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A comparative study of pre-service teachers' understandings of the equal sign

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A comparative study of pre-service teachers’ understandings of the equal sign

by

Julie Ellen (Buddenhagen) Hartzler

A dissertation submitted to the graduate faculty
in partial fulfillment of the requirements for the degree of

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Program of Study Committee:
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Ames, Iowa
2013

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# TABLE OF CONTENTS

LIST OF FIGURES ............................................................................................................................... vi

LIST OF TABLES ...................................................................................................................................... vii

ACKNOWLEDGMENTS ............................................................................................................................ ix

ABSTRACT ................................................................................................................................................. xi

CHAPTER 1. INTRODUCTION .................................................................................................................... 1
  Background of the Study ....................................................................................................................... 1
  Problem ................................................................................................................................................. 2
  Misconceptions ..................................................................................................................................... 3
  Common Core State Standards-Mathematics (CCSS-M) .................................................................... 5
      CCSS-M and the equal sign .............................................................................................................. 5
  Theoretical Framework ....................................................................................................................... 9
  Pre-service Teachers’ Mathematical Knowledge ............................................................................... 12
      Sense-making ............................................................................................................................... 12
      Equal sign understandings ........................................................................................................... 13
  Teacher Knowledge and Mathematical Knowledge for Teaching (MKT) ......................................... 14
      Teacher preparatory coursework ................................................................................................. 16
  Research Questions ............................................................................................................................ 17
  Significance .......................................................................................................................................... 17

CHAPTER 2. LITERATURE REVIEW ........................................................................................................ 19
  Historical Context .............................................................................................................................. 19
      Elementary and middle-school students’ understanding of “=” ................................................. 20
      Elementary ............................................................................................................................... 20
      Middle school .......................................................................................................................... 21
      High school and undergraduate students’ understanding of “=” .............................................. 22
      Cross-cultural data .................................................................................................................... 25
      Curriculum ................................................................................................................................. 26
  Teachers’ interpretation of student work: Subtraction misconceptions versus equal sign misconceptions ......................................................................................................................... 28
  Interventions .................................................................................................................................... 30
  Technology ......................................................................................................................................... 36
  Validity and Reliability ...................................................................................................................... 38
  Summary ............................................................................................................................................ 39

CHAPTER 3. METHODOLOGY ................................................................................................................ 42
  Research Design ................................................................................................................................. 42
  Sample ............................................................................................................................................... 42
  MCC1 and MCC2 ............................................................................................................................... 42
  MMC ................................................................................................................................................... 43
CHAPTER 4. RESULTS AND DISCUSSION

Research Question 1 .................................................................62
Correct vs. Incorrect Tasks .........................................................62
  Equal sign understandings by task ...........................................64
  EQ scores ..............................................................................66
  PSTs who answered all of the tasks correctly .........................68
  PSTs who demonstrated fully relational understanding on the
    EQ but did not get all of the tasks correct ............................70
  PSTs in general ......................................................................71
  Parametric .............................................................................73
  Non-parametric .................................................................75
Research Question 2 ..................................................................76
Participants ..............................................................76
Parametric correlations ......................................................78
  Partial correlations ..............................................................80
Research Question 3 ..................................................................82
Analysis by course: Correct responses .....................................82
  EQ scores by course ..............................................................83
  Parametric procedures .........................................................85
  What impact does a methods course have on PSTs’ understandings of
    the equal sign? .................................................................86
  Correct vs. incorrect ..............................................................86
  EQ scores: Parametric ............................................................87
  Non-parametric .................................................................88
  Two additional post-methods questions .................................88
Summary ..............................................................................90
CHAPTER 5. CONCLUSIONS ...........................................................................................................92
Gaps in the Literature......................................................................................................................92
Directly measuring PSTs’ understandings or the equal sign ..............................................................92
Cross-sectional data .......................................................................................................................92
Undergraduate studies are limited ..................................................................................................93
The EQ (Equal Sign Understanding Questionnaire) ....................................................................94
Summary of Results .......................................................................................................................94
Findings ..............................................................................................................................................95
Definition difficulties persist ..........................................................................................................95
Definition difficulties do not hinder successful task completion ......................................................96
Task context may activate different understandings .........................................................................96
Not all string tasks are created equal .............................................................................................96
Task B’s context may activate an algebra mindset ..........................................................................98
Most PSTs hold a mix of understandings of the equal sign ...............................................................99
Correct responses do not necessarily mean a fully relational Understanding ..................................99
Mix of understandings is common ..................................................................................................99
Equal sign understandings and other variables ............................................................................100
EQ scores and confidence to teach mathematics .........................................................................100
Relationships between EQ and gender .........................................................................................101
Research Question 2 .......................................................................................................................101
Summary of results .........................................................................................................................101
Findings ..............................................................................................................................................102
Discussion .........................................................................................................................................102
Research Question 3 .......................................................................................................................103
Summary of results .........................................................................................................................104
Findings ..............................................................................................................................................104
Task B is resistant to change ..........................................................................................................104
Tasks have consistent difficulty rankings between groups ...........................................................105
MCC1 may impact PSTs’ equal sign understandings ..................................................................106
Mathematics methods course may make an impact ..................................................................107
Summary ...........................................................................................................................................107
Final Thoughts ...............................................................................................................................109
Constraints and limitations ...........................................................................................................110
Future Research ............................................................................................................................110

APPENDIX A. CONSENT FORM: MCC1 AND MCC2 .................................................................111

APPENDIX B. CONSENT FORM MMC .......................................................................................114

APPENDIX C. EQ: MCC1 AND MCC2 .........................................................................................117

APPENDIX D. EQ: MCCpre ............................................................................................................122

APPENDIX E. EQ: MMCpost ..........................................................................................................128
LIST OF FIGURES

Figure 1. Theoretical framework .................................................................10
Figure 2. EQ definition (part I) .................................................................47
Figure 3. EQ tasks (part II) .................................................................47
Figure 4. Scree plot for eigenvalues and variance ........................................60
Figure 5. Distribution of EQ scores for MCC1 and MCC2 PSTs ..................77
Figure 6. Distribution of MKT-NCOP IRT scores for MCC1 and MCC2 PSTs ....77
Figure 7. Distribution of MKT-PFA IRT scores for MCC1 and MCC2 PSTs ..........78
Figure 8. Scatterplot of EQ scores and MKT-PFA IRT scores ......................79
Figure 9. Scatterplot of EQ scores and MKT-NCOP IRT scores ....................79
LIST OF TABLES

Table 1. Characteristics of PSTs by course .................................................................45
Table 2. Characteristics of PSTs by gender .................................................................45
Table 3. Eigenvalues and variance explained ...........................................................59
Table 4. Component matrix (2) ....................................................................................60
Table 5. Component matrix (1) ....................................................................................61
Table 6. Percentage of PSTs who answered the task correctly ...............................63
Table 7. Frequencies for PSTs’ total correct scores ..................................................63
Table 8. Sample responses: Definitions and coding ..................................................64
Table 9. Percent of PSTs with correct UA, UB, and UC responses ...........................65
Table 10. Percent of PSTs with correct UD and UF responses ..................................66
Table 11. Percent of PSTs with correct UE responses ...............................................66
Table 12. Frequencies for PSTs’ EQ scores (n = 268) ...............................................67
Table 13. Frequency of EQ scores for the 86 PSTs with a perfect total score ............68
Table 14. Frequency of non-fully relational responses by task given by participants with a perfect total scores an EQ score of eight* (n = 27) ..........................................................68
Table 15. Frequency of non-fully relational responses by task given by participants with a perfect total scores an EQ score of seven* (n = 16) ..........................................................69
Table.16. Frequency of non-fully relational responses by task given by participants with a perfect total scores an EQ score of six* (n = 12) ..........................................................70
Table 17. A random subgroup of PSTs’ (n = 30) types of understandings ................72
Table 18. Means and standard deviations of EQ scores by characteristic ...............73
Table 19. Parametric test results ..................................................................................74
Table 20. Non-parametric test results .......................................................................75
Table 21. Correlations with and without controls ..............................................................80
Table 22. Percentage of PSTs who answered the tasks correctly.......................................83
Table 23. Percentage of PSTs total correct score by course..................................................83
Table 24. Means and standard deviations by course ...........................................................86
Table 25. Results of between course analyses using Tukey HSD........................................86
Table 26. Percentage of PSTs who answered the task correctly .........................................87
Table 27. Frequency of total correct MMC pre and post .......................................................87
Table 28. Pre- and post-methods EQ scores: Results of the paired t-test.............................88
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ABSTRACT

This multipart study looked at pre-service teachers’ (PSTs’) understandings of the equal sign using an experimental measure, the EQ score. The data were analyzed for PSTs as a group and cross-sectionally at three key points during their elementary teacher education program. Relationships between equal sign understandings and other variables (e.g., endorsement, teaching mathematics confidence ranking, Mathematical Knowledge for Teaching (MKT) scores) were explored. Results indicate that most PSTs vary in their understanding of the equal sign with very few consistently exhibiting a fully relational view of the symbol. PSTs struggled to define the equal sign in a relational manner although this did not necessarily hinder successful task completion. PSTs struggled with “strings” (primarily in the true-false format) indicating that PSTs may have an “answer goes next” (operational) understanding of the equal sign when evaluating other people’s work but may have some form of relational understanding when the task is in a solving format.

Inferential statistics suggest that there were significant difference between PSTs’ understandings of the equal sign based on: (1) mathematical background, (2) pursuing a mathematics endorsement, and (3) confidence to teach mathematics. There also were significant results between PSTs’ EQ scores and their MKT scores, even when controlling for the opposite measure and the influence of their confidence to teach mathematics. PSTs beginning their second mathematics content course outperformed their peers enrolled in their first mathematics content course or methods course. Pre- and post-test analysis does support that completing a mathematics methods course (based on CGI principles of understanding
students’ thinking) did improve PSTs’ understanding of the equal sign. There was no statistical evidence to suggest a relationship between gender and EQ scores.
CHAPTER 1. INTRODUCTION

Background of the Study

In 2000, the National Council of Teachers of Mathematics’ Principles and Standards for School Mathematics (NCTM) recommended that algebraic thinking be emphasized not only at the secondary level but also at the elementary level; this was a call to encourage elementary teachers to cultivate students’ ability to look for patterns, generalizations, and other skills in order to develop foundational knowledge in children that would support algebra learning in the later grades. More recently, the Common Core State Standards – Mathematics (CCSS-M, NGSA, CCSSO, 2010) has reiterated the importance of this type of student thinking by including a domain called Operations and Algebraic Thinking throughout the K-5 standards. Algebra, which is considered a “gatekeeper” course by most (Lott, 2000; Moses & Cobb, 2001), is still a blocked gate for too many of our students, causing some students to avoid future studies and careers that require substantial knowledge of mathematics.

One surprising culprit in students’ quest to learn algebra is the equal sign. According to Webster’s Online Dictionary (2011) the word “equal” comes from the Latin word aequalis, which means level or equal, while the word “sign” has several definitions, of which the most appropriate ones for this discussion are “a mark having a conventional meaning” and a symbol “used in place of words or to represent a complex notion.” Although many assume that the equal sign has a predictable meaning that everyone understands, research suggests that this may not be the case (Behr, Erlwanger, & Nichols, 1976; Falkner, Levi, & Carpenter, 1999; Kieren, 1981). In this chapter I introduce the problem of pre-service teachers’ (PSTs’) understandings of the equal sign. I provide a brief overview of the common
misconceptions of this symbol and the concerns that arise if PSTs hold these misconceptions themselves. Finally, I explain my theoretical framework which is based on the literature dealing with understandings of the equal sign and teachers’ mathematical knowledge specific for teaching.

**Problem**

Researchers have found that elementary students often have misconceptions about the equal sign, typically having a strongly held operational view (i.e., perform the calculation or write the answer next) of the equal sign (Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1976; Behr et al., 1980; Clements, 1982; Falkner et al., 1999; Ginsburg, 1977; Hattikudur & Alibali, 2010; Kieran, 1981; McNeil & Alibali, 2005a, Saenz-Ludlow & Walgamuth, 1998). This is in contrast to a relational view which some describe as an understanding of sameness (i.e., both sides have the same value) and others define as an understanding of balance (without the need to calculate).

The prevalence of the operational view of the equal sign would be less of a concern if students modified their understandings as they progressed through schooling, but research has shown that this misconception can linger beyond the elementary years (Byers & Herscovics, 1977; Clements, 1982; Kieren, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; McNeil & Alibali, 2005b; McNeil, Grandau, Knuth, Alibali, Stephens et al., 2006; McNeil, Rittle-Johnson, Hattikudur, & Petersen, 2010; Weinberg, 2010). Falkner, Levi, and Carpenter asserted that an in-depth understanding of equality will allow students “to reflect on equations and will lay a firm foundation for later learning of algebra” (1999, p. 236). This claim is supported by the research of Kieren (1981) and more recently by Knuth et al. (2006), who found that 6th-8th graders who had a relational understanding of the equal sign were
more likely to solve equations accurately, even when algebra students’ data were excluded. Since success in algebra is critical, and a relational view of the equal sign contributes to algebraic understanding, research in this area is extremely valuable. This is especially true when looking at the equal sign understandings of future elementary teachers. Since elementary teachers will be responsible for helping their students develop algebraic reasoning, it is critical that we delve into PSTs’ understandings of the equal sign. This is particularly important in light of the CCSS-M (2010) as PSTs will need to interpret and implement these standards when they student teach and become practicing teachers.

In order to support this claim, in the following sections I will provide a brief overview of students’ misconceptions of the equal sign using a task typically used in the literature (e.g., $2 + 8 = \underline{\hspace{1cm}} + 3$) and highlight how understandings of the equal sign and the CCSS-M are entwined. Finally, I will discuss the research that informed my theoretical framework. First, I will look at the misconceptions of the equal sign typically held by students.

**Misconceptions**

U.S. elementary students’ misconceptions of the equal sign have been well documented by many researchers (Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1976, 1980; Clements, 1982; Falkner, Levi, & Carpenter, 1999; Ginsburg, 1977; Hattikudur & Alibali, 2010; Kieran, 1981; McNeil & Alibali, 2005a). Most researchers have used open number sentence tasks (e.g., $2 + 8 = \underline{\hspace{1cm}} + 3$) to assess students’ understanding of the equal sign. These studies have found that students typically hold an operational view of the equal sign, meaning they interpret the symbol in a manner that implies “getting an answer” or “putting the answer next.” For example, in the
aforementioned task, students with an operational viewpoint would write 10 or 13 in the blank, either ignoring the 3 or using it as an additional addend (i.e., \(2 + 8 = 10 + 3\); \(2 + 8 = 13 + 3\)). In contrast, students with a “sameness” understanding of the equal sign may find the solution by using arithmetic (\(2 + 8 = 10\) and \(10 - 3 = 7\)); Matthews, Rittle-Johnson, Taylor, and McEldoon (2010) used the term “relational with computational support” to define this type of understanding. Students with a deeper “sameness” understanding, who view the equal sign “like a balance,” would solve this same task by noticing that 3 is one more than 2, so the solution must be 1 less than 8 which is 7 (i.e., \(2 + 8 = 7 + 3\)); Matthews et al. used the term “relational without the need to compute (full relational)” to define this type of understanding.

It is worth noting that some researchers (Jones, 2009a, 2009b; Jones & Pratt, 2011; Jones, Inglis, Gilmore, & Dowens, 2012) identified a substitutionary aspect to this advanced understanding of equivalence. I will attempt to explain how this type of reasoning is different using the same task as in previous examples. Given \(2 + 8 = \_ + 3\) and several number sentences (e.g., \(6 + 2 = 8\), \(2 + 5 = 7\), \(5 + 3 = 8\)) students with this type of understanding would be able to transform the number sentence by subbing in \(5 + 3\) for 8 (which would produce \(2 + 5 + 3 = \_ + 3\)) and then subbing in a 7 for \(2 + 5\) (producing \(7 + 3 = \_ + 3\)). At this point they would know the answer is 7 due to their knowledge of substituting equivalent expression.

Although there is much written about children’s understanding of the equal sign, very little is known about PSTs’ understandings of the equal sign. Only three studies were located that explored teachers’ and PSTs’ understandings of the equal sign via their interpretations of student work (Asquith, Stephens, Knuth, & Alibali, 2007; Stephens, 2006, 2008). No studies were found that directly analyzed PSTs’ personal understandings of the equal sign or
examined whether PSTs at different points in their educational careers had different understandings. Since misunderstandings of this symbol are often found among undergraduates (McNeil, Rittle-Johnson, Hattikudur, & Petersen, 2010; Weinberg, 2010; Wheeler, 2010), it is likely that PSTs may have misconceptions as well. In the next section I’ll provide a brief background on the CCSS-M and highlight aspects of the standards that deal with the equal sign.

**Common Core State Standards-Mathematics (CCSS-M)**

The Common Core State Standards (CCSS) are a set of language arts and mathematics standards that were written to offer a “clear and consistent framework” to prepare American students, K-12, for their future endeavors (NGSA, CCSSO, 2010). The CCSS-M were developed by mathematicians, mathematics education researchers, teachers, administrators, parents, and other stakeholders as part of the CCSS effort led by the National Governors Association and the Council of Chief State School Officers. The purpose of the mathematics standards is to articulate the mathematical knowledge and skills all students will need, regardless of where they live in the United States, to successfully enter the workforce or attend college. Although these standards are state standards, and not national standards, federal Race to the Top funding has been used as an incentive for states to adopt the new standards. As of July 2013, 45 states had adopted the CCSS-M.

**CCSS-M and the equal sign**

When looking at the CCSS-M it is interesting to note that open number sentences typically used in equivalence research are delegated to first grade. This seems problematic as researchers have found misconceptions about the meaning of the equal sign at all grade levels
(Kieran, 1981). It seems probable that the goal of CCSS-M is to develop a strong relational view of the equal sign in kindergarten and first grade so that misunderstandings are eradicated, but if so this is implicit in the document and not explicitly stated for practitioners reading the text. In addition, the CCSS-M does not address that symbolic knowledge should be developed as a mathematical construct even though children do construct their own meanings for symbols, which are often quite different from adults’ (Ginsburg, 1989). It appears that the symbols are perceived to be intuitive and not worthy of a content standard, domain, cluster or even a definition in the glossary, even though the equal sign is found over 200 times within the document.

The CCSS-M does address certain mathematical practices that all students should attain. Two practices, reason abstractly and quantitatively and attend to precision, refer to symbols and the attend to precision standard does point out that students should use the “equal sign consistently and appropriately” yet provides no explanation of what that means (pp. 6-7). The glossary takes time to define whole numbers but does not define any symbols, although the appendix does list all of the properties of equality. The only section in this document that gives an indication as to what “consistently and appropriately” means is the first note underneath Table 1 which states that “…equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean makes or results in but always does mean is the same number as” (p. 88). Thus, the writers of the CCSS-M seem to have assumed that not all readers of the document are familiar with the term whole numbers, which is included in the glossary, but all readers are knowledgeable about the meaning of the equal sign and therefore, a single note under a table is sufficient.
The content from the CCSS-M below, drawn from the K-1 level, models appropriate uses of the equal sign but does not address the equal sign as the focus of understanding.

Operations and Algebraic Thinking K.OA

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

Operations and Algebraic Thinking 1.OA

Understand and apply properties of operations and the relationship between addition and subtraction.

3. Apply properties of operations as strategies to add and subtract. Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)

Add and subtract within 20.

6. Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).

(CCSS-M, 2010, p. 15)

The only area in the content standards of the CCSS-M document that seems to directly address students’ understandings of the equal sign and equivalence is at the first-grade level (see below):

Operations and Algebraic Thinking 1.OA

Work with addition and subtraction equations.

7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.
8. Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 + ? = 11, 5 = □ – 3, 6 + 6 = □.

(CCSS-M, 2010, p. 15)

As highlighted previously, it seems that the CCSS-M writers assume that teachers understand implicitly the unstated meaning of the equal sign. Note that the examples clarify the second part of the sentence “determine if equations involving addition and subtraction are true or false” by providing examples of typical equations used in research, but the first part of the standard “[u]nderstand the meaning of the equal sign” is left to the reader’s interpretation.

The second standard above gives missing number equations written in different formats although the most typical format used in research $a + b = \_\_\_ + c$ is notably missing.

This study of PSTs’ understandings of the equal sign is valuable as PSTs may become teachers who overlook students’ misconceptions of the equal sign (Ding & Li, 2006; Stephens, 2006) and do not fully understand the research-based recommendations in the CCSS-M. For example, according to the CCSS-M mathematical practices, students should be able to “use the equal sign consistently and appropriately” and have the ability to detect “structure” in mathematical problems (2010, p. 15), but it is not clear whether PSTs can themselves use the equal sign and detect structure in the ways described in the CCSS-M. Knuth et al. (2006) recommend that teachers should look for “natural” opportunities within their existing classroom practices to engage students in conversations about the equal sign as well as create “intentional” opportunities (p. 519). Yet, little is known about PSTs’ understandings of these tasks themselves. Hence, it is important that we assess PSTs’ understandings of this symbol to see what they consider an “appropriate” understanding of this symbol to be. In addition, in the CCSS-M mathematical practices (p. 8) we see, through
the lens of equivalence understanding, that students should have a deep relational understanding of the equal sign. This understanding needs to go beyond sameness to include a deeper understanding that allows students to determine values based on the structure of the problem without the need to calculate or solve an algebraic equation. Thus far, we do not know if our future elementary teachers have this knowledge themselves. More research is needed so that researchers, curriculum writers, professional development planners, mathematics coaches, and post-secondary instructors can understand how PSTs perceive this crucial symbol. In order to design a study of PSTs’ understandings of the equal sign, it was important to first develop a theoretical framework.

**Theoretical Framework**

Eisenhart (1991) described a theoretical framework as: “…a structure that guides research by relying on a formal theory” (p. 205). In order to have a “structure” researchers need to build their framework based on the “concepts, assumptions, expectations, beliefs, and theories that support[s] and inform[s]” their study (Maxwell, 2005, p. 33). Therefore, the framework for this study was informed by the existing work on students’ understandings of the equal sign (discussed briefly above and more in depth in the literature review) and teachers’ mathematical knowledge.

Figure 1 provides a visual representation of my theoretical framework. At the bottom I show interlocking ovals, one represents the equal sign understandings PSTs bring with them to college and the other represents the mathematical teacher preparatory coursework that these students complete during their educational program of study. The intersection of these two ovals is labeled *Preservice Teachers’ Sense Making*. This is where the PST takes the
knowledge he/she brings with him/her to college and the new information he/she has been exposed to in his/her coursework and either maintains the same understandings as before or modifies his/her initial understandings by a small degree or to a large degree. The rectangle in the middle represents the different *Understandings of the equal sign*, while the arrow pointing up indicates that I’m looking at how PSTs make sense of the equal sign (based on their background) and which type or types of understandings they exhibit.

