A numerical method of characteristics for solving hyperbolic partial differential equations

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A NUMERICAL METHOD OF CHARACTERISTICS
FOR SOLVING HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

David Lenz Simpson

A Dissertation Submitted to the
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1. INTRODUCTION

In this paper we are considering an algorithm for the solution of quasi-linear hyperbolic partial differential equations by the method of characteristics. The method of characteristics is playing an increasingly important role in the engineering sciences and particularly in propagation type problems. The propagation problems arising in the theory of gas dynamics, the theory of long water waves and the theory of plasticity, to name a few, are well suited to the method of characteristics.

The restriction to quasi-linear hyperbolic partial differential equations is understandable in view of the dependence of the method of characteristics on two sets of real characteristics.

In this work, the method of characteristics is considered only for problems in two independent variables. Although this limitation is implicit in nearly all practical applications of the method, there do exist certain procedures that utilize the special features of the characteristics for problems in more than two independent variables.

The method of characteristics reduces the partial differential equation to a family of initial value problems. And although a substantial knowledge of ordinary differential equations is available the need for discrete variable methods is well known.

Thus the need for a high accuracy numerical method that can be easily applied to a wide range of practical problems seems to be present.

Chapter two deals with the conversion of the quasi-linear, second order, hyperbolic partial differential equation into characteristic normal form. The equivalence of the two problems is remarked upon and the
existence of a solution to the characteristic normal form system along with the differentiability of such solutions is proved.

Chapter three investigates the truncation error involved in the predictor and the corrector. We also affirm the existence and uniqueness of solutions of these two systems used to approximate the true solution of the characteristic normal form system.

In chapter four we discuss the stability or lack of stability of the predictor and corrector as well as the stability of the algorithm. The order and the degree of the algorithm is also defined here.

A starting procedure is recommended in chapter five in addition to a procedure for altering the step-size of the characteristic mesh.

Chapter six illustrates the algorithm with a numerical example.
II. DISCUSSION OF THE ALGORITHM

The second order, quasi-linear, hyperbolic partial differential equation as it arises in most practical situations has the form

\[ au_{xx} + 2bu_{xy} + cu_{yy} + e = 0, \]

where \( u_{xx} = \frac{\partial^2 u}{\partial x^2}, \ u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \ u_{yy} = \frac{\partial^2 u}{\partial y^2} \) and the coefficients \( a, b, c \) and \( e \), due to the quasi-linearity, are functions of \( x, y, u, p \) and \( q; \)
\( (p = u_x, \ q = u_y) \). We mention that due to the hyperbolic nature of the equation we are restricted to a region \( R \) of the \( xy \)-plane where

\[ b^2 - ac > 0. \]

We will however insist on a slightly more restrictive condition, namely that we stay within a region \( R' \) where the discriminant is bounded away from zero:

\[ b^2 - ac > d > 0, \]

for \( d \) some constant.

We also assume that along an initial curve,

\[ y = f(x), \quad a \leq x \leq b, \]

\( u, p \) and \( q \) are known and are given by
Furthermore we assume the initial curve is not a characteristic.

Though the partial differential equation arising in practice is generally of the form (2.1) we find this particular form inappropriate for a formal discussion of convergence, stability and other facets connected with its solution. For this reason we will convert equation (2.1) to its so-called "characteristic normal form,"

\[
\sum_{k=1}^{5} a_{ik} \frac{\partial s^k}{\partial \xi} = 0 \quad i = 1(1)3
\]

(2.6a)

\[
\sum_{k=1}^{5} a_{ik} \frac{\partial s^k}{\partial \eta} = 0 \quad i = 4(1)5,
\]

with

\[
s^1 = x, \ s^2 = y, \ s^3 = u, \ s^4 = p, \ s^5 = q, \text{ and}
\]

(2.6)

\[
(a^k) = \begin{bmatrix}
-\lambda_+ & 1 & 0 & 0 & 0 \\
\frac{e}{a} & 0 & 0 & 1 & \lambda_-
\end{bmatrix}
\]

\[
-\lambda_- & 1 & 0 & 0 & 0 \\
-\lambda_+ & 1 & 0 & 0 & 0 \\
\frac{e}{a} & 0 & 0 & 1 & \lambda_+
\end{bmatrix}
\]

where

\[
\lambda_+ = \frac{b + \sqrt{b^2 - ac}}{a}
\]

\[
\lambda_- = \frac{b - \sqrt{b^2 - ac}}{b}
\]
It is well known ([3], [13]) that a great number of hyperbolic problems in two independent variables — including equation (2.1) as well as the general second order equation

\begin{equation}
Q(x, y, f; f_x, f_y, f_{xx}, f_{xy}, f_{yy}) = 0
\end{equation}

may be reduced to a characteristic normal form.

We will be assuming that the data from the initial curve (2.4)/(2.5) is prescribed on the line

\begin{equation}
\xi + \eta = 1
\end{equation}

in the $\xi\eta$-plane, i.e. $x = x_1(\xi, 1-\xi)$, $y = y_1(\xi, 1-\xi)$, $u = u_1(\xi, 1-\xi)$, $p = p_1(\xi, 1-\xi)$, $q = q_1(\xi, 1-\xi), 0 \leq \xi \leq 1$. We will refer to the initial curve as

\begin{equation}
c = \{(\xi, \eta) | \xi + \eta = 1, 0 \leq \xi \leq 1, 0 \leq \eta \leq 1\}
\end{equation}

Since new characteristic coordinates can be introduced by replacing $\xi$ with any function of $\xi$ and by replacing $\eta$ with any function of $\eta$, and since the initial curve is not itself a characteristic there is no actual loss of generality in taking it to be of the special form (2.8).

It has been shown [7] that equation (2.1) and the characteristic normal form (2.6) are equivalent.

In the characteristic normal form (2.6) the characteristics emanating from the initial curve (2.8) are lines.
For any high-order finite difference algorithm we must be certain that the solutions of (2.6)/(2.8) are sufficiently differentiable. To establish the needed differentiability we adapt a more general theorem by A. Douglas [5].

**Theorem 1.1: (A. Douglas).** Assume

a) \( a^{ik}(S) \in C^4 \) for \( S = (s^1, s^2, s^3, s^4, s^5) \epsilon \overline{U} \),

b) \( |\det(a^{ik}(S))| > d > 0 \) for \( S \epsilon U \),

c) \( s^k \in C^4, k = 1(1)5, \) and \( S \epsilon \overline{U} \) on the initial curve \( c \),

where \( \overline{U} \) is a closed region of the 5-dimensional space \( E^5 \) and let \( D \) be defined as the triangular region enclosed by the initial curve \( c \) and the lines \( \xi = 1, \eta = 1 \).

Then in a certain region \( D^{*} \epsilon \overline{D} \), with \( D^{*} \) containing the initial curve \( c \), there exists a unique set of solutions \( s^k, k = 1(1)5 \) of (2.6)/(2.8) which are four-times differentiable with respect to \( \xi \) and \( \eta \).

In view of later applications we will assume \( D^{*} \) to have the following properties (which are made possible by proceeding to a subregion of the original \( D^{*} \)):

a) \( D^{*} \) is closed

b) Given some \( \epsilon^{*} > 0 \)

\[ (2.11) \quad S^{*} = (s^1(\xi, \eta) + \epsilon^1, \ldots, s^5(\xi, \eta) + \epsilon^5) \epsilon \overline{U} \]

if \( (\xi, \eta) \epsilon D^{*} \) and \( |\epsilon^k| \leq \epsilon^{*}, k = 1(1)5 \).
In the following we will assume that an \( \varepsilon > 0 \) has been chosen and that \( D^* \) is the region just described. Within this region we are investigating the numerical solution of the problem.

The predictor-corrector algorithm we now introduce to solve the system (2.6) with initial conditions described on \( c \) is based upon approximating the ten partial derivatives in (2.6) by forward and backward differences. We impose a square mesh of side \( h \) on the triangular region \( D \). The calculated value at a point \((\xi_1, \eta_1)\) of the characteristic grid depends upon six back points, three along each of the characteristics \( \xi = \xi_1 \) and \( \eta = \eta_1 \).

The partial derivatives \( \frac{\partial f}{\partial \xi} \) and \( \frac{\partial f}{\partial \eta} \) will act as ordinary derivatives when applied along a line \( \xi = c_1 \) or \( \eta = c_2 \). Therefore we will use the following well known ([10],[11]) forward and backward difference operators, truncated after third differences.

\[
\begin{align*}
(2.12a) & \quad f'(x) = \frac{1}{h} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta^i f(x) \\
(2.12b) & \quad f'(x) = \frac{1}{h} \sum_{i=1}^{\infty} \frac{1}{i} \nabla^i f(x).
\end{align*}
\]

We also note the use of \( \Delta_1, \nabla_1 \) when the forward and backward operators are applied to the first variable in a function, \( F(\xi, \eta) \) and \( \Delta_2, \nabla_2 \) when they are applied to the second variable in the argument; i.e.

\[
\begin{align*}
\Delta_1 F(x, y) & = F(x+h, y) - F(x, y) \\
\Delta_2 F(x, y) & = F(x, y+h) - F(x, y)
\end{align*}
\]

with \( \Delta_2, \nabla_2 \) defined analogously.
With the partials in (2.6) replaced by forward differences (2.12a) truncated after third differences we get our predictor system for the solution at the \((\lambda, m)\) point on our characteristic grid:

\[
\sum_{k=1}^{5} a_{ik}(s_{\lambda-3m}) \left[ \left( \Delta_1 - \frac{1}{2} \Delta_1^2 + \frac{1}{3} \Delta_1^3 \right) s^k_{\lambda-3m} \right] = 0, \quad i = 1(1)3,
\]

(2.13)

\[
\sum_{k=1}^{5} a_{ik}(s_{\lambda m-3}) \left[ \left( \Delta_2 - \frac{1}{2} \Delta_2^2 + \frac{1}{3} \Delta_2^3 \right) s^k_{\lambda m-3} \right] = 0, \quad i = 4(1)5.
\]

Expanding the differences this may be written as

\[
\sum_{k=1}^{5} a_{ik}(s_{\lambda-3m}) \left[ \sum_{\nu=0}^{3} \alpha_\nu s^k_{\lambda-\nu m} \right] = 0, \quad i = 1(1)3,
\]

(2.14)

\[
\sum_{k=1}^{5} a_{ik}(s_{\lambda m-3}) \left[ \sum_{\nu=0}^{3} \alpha_\nu s^k_{\lambda m-\nu} \right] = 0, \quad i = 4(1)5
\]

with \(s_{\lambda m} = (s^1_{\lambda m}, s^2_{\lambda m}, s^3_{\lambda m}, s^4_{\lambda m}, s^5_{\lambda m})\),

(2.15) \(\alpha_0 = 1, \alpha_1 = -\frac{9}{2}, \alpha_2 = 9, \alpha_3 = -\frac{11}{2}\) and the \(a_{ik}\)'s as in (1.6b).

