1968

Integrals over proto-rings

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WOODWORTH, Wayne Leon, 1940-
INTEGRALS OVER PROTO-RINGS.

Iowa State University, Ph.D., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan
INTEGRALS OVER PROTO-RINGS

by

Wayne Leon Woodworth

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

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Iowa State University
Ames, Iowa
1968
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I. INTRODUCTION

Integration of functions with respect to a countably additive set function has been treated by numerous persons with Halmos, [6], being perhaps the most standard reference in the area. However, various authors, including Bogdanowicz, [1], von Neumann, [8] and Zaanen, [12] have suggested simplifications for the domain of the set function. Actually, Halmos [6, p. 22, problem 6], also suggests a weaker structure which he calls a semi-ring. The term half-ring in [8, p. 85] means the same thing as semi-ring in [6]. Finally, semi-ring as used in [12] and pre-ring in [1] coincide. The pre-ring structure most closely resembles, and in fact partly suggested, the type of structure considered in this dissertation. In view of this the definition of a pre-ring will be included here.

Definition 1.1. A non-empty collection \( Q \) of subsets of a non-void set \( T \) is said to be a pre-ring if and only if whenever \( E \) and \( F \) are elements of \( Q \), then \( E \cap F \) is in \( Q \) and there exists a finite pairwise disjoint sequence \( \{ E_i \}_{i=1}^n \) of elements of \( Q \) such that \( E - F = \bigcup_{i=1}^n E_i \).

Bogdanowicz, [1], was able to give a very direct construction for a class of vector-valued functions integrable with respect to a real valued countably additive set function defined on a pre-ring. It should be emphasized that the development by Bogdanowicz is only for countably additive set functions. Some of the theory that he develops is for a finitely additive set function, \( \mu \), dominated by a non-negative countably additive set function, \( \nu \), in the sense that \( |\mu(A)| \leq \nu(A) \) for all \( A \) in the domain of \( \mu \) and \( \nu \). However, by a slight modification of a
result obtained by Yosida and Hewitt, [11, Theorem 1.15], this type of dominated condition implies the countable additivity of \( \mu \). Bogdanowicz also takes cognizance of this fact in a latter paper, [2, p. 232].

The problem of developing integration theory for finitely additive set functions seems not to have been treated very extensively. The work by Dunford and Schwartz, [4, pp. 95-125], appears to be about the most exhaustive treatment for the finitely additive case. However, they do require at all times that the finitely additive set function be defined on an algebra (field) of sets, so there remains the question as to what simplifications might be made in the structure of the domain of the set function. Also, they leave some unanswered questions in regard to the \( L_p \) spaces they consider; for example, a characterization of the adjoint spaces for their \( L_p \) spaces. This gap seems to be filled in very nicely by Porcelli, [9]; however, it should be noted that the \( L_p \) spaces in [9] are not the same as those in [4].

The purpose of this dissertation is to give a definition of integral that includes the previous results of Bogdanowicz, [1] and Dunford and Schwartz, [4]. Also, the structure of the domain of the set functions will be of a minimal nature.

**Definition 1.2.** Let \( T \) be a non-void set and \( P \) a non-empty collection of subsets of \( T \). Then \( P \) is said to be a proto-ring if and only if whenever \( E \) and \( F \) are in \( P \), then there exists two finite pairwise disjoint sequences \( \left\{ E_i \right\}_{i=1}^{n} \) and \( \left\{ F_j \right\}_{j=1}^{m} \) of elements of \( P \) such that \( E \cap F \) is \( \bigcup_{i=1}^{n} E_i \) and \( E - F \) is \( \bigcup_{j=1}^{m} F_j \). If \( T \) can be written as a union of a finite number of sets of \( P \), then \( P \) is called a proto-algebra.
The proto-ring will be the basic structure for the domain of the set functions considered in this dissertation. It is not hard to see that an algebra of sets is a proto-algebra of sets, and, more generally then, a proto-ring of sets. This assertion can be seen to be true by the following argument. If \( P \) is an algebra of sets and \( E \) and \( F \) are in \( P \), then \( E \cap F \) and \( E - F \) are in \( P \). Therefore, the two sequences in Definition 1.2 may be taken to be the singleton sequences \( \{ E \cap F \} \) and \( \{ E - F \} \). Finally, \( T \) belongs to \( P \), so \( P \) is a proto-algebra. This shows that the domain for the set functions considered in this dissertation generalizes that of [4] for finitely additive set functions. In this sense the development here is more general than in [4].

If \( P \) is a pre-ring of sets and \( E \) and \( F \) are in \( P \), then \( E \cap F \) is in \( P \). Therefore, the sequence \( \{ E \}_i^{\infty} \), of Definition 1.2 may be taken to be the singleton sequence \( \{ E \cap F \} \). The condition for differences in the definitions of a pre-ring and of a proto-ring are identical. Hence, every pre-ring is a proto-ring, and, in this sense, the development here is a generalization of the work of Bogdanowicz, [1]. It should be noted that the mode of convergence in [1], pointwise almost everywhere, is not the same as the convergence mode to be used in the present development.

However, the class of integrable functions in [1] and this dissertation will coincide. Also, most of the development in [1] could be accomplished with a proto-ring in place of a pre-ring with the appropriate changes relative to intersections.

Following are several examples of proto-rings, some of which are stronger structures. It should be noted that the empty set must belong to
all proto-rings. This follows since a proto-ring $P$ must be non-empty, and if $E$ is in $P$, then $E - E$ is empty and is a union of elements of $P$.

**Example 1.1.** Let $T$ be the real number interval $[0, 1]$ and $P$ be the collection of all singleton subsets of $T$, all open subintervals of the form $(a, b)$ with $0 \leq a < b \leq 1$ and the empty set, $\emptyset$. Suppose that $E$ and $F$ are in $P$. If $E$ is empty or a singleton, and $F$ is any set in $P$, then each of $E \cap F$ and $E - F$ is also empty or a singleton set. If $E$ and $F$ are disjoint, then $E \cap F$ is empty and $E - F$ is $E$. Suppose then that $E$ and $F$ are not disjoint and that $E$ is an open subinterval of $[0, 1]$, say $E = (a, b)$. If $F$ is a singleton set, $F = \{c\}$, then $a < c < b$ and $E \cap F$ is $F$ and $E - F$ is the union of $(a, c)$ and $(c, b)$. If $F$ is an open subinterval, $F = (c, d)$, then there are several cases to be considered. For example, it may be that $a < c < b < d$ and in this case $E \cap F$ is the interval $(c, b)$ while $E - F$ is the interval $(a, c)$. It can easily be seen in each of the other possible cases that $E \cap F$ and $E - F$ can be written as a union of a finite pairwise disjoint sequence of elements of $P$. Hence, $P$ is a proto-ring, and since $T$ is the union of $\{0\}$, $(0, 1]$ and $\{1\}$, $P$ is a proto-algebra. It is actually true that $P$ is a pre-ring.

**Example 1.2.** Let $T$ be the set of positive integers. For each positive integer $q$, let $F_q$ be the subset of $T$ defined by the equation

$$F_q = \{r \in T: r \geq q\}.$$ 

Let $P$ be the collection of subsets of $T$ consisting of the empty set, all singleton subsets of $T$ and the collection of subsets $F_q$ for $q = 1, 2, 3, \ldots$. Let $E$ and $F$ be in $P$. If $E$ and $F$ are disjoint, then $E \cap F$ is empty and $E - F$ is $E$. If $E$ and $F$ are equal, then $E \cap F$ is $E$ and $E - F$
is empty. Suppose then that E and F are distinct and not disjoint. If E is a singleton set, \( E = \{p\} \), then F is an \( F_q \) set for some positive integer q. In this case, \( E \cap F \) is E (since E and F are not disjoint) and \( E - F \) is empty. If E is an \( F_q \) set for some q and F is a singleton set, \( F = \{p\} \), then \( q < p \) since \( E \cap F \neq \emptyset \). Therefore, \( E \cap F \) is F and \( E - F \) is the union of the sets \( \{q\}, \{q + 1\}, \ldots, \{p - 1\}, F_{p+1} \). Finally, if E is the set \( F_p \) and F is the set \( F_p \) for some positive integers p and q, then either E is a subset of F or F is a subset of E. In the first case, \( E \cap F \) is E and \( E - F \) is empty. In the second case, \( E \cap F \) is F and \( E - F \) is the union of the sets \( \{q\}, \{q + 1\}, \ldots, \{p - 1\} \). Therefore, \( E \cap F \) is a proto-ring. Since T is the set \( F_1 \), \( E \cap F \) is actually a proto-algebra. Once more, \( E \cap F \) is in fact a pre-ring.

Example 1.3. Let T be the real number interval \([0, 1]\). Let \( E \) be the collection consisting of the empty set, all singleton subsets of T and all subintervals of the forms \([a, b), (a, b]\) and \((a, b]\) where \( 0 \leq a < b \leq 1 \). The argument that \( E \) is a proto-algebra (T belongs to \( E \)) is essentially the same as for Example 1.1 except that an intersection of elements of \( E \) is not necessarily another element of \( E \). For example if E is the interval \((a, b]\) and F the interval \([c, d)\) with \( a < c < b < d \), then \( E \cap F \) is the interval \([a, b]\). In this case the intersection, \( E \cap F \), can be written as the union of the sets \( \{a\} \) and \( (c, b]\), both of which belong to \( E \). Therefore, \( E \) is a proto-ring and not a pre-ring.

The following two theorems will be used throughout the dissertation. It is assumed throughout the rest of this section that T is a fixed non-void set and \( E \) is a proto-ring of subsets of T.
Theorem 1.1. If $k$ is a positive integer and $B_1, B_2, \ldots, B_k$ and $A$ are elements of $P$, then the difference $A - \left( \bigcup_{i=1}^{k} B_i \right)$ can be written as the union of a finite sequence of pairwise disjoint elements of $P$.

Proof: The proof will be by induction on $k$. If $k$ is 1, then the conclusion follows directly from Definition 1.2. Assume now that the statement is true for $k$ equal to the positive integer $n$, and suppose that $B_1, B_2, \ldots, B_{n+1}$ and $A$ are elements of $P$. By the induction hypothesis, the difference $A - \left( \bigcup_{i=1}^{n} B_i \right)$ is the union of a finite sequence of pairwise disjoint elements of $P$, say $\{C_j\}_{j=1}^{m}$. Then the following equalities follow, with the first and last ones being true by simple set theoretic arguments:

$$A - \left( \bigcup_{i=1}^{n+1} B_i \right) = [A - \left( \bigcup_{i=1}^{n} B_i \right)] - B_{n+1}$$

$$= \left( \bigcup_{j=1}^{m} C_j \right) - B_{n+1}$$

$$= \bigcup_{j=1}^{m} (C_j - B_{n+1}).$$

Now for $j = 1, 2, \ldots, m$, $C_j - B_{n+1}$ is the union of a finite sequence of pairwise disjoint elements of $P$, say $\{D_{ij}\}_{i=1}^{q(j)}$. Also, since the sequence $\{C_j\}_{j=1}^{m}$ is pairwise disjoint, it follows that the sequence $\{C_j - B_{n+1}\}_{j=1}^{m}$ is also a pairwise disjoint sequence. This implies that the collection

$$\{D_{ij}; \ i = 1, 2, \ldots, q(j); \ j = 1, 2, \ldots, m\}$$

of sets of $P$ is a pairwise disjoint collection. Also, it is true that $A - \left( \bigcup_{i=1}^{n+1} B_i \right)$ is given by
\[ A - \left( \bigcup_{i=1}^{n+1} B_i \right) = \bigcup_{j=1}^{m} \bigcup_{i=1}^{q(j)} D_{ij}. \]

Since \( m \) and \( q(j) \) for \( j = 1, 2, \ldots, m \) are finite, this completes the proof.

**Theorem 1.2.** If \( \left\{ E_i \right\}_{i=1}^{p} \) is a sequence of elements of \( P \) with \( p \) a positive integer or \( p = \infty \), then there exists a pairwise disjoint sequence \( \left\{ F_j \right\}_{j=1}^{q} \) of elements of \( P \) such that \( \bigcup_{i=1}^{p} E_i = \bigcup_{j=1}^{q} F_j \), and \( q \) is finite if \( p \) is finite and \( q = \infty \) if \( p = \infty \).

**Proof:** Let \( G_1 \) be the set \( E_1 \), and for \( k = 2, 3, 4, \ldots, p \) (or \( k = 2, 3, 4, \ldots \) in the case \( p = \infty \)) let \( G_k \) be the difference \( E_k \setminus \bigcup_{i=1}^{k-1} E_i \).

Then \( \left\{ G_k \right\}_{k=1}^{p} \) is a pairwise disjoint sequence and \( \bigcup_{k=1}^{p} G_k = \bigcup_{i=1}^{p} E_i \).

By induction and Theorem 1.1 each set \( G_k \) is a union of a finite pairwise disjoint sequence of elements of \( P \). Let \( \left\{ G_{kj} \right\}_{j=1}^{q(k)} \) be this sequence of elements of \( P \), so that \( G_k = \bigcup_{j=1}^{q(k)} G_{kj} \). It follows from this that

\[ \bigcup_{k=1}^{p} E_k = \bigcup_{k=1}^{p} \bigcup_{j=1}^{q(k)} G_{kj}, \]

is a pairwise disjoint collection of elements of \( P \). This completes the proof.

**Definition 1.3.** Suppose that \( Q \) is a collection of subsets of \( T \). The statement that the sequence \( \left\{ F_i \right\}_{i=1}^{m} \) is a \( Q \)-partition for \( T \) means that \( T = \bigcup_{i=1}^{m} F_i \) and \( \left\{ F_i \right\}_{i=1}^{m} \) is a finite sequence of non-empty pairwise disjoint elements of \( Q \).

A \( Q \)-partition is called a \( Q \)-subdivision by some authors.

**Definition 1.4.** Suppose that \( Q \) is a collection of subsets of \( T \) and...
that $\Gamma = \{F_i\}_{i=1}^m$ and $\Delta = \{E_j\}_{j=1}^n$ are Q-partitions of T. The statement that $\Gamma$ refines $\Delta$ means that each $F_i$ is a subset of some $E_j$.

It should be noted that if $\Gamma$ and $\Delta$ are as in Definition 1.4, then each $E_j$ is the union of the elements of some subcollection of $\Gamma$.

Preceding Definition 1.2 it was asserted that a proto-ring is a minimal structure for developing integration theory. The minimal character is in the sense of the following discussion. When one develops a refinement-type integral for functions defined on a set T and over a collection Q of subsets of T, the collection Q should have two properties. They are:

Property 1. There exists a Q-partition of T.

Property 2. The collection of all Q-partitions forms a directed set when ordered by refinement.

The assertion of the next theorem is that proto-algebras and collections satisfying Properties 1 and 2 are the same except for the inclusion of the empty set. It was in fact this observation, along with the pre-ring structure, that suggested the idea of a proto-algebra, and hence the proto-ring as a natural further generalization.

Theorem 1.3. Let T be a set and $P'$ a non-empty collection of subsets of T such that

i) if $E$ is in $P'$, then there exists a $P'$-partition $\Delta$ of T such that $E$ belongs to $\Delta$, and

ii) the collection of all $P'$-partitions of T forms a directed set under refinement. If $P$ is $P'\cup\{\emptyset\}$, then $P$ is a proto-algebra.