The top of the visual is a diagram developed by Ball, Thames, and Phelps (2008), which represents their visualization of Mathematical Knowledge for Teaching (MKT will be discussed in greater detail in its own subsection), a refinement of Shulman’s (1995) work on teacher knowledge. The arrow going down from subject matter knowledge to understandings
of the equal sign represents that understanding the equal sign is a type of subject matter knowledge that teachers need. The arrow is in between both common content knowledge (CCK) and specialized content knowledge (SCK) because all individuals need to understand the equal sign in a functional way but teachers need to have a more robust understanding of this symbol. For instance, ordering decimals would demonstrate CCK while choosing the best list of decimals to have elementary students order would show SCK. In other words, some equal sign tasks the general population should be able to solve correctly (demonstrating CCK) but other tasks for example, where the participant is asked to choose the equal sign task that would best assess students understanding of the equal sign would be a specialized skill only teachers would need to demonstrate (SCK).

In addition, in this study, relationships between PSTs’ subject matter knowledge within the domains of Number concepts and operations and also Patterns, functions, and algebra (both CCK and SCK) and understandings of the equal sign were explored. It is worth noting that in later schematic drawings Ball and Bass (2009) subdivided the far left section of subject matter into CCK on the top half and Horizon Content Knowledge (HCK) on the bottom half. This new knowledge which deals with the teacher being able to see the mathematical big picture has had less attention (Jakobsen, Thames, & Ribeiro, 2013). Since the MKT assessments used in this study were developed before HCK was identified, the older visual was used in my theoretical framework.

The final arrow pointing up to pedagogical content knowledge indicates that PSTs’ understandings of the equal sign may impact their knowledge on this side of the schema. Initially, I was going to explore this area through PST interviews but this part of my study did not come to fruition.
Pre-service Teachers’ Mathematical Knowledge

Sense-making

When exploring the learning of mathematics Schoenfeld (1992) stated that, in order to understand how someone thinks mathematically, one must:

…understand the individual's behavior—e.g. which options are pursued, in which way—one needs to know what mathematical tools the individual has at his or her disposal. Simply put, the issues related to the individual's knowledge base are: What information relevant to the mathematical situation or problem at hand does he or she possess, and how is that information accessed and used? (p. 349).

When analyzing how an individual solves a problem Schoenfeld posited that the individual’s difficulties may be due to issues with metacognition or may be due to the lack of proper “tools.” Therefore,

[f]rom the point of view of the observer or experimenter trying to understand problem solving behavior, then, a major task is the delineation of the knowledge base of individuals who confront the given problem solving tasks. It is important to note that in this context, that knowledge base may contain things that are not true. Individuals bring misconceptions and misremembered facts to problem situations, and it is essential to understand that those are the tools they work with (p. 349).

In the last sentence he emphasized that it is “essential to understand” that misconceptions may be part of the “tools” with which students work, which applies to students of all ages including PSTs. When discussing the Learning Principle set forth in the NCTM Principles and Standards (2000) Stylianides and Stylianides (2007) emphasized how students need to be able to make sense of the mathematics; to have a deep understanding of mathematics; and that those who get “right answers” may still have a “fragile” grasp of the concept. In order to understand how PSTs make sense of the equal sign and to go beyond just assessing correct
answers (with possible “fragile” understanding), this study needed a way to interpret PSTs’ responses.

**Equal sign understandings**

In order to explore PSTs’ equal sign understanding, I looked to the work of Matthews, Rittle-Johnson, Taylor and McEldoon. In their 2010 SREE Conference Abstract these researchers proposed four levels of equivalence understanding based on previous work by Carpenter, Franke, and Levi (2003). These four levels of more sophisticated knowledge (two were discussed earlier and two will be defined shortly) included: *Rigid operational*, *Flexible operational*, *Relational with computational support*, and *Relational without the need to compute (full relational)*.

Later, these same researchers created a *Construct Map for Mathematical Equivalence Knowledge* (Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011) and then, after making minor adaptations, the *Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality* (Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012). The *Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality* was developed to represent the progression of equivalence knowledge but the researchers cautioned that the levels should not be understood as distinct stages due to their continuous nature. In addition, the researchers noted that the concept map was not intended to be all-inclusive of thoughts about mathematical equality but instead to cover the symbolic types of problems typically studied in the research. The levels included: Level 1, *Rigid operational* (*a + b = c* format; any missing amount can be found); Level 2, *Flexible operational* (can also solve *c = a + b* format and accept the *a = a* format); Level 3, *Basic relational* (can also solve
problems with operations on both sides for example, $a + b = c + d$ or $a + b - c = d + e$; Level 4, *Comparative relational* (can solve problems using comparison and compensation). In addition, Levels 3 and 4 contained descriptors in regards to the students’ understanding of the equal sign’s definition.

Due to the nature of my study (e.g., focusing on understandings of college students, rather than K-8 students), I decided to condense the levels of understanding proposed by these researchers into three levels of understanding. Most college students have been exposed to number sentences throughout their academic careers and therefore tasks like $3 + 2 = __$, $8 = 8$, $___ = 2 + 4$ seem trivial and finding differentiation among those with operational knowledge seemed unlikely. In addition, I decided to use similar language from their initial study (Matthews et al., 2010) due to the ease in understanding of the terms. Hence, the following levels of understanding from basic to more advanced understandings were used: *Operational, Relational with Computation,* and *Fully Relational.* Recently, Stephens, Knuth, Blanton, Isler, Gardiner et al. (2013) used the terms operational, relational-computational, and relational-structural in a similar manner. In the next section, I highlight how research related to teacher knowledge and teacher preparatory coursework informed my theoretical framework.

**Teacher Knowledge and Mathematical Knowledge for Teaching (MKT)**

One of the issues in the field of mathematics education involves teacher knowledge and what exactly constitutes the kind of knowledge teachers need to be effective. In 1986, Shulman introduced the idea of “content knowledge for teaching” which entails teachers not only knowing the mathematical “how” but also the mathematical “why.” In her initial delve
into teacher knowledge, Ball (1988) wrote in her dissertation that “…[i]n order to help students develop meaningful understanding of mathematics, teachers themselves need to have explicit and conceptual connected understandings of mathematical concepts and procedures” (p. 191). This early endeavor would eventually lead to what is called *mathematical knowledge for teaching* (MKT) (Hill, Schilling, & Ball, 2004).

In 1999, the Study of Instructional Improvement (SII) began to develop tools to measure the type of knowledge elementary teachers use to teach mathematics (Ball & Bass, 2000). This work grew out of Ball’s (1990) research in which she differentiated “between knowledge of mathematics and knowledge about mathematics, corresponding roughly to knowledge of concepts, ideas, and procedures and how they work, on one hand, and knowledge about ‘doing mathematics’—for example, how one decides that a claim is true, a solution complete, or a representation accurate…” (Hill, Schilling, & Ball, 2004, p.14). The goal was to develop a multiple-choice measurement tool for elementary teachers that would cover CCK (e.g., how to add three-digit whole numbers) but also SKC (e.g., given student-developed algorithms, which ones always work when adding three-digit whole numbers).

Based on existing research and their personal beliefs, the developers decided initially to focus their efforts on three content areas: (1) number concepts; (2) number operations; and (3) patterns, functions, and algebra (Hill, Schilling, Ball, 2004). The first two content areas were chosen because of their prevalence in K-6 curriculum, while the latter was chosen due to its newness in K-6 curriculum. After statistical analyses these three content areas were collapsed into two: (1) *Number concepts and operations* (NCOP) and (2) *Patterns, functions, and algebra* (PFA). According to Hill, Rowan, and Ball (2005), a unique feature of the MKT measure is that “it represents the knowledge teachers *use* in the classrooms, rather than
general mathematical knowledge. To ensure that this was the case, we designed measurement
tasks that gauged proficiency at providing students with mathematical explanations and
representations and working with unusual solution methods” (p. 287). Besides exploring
PSTs’ understandings of the equal sign it seems interesting to explore if there are any
relationships between PSTs’ understandings of the equal sign and their MKT. In addition,
since teacher preparatory classes are one of the methods in which PSTs are educated to teach,
including this component into the framework seemed natural.

**Teacher preparatory coursework**

Does teacher preparatory coursework impact PSTs’ mathematical knowledge needed
for teaching? Battista (1994) found that pedagogical content knowledge could be improved
by well-designed teacher preparatory coursework. Recently, researchers found that those
teachers who took specialized mathematics content courses as opposed to general
mathematics courses had higher mathematical knowledge specific to the elementary
classroom (Matthews, Rech, & Grandgenett, 2010), but does this knowledge include a deep
understanding of the equal sign? Welder and Simonsen (2011) found that PSTs who
completed the first semester of a two-semester mathematics content course geared towards
future elementary educators showed gains in their CCK, SCK, and in two areas of algebra
concepts (as measured by MKT measures). Nevertheless, does this specialized knowledge
impact students’ learning of mathematics? Hill, Rowan, and Ball (2005) found that teachers’
content knowledge for teaching mathematics was significantly related to students’
mathematical gains, thus, supporting the importance of teachers having this type of
mathematical knowledge. Consequently, knowledge about PSTs’ understandings of the equal
sign and the connection between these understandings and specific mathematics teacher preparatory coursework are needed. Do these courses impact PSTs’ understandings of the equal sign? Do students who have completed their mathematics content and methods courses perform better on an assessment of their equal sign understandings than those just beginning their mathematics education journey?

**Research Questions**

Many questions could have been addressed in the proposed study, yet the specific research questions I chose to explore were:

1. What types of understandings of the equal sign do PSTs have?
2. What is the relationship, if any, between PSTs’ understandings of the equal sign and PSTs’ mathematical knowledge for teaching?
3. What types of understandings of the equal sign do PSTs have cross-sectionally at different stages of a teacher preparation program?

**Significance**

The framework for this study (see Figure 1) was based on the work of Mathews and colleagues (2010, 2012) dealing with equal sign understandings, and Hill, Schilling, and Ball’s work (2004) on teacher knowledge. In the spirit of this aforementioned work and additional work, which will be discussed in the literature review, this study explored the use of an experimental equal sign understandings measurement tool and the results of using this tool to measure and compare PSTs’ understandings of the equal sign. The goal of this research was not to critique PSTs in order to belittle their knowledge but to see what type of understandings they had of the symbol and to see what other factors may be at play (e.g.,
teacher preparatory coursework). Students’ misunderstandings of the equal sign have been well documented in the literature. Yet, no large-scale studies have focused directly on PSTs’ understandings of this symbol. As future teachers, these individuals will be expected to foster certain mathematical practices in their students and to adhere to the mathematical goals as outlined in the CCSS-M. The goal of this study was to use a measurement tool to document PSTs’ understandings of the equal sign.
CHAPTER 2. LITERATURE REVIEW

Historical Context

This chapter explores the literature related to students’ understandings of the equal sign. First, I present some of the historic works and then examine the research specific to different age groups of students. Next, I note the difference between recognizing an equal sign misconception versus other mathematical misconceptions and explore the impact of interventions both curricular and technological. Finally, I discuss issues regarding the validity and reliability of past research.

Most research into American students’ lack of understanding of the equal sign can be traced back to the 1970s (Behr et. al, 1976; Weaver, 1971, 1973). One of the most cited of these early works is a research paper written by Behr, Erlwanger, and Nichols’ (1976). In this study, the researchers conducted individual non-structured interviews with students from first to sixth grade in order to ascertain how children think about equality sentences. The researchers found that when first and second graders were given a sentence in the form \(a + b = \square\), they viewed the equal sign as indicating something that needed to be done, “a stimulus calling for an answer to be placed in the box” (p. 2). In regards to \(\square = a + b\), the majority of the students thought the equation was “backwards” and would typically change it to \(a + b = \square\) or change it to \(\square + a = b\) (a third and a sixth grader in the study showed similar response patterns). When discussing statements of the form \(a + b = b + a\), students typically calculated the sum on each side, extended out the statement with an additional equal sign followed by the sum of \(a\) and \(b\), or changed the format to \(a + b + b + a = \square\). Some accepted \(a + b = b + a\) because the numbers were the same (i.e., commutative property of addition) but rejected the
true statement \(a + b = c + d\) (where \(a \neq b \neq c \neq d\)) due to the numbers being different. (A sixth grader who had no issues with the reflexive property of equality, \(a = a\), still showed difficulties with the concepts of sameness and equality.)

Behr et al. (1976) found that most of the students they interviewed had an operational versus a relational view of the equal sign. They found, regardless of age, that when presented with a statement with the equal sign, students felt the need to “do something.” The authors noted the possibility that “children’s concept of equality as an operator symbol, rather than a relational symbol, is symptomatic of their limited understanding and experiences with relational terms in general, such as same, more, less, as many as, etc.” (p.10). The authors warned that merely exposing children to the various formats of these equality statements will not “remove the problem,” noting that “[t]he behaviors uncovered in this investigation suggest a deep-seated mind set which produces rigid reactions, particularly to written number sentences” (p. 10). Although this research gives us insight into students’ understandings, it does not provide insight into how future teachers will view these tasks, especially when found in a policy document like the CCSS-M.

**Elementary and middle-school students’ understanding of “=”**

**Elementary**

The finding by Behr et al. (1976) that U.S. elementary students’ ingrained operational view of the equal sign, has been supported by various researchers during the ensuing years (Alibali, 1999; Baroody & Ginsburg, 1983; Behr et al., 1980; Clements, 1982; Falkner, Levi, & Carpenter, 1999; Ginsburg, 1977; Hattikudur & Alibali, 2010; Kieran, 1981; McNeil & Alibali, 2005a; Saenz-Ludlow & Walsaguth, 1998). Although this operational view is
extremely well-documented among elementary students in the United States, there are exceptions. Seo and Ginsburg (2003) found that, in certain contexts the second graders studied did exhibit an operator viewpoint, but in other contexts they interpreted the equal sign as a relational symbol. When given real life scenarios where no operations were present (i.e., 2 white rods = 1 red rod; 1 dollar = 100 pennies) the majority of the students responded with a relational definition of the equal sign. But when given the symbol in isolation (no context), in canonical form (i.e., \( a + b = c; a - b = c \)), or in a noncanonical form (i.e., \( c = a + b; a = a \)) the majority of the students gave responses consistent with an operator view of the equal sign. Accordingly, these researchers assert that students hold both views of the symbol. A more notable exception is represented in the students who were part of the Measure Up program at the University of Hawaii; they did not hold an operational view of the equal sign (Dougherty, Zenigami, & Okazaki, 2005)). In this program, influenced by the work of Russian researchers El’konin and Davydov (Steffe, 1975), students start with abstract mathematical constructs (using variables) and move to more concrete constructs (numeric) via measurement tasks. According to Dougherty (2003) young students “from their prenumeric beginnings, [understand] that the equal sign is merely a way of showing that two quantities are the same” (p. 20). This program is discussed in more depth in the intervention section.

**Middle school**

Do equal sign understandings improve with age? McNeil (2007) found that the change in understandings by age (based on performance on equivalence tasks) is actually U-shaped, with performance decreasing between 7 and 9 years old and then improving between
9 and 11 years old. Even so, some researchers have found that United States 5th through 8th grade students continue to struggle with an operational understanding of the equal sign (Knuth et al., 2006; Li et al., 2008; McNeil, Grandau, Knuth, Alibali, Stephens, et al., 2006; Oksuz, 2007). Others have found some movement towards a more relational understanding of the equal sign as students progressed through middle school (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007; Gonzalez, Ambrose, & Martinez, 2004; Knuth, Alibali, McNeil, Weinberg, Stephens, 2005; McNeil & Alibali, 2005a). McNeil and Alibali’s (2005a) study found that middle school students who were asked to define the equal sign based on three different contexts (given: $=, a + b + c + d = __$, or $a + b + c = __ + d$) held an operator definition for the first two contexts, but had a relational definition when given the third context, suggesting that “students do not abandon well-established interpretations just because they do not work in a few contexts. Instead, they may view those contexts as exceptions and change their thinking only in those contexts” (p. 301). Knuth and colleagues found that middle school students who gave a relational definition of the equal sign outperformed their peers on a task of mathematical equivalence (2005) and on algebraic equations (2006).

**High school and undergraduate students’ understanding of “=”**

Research in regards to high school and undergraduate students’ understandings of the equal sign, though limited, suggests that some students continue to have misconceptions (Byers & Herscovics, 1977; Clements, 1982; Kieran, 1981; McNeil & Alibali, 2005b, McNeil, Rittle-Johnson, Hattikudur, Peterson, 2010; Weinberg, 2010; Wheeler, 2010).
Research has shown that students, when “showing their work,” often use the equal sign in ways that defy equivalence (Byers & Herscovics, 1977; Clements, 1982; Weinberg, 2010).

In a study conducted by McNeil and Alibali (2005b), undergraduates were randomly assigned to either an activation or control situation. The activation participants participated in a perceptual pattern activation phase (i.e., match operations = answer equations), a concept activation phase (i.e., match target words like total, plus, sum, and add), and a strategy activation phase (i.e., given a target number find pairs of addends whose sum is that number). Those in the control group completed word and color mixing situations to mimic the above phases. In the final phase the researchers had the undergraduates complete two equations in the traditional format (e.g., $5 + 7 + 3 + 5 = ___$ ) followed by eight equations like $7 + 4 + 5 = 7 + __$. Undergraduates were only allowed to view the equations briefly after which they wrote their answer on an answer sheet. The researchers found that when operational patterns were activated “undergraduates could be made to perform like elementary school children” (p. 896).

In another study, McNeil and Alibali (2005a) found that in a task where participants were asked to define the equal sign in three different contexts (given: $=, a + b + c + d = __$, or $a + b + c = __ + d$), undergraduates and graduate students held a relational view in all three contexts. Yet, it should be noted that the generalizability of these results to all undergraduates seems questionable, as all of the undergraduates had taken at least one calculus course in their lifetime and all the graduate students had passed the physics qualifying exam.

A recent study by Weinberg (2010) analyzed the thoughts of calculus students and found that these students viewed “strings” (i.e., $3 + 17 = 20/4 = 5$), or what he termed run-on
statements, as correct in the written part, although in interviews some changed their minds. One-third of the students said $2x + 12 = 6 + x$ was true because they could solve for $x$ thus displaying a more operational understanding. Weinberg concluded that the college students viewed the equal sign as a symbol that could take on different meanings in varied contexts. He also found that even though the students had developed various mental constructs with different views of the equal sign, some students were able to solve the problems correctly even when their mathematical knowledge about the role of the equal sign was faulty. Weinberg concluded that if we view this “symbolization process as a negotiation-between the student, the teacher, formal mathematics, and the context in which the activity is situated-teachers can help their students use these symbols meaningfully in ways that are compatible with standard, formal mathematical notation” (p. 8).

Wheeler’s (2010) dissertation developed, tested, and utilized the Wheeler Test for Comprehension of Equals (WTCE) to measure undergraduates’ understandings of the equal sign; he found that the majority of the students did understand equals as sameness but often did not use this knowledge in differing contexts. The participants, remedial intermediate algebra students, often failed to “recognize the extent of the sameness suggested by an equation” and when students were “focus[ed] on solving, evaluating, or coming up with “the answer” they fail[ed] to recognize the contributions of the equals sign or other indications of the equals relation in a given context” (p. 51). It is worth noting that the initial WTCE had low reliability and, due to the participants’ mathematical background, the generalizability of the results may be limited.

Do vacillating understandings of the equal sign by undergraduates’ matter? What if the undergraduate is pursuing a degree in education and will be teaching mathematics to
children? In the next section I will discuss cross-cultural and curriculum studies. Then I will discuss why having a strong relational understanding of the equal sign is vital for future elementary teachers in interpreting students’ work and in making instructional decisions.

**Cross-cultural data**

Cross-cultural research on sixth graders has shown that Chinese students do not have the same operational view as American students (Li, Ding, Capraro, & Capraro, 2008). Although not as strong as the Chinese, Korean students, and to a lesser extent Turkish students, also have a better understanding of the equal sign when compared to their American counterparts (Capraro, Capraro, Ozel, Kim, & Kucuk, 2010). These cross-cultural patterns seem to hold true with older students as well. In a study that looked at the interference of arithmetic knowledge on U.S. college students’ algebraic abilities, researchers found that when they analyzed their data in light of students’ elementary educational experiences, undergraduates with Asian (i.e., Korea, Singapore, China, Hong Kong, India, and Taiwan) background were six times more likely to answer at least one equation correctly, after the arithmetic treatment (solving simple addition facts prior to the assessment), than U.S. undergraduates (McNeil et al., 2010).

In a recent study by Jones et al. (2012), participants were 11 and 12 year old students from England and from China. In this study students were given 10 minutes to complete 12 definition items that were designed to measure an *operational, sameness-relational,* or *substitutive-relational* meaning of the equal sign. There were three differently worded definition of each type including three distractor definitions; the students were asked to rank the cleverness of each definition (i.e., *not so clever, sort of clever,* and *very clever*). The
researchers found that the English students had a significantly higher mean operational rating and a significantly lower mean substitutive-relational rating than their Chinese counterparts. However, unexpectedly, there was not a statistically significant difference between the two groups’ sameness-relational rating.

**Curriculum**

What do we know about the equal sign in terms of the curricular materials teachers use in the classroom? When looking at four middle school textbook series (two skills-based and two standards-based), McNeil and colleagues (2006) found that the number of times the equal sign was used was higher in eighth grade than in sixth grade. The variability of seeing the equal sign in an *operations equals answer context* was very high between the series. There were more instances of the *operations on both sides context* as the grade level increased but even then only about 5% (on average) of the instances observed were written in this context. Other nonstandard equations were shown the majority of them having no explicit operation like $7 = 7$.

When comparing mathematics methods books in China and the United States, researchers (Li et al., 2008) found a possible cause of the discrepancies between Chinese and U.S. students’ understandings of the equal sign. Of the six U.S. methods books investigated, only two “directly addressed the equal sign, and none include lesson examples or activities to help understand how the equal sign should be taught” (p. 209). The authors concluded that educators “can only be expected to teach their students what they themselves experience and understand” (p. 209) and although few would disagree with this point, one must not assume that methods courses are the only source of mathematical content knowledge for pre-service
teachers. In the United States, mathematics content courses, specifically designed to develop mathematical knowledge, are common for most students seeking an elementary education degree (National Council on Teacher Quality, 2008). Consequently there is need for analysis of how mathematics content textbooks and courses impact PSTs’ understanding of this symbol. Therefore, as part of the present study, I reviewed the text and activity manual used in the PSTs’ mathematics content course to see what kinds of opportunities to learn about the equal sign were available in the course materials.

Li et al.’s (2008) analysis of Chinese elementary curricular materials (teacher guidebooks, the national framework, and student texts) revealed that the materials used various contexts to develop understanding of the equal sign. The guidebooks advocated a relational understanding of the equal sign emphasizing the need to use comparison and “one-to-one” correspondence to teach this concept. The equal sign was typically taught in conjunction with inequality symbols (i.e., < and >) and, in fact, the researchers found that in one guidebook it “clearly states that the goal for the unit is to learn and understand the “=,” “>,” and “<” (Li et al., 2008, p. 204). This is interesting when one contemplates the lack of explicit language about the equal sign in the CCSS-M. The Chinese curricular materials emphasized tasks that involved one-to-one correspondence in order to foster understanding of these symbols. For example, in the article it gives an example where animals are carrying logs as a way in which students could see the one-to-one correspondence of logs to animals and hence understand the equal sign as the same as. In addition, equation tasks used a variety of contexts to develop equal sign understanding. Likewise, contexts were utilized where comparison was stressed (i.e., measurement tasks like length, height, and weight). It seems logical to conclude that U.S. elementary students who only have limited experience with
measurement in general, and measurement comparisons tied to symbolic language specifically, may have misconceptions that other students who have had these experiences may not have. Thus, providing opportunities for PSTs, who have most likely never been exposed to this type of thinking, seems beneficial.