With the partials in (2.6) replaced by backward differences we get our corrector system for the solution at the \((\lambda, m)\) point on our characteristic grid:

\[
\sum_{k=1}^{5} a_{ik}(s_{\lambda m}) \left[ \left( v_1^1 + \frac{1}{2} v_1^2 + \frac{1}{3} v_1^3 \right) s^k_{\lambda m} \right] = 0, \quad i = 1(1)3,
\]

(2.16)
Expanding the differences in (2.16) we get

\[
\sum_{k=1}^{5} a_i^k(s_{\lambda m}) \left[ \sum_{v=0}^{3} \beta_v s_{\lambda m-v}^k \right] = 0, \quad i = 1(1)3
\]

(2.17) \[
\sum_{k=1}^{5} a_i^k(s_{\lambda m}) \left[ \sum_{v=0}^{3} \beta_2 s_{\lambda m-v}^k \right] = 0, \quad i = 4(1)5,
\]

(2.18) \[
\beta_0 = 1, \quad \beta_1 = -\frac{16}{11}, \quad \beta_2 = \frac{9}{11}, \quad \beta_3 = -\frac{2}{11}
\]

Thus we have a linear system (2.14) to solve for the predicted solution \(\tilde{S}_{\lambda m}\) at the \((l, m)\) characteristic grid point. We then use this predicted solution in the non-linear corrector system in an iterative manner to find our final approximate solution to the system of partial differential equations. We will later comment on the recommended number of iterations.
III. DISCUSSION OF LOCAL ERROR, EXISTENCE AND UNIQUENESS THEOREMS

As mentioned in Chapter two, within the triangle $D$ of the $\xi, \eta$-plane we introduce a square lattice with mesh-size $h > 0$. Further we are always assuming $L_0 = \frac{1}{h} > 0$ to be an integer.

We use the following notation and definitions:

a) $Z_{\lambda m} = Z_k(\lambda h, mh)$ for values of the true solution of (2.6) and (2.8).

b) $S_{\lambda m} = S_k(\lambda h, mh)$ for values approximating $Z_{\lambda m}^k$ at the point $(\lambda h, mh)$ of the lattice by the finite difference algorithm, also

\[
Z_{\lambda m} = (Z_{1\lambda m}, Z_{2\lambda m}, Z_{3\lambda m}, Z_{4\lambda m}, Z_{5\lambda m})
\]

\[
S_{\lambda m} = (S_{1\lambda m}, S_{2\lambda m}, S_{3\lambda m}, S_{4\lambda m}, S_{5\lambda m})
\]

and

\[
\frac{1}{||S_{\lambda m}||} = \max_{1 \leq k \leq 5} |S_{\lambda m}^k|.
\]

The $(\lambda, m)$ may take integral values within

\[
D_h = \{(\lambda, m) | \lambda + m \geq 1, 0 \leq \lambda \leq L_0, 0 \leq m \leq L_0\};
\]

$D_h^*$ will denote the part of $D_h$ corresponding to $D^*$, defined in chapter one.

We are assuming the following bounds:

\[
|a_i^k(\xi)| \leq H \text{ and } |a_j^k(\xi)| = \left| \frac{\partial a_i^k}{\partial s^j} \right| \leq H_1,
\]

for $\xi \in \Omega$,

\[
(3.1)
\]

\[
\left| \frac{\partial r_{\xi}^k}{\partial \xi} \right| \leq u_r \text{ and } \left| \frac{\partial r_{\eta}^k}{\partial \eta} \right| \leq u_r,
\]

for $(\xi, \eta) \in D^*, r = 0(1)4$.  

We now want to examine the local error or truncation error of the predictor operator (2.14). To aid us we note a lemma, [10].

**Lemma 3.1:** Let \( f(x) \) have a continuous derivative of order \( q + 1 \) in \( J \). Then for every \( x_\mu, \mu = 0(1)q \) there exists a number \( \alpha \) in the interval containing the \( x_\mu \)'s such that

\[
f'(x_\mu) - p'(x_\mu) = \frac{1}{(q+1)!} f^{(q+1)}(\alpha) L'(x_\mu),
\]

where \( p(x) \) denotes the \( q \)th degree approximating polynomial which agrees with \( f(x) \) at \( x_\mu, \mu = 0(1)q \); and \( L'(x) = \frac{d}{dx} \left( \prod_{i=0}^{q} (x-x_i) \right) \). The proof of lemma 3.1 can be found in a number of books, e.g. [10], [11].

We now state and prove the following lemma concerning the error in using forward differences (2.12a) truncated after third differences to approximate the partial derivatives and •

**Lemma 3.2:** Let \( s^k(\xi, \eta) \in C_4 \) in both \( \xi \) and \( \eta \). Then for \( \xi = \lambda h, \eta = mh, m \) constant, there exists a constant \( \alpha_1 \) on the \( \xi \)-interval \([(\lambda-3)h, \lambda h] \) along the line \( \eta = mh \) such that

\[
(3.2) \quad \frac{\partial s^k(\lambda h, mh)}{\partial \xi} - \frac{1}{h} \left[ \Delta_1 - \frac{1}{2} \Delta_2 + \frac{1}{3} \Delta_3 \right] s^k((\lambda-3)h, mh) = E_{k,\lambda, m}^\xi,
\]

where \( E_{k,\lambda, m}^\xi = \frac{h^3}{4} \frac{\partial^4 s(\alpha_1, mh)}{\partial \xi^4} \). And for \( \xi = \lambda h, \lambda \) constant and \( \eta = mh \) there exists a constant \( \alpha_2 \) on the \( \eta \)-interval \([(m-3)h, mh] \) on the line \( \xi = \lambda h \) such that

\[
(3.3) \quad \frac{\partial s^k(\lambda h, mh)}{\partial \eta} - \frac{1}{h} \left[ \Delta_2 - \frac{1}{2} \Delta_2 + \frac{1}{3} \Delta_2 \right] s^k(\lambda h, (m-3)h) = E_{k,\lambda, m}^\eta,
\]
where \[ E_{k\ell m}^{\xi} = \frac{h^3}{4} \frac{\partial^4 s^k(\xi, \alpha_\mu)}{\partial \xi^4}. \]

**Proof:** Let \( \eta = mh \), \( m \) constant. Then \( f \) is a function of the one variable \( \xi \) and hence we can apply lemma 3.1 with \( q = 3 \) and \( x_\mu = (\ell-3+\mu)h \), \( \mu = O(1) \) and get

\[
E_{k\ell m}^{\xi} = \frac{1}{4!} \frac{\partial^4 s^k(\alpha, mh)}{\partial \xi^4} L'(\xi),
\]

however,

\[
L'(\xi) = (\xi - (\ell-2)h)(\xi - (\ell-1)h)(\xi - \ell h) + \\
(\xi - (\ell-3)h)(\xi - (\ell-1)h)(\xi - \ell h) + \\
(\xi - (\ell-3)h)(\xi - (\ell-2)h)(\xi - \ell h) + \\
(\xi - (\ell-3)h)(\xi - (\ell-2)h)(\xi - (\ell-1)h),
\]

and thus

\[
L'(\ell h) = 3!h^3. \quad \text{Therefore}
\]

\[
E_{k\ell m}^{\xi} = \frac{h^3}{4} \frac{\partial^4 s^k(\alpha, mh)}{\partial \xi^4},
\]

which establishes (3.2). Equation (3.3) is proved in a completely analagous manner.

We now state and prove a theorem giving the truncation error of our predictor operator.

**Theorem 3.1:** Let \( \tilde{Z}(\xi, \eta) \) be the actual solution of (2.6) and assume \( Z \) has continuous fourth partial derivatives. Then the truncation error in using (2.14) to approximate (2.6) and (2.8) at the point \((\ell, m)\) is given by
where $\alpha_1$ is a point on the $\xi$-interval $[(\lambda-3)h, \lambda h]$ on the line $\eta = mh$ and $\alpha_2$ is a point on the $\eta$-interval $[(m-3)h, mh]$ on the line $\xi = \lambda h$.

Proof: Using the results of lemma 3.2 we write

$$\sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) \frac{\partial^2 \tilde{z}_{\lambda-3m}}{\partial \xi} = \sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) (\Delta_1 - \frac{1}{2} \Delta_2 + \frac{1}{3} \Delta_3) \tilde{z}_{\lambda-3m} +$$

$$h \sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) E_{iklm} = 0, \quad i = 1(1)3; \quad (3.5)$$

$$\sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) \frac{\partial^2 \tilde{z}_{\lambda-3m}}{\partial \eta} = \sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) (\Delta_2 - \frac{1}{2} \Delta_2 + \frac{1}{3} \Delta_2) \tilde{z}_{\lambda-3m} +$$

$$h \sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) E_{iklm} = 0, \quad i = 4(1)5.$$

Expanding the differences in (3.5) we get

$$\sum_{k=1}^{5} a_{ik}(\tilde{z}_{\lambda-3m}) \left( \frac{1}{3} z_{\lambda}^k - \frac{3}{2} z_{\lambda-1m}^k + 3 z_{\lambda-2m}^k - \frac{11}{6} z_{\lambda-3m}^k \right) +$$
\[ h \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi-3m}) E_{k\xi m}^E = 0, \quad i = 1(1)3; \]

\[ \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi m-3}) \left( \frac{1}{3} Z_{\xi m} - \frac{3}{2} Z_{\xi m-1} + 3 Z_{\xi m-2} - \frac{11}{6} Z_{\xi m-3} \right) + \]

\[ h \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi m-3}) E_{k\xi m}^H = 0, \quad i = 4(1)5, \]

which can be rewritten as

\[ \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi-3m}) \sum_{v=0}^{3} \alpha_v Z_{\xi-vm}^k + 3h \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi m-3}) E_{k\xi m}^E = 0, \]

\[ i = 1(1)3; \]

\[ \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi m-3}) \sum_{v=0}^{3} \alpha_v Z_{\xi m-v}^k + 3h \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi m-3}) E_{k\xi m}^H = 0, \]

\[ i = 4(1)5, \text{ where } \alpha_0 = 1, \alpha_1 = -\frac{9}{2}, \alpha_2 = 9, \alpha_3 = -\frac{11}{2}. \]

We note that the first term on the right in each of equations (3.6) is precisely our predictor operator operating on the actual solution \( Z \); thus the truncation error is

\[ T \varphi_{\xi m}^i = 3h \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi-3m}) E_{k\xi m}^E \]

\[ + \frac{3}{4} h^4 \sum_{k=1}^{5} a_i k (\hat{Z}_{\xi-3m}) \frac{\partial^4 Z_{\xi m}}{4!}, \]

\[ i = 1(1)3; \]
Using equations (3.1) we note the following bounds for the truncation error of the predictor operator as stated in theorem 3.1:

\[ |TP^i_{\lambda m}| < \frac{3}{4} h^4 \sum_{k=1}^{5} |H_u_4| \leq \frac{15}{4} h^4 H_u_4, \quad i = 1(1)5. \]

We will now investigate the truncation error of the corrector operator (2.17). We first state a lemma concerning the error in using backward differences, (2.12b) truncated after third differences to approximate the partial derivatives. This lemma is analogous to lemma 3.2 for forward differences.