Conversely, if $P$ is a proto-algebra of subsets of T and $P'$ is the collection $P - \{\emptyset\}$, then $P'$ has properties i) and ii).
Proof: Suppose that $P'$ satisfies i) and ii) and that $P = P' \cup \{\emptyset\}$. Let $E$ and $F$ be elements of $P$. If $E$ or $F$ is empty, then it is clear that $E \cap F$ and $E - F$ satisfy the relevant parts of Definition 1.2. Otherwise, $E$ and $F$ are in $P'$, so let $\Delta$ and $\Gamma$ be $P'$-partitions of $T$ with $E$ belonging to $\Delta$ and $F$ to $\Gamma$. Let $\mathring{\mathcal{E}} = \{\mathring{G}_k\}_{k=1}^q$ be a $P'$-partition refining $\Delta$ and $\Gamma$, the existence of $\mathring{\mathcal{E}}$ being assured by ii). Let $K_1$ be the set $\{k: \mathring{G}_k \in \mathring{\mathcal{E}} \text{ and } \mathring{G}_k \subseteq E \cap F\}$ and $K_2$ the set $\{k: \mathring{G}_k \in \mathring{\mathcal{E}} \text{ and } \mathring{G}_k \subseteq E - F\}$. Then $\bigcup_{k \in K_1} \mathring{G}_k$ is a subset of $E \cap F$, and $\bigcup_{k \in K_2} \mathring{G}_k$ is a subset of $E - F$. Since $\mathring{\mathcal{E}}$ is a $P'$-partition of $T$, if $t$ belongs to $E \cap F$, then $t$ is in $\mathring{G}_{k_0}$ for some $\mathring{G}_{k_0}$ in $\mathring{\mathcal{E}}$. However, each $\mathring{G}_k$ in $\mathring{\mathcal{E}}$ is a subset of only one element of $\Delta$ and intersects no other element of $\Delta$. Therefore, $\mathring{G}_{k_0}$ is a subset of $E$. Similarly, $\mathring{G}_{k_0}$ is a subset of $F$, so that $\mathring{G}_{k_0} \subseteq E \cap F$. This implies that $k_0$ is in $K_1$, so that $E \cap F$ is contained in $\bigcup_{k \in K_1} \mathring{G}_k$. A similar argument shows that $E - F$ is the union $\bigcup_{k \in K_2} \mathring{G}_k$. Since each of $K_1$ and $K_2$ are finite, this proves that $P$ is a proto-ring. The fact that $P'$ is non-empty and satisfies condition i) assures that $P$ is a proto-algebra.

Suppose now that $P$ is a proto-algebra and that $P' = P - \{\emptyset\}$. Let $E$ be in $P'$ and $T$ be the union of the finite sequence $\{F_j\}_{j=1}^m$ of pairwise disjoint elements of $P$. The fact that $T$ can be so written is guaranteed by Theorem 1.2 and the fact that $P$ is a proto-algebra. By Theorem 1.1 $E - (\bigcup_{j=1}^m F_j)$ can be written as the union of the elements of a finite sequence $\{E_i\}_{i=1}^n$ of pairwise disjoint elements of $P$. Then the non-empty sets amongst

$$E, E_1, E_2, \ldots, E_n$$
form a P'-partition of T. Since E is non-empty, this P'-partition satisfies condition i). If \( \Delta = \{ E_i \}_{i=1}^{n} \) and \( \Gamma = \{ F_j \}_{j=1}^{m} \) are P'-partitions of T, and if \( E_i \cap F_j \) is non-empty, let \( E_i \cap F_j \) be given by the finite union \( \bigcup_{k=1}^{q(i,j)} G_{ijk} \) of pairwise disjoint non-empty elements of P. Let \( K \) be the set of all ordered pairs of positive integers \((i, j), i = 1, 2, ..., n; j = 1, 2, ..., m\) such that \( E_i \cap F_j \) is non-empty. Define \( \xi \) to be the pairwise disjoint collection

\[
\xi = \{ G_{ijk} : (i, j) \in K \text{ and } k = 1, 2, ..., q(i, j) \}.
\]

Since every t in T belongs to some \( E_i \) and to some \( F_j \), it follows that \( \xi \) is a P'-partition. Clearly, \( \xi \) refines \( \Delta \) and \( \Gamma \). This completes the proof.

The set functions considered in this dissertation are assumed to be defined on the proto-ring \( P \) of subsets of T with range in the real numbers and to be finitely additive. Such a set function will be called a p-volume. A p-volume may, of course, be countably additive, but this is not assumed, unless explicitly stated, at any point in the dissertation. The following notational convenience will be used throughout the development. If \( \mu \) is a countably additive p-volume, then \( D(\mu, P) \) is the collection of all countable sequences of pairwise disjoint elements of \( P \). If \( \mu \) is a p-volume that is only finitely additive, then \( D(\mu, P) \) is the collection of all finite sequences of pairwise disjoint elements of \( P \). Whenever \( \{ F_j \}_{j=1}^{m} \) is asserted to be in \( D(\mu, P) \), it is to be understood that \( m = \infty \) or that \( m \) is a positive integer if \( \mu \) is countably additive and that \( m \) is a positive integer if \( \mu \) is only finitely additive. The following theorem summarizes some properties of non-negative p-volumes.
Theorem 1.4. A non-negative p-volume is finitely subadditive; it is countably subadditive if it is countably additive. If \( \mu \) is a non-negative p-volume, it is monotone in the sense that if \( \{E_i\}_{i=1}^{n} \) is a finite sequence in \( D(\mu, P) \) and \( F \) is in \( P \) with \( \bigcup_{i=1}^{n} E_i \subseteq F \), then \( \sum_{i=1}^{n} \mu(E_i) \leq \mu(F) \).

Proof: Let \( \mu \) be a non-negative p-volume. The monotone property will be demonstrated first. Suppose that \( \{E_i\}_{i=1}^{n} \) and \( F \) are as in the statement of the theorem. Theorem 1.1 implies that \( F - (\bigcup_{i=1}^{n} E_i) \) is the union of a finite sequence \( \{F_j\}_{j=1}^{m} \) in \( D(\mu, P) \). Hence, \( F \) is given by the equation

\[
P = (\bigcup_{i=1}^{n} E_i) \bigcup (\bigcup_{j=1}^{m} F_j),
\]

and the right hand side of this equation is a union of pairwise disjoint elements of \( P \). Therefore,

\[
\mu(F) = \sum_{i=1}^{n} \mu(E_i) + \sum_{j=1}^{m} \mu(F_j) \geq \sum_{i=1}^{n} \mu(E_i),
\]

where the inequality follows from the non-negativity of \( \mu \). This proves that \( \mu \) is monotone in the sense described. It should be noted that as a special case, if \( E \) and \( F \) are in \( P \) with \( EC \), then \( \mu(E) \leq \mu(F) \).

Suppose now that \( \{E_{ij}\}_{i=1}^{m} \) is a sequence of elements of \( P \) and that \( \bigcup_{i=1}^{m} E_i \) belongs to \( P \). It is assumed that \( m \) is a positive integer if \( \mu \) is only finitely additive; otherwise, \( m \) is either a positive integer or \( m = \infty \). As in the proof of Theorem 1.2, there exists a pairwise disjoint sequence \( \{G_k\}_{k=1}^{m} \) such that \( G_k \) is a subset of \( E_k \) for each \( k \) and \( \bigcup_{k=1}^{m} G_k = \bigcup_{k=1}^{m} E_k \). Furthermore, each \( G_k \) is a finite union of pairwise...
disjoint elements of \( P \), \( G_k = \bigcup_{j=1}^{m} G_{kj} \). Therefore, \( \sum_{j=1}^{m} \mu(G_{kj}) \) is less than or equal to \( \mu(E_k) \) and \( \bigcup_{k=1}^{m} \bigcup_{j=1}^{m} G_{kj} \) is \( \bigcup_{k=1}^{m} E_k \). This implies that

\[
\mu\left(\bigcup_{k=1}^{m} E_k\right) = \mu\left(\bigcup_{k=1}^{m} \bigcup_{j=1}^{m} G_{kj}\right) \\
\leq \sum_{k=1}^{m} \sum_{j=1}^{m} \mu(G_{kj}) \\
\leq \sum_{k=1}^{m} \mu(E_k),
\]

so that \( \mu \) is finitely subadditive, and it is countably subadditive if \( \mu \) is countably additive. This completes the proof.

One other property of a non-negative \( p \)-volume \( \mu \) that will be used is the following: if \( \left\{ F_i \right\}_{i=1}^{n} \) is in \( D(\mu, P) \) and \( \left\{ F_j \right\}_{j=1}^{m} \) is a sequence of elements of \( P \) such that \( \bigcup_{i=1}^{n} F_i \) is the same as \( \bigcup_{j=1}^{m} F_j \), then \( \sum_{i=1}^{n} \mu(F_i) \leq \sum_{j=1}^{m} \mu(F_j) \), where \( m \) is a positive integer or \( m = \infty \) if \( \mu \) is countably additive and \( m \) is a positive integer if \( \mu \) is only finitely additive.

This is a fairly immediate consequence of the preceding theorem by the following argument. Let \( \left\{ E_i \right\}_{i=1}^{n} \) and \( \left\{ F_j \right\}_{j=1}^{m} \) be as described. Since \( P \) is a proto-ring, for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \) there exists a finite sequence \( \left\{ G_{ijk} \right\}_{k=1}^{r(i,j)} \) in \( D(\mu, P) \) such that \( E_i \cap F_j = \bigcup_{k=1}^{r(i,j)} G_{ijk} \).

This implies that

\[
E_i = \bigcup_{j=1}^{n} \bigcup_{k=1}^{r(i,j)} G_{ijk} \quad \text{for } i = 1, 2, \ldots, n \\
F_j = \bigcup_{i=1}^{m} \bigcup_{k=1}^{r(i,j)} G_{ijk} \quad \text{for } j = 1, 2, \ldots, m.
\]
The right hand side of the last equation is a union of pairwise disjoint elements of \( P \) since \( \{ E_i \}_{i=1}^n \) is a pairwise disjoint sequence. Therefore, it follows that

\[
\mu(E_i) \leq \sum_{j=1}^m \frac{r(i,j)}{\sum_{k=1}^m \mu(G_{ijk})} \quad \text{for } i = 1, 2, \ldots, n \text{ and }
\]

\[
\mu(F_j) = \sum_{i=1}^n \frac{r(i,j)}{\sum_{k=1}^m \mu(G_{ijk})} \quad \text{for } j = 1, 2, \ldots, m.
\]

Finally, this implies that

\[
\sum_{i=1}^n \mu(E_i) \leq \sum_{i=1}^n \frac{r(i,i)}{\sum_{k=1}^m \mu(G_{ijk})} = \sum_{j=1}^m \frac{r(j,j)}{\sum_{k=1}^m \mu(G_{ijk})} = \sum_{j=1}^m \mu(F_j).
\]

It should be noted that the interchange of the order of summation is permissible since \( \mu(G_{ijk}) \) is a non-negative real number for all values of the subscripts \( i, j \) and \( k \).

One final observation is that a p-volume is zero when evaluated at the empty set \( \emptyset \). This is an immediate consequence of the additivity of a p-volume and the fact that \( A \) is \( A \cup \emptyset \) for any set \( A \).
II. THE INTEGRAL FOR A NON-NEGATIVE P-VOLUME

It is assumed throughout this chapter that $T$ is a non-void set, $P$ is a proto-ring of subsets of $T$ with the property that $\bigcup\{E:E \in P\}$ is $T$ and that $\mu$ is a non-negative $p$-volume defined on $P$. Also, it is assumed that $(X, \|\cdot\|)$ is a real Banach space with norm $\|\cdot\|$. Finally, $\mathbb{R}$ will denote the real numbers with the usual norm. Whenever a function space is referred to as a linear space it is assumed that the operations in the space are pointwise.

The symbol $S_1(P, X)$ stands for the collection of all functions $h$ on $T$ to $X$ that can be written in the form

$$h = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where $n$ is a positive integer, $\{a_i\}_{i=1}^{n}$ is a sequence of elements of $X$, $\{A_i\}_{i=1}^{n}$ is a sequence of elements of $P$ and $\chi_{A_i}$ is the characteristic function of $A_i$. The elements of $S_1(P, X)$ will be called $P$-simple functions. Since $X$ is a linear space over $\mathbb{R}$, it is clear that $S_1(P, X)$ is also a linear space over $\mathbb{R}$.

**Theorem 2.1.** If $h$ belongs to $S_1(P, X)$, then there exists a finite sequence $\{B_j\}_{j=1}^{m}$ in $D(\mu, P)$ and a sequence $\{b_j\}_{j=1}^{m}$ of elements of $X$ such that $h = \sum_{j=1}^{m} b_j \chi_{B_j}$.

**Proof:** Suppose that $h$ is in $S_1(P, X)$ and that $h = \sum_{i=1}^{n} a_i \chi_{A_i}$. The proof will be by induction on the positive integer $n$. The conclusion is obvious if $n$ is 1. Assume that the conclusion holds for $n = k$ and let $h = \sum_{i=1}^{k+1} a_i \chi_{A_i}$. By the induction hypothesis the $P$-simple function

$$h - \sum_{i=1}^{k} a_i \chi_{A_i} = a_{k+1} \chi_{A_{k+1}}$$

is in $S_1(P, X)$. Thus

$$h = \sum_{i=1}^{k} a_i \chi_{A_i} + a_{k+1} \chi_{A_{k+1}}.$$
\[ \sum_{i=1}^{k} a_i \chi_{A_i} \text{ can be written in the form } \sum_{j=1}^{m} b_j \chi_{B_j} \text{ with } \{B_j\}_{j=1}^{m} \text{ a finite sequence of pairwise disjoint elements of } P. \text{ Therefore, } h \text{ is given by the sum } \\
\sum_{j=1}^{m} b_j \chi_{B_j} + a_{k+1} \chi_{A_{k+1}}.\]

By Theorem 1.1, the difference \( A_{k+1} - (\bigcup_{j=1}^{q} B_j) \) is equal to the union of a finite sequence \( \{D_p\}_{p=1}^{q} \) of pairwise disjoint elements of \( P \). By the definition of a proto-ring, for \( j = 1, 2, \ldots, m \), \( B_j - A_{k+1} \) and \( B_j \cap A_{k+1} \) can each be written as the union of a finite sequence of pairwise disjoint elements of \( P \), say

\[
B_j - A_{k+1} = \bigcup_{i=1}^{r(j)} B_{ji}, \quad \text{and} \\
B_j \cap A_{k+1} = \bigcup_{i=1}^{s(j)} C_{ji}.
\]

Therefore, the function \( h \) is given by

\[
h = \sum_{j=1}^{m} b_j \chi_{B_j} - a_{k+1} \chi_{A_{k+1}} + \sum_{j=1}^{m} (b_j + a_{k+1}) \chi_{B_j \cap A_{k+1}} + a_{k+1} \chi_{A_{k+1}} - (\bigcup_{j=1}^{m} B_j) \\
= \sum_{j=1}^{m} \sum_{i=1}^{r(j)} b_{ji} \chi_{B_{ji}} + \sum_{j=1}^{m} \sum_{i=1}^{s(j)} (b_j + a_{k+1}) \chi_{C_{ji}} + \sum_{p=1}^{q} a_{k+1} \chi_{D_p}.\]

The latter is a representation of the type required. This completes the proof.

Suppose that \( h \) is in \( S_1(P, X) \), and that \( h = \sum_{i=1}^{n} a_i \chi_{A_i} \) and \( h = \sum_{j=1}^{m} b_j \chi_{B_j} \) are two representations of the type in the conclusion of Theorem 2.1. That is, each of \( \{A_i\}_{i=1}^{n} \) and \( \{B_j\}_{j=1}^{m} \) are finite sequences...
of pairwise disjoint elements of $P$. Suppose also that none of the $a_i$ nor $b_j$ are the zero element of $X$. Let $A_i \cap B_j$ be equal to the union of the finite sequence $\{C_{ijk}\}_{k=1}^{q(i,j)}$ in $D(\mu, P)$. Note that if $\mu(C_{ijk})$ is not zero, then $C_{ijk}$ is not empty. This implies that $a_i = b_j$ for all $i$ and $j$ for which $\mu(C_{ijk})$ is not zero. Therefore, $a_i \mu(C_{ijk})$ and $b_j \mu(C_{ijk})$ are equal for all $i$ and $j$ and $k = 1, 2, \ldots, q(i,j)$. Also,

$$A_i = \bigcup_{j=1}^{m} \bigcup_{k=1}^{q(i,j)} C_{ijk}, \text{ for } i = 1, 2, \ldots, n, \text{ and}$$

$$B_j = \bigcup_{i=1}^{n} \bigcup_{k=1}^{q(i,j)} C_{ijk}, \text{ for } j = 1, 2, \ldots, m$$

with the right hand side of each equation a union of a finite sequence in $D(\mu, P)$. Hence, the following equations follow:

$$\sum_{i=1}^{n} a_i \mu(A_i) = \sum_{i=1}^{n} a_i \left( \sum_{j=1}^{m} \sum_{k=1}^{q(i,j)} \mu(C_{ijk}) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{q(i,j)} a_i \mu(C_{ijk})$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{q(i,j)} b_j \mu(C_{ijk})$$

$$= \sum_{j=1}^{m} b_j \left( \sum_{i=1}^{n} \sum_{k=1}^{q(i,j)} \mu(C_{ijk}) \right)$$

$$= \sum_{j=1}^{m} b_j \mu(B_j).$$

In view of this, the following definition gives a well-defined function from $S_1(P, X)$ to $X$.