In an extension of this aforementioned works Powell (2012) recently looked at eight US curricula spanning K-5 grade levels. Powell found that seven out of the curricula reviewed did not use nonstandard equation types to help foster a relational understanding of the equal sign. She also found that when reviewing the teacher manuals:

(1) The definitions were typically relational, although no curriculum provided the same definition at all grade levels.
(2) The equal sign is rarely mentioned.
(3) Explanations about the equal sign were typically at the K-1 level (with some extending into second grade).
(4) Some materials gave incorrect definitions of the equal sign like “the sign between the addends and the sum” (p 643).
(5) Only three of the curricula emphasized using nonstandard equations in instruction but only Everyday Mathematics encouraged this in the student textbook.
(6) Half of the curricula used a balance scale to encourage that the both sides of an equation are the same but its use was not consistent.

All in all, Powell noted that “no curriculum provided the complete package of equal sign understanding … a relational definitions of the equal sign (across the school year and across grade levels) and ample opportunities for exposure to nonstandard equations” (p. 643).

**Teachers’ Interpretation of Student Work: Subtraction Misconceptions versus Equal Sign Misconceptions**

An important aspect of a teacher’s job is the ability to analyze students’ work and provide appropriate feedback to help students correct any misconceptions so that they can maximize their learning. Riccomini (2005) looked at elementary teachers’ ability to
determine the nature of two typical subtraction errors and their ability to decide on an instructional plan based on that knowledge. Results indicated that the teachers were successful at identifying student errors but did not change their instructional practices based on the results. In fact, even though the teachers studied correctly identified the error, “17% of the teachers selected other areas to address first during instruction” (p. 239); this is concerning, as the teachers were able to identify the type of error (smaller-from-larger, borrow-across-zero), yet they failed to create an appropriate learning opportunity for students to overcome their errors.

When dealing with subtraction errors, teachers have the advantage of knowing that the difference is incorrect and then they have the opportunity to study the students’ work to determine the nature of the error (e.g., $9 - 2 = 6$). The equal sign poses an even bigger challenge. When dealing with the equal sign, the solution can often be correct based on the format of the task even though the student doesn’t have a relational view of the equal sign. For example, in the task $3 + 4 = \_\_\_$, teachers may assume that all students who answer 7 have a relational understanding of the equal sign. Yet in actuality these students may have an operational view of the equal sign and the teacher may be totally unaware of the students’ misconceptions until they are faced with an open sentence that elicits a more relational view of the equal sign (e.g., $3 + 4 = \_\_\_ + 2$). This idea is supported by Stephens (2006) who found that PSTs were unaware of students’ typical views of the equal sign, with few of the participants indicating that students’ faulty understanding of the equal sign may contribute to an error on the tasks.

Asquith, Stephens, Knuth, and Alibali’s (2007) comparison study looked at the correspondence between the judgments of twenty middle school teachers about their
students’ understanding of the equal sign and variables and their students’ actual performance. In regards to the equal sign, teachers believed that their students would have more of a relational understanding than what was found. The biggest discrepancy was at the seventh grade level where teachers predicted that “73% of students would give a relational definition of the equal sign, but only 37% actually did” (p. 263). This is not surprising given Falkner, Levi, and Carpenter’s (1999) study, where a sixth grade teacher wondered why researchers would want students to complete a problem like $4 + 5 = \Box + 2$, yet when all of her students missed the problem she was compelled to test the rest of the sixth graders in her school. Asquith et al. (2007) noted that the “teachers did not consider whether an operational or relational view of the equal sign might shape students’ thinking” (p. 264) about a certain task even though in the previous task they over-predicted their students’ relational understanding of the equal sign. The authors also noted that five of the twenty teachers did not respond correctly to this aforementioned task and that thirteen did not even consider recognizing equivalence as a valid possibility. Thus, if middle school teachers have difficulty in this area, then it is likely that elementary PSTs may have issues in this area as well.

Interventions

If a teacher does recognize students’ misconceptions about the equal sign, what instructional strategies, or interventions, may help his/her students gain a deeper understanding of this symbol? Falkner, Levi, and Carpenter (1999) used number sentences and true/false statements to develop a more relational view of the equal sign in a 1st and 2nd grade mixed classroom over 1.5 years. When Falkner, the classroom teacher, first gave the problem $8 + 4 = \Box + 5$ to her students she was surprised when the majority answered 12
while others created a “string” by writing 12 in the box, adding another equal sign behind the 5 with the number 17 following. Class discussion took place where students debated the merits of their solutions. One student suggested that the answer had to be 7 “[b]ecause you have to have the same amount on each side”; after careful consideration the teacher supported a definition purported by a few of the students that meant sameness (p. 233). Next, based on the work of Robert Davis (1964), she gave her students true and false statements (e.g., $2 + 8 = 10$, $3 + 4 = 7 + 2$, $5 = 3 + 2$, $9 = 9$) to discuss. During the final weeks of the school year she provided opportunities for her students to work on number sentences where the equal sign was in a variety of positions. The following year the teacher continued to integrate number sentences with boxes in strategic locations and true/false statements. She also had students create their own true/false statements. As the year progressed she also introduced variables. For example, students were asked if $a$ or $b$ was larger given $a = b + 2$.

The authors indicate that students with an operator view of the equal sign would have difficulty answering this question where those with a relational view would not. If teachers are to use tasks appropriately they need to understand the purpose of the task and see the potential the task may have for drawing out certain types of knowledge from students.

In another intervention study the researchers (Molina & Ambrose, 2006) acted as guest teachers and worked with third grade students on a weekly basis. The researchers presented open sentence tasks and found that “[t]he use of the equals sign in these sentences was unnatural to the students” (p. 112). They also used true/false sentences which caused the students to study the problems rather than immediately compute an answer. Across sessions spread out over several months, and using a variety of tasks, the teacher researchers were able to see most students progress from “get an answer” to the right, to accepting backwards
sentences (still operator) and to understanding the = as a symbol of equivalent expressions.

According to the authors “[a]sking the students to write their own sentences [e.g., □ + □ = □ + □] was particularly beneficial in helping students assimilate new information and consolidate their broadening conceptions, because they had to use the sign themselves rather than evaluate someone else’s use of it” (p. 116). The authors felt that they were only moderately successful in getting students to think in a relational way. Specifically they found that sentences like 12 + 3 = 5 + ___ caused students to stop and contemplate due to the fact that 12 + 3 = 15. The authors believed that the class discussions were extremely beneficial as these opportunities allowed students to hear the thoughts of other students in the class. They also found that some students oscillated between their older and newer understandings and thus claimed that “developing a robust understanding of the equals sign can take considerable time” (p. 117).

Instead of being reactive to students’ learning difficulties mathematics, researchers at the University of Hawaii (Dougherty, 2003; Dougherty & Slovin, 2004) decided to be proactive and develop a program for elementary students that would better prepare them to deal with the more complicated mathematics found in the middle grades. Based on the work of Davydov and Vygotsky, these researchers developed a program where students started with abstractions using comparison (of continuous measures) versus counting (of discrete objects) and worked their way to a more generalized knowledge of number. In these measurement comparison tasks, dealing with length, area, volume, and mass, students record their thinking using relational statements, such as $W > S$, to describe the situations they encounter. Students at the prenumeric stage would be focused on making unequal quantities equal or equal quantities unequal (e.g., if $W > S$, then $W - T = S$).
Unlike many U.S. primary students, the Measure Up students were extremely comfortable with variables representing quantities and were flexible in using symbolic language (i.e., <, >, =) (Dougherty, Zenigami, & Okazaki, 2005). When analyzing student work the researchers found that “[s]tudent solution methods strongly suggest that young children are capable of using algebraic symbols …” (Dougherty & Slovin, 2004, p. 301).

Interview data also supported this claim:

While interviewing students about equality, Dougherty asked, “How would you explain to a kindergartner what the equal sign is?” After the first-graders finished rolling their eyes, they answered in these exact words, “that the quantities on both sides of the equal sign are the same amount.” When we ask children who have not been in a program …they tell us, ‘this means that you have to find an answer.’” (Boynton, 2003, p. 65)

[W]hen asked to describe what 5 = 5 meant, “It’s probably true unless you have a big 5 and a little 5. Like 5 big units and 5 small units, then it isn’t true.” (Dougherty, 2003, p. 20)

A recent quantitative study (Hattikudur & Alibali, 2010) did not utilize measurement (Dougherty, Zenigami, & Okazaki, 2005), but did look at the effects of an intervention lesson that included the use of three comparative symbols (i.e., <, >, =). In this study 3rd and 4th graders were randomly assigned to different treatment lessons: equal sign only, multiple symbols (equal sign with less than and greater than symbols), and the pick the biggest number (the control). The researchers found that the multiple symbols group outperformed the other groups in the conceptual measure (i.e., determining if numeric equations like 8 = 8 or 10 = 3 + 7 were correct or incorrect) and the symbol sort (which symbols are alike). In the problem solving measure (i.e., 2 + 3 + 6 = 2 + __) there was no significant difference among the three groups. When asked to solve problems involving inequalities, the multiple symbols group outperformed the other two groups.
Hattikudur and Alibali (2010) found that comparison can enhance the learning of mathematics. However, because this was a one-day study consisting of one thirty-minute lesson, it is hard to discern if the knowledge gained was retained by the students over the long term. In addition, because there was no significant improvement in the completion of tasks that require relational understanding (i.e., problem solving measure) it is hard to ascertain what kind of knowledge the students gained. If the students did attain some type of relational view (as measured by symbol sorting) and conceptual knowledge (as measured by successfully labeling numeric equations as incorrect and correct) of the symbol, then it is possible that the students are in the process of developing a relational view of the equal sign. Yet, as the authors note, it may take more time for students to move from a conceptual to a problem solving understanding of the equal sign, which would be consistent with Behr et al. (1976) who found that students had entrenched operator responses to open number equations. Regardless, in the same amount of intervention time, the multiple symbol group either matched or outperformed the other two groups. Therefore, this research and other studies support the use of multiple relational symbols for a deeper understanding of mathematical symbols (Dougherty, Zenigami, & Okazaki, 2005; Hattikudur & Alibali, 2010; Li et al., 2008).

Powell and Fuchs (2010) studied third graders with mathematics difficulties who were divided into three different groups: those who received word-problem tutoring, word-problem tutoring and equal sign instruction, and no tutoring. In their research, non-standard equations (i.e., \(8 + 5 = x + 2\)) were not given to any of the three groups and yet they found that on these types of equations the benefits of equal-sign instruction were prominent. They concluded that a “relational understanding of the equal sign carries important transfer effects
to solving nonstandard equations” (p. 392). Another study with seven and eight year olds had similar results (McNeil, Fyfe, Petersen, Dunwiddie, & Brletic-Shipley, 2011). In this study children were also divided into three different groups: those who received practice on traditional problems (e.g., $2 + 3 = \_\_\_\_$), practice on nontraditional problems (e.g., $\_\_\_\_ = 2 + 5$), and a control group that did not receive additional practice. Practice, in both cases, consisted of one-on-one tutoring in which the problems were consistent with the non-traditional or traditional assignment. During the three sessions participant and the tutor played a card or dice game, practiced flashcards, and played a computer game. In addition, brief worksheets were given as homework between practice sessions. McNeil et al. found that those in the nontraditional group demonstrated a better understanding of equivalence than the other two groups although the gains were not as pronounced as one might expect with these students only answering half of the equivalence problems correctly. In addition, a little less than a fourth of these students defined the equal sign relationally. The difficulties of these 7 and 8 year olds with the definition task are not surprising when one considers Rittle-Johnson et al.’s (2011) findings. They found that “generating a relational definition of the equal sign was much harder than solving or evaluating equations with operations on both sides. Rather, generating a relational definition was as hard as recognizing that a relational definition is the best definition of the equal sign (a Level 4 item)” on their *Construct Map for Mathematical Equivalence Knowledge* (p. 97).

McNeil et al. (2011) posited that “…learning difficulties arise when to-be-learned information overlaps with, but does not map directly onto, entrenched patterns” (p. 1621). This change-resistance account purports that “difficulties with mathematical equivalence stem not from general conceptual or working memory limitations in childhood, but from
children’s representations of patterns routinely encountered” in their early years of education (McNeil & Alibali, 2005b; McNeil, 2007, p. 1621). This conclusion is similar to that of MacGregor and Stacey (1997), who studied 11-15 year olds’ understandings of variable and found that the origins of students’ misconceptions were associated with “intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials” and not necessarily cognitive level (p. 1).

**Technology**

In regards to using technology to improve students’ understandings of the equal sign, the research is sparse. Some of the most innovative approaches to looking at students’ understanding of the equal sign and how to improve their understanding involve the use of technology “microworlds” where students can construct new knowledge via the technology (Jones, 2009a, 2009b; Jones & Pratt, 2005). Jones and Pratt looked at the responses of three pairs of eight and nine year olds when operating in a relational calculator “microworld”. The calculator had two sets of keypads with three screens displayed. Above the keypads on the far left, the numeric expression typed in via the left keypad were displayed, while on the far right, the numeric expression typed in via the right keypad were displayed. In the middle between the two expressions was a blank box. If students clicked on an operator in one of these expressions, a number solution appeared in that spot (i.e., $2 \cdot 3 + 32$ changed to $6 + 32$ if the $\cdot$ was clicked). So, for example, a student could type in $3 + 4 - 2$ using the left keypad and type in $7 - 2$ on the right keypad and subsequently click on each operator until the number 5 appeared on both sides, thus affording the student the opportunity to see that both
numeric expressions produced equivalent numbers. The authors found that only one of the three pairs of students in the trials seemed to recognize the equivalence aspect of the calculator. These authors “see the pedagogic challenge not as one of eliminating the operational utility but as providing new experiences, carefully designed, to optimize the possibility that the child may construct new utilities for the equal sign such as that of equivalence” (p. 192). Whether involving technology or not researchers and other stakeholders need to gain insight into whether PSTs and ISTs are up for this “pedagogic challenge.” Are they aware of students’ understandings of the equal sign?

Jones (2009a, b) and colleagues (Jones & Pratt, 2011; Jones et al., 2012) believed that, for students to have a complete understanding of equivalence as a relation, students need not only a “is the same as” understanding of the equal sign but also a “can be substituted for” understanding as well (2009a, p. 257). In Jones’ aforementioned study the software provides the students with a “puzzle,” a numeric expression (only sums) with statements of equality below (addition only). The students’ task was to select successive statements that allowed them to transform the initial expression into a single value. One issue with the software was that students gave little thought to the concept of balance. For example, when false statements (i.e., $22 + 12 = 30$) were listed as possible choices, students did not seem to notice the false statements and their presence did not impact the processes they used to solve the problems. The author noted that, “[a]nalogous to algebraic symbol manipulation, there is simply no advantage to considering numerical balance or conservation of quantity when working towards the task goal of transforming the boxed term into a single numeral” (2009a, p. 259). In order to deal with this issue, Jones conducted another study where a pair of students, who were part of the initial trial, was instructed to develop their own “puzzles” in
which they created the initial numeric expression and the statement of equalities using software tools. The children were able to correctly create multi-step puzzles and thus showed understanding of both balance and substitution. However, in a follow up study Jones and Pratt (2011) found that only high achievers were able to activate both the basic relational meaning (sameness) and substitutionary meaning of the equal sign in a coordinated manner. Those who were considered medium achievers could oscillate between the two understandings but could not bring together the two meanings in order to successful develop their own puzzle. Jones hypothesized that fostering this “duality of meanings for the equal sign” may help students progress from arithmetic to algebra (2009b, p. 263).

**Validity and Reliability**

The aforementioned studies typically used researcher-made tasks in which little is known about the validity and reliability of the measures used, a situation not atypical in mathematics education research (Hill & Shih, 2009). In contrast, it is worth noting that Rittle-Johnson et al. (2011) developed a measure for elementary students’ equality understandings, and recently Matthews, Rittle-Johnson, McEldoon, and Taylor (2012) replicated the original study with a different population of students (public, versus parochial) reaffirming the reliability and validity of their measurement tool. As mentioned previously, the initial version of the WTCE (Wheeler, 2010) developed to measure undergraduates’ understandings of the equal sign had low reliability (alpha = 0.34) and neither item analysis nor reliability testing was conducted after improvements were made so, little is known about the validity of the final measure. Consequently, the research suggests the need to create a better measure and to address the validity of the measure used.
Summary

Research has repeatedly documented American elementary students’ operational view of the equal sign (Alibali, 1999; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1976; Behr et al., 1980; Clements, 1982; Falkner, Levi, & Carpenter, 1999; Ginsburg, 1977; Hattikudur & Alibali, 2010; Kieran, 1981; McNeil & Alibali, 2005a). Some have found this same view lingering into the middle school and college years (Byers & Herscovics, 1977; Clements, 1980; Kieren, 1981; Knuth et al., 2006; Li et al., 2008; McNeil & Alibali, 2005b; McNeil, Grandau, Knuth, Alibali, Stephens, et al., 2006; McNeil, Rittle-Johnson, Hattikudur, Peterson, 2010; Oksuz, 2007; Weinberg, 2010; Wheeler, 2010). This is discouraging as a relational view of the equal sign is essential for algebra success (Knuth et al., 2006; Matthews et al., 2012) thus implying that equal sign misconceptions may contribute to the “closing” of the algebra “gate.” Given the limited understanding of the equal sign, the need to further explore this area of study is crucial.

The majority of studies in the U.S. have focused on preschool through 6th grade students. There were only five studies found that included 7th and 8th grade students (Alibali, Knuth, Hattikudur, McNeil, & Stevens, 2007; Knuth et al., 2005, Knuth et al., 2006; McNeil & Alibali, 2005a; Rittle-Johnson & Star, 2009). At the undergraduate level there were only five studies located that directly explored undergraduates’ understanding of the equal sign (McNeil & Alibali, 2005a, 2005b; McNeil et. al, 2010; Weinberg, 2010; Wheeler, 2010). Yet, it is hard to generalize these findings to all undergraduates as four of these studies used more advanced mathematics students (McNeil & Alibali; 2005a, 2005b; McNeil et. al, 2010; Weinberg, 2010) while the other used students enrolled in developmental mathematics (Wheeler, 2010). In my review of the literature I found no research that represents a more
“average” undergraduate’s understanding of the equal sign. In order to add to this limited body of literature and to explore more typical undergraduate students this study is warranted.

Stephen (2006) stated that, in regards to elementary PSTs’ and ISTs’ knowledge, “very little research is focused specifically on early algebra” (p. 250), indicating a need for more research in this area. Asquith et al. (2007) was the only study found that analyzed in-service middle school teachers’ beliefs about students’ understandings of the equal sign; Stephens (2006, 2008) studied pre-service elementary teachers’ views of student work dealing with the equal sign and equations, although neither of these studies explicitly set out to analyze the teachers’ understanding of the equal sign. All three of these studies had samples sizes of 30 or less, indicating the need for large sample studies in this area.

In regards to the equal sign, the CCSS-M seems to assume that teachers have a deep relational view of the equal sign, are aware of students’ typical misconceptions, and understand the implications of the provided examples. It is crucial that researchers delve deeper into what kinds of understandings PSTs hold. Stephens’ (2006, 2008) work, similar to the previous undergraduate studies, looked at higher ability students (i.e., 9 of her 30 participants had taken at least one calculus course, 12 of the 30 indicated that mathematics was one of the subject(s) they really wanted to teach), thus indicating the need to look at the understandings of PSTs who have a more typical background in mathematics. In addition, when reading the literature I found no studies that looked cross-sectionally at PSTs’ understandings of the equal sign. Nor were there studies that connected PSTs’ specialized knowledge needed to teach mathematics (i.e., their MKT) and their understandings of the equal sign. In addition, there were no studies found that looked at PSTs’ understandings of the equal sign as they progress through their teacher preparatory coursework. This is crucial
as “[u]ncovering preservice teachers’ thinking about early algebraic ideas is thus an important part of informing the design of teacher education in this area” (Stephens, 2006, p. 250). Finally, there are few studies (Hill & Shih, 2009) that deal with validity and reliability of the measures used, so this study will address this issue prior to statistical analyses. The current study addressed these specific questions in order to help fill the gaps in the literature. The design and methodology are described in the next chapter.
CHAPTER 3. METHODOLOGY

Research Design

In this chapter I discuss the design and methodology used in my study. I describe the participants and some of their characteristics. I also highlight the tasks in my researcher-developed questionnaire, the Equal Sign Questionnaire (EQ), which I developed to explore elementary pre-service teachers’ (PSTs’) understanding of the equal sign. In this section I analyze this instrument and explain how it was used to examine PSTs’ understandings of the equal sign using both descriptive and inferential statistics.

Sample

The sample for this study was comprised of PSTs enrolled in three different teacher preparation courses at a land grant university in the Midwest. The courses in which they were enrolled included both courses in a two-semester sequence of mathematics content geared specifically for future elementary teachers (hence referred to as MCC1 and MCC2) and a mathematics methods course (MMC) designed for elementary education majors not pursuing an endorsement in early childhood education. Participation was voluntary and of those enrolled in the three courses (four sections of MCC1, four sections of MCC2, and two sections of MMC), 93.1% (n = 268) of the students were included in the main study. Subsets of this sample, which will be addressed later, were also used in analyses.

MCC1 and MCC2

In order to be enrolled in MCC1 students have to meet the prerequisites of a satisfactory math placement exam score, two years of high school algebra, and one year of
high school geometry. According to the mathematics department, MCC1 used the Beckmann (2011) text. According to the course description, this course focuses on “theoretical and hands-on models; standard and non-standard algorithms and properties related to whole numbers and whole number operations”. In section 3.2 of the course text, Beckmann wrote about the proper use of the equal sign stating that it is “common for students at all levels (including college) to make careless, incorrect use of the equal sign. Because the proper use of the equal sign is essential in algebra, and because elementary school mathematics lays the foundation for learning algebra, it is especially important for you to use the equal sign correctly” (p. 105). An instructor of the course stated that, due to the emphasis the Beckmann text places on the correct use of the equal sign (versus the previous text), course instructors now deduct a set number of points every time a PST misuses the symbol in his/her work. Students enrolled in MCC2 use the same text and are also required to meet the prerequisite of a C- or above in MCC1. According to the course description, this course covers “two-and three-dimensional measurement, probability, data fitting, statistics, operations and algorithms for computing with integers, fractions, and decimals.”