**Lemma 3.3:** Let \( s^k(\xi, \eta) \) have continuous fourth partial derivatives. Then for \( \xi = \lambda h, \eta = mh, m \) constant, there exists a constant \( \alpha_1 \) on the \( \xi \)-interval \([(\lambda-3)h, \lambda h]\) along the line \( \eta = mh \) such that

\[ \frac{\partial s^k(\lambda h,mh)}{\partial \xi} = \frac{1}{h} \left[ v_1 + \frac{1}{2} v_2 + \frac{1}{3} v_3 \right] s^k(\lambda h,mh) = Ec^\xi_{k\lambda m}, \]

where \( Ec^\xi_{k\lambda m} = \frac{4}{h^4} \frac{\partial s^k(\alpha_1,mh)}{\partial \xi^4} \). And for \( \xi = \lambda h, \lambda \) constant and \( \eta = mh \) there exists a constant \( \alpha_2 \) on the \( \eta \)-interval \([(m-3)h,mh]\) on the line \( \xi = \lambda h \) such that

\[ \frac{\partial s^k(\lambda h,mh)}{\partial \eta} = \frac{1}{h} \left[ v_2 + \frac{1}{2} v_2 + \frac{1}{3} v_3 \right] s^k(\lambda h,mh) = Ec^\eta_{k\lambda m}, \]
where
\[ EC_{klm} = \frac{h^4}{4} \frac{\partial^4 Z^k(\xi, \eta, \alpha_2)}{\partial \eta^4}. \]
The proof is along the same lines as lemma 3.2 and will be omitted.

**Theorem 3.2:** Let \( \tilde{Z}(\xi, \eta) \) be the actual solution of (2.6) and assume \( \tilde{Z} \) has continuous fourth partial derivatives. Then the truncation error in using (2.17) to approximate (2.6) and (2.8) for the solution at the \((\lambda, m)\) point is given by

\[
TC_{\lambda m}^i = \frac{3h^4}{22} \sum_{k=1}^{5} a_{ik}(\tilde{Z}_{\lambda m}) \frac{\partial^4 Z^k}{\partial \eta^4}, \quad i = 1(1)3;
\]

\[
TC_{\lambda m}^i = \frac{3h^4}{22} \sum_{k=1}^{5} a_{ik}(\tilde{Z}_{\lambda m}) \frac{\partial^4 Z^k}{\partial \eta^4}, \quad i = 4(1)5,
\]

where \( \alpha_1 \) is a point on the \( \xi \)-interval \([(\lambda-3)h, \lambda h] \) on the line \( \eta = mh \) and \( \alpha_2 \) is a point on the \( \eta \)-interval \([(m-3)h, mh] \) on the line \( \xi = \lambda h \). The proof follows the same lines as the proof of Theorem 3.1.

Again using equations (3.1) we can establish bounds on the corrector truncation error as

\[
TC_{\lambda m}^i \leq \frac{3h^4}{22} \sum_{k=1}^{5} H_u^4 \leq \frac{15}{22} h^4 H_u^4, \quad i = 1(1)5.
\]

We remarked earlier that the predictor operator yields a linear system to solve. In the next lemma we deal with the solvability of the system (2.14).

**Lemma 3.4:** Given that

a) (3.1) holds,
b) \(|\det(a^i_k(Z))| \geq d > 0 \) for \(Z \in \overline{U}\),

c) \(\|\hat{Z}_{m'} - \hat{Z}_{m'}\| < ch^p \) with \(p \geq 1\) for \(\lambda' + m' < \lambda + m\)

then (2.14) is solvable for the \(s_{l,m}^k\) if \(h\) is sufficiently small.

**Proof:** (Due to a proof of a more general lemma by Stetter, [13]).

Assumption b) guarantees the existence of an \(e > 0\) for which

\[
(3.12) \quad |\det(a^i_k(Z) + e^{i_k})| \geq \frac{d}{2}
\]

if \(|e^{i_k}| \leq e, Z \in \overline{U}\). We now consider \(a^i_k(\hat{Z}_{m-3}), i = 1(1)3\) and \(a^i_k(\hat{Z}_{m-3}), i = 4(1)5\). By the differentiability of \(a^i_k\) we have

\[
|a^i_k(\hat{Z}_{m-3}) - a^i_k(\hat{Z}_{m-3})| \leq |V a^i_k(Z) \cdot (\hat{Z}_{m-3} - \hat{Z}_{m-3})|.
\]

Then by a) and c) we get

\[
|a^i_k(\hat{Z}_{m-3}) - a^i_k(\hat{Z}_{m-3})| \leq 5H_1 c h^p.
\]

And thus for some \(h_1 > 0\) and \(h < h_1\) we have

\[
|a^i_k(\hat{Z}_{m-3}) - a^i_k(\hat{Z}_{m-3})| < \frac{e}{2}, \quad i = 1(1)3, k = 1(1)5.
\]

In a completely analogous manner we get

\[
|a^i_k(\hat{Z}_{m-3}) - a^i_k(\hat{Z}_{m-3})| < \frac{e}{2}, \quad i = 4(1)5, k = 1(1)5
\]

for some \(h_2 > 0\) and \(h < h_2\). Also since \(\hat{Z}\) is continuous there exists an \(h_3 > 0\) such that for \(h < h_3\) we have

\[
\|\hat{Z}_{m-3} - \hat{Z}_{m-3}\| < \frac{e}{2H_1}
\]

and thus

\[
|a^i_k(\hat{Z}_{m-3}) - a^i_k(\hat{Z}_{m-3})| \leq |V a^i_k(Z) \cdot (\hat{Z}_{m-3} - \hat{Z}_{m-3})|\]

\[
\leq H_1 \frac{e}{2H_1} = \frac{e}{2}, \quad i = 4(1)5, k = 1(1)5.
\]

Now for \(h < h_4 = \min(h_1, h_2, h_3)\) consider.
In determinant (3.13) we add and subtract a term from each element as follows:

\[
\begin{pmatrix}
 a_1^{(i)}(x_{lm}^m) & \ldots & a_k^{(i)}(x_{lm}^m)
\end{pmatrix}
\]

In chapter one we mentioned that the non-linear system arising from our corrector (2.17) would be solved for \( \hat{x}_{lm}^m \) by iteration using a predicted value \( \hat{x}_{lm}^m \) from (2.14). We will prove the existence of a unique solution of (2.17) in the vicinity of \( \hat{x}_{lm}^m \). The following lemma is adapted from a general theorem by [13] who in turn used a fixed point theorem of the type due to Weisinger, ([2] pp 36-38).
Lemma 3.5: Given that

a) (3.1) holds,

b) \[ \| \hat{S}_{\varphi}^{m'} - \hat{z}_{\varphi}^{m'} \| \leq c h^p \text{ with } p \geq 1 \text{ for } \varphi + m' < \varphi + m, \]

c) \[ |\text{det}(a^{ik}(\hat{z}))| \geq d > 0 \text{ for } \hat{z} \in \vec{U}, \]

d) \[ \| \hat{S}_{\varphi}^{m} - \hat{z}_{\varphi}^{m} \| \leq c_0 h^p \text{ with } 1 \leq p_0 \leq p. \]

Then within a certain neighborhood, \( \vec{E}_{\varphi}^{m} \) of \( \hat{z}_{\varphi}^{m} \), there exists a unique solution \( \hat{S}_{\varphi}^{m} \) of (2.17) if \( h \) is sufficiently small.

**Proof:** Using the abbreviations

\[ U(\hat{S}) = (a^{ik}(\hat{S})) \]
\[ U_{ik}^{jk}(\hat{S}) = 0 \ldots 0 \]
\[ = \begin{array}{ccc}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & & \ddots \\
\end{array} \]
\[ a^{41}(\hat{S}) \ldots a^{45}(\hat{S}) \]
\[ a^{51}(\hat{S}) \ldots a^{55}(\hat{S}) \]

\[ \tilde{U}(\hat{S}) = - \left[ U(\hat{S}) \right]^{-1} U_{ik}(\hat{S}) = (\hat{a}^{ik}(\hat{S})) \]

we may write (2.17) as

\[ \hat{S}_{\varphi}^{m} = - \sum_{v=1}^{3} \beta_v \hat{S}_{\varphi}^{m-v} + U(\hat{S}_{\varphi}^{m}) \Delta \hat{S}_{\varphi}^{m} = \vec{E}_{\varphi}^{m} \left( \hat{S}_{\varphi}^{m} \right) \]

with

\[ \Delta \hat{S}_{\varphi}^{m} = \sum_{v=1}^{3} \beta_v (\hat{S}_{\varphi}^{m-v} - \hat{S}_{\varphi}^{m-v}). \]

We note that

\[ \| \Delta \hat{S}_{\varphi}^{m} \| \leq c_\Delta h, \text{ } h < h_1 \text{ for some } h_1 > 0. \]

The proof of this inequality depends on hypothesis a) i.e.
\[
\left\| \Delta \bar{s}_{\lambda m} \right\| \leq \sum_{v=1}^{3} |\beta_v| \left( \left\| \bar{s}_{\lambda m} - \bar{s}_{\lambda v m} \right\| + \left\| \bar{s}_{\lambda v m} - \bar{s}_{\lambda v m} \right\| \right) \\
\leq \sum_{v=1}^{3} |\beta_v| \left( \frac{\partial \bar{s}_{\lambda m}}{\partial v} \right) |v| \| + \left( \frac{\partial \bar{s}_{\lambda v m}}{\partial v} \right) |v| \| \\
\left\| \Delta \bar{s}_{\lambda m} \right\| \leq \sum_{v=1}^{3} |\beta_v| 2u_1 |v| \\
\left\| \Delta \bar{s}_{\lambda m} \right\| \leq c_\Delta h.
\]

Thus if \( C_\Delta = \sum_{v=1}^{3} |\alpha_v| 2u_1 |v| \) then \( \left\| \Delta \bar{s}_{\lambda m} \right\| \leq c_\Delta h. \)

The existence of \( U^{-1} \) for sufficiently small \( h \), i.e. \( h < h_2 \) for \( h_2 > 0 \) and \( \| S_{\lambda m} - Z_{\lambda m} \| < c_1 h \) is proved like lemma 3.5. Assumptions a) and b) establish the boundedness of the \( a_{ik} \)'s and their partial derivatives. The bound on the partials is denoted by \( H_1 \) and on the elements by \( H_1 \), also we will assume \( H_1 > H_1 \).