**Definition 2.1.** Suppose that $h$ is in $S_1(P, X)$ and that $h$ is
\[ \sum_{i=1}^{n} a_i X_{A_i} \text{ with } \{A_i\}_{i=1}^{n} \text{ a finite sequence in } D(\mu, P). \]

The vector 
\[ \sum_{i=1}^{n} a_i \mu(A_i) \text{ in } X \text{ is said to be the integral over } T \text{ of } h \text{ with respect to } \mu, \text{ or in symbols } \int_{T} h \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i). \]

It is clear that the integral is homogeneous, so to see that it is a linear transformation it is necessary only to show that the integral is additive. Suppose then that 
\[ h = \sum_{i=1}^{n} a_i X_{A_i} \text{ and } g = \sum_{j=1}^{m} b_j X_{B_j} \]

are \( P \)-simple functions with \( \{A_i\}_{i=1}^{n} \) and \( \{B_j\}_{j=1}^{m} \) finite sequences in \( D(\mu, P) \).

Now Theorem 1.1 implies that \( A_i \cap (\bigcup_{j=1}^{m} B_j) \) is the union of a finite sequence \( \{C_{ik}\}_{k=1}^{p(i)} \) in \( D(\mu, P) \) for \( i = 1, 2, \ldots, n \). Similarly, \( B_j \cap (\bigcup_{i=1}^{n} A_i) \) is the union of a finite sequence \( \{D_{js}\}_{s=1}^{q(j)} \) in \( D(\mu, P) \) for \( j = 1, 2, \ldots, m \). The definition of proto-ring implies that \( A_i \cap B_j \) is also the union of a finite sequence \( \{E_{ijk}\}_{k=1}^{r(i,j)} \) in \( D(\mu, P) \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \). Since the sum \( h + g \) can be written in the form

\[
\begin{align*}
    h + g &= \sum_{i=1}^{n} a_i X_{A_i} - (\bigcup_{j=1}^{m} B_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) X_{A_i \cap B_j} \\
    &\quad + \sum_{j=1}^{m} b_j X_{B_j} - (\bigcup_{i=1}^{n} A_i),
\end{align*}
\]

the preceding three statements imply that the sum can be written as

\[
\begin{align*}
    h + g &= \sum_{i=1}^{n} \sum_{k=1}^{p(i)} a_i X_{C_{ik}} + \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} (a_i + b_j) X_{E_{ijk}} \\
    &\quad + \sum_{j=1}^{m} \sum_{s=1}^{q(j)} b_j X_{D_{js}}.
\end{align*}
\]
Furthermore, the collection of all of those sets in $P$ appearing on the right hand side of this equation is a pairwise disjoint collection. Also, those same three statements imply that

$$A_i = \bigcup_{k=1}^{p(i)} \bigcup_{j=1}^{m} C_{i,k} \bigcup_{j=1}^{m} E_{i,j,k}$$
for $i = 1, 2, \ldots, n$, and

$$B_j = \bigcup_{s=1}^{g(j)} \bigcup_{j=1}^{n} D_{s,j} \bigcup_{i=1}^{n} E_{i,j,k}$$
for $j = 1, 2, \ldots, m$.

The right hand side of each of the last two equations is a finite union of pairwise disjoint elements of $P$. Therefore, it follows that

$$\mu(A_i) = \sum_{k=1}^{p(i)} \mu(C_{i,k}) + \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} \mu(E_{i,j,k})$$
for $i = 1, 2, \ldots, n,$ and

$$\mu(B_j) = \sum_{s=1}^{g(j)} \mu(D_{s,j}) + \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} \mu(E_{i,j,k})$$
for $j = 1, 2, \ldots, m$.

Finally then the integral of the sum $h + g$ is given by

$$\int_T (h + g) d\mu = \sum_{i=1}^{n} a_i \mu(C_{i,k}) + \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} (a_i + b_j) \mu(E_{i,j,k})$$

$$+ \sum_{j=1}^{m} \sum_{s=1}^{q(j)} b_j \mu(D_{s,j}) = \sum_{i=1}^{n} \sum_{k=1}^{p(i)} a_i \mu(C_{i,k}) + \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} a_i \mu(E_{i,j,k})$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{r(i,j)} b_j \mu(E_{i,j,k}) + \sum_{j=1}^{m} \sum_{s=1}^{q(j)} b_j \mu(D_{s,j})$$

$$= \sum_{i=1}^{n} a_i \left( \sum_{k=1}^{p(i)} \mu(C_{i,k}) + \sum_{j=1}^{m} \sum_{k=1}^{r(i,j)} \mu(E_{i,j,k}) \right)$$

$$+ \sum_{j=1}^{m} b_j \left( \sum_{s=1}^{q(j)} \mu(D_{s,j}) + \sum_{i=1}^{n} \sum_{k=1}^{r(i,j)} \mu(E_{i,j,k}) \right)$$
Therefore, the integral is a linear operator from $S_1(P, X)$ to $X$.

If $f$ is a function on $T$ to $X$, then $|f(\cdot)|$ is the function on $T$ to the non-negative reals $\mathbb{R}^+$ defined by

$$|f(\cdot)|(t) = |f(t)|, \forall t \in T.$$ 

In particular, if $f$ is a $P$-simple function, $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, with $\{A_i\}_{i=1}^{n}$ a finite sequence in $D(\mu, P)$, then $|f(\cdot)|$ is given by the sum $\sum_{i=1}^{n} |a_i| \chi_{A_i}$. This shows that if $f$ is in $S_1(P, X)$, then $|f(\cdot)|$ is an element of $S_1(P, \mathbb{R})$.

In particular, if $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ as in the immediately preceding discussion, then the integral of $|f(\cdot)|$ is $\sum_{i=1}^{n} |a_i| \mu(A_i)$. Let $\sigma$ be the function defined on $S_1(P, X)$ into $\mathbb{R}$ by

$$\sigma(f) = \int_T |f(\cdot)| \, d\mu, \forall f \in S_1(P, X).$$ 

Since $\mu$ is a non-negative $p$-volume, the function $\sigma$ is a mapping into $\mathbb{R}^+$. Since $\| \|$ is a norm for $X$, if $\alpha$ is in $\mathbb{R}$, then $|\alpha \cdot f(t)|$ equals $|\alpha| \cdot |f(t)|$ for all $t$ in $T$. Therefore, it follows that

$$\sigma(\alpha \cdot f) = \int_T |\alpha \cdot f(\cdot)| \, d\mu = \int_T |\alpha| \cdot |f(\cdot)| \, d\mu = |\alpha| \cdot \sigma(f),$$

where $\sigma$ is the function defined above.
and σ is positively homogeneous. Now if h is in \( S_1(P, R) \) and \( h(t) \) is non-negative for all \( t \) in \( T \), then \( \int_T h d\mu \) is non-negative since \( \mu \) is non-negative. It follows from this observation and the linearity of the integral that if \( h \) and \( k \) are in \( S_1(P, R) \) and if \( h(t) \) is greater than or equal to \( k(t) \) for all \( t \) in \( T \), then

\[
\int_T h d\mu \geq \int_T k d\mu.
\]

In particular, this implies that the following relations hold for \( f \) and \( g \) in \( S_1(P, X) \):

\[
\sigma(f + g) = \int_T |f(\cdot) + g(\cdot)|\ d\mu
\]

\[
\leq \int_T (|f(\cdot)| + |g(\cdot)|)\ d\mu
\]

\[
= \int_T |f(\cdot)|\ d\mu + \int_T |g(\cdot)|\ d\mu
\]

\[
= \sigma(f) + \sigma(g).
\]

Hence, \( \sigma \) is a seminorm for \( S_1(P, X) \). Since there may be non-void sets \( E \) in \( P \) for which \( \mu(E) \) is zero, it is not necessarily true that \( \sigma \) vanishes only at the zero function in \( S_1(P, X) \). Therefore, \( \sigma \) is not in general a norm. In order to obtain a normed linear space, an equivalence relation will be defined on the set of all functions from \( T \) to \( X \) in such a manner that if \( f \) and \( g \) are equivalent \( P \)-simple functions, then \( \int_T f d\mu \) and \( \int_T g d\mu \) have the same value. It is then possible to define the integral on the equivalence classes of \( P \)-simple functions, and then to define a norm in the manner suggested by the definition of \( \sigma \). Finally,
the equivalence relation will be defined in such a way as to allow a natural extension of the integral to a larger class of functions (actually, to a larger set of equivalence classes of functions). The next several definitions and theorems will formalize this discussion.

**Definition 2.2.** A subset $B$ of $T$ is said to be a $\mu$-null set if and only if for every positive real number $\epsilon$ there exists a sequence $\{F_j\}_{j=1}^m$ in $D(\mu, P)$ such that $B$ is a subset of $\bigcup_{j=1}^m F_j$ and $\sum_{j=1}^m \mu(F_j) < \epsilon$.

**Theorem 2.2.** If $E$ is in $P$, then $E$ is $\mu$-null if and only if $\mu(E) = 0$.

Proof: Suppose $E$ is in $P$. If $\mu(E)$ is zero, then it is obvious that $E$ is $\mu$-null. The proof will be concluded by showing that $E$ is not $\mu$-null if $\mu(E)$ is not zero. Therefore, suppose that $\mu(E)$ is not zero and let $\epsilon$ in $R$ be such that $0 < \epsilon < \mu(E)$. If $\{F_j\}_{j=1}^m$ is in $D(\mu, P)$ and $E$ is contained in $\bigcup_{j=1}^m F_j$, then it is true that

$$E = \bigcup_{j=1}^m (F_j \cap E).$$

Since $P$ is a proto-ring, $F_j \cap E$ is a union of a finite sequence $\{B_{jk}\}_{k=1}^{q(j)}$ in $D(\mu, P)$, so that

$$E = \bigcup_{j=1}^m (F_j \cap E) = \bigcup_{j=1}^m \bigcup_{k=1}^{q(j)} B_{jk}.$$

The subadditivity and the monotonicity of $\mu$ imply that

$$\epsilon < \mu(E) \leq \sum_{j=1}^m \sum_{k=1}^{q(j)} \mu(B_{jk}) = \sum_{j=1}^m \mu(F_j).$$

Hence, $E$ is not $\mu$-null and this completes the proof.

**Definition 2.3.** If $f$ and $g$ are functions on $T$ to $X$, then $f$ is said
to be $\mu$-equivalent to $g$, written $f \sim g$, if and only if for every positive real number $\varepsilon$, there exists $\{F_j\}_{j=1}^m$ in $D(\mu, P)$ such that $\sum_{j=1}^m \mu(F_j) < \varepsilon$ and

$$|f(t) - g(t)| < \varepsilon, \forall t \in T - \bigcup_{j=1}^m F_j.$$ 

Since $\|\|$ is a norm for $X$, it is clear that the relation $\sim$ is reflexive and symmetric. Suppose now that $f \sim g$ and $g \sim h$ and that $\varepsilon$ is a positive real number. Pick $\{F_j\}_{j=1}^m$ and $\{E_i\}_{i=1}^n$ in $D(\mu, P)$ in accordance with Definition 2.3 so that $\sum_{j=1}^m \mu(F_j)$ and $\sum_{i=1}^n \mu(E_i)$ are both less than $\varepsilon/2$ and

$$|f(t) - g(t)| < \frac{\varepsilon}{2}, \forall t \in T - \bigcup_{j=1}^m F_j, \text{ and}$$

$$|g(t) - h(t)| < \frac{\varepsilon}{2}, \forall t \in T - \bigcup_{i=1}^n E_i.$$ 

By Theorem 1.2 there exists $\{G_k\}_{k=1}^p$ in $D(\mu, P)$ such that $\bigcup_{k=1}^p G_k = (\bigcup_{j=1}^m F_j) \cup (\bigcup_{i=1}^n E_i)$. Therefore, the sum $\sum_{k=1}^p \mu(G_k)$ satisfies the inequality

$$\sum_{k=1}^p \mu(G_k) \leq \sum_{j=1}^m \mu(F_j) + \sum_{i=1}^n \mu(E_i) < \varepsilon.$$ 

Also, if $t$ is in $T - \bigcup_{k=1}^p G_k$, then

$$|f(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

Hence, $f$ is $\mu$-equivalent to $h$, and $\mu$-equivalence is an equivalence relation on $X_T$. 
If }f\text{ is a function on }T\text{ to }X,\text{ then }\overline{f}\text{ shall denote the equivalence class of all functions on }T\text{ to }X\text{ that are }\mu\text{-equivalent to }f,\text{ and the symbol }X^T/\overline{\mu}\text{ denotes the set of all such equivalence classes of functions. The notation }S'(\mu, P, X)\text{ is used for the linear space of all equivalence classes determined by }\overline{\mu}\cap(S_1(P, X) \times S_1(P, X)).\text{ Following the usual convention, the word "function" will be used quite often in the remainder of this paper when what is meant is "equivalence class of functions." In particular, the elements of }S'(\mu, P, X)\text{ will quite often be called }P\text{-simple functions.}

**Theorem 2.3.** If }f\text{ and }g\text{ are in }S_1(P, X)\text{ and }f\text{ is }\mu\text{-equivalent to }g,\text{ then }\int_T f d\mu = \int_T g d\mu.

**Proof:** Since }|f(t) - g(t)| = |f(t) - g(t)| - 0|,\text{ it is a trivial observation that }f\text{ and }g\text{ are }\mu\text{-equivalent if and only if }f - g\text{ is }\mu\text{-equivalent to the zero function of }S_1(P, X).\text{ This, together with the linearity of the integral on }S_1(P, X),\text{ makes it clear that it is sufficient to show that if }f\text{ is }\mu\text{-equivalent to the zero function in }S_1(P, X),\text{ then }\int_T f d\mu\text{ is zero. This is certainly true if }f\text{ is the zero function, so assume this is not the case but that }f\text{ in }S_1(P, X)\text{ is }\mu\text{-equivalent to the zero function. Let }f\text{ be given by}

\[ f = \sum_{i=1}^{n} a_i \chi_{A_i} \]

with }\{A_i\}_{i=1}^{n}\text{ a finite sequence in }D(\mu, P)\text{ and }a_i\text{ not the zero element of }X\text{ for }i = 1, 2, \ldots, n.\text{ The method of proof is to show that each }A_i\text{ is }\mu\text{-null and then use Theorem 2.2 to conclude that }\mu(A_i) = 0\text{ for }i = 1, 2, \ldots, n.\text{ This clearly implies that }\int_T f d\mu\text{ is zero.
Given a positive real number $\varepsilon$, pick $\varepsilon_1$ to be a positive real number less than the minimum of $\varepsilon$, $|a_1|$, $|a_2|$, ..., $|a_n|$. Since $f$ is $\mu$-equivalent to the zero function, there exists a sequence $\{F_j\}_{j=1}^m$ in $D(\mu, F)$ such that $\sum_{j=1}^m \mu(F_j) < \varepsilon_1$ and
\[
|f(t) - 0| < \varepsilon_1, \forall t \in T - \left( \bigcup_{j=1}^m F_j \right).
\]
The assertion now is that $A_i$ is a subset of $\bigcup_{j=1}^m F_j$ for $i = 1, 2, ..., n$. If this were not true for some $A_i$, then there exists $t_0$ in $A_i - \left( \bigcup_{j=1}^m F_j \right)$. However, then $t_0$ is in $T - \left( \bigcup_{j=1}^m F_j \right)$, and it must be true that $|f(t_0)| = |a_i|$ is less than $\varepsilon_1$. This contradicts the choice of $\varepsilon_1$. Therefore, for $i = 1, 2, ..., n$, $A_i$ is a subset of $\bigcup_{j=1}^m F_j$ and
\[
\sum_{j=1}^m \mu(F_j) < \varepsilon_1 < \varepsilon.
\]
Hence, $A_i$ is $\mu$-null for $i = 1, 2, ..., n$ and this completes the proof.

