \quad n \geq 1.$$

The $W$ transformation is then defined by the relations

$$W = w_0 + w_1 + w_2 + \ldots + w_n + \ldots,$$

$$W_n = w_0 + w_1 + w_2 + \ldots + w_n,$$

$$= \frac{T_n - P_n S_n}{1 - P_n}. \quad (118)$$

The conditions under which the $W$ transformation is applicable are given in the following theorem as given by Lubkin (14):

**Theorem.** If $S$ converges, $Q$ exists, $\neq 1$, $n(Q_n - Q_{n-1}) \rightarrow 0$, and $n[(n+1)(Q_{n+1} - Q_n) - n(Q_n - Q_{n-1})] \rightarrow 0$ as $n$ tends to infinity, then $W$ converges more rapidly than $S$ and has the same sum.

Examination of the numerical values obtained for the series given in Equation 114 has revealed that the
transformations appearing in Equations 117 and 118 lead to results which are not consistent with the partial sums of the series itself. The complexity of the general term in the series makes it rather difficult to show that the conditions of the appropriate theorems are actually fulfilled.

Salzer (15) discusses a method of summing certain slowly convergent series which is well-suited for machine computation. The application of Salzer's technique may be widespread since it involves a purely numerical device which is employed without any specific analytic work upon the series. The basic idea of this approach is to consider \( S_n \) (see Equation 116) as the tabulated value of a certain function of \( x \), say \( S(x) \), at \( x = n \), from which one would like to calculate \( S(\infty) \) by the \( m \)-point Lagrangian interpolation polynomial. Then to calculate the limit \( S \) of a sequence \( S_1, S_2, \ldots, S_n \), using the \( m \)-point extrapolation formula, one multiplies each of the last \( m \) terms, \( S_{n-1}, i = 1, 2, \ldots, m-1 \), by the corresponding extrapolation coefficient \( B_{n,n-1}^{(m)} / D_n^{(m)} \), and sums; thus

\[
S \sim \left( \frac{1}{D_n^{(m)}} \right) \sum_{i=0}^{m-1} B_{n,n-1}^{(m)} S_{n-i}. \tag{119}
\]

The coefficients required in Equation 119 are listed in Table I in Salzer's paper.

For machine purposes, numerical values of the Bessel functions involved in various phases of the problem can be obtained by writing a machine language subroutine following a recent paper by Gautschi (16). This procedure evaluates to
significant digits the Bessel functions $J_{a+n}(x)$ for $n = 0, 1, 2, \ldots$, $0 \leq a < 1$, and $x > 0$. The method of computation is a variant of the backward recurrence algorithm of J.C.P. Miller as discussed by Gautschi (17). The algorithm is most efficient when $x$ is small or moderately large, although near a zero of one of the Bessel functions generated, the accuracy of that particular Bessel function may deteriorate to less than $d$ significant digits.

Abramowitz and Stegun (18) have given polynomial approximations for $J_0(x)$ and $J_1(x)$ for small argument $x$ as well as large. In particular, for $|x| \leq 3$,

$$J_0(x) = 1 - 2.24999 97 \left(\frac{x}{3}\right)^2 + 1.26562 08 \left(\frac{x}{3}\right)^4 - 0.31638 66 \left(\frac{x}{3}\right)^6$$

$$\quad + 0.04444 79 \left(\frac{x}{3}\right)^8 - 0.00394 44 \left(\frac{x}{3}\right)^{10} + 0.00021 00 \left(\frac{x}{3}\right)^{12} + e,$$

where $|e| < 5 \times 10^{-8}$, whereas for $3 \leq x < \infty$,

$$J_0(x) = x^{-\frac{1}{2}} f_0(x) \cos \theta_0(x),$$

where

$$f_0(x) = 0.79788 456 - 0.00000 077(3/x) - 0.00552 740(3/x)^2$$

$$- 0.00009 512(3/x)^3 + 0.00137 237(3/x)^4 - 0.00072 805(3/x)^5$$

$$+ 0.00014 476(3/x)^6 + e,$$

with $|e| < 1.6 \times 10^{-8}$ and

$$\theta_0(x) = x - 0.78539 816 - 0.04166 397(3/x) - 0.00003 954(3/x)^2$$

$$+ 0.00262 573(3/x)^3 - 0.00054 125(3/x)^4 - 0.00029 333(3/x)^5$$

$$+ 0.00013 558(3/x)^6 + e,$$
Similar expressions are given for $J_1(x)$. They are, for $|x| \leq 3$,

\[
x^{-1} J_1(x) = \frac{1}{2} - 0.56249 \ 985(x/3)^2 + 0.21093 \ 573(x/3)^4 \\
- 0.03954 \ 289(x/3)^6 + 0.0043 \ 319(x/3)^8 - 0.00031 \ 761(x/3)^{10} \\
+ 0.00001 \ 109(x/3)^{12} + \epsilon,
\]

$|\epsilon| < 1.3 \times 10^{-8}$

and for $3 \leq x < \infty$,

\[
J_1(x) = x^{-\frac{1}{2}} f_1(x) \cos \theta_1(x),
\]

where

\[
f_1(x) = 0.79788 \ 456 + 0.00000 \ 156(3/x) + 0.01659 \ 667(3/x)^2 \\
+ 0.00017 \ 105(3/x)^3 - 0.00249 \ 511(3/x)^4 + 0.00113 \ 653(3/x)^5 \\
- 0.00020 \ 033(3/x)^6 + \epsilon,
\]

$|\epsilon| < 4 \times 10^{-8}$

and

\[
\theta_1(x) = x - 2.35619 \ 449 + 0.12499 \ 612(3/x) + 0.00005 \ 650(3/x)^2 \\
- 0.00637 \ 879(3/x)^3 + 0.00074 \ 348(3/x)^4 + 0.00079 \ 824(3/x)^5 \\
- 0.00029 \ 166(3/x)^6 + \epsilon,
\]

$|\epsilon| < 9 \times 10^{-8}$.