To show that
\[
\bar{s}_{\lambda m} = \bar{P}_{\lambda m}(\bar{s}_{\lambda m})
\]
has a unique solution in a certain region \( \bar{\Omega}_{\Delta m} \), we remark that

\[
\left\| \vec{F}_{\Delta m}(s_{\Delta m}) - \vec{S}_{\Delta m} \right\| \leq c^0 h
\]

for \( h < h_3 \) for some \( h_3 > 0 \), and some \( c^0 > 0 \). To see this consider

\[
\left\| \vec{F}_{\Delta m}(s_{\Delta m}) - \vec{S}_{\Delta m} + \vec{Z}_{\Delta m} - \vec{F}_{\Delta m}(\vec{Z}_{\Delta m}) - \vec{C}_{\Delta m} \right\|
\]

\[
\leq \left\| \vec{F}_{\Delta m}(s_{\Delta m}) - \vec{F}_{\Delta m}(\vec{Z}_{\Delta m}) \right\| + \left\| \vec{S}_{\Delta m} - \vec{Z}_{\Delta m} \right\| + \left\| \vec{C}_{\Delta m} \right\|
\]

\[
\leq \left\| - \sum_{v=1}^{3} \beta_v S_{\Delta - \Delta m} + \vec{u}(s_{\Delta m}) \right\| + \left\| \vec{u}(s_{\Delta m}) \right\| + \sum_{v=1}^{3} \beta_v S_{\Delta - \Delta m} - \vec{u}(\vec{Z}_{\Delta m}) \Delta \vec{Z}_{\Delta m} \right\|
\]

\[
\leq \sum_{v=1}^{3} |\beta_v| \left\| \vec{Z}_{\Delta - \Delta m} - \vec{S}_{\Delta - \Delta m} \right\| + \left\| \vec{u}(s_{\Delta m}) \right\| \left\| \Delta \vec{S}_{\Delta m} \right\| + \left\| \vec{u}(\vec{Z}_{\Delta m}) \right\| ch^p + \frac{15}{22} h^4 \mu_4
\]

\[
\leq C_0 h^p + \frac{15}{22} h^4 \mu_4
\]

\[
\leq Ch^p \sum_{v=1}^{3} |\beta_v| + 5H_1 C_{\Delta h} + 5H_1 C_{\Delta h} + C_0 h^p + \frac{15}{22} h^4 \mu_4
\]

\[
\leq C^0 h
\]

where

\[
c^0 = Ch^{p-1} \sum_{v=1}^{3} |\beta_v| + 10H_1 C_{\Delta} + C_0 h^{p-1} + \frac{15}{22} h^3 \mu_4.
\]

We now choose for the neighborhood \( \bar{\Omega}_{\Delta m} \in E^5 \) the complete region
\[ \overline{p}_{\lambda m} = \left\{ \tilde{s} | \| \tilde{s} - \overline{s}_{\lambda m} \| \leq (c_0 + 3c^0)h_v \right\} \]

where \( h_v \) satisfies

\( \text{a)} \quad h_v = \min(h_1, h_2, h_3, h_4, h_5) \)

\( \text{b)} \quad (c_0 + 3c^0)h_4 \leq \varepsilon^* \quad \text{i.e.} \quad \overline{p}_{\lambda m} \subset U, \)

\( \text{c)} \quad 5h_1c_h \leq L < \frac{1}{2}. \)

Then \( \overline{F}_{\lambda m} \) satisfies the Lipschitz condition in \( \overline{p}_{\lambda m} \) with Lipschitz constant \( L \), i.e. if \( \overline{s}_{\lambda m} \) and \( \overline{s}_{\lambda m}^{(0)} \) are two arbitrary vectors in \( \overline{p}_{\lambda m} \) then

\[ \| F_{\lambda m}^{(1)}(\overline{s}_{\lambda m}) - F_{\lambda m}^{(2)}(\overline{s}_{\lambda m}) \| \leq \| (U(\overline{s}_{\lambda m}) - U(\overline{s}_{\lambda m}^{(0)}) \Delta \overline{s}_{\lambda m} \| \]

\[ \leq \| U(\overline{s}_{\lambda m}) - U(\overline{s}_{\lambda m}^{(0)}) \| \| \Delta \overline{s}_{\lambda m} \| \]

\[ = \| (\nabla \overline{F}(\overline{s}_{\lambda m}^{(0)}) \cdot (\overline{s}_{\lambda m} - \overline{s}_{\lambda m}^{(0)}) \| \| \Delta \overline{s}_{\lambda m} \| \]

\[ \leq \max_{1 \leq i \leq 5} \sum_{j=1}^{5} |\nabla \overline{F}_{ij}(\overline{s}_{\lambda m}^{(0)}) \cdot (\overline{s}_{\lambda m} - \overline{s}_{\lambda m}^{(0)}) | c_{\Delta h_v} \]

\[ \leq \max_{1 \leq i \leq 5} \sum_{j=1}^{5} \| \nabla \overline{F}_{ij}(\overline{s}_{\lambda m}^{(0)}) \| \| \overline{s}_{\lambda m} - \overline{s}_{\lambda m}^{(0)} \| c_{\Delta h_v} \]

\[ \leq 5h_1 \| \overline{s}_{\lambda m} - \overline{s}_{\lambda m}^{(0)} \| c_{\Delta h_v} \leq L \| \overline{s}_{\lambda m} - \overline{s}_{\lambda m}^{(0)} \|. \]

We define the matrix norm as \( \| A \| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|. \)

We also note that

\[ \| (n+1) \overline{s}_{\lambda m} - (n) \overline{s}_{\lambda m} \| = \| F_{\lambda m}^{(n)}(\overline{s}_{\lambda m}) - F_{\lambda m}^{(n-1)}(\overline{s}_{\lambda m}) \| \leq L \| (n) \overline{s}_{\lambda m} - (n-1) \overline{s}_{\lambda m} \| \]

\[ \leq L^n \| (1) \overline{s}_{\lambda m} - (0) \overline{s}_{\lambda m} \|. \]
hence
\[ \left\| \begin{pmatrix} (n+1) & (n) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \leq L^n \left\| \begin{pmatrix} (0) & (0) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \leq L^n c_{\text{h}}. \]

Thus for \( m > n \) we have
\[ \left\| \begin{pmatrix} (m) & (n) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \leq \sum_{i=n}^{m-1} \left\| \begin{pmatrix} (i+1) & (i) \\ S - S \end{pmatrix} \right\| \leq \sum_{i=n}^{m-1} L^i \left\| \begin{pmatrix} (1) & (0) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \]
\[ \leq c_{\text{h}} \sum_{i=0}^{m-n-1} L^{i+n}; \text{ however since } L < \frac{1}{2} \]
\[ \left\| \begin{pmatrix} (m) & (n) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \leq c_{\text{h}} L^{n-1} \frac{L}{1-L}. \]

Hence
\[ \lim_{m,n \to \infty} \left\| \begin{pmatrix} (m) & (n) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| = 0; \]

and a solution \( S_{\text{lm}} \) exists.

We also note that if \( n = 1 \) we have
\[ \left\| \begin{pmatrix} (m) & (1) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \leq c_{\text{h}} \frac{L}{1-L} < 2 c_{\text{h}}. \]

\[ \text{hence } \left\| \begin{pmatrix} (m) & (1) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| = \left\| \begin{pmatrix} (m) & (1) & (1) & (0) & (0) \\ S_{\text{lm}} - S_{\text{lm}} + S_{\text{lm}} - S_{\text{lm}} + S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| \]
\[ \leq \left\| \begin{pmatrix} (m) & (1) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} (0) & (1) & (0) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} (1) & (0) \\ S_{\text{lm}} - S_{\text{lm}} \end{pmatrix} \right\|. \]
\[ \leq 2C^0h_v + C_0h_v + C^0h_v \leq (C_0 + 3C^0)h_v \]

and therefore all iterates stay within the neighborhood \( \bar{B}_{\lambda m} \).

To establish the uniqueness of the solution \( \bar{S}_{\lambda m} \) we assume a second solution \( \bar{T}_{\lambda m} \) exists. Then we have \( \bar{T}_{\lambda m} = \bar{F}_{\lambda m}(\bar{T}_{\lambda m}) \) and \( \bar{S}_{\lambda m} = \bar{F}_{\lambda m}(\bar{S}_{\lambda m}) \).

However,

\[ ||\bar{T}_{\lambda m} - \bar{S}_{\lambda m}|| = ||\bar{F}_{\lambda m}(\bar{T}_{\lambda m}) - \bar{F}_{\lambda m}(\bar{S}_{\lambda m})|| \leq L \left| \bar{T}_{\lambda m} - \bar{S}_{\lambda m} \right| \]

and since \( L < \frac{1}{2} \) we must have \( \bar{T}_{\lambda m} = \bar{S}_{\lambda m} \).
IV. STABILITY OF THE PREDICTOR, CORRECTOR AND ALGORITHM

Chapter four is primarily concerned with the stability of our algorithm, however before stating or proving a stability theorem we will define several terms.