If $h$ is an element of $S_1(P, X)$, then the symbol $[h]$ will be used for the equivalence class of all $P$-simple functions that are $\mu$-equivalent to $h$. Theorem 2.3 then says that if $[h]$ is in $S'(\mu, P, X)$ and if the integral over $T$ of $[h]$ with respect to $\mu$, in symbols $\int_T [h]d\mu$, is defined to be the vector $\int_T hd\mu$ in $X$, then the integral is a well-defined function on $S'(\mu, P, X)$ into $X$. If $[f]$ and $[g]$ are elements of $S'(\mu, P, X)$ and $\alpha$ and $\beta$ are in $R$, then $\alpha [f] + \beta [g]$ is the element $[\alpha f + \beta g]$ of $S'(\mu, P, X)$. It follows from this and the linearity of the integral on $S_1(P, X)$ that the following equations hold:
\[
\int_T (\alpha f + \beta g) \, d\mu = \int_T (\alpha f + \beta g) \, d\mu \\
= \int_T (\alpha f + \beta g) \, d\mu \\
= \alpha \int_T f \, d\mu + \beta \int_T g \, d\mu \\
= \alpha \int_T [f] \, d\mu + \beta \int_T [g] \, d\mu.
\]

Therefore, the integral is a linear transformation on \( S'(\mu, P, X) \) into \( X \).

Now if \( f \) and \( g \) in \( S_1(P, X) \) are \( \mu \)-equivalent, then the inequality

\[
|\tau(f(t)) - \tau(g(t))| \leq |f(t) - g(t)|, \quad \forall t \in T
\]

implies that \( |f(\cdot)| \) and \( |g(\cdot)| \) in \( S_1(P, R) \) are \( \mu \)-equivalent. This in turn implies that the function \( \tau \) defined on \( S'(\mu, P, X) \) into \( R \) by the equation

\[
\tau([f]) = \int_T |f(\cdot)| \, d\mu, \quad \forall [f] \in S'(\mu, P, X)
\]

is well-defined. In particular, it should be noted that \( \tau([f]) = \sigma(f) \) where \( \sigma \) is the seminorm on \( S_1(P, X) \) discussed previous to Definition 2.2. This, together with the definitions of scalar multiplication and addition in \( S'(\mu, P, X) \), gives as an immediate consequence that \( \tau \) is also a seminorm. It then follows by Theorem 2.3 that \( \tau \) is a norm for \( S'(\mu, P, X) \). If \([f]\) is in \( S'(\mu, P, X) \), then the value of \( \tau \) at \([f]\) will be denoted by \( \|f\|' \). Therefore, the following equation is valid:
It should be noted that quite often throughout the rest of this paper, the elements of \( S'(\mu, P, X) \) will be written as though they were functions. For example, it may be stated that "f is in \( S'(\mu, P, X) \)," and one should then think of \( f \) as being the representative of an equivalence class of \( P \)-simple functions. Also, if \([f] \) is in \( S'(\mu, P, X) \), then the integral evaluated at \([f] \) will be written as \( \int f \, d\mu \) rather than \( \int [f] \, d\mu \), unless this might lead to confusion.

**Theorem 2.4.** If \( S'(\mu, P, X) \) is given the topology induced by the norm function \( || \cdot ||' \), then the integral on \( S'(\mu, P, X) \) into \( X \) is a continuous linear transformation.

Proof: Suppose that \([f] \) is in \( S'(\mu, P, X) \) and that \( f \) is given by

\[
f = \sum_{i=1}^{n} a_i X_{A_i},
\]

where \( \{X_{A_i}\}_{i=1}^{n} \) is a finite sequence in \( D(\mu, P) \). Since \( || \cdot || \) is a norm for \( X \) and \( \mu \) is non-negative, it follows that

\[
\left| \int f \, d\mu \right| = \left| \sum_{i=1}^{n} a_i \mu(A_i) \right|
\]

\[
\leq \sum_{i=1}^{n} |a_i| \mu(A_i)
\]

\[
= \int |f(\cdot)| \, d\mu
\]

\[
= ||[f]||'.
\]
Therefore, the integral is a bounded linear transformation on $S'(\mu, P, X)$ into $X$, and it is well known that a linear transformation between normed linear spaces is continuous if and only if it is bounded. This completes the proof.

The next step in this development of the integral is to extend its domain of definition to a larger set of (equivalence classes of) functions. Actually, since the enlarged domain will be a subset of $X^T/\sim$ and since the equivalence classes belonging to $S'(\mu, P, X)$ contain only $P$-simple functions, the present domain of definition for the integral will not be a subset of the enlarged domain. In this sense the extension of the integral will not be an extension in the usual sense. However, it is not hard to see that there is a one-to-one correspondence between $S'(\mu, P, X)$ and a subset of $X^T/\sim$, namely, the subset of $T$ given by

$$\{f \in X^T/\sim : f \text{ is } \mu\text{-equivalent to a } P\text{-simple function}\}.$$ 

Of course, one could simply define the integral for any $\bar{f}$ in $X^T/\sim$ for which $f$ is $\mu$-equivalent to a $P$-simple function $g$ on $T$ to $X$ to be $\int_T [g]d\mu$. However, this will not be done here since it will be a result of the extension procedure that the integral of such an $\bar{f}$ will be $\int_T [g]d\mu$.

**Definition 2.4.** Let $f, f_1, f_2, f_3, \ldots$ be functions on $T$ to $X$. The statement that the sequence $\{f_n\}_{n=1}^\infty$ $\mu$-converges to $f$, written $f_n \rightarrow f$, means that for every positive real number $\varepsilon$, there exists a positive integer $N$ such that if $n$ is a positive integer and $n \geq N$, then there
exists a sequence \( \{E_i(n)\}_{i=1}^{\infty} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{q(n)} \mu(E_i(n)) < \varepsilon \) and

\[
|f_{n}(t) - f(t)| < \varepsilon, \forall t \in T - \bigcup_{i=1}^{\infty} E_i(n).
\]

Suppose that \( \{f_n\}_{n=1}^{\infty} \) is a sequence of functions on \( T \) to \( X \) that \( \mu \)-converges to the function \( f \) on \( T \) to \( X \). Suppose further that \( \{f_{k(n)}\}_{n=1}^{\infty} \) is a subsequence of the sequence \( \{f_n\}_{n=1}^{\infty} \). Given a positive element \( \varepsilon \) in \( R \), let \( N(\varepsilon) \) be the positive integer \( N \) of Definition 2.4. If \( k(n) \) is a positive integer greater than \( N(\varepsilon) \), then \( f_{k(n)} \) is an element of the sequence \( \{f_n\}_{n=1}^{\infty} \). Since \( k(n) \) is greater than \( N(\varepsilon) \) and \( f_{n} \rightarrow f \), there exists a sequence \( \{E_i(k(n))\}_{i=1}^{\infty} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{q(k(n))} \mu(E_i(k(n))) < \varepsilon \) and

\[
|f_{k(n)}(t) - f(t)| < \varepsilon, \forall t \in T - \bigcup_{i=1}^{\infty} E_i(k(n)).
\]

Therefore, the sequence \( \{f_{k(n)}\}_{n=1}^{\infty} \) \( \mu \)-converges to \( f \).

Suppose now that \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are sequences of functions on \( T \) to \( X \) that \( \mu \)-converge to the functions \( f \) and \( g \) on \( T \) to \( X \), respectively. Let \( \alpha \) be a real number. If \( \alpha \) is zero, then it is clear that \( \{\alpha f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( \alpha f \). Assume then that \( \alpha \) is not zero and let \( \varepsilon \) be a positive real number. Pick \( \delta \) to be the minimum of \( \varepsilon \) and \( \varepsilon/|\alpha| \). Then there exists a positive integer \( N \) such that if \( n \) is a positive integer and \( n > N \), then there exists a sequence \( \{E_i(n)\}_{i=1}^{\infty} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{q(n)} \mu(E_i(n)) < \delta \leq \varepsilon \) and
\[ |f_n(t) - f(t)| < \delta \leq |a_i|, \forall t \in \mathbb{T} - (\bigcup_{i=1}^{q(n)} E_i(n)). \]

This last inequality implies that
\[ |\alpha f_n(t) - \alpha f(t)| < \varepsilon, \forall t \in \mathbb{T} - (\bigcup_{i=1}^{q(n)} E_i(n)), \]
so that the sequence \( \{\alpha f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( \alpha f \).

Also, there exist positive integers \( N_1 \) and \( N_2 \) such that if \( n \) is a positive integer greater than \( N_1 \) and greater than \( N_2 \), then there exist sequences \( \{E_i(n)\}_{i=1}^{q(n)} \) and \( \{F_j(n)\}_{j=1}^{p(n)} \) in \( D(\mu, P) \) such that each of the sums \( \sum_{i=1}^{q(n)} \mu(E_i(n)) \) and \( \sum_{j=1}^{p(n)} \mu(F_j(n)) \) is less than \( \varepsilon/2 \) and
\[ |f_n(t) - f(t)| < \frac{\varepsilon}{2}, \forall t \in \mathbb{T} - (\bigcup_{i=1}^{q(n)} E_i(n)), \]
and
\[ |g_n(t) - g(t)| < \frac{\varepsilon}{2}, \forall t \in \mathbb{T} - (\bigcup_{j=1}^{p(n)} F_j(n)). \]

Let \( N \) be the maximum of \( N_1 \) and \( N_2 \) and for each positive integer \( n \) greater than \( N \), let \( \{G_k(n)\}_{k=1}^{r(n)} \) be a sequence in \( D(\mu, P) \), guaranteed by Theorem 1.2, such that
\[ \bigcup_{k=1}^{r(n)} G_k(n) = (\bigcup_{i=1}^{q(n)} E_i(n)) \bigcup (\bigcup_{j=1}^{p(n)} F_j(n)). \]

Then the sum \( \sum_{k=1}^{r(n)} \mu(G_k(n)) \) is less than \( \varepsilon \) and
\[ |(f_n(t) + g_n(t)) - (f(t) + g(t))| \leq |f_n(t) - f(t)| + |g_n(t) - g(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall t \in \mathbb{T} - (\bigcup_{k=1}^{r(n)} G_k(n)). \]
for all integers \( n \) greater than \( N \). Therefore, the sequence \( \{f_n + g_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( f + g \). These results are summarized in the next theorem.

\textbf{Theorem 2.5.} If \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are sequences of functions on \( T \) to \( X \) \( \mu \)-converging to the functions \( f \) and \( g \) on \( T \) to \( X \), respectively, and if \( \alpha \) and \( \beta \) are real numbers, then the sequence \( \{\alpha f_n + \beta g_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( \alpha f + \beta g \).

The next two theorems show that \( \mu \)-convergence can be defined in a meaningful manner on the set of all equivalence classes determined by \( \tilde{\mu} \) of functions on \( T \) to \( X \).

\textbf{Theorem 2.6.} If \( \{f_n\}_{n=1}^{\infty} \) is a sequence of functions on \( T \) to \( X \) that \( \mu \)-converges to the function \( f \) on \( T \) to \( X \), then \( f_n \) \( \mu \)-converges to the function \( g \) on \( T \) to \( X \) if and only if \( f \) and \( g \) are \( \mu \)-equivalent.

\textbf{Proof:} Assume first that \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to the function \( g \). Let \( \varepsilon \) be a positive real number. Since \( f \rightarrow f \) and \( f_n \rightarrow g \), there exists a positive integer \( N \) such that if \( n \) is a positive integer, \( n \geq N \), then there exist sequences \( \{E_i^{(n)}\}_{i=1}^{q(n)} \) and \( \{F_j^{(n)}\}_{j=1}^{p(n)} \) in \( D(\mu, P) \) such that each of the sums \( \sum_{i=1}^{q(n)} \mu(E_i^{(n)}) \) and \( \sum_{j=1}^{p(n)} \mu(F_j^{(n)}) \) is less than \( \varepsilon/2 \), and

\[ |f_n(t) - f(t)| < \frac{\varepsilon}{2}, \forall t \in T - \left( \bigcup_{i=1}^{q(n)} E_i^{(n)} \right), \text{ and} \]

\[ |f_n(t) - g(t)| < \frac{\varepsilon}{2}, \forall t \in T - \left( \bigcup_{j=1}^{p(n)} F_j^{(n)} \right). \]

Fix \( n \) greater than \( N \) and let \( \{G_k\}_{k=1}^{r} \) be a sequence in \( D(\mu, P) \) such that

\[ \bigcup_{k=1}^{r} G_k = \left( \bigcup_{i=1}^{q(n)} E_i^{(n)} \right) \cup \left( \bigcup_{j=1}^{p(n)} F_j^{(n)} \right). \]
Then it follows that

\[ \sum_{k=1}^{r} \mu(G_k) < \varepsilon, \quad \text{and} \]

\[ |f(t) - g(t)| \leq |f(t) - f_n(t)| + |f_n(t) - g(t)| \]

\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall t \in \mathcal{T} - \left( \bigcup_{k=1}^{r} G_k \right). \]

Therefore, \( f \) and \( g \) are \( \mu \)-equivalent.

Suppose now that \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( f \) and that \( f \) and \( g \) are \( \mu \)-equivalent. Let \( \varepsilon \) be a positive real number. Since \( f_n \to f \), there exists a positive integer \( N \) such that if \( n \) is a positive integer with \( n \geq N \), then there exists a sequence \( \{E_i(n)\}_{i=1}^{q(n)} \) in \( D(\mu, P) \) such that the sum

\[ \sum_{i=1}^{q(n)} \mu(E_i(n)) \]

is less than \( \varepsilon/2 \) and

\[ |f_n(t) - f(t)| < \frac{\varepsilon}{2}, \quad \forall t \in \mathcal{T} - \left( \bigcup_{i=1}^{q(n)} E_i(n) \right). \]

The \( \mu \)-equivalence of \( f \) and \( g \) implies the existence of a sequence \( \{F_j\}_{j=1}^{m} \) in \( D(\mu, P) \) such that the sum

\[ \sum_{j=1}^{m} \mu(F_j) \]

is less than \( \varepsilon/2 \) and

\[ |f(t) - g(t)| < \frac{\varepsilon}{2}, \quad \forall t \in \mathcal{T} - \left( \bigcup_{j=1}^{m} F_j \right). \]

For each positive integer \( n \), \( n \geq N \), let \( \{G_k(n)\}_{k=1}^{p(n)} \) be a sequence in \( D(\mu, P) \) such that

\[ \bigcup_{k=1}^{p(n)} G_k(n) = \left( \bigcup_{i=1}^{q(n)} E_i(n)\right) \cup \left( \bigcup_{j=1}^{m} F_j \right). \]

Hence, if \( n \) is a positive integer greater than \( N \), then
Therefore, the sequence \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( g \). This completes the proof.

Suppose now that \( \{f_n\}_{n=1}^{\infty} \) is a sequence of functions on \( T \) to \( X \) and that \( f_n \) is \( \mu \)-equivalent to the zero function on \( T \) to \( X \) for \( n = 1, 2, 3, \ldots \). Let \( \epsilon \) be a positive real number and let \( n \) be an arbitrary positive integer. Since \( f_n \sim 0 \), there exists a sequence \( \{E_i(n)\}_{i=1}^{q(n)} \) in \( D(\mu, F) \) such that the sum \( \sum_{i=1}^{q(n)} \mu(E_i(n)) \) is less than \( \epsilon \) and

\[
\left| f_n(t) - 0 \right| < \epsilon, \forall t \in T - \left( \bigcup_{k=1}^{p(n)} G_k(n) \right).
\]

Hence, the sequence \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to the zero function on \( T \) to \( X \).

If \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are sequences of functions on \( T \) to \( X \) with the property that

\[ f_n \sim g_n, \text{ for } n = 1, 2, 3, \ldots, \]

then it follows that \( g_n - f_n \) is \( \mu \)-equivalent to the zero function for \( n = 1, 2, 3, \ldots \). Therefore, the sequence \( \{g_n - f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to the zero function on \( T \) to \( X \). Hence, if \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to a function \( f \) on \( T \) to \( X \), then Theorem 2.5 implies that the sequence \( \{g_n\}_{n=1}^{\infty} \) \( \mu \)-converges.
to $f$, also. These results constitute a proof of the next theorem.

**Theorem 2.7.** If the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $T$ to $X$ \(\mu\)-converges to the function $f$ on $T$ to $X$ and if $\{g_n\}_{n=1}^{\infty}$ is a sequence of functions on $T$ to $X$ with the property that $g_n$ is $\mu$-equivalent to $f_n$ for $n = 1, 2, 3, \ldots$, then the sequence $\{g_n\}_{n=1}^{\infty}$ \(\mu\)-converges to $f$.