MMC

MMC students are required to have completed MCC1 and MCC2, or the equivalent at another institution, with a C- or above and to be enrolled concurrently in a mathematics practicum, a reading/language arts methods course, and a reading/language arts practicum. The course description states that this course focuses on the “[s]tudy, development, and application of current methods for providing appropriate Mathematical learning experiences for primary and intermediate children. Includes critical examination of factors related to the
teaching and learning of Mathematics.” According to one of the instructors, the course piloted a Cognitively Guided Instruction (Carpenter et al., 1999)-based methods book developed by Drake, Land, Franke, Johnson, and Sweeney (in press, 2013). This book, unlike most methods texts, was not organized around content strands but was “organized around a specific classroom structure and particular teaching practices that … are critical for teaching through problem solving” (Drake et al., 2013, p. 2).

**Characteristics of the Participants**

Of the 268 PSTs in the sample, approximately 45% were enrolled in MCC1, 38% were enrolled in MCC2, with the remaining students enrolled in MMC. The gender breakdown was roughly 90% female and 10% male. In regards to their confidence level to teach mathematics, 25.7% indicated that they lacked confidence or were not very confident; 54.5% indicated that they were somewhat confident; and 18.7% reported that they were very confident (with three students not responding to this question). As for their educational background, 40.7% had never taken a pre-calculus or calculus course in high school or college, while 58.6% indicated that they had taken this type of course during high school or college (two students did not respond). In regards to endorsements, 23.5% of the PSTs indicated that they were pursuing a mathematics endorsement. The breakdown of these categories by course and gender is shown below in Tables 1 and 2.
Table 1. Characteristics of PSTs by course

<table>
<thead>
<tr>
<th>Characteristics of PSTs by course</th>
<th>MCC1 (n = 120)</th>
<th>MCC2 (n = 101)</th>
<th>MMC (n = 47)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gender</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Female (n = 240)</td>
<td>85.8%</td>
<td>93.1%</td>
<td>91.5%</td>
</tr>
<tr>
<td>Male (n = 28)</td>
<td>14.2%</td>
<td>6.9%</td>
<td>8.5%</td>
</tr>
<tr>
<td><strong>Confidence to teach mathematics</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very confident</td>
<td>17.5%</td>
<td>21.8%</td>
<td>14.9%</td>
</tr>
<tr>
<td>Somewhat confident</td>
<td>48.3%</td>
<td>61.4%</td>
<td>55.3%</td>
</tr>
<tr>
<td>Lacked confidence or not very confident</td>
<td>31.7%</td>
<td>16.8%</td>
<td>29.8%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>2.5%</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Mathematics background: pre-calculus or calculus in high school or college</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Completed this type of course</td>
<td>56.7%</td>
<td>59.4%</td>
<td>61.7%</td>
</tr>
<tr>
<td>Did not complete this type of course</td>
<td>41.7%</td>
<td>40.6%</td>
<td>38.3%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>1.7%</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Where MCC1 was completed</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>At the university studied</td>
<td>90.1%</td>
<td></td>
<td>66.0%</td>
</tr>
<tr>
<td>At a different four year institution</td>
<td></td>
<td>2.1%</td>
<td></td>
</tr>
<tr>
<td>At a community college</td>
<td>9.9%</td>
<td></td>
<td>31.9%*</td>
</tr>
<tr>
<td><strong>Where MCC2 was completed</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>At the university studied</td>
<td></td>
<td></td>
<td>78.7%</td>
</tr>
<tr>
<td>At a different four year institution</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>At a community college</td>
<td></td>
<td></td>
<td>21.3%*</td>
</tr>
<tr>
<td><strong>Endorsement</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeking a mathematics endorsement</td>
<td>17.5%</td>
<td>25.7%</td>
<td>34.0%</td>
</tr>
<tr>
<td>Not seeking a mathematics endorsement</td>
<td>81.7%</td>
<td>74.3%</td>
<td>66.0%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>0.4%</td>
<td></td>
<td>3.6%</td>
</tr>
<tr>
<td><strong>Note:</strong> Percents may not add up to 100% due to rounding.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>* Six PSTs completed both courses at a community college.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: PSTs Characteristics by Gender

<table>
<thead>
<tr>
<th>Characteristics of PSTs by gender</th>
<th>Female (n = 240)</th>
<th>Male (n = 28)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Confidence to teach mathematics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very confident</td>
<td>17.5%</td>
<td>28.6%</td>
</tr>
<tr>
<td>Somewhat confident</td>
<td>53.3%</td>
<td>64.3%</td>
</tr>
<tr>
<td>Lacked confidence or not very confident</td>
<td>28.8%</td>
<td>7.1%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>0.4%</td>
<td>7.1%</td>
</tr>
<tr>
<td><strong>Mathematics background: pre-calculus or calculus in high school or college</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Completed</td>
<td>57.1%</td>
<td>71.4%</td>
</tr>
<tr>
<td>Did not complete</td>
<td>42.5%</td>
<td>25.0%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>0.4%</td>
<td>3.6%</td>
</tr>
<tr>
<td><strong>Endorsement</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeking a mathematics endorsement</td>
<td>22.1%</td>
<td>35.7%</td>
</tr>
<tr>
<td>Not seeking a mathematics endorsement</td>
<td>77.9%</td>
<td>60.7%</td>
</tr>
<tr>
<td>Did not respond</td>
<td>3.6%</td>
<td></td>
</tr>
<tr>
<td><strong>Note:</strong> Two males completed MCC1 at a community college the rest of the PSTs who completed MCC1 and MCC2 at different institutions were female.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Instruments

Equal-sign Questionnaire (EQ): Part I

The first part of the questionnaire (see Figure 2; the precise forms used in this study are located in the appendices and are listed in the procedure section) asked PSTs to define the equal sign. Asking students to define the equal sign either verbally or in writing has been used extensively in the research (Alabali et al., 2007; Asquith et al., 2007; Behr et al., 1976; Hattikudur & Alibali, 2010; Knuth et al., 2006; Knuth et al., 2008; Knuth, Stephens, & McNeil, 2006; Matthews et al., 2012; McNeil & Alibali, 2005, McNeil et al., 2006; McNeil et al., 2011; Oksuz, 2007; Powell & Fuchs, 2010; Rittle-Johnson & Alabali 1999; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011; Seo & Ginsburg, 2003; Stephens, 2006, Stephens et al., 2013). The format for this task (see Figure 2), an arrow pointing at the equal sign followed by three questions, has been widely used and was adapted from Knuth and his colleagues (2006). Although not all of these studies showed the symbol in isolation (i.e., most use $3 + 4 = 7$), I chose to use this format as researchers have found that context may elicit certain types of understandings of the equal sign (McNeil & Alibali, 2005a; McNeil et al., 2006). Based on Knuth et al. (2006), the first question is used to minimize the possibility of students answering question two with “equals” and the third question is given as participants often provide more than one definition when asked. This page was to be submitted prior to the student completing Part II, the six-task questionnaire.
What is the name of the symbol that the arrow is pointing at?

What does that symbol mean?

Can you write down another definition? If so, what would it be?

---

**Figure 2. EQ definition**

**Equal-sign Questionnaire (EQ): Part II**

The second part of the questionnaire (Figure 3) was designed to assess PSTs’ understandings of the equal sign utilizing problems that were drawn from previous studies and also new problems to contribute to the field. The assessment consists of six tasks: three are open sentences where the answers are numeric, two are true/false, and one is an open-ended question asking for a numeric answer.

A. \[267 + 85 = \_\_ + 83\]

B. \[3x + 6 = 60\] is true. Is \[3x + 6 - 7 = 60 - 7\] TRUE or FALSE? Why?

C. \[\_\_ + 21 = 25 + 32 = \_\_\]

D. Knowing that the sum of 79 and 148 is 227, can you find the sum of 149 and 82? If so, what is it?

E. \[3 (\_\_) + 8 = 2 (\_\_) + 8\]

F. Circle whether the following statements are true, false, or not enough information to tell.

\[12 + 5 = 17 + 2 = 19\] TrueFalseNot Enough Information

\[2x + 14 = 7 + x\] TrueFalseNot Enough Information*

* Not coded.

---

**Figure 3. EQ tasks**
Task A (see Figure 3) is commonly used in the research to test for equal sign understanding (Carpenter & Levi, 2000; Falkner et al., 1999; Gonzalez, Ambrose, & Martinez, 2004; Molina & Ambrose, 2008; Powell & Fuchs, 2010; Saenz-Ludlow & Walgamuth, 1998; Stephens, 2006). If participants fill in the blank with 352, then one would conclude an operational view of the equal sign; a solution of 269 with “work” would indicate that the participant calculated the answer (i.e., $267 + 85 - 83$) and thus has a relational with computation understanding; while a solution of 269 with no work would signify relational. Although it is possible that a participant could calculate the solution mentally, the numbers were chosen strategically to require regrouping.

Task B (see Figure 3) is also based on the work of several researchers (Alibali et al., 2007; Asquith et al., 2007; Knuth et al., 2008; Matthews et al., 2012; Rittle-Johnson et al., 2011; Steinberg, Sleeman & Krorza, 1990; Stephens, 2006; Stephens et al., 2013). This task is designed to determine if students need to solve for $x$ to see if the second equation is true or if they can just “look” at the second equation and tell if it is true based on relational understanding. Therefore, if a participant responds with something like “True, because both equations are the same” (with no calculations) this would demonstrate a fully relational understanding, while a response of “True, because $x$ is the same in both equations” (with mathematical work solving for $x$) would be evidence for a relational with computation understanding. Finally, a student who responds with “False” with no mathematical work would suggest an operational understanding. Note that a non-standard transformation was used so that it would not fit the traditional solving format (Alibali et al., 2007).

Task C (see Figure 3) was based on a task used by Li, Ding, Capraro, and Capraro (2008); in this case two-digit numbers were used to discourage mental calculations. This task
is designed to assess whether students view “strings” or “run-ons” as an accurate way to denote mathematical work. Weinberg (2010) found that this error was common among undergraduate calculus students. In this task, students with an operational view of the equal sign may fill in the first blank with four and the second blank with 57. Those with a relational understanding would look at the structure of the problem and put 36 in the first blank and 57 in the second, while others may use computation to find these same values. Another “string” is Task F (see Figure 3), which is written in a true/false format. Several researchers use true/false questions to assess students’ understandings of the equal sign but this task was based on work by Oksuz (2007) and Weinberg (2010).

I wrote Task D (see Figure 3) in order to measure a “can be substituted for” meaning as advocated by the work of Jones (2009a, 2009b) and colleagues (Jones et al., 2012; Jones & Pratt, 2011). In this case, some students will realize that four more than 227 is the sum which would indicate a substitutionary/relational understanding where those who simply add 149 and 82 would demonstrate an operational understanding.

Task E was based on the work of Oksuz (2007) who used a “boxes on both sides format” (i.e., $26 + \square = 12 + \square$). I developed this item to see if participants would use the commutative property, multiplication by zero, or other values to cause equivalence. The second equation on task F (i.e., $2x + 14 = 7 + x$) was not assessed. This item was based on the work of Weinberg (2010) but when I was editing my form I inadvertently left off the why or why not part of the question.
Equal-sign Questionnaire (EQ): Additional questions

Additional questions were asked to understand the characteristics of the PSTs studied. These questions included information about their gender, mathematical background, endorsement area(s), confidence to teach mathematics and other similar questions.

Mathematical Knowledge for Teaching (MKT)

Researchers who have attended training can access the MKT measurement tools as part of the Learning Mathematics for Teaching project (LMT, a sister project to the Study of Instructional Improvement (SII)). Currently there are premade assessments that measure both common knowledge of content (CKC) and specialized knowledge of content (SKC) within one form. These well developed and validated forms exist in the following domains:

(1) Number concepts and operations (K-6, 6-8); (2) Patterns, functions, and algebra (K-6, 6-8); and (3) Geometry (3-8). There is also an assessment that measures teachers’ knowledge of content and students (KCS) within the domain of Number concepts and operations. They also have measurement tools based on topics covered at the 4-8 grade levels in: (1) Rational number; (2) Proportional reasoning; (3) Geometry; (4) Data, probability, and statistics. In addition, there is a “use at your own risk” elementary place value form for which no statistical analyses have occurred. Researchers can use these already developed assessments or can select tasks to develop their own measures. According to the LMT website, the…

[i]tems in each category capture whether teachers can not only answer the mathematics problems they assign students, but also how teachers solve the special mathematical tasks that arise in teaching, including evaluating unusual solution methods, using mathematical definitions, representing mathematical content to students, and identifying adequate mathematical explanations. [In addition], [e]ach elementary (K-6) item has each been piloted with over 600 elementary teachers, yielding information about item characteristics and overall scale reliabilities for piloted forms.
These items are not simply mathematical problems where the participants choose a correct answer out of a set of numeric answers. An example of a task from their released items states:

*Imagine that you are working with your class on multiplying large numbers. Among your students’ papers, you notice that some have displayed their work in the following ways: [the work of three different students is shown]. Which of these students would you judge to be using a method that could be used to multiply any two whole numbers? (Hill, Schilling, & Ball, 2004, p. ___)*

After posing the question the researchers asked the teacher to make a judgment about each solution path using the responses: *Method would work for all whole numbers, Method would NOT work for all whole numbers, or I’m not sure.* Thus, the tasks go beyond merely measuring mathematical knowledge, but instead measure *mathematical knowledge for teaching.*

Researchers at the studied institution involved in investigating ways to measure the impact of mathematics teacher education (subsequently referred to as the MIMTE study) attended the required training. These researchers use the MKT measures longitudinally to assess the effectiveness of a teacher preparatory program at a land grant university. In this study, which took place at the same land grant institution, data were linked to the MIMTE study to see what, if any, relationship exists between PSTs’ understandings of the equal sign and their scores on the MKT measures. As part of the MIMTE study, existing MKT forms were used, one dealing with number concepts and operation (MKT-NCOP-CK 2004A) and one dealing with patterns functions and algebra (MKT-PFA-CK 2006A).

These assessments are the most recent versions advanced from the researchers initial 2001 work (Hill, Schilling, & Ball, 2004). In this aforementioned work the researchers’ used exploratory factor analyses, factor analyses with some of the items removed, and bi-factor
analyses (a type of factor analysis where there is a general factor and several group factors). They found that the “reliabilities for patterns, functions, and algebra scales, as well as for scales that combined number and operations within each domain [i.e., knowledge of content and knowledge of students and content], were good to excellent, ranging from 0.71 to 0.84” (p. 25). For the forms used in this study, the supporting materials provided by LMT state that the reliabilities for MKT-NCOP-CK 2004A and for PFA-CK 2006A (using a one parameter model) are 0.79 and 0.84 respectively. In addition, cognitive interviews, videos of teachers teaching, and discussions with mathematicians have also been used to validate that these measurements are measuring what they are designed to measure. According to the LMT website the measures were developed in such a way that 50% of the tasks would be answered correctly by ISTs. The LMT project standardized the tests and requested that IRT scores ($M = 0, SD = 1$) be used when reporting data.

**Pilot Study**

The EQ, which initially consisted of ten tasks plus additional questions, was piloted with a small number of undergraduates ($n = 3$) prior to the study. Participants offered feedback as to the clarity of the instructions, their understandings of the tasks, and any thoughts they wanted to share about the forms. The time needed for the completion of the materials was also noted and, prior to sending the survey to the Institutional Review Board (IRB), the questionnaire was edited based on this feedback. Following additional feedback from the IRB review, the original 10 tasks developed for the instrument were edited to the six discussed previously.
Procedures

This study took place in conjunction with the MIMTE study. IRB approval was obtained and the EQ was given to students in the targeted courses during the first two weeks of the semester during a normal class period. On the day of the assessment students were given two consent forms, for the MIMTE study and for this study, which were subsequently explained by the administrator of the surveys. There were two administrators of the EQ; neither was an instructor of the course being assessed nor the researcher in this study. After the consent forms were explained (see Appendices A and B), the administrator gave the PSTs the two MKT measures. After completing the MKT assessments, the PSTs were given the two-part EQ. MCC1 and MCC2 students received the same questionnaire (see Appendix C), while MMC received a slightly different questionnaire (i.e., it asked where they completed both MCC1 and MCC2 versus if they had taken MCC1 and if so, where; see Appendix D). All questionnaires asked the students to: define the equal sign (this page was collected after completion); complete six mathematical tasks dealing with equal sign knowledge (Tasks A-F above); and answer additional questions about their mathematical background in high school, additional mathematics courses taken in college, endorsements, grade level teaching preference, confidence level in teaching mathematics, gender. MCC2 and MMC students were asked to report where they took MCC1 and MCC2 (if not currently enrolled) and to report the grade earned in the course(s) completed. At the end of the semester the MMC students were assessed a second time. This questionnaire contained the definition task, the six initial tasks, and an additional page with two additional questions: one question dealing with additional solutions to item E and another question dealing with using true/false equations from the CCSS-M (see Appendix E). These questions were designed more to guide
future research than to be evaluated as part of this study. I will discuss these two questions at the end of this section. In MCC1 and MCC2 the same participant codes were used for this study and the MIMTE study so that that data could be linked at the individual level. Different codes were used to link the pre and post data in MMC.

**Data Analysis**

**Removal of participants from the sample**

There were 288 students enrolled in these courses at the time of the study. Of these 273 students participated in the study. I removed five students from the study for the following reasons: one student was not going into teaching but merely taking the course to raise his/her GPA, one took an extreme amount of time on the MKT measures and lacked sufficient time to complete the EQ, two had definitions which did not fit the context and could not be coded, and one had the researcher as an MCC1 instructor at a different institution. There were two students who did not turn in the definition task prior to completing the second part of the survey. In both cases the students had strong relational understandings and it was deemed that the discrepancies with the standard procedure did not affect the validity of the results.

Those PSTs who were removed by the researcher showed various types of understandings of the equal sign. There is no way to discern the equal sign understandings of those who didn’t sign the consent or were absent.
Coding

The equal sign definition and the six tasks were coded by the researcher and a second coder. Matthews, Rittle-Johnson, Taylor, and McEldoon (2010), informed by earlier work by Carpenter, Franke, and Levi (2003), defined four levels of increasingly complex understandings of the equal sign that students hold. “Rigid operational” (Level One) is when students can only solve problems in the traditional $a + b = \_\_\_$ format. “Flexible operational” (Level Two) is when students have an operational view but can solve some other formats, like $\_\_\_ = a + b$. “Relational with computational support” (Level Three) is when students have emerging relational understandings combined with their existing operational views, which allows them to solve tasks like $a + b + c = \_\_\_ + d$. Finally, “Fully relational” (Level Four) is when students can solve the tasks without the need to calculate and show understandings of equivalence properties. In this study the same terminology will be used with the term “operational” representing both rigid operational and flexible operational understandings.

Questions were coded for accuracy (correct = 1, incorrect = 0) and for the type of equal sign understanding the PST demonstrated using a coding guide (see Appendix F). Tasks were referred to by their letter name when coding for correctness (e.g., A); when coding for understanding they were labeled with the prefix of “U” for understanding (e.g. UA). The definition task and the six tasks were coded by the researcher, who coded original documents, and a second coder, who coded scans of the documents. After coding the MCC1, MCC2, and MMC pretest data separately there was a 93% agreement between the two coders. At this point the coders met face-to-face (with the original data) and went through all discrepancies until 100% agreement was met. Most discrepancies were the result of human
error such as typographical errors or the inability to see students’ pencil work due to the
degree of the scans. This process was also used at the end of the semester with MMC posttest
data, although these differences were discussed over the phone until consensus was reached.

When coding skipped tasks, the coders assigned a score of zero (operational) since
there was no evidence to suggest a more advanced understanding of equivalence. One
student, who lacked time to sufficiently complete the questionnaire, as noted by the
administrator, was removed from the study. When coding tasks that were partially
completed, evidence was used to support the coding. For example, a PST who answered True
on task B but did not provide a reason was coded as fully relational as the evidence (i.e.,
True) supports this coding and there is no evidence to suggest otherwise. The additional
questions were left blank if the participant did not answer them and are reported as “no
response” in the analyses. The two MKT measures, administered for the MIMTE study, were
scored by the graduate student who administered most of the MKT assessments and the EQs.
Both the total correct and the corresponding IRT scores, based on information provided by
the MKT authors, were shared with the researcher by MIMTE project staff.

**Statistical software and tests**

The IBM SPSS version 19 (hence referred to as SPSS) was used to analyze the data.
Data were analyzed as a group, by subgroups, and between groups. Descriptive statistics
were used to report means, percentages correct, and other relevant findings. Factor analysis
was used to explore the validity of the EQ. Even though the distributions were not perfectly
normal, parametric procedures were used due to the robustness of parametric procedures for
large samples (Gravetter & Wallnau, 2009).
In order to see if there was a significant difference between the mean EQ scores based on two different groups, an independent-samples t-test was used to explore the relationship between EQ scores and gender and also EQ scores and pre-calculus/calculus background. In order to compare the means of three groups one-way analysis of variance (ANOVA) was appropriate, thus this statistical test was used to explore the relationship between PSTs’ EQ scores and their confidence to teach mathematics. In addition, ANOVA was used when comparing the mean EQ scores between PSTs enrolled in the three different courses (MCC1, MCC2, and MMC). When looking for a possible relationship between PSTs’ EQ scores and their MKT-NCOP scores, and also the relationship between the PSTs’ EQ scores and their MKT-PFA scores, correlation analysis was used. Partial correlation was also used to control for additional variables (gender, pre-calculus/calculus, and confidence level). When looking for change in PSTs’ understandings due to the intervention of a mathematics methods course a paired t-test was used analyze the data.

Although parametric tests were used, I also performed non-parametric tests, when possible, in order to further validate the results. Since multiple statistical tests were run a Bonferroni adjustment was made to the 0.05 alpha level, resulting in a new alpha level of 0.006. This alpha was used for all of the tests run on the PSTs as a group. For the last two statistical tests, one which looked at the data cross-sectionally and the other that looked at pre-post-EQ scores for the PSTs enrolled in the methods course, an alpha level of 0.05 was used.
Measurement: Developing the EQ measure

One goal of this study was to quantify PSTs’ understandings of the equal sign so that comparisons based on course, gender, MKT scores, and other variables could be made. In order to accomplish this goal a measurement tool was needed. Initially, the sum of the understanding codes was used as an index of PSTs’ understanding of the equal sign score (EQ score). Factor analysis was used to determine if this was a suitable measure and adjustments were made, as described below.

Reliability

Due to the time constraints placed upon the assessment (approximately ten minutes) it was determined that only seven equal sign tasks plus additional background questions would be used. A Cronbach’s alpha of 0.63 was found. This level, although below 0.7 (Nunnally, 1978), was encouraging given the small number of tasks. When looking at the output it was noted that dropping item UE would increase the Cronbach’s alpha level to 0.64. In addition, if item UB were also removed to the resulting alpha would have been 0.65. In order to decide if the deletion of either task was appropriate, I next conducted factor analysis.