In a characteristic finite-difference method - as we are considering - only values along $x = l_o$ and $n = m_o$ are used for the computation of a value at $(l_o, m_o)$.

Definition. Degree N: We call a characteristic finite-difference method of "degree N" if only values at the N preceding points on each characteristic contribute to the value at $(l_o, m_o)$.

From the description of our algorithm in chapter two we see that it is a method of "degree 3."

At this time we state a general finite-difference technique from which both our predictor and our corrector may be obtained. We then prove a stability theorem for this general technique from which we can evaluate the stability of our predictor and corrector.

\[
\sum_{k=1}^{k} a_{ik} \left( \sum_{M=0}^{N_1} \gamma_{\mu} S_{l-M}^{\mu} \right) \left[ \sum_{v=0}^{N_2} \alpha_{v} S_{l-v}^{k} \right] = 0, \quad i = 1(1)k^{'}
\]

\[
\sum_{k=1}^{k} a_{ik} \left( \sum_{M=0}^{N_1} \gamma_{\mu} S_{l-M}^{\mu} \right) \left[ \sum_{v=0}^{N_2} \alpha_{v} S_{l-M-v}^{k} \right] = 0, \quad i = k'(1)k.
\]

We assume the following relation holds for the coefficients $\alpha_{v}$, with $f(x)$ sufficiently differentiable

\[
\sum_{v=0}^{N_2} \alpha_{v} f(x-vh) - hf'(x) = \sum_{p_1}^{P_1} \left[ p_1 \right] f(x_1), \quad p_1 \geq 2,
\]
with \( x^* \) in the interval \([x - N\varphi h, x]\).

We assume that already the starting values for the computation contain
numerical errors:

\[
(4.3) \quad S^k_{\lambda\mu} = Z^k_{\lambda\mu} + \Theta^k_{\lambda\mu},
\]

for \( \lambda + \mu = 1 + \varphi h, \lambda = 1(1)5, \mu = n - 1, n - 2, \ldots, 0 \) if we are com-
puting the value on the \( n^{th} \) diagonal.

Note: We will refer to the initial curve as the zero diagonal, hence the
first values actually computed are on the third diagonal, and the point
\((\lambda, \mu)\) is on the \( \lambda + \mu - L^* \) diagonal where \( L^* = \frac{1}{h^*} \).

Furthermore, the numerical solution of (4.1) will introduce errors
which we denote by

\[
(4.4) \quad \phi^i_{\lambda\mu} = \sum_{k=1}^{K} a_{i^k} \left( \sum_{\mu=0}^{N_1} \gamma_{\mu} S^k_{\lambda-\varphi\mu} \right) \left[ \sum_{\nu=0}^{N_2} \alpha_{\nu} S^k_{\lambda-\varphi\nu} \right], \quad i = 1(1)k^i
\]

\[
(4.4a) \quad |\phi^i_{\lambda\mu}| \leq kh^{p+1} \quad \text{and} \quad |\Theta^i_{\lambda\mu}| \leq kh^{p'}.
\]

Definition. Class \( \beta(k, p') \): The numerical result of an integration
toolgorithm for a given problem e.g. \((2.6)/(2.8)\) is called of "class \( \beta(k, p') \)";
k > 0, \( p' > 0 \) if the following relations hold for \( 0 < h < h_0 \) (for some \( h_0 > 0 \)):

\[
|\theta_{\lambda m}^i| \leq kh^{p'}, \quad |\phi_{\lambda m}^i| \leq kh^{p'+1},
\]
for all \( (\lambda, m) \in D_h^* \).

The following definition of "Stable convergence" was introduced in a similar manner by Dahlquist [4] for ordinary differential equations.

**Definition. Stable convergence:** Let \( S_{\lambda m}^k \) be obtained from the initial values (4.3) by the procedure (4.1) with a mesh size \( h \), with computing errors (4.4). Then the algorithm is called stably convergent if there exists a function \( P^*(h; k, p') \), defined for \( 0 \leq h \leq h_0 \), continuous and increasing with respect to \( h \), for which

\[
\sup \sup \| S_{\lambda m}^k - Z_{\lambda m} \| \leq P^*(h; k, p')
\]

and

\[
\beta(k, p') (\lambda, m) \in D_h^*
\]

and

\[
\lim_{h \to 0^+} P^*(h; k, p') = 0.
\]

**Note:** The first supremum is extended over all results of class \( \beta(k, p') \); however since our algorithm is assumed to be using back values of order three and since both our predictor and corrector are of order three we are only interested in results of class \( \beta(k, 3) \) with \( k \) the maximum of the constants, \( \frac{15}{4} HU_4 \) and \( \frac{5}{2} HU_2 \).

**Definition. Stable convergence of order \( p \):** A method is called stably convergent of order \( p \), if it is stably convergent and if there exists a positive number \( p \) and a constant \( P > 0 \), for which \( P^*(h; k, p') \leq P \cdot h^p \) for
\[ p' \leq p, \quad 0 \leq h \leq h_0, \quad k \leq k_0 \quad \text{(for some } k_0) \]

This situation is often denoted by
\[ \|S - Z\| \leq \overline{P} h^p, \quad \overline{P} \text{ constant.} \]

As a final preliminary before we state and prove our stability theorem consider:

**Lemma 4.1:** Let \( B \) be a linear difference operator with constant coefficients:
\[
B x_n = \sum_{v=0}^{N} a_v x_{n-v},
\]
and let \( x_{\ell_0}, \ell \geq N \) be the solution of
\[
B x_n = e_n
\]
with the initial values \( x_{\ell_0} = \overline{x}_0, \quad \ell = 0(1)N-1 \). Then, if the eigenvalues, \( Z \mu \) of \( B \), i.e. the zeros of the polynomial
\[
B_0 (Z) = \sum_{v=0}^{N} a_v Z^{N-v},
\]
are such that their magnitude does not exceed unity and the zeros whose magnitude equals unity are simple,

\[
|x_{\ell_0}| \leq M_1 \sum_{v=0}^{N-1} |\overline{x}_v| + \sum_{n=N}^{\ell} |e_n|,
\]
for some constant \( M_1 \).

The proof of this lemma may be found in [8] and will be omitted.

The following stability theorem is adapted from a more general
Theorem 4.1: Let the following assumptions hold:

a) with regard to the problem (2.6)/(2.8)
   a1) \( Z^k \in C^1 \) and \( Z \in \overline{U} \) on the initial curve, \( C \);
   a2) \( a_i^k(Z) \in C^1 \) for \( Z \in \overline{U} \), with \( p \) as defined in (4.2)
   a3) \( |\det(a_{ik}(\overline{Z}))| > d > 0 \) for \( \overline{Z} \in \overline{U} \);

b) with regard to the finite-difference method (4.1) the approximation relations (4.2) hold.

Then the following condition is both sufficient and necessary for the stable convergence of the numerical procedure (4.1):

The zeros \( Z_{\mu}, \mu = 1(1)N \) of

\[
P(Z) = \sum_{v=0}^{N} a_v Z^{N-v}
\]

satisfy \( |Z_{\mu}| < 1 \), and \( P'(Z_{\mu}) = 0 \) if \( |Z_{\mu}| = 1 \). Furthermore, if the result of the computations is of class \( \beta(k,p') \) with \( p' > p = (p_1-1) \) then the convergence is of order \( p \).

Note: The method of proof of sufficiency is that of displaying a function \( P^*(h; k,p') \) as described in the definition of stable convergence. The proof of necessity takes the form of counterexamples.

Proof: (We prove the more general result concerning the order of the stability).

We use the following for difference operators along a characteristic:

\[
A_k^l S_{lm}^k \equiv \sum_{v=0}^{N_2} \alpha_v S_{l-vm}^k, \quad \overline{A_k^l S_{lm}^k} \equiv \frac{\nu_o}{\alpha_o} \sum_{v=1}^{N_2} \alpha_v S_{l-vm}^k.
\]
The meaning of $A_m$, $A_m$, $\Gamma_m$, $\Gamma_m$ is analogous. Furthermore we set

\begin{equation}
\Phi^k_{\lambda m} = S^k_{\lambda m} - Z^k_{\lambda m}
\end{equation}

"Interior points" in the Taylor formula sense are marked by $\wedge$.

We deal only with the equations along $n = \text{constant}$ ($i \leq k'$) since the corresponding relations for $\xi = \text{constant}$ arise in an analogous way. Since the second subscript remains constant and since we are dealing only with the sub-$\lambda$ operations of (4.7) we do not write the subscript i.e.

$S^k_{\lambda m} = S^k_\lambda$, $A_\lambda = A$.

We form the difference

\begin{equation}
\Delta^i_\lambda = \sum_{k=1}^{K} \left\{ a^{ik}(\Gamma S_\lambda) (A S^k_\lambda) - a^{ik}(\Gamma Z^k_\lambda) (A Z^k_\lambda) \right\}
\end{equation}

\begin{equation}
= \sum_{k=1}^{K} \left\{ [a^{ik}(\Gamma S_\lambda) - a^{ik}(\Gamma Z^k_\lambda)] A Z^k_\lambda + a^{ik}(\Gamma \wedge_\lambda) [A S^k_\lambda - A Z^k_\lambda] \right\}
\end{equation}

\begin{equation}
= \sum_{k=1}^{K} \left\{ [A a^{ik}(\wedge_\lambda) + \Gamma b^{ik}_\lambda] \Phi^k_\lambda \right\},
\end{equation}

where

\begin{equation}
a^{ik} = a^{ik}(\Gamma S_\lambda) \quad \text{and} \quad b^{ik}_\lambda = \sum_{j=1}^{k} v_{a}^{ij}(\Gamma S_\lambda) A Z^j_\lambda.
\end{equation}
Note: We use the bracket around the subscript to mean that the operator ignores this term.

Now we define

\[ (4.11) \quad c_{i k} = a_{i k} + \frac{\gamma}{\alpha} b_{i k} \]

and equation (4.9) becomes

\[ (4.12) \quad \Delta_l = A \sum_{k=1}^{K} c_{i k} v_{l}^k - \sum_{k=1}^{K} b_{i k} v_{l}^k + \sum_{k=1}^{K} b_{i k} v_{l}^k. \]

Considering the errors (4.3) and (4.4) we write

\[ (4.13) \quad \Delta_l = \phi_l + \theta_l = \sigma_l. \]

We use (4.13) to rewrite (4.12) as

\[ (4.12) \quad A \sum_{k=1}^{K} c_{i k} v_{l}^k = \sum_{k=1}^{K} [(\bar{A} - \bar{A}) b_{i k} v_{l}^k] + \sigma_l, \quad i = 1(1)K. \]

Note that the right side of (4.14) does not contain \( v_{l}^k \).