**Definition 2.5.** If $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of $X^T/\mu$ and if $\bar{f}$ is in $X^T/\mu$, then the statement that $\{f_n\}_{n=1}^{\infty}$ \(\mu\)-converges to $\bar{f}$, in symbols $f_n \xrightarrow{\mu} \bar{f}$, means that the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $T$ to $X$ \(\mu\)-converges to the function $f$ on $T$ to $X$.

Theorem 2.6 shows that the limit under \(\mu\)-convergence of a sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $X^T/\mu$ is unique. Theorem 2.7 shows that if the sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $X^T/\mu$ \(\mu\)-converges to $\bar{f}$ in $X^T/\mu$ and if $\bar{g}_n$ is in $X^T/\mu$ with $\bar{g}_n = \bar{f}_n$ for $n = 1, 2, 3, \ldots$, then $\{\bar{g}_n\}_{n=1}^{\infty}$ also \(\mu\)-converges to $\bar{f}$.

**Definition 2.6.** If $\{[f_n]\}_{n=1}^{\infty}$ is a sequence of elements of $S'(\mu, P, X)$ and if $\bar{f}$ is in $X^T/\mu$, then the statement that $\{[f_n]\}_{n=1}^{\infty}$ \(\mu\)-converges to $\bar{f}$, in symbols $[f_n] \xrightarrow{\mu} \bar{f}$, means that the sequence $\{f_n\}_{n=1}^{\infty}$ of elements of $X^T/\mu$ \(\mu\)-converges to $\bar{f}$.

Using Theorem 2.6 again, the limit under \(\mu\)-convergence of a sequence $\{[f_n]\}_{n=1}^{\infty}$ of elements of $S'(\mu, P, X)$ is unique. Theorem 2.7 shows that if the sequence $\{[f_n]\}_{n=1}^{\infty}$ of elements of $S'(\mu, P, X)$ \(\mu\)-converges to $\bar{f}$ in $X^T/\mu$ and if $[g_n] = [f_n]$ for $n = 1, 2, 3, \ldots$, then $\{[g_n]\}_{n=1}^{\infty}$ \(\mu\)-converges to $\bar{f}$.

The following theorem will be needed in order to extend the domain of definition of the integral presently defined on $S'(\mu, P, X)$ to a
larger space. The proof given here is similar to the proof given by Bogdanowicz in [1] for Lemma 3. Some differences do occur since Bogdanowicz deals only with countably additive set functions defined on a pre-ring of sets.

**Theorem 2.8.** If the sequence \( \{f_n\}_{n=1}^{\infty} \) of elements of \( S'(\mu, P, X) \) is norm Cauchy and \( \mu \)-convergent to the zero element of \( X^{T/\mu} \), then

\[
\lim_{n \to \infty} \|f_n\| = 0.
\]

**Proof:** The fact that if \( g \) is in \( S_1(P, R) \) and if \( f \) is in \( S_1(P, X) \), then \( g \cdot f \) is in \( S_1(P, X) \) will be used in the proof. In order to see that this is true, consider the following argument. Let \( g \) in \( S_1(P, R) \) be given by the equation

\[
g = \sum_{i=1}^{n} \alpha_i X_{E_i}, \quad \{E_i\}_{i=1}^{m} \text{ a finite sequence in } D(\mu, P),
\]

and \( f \) in \( S_1(P, X) \) be given by

\[
f = \sum_{j=1}^{m} \beta_j X_{F_j}, \quad \{F_j\}_{j=1}^{m} \text{ a finite sequence in } D(\mu, P).
\]

Since \( P \) is a proto-ring, for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \), there exists a finite sequence \( \{B_{i,j,k}\}_{k=1}^{q(i,j)} \) in \( D(\mu, P) \) such that

\[
E_i \cap F_j = \bigcup_{k=1}^{q(i,j)} B_{i,j,k}.
\]

Then \( g \cdot f = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{q(i,j)} \alpha_i \beta_j \chi_{B_{i,j,k}} \), and it is clear from this that \( g \cdot f \) is in \( S(P, X) \).

Suppose then that \( \{f_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to the zero element of \( X^{T/\mu} \). Let \( g_n \) be the function \( |f_n(\cdot)| \) in \( S_1(P, R) \) for \( n = 1, 2, 3, \ldots \) and let \( \varepsilon \) be a
positive real number. In view of the relation
\[ |g_n(t) - g_m(t)| = |f_n(t) - f_m(t)| \]
\[ \leq |f_n(t) - f_m(t)|, \forall \tau \in T, \]

it follows that
\[ \|g_n - g_m\| \leq \|f_n - f_m\|. \]

This implies that the sequence \( \{g_n\}_{n=1}^{\infty} \) is norm Cauchy, so there exists a positive integer \( k \) such that

\[ (a) \int_{T} |g_n(\cdot) - g_k(\cdot)| d\mu = \|g_n - g_k\| < \epsilon, \forall n \geq k. \]

Since each function \( g_n \) is a \( P \)-simple function, if

\[ A = \bigcup_{j=1}^{k} \{\tau \in T: g_j(\tau) > 0\}, \]

then there exists a finite sequence \( \{A_i\}_{i=1}^{r} \) in \( D(\mu, P) \) such that

\[ (b) A = \bigcup_{i=1}^{r} A_i. \]

The functions \( \chi_A \) and \( g_n, n = 1, 2, 3, \ldots \), are in \( S_1(P, R) \), so it follows that \( \chi_A \cdot g_n \) is also in \( S_1(P, R) \) for \( n = 1, 2, 3, \ldots \). Since \( S_1(P, R) \) is a linear space, this implies that \( g_n - \chi_A \cdot g_n = (1 - \chi_A)g_n \) is in \( S_1(P, R) \). If \( n \) is an integer satisfying \( 1 \leq n \leq k \), then \( (1 - \chi_A(\tau))g_n(\tau) \) is zero for all \( \tau \) in \( T \). Therefore, if \( n \) is an integer, \( 1 \leq n \leq k \), then

\[ \int_{T} (1 - \chi_A)g_n d\mu = 0. \]

If \( n \) is an integer greater than \( k \), then
\[(1 - \chi_A)g_n = (1 - \chi_A)g_n - (1 - \chi_A)g_k\]
\[= (1 - \chi_A)(g_n - g_k)\]
\[\leq \left| g_n(\cdot) - g_k(\cdot) \right|,\]

where the inequality holds pointwise on \(T\). Statement (a) then implies that for all integers \(n\) that are greater than \(k\),

\[
\int_T (1 - \chi_A)g_n \, d\mu \leq \|g_n\| - \|g_k\| < \epsilon.
\]

Therefore,

\[
(c) \int_T (1 - \chi_A)g_n \, d\mu < \epsilon \text{ for } n = 1, 2, 3, \ldots.
\]

Since \(g_n\) is a \(P\)-simple function for \(n = 1, 2, 3, \ldots\), it follows that the supremum of the set

\[
\{g_j(t) : t \in T \text{ and } 1 \leq j \leq k\}
\]

is a finite real number. Also, \(\sum_{i=1}^{r} \mu(A_i)\) is a finite real number. Let \(m\) be a real number greater than each of these two real numbers.

Suppose now that \(\{B_j\}_{j=1}^{q}\) is any finite sequence in \(D(\mu, P)\) and that

\[
(d) \quad C = \bigcup_{j=1}^{q} B_j \text{ and } \sum_{j=1}^{q} \mu(B_j) < \frac{\epsilon}{m}.
\]

If \(n\) is an integer and \(1 \leq n \leq k\), then the following relations hold:

\[
\int_T \chi_C g_n \, d\mu \leq \int_T m \cdot \chi_C \, d\mu
\]
\[
< m \cdot \frac{\epsilon}{m} = \epsilon.
\]
If \( n \) is an integer and \( n > k \), then it follows that

\[
\chi_c \cdot g_n = \chi_c(g_n - g_k) + \chi_c g_k \leq |g_n(\cdot) - g_k(\cdot)| + \chi_c g_k
\]

where the inequality holds pointwise on \( T \). This implies

\[
(e) \int_T \chi_c g_n \, d\mu < 2\varepsilon \text{ for } n = 1, 2, 3, \ldots.
\]

Since \( \{\varepsilon_n\}_{n=1}^\infty \) \( \mu \)-converges to zero, there exists a positive integer \( K \) such that if \( n \) is an integer, \( n \geq K \), then there exists a sequence \( \{F_j(n)\} s(n) \) in \( D(\mu, P) \) such that \( \sum_{j=1}^{s(n)} \mu(F_j(n)) < \varepsilon/m \) and

\[
(f) \text{ if } t \notin \bigcup_{j=1}^{s(n)} F_j(n), \text{ then } |f_n(t)| < \frac{\varepsilon}{m}.
\]

Again using that each of the functions \( g_n \) is \( P \)-simple, the set

\[
C_n = \{ t \in T : g_n(t) \geq \frac{\varepsilon}{m} \}
\]

is the union of a finite sequence \( \{B_i(n)\}_{i=1}^{m(n)} \) in \( D(\mu, P) \). If \( n \) is an integer greater than \( K \), then it follows that

\[
B_i(n) \subseteq C_n \subseteq \bigcup_{j=1}^{s(n)} F_j(n), \text{ for } i = 1, 2, \ldots, m(n).
\]

Let the intersection \( B_i(n) \cap F_j(n) \) be given by

\[
B_i(n) \cap F_j(n) = \bigcup_{k=1}^{p(i,j,n)} D_k(i,j,n)
\]

where \( \{D_k(i,j,n)\}_{k=1}^{p(i,j,n)} \) is a finite sequence in \( D(\mu, P) \). Then \( B_i(n) \) is equal to the union \( \bigcup_{j=1}^{s(n)} \bigcup_{k=1}^{p(i,j,n)} D_k(i,j,n) \) and this implies that
Therefore, by statements (d) and (e) it follows that

\[(g) \int_T \chi_{C_n} g_n \, d\mu < 2\varepsilon \text{ for all integers } n \geq K.\]

Also, if \(n\) is an integer, \(n \geq K\), then \(C_n\) is a subset of \(A\), and it follows that \(\chi_A(t)(1 - \chi_{C_n}(t)) \cdot g_n(t)\) is zero if \(t\) is not in \(A - C_n\) and is \(g_n(t)\) if \(t\) is in \(A - C_n\). Since \(g_n(t)\) is less than \(\varepsilon/m\) for \(n \geq K\) and \(t\) in \(T\), it follows that

\[\chi_A(1 - \chi_{C_n}) g_n \leq \frac{\varepsilon}{m} \chi_A, \quad \forall n \geq K.\]

Therefore,

\[(h) \int_T \chi_A(1 - \chi_{C_n}) g_n \, d\mu < \varepsilon, \quad \forall n \geq K.\]

Now it follows from the definition of the norm, that

\[\|f_n\| = \int_T g_n \, d\mu\]

\[\leq \int_T (1 - \chi_A) g_n \, d\mu + \int_T \chi_A(1 - \chi_{C_n}) g_n \, d\mu + \int_T \chi_A \chi_{C_n} g_n \, d\mu,\]

and if \(n\) is an integer greater than \(K\), the first integral on the
dominant side of the inequality is less than ε by (c), the second one is less than ε by (h) and the last one is less than 2ε by (g). Therefore, if \( n \) is a positive integer greater than \( K \), then \( \|f_n\|' \) is less than \( 4ε \). Hence, \( \lim_{n \to \infty} \|f_n\|' = 0 \), and this completes the proof.

Suppose now that \( \{f_n\}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to the element \( \bar{f} \) of \( X^T/\mu \). Using the linearity and then the continuity of the integral on \( S'(\mu, P, X) \) one obtains the inequality

\[
\left| \int_T [f_n]d\mu - \int_T [f_m]d\mu \right| \leq \|f_n\| - \|f_m\|'.
\]

This implies that the sequence of vectors \( \left\{ \int_T [f_n]d\mu \right\}_{n=1}^\infty \) in the Banach space \( (X, \| \cdot \|) \) is norm Cauchy and so has a limit in \( X \). Suppose now that \( \{g_n\}_{n=1}^\infty \) is a second norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( \bar{f} \). Since the difference of norm Cauchy sequences is norm Cauchy, the sequence \( \{[f_n] - [g_n]\}_{n=1}^\infty \) of elements of \( S'(\mu, P, X) \) is norm Cauchy, and Theorem 2.5 implies that this sequence \( \mu \)-converges to the zero function on \( T \) to \( X \). Therefore, Theorem 2.8 implies that

\[
\lim_{n \to \infty} \|f_n - g_n\|' = 0.
\]

The linearity and continuity of the integral on \( S'(\mu, P, X) \) imply that for every positive integer \( n \),

\[
\left| \int_T [f_n]d\mu - \int_T [g_n]d\mu \right| \leq \|f_n\| - \|g_n\|'.
\]

Hence, it follows that
\[ \lim_{n \to \infty} \int_T [f_n]d\mu - \lim_{n \to \infty} \int_T [g_n]d\mu = \lim_{n \to \infty} \int_T [f_n]d\mu \]

\[ - \int_T [g_n]d\mu \leq \lim_{n \to \infty} \|[f_n] - [g_n]\|' = 0, \]

and this implies that \( \lim_{n \to \infty} \int_T [f_n]d\mu \) and \( \lim_{n \to \infty} \int_T [g_n]d\mu \) are the same element of \( X \). It should be noted that if \( g \) is a function on \( T \) to \( X \) that is \( \mu \)-equivalent to the \( P \)-simple function \( h \) on \( T \) to \( X \) and if one defines \( k_n \) to be \( h \) for \( n = 1, 2, 3, \ldots \), then \( \{[k_n]\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( g \) in \( X^T/\mu \) and

\[ \lim_{n \to \infty} \int_T [k_n]d\mu = \int_T [h]d\mu. \]

**Definition 2.7.** If \( f \) is an element of \( X^T/\mu \) with the property that there exists a norm Cauchy sequence \( \{[f_n]\}_{n=1}^{\infty} \) of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( f \), then the element of \( X \) given by \( \lim_{n \to \infty} \int_T [f_n]d\mu \) is said to be the integral over \( T \) of \( f \) with respect to \( \mu \), or in symbols,

\[ \int_T f d\mu = \lim_{n \to \infty} \int_T [f_n]d\mu. \]

The symbol \( K(\mu, P, X) \) will be used for the set of all elements \( \overline{f} \) in \( X^T/\mu \) such that there exists a norm Cauchy sequence \( \{[f_n]\}_{n=1}^{\infty} \) of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( \overline{f} \). The discussion preceding Definition 2.7 shows that if \( \overline{f} \) is in \( K(\mu, P, X) \), then \( \int_T \overline{f} d\mu \) exists. The last statement preceding Definition 2.7 shows that if \( \overline{f} \) is \( \mu \)-equivalent to a \( P \)-simple function \( g \) on \( T \) to \( X \), then
\[
\int_T \overline{f} d\mu = \int_T \overline{[g]} d\mu.
\]

The symbol \( S(\mu, P, X) \) will be used for the set of all \( \overline{f} \) in \( K(\mu, P, X) \) such that \( f \) is \( \mu \)-equivalent to some \( P \)-simple function on \( T \) to \( X \). It is clear that \( S'(\mu, P, X) \) and \( S(\mu, P, X) \) are isomorphic as linear spaces. In this sense, the integral as defined on \( K(\mu, P, X) \) is an extension of the integral previously defined on \( S'(\mu, P, X) \).

Suppose \( f \) and \( g \) are in \( K(\mu, P, X) \) and \( \{[f_n]\}_{n=1}^{\infty} \) and \( \{[g_n]\}_{n=1}^{\infty} \) are norm Cauchy sequences of elements of \( S'(\mu, P, X) \) that \( \mu \)-converge to \( \overline{f} \) and \( \overline{g} \), respectively. Let \( \alpha \) and \( \beta \) be real numbers. Then \( \{\alpha[f_n] + \beta[g_n]\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( \overline{\alpha f + \beta g} \), so that

\[
\int_T (\alpha \overline{f} + \beta \overline{g}) d\mu = \lim_{n \to \infty} \int_T (\alpha[f_n] + \beta[g_n]) d\mu
\]

\[
= \alpha \lim_{n \to \infty} \int_T [f_n] + \beta \lim_{n \to \infty} \int_T [g_n] d\mu
\]

\[
= \alpha \int_T \overline{f} d\mu + \beta \int_T \overline{g} d\mu.
\]

It follows from this that \( K(\mu, P, X) \) is a linear space and that the integral defined on \( K(\mu, P, X) \) is a linear transformation of \( K(\mu, P, X) \) into \( X \).