Factor analysis

The seven tasks that were part of the Equal-sign Questionnaire (EQ) were analyzed using SPSS to determine if the assessment was measuring one category. First, to determine if factor analysis was appropriate for the data, assumptions needed to be addressed. Due to the large sample size, \( n = 268 \), and a strong ratio of participants to items (above 10:1) factor analysis seemed plausible (Pallant, 2010). The Kaiser-Meyer-Olkin Measure of Sampling Adequacy (KMO) and the Bartlett’s Test of Sphericity values were determined to test the
factorability of the data. For these data the KMO value was 0.73, which meets the .6 or above criterion and the result of the Bartlett’s test was significant (less than or equal to 0.05). In addition, some of the coefficients in the correlation matrix were 0.3 or above, also meeting the criteria for factor analysis.

Next, the items were analyzed using principal components analysis (PCA). When looking at the number of components to extract, several tests were used. The Kaiser criterion indicated there may be two components (two eigenvalues of one or more, see Table 3 second column). These two components explained 47.73% of the variance. Although, when the Scree plot (see Figure 4) was analyzed it indicated that one component was to be retained (i.e., one point above the “elbow”). Parallel analysis involves “comparing the size of the eigenvalues with those obtained from a randomly generated data set of the same size. Only those eigenvalues that exceed the corresponding values form the random data set are retained” (Pallant, p. 184). In this analysis there is only one eigenvalue higher than the criterion value (see Table 3 fourth column). When looking at the component matrix in Table 4, most items loaded strongly with values above 0.4 under one component, thus supporting the one component model.

Table 3. Eigenvalues and variance explained

<table>
<thead>
<tr>
<th>Component</th>
<th>Variance (%)</th>
<th>Eigenvalue</th>
<th>Criterion value* (decision)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.229</td>
<td>32.839</td>
<td>1.229 (accept)</td>
</tr>
<tr>
<td>2</td>
<td>1.042</td>
<td>14.889</td>
<td>1.137 (reject)</td>
</tr>
<tr>
<td>3</td>
<td>0.955</td>
<td>13.644</td>
<td>1.066 (reject)</td>
</tr>
<tr>
<td>4</td>
<td>0.779</td>
<td>11.135</td>
<td>0.998 (reject)</td>
</tr>
<tr>
<td>5</td>
<td>0.733</td>
<td>10.470</td>
<td>0.926 (reject)</td>
</tr>
<tr>
<td>6</td>
<td>0.710</td>
<td>10.144</td>
<td>0.864 (reject)</td>
</tr>
<tr>
<td>7</td>
<td>0.482</td>
<td>6.879</td>
<td>0.779 (reject)</td>
</tr>
</tbody>
</table>

Note: Extraction method: Principal Component Analysis

*MonteCarlo PCA for Parallel Analysis (Watkins, M. W., 2000).
Figure 4. Scree plot for eigenvalues and variance

Table 4. Component matrix (2)

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>0.741</td>
<td></td>
</tr>
<tr>
<td>UA</td>
<td>0.692</td>
<td></td>
</tr>
<tr>
<td>UF</td>
<td>0.618</td>
<td></td>
</tr>
<tr>
<td>UD</td>
<td>0.611</td>
<td></td>
</tr>
<tr>
<td>UB</td>
<td>0.420</td>
<td></td>
</tr>
<tr>
<td>UE</td>
<td></td>
<td>0.766</td>
</tr>
<tr>
<td>Defn</td>
<td>0.503</td>
<td>-0.506</td>
</tr>
</tbody>
</table>

Note: Extraction method: Principal Component Analysis

Consequently, this analysis indicates that six of the items, not including UE, are strongly correlated to one factor. When the above process was repeated without UE and one factor extracted, UB had the lowest loading (see Table 5). Because this is still a sizeable value and because previous researchers (Alibali et al., 2007; Asquith et al., 2007; Knuth et al., 2008; Matthews et al., 2012; Rittle-Johnson et al., 2011; Stephens, 2006; Steinberg, Sleeman & Krorza, 1990) have used similar tasks to explore students’ understandings of the
equal sign I decided to retain UB. Consequently, the Cronbach’s Alpha was 0.64. It is worth noting that in addition to a small number of tasks, multidimensional data can also be a cause of a low Cronbach’s alpha (Cortina, 1993). Regardless, this alpha places the reliability of the instrument into the questionable category (George & Mallery, 2003). This is not ideal but given the low number of tasks and the inability to give a lengthy assessment during class time this level was not surprising and is certainly a limitation worth noting. Thus, this questionnaire is viewed as experimental in nature with future research needed to add a minimal number of well written tasks so that a Cronbach’s Alpha of 0.7 or higher can be achieved.

Table 5. Component matrix (1)

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>0.746</td>
</tr>
<tr>
<td>UA</td>
<td>0.687</td>
</tr>
<tr>
<td>UF</td>
<td>0.627</td>
</tr>
<tr>
<td>UD</td>
<td>0.604</td>
</tr>
<tr>
<td>Defn</td>
<td>0.529</td>
</tr>
<tr>
<td>UB</td>
<td>0.425</td>
</tr>
</tbody>
</table>

Note: Extraction method: Principal Component Analysis

Summary

In this chapter I discussed the design and methodology used in my study. I explained the intentions behind each task and how the EQ was developed. I analyzed the measure for validity and reliability. In the next chapter I describe the results of using the EQ to measure PSTs’ understandings of the equal sign using both descriptive and inferential statistics.
CHAPTER 4. RESULTS AND DISCUSSION

In this study both descriptive and inferential statistics were used to answer the research questions. First, descriptive statistics were used to explore PSTs’ responses on the tasks and their understandings of the equal sign as measured by their EQ scores. I not only looked at the group as a whole but I also explored the data of subgroups (e.g., the equal sign understandings of those who correctly answered all of the tasks). Second, I used inferential statistics to explore the relationship between various factors and PSTs’ understandings of the equal sign. Third, I investigated the data cross-sectionally. Next, I looked at the participants who were enrolled in MCC1 and MCC2 to see if there was a relationship between their understandings of the equal sign and their MKT scores even when controlling for certain characteristics. Finally, I looked at the impact of the MMC on PSTs’ understanding of the equal sign.

Research Question 1

*What types of understandings of the equal sign do PSTs have?*

- *Is there a relationship between gender and PSTs’ understandings of the equal sign?*
- *Is there a relationship between pre-calculus/calculus coursework and PSTs’ understandings of the equal sign?*
- *Is there a relationship between pursuing a mathematics endorsement and PSTs’ understandings of the equal sign?*
- *Is there a relationship between level of confidence to teach mathematics and PSTs’ understandings of the equal sign?*
Correct vs. Incorrect Tasks

The six tasks on the EQ were first coded as 0 for incorrect/skipped and 1 for correct; Table 6 shows the percentage of students who correctly answered each task. In every instance the majority of the PSTs answered the task correctly. Task F was the most missed question, with only 59% of the PSTs answering the task correctly. These data are consistent with Weinberg (2010) who found that college students had misconceptions about the validity of “strings.”

Table 6. Percentage of PSTs who answered the task correctly

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C1</th>
<th>C2</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>84.3%</td>
<td>74.3%</td>
<td>79.5%</td>
<td>94.0%</td>
<td>81.0%</td>
<td>91.8%</td>
<td>58.6%</td>
</tr>
</tbody>
</table>

There were seven total points, when one splits task C into C1 and C2. The frequency for the total correct is shown below (see Table 7). The distribution of the results was not normally distributed and had a mean of 5.63, and a standard deviation of 1.34.

Table 7. Frequencies for PSTs’ total correct scores (n = 268)

<table>
<thead>
<tr>
<th>Points based on accuracy</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.7%</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1.9%</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>14.9%</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>11.9%</td>
</tr>
<tr>
<td>5</td>
<td>51</td>
<td>19.0%</td>
</tr>
<tr>
<td>6</td>
<td>79</td>
<td>29.5%</td>
</tr>
<tr>
<td>7</td>
<td>86</td>
<td>32.1%</td>
</tr>
</tbody>
</table>
It is noteworthy that 32% of the PSTs scored a perfect seven. Nevertheless, one must be cognizant that this score only represents correct vs. incorrect responses and not the type of equal sign understandings the PSTs utilized to attain those correct responses. In other words, a PST could have a strong relational with computation understanding of the equal sign and still get all seven points under this scoring paradigm. In addition, the PST may have used relational thinking to answer the task but due to a minor error (e.g., compensate by adding three instead of four) he/she received a score of zero on the task due to the incorrect response. These possibilities supported the need for a more sophisticated coding strategy and measurement tool in order to go beyond PSTs’ accuracy to their understanding.

**Equal sign understandings by task.** Definition tasks were coded as 0 for operational and 1 for relational. If students gave both types of definition, then the higher coding was assigned. In MMC1, two participants gave definitions that were not related to the equal sign, one dealing with place value and the other dealing with base-ten blocks. Both of these PSTs were removed from the sample; although this is not a perfect solution I did not feel comfortable in looking at their other data to “decide” what type of definition they may have given had they understood the context. Therefore, 23% of the participants gave strictly an operational definition, while 77% gave a relational definition (or both). Examples of student responses are given in Table 8.

Tasks UA-UC were coded for three levels of understanding. Item UD was scored as 1 relational/substitutionary or 0 operational and UF was scored as 1 relational or 0 operational. Table 9 shows the percentage of PSTs using the various types of understandings to answer each of these tasks. It is interesting to note that UC had more students using a
Table 8. Sample responses: Definitions and coding

“It shows that this is the end of the problem and it’s the final answer. When #’s are put together to make a new #, you must have the equal sign to show your results” (operational)

“An equal sign means the sum, product, etc., or answer to a problem. The result of a math procedure or equation.” (operational)

“It means the answer.” (operational)

“In a mathematical question it means the answer to the problem comes after.” (operational)

“Balance, a symbol that denotes balance in an equation.” (relational)

“It means that the two things on the left and on the right of the sign are the same. Whatever is shown on the left and on the right of the sign are the same.” (relational)

The two values are identical. (relational)

“Whatever expressions are on either side of the symbol are equal to each other. The equal sign shows that the expressions to the right and left of it have the same value.” (relational)

Table 9. Percent of PSTs with correct UA, UB, and UC responses

<table>
<thead>
<tr>
<th>Responses</th>
<th>UA</th>
<th>UB</th>
<th>UC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operational understanding</td>
<td>6.7%</td>
<td>22.0%</td>
<td>16.0%</td>
</tr>
<tr>
<td>Relational with computation understanding</td>
<td>34.0%</td>
<td>26.9%</td>
<td>46.6%</td>
</tr>
<tr>
<td>Fully relational understanding</td>
<td>59.3%</td>
<td>51.1%</td>
<td>37.3%</td>
</tr>
</tbody>
</table>

relational with computational understanding than a fully relational understanding, indicating that they added 25 and 32 to get the sum of 57 and then subtracted 21 to find the value of the first blank, 36. It is also noteworthy that slightly more than a fifth of the PSTs demonstrated an operational view of the equal sign on task UB meaning that they were not able to answer the question by noticing that equivalence was maintained, by using substitution, nor by solving the equation(s). Of those demonstrating a fully relational understanding the majority gave a reason like this PST who wrote “[t]rue, because you are subtracting 7 from both sides of the equation, therefore making it equal.” There were a few (n = 7) PSTs who gave a substitutionary answers like this PST who stated that the task was “[t]rue, [because] when substituting 54 for 3x you get 60 – 7 on both sides of the equation”. Items UD and UF were
coded for two levels of understanding. Table 10 shows the percentage of PSTs who demonstrated these two types of understanding.

Task UE was ultimately left off of the total EQ score due to the results of the factor analysis (discussed previously) but it is interesting to consider the results from an exploratory perspective. The overwhelming majority answered using their knowledge of the commutative property of multiplication and yet, roughly 8% of PSTs were unable to find values that would make the number sentence true (see Table 11).

Table 10. Percent of PSTs with correct UD and UF responses

<table>
<thead>
<tr>
<th>Responses</th>
<th>UD</th>
<th>UF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Operational or relational with computation understanding</td>
<td>33.6%</td>
<td>41.4%</td>
</tr>
<tr>
<td>Fully relational understanding</td>
<td>66.4%</td>
<td>58.6%</td>
</tr>
</tbody>
</table>

Table 11. Percent of PSTs with correct UE responses

<table>
<thead>
<tr>
<th>Responses</th>
<th>UE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Values beside 2, 3 (e.g., 0, 0 or 4, 6)</td>
<td>10.1%</td>
</tr>
<tr>
<td>Commutative (i.e., 2, 3)</td>
<td>81.7%</td>
</tr>
<tr>
<td>Incorrect/Blank</td>
<td>8.2%</td>
</tr>
</tbody>
</table>

**EQ scores.** After factor analysis, it was determined that the EQ score would consist of the sum of the understanding codes for the definition task, UA, UB, UC, UD, and UF and hence, have a maximum score of nine. The mean of the EQ scores was 6.05 with a standard deviation of 2.15 which is in contrast to the standard deviation of 1.34 for the total correct score. The distribution of the EQ scores (see Table 12) is dissimilar to the total correct distribution in that it has a more normal distribution, although skewed to the left. Skewed data are not unusual in educational research where one would expect most people to have
some type of knowledge of the subject being studied (Pallant, 2010), in this case the equal
sign.

Next, the frequency data will be discussed (see Table 12). Based on the coding
scheme, a PST who answered each question in an operational manner would receive an EQ
score of zero and those who answer in a fully relational manner would end up with a score of
nine. A PST who consistently answered the tasks with a relational with computation
understanding would have scored a five. When looking at the 40 PSTs who scored a 5, there
were only 8 who scored a 5 due to a consistent relational with computation understanding. Of
the remaining 32, 11 scored relational with computation on tasks UA, UB and UC, but
relational on UD and operational on UF. The final 21 PSTs in this group showed some
evidence of fully relational thinking, 15 of them showed this type of understanding on one of
UA, UB, or UC combined with an operational meaning on two other tasks, while 6 of them
showed fully relational thinking on two of these tasks which would mean they would have
answered three other tasks in an operational manner. It is noteworthy that of these 40 PSTs
24 of them answered task UF in an operational manner. At the extremes, 13.4% of the PSTs

Table 12. Frequencies for PSTs’ EQ scores ($n = 268$)

<table>
<thead>
<tr>
<th>EQ score</th>
<th>Frequency</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3.4%</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>4.5%</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>5.6%</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>9.0%</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>14.9%</td>
</tr>
<tr>
<td>6</td>
<td>44</td>
<td>16.4%</td>
</tr>
<tr>
<td>7</td>
<td>44</td>
<td>16.4%</td>
</tr>
<tr>
<td>8</td>
<td>44</td>
<td>16.4%</td>
</tr>
<tr>
<td>9</td>
<td>36</td>
<td>13.4%</td>
</tr>
</tbody>
</table>
had a fully relational understanding and none of the PSTs had a completely operational understanding of the equal sign.

**PSTs who answered all of the tasks correctly.** Next, I decided to delve deeper in the equal sign understandings of the 86 PSTs (see Table 13) who received a perfect score on the total correct measure. Of these PSTs, 22 demonstrated a fully relational understanding of the equal sign based on an EQ score of nine. Frequency data for the entire group indicated that 36 PSTs showed this type of understanding, revealing that there were 14 PSTs who did not respond correctly to all of the tasks but demonstrated fully relational thinking. I will explore their data in the next section but for now will focus on the 64 PSTs who received perfect scores on the total correct measure but had EQ scores less than nine.

Table 13. Frequency of EQ scores for the 86 PSTs with a perfect total score

<table>
<thead>
<tr>
<th>EQ scores</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td># of PSTs</td>
<td>22</td>
<td>27</td>
<td>16</td>
<td>12</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

When looking at these PSTs’ understandings I found that they demonstrated a mix of understandings with fully relational/relational with computation being the most prevalent. Of the 27 PSTs (see Table 14) who had EQ scores of 8, the majority of them used a relational with computation understanding to solve either task UA, UB, or UC. Four of the PSTs used relational understanding on all of the tasks but failed to give a relational definition, while the remaining three answered UD in an operational manner.
Next I looked at PSTs who had a perfect total correct score of 7 and an EQ score of 7 (see Table 15). In order to score a seven PSTs had to lose 2 points; one individual used operational thinking on task UB and, therefore, lost both points on that one task (this is denoted in the last row of the table). When looking at the table, we see a higher proportion of PSTs stating an operational definition (6 out of 16), although only 3 of the 16 lost points due to two types of operational responses (definition with UE), again indicating that these students typically had a mix of understandings.

Table 14. Frequency of non-fully relational responses by task given by participants with a perfect total score and an EQ score of eight* ($n = 27$)

<table>
<thead>
<tr>
<th>Component</th>
<th>Operational</th>
<th>Relational with computation*</th>
</tr>
</thead>
<tbody>
<tr>
<td>UA</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>UB</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>UC</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td>UD</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Definition</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

*Understanding that resulted in an EQ score of eight (i.e., the loss of one point)

Table 15. Frequency of non-fully relational responses by task given by participants with a perfect total score and an EQ score of seven* ($n = 16$)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Operational</th>
<th>Relational with computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Definition</td>
<td>and UB</td>
</tr>
<tr>
<td>2</td>
<td>Definition</td>
<td>and UC</td>
</tr>
<tr>
<td>3</td>
<td>Definition, UD</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>UA, UB</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>UA, UC</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>UB, UC</td>
</tr>
<tr>
<td>2</td>
<td>UD</td>
<td>and UC</td>
</tr>
<tr>
<td>1</td>
<td>UB**</td>
<td></td>
</tr>
</tbody>
</table>

*This student showed an operational understanding on this task which was, therefore, coded as a 0 (a loss of two points).
PSTs who had a perfect total correct score of seven but had an EQ score of six results are below (see Table 16). A little more than half of the PSTs scored at this level due to relational with computation responses on task UA-UC. The remaining PSTs showed this type of understanding on two of these aforementioned tasks coupled with an operational understanding on either task UD or the definition task.

Table 16. Frequency of non-fully relational responses by task given by participants with a perfect total score and an EQ score of Six* (n = 12)

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Operational</th>
<th>Relational with computation</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td></td>
<td>UA, UB, UC</td>
</tr>
<tr>
<td>2</td>
<td>UD</td>
<td>and UA, UC</td>
</tr>
<tr>
<td>1</td>
<td>UD</td>
<td>and UB, UC</td>
</tr>
<tr>
<td>1</td>
<td>Definition</td>
<td>and UA, UC</td>
</tr>
<tr>
<td>1</td>
<td>Definition</td>
<td>and UB, UC</td>
</tr>
</tbody>
</table>

* Understanding that resulted in an EQ score of seven (i.e., the loss of three points)

When examining the data from the eight PSTs who scored a five on the EQ (given a seven on the total correct), six of them used relational with computation understandings on tasks UA, UB, UC and an operational understanding on task UD. The other two PSTs used relational with computational understanding on UA and UC with one demonstrating operational understanding on UB and the other showing operational understandings on the definition and UD. The final PST in this group who correctly answered the questions had a four on the EQ. This PST used relational with computational understandings on tasks UA, UB, and UC; gave an operational definition and used operational understanding on task UD.
PSTs who demonstrated fully relational understanding on the EQ but did not get all of the tasks correct. Next, I analyzed the reverse situation, PSTs who had perfect EQ scores but did not answer all of the tasks correctly. When looking at the data for these 14 individuals it is noteworthy that 12 received a total correct score of six, with the additional 2 PSTs scoring five. This seems to indicate that those with a relational understanding are more likely to successfully complete the mathematical tasks, since of the 36 PSTs who demonstrated a fully relational understanding all but 2 of them scored a six or a seven on the total correct measure. When looking at these 14 PSTs, task D was the most frequent task completed incorrectly with all but 2 of the students making an error even while using relational thinking. The breakdown of the tasks missed include: one PST missed A, another missed C1, 10 missed task D, one missed both A and D, and another missed B and D. It may be that the students were rushed for time and made minor errors in their thinking or what they recorded on the questionnaire.

In summary, when analyzing the responses of the PSTs who correctly completed all of the tasks it appears that most PSTs demonstrated a mix of understandings. Of the 64 PSTs who answered all of the tasks correctly, 22 were fully relational, 13 had a mix of relational and relational with computation, and 29 showed a mix of all three. Based on the data of those who had fully relational understandings it seems likely that this in depth of understanding plays a part in their successful completion of the tasks given. Next, I look at the understandings of a randomly selected group of PSTs from the sample.
PSTs in general

The aforementioned data are interesting in that they provide insight to the equal sign understandings of those who successfully completed all of the tasks correctly and yet still demonstrated a mix of understandings. Given the proportion of these PSTs who showed a mix of understandings it seems likely that the majority of PSTs would also show a mix of understandings, but what types of mixtures should we expect? Due to the large sample size \( n = 268 \), I decided to look at the understandings of a random group of PSTs \( n = 30 \) from my sample. Using a random number generator online (random.org) I selected 30 different PSTs and analyzed the type of understandings they exhibited. I coded the understandings they demonstrated as Fully Relational (5), Fully Relational/Relational with Computation (4), Mix of all three (3), Relational with computation/Operational (2), and Operational (1). The mean and standard deviation of the group’s EQ scores were 6.03 and 1.87 respectively. When looking at the data (see Table 17) note that 90% of the PSTs showed a mix of understandings. Thus, these data suggest that most PSTs have varied types of understandings of the equal sign, including an operational understanding.

At this point, I have looked at various descriptive statistics on the nature of PSTs’ understandings of the equal sign. I have found that based on their EQ scores the majority of

<table>
<thead>
<tr>
<th>PST type</th>
<th>( n )</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mix of Operational and Relational with Computation</td>
<td>6</td>
<td>20%</td>
</tr>
<tr>
<td>Mix of Fully Relational and Relational with Computation</td>
<td>15</td>
<td>50%</td>
</tr>
<tr>
<td>Mix of all three</td>
<td>6</td>
<td>20%</td>
</tr>
<tr>
<td>Fully Relational</td>
<td>3</td>
<td>10%</td>
</tr>
</tbody>
</table>
the PSTs studied had varied understandings of the equal sign and often vacillated between operational and relational views of the symbol. Next, I explored the second part of Research Question 1 by looking at the impact of certain variables on PSTs’ understandings of the equal sign:

What relationship do gender, pre-calculus/calculus coursework, pursuit of a mathematics endorsement, and level of confidence to teach mathematics have on PSTs’ understandings of the equal sign?

**Parametric**

Three independent-samples t-tests were conducted in order to compare EQ scores by gender, mathematical background, and endorsement (see Tables 18 and 19). Given the results of the Levene’s tests, equal variances were assumed for both gender and endorsement but not for mathematical background. Cohen’s $d$ was calculated and evaluations of effect size were based on values stated in Gravetter and Wallnau (i.e., $d = 0.2$ small, $d = 0.5$ medium, and $d = 0.8$ large, 2009). There was no statistically significant difference in the EQ scores of males versus females. There was statistical evidence to suggest that taking a pre-calculus/calculus course in high school or college may have an impact on PSTs’ understanding of the equal sign as measured by their EQ score. There was also statistical evidence to suggest that PSTs who seek a mathematics endorsement may have different understandings than those not seeking this endorsement. In both cases the effect size was moderate based on Cohen’s $d$.