An expression similar to (4.14) would be obtained along the characteristic \( \xi = \text{constant} \).

We note that the use of the mean value theorem in (4.9) assumes that the \( S_{l} \) also lie in \( \bar{U} \). That this is the case is shown later.

We now regard (4.14) as a system of linear difference equations for the linear combinations

\[ \sum_{k=1}^{K} c_{i k} v_{l}^k; \]
with $n = \lambda$ along the $\eta = \text{constant characteristic}$, $i = 1(1)k'$ and $n = m$ along the $\xi = \text{constant characteristic}$, $i = k'(1)k$. The inhomogeneities $e^i_{\lambda,m}$ contain $V^k_{\lambda',m'}$ with $\lambda' + m' < \lambda + m$ only, i.e. only values of $V^k$ on the diagonals preceding the $\xi + \eta = (\lambda + m - L^*)$ diagonal.

We solve the finite-difference equations (4.15) by superposition of solutions of a set of finite-difference equations with constant coefficients. To see this let

$$
A^i_{\lambda,m} = \sum_{k=1}^{K} C^i_{\lambda,m-n} V^k_{\lambda,n}
$$

(4.16)

$$
A^{-i}_{\lambda,m} = \sum_{k=1}^{K} C^{-i}_{m-n,m} V^k_{m+n,m}
$$

where $n$ denotes the diagonal on which $V^k$ is evaluated. Thus for $n = \lambda + m - L^*$

$$
A^i_{\lambda,m-L^*} = e^i_{\lambda,m}
$$

(4.17)

$$
A^{-i}_{\lambda,m-L^*} = e^{-i}_{\lambda,m}
$$

are precisely the difference equations we want to solve. However, as noted above the inhomogeneities contain $V^k_{\lambda',m'}$'s, $\lambda' + m' < \lambda + m$ and the only such values that we know are those to the left of the first computed diagonal $N = \max(N_1, N_2)$. Hence, we first solve the homogeneous difference-
equations

\[ A^i_{\lambda(N+1, \lambda m)} = 0, \quad i = 1(1)K', \]

\[ A^i_{m(N+1, \lambda m)} = 0, \quad i = K'(1), \]

with initial conditions

\[ N^i_{\lambda, \lambda m} = \sum_{k=1}^{K} C^{i k}_{\lambda, \lambda-N} V^{k}_{\lambda, \lambda-N'}, \quad i = 1(1)K', \]

and

\[ N^{i}_{\lambda, \lambda m} = \sum_{k=1}^{K} C^{i k}_{m-N, m} V^{k}_{m+N, m}, \quad i = K'(1)K, \]

\[ v \in [N - N_2](1)[N - 1], \]

respectively. We note that the initial conditions only involve \( V^k_{\lambda m} \)'s from diagonals to the left of the \( N \) diagonal and thus are assumed known. We, therefore can solve these difference-equations and thus we know

\[ N^i_{\lambda, \lambda m} = \sum_{k=1}^{K} C^{i k}_{\lambda, \lambda-N} V^{k}_{\lambda, \lambda-N'}, \quad i = 1(1)K', \]

\[ N^{i}_{\lambda, \lambda m} = \sum_{k=1}^{K} C^{i k}_{m-N, m} V^{k}_{m+N, m}, \quad i = K'(1)K, \]

for all points \((\lambda, m)\) on the \( N \) diagonal. Using these \( k \) equations we solve for the \( k \) unknown values of \( V^k_{\lambda m} \) on the \( N \) diagonal. Then we consider the finite-difference equations

\[ A^i_{\lambda(N+1, \lambda m)} = 0, \quad i = 1(1)K', \]

\[ A^i_{m(N+1, \lambda m)} = 0, \quad i = K'(1)K, \]

with initial conditions
and \[ N+1^i_{\lambda \nu} = \sum_{k=1}^{K} c_{ik} \varepsilon_{m+n,m} \varepsilon_{m+n,m} \]

\[ i = K(1)K'. \]

\( v = [N - N_2 + 1](1)[N], \) respectively. We note that the initial conditions involve the values of \( \varepsilon_{\lambda \mu} \) on the \( N \) diagonal but these we have just found in the preceding problem. Therefore this problem is solvable and we use the solutions to find the \( \varepsilon_{\lambda \mu} \)'s on the \( N + 1 \) diagonal. Continuing in this manner we find the \( \varepsilon_{\lambda \mu} \)'s on all diagonals through the \( (J_m - 1)^{st} \). We then consider the finite difference equations

\[ A_\lambda (\varepsilon_{\lambda + m - L^{st}}^i) = e_{im} \]

\[ A_\lambda (\varepsilon_{\lambda + m - L^{st}}^i) = e_{im} \]

These are the equations we actually wanted to solve and since we now know all the \( \varepsilon_{\lambda \mu} \)'s on preceding diagonals the \( e_{im} \) term is a known constant and the problem is solvable. The above paragraphs concerning the solution of (4.15) can be written analytically:

Solve

\[ A_\lambda (\varepsilon_{\lambda + m - L^{st}}^i) = e_{im} \]

\[ A_\lambda (\varepsilon_{\lambda + m - L^{st}}^i) = e_{im} \]

with initial conditions

\[ n^i_{\lambda \nu} = \sum_{k=1}^{K} c_{ik} \varepsilon_{\lambda - n} \varepsilon_{\lambda - n} \]

\[ i = 1(1)K'. \]

\[ n^i_{\lambda \nu} = \sum_{k=1}^{K} c_{ik} \varepsilon_{m+n,m} \varepsilon_{m+n,m} \]

\[ i = K'(1)K. \]
\[ v = [n - N_2](n - 1); \quad n = [N](1)[\ell + m - L^*] \]

Then we can write the solution of (4.15) as

\[
(4.18) \quad \sum_{k=1}^{K} C_{(\ell m)}^{i k} \psi_{\ell m}^k = \begin{cases} 
\sum_{n=N}^{\ell + m - L^*} n^i t_n^i & i = 1(1)k', \\
\sum_{n=N}^{\ell + m - L^*} n^i t_n^i & i = k'(1)k.
\end{cases}
\]

Once again we return to only considering the equations along \( \eta = \text{constant} \), \( i = 1(1)k^1 \) and also delete certain subscripts as before.

Applying lemma (4.1) and the hypothesis of the theorem concerning the zeros of the polynomial, \( P(z) \) to the solution (4.18) of (4.15) we get

\[
(4.19) \quad \left| \sum_{k=1}^{K} C_{(\ell m)}^{i k} \psi_{\ell m}^k \right| \leq \sum_{n=N}^{\ell + m - L^*} |n| t_n^i \\
\leq M_1 \sum_{j=N}^{\ell + m - L^*} \left[ \sum_{v=j-N_2}^{j-1} \left| j t_v^i \right| + \left| e_j^i \right| \right]
\]

where \( M_1 \) is some positive constant.

We introduce bounds for the errors:

\[
(4.20) \quad |\psi_{\ell m}^k| \leq V_n, \quad \ell + m - L^* = n, \quad k = 1(1)k'.
\]

The sequence of \( V_n \) is assumed to be non-decreasing.

By using (4.2) and (3.1) and assuming the truncation errors \( \phi_{\ell m}^i \) are such that

\[ |\phi_{\ell m}^i| \leq th^{p+1} \quad \text{for} \quad h < h_1, \]
(for some $h_1 > 0$), $P = p_1 - 1$ we obtain the following estimates for sufficiently small $h$, $h < h_2$, (for some $h_2 > 0$), with each bound $H$ and $J$ of order one:

\[(4.21) \quad |b^{ik}_n| \leq hH_b, \quad |c^{ik}_n| \leq H_c, \quad |e^{i}_n| \leq Jh^{p+1}.\]

The estimates on $e^{i}_n$ make use of the assumption $p' \geq p$ from the definition of stable convergence of order $p$.

From these estimates we get

\[
(4.22) \quad \sum_{v=j-N_2}^{j-1} |j^iv^i| \leq \sum_{v=j-N_2}^{j-1} KH_v^j < N_2KH_v^j < K \leq 2K \leq 2K \frac{1}{k=1}^{j} c^{ik}_n < \sum_{v=j-N_2}^{j-1} KH_v^j < N_2KH_v^j < K \leq 2K \leq 2K \]

The estimates for $|v^i|^2$ obtained by introducing the above estimates into (4.16) may be turned into estimates for the $v^{k}_{lm}$ themselves in the manner in which we obtained the $V_{lm}$'s in solution of (4.15).

By lemma 3.4 and hypothesis A3) the determinant of $(a^{ik}_{lm})$ is bounded away from zero for sufficiently small $h$ i.e. $h < h_3$ (for some $h_3 > 0$) and if $\|s_{lm} - z_{lm}\| \leq R_1h$, for $R_1$ some constant in case $\gamma = 0$. Let $|c^{ik}_{lm} - a^{ik}_{lm}| \leq R_2h$, for $R_2$ some constant by (4.21), (3.1) and the definition of $c^{ik}_{lm}$, therefore we have for sufficiently small $h$, $h < h_4$ (for some $h_4 > 0$), $|\det(c^{ik}_{lm})| \geq d' > 0$ for $(l,m) \in D_h$. Hence $(c^{ik}_{lm})^{-1}$ exists and we need only multiply the estimates for $\sum_{k=1}^{K} c^{ik}_{lm} v^{k}_m$ by the bound for its norm (we are using the "maximum row sums" norm) to obtain the desired estimate for the
Combining terms of equal structure we finally have for sufficiently small \( h \), i.e. \( h < h^* \) (for some \( h^* > 0 \)), with \( D_i \)'s constants of order one, and for \( N \leq \lambda + m - L^* \leq L = \max(\lambda + m - L^*) < \frac{1}{h} \),

\[
(4.23) \quad V_{\lambda - m} \leq hD_2 V_{\lambda - m - 1} + D_1 \sum_{j=N}^{\lambda + m - L^*} V_{j-1} + D_0 h^{p+1}.
\]

The proof of sufficiency is completed when we show that there exist constants \( E_0, E_1 \), such that

\[
(4.24) V_{\lambda + m - L^*} \leq E_0 h^p E_1 h^L
\]

for sufficiently small \( h < h_6 \) (for some \( h_6 > 0 \)) and \( \lambda - m \leq L^* \), because we will choose

\[
P^*(h; \epsilon, p') = E_0 h^p E_1 h^L
\]

which will be a non-decreasing function of \( h \) and

\[
P^*(h; \epsilon, p') \leq E_0 h^p \leq \tilde{P} h^p,
\]

and

\[
\lim_{h \to 0^+} P^*(h; \epsilon, p') = 0.
\]

We prove the existence of \( E_0, E_1 \) by induction:

Let

\[
E_0 = K_0, \quad E_1 = \frac{\lambda n (1 + 2e^{h(N-1)} hD_2 + D_1)}{h}
\]

where \( K_0 \) is defined in the definition of stable convergence of order \( p \).
Setting

\[(4.26) \quad h_7 \leq \frac{E_0}{2D_0}, \]

\[(4.27) \quad h_8 \leq \frac{\epsilon^*}{E_0 E_2}, \quad E_2 = E_0 e^{2E_1}, \]

with \( \epsilon^* \) defined as in (2.11),

\[h_9 \leq \frac{1}{4D_2}, \]

we then define

\[h = \min[h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9] \]

with the \( h_i \)'s defined above.