If \( \{[f_n]\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \), then the relations

\[
\| [f_n] \|' - \| [f_m] \|' = \int_T |f_n(\cdot)| d\mu - \int_T |f_m(\cdot)| d\mu.
\]
show that \( \{\|f_n\|\}_{n=1}^{\infty} \) is a norm Cauchy sequence of real numbers, and therefore, has a limit in the real numbers. If \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are both norm Cauchy sequences of elements of \( S'(\mu, P, X) \) and if both sequences \( \mu \)-converge to \( \mathbf{f} \) in \( K(\mu, P, X) \), then it follows that the sequence \( \{f_n - g_n\}_{n=1}^{\infty} \) in \( S'(\mu, P, X) \) is norm Cauchy and \( \mu \)-convergent to the zero element of \( X^T/\mu \). Theorem 2.8 then implies that \( \lim_{n \to \infty} \|f_n\| = 0. \) Hence, because of the relation

\[ \|\|f_n\||' - \|\|g_n\||' \leq \|f_n - g_n\||' \]

for all positive integers,

it follows that \( \lim_{n \to \infty} \|f_n\||' \) and \( \lim_{n \to \infty} \|g_n\||' \) coincide. Suppose now that \( \mathbf{f} \) and \( \mathbf{g} \) are in \( K(\mu, P, X) \) and that \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are norm Cauchy sequences of elements of \( S'(\mu, P, X) \) \( \mu \)-converging to \( \mathbf{f} \) and \( \mathbf{g} \), respectively. Since \( \|f_n\||' \) is non-negative for \( n = 1, 2, 3, \ldots \), it is clear that \( \lim_{n \to \infty} \|f_n\||' \) is also non-negative. If \( \alpha \) is a real number, then \( \|\alpha f_n\||' = |\alpha| \cdot \|f_n\||' \) for \( n = 1, 2, 3, \ldots \), so that \( \lim_{n \to \infty} \|\alpha f_n\||' \) is given by \( |\alpha| \cdot \lim_{n \to \infty} \|f_n\||' \). Since \( \{f_n + g_n\}_{n=1}^{\infty} \) is norm Cauchy and \( \mu \)-convergent to \( \mathbf{f} + \mathbf{g} \), it follows that the limit, \( \lim_{n \to \infty} \|f_n + g_n\||' \), exists, and
Theorem 2.8 implies that if $\bar{f}$ is the zero element of $K(\mu, P, X)$, then $\lim_{n \to \infty} \| [f_n] \|' = 0$. The converse is also true; that is, if $\lim_{n \to \infty} \| [f_n] \|' = 0$, then $\bar{f}$ is the zero element of $K(\mu, P, X)$. This can be proved as follows. Suppose that $\lim_{n \to \infty} \| [f_n] \|' = 0$ and let $\varepsilon$ be a positive real number.

For each positive integer $n$, since $f_n$ is a $P$-simple function, there exists a finite sequence $\{H_i(n)\}_{i=1}^s(n)$ in $D(\mu, P)$ such that if $G_1(\varepsilon) \subseteq \left\{ t \in T : |f_n(t)| \geq \varepsilon \right\}$, then $G_1(\varepsilon)$ is the union $\bigcup_{i=1}^{s(n)} H_i(n)$. In particular, the function $\chi_{G_1}(\varepsilon)$ is in $S_1(P, R)$. Since $\lim_{n \to \infty} \| [f_n] \|' = 0$, there exists a positive integer $N$ such that for $n \geq N$,

$$\varepsilon^2 > \| [f_n] \|' = \int_T |f_n(\cdot)| \, d\mu \geq \int_T \varepsilon \cdot \chi_{G_1}(\varepsilon) \, d\mu = \varepsilon \cdot \sum_{i=1}^{s(n)} \mu(H_i(n)).$$

Therefore, the sum $\sum_{i=1}^{s(n)} \mu(H_i(n))$ is less than $\varepsilon$ for all integers $n$ that exceed $N$, and

$$|f_n(t) - 0| < \varepsilon, \forall t \in T - \left( \bigcup_{i=1}^{s(n)} H_i(n) \right).$$

This implies that $f_n \to 0$; hence, $\bar{f}$ is the zero element of $K(\mu, P, X)$, and this proves the assertion. Finally, if $f$ is $\mu$-equivalent to the $P$-simple function $g$ on $T$ to $X$ and if $k_n = g$ for $n = 1, 2, 3, \ldots$, then $\{[k_n]\}_{n=1}^\infty$ is a norm Cauchy sequence of elements of $S'(\mu, P, X)$ that
\[ \mu \text{-converges to } \bar{f} \text{ and } \lim_{n \to \infty} \| f_n \|' = \| f \|'. \] This discussion then shows that the functional \( \| \| \) defined on \( K(\mu, P, X) \) in the next definition is a norm for \( K(\mu, P, X) \) and extends the norm previously defined on \( S'(\mu, P, X) \) in the same sense that the integral on \( K(\mu, P, X) \) is an extension of the integral on \( S'(\mu, P, X) \).

**Definition 2.8.** If \( \bar{f} \) is in \( K(\mu, P, X) \) and if \( \{ f_n \}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( \bar{f} \), then the norm function \( \| \| \) is defined at \( \bar{f} \) to be the real number \( \lim_{n \to \infty} \| f_n \|' \), or in symbols \( \| \bar{f} \| = \lim_{n \to \infty} \| f_n \|' \).

In particular, it should be noted that the discussion preceding Definition 2.8 and the previous observation that the spaces \( S'(\mu, P, X) \) and \( S(\mu, P, X) \) are isomorphic shows that these spaces are isometrically isomorphic. Also, since the difference of two functions \( \mu \)-equivalent to \( P \)-simple functions is a function that is \( \mu \)-equivalent to a \( P \)-simple function, the space \( S(\mu, P, X) \) is a subspace of \( K(\mu, P, X) \). Henceforth, the elements of \( S(\mu, P, X) \) will be called \( P \)-simple functions.

If \( \bar{f} \) is in \( K(\mu, P, X) \) and \( \{ f_n \}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S'(\mu, P, X) \) that \( \mu \)-converges to \( \bar{f} \), then \( \left| \int_T [f_n] d\mu \right| \leq \| f_n \|' \) since the integral on \( S'(\mu, P, X) \) is a continuous linear transformation. This implies that

\[
\left| \int_T \bar{f} d\mu \right| = \lim_{n \to \infty} \left| \int_T [f_n] d\mu \right| = \lim_{n \to \infty} \left| \int_T [f_n] d\mu \right|
\]
so the integral defined on $K(\mu, P, X)$ is a continuous linear transformation into $X$ relative to the norm topology.

With $\overline{f}$ and $\{[f_n]\}_{n=1}^{\infty}$ defined as in the preceding paragraph, the inequalities

$$|f_n(\cdot)| - |f_m(\cdot)| \leq |f_n(\cdot) - f_m(\cdot)|,$$

and

$$|f_n(\cdot) - f(\cdot)| \leq |f_n(\cdot) - f(\cdot)|$$

imply that $\{[f_n(\cdot)]\}_{n=1}^{\infty}$ is a norm Cauchy sequence of elements of $S'(\mu, P, R)$ that $\mu$-converges to $[\overline{f}(\cdot)]$. For notational convenience, the symbol $[\overline{f}(\cdot)]$ is used for the element of $K^T/\sim$ containing $[f(\cdot)]$. Therefore, if $\overline{f}$ is in $K(\mu, P, X)$, then $[\overline{f}(\cdot)]$ is in $K(\mu, P, R)$. Also, the norm of $\overline{f}$ is given by

$$\|\overline{f}\| = \lim_{n \to \infty} \|[f_n]\|' = \lim_{n \to \infty} \int_T \overline{[f_n(\cdot)']} d\mu = \int_T |\overline{f}(\cdot)| d\mu.$$

It should be noted that in the preceding paragraph, the symbol $[\overline{f}(\cdot)]$ is used for the equivalence class $\overline{[f(\cdot)]}$. This notation will be used consistently throughout the dissertation.

The following theorem summarizes the results obtained so far for the space $K(\mu, P, X)$.

**Theorem 2.9.** (a) The space $K(\mu, P, X)$ with the topology induced by
the norm function \( \| \| \) is a normed linear space. (b) The space \( S(\mu, P, X) \) is a subspace of \( K(\mu, P, X) \). (c) Relative to the norm topology, the integral on \( K(\mu, P, X) \) is a continuous linear transformation into \( X \).

(d) If \( \widetilde{f} \) is in \( K(\mu, P, X) \), then \( |\widetilde{f}(\cdot)| \) is in \( K(\mu, P, R) \) and

\[
\int_T |\widetilde{f}(\cdot)| \, d\mu.
\]

Suppose now that \( f \) is a function on \( T \) to \( R \) such that \( f \) and \( |f(\cdot)| \) are \( \mu \)-equivalent. If \( \varepsilon \) is a positive real number, then there exists \( \{ E_i \}_{i=1}^q \) in \( D(\mu, P) \) such that \( \sum_{i=1}^q \mu(E_i) < \varepsilon \), and

\[
|f(t) - |f(t)|| < \varepsilon, \quad \forall t \in T - (\bigcup_{i=1}^q E_i).
\]

It follows from the last inequality that

\[
f(t) > -\varepsilon + |f(t)| > -\varepsilon, \quad \forall t \in T - (\bigcup_{i=1}^q E_i).
\]

On the other hand, suppose \( f \) is a function on \( T \) to \( R \) with the property that if \( \varepsilon \) is a positive real number, then there exists a sequence \( \{ F_j \}_{j=1}^m \) in \( D(\mu, P) \) such that the sum \( \sum_{j=1}^m \mu(F_j) \) is less than \( \varepsilon \) and

\[
f(t) > -\varepsilon, \quad \forall t \in T - (\bigcup_{j=1}^m F_j).
\]

Then let \( \varepsilon \) be a positive real number and pick \( \{ E_i \}_{i=1}^q \) in \( D(\mu, P) \) such that the sum \( \sum_{i=1}^q \mu(E_i) \) is less than \( \varepsilon/2 \) and

\[
f(t) > -\frac{\varepsilon}{2}, \quad \forall t \in T - (\bigcup_{i=1}^q E_i).
\]

It follows from this that

\[
|f(t) - |f(t)|| < \varepsilon, \quad \forall t \in T - (\bigcup_{i=1}^q E_i)
\]
and that \( \sum_{i=1}^{q} \mu(E_i) < \varepsilon \). Hence, \( f \) and \( |f(\cdot)| \) are \( \mu \)-equivalent.

**Definition 2.9.** If \( \bar{f} \) and \( \bar{g} \) are elements of \( K(\mu, P, R) \), then \( \bar{f} \) is said to be greater than or equal to \( \bar{g} \), in symbols \( \bar{f} \geq \bar{g} \), if and only if \( |f(\cdot) - g(\cdot)| \) is \( \mu \)-equivalent to \( f - g \).

The discussion preceding Definition 2.9 shows that an alternate characterization of the relation \( \geq \) is given by the following. If \( \bar{f} \) and \( \bar{g} \) are in \( K(\mu, P, R) \), then \( \bar{f} \geq \bar{g} \) if and only if for every positive real number \( \varepsilon \), there exists a sequence \( \{F_j\}_{j=1}^{m} \) in \( D(\mu, P) \) such that

\[
\sum_{j=1}^{m} \mu(F_j) < \varepsilon \quad \text{and} \quad f(t) - g(t) > -\varepsilon, \quad \forall t \in T - (\bigcup_{j=1}^{m} F_j).
\]

If \( \bar{f} \) is in \( K(\mu, P, R) \), then \( \bar{f} \geq \bar{f} \) since \( f(t) - f(t) = 0 \) is greater than \( -\varepsilon \) for any positive real number \( \varepsilon \) and all \( t \) in \( T \). Suppose that \( \bar{f} \) and \( \bar{g} \) are in \( K(\mu, P, R) \) and that \( \bar{f} \geq \bar{g} \) and \( \bar{g} \geq \bar{f} \). Let \( \varepsilon \) be a positive real number and let \( \{E_i\}_{i=1}^{n} \) and \( \{F_j\}_{j=1}^{m} \) be sequences in \( D(\mu, P) \) such that

\[
\sum_{i=1}^{n} \mu(E_i) < \varepsilon/2, \quad \sum_{j=1}^{m} \mu(F_j) < \varepsilon/2, \quad \text{and}
\]

\[
f(t) - g(t) > -\varepsilon, \quad \forall t \in T - (\bigcup_{i=1}^{n} E_i),
\]

\[
g(t) - f(t) > -\varepsilon, \quad \forall t \in T - (\bigcup_{j=1}^{m} F_j).
\]

Let \( \{G_k\}_{k=1}^{q} \) be a sequence in \( D(\mu, P) \) such that

\[
\bigcup_{k=1}^{q} G_k = \left( \bigcup_{i=1}^{n} E_i \right) \cup \left( \bigcup_{j=1}^{m} F_j \right).
\]

Then it follows that \( \sum_{k=1}^{q} \mu(G_k) < \varepsilon \) and
\[ |f(t) - g(t)| < \epsilon, \forall t \in \mathcal{T} - (\bigcup_{k=1}^{q} G_k). \]

Therefore, \( f \) is \( \mu \)-equivalent to \( g \) and \( \overline{f} = \overline{g} \). Suppose now that \( \overline{f}, \overline{g} \) and \( \overline{h} \) are in \( K(\mu, P, R) \) and that \( \overline{f} \geq \overline{g} \) and \( \overline{g} \geq \overline{h} \), and let \( \epsilon \) be a positive real number. Then there exists sequences \( \{E_i\}_{i=1}^{n} \) and \( \{F_j\}_{j=1}^{m} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{n} \mu(E_i) < \epsilon/2 \), \( \sum_{j=1}^{m} \mu(F_j) < \epsilon/2 \) and

\[ f(t) - g(t) > - \frac{\epsilon}{2}, \forall t \in \mathcal{T} - (\bigcup E_i), \text{ and} \]

\[ g(t) - h(t) > - \frac{\epsilon}{2}, \forall t \in \mathcal{T} - (\bigcup F_j). \]

Let \( \{G_k\}_{k=1}^{q} \) be a sequence in \( D(\mu, P) \) such that

\[ \bigcup_{k=1}^{q} G_k = \bigcup_{i=1}^{n} E_i \bigcup_{j=1}^{m} F_j. \]

Then it follows that \( \sum_{k=1}^{q} \mu(G_k) < \epsilon \), and

\[ f(t) - h(t) > - \epsilon, \forall t \in \mathcal{T} - (\bigcup_{k=1}^{q} G_k), \]

so that \( \overline{f} \geq \overline{h} \). Therefore, the relation \( \geq \) on \( K(\mu, P, R) \) is a partial order relation.

**Theorem 2.10.** If \( \overline{f} \) and \( \overline{g} \) are in \( K(\mu, P, R) \) and \( \overline{f} \geq \overline{g} \), then

\[ \int_{\mathcal{T}} \overline{f} d\mu \geq \int_{\mathcal{T}} \overline{g} d\mu. \]

**Proof:** If \( \overline{f} \) and \( \overline{g} \) are in \( K(\mu, P, R) \) and if \( \overline{f} \geq \overline{g} \), then \( |f(.) - g(.)| \) is \( \mu \)-equivalent to \( f - g \). Therefore, it follows that

\[ \int_{\mathcal{T}} (\overline{f} - \overline{g}) d\mu = \int_{\mathcal{T}} |f(.) - g(.)| d\mu. \]

Theorem 2.9.d and the fact that \( \overline{f} - \overline{g} \) is \( \overline{f} - \overline{g} \) imply that
\[
\int_T (\bar{f} - \bar{g}) d\mu = \int_T (\bar{f} - \bar{g}) d\mu \\
= \|\bar{f} - \bar{g}\| \\
= \|\bar{f} - \bar{g}\| \\
\geq 0.
\]

Therefore, the linearity of the integral gives the result that
\[
\int_T \bar{f} d\mu \geq \int_T \bar{g} d\mu,
\]
and this completes the proof.