A one-way analysis of variance (ANOVA) between groups was used to investigate the impact of confidence level to teach mathematics on level of equal sign understandings, as indicated by the EQ score. PSTs were divided into three groups based on their self-reported confidence level (lacks confidence/not very confident, somewhat confident, and very
Table 18. EQ score means and standard deviations by characteristic

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>N</th>
<th>M</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Gender</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Female</td>
<td>240</td>
<td>5.98</td>
<td>2.19</td>
</tr>
<tr>
<td>Male</td>
<td>28</td>
<td>6.64</td>
<td>1.70</td>
</tr>
<tr>
<td><strong>Mathematics background: pre-calculus in high school or college</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Completed this type of course</td>
<td>157</td>
<td>6.48</td>
<td>1.96</td>
</tr>
<tr>
<td>Did not complete this type of course</td>
<td>109</td>
<td>5.41</td>
<td>2.27</td>
</tr>
<tr>
<td><strong>Endorsement</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seeking a mathematics endorsement</td>
<td>63</td>
<td>6.81</td>
<td>1.92</td>
</tr>
<tr>
<td>Not seeking a mathematics endorsement</td>
<td>204</td>
<td>5.81</td>
<td>2.17</td>
</tr>
<tr>
<td><strong>Confidence to teach mathematics</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Very confident</td>
<td>50</td>
<td>7.16</td>
<td>1.73</td>
</tr>
<tr>
<td>Somewhat confident</td>
<td>146</td>
<td>6.30</td>
<td>1.87</td>
</tr>
<tr>
<td>Lacked confidence or not very confident</td>
<td>69</td>
<td>4.71</td>
<td>2.36</td>
</tr>
</tbody>
</table>

Table 19. Parametric test results

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>df</th>
<th>Test statistic</th>
<th>p</th>
<th>Effect size</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gender</td>
<td>266</td>
<td>( t = 1.54 )</td>
<td>0.124</td>
<td>Cohen’s ( d = 0.5 )</td>
</tr>
<tr>
<td>Mathematics background</td>
<td>209.284</td>
<td>( t = -4.00 )</td>
<td>(&lt; 0.001^*)</td>
<td>Cohen’s ( d = 0.5 )</td>
</tr>
<tr>
<td>Endorsement</td>
<td>265</td>
<td>( t = -3.27 )</td>
<td>(&lt; 0.001^*)</td>
<td>Cohen’s ( d = 0.5 )</td>
</tr>
<tr>
<td>Confidence to teach mathematics</td>
<td>2, 115.565</td>
<td>Welch’s ( F = 21.53 )</td>
<td>(&lt; 0.001^*)</td>
<td>Eta(^2) = 0.16</td>
</tr>
</tbody>
</table>

\(^*\) Level of significance: \( p < 0.006 \).
Note: Cohen’s \( d \) was calculated using the effect size calculator at http://www.uccs.edu/~lbecker/

confident); these data were re-coded from the initial four categories into three, due to small number of PSTs who marked the “lacks confidence” category. Unequal variance was indicated by the Levene’s test and therefore the Welch’s test statistic was used. Eta squared was calculated, due to the comparison between three groups, and evaluations of effect size were based on values stated in Pallant (i.e., 0.01 small, 0.06 medium, and 0.14 large, 2010).
In Table 19, we can see that there was a statistically significant result with a large effect size for confidence to teach mathematics. Post-hoc comparison using the Tukey HSD test indicated that the mean scores were significantly different between all three levels of confidence.

**Non-parametric**

In order to further validate these results the non-parametric alternative procedures, Mann-Whitney U test and Kruskal-Wallis test, were used. The result of these tests supported the previous results (see Table 20).

<table>
<thead>
<tr>
<th>Test/Characteristic</th>
<th>Mann-Whitney</th>
<th>Kruskal-Wallis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>df</td>
</tr>
<tr>
<td>Gender</td>
<td>U</td>
<td>z</td>
</tr>
<tr>
<td>Mathematics background</td>
<td>2827.000</td>
<td>-1.387</td>
</tr>
<tr>
<td>Endorsement</td>
<td>6190.500</td>
<td>-3.873</td>
</tr>
<tr>
<td></td>
<td>4692.000</td>
<td>-3.269</td>
</tr>
<tr>
<td></td>
<td></td>
<td>df</td>
</tr>
<tr>
<td>Confidence to teach math.</td>
<td></td>
<td>2</td>
</tr>
</tbody>
</table>

* Level of significance: $p < 0.006$.

In summary, gender had no significant relationship with PSTs’ understandings of the equal sign. This result should be viewed cautiously as it may be due to discrepancies in the sample sizes (i.e., few men are pursing degrees in elementary education). Pursuing an endorsement in mathematics and mathematical background each had statistically significant relationships with EQ scores with moderate effect sizes. This indicates that the difference between the EQ score means of those who are seeking a mathematics endorsement and those who are not is approximately half of a standard deviation. This would also be true for the differences between those who have taken pre-calculus/calculus courses in high school or
college. Students’ self-reported confidence level had a large effect size. The eta squared value indicates that sixteen percent of the variance in equal sign understanding is explained by the PSTs’ confidence to teach mathematics.

Research Question 2

Is there a relationship between PSTs’ understandings of the equal sign and their performance on the MKT-NCOP and the MKT-PFA measures?
- After controlling for participants’ score on one MKT measure, is there still a significant relationship between the non-controlled MKT measure and PSTs’ EQ score?
- After controlling for participants’ confidence score, is there still a significant relationship between the MKT measures and PSTs’ EQ score?
- After controlling for participants’ confidence score and the alternate MKT measure, is there still a significant relationship between the other MKT measure and PST’s EQ score?

Participants

When looking for connections between the MKT measures and PSTs’ understandings of the equal sign the sample consisted of 219 PSTs which is 91.6% of those enrolled. This sample consisted of PSTs enrolled in either MCC1 or MCC2 who signed consents for this study and the MIMTE study; linked data for the methods students was not shared at the individual level, thus they are not included in these group analyses. As in the original sample, the EQ score data were skewed to the left (see Figure 5) with a mean of 6.1 and a standard deviation of 2.16. MKT-NCOP and MKT-PFA distributions were somewhat normal (see Figures 6 and 7), with the MKT-NCOP distribution showing more scores to the left of the mean.
Figure 5. Distribution of EQ scores for MCC1 and MCC2 PSTs

Figure 6. Distribution of MKT-NCOP IRT scores for MCC1 and MCC2 PSTs
Parametric correlations

The scatterplots (see Figures 8 and 9) looked somewhat linear; outliers were not excluded from the correlation analyses. The correlation coefficients were interpreted based on ranges recommended by Pallant (i.e., 0.10-0.29 small, 0.30-0.49 medium, and 0.50-1.0 large, 2010). Correlation analyses between EQ scores and the MKT measures were conducted both with and without controlling for other variables; the results for all of the tests are displayed in Table 21.

When looking at the relationship between EQ scores and the MKT measures without controls I found that in both cases $r > 0.5$ which indicated that EQ scores were strongly correlated with both of the MKT measures. It is also interesting to note that the MKT-NCOP had a strong correlation with the MKT-PFA. Ball, Schilling, and Hill (2004) analyzed the MKT measures using bi-factor analysis, and found that:
Figure 8. Scatterplot of EQ scores and MKT-PFA IRT scores

Figure 9. Scatterplot of EQ scores and MKT-NCOP IRT scores

...the general factor explained between 72-77% of the overall variation in teachers’ responses to items on each of the three forms: (1) Number concepts and operations (knowledge of content), (2) Number concepts and operations (knowledge of students and content), and (3) Patterns, functions, algebra (knowledge of content). ...This factor can be interpreted as common knowledge of content (CKC), and suggesting an influence of general grasp of mathematics in patterning teachers’ responses to items. (p. 21)
Hence, by taking the square root of each of the percentages, we can determine that the researchers found $r$ values between 0.819 and 0.877 when comparing the three measures.

Table 21. Correlations with and without controls

<table>
<thead>
<tr>
<th>Situation analyzed</th>
<th>$n$</th>
<th>$df$</th>
<th>$r$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EQ Score &amp; MKT-NCOP</td>
<td>219</td>
<td></td>
<td>.53</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for MKT-PFA</td>
<td>216</td>
<td></td>
<td>.29</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for confidence level</td>
<td>213</td>
<td></td>
<td>.44</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for both</td>
<td>212</td>
<td></td>
<td>.23</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>EQ Score &amp; MKT-PFA</td>
<td>219</td>
<td></td>
<td>.54</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for MKT-NCOP</td>
<td>216</td>
<td></td>
<td>.30</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for confidence level</td>
<td>213</td>
<td></td>
<td>.45</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>After controlling for both</td>
<td>212</td>
<td></td>
<td>.27</td>
<td>&lt;.001*</td>
</tr>
<tr>
<td>MKT-NCOP &amp; MKT-PFA</td>
<td>219</td>
<td></td>
<td>.65</td>
<td>&lt;.001*</td>
</tr>
</tbody>
</table>

* Level of significance: $p < 0.006$.

**Partial correlations.** It seemed imperative to explore this relationship further to ascertain if the EQ score merely measures CKC or if there is some other relationship present. In order to mimic a control for CKC I decided to explore the relationship between PSTs’ equal sign understanding and each of the assessments while controlling for the other assessment; the results are displayed in Table 20. When looking at the correlation between the MKT-NCOP score and EQ score while controlling for MKT-PFA score the correlation was still significant but dropped to a more modest level (on the cusp between a small and medium correlation). These results were identical when looking at the correlation between the MKT-PFA score and EQ score while controlling for the MKT-NCOP score. These results indicate that PSTs’ understandings of the equal sign may help to explain approximately 9%
of the variance in the PSTs’ scores on the MKT-PFA or on the MKT-NCOP even when controlling for CKC.

In the previous section the impact of the PSTs’ confidence to teach mathematics on their EQ scores was explored. Since a strong effect size was found, it seemed logical to control for this variable to see how it affected the relationship between the EQ score and the MKT measures (see Table 20). The correlation results indicated there was statistically significant evidence that a moderate correlation exists between PSTs’ understandings of the equal sign, as measured by the EQ score and each of the MKT measures, even when confidence to teach mathematics was controlled. It is worth noting that controlling for confidence did not have as much of an impact on the correlation as did controlling for the opposite MKT. Finally, when controlling for both confidence and the MKT-PFA score a small correlation was found; this was also the case when confidence and MKT-NCOP score were controlled. These findings suggest that PSTs’ understandings of the equal sign alone may be responsible for 5.3% and 7.2% of the variance in the PSTs’ scores on the MKT-NCOP and MKT-PFA, respectively. Since there is still a statistically significant correlation after these controls were in place it seems likely that the EQ does measure something different than overall mathematical ability (i.e., equal sign understandings). In addition, in the partial correlation analyses, the EQ had a slightly stronger correlation with the MKT-PFA than the MKT-NCOP which supports the works of others who have found a link between equal sign understanding and algebraic success (Knuth et al., 2006).

In summary, there is statistically significant evidence to suggest that PSTs’ understandings of the equal sign, as measured by their EQ scores and PSTs’ performance on the MKT measures, are positively correlated. This relationship holds even when controlling
for the alternate MKT measure; although at more modest levels. This relationship also holds when controlling for both the alternate MKT measure and confidence although at a low level. It is worth noting that the Learning Mathematics for Teaching (LMT) website cautions about the validity of using individual MKT scores to make judgments about individual teacher knowledge (based on the reliability of the instrument), instead suggesting that groups of teachers be used for this type of analysis. Therefore, since individual data were used to see if there was some type of relationship between how the PSTs performed on all three measures these results should be considered exploratory in nature.

**Research Question 3**

*What type of understandings of the equal sign do PSTs have cross-sectionally across a teacher preparation program? What impact does a methods course have on PSTs’ understandings of the equal sign?*

**Analysis by course: Correct responses**

When looking at the data for PSTs in MCC1 and MCC2, it is interesting to note that the percentages of PSTs who answered the individual tasks correctly were typically higher for those beginning MCC2 (see Table 22). This result held true for all of the tasks except task D. MMCpre PSTs had higher percentage of correct responses on tasks A, B, and D but MCC2 PSTs had higher percentages on the other four tasks. When looking at the total correct score (seven points total), MCC2 had a smaller percentage of PSTs scoring four or below and likewise, a higher percentage scoring five and above (see Table 23). Tasks C2 and E were the most likely to be answered correctly and tasks B and F were the least likely to be answered correctly this result was consistent for PSTs in all three courses. Task F had the lowest rate of
Table 22. Percentage of PSTs who answered the tasks correctly

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C1</th>
<th>C2</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCC1</td>
<td>82.5</td>
<td>74.2</td>
<td>75.8</td>
<td>91.7</td>
<td>80.8</td>
<td>89.2</td>
<td>53.3</td>
</tr>
<tr>
<td>MCC2</td>
<td>85.1</td>
<td>74.3</td>
<td>84.2</td>
<td>96.0</td>
<td>78.2</td>
<td>95.0</td>
<td>69.3</td>
</tr>
<tr>
<td>MMCpre</td>
<td>87.2</td>
<td>74.5</td>
<td>78.7</td>
<td>95.7</td>
<td>87.2</td>
<td>91.5</td>
<td>49.8</td>
</tr>
</tbody>
</table>

Table 23. Percentage of PSTs total correct score by course

<table>
<thead>
<tr>
<th>Total correct score</th>
<th>MCC1</th>
<th>MCC2</th>
<th>MMCpre</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.5%</td>
<td>2.0%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>7.5%</td>
<td>1.0%</td>
<td>6.4%</td>
</tr>
<tr>
<td>4</td>
<td>10.8%</td>
<td>10.9%</td>
<td>17.0%</td>
</tr>
<tr>
<td>5</td>
<td>20.8%</td>
<td>17.8%</td>
<td>17.0%</td>
</tr>
<tr>
<td>6</td>
<td>25.8%</td>
<td>35.6%</td>
<td>25.5%</td>
</tr>
<tr>
<td>7</td>
<td>30.8%</td>
<td>32.7%</td>
<td>34.0%</td>
</tr>
</tbody>
</table>

Note: Percents may not add up to 100% due to rounding.

being answered correctly, despite the course, with less than 70% of the students answering the task correctly.

**EQ scores by course**

When looking at the distributions by course for equal sign understanding as measured by the PSTs’ EQ scores, we see distributions that are skewed to the left (see Figures 10-12). Again, as stated earlier, due to the large sample size in each of the three groups ($n >30$), the lack of perfectly normal distributions is not a concern.
Figure 10. EQ score distribution for MCC1

Figure 11. EQ distribution for MCC2
In order to look at whether the students enrolled in the different teacher preparation courses exhibited different understandings of the equal sign, a one-way between-groups analysis of variance was conducted. The groups consisted of three teacher preparation courses: Math Content Course 1 (MCC1), Math Content Course 2 (MCC2), and Math Methods Course (MMC-pre); the means and standard deviations of their EQ scores are displayed in Table 24. In order to determine if the use of ANOVA was appropriate, a test of homogeneity of variances was used; Levine’s test produced a significance value of 0.210 (> 0.05), which shows that the data did not violate the assumption of homogeneity of variance. Eta squared was calculated, due to the comparison between three groups, and evaluations of effect size were based on values stated in Pallant (2010). The results indicate that there was a statistically significant result within the groups: $F (2, 265) = 5.380, p = 0.005 (<0.05)$.
Table 24. Means and standard deviations by course

<table>
<thead>
<tr>
<th>Course</th>
<th>N</th>
<th>M</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCC1</td>
<td>120</td>
<td>5.69</td>
<td>2.23</td>
</tr>
<tr>
<td>MCC2</td>
<td>101</td>
<td>6.59</td>
<td>1.98</td>
</tr>
<tr>
<td>MMC</td>
<td>47</td>
<td>5.81</td>
<td>2.10</td>
</tr>
</tbody>
</table>

Table 25. Results of between course analyses using Tukey HSD

<table>
<thead>
<tr>
<th>Courses</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>MCC1 and MCC2</td>
<td>.005*</td>
</tr>
<tr>
<td>MCC1 and MMC</td>
<td>.95</td>
</tr>
<tr>
<td>MCC2 and MMC</td>
<td>.09</td>
</tr>
</tbody>
</table>

* Level of significance: $p < 0.05$.

although with an eta squared of 0.039 indicating a small effect size. Post-hoc evaluation using the Tukey HSD (see Table 25) showed a statistically significant difference between MCC1 and MCC2.

Therefore, there is statistical evidence that supports the conclusion that PSTs who are enrolled in MCC2 have a better understanding of the equal sign than those enrolled in MCC1. In addition, these findings also indicate that there is no significant difference in EQ scores between those beginning their first math content course and those beginning their only math methods course.

What impact does a methods course have on PSTs’ understandings of the equal sign?

Correct vs. incorrect. When looking at the pre-post methods data, there was a sample of 43 students which is approximately 90% of the students enrolled (due to a student
dropping, lack of consent for both studies, removal due having the researcher as an instructor at a different institution, and absenteeism). When looking at Table 26 we see the percentage of PSTs who answered the individual tasks correctly improved from pre-test to post-test. We also see that the percentage of students scoring either a 6 or a 7 increased by approximately 20% between the pre and the post measurement (see Table 27).

Table 26. Percentage of PSTs who answered the task correctly

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C1</th>
<th>C2</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMC (pre)</td>
<td>87.2</td>
<td>74.5</td>
<td>78.7</td>
<td>95.7</td>
<td>87.2</td>
<td>91.5</td>
<td>48.9</td>
</tr>
<tr>
<td>MMC(post)</td>
<td>93.0</td>
<td>83.7</td>
<td>88.4</td>
<td>97.7</td>
<td>90.7</td>
<td>100</td>
<td>76.7</td>
</tr>
</tbody>
</table>

Table 27. Frequency of total correct MMC pre and post

<table>
<thead>
<tr>
<th># Correct items</th>
<th>MMC pre</th>
<th>MMC post</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
<td>24</td>
</tr>
</tbody>
</table>

**EQ scores: Parametric.** In order to determine if there was statistical evidence to support an increase in PSTs’ understandings of the equal sign due to the intervention to the mathematics methods course a paired-samples t-test was used. As illustrated in Table 28, the results show a statistically significantly increase in the scores from the beginning of the semester to the end of the
Table 28. Pre- and post-methods EQ scores: Results of the paired t-test

<table>
<thead>
<tr>
<th></th>
<th>Cohen’s $d$</th>
<th>$N$</th>
<th>$M$</th>
<th>$SD$</th>
<th>$df$</th>
<th>$t$</th>
<th>$p$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td></td>
<td>43</td>
<td>5.74</td>
<td>2.07</td>
<td>42</td>
<td>6.550</td>
<td>&lt; 0.001</td>
<td>0.587</td>
</tr>
<tr>
<td>Post</td>
<td></td>
<td>43</td>
<td>7.47</td>
<td>1.61</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Note: Cohen’s $d$ was calculated using $r = 0.587$ and the effect size calculator at www.cognitiveflexibility.org/effectsize/.

semester. The effect size was large as indicated by the Cohen’s $d$ (based on values reported in Gravetter & Wallnau, 2009).

**Non-parametric.** A non-parametric test was used to verify the previous results. A Wilcoxon Signed Rank Test was used due to the fact that the data consist of several values that were repeated. This test supported the conclusion that PSTs’ understanding of the equal sign, as measured by the EQ, improved after completing the elementary mathematics methods course, $z = -4.793, p = 0.0000016$ ($p < 0.05$), and a large effect size of 0.52. The median score on the EQ increased from the start of the semester ($Md = 6$) to the end of the semester ($Md = 8$).

**Two additional post-methods questions.** Although task E was used in the total correct measure it was ultimately left off the EQ score following factor analysis as it did not load with the other variables. The PSTs in the pre- and post-methods analysis answered this task correctly both times. But I was curious whether PSTs could generate other answers when asked. So in the post-test (see Appendix E) I asked the methods students to either generate three more solutions or explain why there was only one unique answer. The majority of the PSTs found additional solutions although some had mathematical errors. Only two PSTs believed that 2, 3 was the only solution, but there were an additional four PSTs who could only generate examples that were equivalent to 2, 3 (e.g., $1 + 1$ and $2 + 1$).
Since I developed this task I was hoping to determine if it had any merit in future research. At this point, I’m wondering if this task is more of a proxy for algebra understanding and was coded inversely, that in fact those who answer 2, 3 have a stronger understanding of the equal sign due to their knowledge of algebra (i.e., knowledge of properties) and therefore should have been coded higher than those who answered 8, 12.

There were two other questions that were asked in order to shape future research. Initially I was going to interview a group of pre- and post- methods students consisting of 6-9 PSTs with a range of equal sign understandings. Unfortunately, few students were willing to participate. At the post interview only two students participated and both of them had taken their math content courses at a community college and both of them had high EQ scores. In addition, one of them was leaving the teacher education program due to health issues. So, I decided to ask these questions on the post-test to see if there seemed to be a tie between EQ and PSTs’ understandings of the recommendations regarding the equals sign in the CCSS-M.

The first question asked the post methods PSTs to identify the purpose of true/false number sentences, as outlined in the CCSS-M. Not surprisingly, the majority of the students stated that it was to make sure students understood the equal sign but there were three students who deviated from that answer. These students wrote:

“*That there is more than one way to get a certain #*”. PST’s Post Methods Response, Pre EQ – 3, Post – EQ 8

“To see if students can solve simple math.” PST’s Post Methods Response, Pre EQ – 6, Post – EQ 7

“Purpose of seeing numbers mixed and orders changed to higher math.” PST’s Post Methods Response, Pre EQ – 3, Post – EQ 7

The follow up question asked why the CCSS-M might be recommending true/false tasks versus problems like 5 + 4 = ____. Although several students brought up sameness there
were five students that seemed to have a keen understanding of the intent of the question stating:

“A fill in the blank equation would reinforce the false thinking of an equal sign telling you to compute the answer.” PST’s Post Methods Response, Pre EQ – 9, Post – EQ 9

“It would take the focus away from the equal sign, and focus more on finding a correct numerical answer.” PST’s Post Methods Response, Pre EQ – 6, Post – EQ 9

“They don’t want the child to think = means solve.” PST’s Post Methods Response, Pre EQ – 8, Post – EQ 8

“They could just write 9 and reaffirm the misconception that = is a prompt for the answer.” PST’s Post Methods Response, Pre EQ – 9, Post – EQ 9

“True/false give a better understanding for if the student used the equal sign correctly.” PST’s Post Methods Response, Pre EQ – 5, Post – EQ 9

“Students would not be able to understand the equals sign necessarily. They would be just adding the numbers not really thinking about why.” PST’s Post Methods Response, Pre EQ – 6, Post – EQ 8

I believe these questions are useful and that interview data would shed more light on the link between PSTs’ personal knowledge about the equal sign and their interpretation of policies regarding students’ knowledge about that same symbol.