From (4.4a) and \( p' \geq P \) we have that (4.24) holds for \( V_{N-V}, \quad v = 1(1)N-1 \).

We now make the induction assumption that (4.24) holds for \( V_n', \quad n \leq \ell - m - 1 \) and show that this assumption implies that it holds for \( V_n, \quad n = \ell - m \).

From (4.27) and (2.11), we conclude \( \tilde{S}_{\ell m} \in U \) for \( \ell + m' \leq \ell + m - 1 \).

In case \( \gamma_6 = 0 \), we use lemma 3.5 to conclude that \( \tilde{S}_{\ell m} \in \mathcal{B}_{\ell m} \) for \( \ell' + m' = \ell + m \). These are facts we assumed in using the mean value theorem earlier.

Furthermore, using the fact that the \( V_n \) are non-decreasing along with (4.23) and (4.24) we get

\[V_{\ell + m - L^*} \leq \frac{e^{h(N-1)}h^{p+1}D_0 E_0}{1 - e^{E_1 h}} \frac{1 - e^{E_1 h(\ell + m - L^* - 1)}}{1 - e^{E_1 h}} + \frac{D_1 E_0}{1 - e^{E_1 h}} \frac{h^{p+1}E_0 h^{(N-1)}}{1 - e^{E_1 h}} \]

\[V_{\ell + m - L^*} \leq e^{h(N-1)} \frac{hD_2 + D_1}{e^{E_1 h} - 1} \frac{E_0 h^{p+1}E_1 (\ell + m - L^*)}{1 - e^{E_1 h}} \frac{E_0}{2} \frac{h^{p+1}E_1 (\ell + m - L^*)}{1 - e^{E_1 h}} \]
The necessity of the theorem was demonstrated by Dahlquist, [4] by showing examples of ordinary differential equations that were unstable when the hypothesis was violated.

**Theorem 4.2:** Let the following assumptions with regard to the problem (2.6)/(2.8) hold:

a) \( Z^k \in \mathcal{C}^h \) and \( Z \in \mathcal{U} \) on the initial curve \( \mathcal{C} \),

b) \( a^k(Z) \in \mathcal{C}^h \) for \( Z \in \mathcal{U} \),

c) \( |\text{det}(a^k(Z))| \geq d > 0 \) for \( Z \in \mathcal{U} \).

Then the predictor (2.14) is not stably convergent but the corrector (2.17) is stably convergent.

**Proof:** The predictor (2.14) is written as

\[
\sum_{k=1}^{5} a^k(Z) = 0, \quad i = 1(1)3
\]

\[
\sum_{k=1}^{5} a^k(Z) = 0, \quad i = 4(1)5.
\]

We note that this is the general operator (4.4) if we take \( N_1 = N_2 = 3 \), \( K' = 3 \), \( K = 5 \). We also note that \( \gamma_0 = \gamma_1 = \gamma_2 = 0, \gamma_3 = 1 \) but more important \( \alpha_0 = 1, \alpha_1 = -\frac{9}{2}, \alpha_2 = 9, \alpha_3 = -\frac{11}{2} \) and thus the polynomial \( P(Z) \) written as
\[ P(Z) = z^3 - \frac{9}{2} z^2 + 9z - \frac{11}{2} \]

has zeros \( Z_1 = 1, Z_2 = \frac{7 + \sqrt{391}}{2}, Z_3 = \frac{7 - \sqrt{391}}{2} \). And since \( |Z_1| = |Z_2| > 1 \)
we have by theorem 4.1 that (2.14) is not stably convergent.

The corrector (2.17) has the form

\[
\sum_{k=1}^{5} a_i k (s_{2m}) \sum_{v=0}^{3} \beta_v s_{2-v,m}^k, \quad i = 1(1)3,
\]

\[
\sum_{k=1}^{5} a_i k (s_{2m}) \sum_{v=0}^{3} \beta_v s_{2m-v}^k = 0, \quad i = 4(1)5.
\]

This has the form of the general operator (4.4) also if we take \( N_1 = 0, N_2 = 3, K' = 3; K = 5. \) We see that \( \gamma_0 = 1, \beta_0 = 1, \beta_1 = -\frac{18}{11}, \beta_2 = \frac{9}{11}, \beta_3 = -\frac{2}{11}. \) Also by lemma 3.3 we have that our coefficients \( \beta_i \) satisfy

(4.2) with \( p_1 = 4. \) Assuming sufficiently accurate starting values, we
have by lemmas 3.2 and 3.3 that the numerical result of (2.17) is \( \beta(K, 3), \)
i.e. \( p' = 3. \) The polynomial of theorem 4.1 is

\[ P(Z) = \sum_{v=0}^{3} \beta_v Z^{3-v} \]

or

\[ P(Z) = z^3 - \frac{18}{11} z^2 + \frac{9}{11} z - \frac{2}{11} \]

and has zeros

\[ Z_1 = 1, Z_2 = \frac{7 + \sqrt{391}}{22}, Z_3 = \frac{7 - \sqrt{391}}{22} \]

and \( |Z_i| \leq 1, i = 1(1)3; \) therefore the hypothesis of theorem 4.1 are satis-
fied and (2.17) is stably convergent of order three.
We mentioned earlier that we would make a statement concerning the recommended number of iterations of the corrector. This is often an important consideration when discussing so-called numerical stability.

Since the corrector (2.17) is stably convergent of order 3 this means that it would be wise to continue our iterations only until

\[(4.29) \quad \|S_{\ell m}^{(k)} - S_{\ell m}\| \leq M h^4\]

for some constant \(M\), where \(S_{\ell m}^{(k)}\) denotes the \(k^{th}\) iterative value of the corrector while \(S_{\ell m}\) denotes the actual solution of the corrector. The number of iterations needed to bring this about is noted in the following lemma.

Lemma 4.2: Under the assumptions of lemma 3.5, using (2.14) as predictor and (2.17) as corrector, (4.28) holds after one iteration.

Proof:

\[
\begin{align*}
\|S_{\ell m}^{(k)} - S_{\ell m}\| &= \|F_{\ell m}(S_{\ell m}) - F(S_{\ell m})\| \\
&\leq 25H_1C_\Delta h^{(k-1)} \|S_{\ell m}^{(k)} - S_{\ell m}\| \\
&\leq (25H_1C_\Delta)^K h^K \|S_{\ell m}^{(0)} - S_{\ell m}\| \\
&\leq (25H_1C_\Delta)^K h^K \|S_{\ell m} - Z_{\ell m} + Z_{\ell m} - S_{\ell m}\| \\
&\leq (25H_1C_\Delta)^K h^K (C_0 h^3 + B h^3) \\
&\leq M h^{K+3}.
\end{align*}
\]

Hence if we choose \(K = 1\) we have the desired result (4.28).
V. DISCUSSION OF STARTING TECHNIQUES AND CHANGE OF STEP-SIZE

A number of possibilities for acquiring the needed starting values for the algorithm present themselves. A "Runge-Kutta type" was briefly considered but the high number of functional evaluations that appeared necessary for a Runge-Kutta adaptation to the problem caused us to look elsewhere. A number of predictor-corrector type starting routines using lower order operators which were more in the spirit of our algorithm were outlined in [2]; however in keeping with our desire to maintain an "easily applied" high accuracy method we suggest a simple one-step starting technique. By a one step starting method we mean a method using only the information one diagonal behind the desired diagonal, e.g. [6] pp. 212-214, [15]. The advantages of using such a starting technique include:

a) the ease of application,
b) the known convergence and stability of many one step methods,
c) the availability of numerical programs for their application.

The main disadvantage is the large number of grid points, i.e. small step-size, necessary to give back values of sufficient accuracy. For instance, assume we want a step-size of \( h \) for our main predictor-corrector technique and assume we are using a one step starting procedure which uses first forward differences. That is partial derivatives will be approximated by

\[
(5.1) \quad f'(x^p) \approx \frac{1}{h_s} \Delta f(x^p_{p-1}).
\]

The truncation error \( E_T \) in such a procedure would be of the form

\[
(5.2) \quad |E_T| \leq Mh_s^2.
\]
Hence to achieve sufficient accuracy in our back values we would choose \( h_s = h^2 \). The number of calculations necessary to get started even with a step-size of \( h = .1 \) for the main predictor-corrector would be

\[
\sum_{i=0}^{20} (100-i) = 1890.
\]

This would take us out to the second diagonal \( \xi + \eta = 1 + .2 \) from where the main predictor-corrector would take over. As grim as this may seem we note that the number of calculations needed to start may be greatly reduced as follows.

Assume the overall step-size desired is \( h \). Then we choose for a starting step-size on the initial curve

\[
h_s = \frac{h}{4^i},
\]

where \( i \) is such that \( 4^i > h \). This particular choice accomplishes two things; one is that

\[ h_s < h^2 \]

and thus the desired accuracy in back values is achieved. The other will be discussed below.