In order to investigate the properties of \(K(\mu, P, X)\) more fully, it is necessary to study relations between norm and \(\mu\)-convergence more deeply. In the following theorems, the statement that a condition is "almost uniform" means that for every positive real number \(\varepsilon\), there exists a sequence \(\{E_n\}_{n=1}^\infty\) in \(D(\mu, P)\) such that \(\sum_{i=1}^n \mu(E_i) < \varepsilon\) and the condition holds uniformly on \(T - (\bigcup_{i=1}^n E_i)\). A statement is true "almost everywhere" if it is true on the complement of a \(\mu\)-null set.

If more than one \(p\)-volume is under consideration, then, of course, it is necessary to specify the almost uniform and almost everywhere properties with respect to the proper \(p\)-volume.

**Theorem 2.11.** If \(\{f_n\}_{n=1}^\infty\) in \(S(\mu, P, X)\) is norm Cauchy, then \(\{f_n\}_{n=1}^\infty\) is \(\mu\)-Cauchy.

**Proof:** Note that if \(\{h_n\}_{n=1}^\infty\) is a \(\mu\)-Cauchy sequence of functions on \(T\) to \(X\) and if \(\{g_n\}_{n=1}^\infty\) is a sequence of functions on \(T\) to \(X\) such that \(g_n \sim h_n\) for \(n = 1, 2, 3, \ldots\), then \(\{g_n\}_{n=1}^\infty\) is also \(\mu\)-Cauchy. The truth of this statement may be seen by the following argument. Let \(\varepsilon\) be a positive real number. For every positive integer \(n\) let \(\{E_{i_{(n)}}\}_{i=1}^q(n)\)
be a sequence in $D(\mu, P)$ such that $\sum_{i=1}^{q(n)} \mu(\mathcal{E}_i^{(n)}) < \varepsilon/3$ and

$$|h_n(t) - g_n(t)| < \frac{\varepsilon}{3}, \forall t \in T - \left( \bigcup_{i=1}^{q(n)} \mathcal{E}_i^{(n)} \right).$$

Let $N$ be a positive integer such that $n, m > N$ implies there exists a sequence $\{F_{j}^{(n,m)}\}_{j=1}^{p(n,m)}$ in $D(\mu, P)$ such that $\sum_{j=1}^{p(n,m)} \mu(F_{j}^{(n,m)}) < \varepsilon/3$ and

$$|h_n(t) - h_m(t)| < \frac{\varepsilon}{3}, \forall t \in T - \left( \bigcup_{j=1}^{p(n,m)} F_{j}^{(n,m)} \right).$$

For each pair of integers $n, m \geq N$, let $\{G_{k}^{(n,m)}\}_{k=1}^{r(n,m)}$ be a sequence in $D(\mu, P)$ such that

$$r(n,m) = \bigcup_{k=1}^{r(n,m)} G_{k}^{(n,m)} = \bigcup_{i=1}^{q(n)} \mathcal{E}_i^{(n)} \cup \bigcup_{i=1}^{q(m)} \mathcal{E}_i^{(m)} \cup \bigcup_{j=1}^{p(n,m)} F_{j}^{(n,m)}. $$

If $n, m \geq N$, then it follows that $\sum_{k=1}^{r(n,m)} \mu(G_{k}^{(n,m)}) < \varepsilon$, and

$$|g_n(t) - g_m(t)| \leq |g_n(t) - h_n(t)| + |h_n(t) - h_m(t)|$$

$$+ |h_m(t) - g_m(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

$$\forall t \in T - \left( \bigcup_{k=1}^{r(n,m)} G_{k}^{(n,m)} \right).$$

Therefore, the sequence $\{g_n\}_{n=1}^{\infty}$ is $\mu$-Cauchy.

Let $\{f_n\}_{n=1}^{\infty}$ be a norm Cauchy sequence of elements of $S(\mu, P, X)$. For $n = 1, 2, 3, \ldots$, let $g_n$ be an element of $S_1(P, X)$ such that $g_n \sim f_n$. The note at the beginning of this proof makes it clear that it is sufficient to show that $\{g_n\}_{n=1}^{\infty}$ is $\mu$-Cauchy to complete the proof. Therefore, suppose $\varepsilon$ is a positive real number and that for each pair of positive integers $n$ and $m$, $F_{nm}$ is the set given by
\[ F_{nm} = \{ t \in T : |g_n(t) - g_m(t)| \geq \epsilon \}. \]

Since each \( g_n \) is in \( S_1(P, X) \), there exists a sequence \( \{g_{(n,m)}\}_{k=1}^{s(n,m)} \) in \( D(\mu, P) \) such that \( F_{nm} = \bigcup_{k=1}^{s(n,m)} G_k \). Then \( \chi_{F_{nm}} \) is in \( S_1(P, X) \) and, pointwise on \( T \),

\[ \epsilon \cdot \chi_{F_{nm}} \leq |g_n(\cdot) - g_m(\cdot)|, \text{ for all } n \text{ and } m. \]

There exists an integer \( N \) such that if \( n, m \geq N \), then

\[ \epsilon^2 > \| \bar{f}_n - \bar{f}_m \| = \| g_n - \bar{g}_m \| \]

\[ = \int_T |g_n(\cdot) - g_m(\cdot)| \, d\mu \]

\[ \geq \epsilon \int_T \chi_{F_{nm}} \, d\mu \]

\[ = \epsilon \cdot \sum_{k=1}^{s(n,m)} \mu(G_k). \]

Hence, for \( n, m \geq N \), it follows that \( \sum_{k=1}^{s(n,m)} \mu(G_k) < \epsilon \) and that

\[ |g_n(t) - g_m(t)| < \epsilon, \forall t \in T - \bigcup_{k=1}^{s(n,m)} G_k. \]

Therefore, \( \{g_n\}_{n=1}^{\infty} \) is \( \mu \)-Cauchy, and this completes the proof.

**Theorem 2.12.** If \( \mu \) is countably additive and if \( \{f_n\}_{n=1}^{\infty} \) in \( S(\mu, P, X) \) is \( \mu \)-norm Cauchy, then there is a subsequence of \( \{f_n\}_{n=1}^{\infty} \) that is almost uniformly Cauchy.

**Proof:** Let \( k \) be a positive integer. By Theorem 2.11, there exists an integer \( \overline{n}(k) \) such that if \( p, q \geq \overline{n}(k) \), then there exists a sequence \( \{g_{(p,q)}\}_{j=1}^{s_{(p,q)}} \) in \( D(\mu, P) \) such that \( \sum_{j=1}^{s_{(p,q)}} \mu(g_{(p,q)}) < 2^{-k} \) and
Let \( n_1 \) be \( n(1) \) and for \( j = 2, 3, 4, \ldots \), let \( n_j \) be the maximum of \( n_{j-1} + 1 \) and \( n(j) \). Then \( n_1 < n_2 < n_3 < \ldots \) and \( \{ f_{n_k} \}_{k=1}^{\infty} \) is a subsequence of \( \{ f_{n} \}_{n=1}^{\infty} \). Let \( \{ g_i \}_{i=1}^{\infty} \) be the sequence \( \{ (n_i, n_{i+1}) \}_{i=1}^{\infty} \) in \( D(\mu, P) \). Let \( \epsilon \) be a positive real number and pick \( k \) to be a positive integer such that \( 2^{-(k-1)} < \epsilon \). Then

\[
\sum_{m=k}^{\infty} \sum_{j=1}^{\infty} \mu(g_{j}^{(m)}) < \sum_{m=k}^{\infty} 2^{-m} = 2^{-(k-1)} < \epsilon,
\]

and if \( j > i > k \), then

\[
|f_{n_i}^{(j)}(t) - f_{n_j}^{(i)}(t)| \leq \sum_{m=i}^{\infty} |f_{n_{m}}^{(j)}(t) - f_{n_{m-1}}^{(i)}(t)|
\]

\[
< 2^{-(i-1)}
\]

\[
< 2^{-(k-1)} < \epsilon, \ \forall t \in T - \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\infty} g_{j}^{(m)}.
\]

Therefore, the sequence \( \{ f_{n_k} \}_{k=1}^{\infty} \) is almost uniformly Cauchy, and this completes the proof.

Suppose now that \( \{ f_{n} \}_{n=1}^{\infty} \) is a sequence of elements of \( S(\mu, P, X) \) such that \( \{ f_{n} \}_{n=1}^{\infty} \) is almost uniformly Cauchy. For each positive integer \( k \), let \( \{ g_{j}^{(k)} \}_{j=1}^{m(k)} \) be a sequence in \( D(\mu, P) \) such that \( \sum_{j=1}^{m(k)} \mu(g_{j}^{(k)}) < 1/k \)

and if \( F_k = \bigcap_{j=1}^{m(k)} g_{j}^{(k)} \), then \( f_{n} \) is uniformly Cauchy on \( T - F_k \). Let \( F = \bigcap_{k=1}^{\infty} F_k \), so that \( F \) is \( \mu \)-null. Since \( \{ f_{n} \}_{n=1}^{\infty} \) is uniformly Cauchy on \( T - F \) for each \( k \) and since \( T - F \) is equal to \( \bigcup_{k=1}^{\infty} (T - F_k) \), then \( \{ f_{n} \}_{n=1}^{\infty} \) is Cauchy almost everywhere. Also, if the function \( f \) on \( T \) to \( X \) is
defined by

\[ f(t) = \begin{cases} 
\lim_{n \to \infty} f_n(t), & \forall t \in T - F, \\
0, & \forall t \in F 
\end{cases} \]

then \( f(t) = \lim_{n \to \infty} f_n(t) \) for all \( t \) in \( T - F_k \) for \( k = 1, 2, 3, \ldots \). Since the sequence \( \{f_n\}_{n=1}^{\infty} \) is uniformly Cauchy on \( T - F_k \), it uniformly converges to \( f \) on \( T - F_k \) for \( k = 1, 2, 3, \ldots \). Therefore, the sequence \( \{f_n\}_{n=1}^{\infty} \) converges uniformly to \( f \). Hence, if \( \epsilon > 0 \) is a positive real number, then there exists a sequence \( \{E_i\}_{i=1}^{q} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{q} \mu(E_i) < \epsilon \) and there exists a positive integer \( N \) such that if \( n \geq N \), then

\[ |f_n(t) - f(t)| < \epsilon, \forall t \in T - (\bigcup_{i=1}^{q} E_i). \]

This implies that the sequence \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( f \).

Suppose now that \( \mu \) is countably additive and that \( \{f_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S(\mu, P, X) \). Theorem 2.11 implies that the sequence \( \{f_n\}_{n=1}^{\infty} \) is \( \mu \)-Cauchy, and Theorem 2.12 implies there exists a subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) of \( \{f_n\}_{n=1}^{\infty} \) that is almost uniformly Cauchy.

Therefore, there exists a function \( f \) on \( T \) to \( X \) such that \( \{f_{n_k}\}_{k=1}^{\infty} \) \( \mu \)-converges to \( f \). Let \( \epsilon > 0 \) be a positive real number. There exists a positive integer \( N_1 \) such that if \( n, m \geq N_1 \), then there exists

\[ \{G_{k}(n,m)\}_{k=1}^{s(n,m)} \in D(\mu, P) \text{ such that } \sum_{k=1}^{s(n,m)} \mu(G_{k}(n,m)) < \epsilon/2 \text{ and } \]

\[ |f_n(t) - f_m(t)| < \frac{\epsilon}{2}, \forall t \in T - (\bigcup_{k=1}^{s(n,m)} G_{k}(n,m)). \]

There exists a positive integer \( N_2 \) such that if \( m \geq N_2 \), then there
exists \( \{f_j^{(m)}\}_{j=1}^{r(m)} \) in \( D(\mu, P) \) such that \[ \sum_{j=1}^{r(m)} \mu(f_j^{(m)}) < \varepsilon/2 \] and \[ |f_{n_m}^{(m)}(t) - f(t)| < \varepsilon/2, \forall t \in T - (\bigcup_{j=1}^{r(m)} F_j^{(m)}). \]

Let \( N \) be the maximum of \( N_1 \) and \( N_2 \). If \( m \geq N \), let \( \{E_i^{(m)}\}_{i=1}^{q(m)} \) in \( D(\mu, P) \) be such that \[ \bigcup_{i=1}^{q(m)} E_i^{(m)} = (\bigcup_{k=1}^{r(m)} C_k^{(m)}) \bigcup (\bigcup_{j=1}^{r(m)} F_j^{(m)}). \]

It follows that \[ \sum_{i=1}^{q(m)} \mu(E_i^{(m)}) < \varepsilon \] and \[ |f_m(t) - f(t)| \leq |f_m(t) - f_{n_m}(t)| + |f_{n_m}(t) - f(t)| \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall t \in T - (\bigcup_{i=1}^{q(m)} E_i^{(m)}). \]

Hence, the sequence \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to the function \( f \) on \( T \) to \( X \).

The next theorem summarizes these results.

**Theorem 2.13.** If \( \mu \) is countably additive and if \( \{f_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S(\mu, P, X) \), then there exists an element \( \overline{f} \) in \( X^{T}/\mu \) such that \( f_n \rightarrow \overline{f}. \)

The next theorem establishes the relationship between norm convergence and \( \mu \)-convergence. It also shows that relative to the norm topology, \( S(\mu, P, X) \) is a dense subspace of \( K(\mu, P, X) \).

**Theorem 2.14.** Suppose that \( \{f_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( K(\mu, P, X) \) and that \( \overline{f} \) is in \( K(\mu, P, X) \). Then the sequence \( \{f_n\}_{n=1}^{\infty} \) norm converges to \( \overline{f} \) if and only if it \( \mu \)-converges to \( \overline{f} \).

**Proof:** Suppose first that \( \{f_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S(\mu, P, X) \) that \( \mu \)-converges to \( \overline{f} \) in \( K(\mu, P, X) \). Let \( n(1) \)
be a positive integer such that

\[ \| f_n - f_m \| < 2^{-1}, \forall n, m \geq n(1). \]

For \( k = 2, 3, 4, \ldots \), let \( n(k) \) be a positive integer greater than \( n(k - 1) \) such that

\[ \| f_n - f_m \| < 2^{-k}, \forall n, m \geq n(k). \]

Then \( n(1) < n(2) < n(3) < \ldots \), so that \( \{ f_{n(k)} \}_{k=1}^\infty \) is a subsequence of \( \{ f_n \}_{n=1}^\infty \) and

\[ \| f_{n(k)} - f_{n(m)} \| < 2^{-k}, \forall m \geq k. \]

Fix \( k \) and let \( g_m = f_{n(k)} - f_{n(m)} \). Then \( \{ g_m \}_{m=1}^\infty \) is a norm Cauchy sequence of elements of \( S(\mu, P, X) \) that \( \mu \)-converges to \( f_{n(k)} - f \).