**Summary**

In summary, it appears that PSTs’ understandings of the equal sign are mixed with very few PSTs demonstrating a fully relational understanding of the equal sign. The EQ measure is experimental in nature but was used to quantify PSTs’ understandings of the equal sign. Using this measure and inferential statistics it was found that gender did not seem to have a relationship with PSTs’ EQ scores but endorsement and mathematical background did show significant results with moderate effect sizes. In addition, PSTs’ confidence levels to teach mathematics and their EQ scores were statistically significant with a strong effect size.
Initially, there was a strong correlation between PSTs’ EQ score and their MKT scores. However, after controlling for the alternate MKT measure the correlations fell to low and moderate levels. There was also a statistically significant correlation between the EQ score and each of the MKT measures, when controlling for the other MKT measure and the PSTs’ confidence level, although with a small effect size. When looking at the cross-sectional data, PSTs beginning their second mathematics content course have a more relational understanding of the equal sign than those enrolled in their first mathematics content course. Longitudinally, PSTs enrolled in the mathematics methods course showed statistically significant improvement in their pre and post EQ scores.
CHAPTER V. CONCLUSIONS

The purpose of this study was to explore the nature of PSTs’ understandings of the equal sign. In addition, I wanted to determine if there were any relationships between their understandings of this important symbol and other variables including their MKT (Hill et al., 2004). I also wanted to explore connections between elementary mathematics teacher preparatory coursework and PSTs’ understandings of the equal sign.

Gaps in the Literature

Directly measuring PSTs’ understandings or the equal sign

As outlined in my literature review, no studies were found that directly measured PSTs’ understandings of the equal sign. Stephens (2006, 2008) did explore PSTs’ cognizance of equivalence and relational thinking and their interpretations of what is and is not algebra; however, she did not measure the PSTs’ understandings of the equal sign directly. To fill this gap I developed the EQ and used both descriptive and inferential statistics to analyze the understandings of the PSTs.

Cross-sectional data

When evaluating the literature no studies were found that looked at PSTs’ understandings of the equal sign cross-sectionally. In order to address this gap, I decided to explore PSTs’ understandings at three key points in their educational journey: at the beginning of their first mathematics content course, at the beginning of their second mathematics content course, and at the beginning of their mathematics methods course.
Undergraduate studies are limited

When reviewing recent studies dealing with undergraduates’ understandings of the equal sign I noticed that the participants studied were enrolled in a few different courses (i.e., introductory psychology, elementary mathematics methods, calculus, and remedial intermediate algebra) with sample sizes below 230 (McNeil & Alibali, 2005a; McNeil & Alibali, 2005b; McNeil, Rittle-Johnson, Hattikudur, and Peterson, 2010; Stephens, 2006, 2008; Weinberg, 2010; Wheeler, 2010). The undergraduate participants in some of these studies were more academically advanced than a “typical” college student. For example undergraduates who: were enrolled in calculus (Weinberg, 2010), had taken at least one calculus course in their lifetime (McNeil & Alibali, 2005a), had a mean ACT/SAT in the 89th percentile (McNeil & Alibali, 2005b), or were taken from a pool of undergraduates with the aforementioned mean ACT/SAT (McNeil, Rittle-Johnson, Hattikudur, and Peterson, 2010).

In my study, I looked at undergraduates who were enrolled in courses who had not previously been studied (i.e., elementary mathematics content courses) and who may be of more “typical” mathematical ability with 38% to 42% of my participants (enrolled in MCC1, MCC2, MMCpre) never have taken a pre-calculus or calculus course in their lifetime.

My research has addressed these gaps in the literature and in this chapter I provide a brief summary of my work and discuss my conclusions based on the results of my analyses. Finally, I conclude this chapter with the constraints and limitations of this study, and my recommendations for future research in this area.
The EQ (Equal Sign Understanding Questionnaire)

I developed a short equal sign understanding questionnaire (called the EQ) using tasks gleaned from the literature. Factor analysis was used to see if the questionnaire was truly measuring one construct. Due to factor analysis one task was removed and the final EQ had 6 items, a maximum score of 9 and a Cronbach’s alpha of 0.64. Factor analysis supports that the EQ measures one construct, although it does not indicate what this construct is. Both the nature of the tasks and the results of statistical tests run to rule out just measuring overall mathematics ability seem to suggest the EQ does in fact measure equal sign understandings.

The specific research questions I explored were:

1. What types of understandings of the equal sign do PSTs have?
2. What is the relationship, if any, between PSTs’ understandings of the equal sign and PSTs’ mathematical knowledge for teaching?
3. What types of understandings of the equal sign do PSTs have cross-sectionally at different stages of a teacher preparation program?

Summary of Results

Descriptive statistics indicated that the majority of PSTs hold a mix of understandings with fully relational and relational with computation being the most prevalent although mixes with an operational understanding were present too. Inferential statistics were used to explore any possible differences between PSTs’ understandings of the equal sign based on these variables: gender, mathematical background, seeking a mathematics endorsement, and their level of confidence to teach mathematics. I found that every variable was significant ($p <$
0.001) except for gender. Mathematical background and endorsement both had a moderate effect size while confidence to teach had a strong effect size.

**Findings**

**Definition difficulties persist**

In this study I found about one fourth of the PSTs struggled defining the equal sign relationally. This supports the conclusion that even undergraduates struggle with this higher-level item on the *Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality* (Matthews et al., 2012). Since the symbol was given in isolation it is very unlikely that the context would elicit an operational understanding based on context alone (McNeil & Alibali, 2005b). In contrast to McNeil and Alibali’s work in which all of the thirty-five undergraduates gave relational definitions (despite context) I found that 23% of my participants gave operational definitions. It may be that introductory psychology students have a more robust understanding of the equal sign than PSTs but that does not seem probable. Most likely this discrepancy is due to the fact that all of their undergraduate participants had taken a minimum of one calculus class where in my study around 60% had taken some type of pre-calculus/calculus course.

Rittle-Johnson et al. (2011) found that for 7 and 8 year olds “generating a relational definition of the equal sign was much harder than solving or evaluating equations with operations on both sides” (p. 97) as did Stephens et al. (2013) with third through fifth graders. Knuth et al. (2005) and McNeil et al. (2006) also found that sixth through eighth graders struggle defining the equal sign in a relational manner. My work combined with the aforementioned research suggests that although over time most students are able to move
from giving an operational definition to producing a relational definition this type of task proves challenging even to some college students. Knuth and colleagues’ (2005, 2006) research involved middle school students. They stated that:

…an understanding of equivalence is a pivotal aspect of algebraic reasoning and development. Consequently, students’ preparation for and eventual success in algebra may be dependent on efforts to enhance their understanding of mathematical equivalence and the meaning of the equal sign. (2005, p. 7)

Since teachers are the ones fostering this algebraic understanding in children, those involved in teacher preparatory efforts need to create opportunities where all PSTs can develop a robust understanding of the equal sign.

**Definition difficulties do not hinder successful task completion**

Difficulties with defining the equal sign did not mean the PSTs could not navigate problems with the symbol, as tasks A – E had successful completion rates between 74% and 94% (note: task F is discussed in the next section) and on the assessment as a whole about one-third of the PSTs answered all of the tasks correctly. Of those answering all of the tasks correctly roughly 16% gave an operational definition. Hence, similar to Weinberg’s (2010) work with college students I found that some PSTs were able to answer tasks correctly even though their knowledge about the role of the equal sign was faulty.

**Task context may activate different understandings**

**Not all string tasks are created equal**

Task F was the most missed task with only 59% of the PSTs answering it correctly. Research has suggested using true/false number sentences with children is beneficial (Faulkner et. al, 1999; Molina & Ambrose, 2006) and my research suggests that using these
tasks with PSTs may be beneficial as well. The low success rate on task F supports the work of Weinberg (2010) who found that college students struggled with “strings” but probably not to the extent that this result suggests.

If college students always struggle with strings one would expect a similar correct response rate to any type of string task but that is not the case. Tasks C and F are very similar in format except task C (i.e., __ + 21 = 25 + 32 = __ ) has a fill-in-the-blank format and task F (i.e., 12 + 5 = 17 + 2 = 19) has a true/false format. The first blank in task C was answered correctly by 80% of the PSTs (C1) indicating that these students have some type of relational understanding of the equal sign. This is in contrast to the 59% correct response rate for task F. This result suggests that the tasks themselves may trigger different types of responses. It may suggest that the PSTs hold several meanings of the equal sign internally and apply different meanings based on the format of the task. In other words, the structure of task C may trigger in the majority of PSTs a relational with computation understanding of the equal sign or a fully relational understanding just by how it is written, while task F may elicit an “answer comes next” (operational) definition of the equal sign.

PSTs may view the equal sign in one task meaning something completely different than in another task. Given today’s culture one may understand why this may not be problematic for students. For example, when sending texts people often use “u” for “you” or “4” for “four” or they just blatantly misspell words and break the rules of grammar because “it is just a text” and they know that the recipient will “know what he/she means”. Yet in a term paper the same individual would use correct spelling and grammar because the context has changed. It appears that task F may fall into this type of category, where since the numbers are already written out it is not a solving task but a confirmation of someone else’s
thinking task and since the PST “knows what the person means” it is okay or true. This is concerning as teachers often spend time evaluating students’ mathematical work and offering feedback to help their students learn. Using tasks where PSTs are asked to evaluate hypothetical student work involving “strings” should be integrated into teacher preparatory coursework to help them confront this type of understanding of the equal sign.

**Task B’s context may activate an algebra mindset**

In the previous paragraphs I pointed out that tasks C and F were designed to measure how PSTs deal with “strings” and yet the two tasks had very different successful completion rates. Likewise, tasks B and D had some analogous traits as in both cases students could use substitution or a fully relational understanding to answer the tasks. Nevertheless, task D had a higher successful completion rate than B (81% compared to 74%). In addition, task D had a higher percentage of students using fully substitution/relational understanding than task B (66% to 51%). It may be that PSTs are well trained in the typical algebra mentality (Kieran, 1992; Stephens, 2008) so that when they see a variable in an equation they think *What rule or procedure should I use?* versus *How should I think about this problem relationally?* It is noteworthy that for those who did think about task B in a fully relationally way the majority of them gave a reason like “[t]rue, because you are subtracting 7 from both sides of the equation ….” However, there were a handful of PSTs (*n* = 7) who gave a substitutionary reason like “True; since 3x + 6 = 60, then 60 − 7 = 60 – 7. They are equivalent.” These responses suggest that this task may be useful at uncovering multiple understandings of the equal sign if PSTs are asked to provide multiple reasons as to why the answer is true or false.
It also supports the work of those who argue for a substitutionary meaning of the equal sign (Jones & Pratt, 2011; Jones et al., 2012).

Most PSTs hold a mix of understandings of the equal sign

Correct responses do not necessarily mean a fully relational understanding

When looking at task A 84% of the PSTs answered it correctly but only 59% demonstrated a fully relational understanding of the equal sign (UA). Similarly, 74% of the PSTs answered task B correctly but only 51% showed a fully relational understanding of the equal sign (UB). When looking at PSTs’ understandings of the equal sign “getting the right answer” and having a fully relational understanding of the equal sign are not the same thing. There were 86 PSTs with perfect total correct scores and yet only 26% of these PSTs had a perfect EQ score indicating a fully relational understanding of the equal sign. Thirty-nine percent of the PSTs with a perfect correct score demonstrated a mix of fully relational and relational with computation understandings. Thirty-five percent of those with a perfect total correct score still demonstrated an operational understanding, these were typically limited to task D (where they just added) and/or the definition task.

Mix of understandings is common

Eighty-seven percent of the PSTs (n = 268) did not have a perfect score on the EQ and therefore demonstrated a mix of understandings. In the random sub-sample (n = 30), only a tenth of the PSTs held a fully relational view of the equal sign. The most common mix was a mix of fully relational and relational with computation, although 40% of the PSTs in this random sample exhibited some type of operational understanding. Initially I went into this
study thinking that PSTs would demonstrate a variety of understandings of the equal sign corporally but not necessarily at the individual level. I believed that some students would demonstrate a mix of understandings but not to the extent that I found. Again, evidence suggests that the majority of PSTs who can successful navigate the mathematics may hold multiple meanings of the equal sign. On the Construct Map for Knowledge of the Equal Sign as Indicator of Mathematical Equality (Matthews et al., 2012) the majority of PSTs are at Level 3 or Level 4 but certain tasks seem to activate an operational understanding of the equal sign indicating that some PSTs may have entrenched operational understandings that are resistant to change (McNeil & Alibali, 2005b). Unlike the work of McNeil and Alibali I did not purposefully activate an operational understanding prior to completing the tasks, but it is possible that the tasks themselves may have activated this type of understanding. Therefore, those involved in teacher preparatory efforts need to strategically choose tasks that will reveal their PSTs’ understandings of the equal sign. Based on this study the definition task, task B, and task F may be more illuminating than the other tasks.

**Equal sign understandings and other variables**

**EQ scores and confidence to teach mathematics**

Mathematical background and endorsement both had statistically significant results with moderate effect size which is not surprising. But what was surprising was the strong effect size found when analyzing the PSTs’ equal sign understandings and confidence to teach mathematics. This categorical variable was based on the PSTs’ personal beliefs about their abilities as future mathematics teachers. I wasn’t expecting such a strong effect size for this variable. I wonder if this variable is in some way a proxy question that measures how the
PST feels about his/her conceptual understanding of mathematics (i.e., strong conceptual understanding, surface level understanding, or poor understanding). It may be that these PSTs were very aware of whether they have a “deep and profound” (Ma, 1999) understanding of mathematics, in this case of the equal sign, and this link between confidence to teach and their equal sign understandings was an indicator of whether or not they had this deep mathematical knowledge.

**Relationships between EQ and gender**

When moving beyond the descriptive statistics I found that there are several statistically significant relationships between the PSTs’ EQ scores and these other variables. Although gender was not one of these variables I think this is due to the small number of the males in the study and not necessarily conclusive evidence for the lack of a relationship.

**Research Question 2**

*Is there a relationship between PSTs’ understandings of the equal sign and their performance on the MKT-NCOP and the MKT-PFA measures?*

- After controlling for participants’ score on one MKT measure, is there still a significant relationship between the non-controlled MKT measure and PSTs’ EQ score?
- After controlling for participants’ confidence score, is there still a significant relationship between the MKT measures and PSTs’ EQ score?
- After controlling for participants’ confidence score and the alternate MKT measure, is there still a significant relationship between the other MKT measure and PST’s EQ score?

**Summary of results**

Correlation analyses were used to study the relationship between PSTs’ understandings of the equal sign and their MKT. Two MKT measures were used - one that measured the PSTs’ MKT in the domain of *Number Concepts and Operations* (NCOP) and
another in the domain of *Patterns, Functions, and Algebra* (PFA). Only the PSTs enrolled in MCC1 and MCC2 \( (n = 219) \) were included in the analyses of MKT scores as I did not have access to MCC students’ MKT scores. The \( r \) value comparing these two MKT measures in my study \( (r = .65) \), although lower than Hill et al. (2004) found, does not seem unreasonable. This result may be due to the fact that these measures were developed and validated for the use with in-service teachers and yet, were used here to measure the understandings of PSTs.

Partial correlation analyses indicated that when analyzing the relationship between the EQ scores and MKT-NCOP while controlling for other variables (MKT-PFA, confidence level) that the results were still statistically significant \( (p < 0.001) \) but with small or moderate correlations. While controlling for both of these variables the Pearson’s correlation coefficient dropped to 0.23. Similarly, the EQ scores and MKT-PFA had similar statistically significant \( (p < 0.001) \) results when controlling for the opposite measure (MKT-NCOP) and for confidence level. Although, while controlling for both the Pearson’s correlation coefficient was 0.27.

**Findings**

**Discussion**

MKT measures were designed to measure teacher knowledge, in this case both the mathematical content and the specialized content knowledge that teachers need in regards to: (1) Number Concepts and Operations and (2) Patterns, Functions, and Algebra. The main reason I wanted to explore this area was to explore if there was some type of relationship between PSTs’ understandings of the equal sign and their MKT. If a relationship exists then future research involving regression analyses may be warranted. Based my results it is
possible that the equal sign understanding a PST holds may impact his/her MKT but without future analysis this is just speculation.

The MKT measures also helped me validate the EQ. Although factor analysis supported that the understanding tasks worked together as one entity, it did not (and cannot) tell what exactly that component is. Most of these tasks were designed to measure equal sign understandings and were based on previous research but that does not necessarily guarantee that the tasks really measure equal sign understanding. In a sense, looking to see if there is a relationship between these two measures helped clarify if the EQ really measures equal sign understandings. Since the EQ score had a strong correlation with both MKT measures it is possible that is because they both measure the same thing, in other words maybe the EQ really just measures number and operation ability and algebra ability. Or maybe the EQ just measures overall mathematics ability (CCK). The strong correlations between the EQ score and both measures really does not help establish statistically what the EQ measures but by controlling for the opposite measure I attempted to control for overall mathematical ability. I also controlled for confidence level as this had a strong relationship with the EQ and I wanted to make sure that I was not measuring that variable. These results add credence to my claim that the EQ does measure equal sign understandings as it was designed to do.

**Research Question 3**

*What type of understandings of the equal sign do PSTs have cross-sectionally across a teacher preparation program? What impact does a methods course have on PSTs’ understandings of the equal sign?*
Summary of results

Descriptive statistics indicate that when comparing the PSTs enrolled in MCC1, MCC2, and MMCpre that MCC1 typically had a lower percentage of PSTs answering the tasks correctly compared to the other two groups. There were statistically significant results showing that the PSTs enrolled in MCC2 had a better understanding of the equal sign than those enrolled in MCC1 (although with a small effect size); there were no significant results when comparing the other groups. There were also statistically significant results when comparing MMCpre and MMCpost PSTs (with a large effect size). These results indicate that the methods course did impact the PSTs’ understandings of the equal sign.

Findings

Task B is resistant to change

It is interesting to note that essentially all three classes had the same percentage of PSTs answering task B correctly (approximately 74%). Again, this is the task that looked more algebraic than the others. This may indicate that beliefs about what is and is not algebra and how students approach algebra tasks are very resistant to change. This finding is interesting when coupled with Asquith et al.’s (2007) work in which five out of twenty teachers failed to provide an answer in a similar type task (e.g., Is the number that goes in the box the same number in the following two equations?) and only 65% of the teachers in the study indicated that a “recognize equivalence strategy” could be used by middle school students to solve the task. Combined, these results seem to suggest that a task in the $ax + b = c$ with $ax + b - d = c - d$ format would have a similar response rate with about 25% of PSTs and ISTs struggling to answer the task correctly. In my task it may be due to the presence of
a variable but in Asquith et al.’s work a box was used (in place of the variable) and ISTs still struggled.

**Tasks have consistent difficulty rankings between groups**

Although the percentage of PSTs correctly responding varied by course, the rank order varied very little between these groups. The tasks’ rank order from the highest percentage of PSTs answering correctly to the lowest were as follows: the second blank in task C (C2), task E, task A, followed by task D then the first blank in task C (C1) or vice versa, task B, and finally task F. The second part of task C (C2) was the easiest for the PSTs, which considering its $a + b = c$ format is not surprising. The T/F tasks were the most difficult and the fill in the blank tasks were the easiest. It is interesting that MCC2 students had a higher success rate, answering the string tasks correctly (C1, C2, and F) while MMCpre students having the lowest percentage of correct rates on both parts of task C but still higher than the percentage correct for MCC1. This was not true for task F which was answered correctly by 53% of MCC1 students, 69% of MCC2 students, and only 50% of the MMCpre students. This indicates that all of the PSTs struggled with the true/false “string” but also that those enrolled in the MMCpre may not be retaining knowledge about the use of the equal sign in an appropriate manner. Again, these data are just descriptive and cross-sectional in nature but it does raise the question of why the PSTs who have completed two mathematics content courses have a lower success rate on a true/false number sentence compared to those PSTs who have had less teacher preparatory coursework. Was it a change in curricular materials? More lower ability students? Large gaps between the content course completion and enrollment in the methods course? Or does this type of task activate a deep seated
understanding of the equal sign that even the best of content course experiences have difficulty eradicating?

**MCC1 may impact PSTs’ equal sign understandings**

It appears that PSTs may gain a deeper understanding of the equal sign due to taking MCC1, but that those gains may not be retained. Since these are cross-sectional data I cannot necessarily say that the PSTs improved due to taking MCC1 even though this may be the case. It may be possible that the drop in scores is due to the fact that 30% of the MMC students completed at least one of the content courses at another institution (typically a community college) or that these students completed MCC1 when a different curriculum was used. It may also be the case that those enrolled in MCC2 performed better on the tasks due to those with weaker understandings dropping out of the teacher education program after completing MCC1 or that due to random chance the MCC2 students were better students than those enrolled in the other courses. Longitudinal research is needed to study these possibilities.

Despite these possibilities, it seems plausible based on the statistical results, the curriculum used, and the instructors’ emphasis on the role of the equal sign, that PSTs gain a deeper understanding of the equal sign while taking MCC1 and that this knowledge is demonstrated at the beginning of their second content course, but after time they may revert to their old understandings. Although interview data were not ultimately used in this study, it is worth noting anecdotally that those MCC PSTs who were interviewed at the beginning of the semester mentioned how MCC1 helped them have a deeper understanding of mathematics.
Mathematics methods course may make an impact

There was statistical evidence to support that a mathematics methods course can impact PSTs’ understandings of the equal sign. Pre- and post-data suggest that the intervention of a methods course is responsible for a 1.2 standard deviation increase in PSTs’ EQ score. This is not surprising given that this course: (1) exposed the PSTs to examples of student work; (2) used videos of elementary students thinking mathematically; and (3) provided tutoring sessions in which the PSTs received first-hand experience with elementary students’ thought processes, including specific tasks targeting elementary students’ understandings of the equal sign. This course is specifically designed to help PSTs understand how elementary students think mathematically. Therefore, it seems plausible that a course with this focus would help PSTs focus on their own understandings as well. Of course, it is possible that the improvement is merely due to practice as the same tasks were given at both the beginning and end of the semester. It is also possible that the four students who were not included in the pre/post comparison may have not shown growth and thus, by excluding these cases, the results look more promising than they actually are. Notwithstanding, the results are promising and show that a methods course focused on teaching PSTs to think about their future students’ thinking may in fact improve their own mathematical understandings.

Summary

In this study I found that most PSTs have a mix of understandings of the equal sign with very few consistently exhibiting a fully relational view of the symbol. Most PSTs have a mix of fully relational and relational with computation understandings of the equal sign but
operational answers were common as well, especially when defining the symbol or answering a true/false “string.” PSTs struggled to define the equal sign in a relational manner although this did not necessarily hinder successful task completion. The rankings of the tasks by difficulty (i.e., successful completion) were similar across all three courses indicating that certain tasks, specifically, the true-false tasks, proved to be more difficult than others. PSTs struggled with “strings” (primarily in the true-false format), indicating that PSTs may have an “answer goes next” (operational) understanding of the equal sign when evaluating other people’s work but may have some form of relational understanding when the task is in a solving format. Task B had similar successful response rates across all the courses which may signify that this format may demand a more sophisticated understanding of the equal sign that may be resistant (McNeil & Alibali, 2005) to the efforts of teacher preparatory coursework or it may be that the format of the task elicits an “algebra” mindset (Kieran, 1992; Stephens, 2008) that hinders success.