Using the \( h_s \) step-size on the initial curve we use the one step technique to compute the solutions at the \( (h_s)^{-1} \) points on the diagonal \( \xi + \eta = 1 + h_s \) and also at the \( (h_s)^{-1} - 1 \) points on the diagonal \( \xi + \eta = 1 + 2h_s \). We then have three back values on each characteristic.
and therefore we shift to our predictor-corrector algorithm. Using the predictor-corrector technique we compute the \((h_s)^{-1} - 2, (h_s)^{-1} - 3\) points on the diagonals \(\xi + \eta = 1 + 3h_s, \xi + \eta = 1 + 4h_s\) respectively. Hereafter we will refer to the diagonal \(\xi + \eta + 1 = kh_s\) as the \(kh_s\) diagonal. So far all the calculations have been with the step-size \(h_s\) while the desired step-size with the algorithm is \(h = \sqrt{h_s}\). Hence we now discard the \(h_s\) diagonal and use only the values on the initial curve, the \(2h_s\) and \(4h_s\) diagonals to compute the values on the \(6h_s\) diagonal, then use the \(2h_s, 4h_s, 6h_s\) values to compute the \(8h_s\) values. At this point we can discard the \(2h_s\) and \(3h_s\) diagonals and use the initial curve, the \(4h_s\) and \(8h_s\) diagonals to compute the values on the \(12h_s\) diagonals, then in turn use the \(4h_s, 8h_s, 12h_s\) diagonals to compute values on the \(16h_s\) diagonals. We continue to extend the step-size in this manner until we achieve the \(h\) step-size. It now becomes evident why we choose \(h_s\) in the manner of (5.4); i.e. so the expanding step method will pass through the diagonals \(\xi + \eta = 1 + h\) and \(\xi + \eta = 1 + 2h\). Once we have reached the \(2h\) diagonal in our expanding step process we quit expanding and simply continue with the step-size \(h\).

We note the number of calculations required for the \(h = .1\) step is 576 as opposed to the 1890 in (5.3).

If at any time during the calculations we wish to increase, (double it or more), this process may be employed again.

We have advocated our algorithm as a practical, easy to apply method for the solution of certain hyperbolic partial differential equations and for this reason we feel a word of explanation is in order.

In the above procedure if a step-size of \(h = .01\) is desired for the over-all algorithm we would require
\[ h_s = \frac{h}{4^i}, \]

\[ 4^i > \frac{i}{h}, \]

Hence for \( h = .01 \), we need \( i = 4 \) and thus

\[ h_s = \frac{1}{25,600}. \]

Therefore we would require 25,601 data points on the initial curve to get started. This is a somewhat prohibitive number but we mention now that the choice for the number of points was arrived at so as to satisfy certain "sufficiency" requirements within the development of our theory and it is very possible that a much smaller number would be adequate. We would suggest however that the method of choosing the starting values should continue to contain the factor to make the expanding step size pass through the \( h \) and \( 2h \) diagonals. This is accomplished by simply choosing \( i \) less than required by 5.4. We also mention that due to the high accuracy of the method a step-size in the overall technique as small .01 may not be necessary in most practical situations.

If at a certain point in our calculations the difference between the predicted value and the corrected value becomes so large as to warrant a change in the step-size we suggest the following procedure. From the new desired step size we calculate a new starting step size \( h_s \) then use an interpolation technique e.g. Lagrangian interpolation, to find newly spaced data points on the last satisfactorily computed diagonal. From these points we use the one step technique to compute two additional diagonals, then shift to predictor-corrector again as described earlier in this chapter.
VI. NUMERICAL EXAMPLES

To illustrate our algorithm we solve two hyperbolic partial differential equations. The first of which is linear and the second a genuine quasi-linear partial differential equation. Both problems are rather simple in nature but this facilitates the finding of the transformation to the \( \xi \eta \)-plane so that we can compare actual and calculated solutions.

**Example 1.** Consider the partial differential equation

\[
(6.1) \quad u_{xx} - 4x^2 u_{yy} = 6x.
\]

This has the general form of the partial differential equation (2.1) with \( a = 1, b = 0, c = -4x^2, e = -6x \). We have

\[
(6.2) \quad b^2 - ac = 4x^2 > 0,
\]

for all \( x \neq 0 \) hence the equation is hyperbolic away from the origin.

We consider the initial curve as

\[
(6.3) \quad C = \{ (x,y) \mid 1 \leq x \leq 2, y = 0 \}
\]

with \( U = 3x^2, p = 6x, q = 2 \) on \( C \). Solving analytically, the slope of the characteristic is given by

\[
\frac{dy}{dx} = \frac{b^2 \pm \sqrt{b^2 - ac}}{a} = \pm 2x.
\]

and therefore the characteristics are

\[
(6.4) \quad y = x^2 + c_1
\]

\[
 y = c_2 - x^2.
\]
Hence we make the transformation
\[ \xi' = y + x^2, \quad \eta' = y - x^2. \]
Which in turn changes (6.1) into
\[ 2u_{\xi} - 2u_{\eta} - 8(\xi - \eta)u_{\xi\eta} = \frac{6}{\sqrt{2}} \sqrt{\xi - \eta} \]
or
\[ (6.5) \quad u_{\xi\eta} = \frac{3}{\sqrt{2}} \frac{(\xi - \eta)^{1/2} - u_{\xi} + u_{\eta}}{4(\xi - \eta)} \]

The actual solution of (6.1) with initial conditions (6.3) is
\[ (6.6) \quad U(x, y) = x^3 + 2y, \]
which we should get by using a method such as Picard's method of successive approximations on (6.5).

The transformations necessary to put the problem into the form suggested in chapter two i.e. initial curve \( \xi + \eta = 1 \) are given by:
\[ (6.7) \quad \xi = \sqrt{x^2 + y - 1} \]
\[ \eta = -\sqrt{x^2 - y + 2}. \]

Also then
\[ x = \left[ \frac{(\xi + 1)^2 + (\eta - 2)^2}{2} \right]^{1/2} \]
\[ y = \frac{(\xi + 1)^2 - (\eta - 2)^2}{2}. \]
The actual values of $x$, $y$, $u$, $p$, $q$ and the computed values are compared in Table 10.1 of the appendix.

We note that the I.B.M. 360 Model 50 was used and the machine time needed for the actual calculation of the five variables at 36 grid points was 5.96 seconds. It should be mentioned that this time also includes the evaluation of the actual solutions using (6.8) and (6.6).

**Example 2.** Consider the partial differential equation

\[
(6.9) \quad 3u_{xx} - 2u_{xy} - u_{yy} = -3 \sin x.
\]

This has the general form of the partial differential equation (2.1) with $a = 3$, $b = -u$, $c = -u^2$, $e = 3 \sin x$.

\[
(6.10) \quad b^2 - ac = u^2 + 3u^2 = 4u^2 > 0,
\]

for all $u \neq 0$, hence (6.9) is hyperbolic away from $u = 0$.

We consider the initial curve as

\[
(6.11) \quad c = \{(x,y) \mid \frac{\pi}{2} \leq x \leq \pi, y = 0\}
\]

with $u = \sin x$, $p = \cos x$, $q = 0$ on $c$. The actual solution is

\[
(6.12) \quad u = \sin x.
\]

We note that attempting to solve analytically immediately leads to trouble since the slope of the characteristics will be given by
Knowing the solution we see that the characteristics are given as

\[ y = -\cos x + c_1, \]

\[ y = \frac{1}{3} \cos x + c_2. \]

The transformation necessary to put the problem in the form (2.6)/(2.8) is

\[ \xi = \frac{2}{\pi} \cos^{-1} (\cos x - y)^{-1} \]
\[ \eta = 2 - \frac{2}{\pi} \cos^{-1} (3y + \cos x). \]

The actual values of \( x, y, u, p, q \) and the computed values are compared in Table 10.2 of the appendix.

The machine time for the calculation of the five variables at the 36 grid points is 5.91 seconds. Again this time includes the time necessary to compute the actual results using (6.12) and \( x \) and \( y \) found from (6.15).

In both examples a step-size of \( h = .1 \) in the \( (\xi, \eta) \)-plane was used while in addition example two was also run using a step of \( h = .01 \).
VII. SUMMARY

We have demonstrated an algorithm which we feel is easy to apply and yet gives high enough accuracy so a somewhat realistic step-size may be used. The truncation error has been exhibited as has the stability of the method. Also procedures for obtaining starting values and for increasing or decreasing the step-size of the computation have been discussed. The examples of chapter six, though admittedly simple in nature, do show the method to be applicable.

To assume that the solving of quasi-linear hyperbolic second order partial differential equations is now complete would be foolish of course. The physical situations which often give rise to the hyperbolic equations also cause many difficulties which our method may not be able to handle.

Therefore, though we admit shortcomings in our method we feel that it can be a significant contribution to the solving of many hyperbolic partial differential equations. And its simplicity and ease of application should make it especially desirable for the technicians and engineers who must find solutions to hyperbolic partial differential equations as they arise in their various applied fields.

Future work in this area could include a process for determining a realistic step-size which in turn would not cause great instability, also some possible adaptation of the technique for systems of partial differential equations. Looking into certain perturbations of the coefficients of the predictor to cause it to become stable might prove interesting. Further processes for changing step-size would always be welcome.
VIII. BIBLIOGRAPHY


IX. ACKNOWLEDGMENT

The author wishes to express his appreciation to his major professor, Dr. R. J. Lambert for the inspiration and suggestions given him during the preparation of this thesis. The author also wishes to thank his wife, Marlys, for her patience, encouragement and helpfulness not only during the preparation of this thesis but throughout his graduate studies.

Sincere thanks must also be given to Dr. T. R. Rogge for suggesting the area of study and to Martha Marks who prepared the Fortran program used to compute the illustrations.
X. APPENDIX

Table 10.1 gives selected values of the solution of example 1. The arrangement in the table has the predicted solution on the first line, the corrected solution on the second line followed by the true solution on line three. The step-size is $h = .1$ in the $\xi\eta$-plane which for this example also corresponded to $h = .1$ in the $xy$-plane.

Table 10.2 gives selected values of the solution of example 2. The arrangement differs from table 10.1 in that the solutions were calculated for two different step sizes. Therefore the first three lines are predicted, corrected, and true values corresponding to a step-size of $h = .1$ in the $\xi\eta$-plane while the next two lines give first the corrected value then the predicted value for a step-size of .05 in the $\xi\eta$-plane.
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Figure 1. Characteristics
Figure 2. Characteristics