When $v$ is real, the function $J_v^1(x)$ has an infinite number of real zeros, all of which are simple with the possible exception of $x = 0$. For non-negative $v$ the $m$th positive zero of this function is denoted by $J_v^{1,m}$. Large zeros may be obtained from McMahon's expansion as given by Abramowitz and Stegun (18); i.e., when $v$ is fixed, $m \gg v$, and $u = 4v^2$, then
\[ j_{v,m} = b \frac{u+3}{8b} - \frac{4(7u^2+82u-9)}{3(8b)^3} - \frac{32(83u^3+2075u^2-3039u+3537)}{15(8b)^5} \]
\[ - \frac{64(6949u^4+296,492u^3-1,248,002u^2+7,414,380u-5,853,627)}{105(8b)^7} \ldots, \]

where \( b = (m+v/2-3/4) \pi \). On the other hand, the well-known Newton-Raphson technique may be employed to obtain the small zeros as well as the large. This method involves an iterative process in which an initial approximation \( x_0 \) to a desired real root is obtained, by rough graphical methods or otherwise, and the recurrence relation

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

is used to generate a sequence of successive approximations \( x_1, x_2, \ldots, x_n, \ldots \) which converges to the desired root. Starting with

\[ f(x) = xJ'_1(x) = xJ_0(x) - J_1(x), \]

it is easy to show that the zeros of \( J'_1(x) = 0 \) can be obtained from

\[ x_{n+1} = \frac{x_n[(x_n^2-2)J_1(x_n)+x_nJ_0(x_n)]}{(x_n^2-1)J_1(x_n)} \]

Thus, once the first zero \( J_1^{1,1} \) has been determined to the desired accuracy, the next, \( J_1^{1,2} \), can be obtained by repeating the above process starting with the initial approximation \( J_1^{1,2} = J_1^{1,1} + \pi \), since the zeros of \( J_1(x) \) are separated roughly by \( \pi \). This process has been utilized to obtain the first forty zeros \( J_1^{m,m} = \xi_m, (m=1,2,3,\ldots,40) \), which are tabulated along with the values of \( J_1(\xi_m) \) in Table 1.
Table 1. Some zeros of $J_1(x)$ and values of $J_1(\xi_m)$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\xi_m$</th>
<th>$J_1(\xi_m)$</th>
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<tr>
<td>1</td>
<td>1.84118</td>
<td>0.58186 512</td>
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<tr>
<td>2</td>
<td>5.33144</td>
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<td>3</td>
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<td>0.27329 993</td>
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<td>4</td>
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<td>5</td>
<td>14.86359</td>
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</tr>
<tr>
<td>6</td>
<td>18.01553</td>
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</tr>
<tr>
<td>7</td>
<td>21.16437</td>
<td>0.17345 904</td>
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<tr>
<td>8</td>
<td>24.31133</td>
<td>-0.16183 821</td>
</tr>
<tr>
<td>9</td>
<td>27.45705</td>
<td>0.15228 206</td>
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<tr>
<td>10</td>
<td>30.60192</td>
<td>-0.14424 289</td>
</tr>
<tr>
<td>11</td>
<td>33.74618</td>
<td>0.13735 718</td>
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<td>12</td>
<td>36.88999</td>
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<tr>
<td>13</td>
<td>40.03344</td>
<td>0.12610 881</td>
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<tr>
<td>14</td>
<td>43.17663</td>
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VII. CONCLUSION

This paper considers the irrotational motion of an incompressible, inviscid liquid contained in a partially filled cylindrical tank on the vertical wall of which is mounted a thin annular ring for the purpose of damping the free surface oscillations of the liquid. The tank is subjected to both transverse and rotational harmonic vibrations. In the solution of Laplace's equation for the velocity potential a dual Dini series arises because the boundary conditions in the plane of the annular baffle are of the mixed type. Following a method due to Tranter and Cooke (8), several forms of the formal solution of the dual series are given which, in every case, lead to an infinite system of linear algebraic equations. From a numerical point of view, many of these systems are plagued with an ill-conditioned coefficient matrix, and, also, it should be pointed out that in certain cases great care must be taken to obtain accurately the elements of these matrices since they are obtained by summing rather slowly convergent infinite series. Some methods of speeding the convergence of these series are discussed. For the numerical solution of the dual series, since the zeros of the transcendental equation \( J_1'(x) = 0 \) and numerical values of Bessel functions of the first kind of various orders and arguments are needed, special formulas are tabulated and machine language algorithms are referenced.
VIII. REFERENCES


IX. ACKNOWLEDGMENT

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