Inferential statistics suggest that there were significant difference between PSTs’ understandings of the equal sign based on: (1) mathematical background, (2) pursuing a mathematics endorsement, and (3) confidence to teach mathematics. Those who had taken a pre-calculus or calculus course scored about half a standard deviation higher than those who had a less advanced mathematics background, as did those who were seeking a mathematics endorsement versus those who were not. Although these results were moderate, the confidence to teach mathematics had a strong effect size with 16% of the variance in equal sign understandings explained by this variable. There also were significant relationships between PSTs’ EQ scores and their MKT scores, even when controlling for the opposite measure and the influence of their confidence to teach mathematics. After both these controls
about 7.2% of the variance in PSTs’ MKT-PFA scores could be explained by equal sign understanding which was slightly higher than the MKT-NCOP measure (5.3% of the variance), supporting the work of others (Knuth et. al, 2006) who have found a link between equal sign understanding and algebra success. PSTs beginning their second mathematics content course outperformed their peers enrolled in their first mathematics content course or methods course, hinting that PSTs may gain equal sign understanding during their first content course but that they may revert to weaker understandings prior to completing their methods course. But without longitudinal data this is just speculation. Pre- and post-test analysis does support that completing a mathematics methods course (based on CGI principles of understanding students’ thinking) did improve PSTs’ understanding of the equal sign. There was no statistical evidence to suggest a relationship between gender and EQ scores.

**Final Thoughts**

If algebra is a gatekeeper, then mathematics teachers are some of the keepers of the gate. Since very few PSTs hold a fully relational view of the equal sign and some PSTs even have a “fragile” understanding, fostering a strong relational understanding of the equal sign needs to be integrated throughout mathematics teacher preparatory classes. Understandings of the equal sign seem to be resistant to change (McNeil & Alibali, 2005b) so exposing PSTs to true/false tasks, practitioner articles on this topic, and student thinking (either through hypothetical or real experiences) throughout their teacher preparatory courses is pivotal.
Constraints and limitations

As in many educational studies time was a constraint. Due to time constraints the EQ had a limited number of tasks which impacted its reliability. Cross-sectional data give a snapshot of PSTs at specific points in time, but it does not allow for statistical conclusions on growth or lack of growth. Also, because the MKT measures were designed for in-service teachers, so it may be that the MKT scores are not valid for PSTs and the scale may need to be re-standardized to fit this population. Finally, due to lack of participation interview data were not used in this study thus conclusions are based on a written assessment(s).

Future research

As is often the case, exploring certain questions and data can bring more questions to light. First of all, the EQ needs additional well-written tasks in order to increase its reliability while not burdening the participants with a time-consuming assessment. Second of all, future research is needed to explore how PSTs with varied understandings of the equal sign interpret specific tasks recommended in the CCSS-M. The additional questions on the MMC post-test seem to suggest that some students with a relational understanding of the equal sign may not be aware of the purposes of some of the tasks given in the CCSS-M. Interview data would help clarify PSTs’ written responses on the EQ and would allow for an opportunity to explore their understandings of the examples in the CCSS-M. Finally, longitudinal data are needed to see if PSTs’ understandings of the equal sign change during their educational journey and what courses and opportunities are most pivotal in developing a relational understanding of the equal sign.
APPENDIX A. CONSENT FORM: MCC1 AND MCC2

INFORMED CONSENT DOCUMENT (Math 195/196)

Title of Study: Pre-service Teachers' Mathematical and Policy Understandings Explored (part 1)

Investigator:

Julie Hartzler, Doctoral Candidate in Curriculum and Instruction (Math Ed), Iowa State University, Ames, Iowa; Assistant Professor of Mathematics, Grand View University, Des Moines, Iowa

Heather Bolles, ISU Professor

Alejandro Andreotti, ISU Professor

Brenda Diesslin, Instructor

Gail Johnston, Instructor

Neil Seely, Graduate Student in CI (Math Ed)

Mary Gichobi, Doctoral Candidate in CI (Math Ed)

This is a research study. Please take time to consider if you would like to participate and feel free to ask questions at any time.

INTRODUCTION

The purpose of this study is to learn more about how pre-service teachers think about mathematics. You are being invited to participate in this study because you are a pre-service teacher enrolled in a mathematics content course (Math 195/196).

DESCRIPTION OF PROCEDURES

If you agree to participate in this study, your participation will last for approximately fifteen minutes at the beginning of the semester. During the study you may expect the following events to take place: you will be asked to complete questions dealing with a mathematical concepts, mathematical teaching beliefs and demographic information. When completing the questionnaire you may skip any question that you do not wish to answer or that makes you feel uncomfortable. All data is confidential and individual data will not be disclosed. Paper copies will be kept in a locked cabinet for up to two years after the end of the research study at which time they will be destroyed.

RISKS

There are no known risks to participating in this study.
BENEFITS

If you decide to participate in this study there may or may not be a direct benefit to you. It is hoped that the information gained by this study will benefit elementary students by better understanding pre-service teachers’ mathematical knowledge.

COSTS AND COMPENSATION

You will not have any costs from participating in this study. No compensation will be given.

PARTICIPANT RIGHTS

Your participation in this study is completely voluntary and you may refuse to participate or leave the study at any time. If you decide to not participate in the study or leave the study early, it will not result in any penalty or loss of benefits to which you are otherwise entitled. You can skip any questions that you do not wish to answer.

CONFIDENTIALITY

Records identifying participants will be kept confidential to the extent permitted by applicable laws and regulations and will not be made publicly available. However, federal government regulatory agencies, auditing departments of Iowa State University, and the Institutional Review Board (a committee that reviews and approves human subject research studies) may inspect and/or copy your records for quality assurance and data analysis. These records may contain private information.

To ensure confidentiality to the extent permitted by law, the following measures will be taken: after the research is conducted the data will be coded by number. Only the researcher will have access to the participants’ names. Other key personnel will only have access to the coded data. The names of all participants and the results of their questionnaires will be protected by the researcher. Paper copies will be kept in a locked cabinet for up to two years after the end of the research study at which time they will be destroyed. All electronic information will be on a password protected computer. The researcher will also exercise discretion by primarily working on this project at home or in her office with the door locked. If the results are published, your identity will remain confidential.

QUESTIONS OR PROBLEMS

You are encouraged to ask questions at any time during this study.

- For further information about the study contact Julie Hartzler at (515) 964-5978 or jhartz@iastate.edu. Co-major professor, Dr. Anne Foegen at (515) 294-8373 or afoegen@iastate.edu.
- If you have any questions about the rights of research subjects or research-related injury, please contact the IRB Administrator, (515) 294-4566, IRB@iastate.edu, or Director, (515) 294-3115, Office for Responsible Research, Iowa State University, Ames, Iowa 50011.
Please print this off and sign it for your records.

PARTICIPANT SIGNATURE

Your signature indicates that you voluntarily agree to participate in this study, that the study has been explained to you, that you have been given the time to read the document, and that your questions have been satisfactorily answered. You will receive a copy of the written informed consent prior to your participation in the study.

Participant’s Name (printed) ________________________________

_____________________________  ________________

(Participant’s Signature)  (Date)
APPENDIX B. CONSENT FORM MMC

INFORMED CONSENT DOCUMENT (CI 448)

Title of Study: Pre-service Teachers' Mathematical and Policy Understandings Explored (part 1)

Investigator:

Julie Hartzler, Doctoral Candidate in Curriculum and Instruction (Math Ed), Iowa State University, Ames, Iowa; Assistant Professor of Mathematics, Grand View University, Des Moines, Iowa

Heather Bolles, ISU Professor Alejandro Andreotti, ISU Professor

Brenda Dießlin, Instructor

Gail Johnston, Instructor

Neil Seely, Graduate Student in CI (Math Ed)

Mary Gichobi, Doctoral Candidate in CI (Math Ed)

This is a research study. Please take time to consider if you would like to participate and feel free to ask questions at any time.

INTRODUCTION

The purpose of this study is to learn more about how pre-service teachers think about mathematics. You are being invited to participate in this study because you are a pre-service teacher enrolled in a mathematics methods course (CI 448).

DESCRIPTION OF PROCEDURES

If you agree to participate in this study, your participation will last for approximately fifteen minutes at the beginning of the semester and fifteen minutes at the end of the semester. During the study you may expect the following events to take place: you will be asked to complete questions dealing with a mathematical concepts, mathematical teaching beliefs and demographic information. When completing the questionnaire you may skip any question that you do not wish to answer or that makes you feel uncomfortable. All data is confidential and individual data will not be disclosed. Paper copies will be kept in a locked cabinet for up to two years after the end of the research study at which time they will be destroyed.

RISKS

There are no known risks to participating in this study.
BENEFITS
If you decide to participate in this study there may or may not be a direct benefit to you. It is hoped that the information gained by this study will benefit elementary students by better understanding pre-service teachers’ mathematical knowledge.

COSTS AND COMPENSATION
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PARTICIPANT RIGHTS
Your participation in this study is completely voluntary and you may refuse to participate or leave the study at any time. If you decide to not participate in the study or leave the study early, it will not result in any penalty or loss of benefits to which you are otherwise entitled. You can skip any questions that you do not wish to answer.

CONFIDENTIALITY
Records identifying participants will be kept confidential to the extent permitted by applicable laws and regulations and will not be made publicly available. However, federal government regulatory agencies, auditing departments of Iowa State University, and the Institutional Review Board (a committee that reviews and approves human subject research studies) may inspect and/or copy your records for quality assurance and data analysis. These records may contain private information.

To ensure confidentiality to the extent permitted by law, the following measures will be taken: after the research is conducted the data will be coded by number. Only the researcher will have access to the participants’ names. Other key personnel will only have access to the coded data. The names of all participants and the results of their questionnaires will be protected by the researcher. Paper copies will be kept in a locked cabinet for up to two years after the end of the research study at which time they will be destroyed. All electronic information will be on a password protected computer. The researcher will also exercise discretion by primarily working on this project at home or in her office with the door locked. If the results are published, your identity will remain confidential.

QUESTIONS OR PROBLEMS
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- For further information about the study contact Julie Hartzler at (515) 964-5978 or jhartz@iastate.edu. Co-major professor, Dr. Anne Foegen at (515) 294-8373 or afoegen@iastate.edu.
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************************************************************************
Please print this off and sign it for your records.
PARTICIPANT SIGNATURE

Your signature indicates that you voluntarily agree to participate in this study, that the study has been explained to you, that you have been given the time to read the document, and that your questions have been satisfactorily answered. You will receive a copy of the written informed consent prior to your participation in the study.

Participant’s Name (printed) ________________________________

______________________________  _________________

(Participant’s Signature)  (Date)
APPENDIX C. EQ: MCC1 AND MCC2

Pre-service Teachers' Mathematical and Policy Understandings Explored Questionnaire
Math 195 and Math 196 ONLY

Please write your code on this page and on the following page before beginning this questionnaire. This is the same code that you used earlier.

Code: ________________________________

=  

↑

1. What is the name of the symbol that the arrow is pointing at?

2. What does that symbol mean?

3. Can you write down another definition? If so, what would it be?
Pre-service Teachers' Mathematical and Policy Understandings Explored Questionnaire, Math 195 and Math 196

Code: ________________________________

Please complete each problem without the use of a calculator. You can write on this assessment to complete these problems if you would like.

A. 267 + 85 = _____ + 83

B. 3x + 6 = 60 is true. Is 3x + 6 – 7 = 60 – 7 TRUE or FALSE? Why?

C. _____ + 21 = 25 + 32 = __________

D. Knowing that the sum of 79 and 148 is 227, can you find the sum of 149 and 82? If so, what is it?

E. 3 (___) + 8 = 2 (___) + 8
F. Circle whether the following statements are true, false, or not enough information to tell.

\[ 12 + 5 = 17 + 2 = 19 \]  True    False    Not Enough Information

\[ 2x + 14 = 7 + x \]  True    False    Not Enough Information

Circle all the math courses you took in high school.

Consumer Mathematics
Algebra I
Algebra II
Geometry
Pre-calculus (i.e., Trigonometry, College Algebra)
Statistics (not for Advanced Placement credit)
AP Statistics
Calculus (not for Advanced Placement credit)
AP Calculus
Additional Mathematics courses (please list): _______________________________________

What math courses have you taken since graduating high school (not including Math 195 or Math 196)?

____________________________________
Circle the grade levels you would you most like to teach.

First choice:     K  1st  2nd  3rd  4th  5th  6th  7th  8th
Second choice:   K  1st  2nd  3rd  4th  5th  6th  7th  8th
Third choice:    K  1st  2nd  3rd  4th  5th  6th  7th  8th

What endorsement(s) are you seeking? ____________________________

How confident are you in your ability as a future teacher to teach mathematics effectively? Please check the box that best describes you.

| I lack confidence to teach math. I’ll do my best but I’m not sure how good I’ll be. | I’m not very confident. I work hard on math but it still does not come easy to me. I know I’ll be well prepared but I’m afraid I’ll make mistakes in front of the students. | I’m somewhat confident. I think I’ll be a good math teacher. I think I’ll understand the math I teach and will be able to explain it well to students. | I’m very confident. I will be a strong math teacher. I understand math and can easily detect errors in my students thinking. |

What is your gender (circle one)?     Female     Male

Where did you complete Math 195? Check the appropriate box and fill in the blanks if given.

☐ I’m currently enrolled in this course.

☐ I completed this course at ISU with a grade of _______ in the____________________ _______ (semester/year).
I completed this course at __________________________ (name of institution) with a grade of __________ in ___________________ (semester and year).

Thank you for your time!
APPENDIX D. EQ: MCCpre

Pre-service Teachers' Mathematical and Policy Understandings Explored Questionnaire,

CI 448 ONLY

Please fill in this information for coding purposes.

Mom’s first name: ____________ Dad’s first name: ____________ First two letters of your hometown: ________ Room number or house number of where you currently live: ________

1. What is the name of the symbol that the arrow is pointing at?

2. What does that symbol mean?

3. Can you write down another definition? If so, what would it be?
Please fill in this information for coding purposes.
Mom’s first name: ___________  Dad’s first name: ___________
First two letters of your hometown: __________  Room number or house number of where you currently live: ______

Please complete each problem without the use of a calculator. You can write on this assessment to complete these problems if you would like.

A.  267 + 85 = _______ + 83

B.  3x + 6 = 60 is true. Is 3x + 6 – 7 = 60 – 7 TRUE or FALSE? Why?

C.  _______ + 21 = 25 + 32 = __________

D.  Knowing that the sum of 79 and 148 is 227, can you find the sum of 149 and 82? If so, what is it?

E.  3 (____) + 8 = 2 (____) + 8
F. Circle whether the following statements are true, false, or not enough information to tell.

12 + 5 = 17 + 2 = 19

True    False    Not Enough Information

2x + 14 = 7 + x

True    False    Not Enough Information

Circle all the math courses you took in high school.

Consumer Mathematics
Algebra I
Algebra II
Geometry
Pre-calculus (i.e., Trigonometry, College Algebra)
Statistics (not for Advanced Placement credit)
AP Statistics
Calculus (not for Advanced Placement credit)
AP Calculus
Additional Mathematics courses (please list): ________________________________

What math courses have you taken since graduating high school (not including Math 195 or Math 196)?
Circle the grade levels you would you most like to teach.

First choice: K 1st 2nd 3rd 4th 5th 6th 7th 8th
Second choice: K 1st 2nd 3rd 4th 5th 6th 7th 8th
Third choice: K 1st 2nd 3rd 4th 5th 6th 7th 8th

What endorsement(s) are you seeking? ______________________

How confident are you in your ability as a future teacher to teach mathematics effectively? Please check the box that best describes you.

<table>
<thead>
<tr>
<th>I lack confidence to teach math. I’ll do my best but I’m not sure how good I’ll be.</th>
<th>I’m not very confident. I work hard on math but it still does not come easy to me. I know I’ll be well prepared but I’m afraid I’ll make mistakes in front of the students.</th>
<th>I’m somewhat confident. I think I’ll be a good math teacher. I think I’ll understand the math I teach and will be able to explain it well to students.</th>
<th>I’m very confident. I will be a strong math teacher. I understand math and can easily detect errors in my students thinking.</th>
</tr>
</thead>
</table>

What is your gender (circle one)? Female Male

Where did you complete Math 195? Check the appropriate box and fill in the blanks if given.

☐ I completed this course at ISU with a grade of _________ in the_________________________ _______ (semester/year).
I completed this course at _____________________________ (name of institution) with a grade of _______ in ________________ (semester and year).

Where did you complete Math 196? Check the appropriate box and fill in the blanks if given.

☐ I completed this course at ISU with a grade of _______ in the ____________ (semester/year). Please continue to the last page.

☐ I completed this course at _____________________________ (name of institution) with a grade of _______ in ________________ (semester and year).
Please continue to the next page.

If you would be willing to consider participating in two interviews lasting 60-90 minutes (one at the beginning of the semester and one at the end of the semester) please leave your contact information below. Participants will receive a $10 gift card after each interview as a token of appreciation.

☐ Yes, I may be interested (please complete the information below)

Name: _______________________________________________________

Mom’s first name: ___________ Dad’s first name: ___________ First two letters of your hometown: ________ Room number or house number of where you currently live: ________

Phone number(s): ________________________________

Email(s): ________________@iastate.edu ________________________________

☐ No, thank you.

Thank you for your time!
APPENDIX E. EQ: MMCpost

Pre-service Teachers' Mathematical and Policy Understandings Explored Questionnaire, CI 448 ONLY (end of year)

Please fill in this information for coding purposes.

Mom’s first name: _______________ Dad’s first name: _______________
First two letters of your hometown: _______ Room number or house number of where you currently live: ______

= 
↑

1. What is the name of the symbol that the arrow is pointing at?

2. What does that symbol mean?

3. Can you write down another definition? If so, what would it be?
Pre-service Teachers' Mathematical and Policy Understandings Explored Questionnaire, CI 448 (end of year)

**Please fill in this information for coding purposes.**

Mom’s first name: ____________ Dad’s first name: ____________ First two letters of your hometown: ________ Room number or house number of where you currently live: ________

**Please complete each problem without the use of a calculator. You can write on this assessment to complete these problems if you would like.**

A. \( 267 + 85 = \) _____ + 83

B. \( 3x + 6 = 60 \) is true. Is \( 3x + 6 - 7 = 60 - 7 \) TRUE or FALSE? Why?

C. _____ + 21 = 25 + 32 = ____________

D. Knowing that the sum of 79 and 148 is 227, can you find the sum of 149 and 82? If so, what is it?

E. \( 3 (\_\_) + 8 = 2 (\_\_) + 8 \)
F. Circle whether the following statements are true, false, or not enough information to tell.

\[
12 + 5 = 17 + 2 = 19 \quad \text{True} \quad \text{False} \quad \text{Not Enough Information}
\]

\[
2x + 14 = 7 + x \quad \text{True} \quad \text{False} \quad \text{Not Enough Information}
\]

G. In task E you were given the problem: \[3 \, (\_ \_ \_) + 8 \, = \, 2 \, (\_ \_ \_) + 8\]

Write three more solution to this problem.

\[
3 \, (\_ \_ \_) + 8 \, = \, 2 \, (\_ \_ \_) + 8 \\
3 \, (\_ \_ \_) + 8 \, = \, 2 \, (\_ \_ \_) + 8 \\
3 \, (\_ \_ \_) + 8 \, = \, 2 \, (\_ \_ \_) + 8
\]

☐ If you do not believe there are additional solutions check the box and explain why:

H. The Common Core State Standards – Mathematics indicates that at the first grade level students should work with addition and subtraction equations and give the following example as a useful problem.

Which of the following equations are true and which are false?
What purpose do you think this task would serve? What type of understandings is it trying to get at?

Why do you think the writers of the Common Core State Standards – Mathematics used true/false equations versus a fill in the blank equation like: $5 + 4 = \underline{\phantom{1}}$?
APPENDIX F. CODING GUIDE

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relational (1): Dealing with sameness; equivalence; both sides are equal</td>
</tr>
<tr>
<td>Operational (0): Dealing with getting an answer; calculating an answer; only writes “equals”</td>
</tr>
</tbody>
</table>

*If two definitions are given, one operational and one relational, then the code would be a the higher of the two (a 1).*

**If the definition does not make sense (i.e., thinks the question is about a different mathematical topic) or is left blank, then the coding will be a blank.**

**Task A:** $267 + 85 = ____ + 83$

- **Fully Relational (2):** Fill in the blank with 269 with no work shown or if work is shown it supports adding 2 to 267 to fill in the blank. If they make an error but show FR thinking it is still marked as a 2.

- **Relational with Computation (1):** The student uses computation to solve the problem (i.e., $267+85-83$). If they make a computational error but show RC thinking it is still marked as a 1.

- **Operational (0):** The student adds straight across and fills in the blank with the sum of 267 and 85.

**Task B:** $3x + 6 = 60$ is true. Is $3x + 6 - 7 = 60 - 7$ True or False? Why?

- **Fully Relational (2):** No computation is shown and the student mentions that it is true because the same thing is being done to both sides OR that it is true because if they sub in 60 for $3x - 7$ they get $60 - 7 = 60 - 7$.

- **Relational with Computation (1):** The student solves the first equation for $x$ ($x=18$) and then subs the value into the second equation to see if it works. Explanations may include some of the ideas from above but computation occurred.

- **Operational (0):** The solution doesn’t make sense, answers T with no reason, or is left blank.
Task C: \[ \_ + 21 = 25 + 32 = \_
\]

Fully Relational (2): Fills in the first blank with 36 and the second with 57 without any work. If there is an error but they show this type of understanding it is still marked at FR.

Relational with Computation (1): The student shows computational work (i.e., 25+32-21).

Operational: The first the blank has a 5 and the other blank may be 57 or some other value.

Task D: Given the sum of … find the sum of 149 and 82?

Fully Relational/Substitutionary (1): The student writes the sum of 231 with no work OR shows that adding 4 to 227 produces the correct sum. If they make an error but show FR thinking it is still marked as a 1. If they show both the sum of 149 and 82 AND the relational idea of +4, then it will still be marked as a 1.

Operational or Blank(0): If they simply add or leave blank.

Task E: \[ 3(\_ ) + 8 = 2 ( \_ ) + 8 \]

Advanced (2): Use numbers other than 2 and 3 (i.e., 0 and 0, 4 and 6, etc.).

Commutative (1): Fill in the first blank with 2 and the second blank with 3.

Incorrect Numbers or Blank (0): The two values do not produce a correct numeric equation or are left blank.

Task F: \[ 12 + 5 = 17 + 2 = 19 \]

(True, False, Not Enough Information)

(IGNORE THE SECOND EQUATION)

Relational (1): Correctly answer FALSE.

Operational or Blank (0): True
Other Codings

Correct (1)
Incorrect (0)

Confidence Level: Very (3), Somewhat (2), Not Very (1), Lack Confidence (0)

Took a pre-calculus course or a calculus course in high school or college (1)
Did not (0)
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