Therefore, it follows that

\[ \| f_{n(k)} - f \| = \lim_{m \to \infty} \| g_m \| = \lim_{m \to \infty} \| f_{n(k)} - f_{n(m)} \| \leq 2^{-k}. \]

This implies that \( \lim_{k \to \infty} \| f_{n(k)} - f \| = 0 \). Since \( \{ f_n \}_{n=1}^\infty \) is norm Cauchy and

\[ \| f_n - f \| \leq \| f_n - f_{n(k)} \| + \| f_{n(k)} - f \|, \]

it then follows that \( \lim_{n \to \infty} \| f_n - f \| = 0. \)

Suppose now that \( \{ f_n \}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( K(\mu, P, X) \) that \( \mu \)-converges to \( f \) in \( K(\mu, P, X) \). For each positive integer \( n \), there exists a norm Cauchy sequence \( \{ h_k \}_{k=1}^\infty \) in \( S(\mu, P, X) \)
that \( \mu \)-converges to \( \overline{f}_n \). By the first part of this proof, \( \lim_{k \to \infty} \| \overline{h}_k \| = 0 \). Hence, let \( \overline{g}_n \in S(\mu, P, X) \) and \( \{ \overline{g}_i \}_{i=1}^{q(n)} \) in \( D(\mu, P) \) be chosen such that \( \| \overline{g}_n - \overline{f}_n \| < 2^{-n}, \sum_{i=1}^{q(n)} \mu(E_i(n)) < 2^{-n} \) and

\[
\| \overline{g}_n(t) - \overline{f}_n(t) \| < 2^{-n}, \forall t \in \mathcal{T} - \bigcup_{i=1}^{q(n)} E_i(n).
\]

The inequalities

\[
\| \overline{g}_n - \overline{g}_m \| \leq \| \overline{g}_n - \overline{f}_n \| + \| \overline{f}_n - \overline{f}_m \| + \| \overline{f}_m - \overline{g}_m \|, \quad \text{and}
\]

\[
\| \overline{g}_n(t) - \overline{f}(t) \| \leq \| \overline{g}_n(t) - \overline{f}_n(t) \| + \| \overline{f}_n(t) - \overline{f}(t) \|, \forall t \in \mathcal{T}
\]

together with the properties of the sequence \( \{ \overline{f}_n \}_{n=1}^{\infty} \) and the choice of the sequence \( \{ \overline{g}_n \}_{n=1}^{\infty} \), imply that \( \{ \overline{g}_n \}_{n=1}^{\infty} \) is norm Cauchy and \( \mu \)-convergent to \( \overline{f} \). Since each \( \overline{g}_n \) is in \( S(\mu, P, X) \), the first part of the proof implies that \( \lim_{n \to \infty} \| \overline{g}_n - \overline{f} \| = 0 \). Therefore, it follows that

\[
\lim_{n \to \infty} \| \overline{g}_n - \overline{f} \| \leq \lim_{n \to \infty} (\| \overline{f}_n - \overline{g}_n \| + \| \overline{g}_n - \overline{f} \|)
\]

\[
= \lim_{n \to \infty} \| \overline{f}_n - \overline{g}_n \| + \lim_{n \to \infty} \| \overline{g}_n - \overline{f} \| = 0.
\]

Suppose now that \( \{ \overline{f}_n \}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( K(\mu, P, X) \), that \( \overline{f} \) is in \( K(\mu, P, X) \) and that \( \lim_{n \to \infty} \| \overline{f}_n - \overline{f} \| = 0 \). Let \( \overline{k}_n \) be \( \overline{f}_n - \overline{f} \) for \( n = 1, 2, 3, \ldots \). Then \( \{ \overline{k}_n \}_{n=1}^{\infty} \) is a norm Cauchy sequence and \( \lim_{n \to \infty} \| \overline{k}_n \| = 0 \). It must be shown that \( \{ \overline{k}_n \}_{n=1}^{\infty} \) \( \mu \)-converges to zero in order to complete the proof. Let \( \overline{h}_n \in S_1(P, X) \) and \( \{ \overline{f}_j \}_{j=1}^{q(n)} \) in \( D(\mu, P) \) be chosen such that \( \| \overline{h}_n - \overline{k}_n \| < 2^{-n} \),
\[
\sum_{j=1}^{q(n)} \mu(F_j^{(n)}) < 2^{-n} \quad \text{and} \quad |k_n(t) - h_n(t)| < 2^{-n}, \forall t \in \mathcal{T} - \left( \bigcup_{j=1}^{q(n)} F_j^{(n)} \right).
\]

Since \( \|h_n\| < \|h_n - k_n\| + \|k_n\| \), it follows that \( \lim_{n \to \infty} \|h_n\| = 0 \). Let \( \varepsilon \) be a positive real number. Since each \( h_n \) is an element of \( S_1(P, X) \), the set

\[
G_n(\varepsilon) = \left\{ t \in \mathcal{T} : |h_n(t)| \geq \varepsilon \right\}
\]

is equal to the union of a finite sequence \( \{H_i^{(n)} \}_{i=1}^{s(n)} \) in \( D(\mu, P) \).

There exists a positive integer \( N \) such that \( n > N \) implies that

\[
\varepsilon^2 > \|h_n\| = \int_{\mathcal{T}} |h_n(t)| \, d\mu \geq \varepsilon \int_{\mathcal{T}} \chi_{G_n(\varepsilon)} \, d\mu = \varepsilon \cdot \sum_{i=1}^{s(n)} \mu(H_i^{(n)}).
\]

Therefore, if \( n > N \), then \( \sum_{i=1}^{s(n)} \mu(H_i^{(n)}) < \varepsilon \) and

\[
|h_n(t) - 0| < \varepsilon, \forall t \in \mathcal{T} - \left( \bigcup_{i=1}^{s(n)} H_i^{(n)} \right).
\]

For each positive integer \( n \), let \( \{E_i^{(n)} \}_{i=1}^{p(n)} \) be a sequence in \( D(\mu, P) \) such that

\[
\bigcup_{i=1}^{p(n)} E_i^{(n)} = \left( \bigcup_{j=1}^{q(n)} F_j^{(n)} \right) \cup \left( \bigcup_{i=1}^{s(n)} H_i^{(n)} \right).
\]

Then it follows that \( \sum_{i=1}^{p(n)} \mu(E_i^{(n)}) < 2^{-n} + \varepsilon \) and
Therefore, the sequence \( \{k_n(\tau)\}_{n=1}^{\infty} \) \( \mu \)-converges to zero. This completes the proof.

Suppose now that \( \{L_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( K(\mu, P, X) \) that \( \mu \)-converges to \( \bar{f} \) in \( X^T/\mu \). As in the proof of the second half of Theorem 2.14, there exists a norm Cauchy sequence \( \{T_n\}_{n=1}^{\infty} \) in \( S(\mu, P, X) \) such that \( \|L_n - \bar{T}_n\| < 2^{-n} \) and \( \{T_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( \bar{f} \).

It follows from this that \( \bar{f} \) is in \( K(\mu, P, X) \), and then Theorem 2.14 implies that \( \{T_n\}_{n=1}^{\infty} \) norm converges to \( \bar{f} \).

If \( \mu \) is countably additive and \( \{L_n\}_{n=1}^{\infty} \) in \( K(\mu, P, X) \) is a norm Cauchy sequence, then let \( \{g_n\}_{n=1}^{\infty} \) be a sequence in \( S(\mu, P, X) \) such that \( \|L_n - \bar{g}_n\| < 2^{-n} \) for \( n = 1, 2, 3, \ldots \). Then \( \{g_n\}_{n=1}^{\infty} \) in \( S(\mu, P, X) \) is norm Cauchy so that Theorem 2.13 implies there exists an element \( \bar{f} \) in \( X^T/\mu \) such that \( \{g_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( \bar{f} \). The previous paragraph then shows that \( \bar{f} \) is in \( K(\mu, P, X) \) and \( \lim_{n \to \infty} \|g_n - \bar{f}\| = 0 \). Therefore, it follows that

\[
\lim_{n \to \infty} \|L_n - \bar{f}\| \leq \lim_{n \to \infty} \|L_n - \bar{T}_n\| + \lim_{n \to \infty} \|\bar{T}_n - \bar{f}\| = 0.
\]

Hence, this proves the next theorem.

**Theorem 2.15.** If \( \mu \) is a non-negative countably additive p-volume on \( P \), then \( K(\mu, P, X) \) is a complete normed linear space.

If \( \mu \) is not countably additive it is not necessarily true that \( K(\mu, P, X) \) is complete. The following example is a \( K(\mu, P, X) \) space that is not complete.
Example 2.1. Let $T$ be the set of all positive integers. Let $P$ be the collection of all subsets $E$ of $T$ such that $E$ is empty, $E$ is finite, $T - E$ is finite or $T - E$ is empty. The union of two finite sets is finite, while the union of the complement of a finite subset of $T$ with any element of $P$ is the complement of a finite or empty subset of $T$. Hence, $P$ is closed under the operation of union. A similar argument shows that $P$ is closed under differences, so $P$ is an algebra. Hence, it is certainly a proto-ring. Define the set function $\mu$ on $P$ by the following equations:

\[
\mu(\emptyset) = 0, \quad \mu(\{n_i\}_{i=1}^q) = \sum_{i=1}^q \frac{1}{2^{n_i}}, \quad q \text{ finite and } \{n_i\}_{i=1}^q \in P, \text{ and }
\]

\[
\mu(T - E) = 2 - \mu(E) \text{ if } E \text{ is finite or empty.}
\]

Since $P$ is an algebra of sets, it is only necessary to check the additivity of $\mu$ on pairs of disjoint elements of $P$. Let $A$ and $B$ be disjoint elements of $P$. If each of $A$ and $B$ is finite, then it is clear that $\mu(A \cup B) = \mu(A) + \mu(B)$. If one of the sets, $A$ for example, is the complement of a finite set $\{n_i\}_{i=1}^q$ in $P$, then $B$ must be finite since $A \cap B$ is empty. Let $B$ be the set $\{m_j\}_{j=1}^p$ in $P$. Since $A$ and $B$ are disjoint, it follows that $\{m_j\}_{j=1}^p$ is a subset of $\{n_i\}_{i=1}^q$. Therefore, $A \cup B$ is given by

\[
A \cup B = T - (\{n_i\}_{i=1}^q - \{m_j\}_{j=1}^p),
\]

so that

\[
\mu(A \cup B) = 2 - \mu(\{n_i\}_{i=1}^q - \{m_j\}_{j=1}^p).
\]
Therefore, \( \mu \) is finitely additive; it is not countably additive since 
\( \mu(T) \) is 2, \( T \) is \( \bigcup_{i=1}^{\infty} \{i\} \) and \( \sum_{i=1}^{\infty} \mu(\{i\}) \) is 1. Let \( f_n \) be the function on 
\( T \to R \) given by

\[
\begin{align*}
  f_n &= \chi_{\{i \in T : i \leq n\}}, \quad n = 1, 2, 3, \ldots
\end{align*}
\]

If \( n \) is greater than \( m \), then \( f_n - f_m = \chi_{\{i \in T : m+1 \leq i \leq n\}} \), so that

\[
\| f_n - f_m \| = \int_T |f_n(\cdot) - f_m(\cdot)| \, d\mu = \sum_{i=m+1}^{n} \frac{1}{2^i}.
\]

Hence, the sequence \( \{f_n\}_{n=1}^{\infty} \) is norm Cauchy. Suppose that for some \( f \)
in \( K(\mu, P, R) \),

\[
\lim_{n \to \infty} \| f - f_n \| = 0.
\]

Theorem 2.14 implies that \( f_n \to f \). Let \( n_0 \) be in \( T \) and pick \( \delta \) to be a 
real number such that \( 0 < \delta < 1/2^n_0 \). There exists a positive integer

\( N \), greater than \( n_0 \), such that if \( n \geq N \), then there exists a finite

sequence \( \{F_{j}^{(n)}\}_{j=1}^{q(n)} \) in \( D(\mu, P) \) such that \( \sum_{j=1}^{q(n)} \mu(F_{j}^{(n)}) < \delta \) and

\[
|f_n(m) - f(m)| < \delta, \quad \forall m \in T - \bigcup_{j=1}^{q(n)} F_{j}^{(n)}.
\]
Since $\mu(\{n_0\}) = 1/2^{n_0} > \delta$, $n_0$ is in $T - \bigcup_{j=1}^{q(n)} F_j(n)$ for all $n \geq N$ so that

$$|f_n(n_0) - f(n_0)| < \delta, \forall n \geq N.$$ Since $N$ is greater than $n_0$, if $n \geq N$, then $f_n(n_0)$ is $1$. Hence, it follows that

$$|1 - f(n_0)| < \delta.$$ Since $\delta$ is any positive real number less than $1/2^{n_0}$, this implies that $f(n_0)$ is $1$. Since $n_0$ was arbitrary, it follows that $f(n)$ is $1$ for all $n$; that is, $f$ is the function $\chi_T$. However, $\chi_T - f_n$ is $\chi_{\{i: i > n\}}$ and this implies that

$$\|f - f_n\| = \|\chi_T - f_n\| = \mu(\{i: i > n\}) = 2 - \mu(\{i: i \leq n\}) = 2 - \sum_{i=1}^{n} \frac{1}{2^i}.$$ Hence, it follows that

$$\lim_{n \to \infty} \|f - f_n\| = 1,$$ and this is a contradiction. Therefore, $\{f_n\}_{n=1}^{\infty}$ does not converge, and $K(\mu, P, R)$ is not complete.

The next two theorems give some conditions for $f$ to be in $K(\mu, P, X)$ when $\mu$ is only finitely additive. The second theorem also has the restriction that $P$ is a proto-algebra.

Theorem 2.16. If $\mu$ is only finitely additive and if $f$ is in $K(\mu, P, X)$, then for every positive real number $\epsilon$, there exist sequences
and in $D(\mu, P)$ and a positive real number $S(\epsilon)$ such that is empty for all $i$ and $j$, and

a) \[
\sum_{j=1}^{m} \mu(G_j(\epsilon)) < \epsilon,
\]

b) for $i = 1, 2, \ldots, q$, \[|f(t_i) - f(t_j)| < \epsilon, \quad \forall t, t_2 \in E_i(\epsilon),\]

c) \[|f(t)| < \epsilon, \quad \forall t \in T - \left( \bigcup_{i=1}^{q} E_i(\epsilon) \bigcup_{j=1}^{m} G_j(\epsilon) \right),\]

\[\text{and}\]

d) \[|f(t)| < S(\epsilon), \quad \forall t \in T - \left( \bigcup_{j=1}^{m} G_j(\epsilon) \right).\]

Proof: Suppose that $f$ is in $K(\mu, P, X)$ and let $\epsilon$ be a positive real number. Let $g$ be an element of $S_1(P, X)$ and \[\left\{ G_j(\epsilon) \right\}_{j=1}^{m} \] a sequence in $D(\mu, P)$ such that \[\sum_{j=1}^{m} \mu(G_j(\epsilon)) < \epsilon\] and

\[|f(t) - g(t)| < \frac{\epsilon}{2}, \quad \forall t \in T - \left( \bigcup_{j=1}^{m} G_j(\epsilon) \right).\]

Let the $P$-simple function $g$ be given by \[\sum_{i=1}^{s} a_i \chi_{A_i}\] where \(\left\{ A_i \right\}_{i=1}^{s}\) is in $D(\mu, P)$. Since \[\left\{ G_j(\epsilon) \right\}_{j=1}^{m}\] is a finite sequence, Theorem 1.1 implies that $A_i - \left( \bigcup_{j=1}^{m} G_j(\epsilon) \right)$ is the union of a (finite) sequence \(\left\{ H_{i,k} \right\}_{k=1}^{p(i)}\) in $D(\mu, P)$ for $i = 1, 2, \ldots, s$. Let \(\left\{ E_i(\epsilon) \right\}_{i=1}^{q}\) be the collection \(\left\{ H_{i,k} : i = 1, 2, \ldots, s; j = 1, 2, \ldots, p(i) \right\}\) ordered as a sequence. If $t_1$ and $t_2$ are in $E_i(\epsilon)$ for $1 \leq i \leq q$, then

\[|f(t_1) - f(t_2)| \leq |f(t_1) - g(t_1)| + |g(t_1) - g(t_2)| + |g(t_2) - f(t_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\]

Since $g$ is in $S_1(P, X)$, the supremum $M$ of the set \(\left\{ |g(t)| : t \in T \right\}\) is
finite. Let $S(e)$ be $M + e$. Then it follows that

$$|f(t)| \leq |f(t) - g(t)| + |g(t)|$$

$$< e + M = S(e), \forall t \in T \subseteq \left( \bigcup_{j=1}^{m} G_j(e) \right).$$

Also, if $t$ belong to $T - \left[ (\bigcup_{i=1}^{q} E_i(e)) \bigcup (\bigcup_{j=1}^{m} G_j(e)) \right]$, then $g(t)$ is zero, so that

$$|f(t)| = |f(t) - g(t)| < \varepsilon.$$ 

This completes the proof.

Theorem 2.17. If $P$ is a proto-algebra, $\mu$ is only finitely additive and $f$ is $\mu$-equivalent to a bounded function, then $f$ is in $K(\mu, P, X)$ if and only if for every positive real number $\varepsilon$, there exist sequences $\Delta_1 = \left\{ G_j(e) \right\}_{j=1}^{m}$ and $\Delta_2 = \left\{ E_i(e) \right\}_{i=1}^{n}$ in $D(\mu, P)$ such that the non-empty elements of $\Delta_1 \cup \Delta_2$ form a $P$-partition of $T$ and such that

i) $\sum_{j=1}^{m} \mu(G_j(e)) < \varepsilon$, and

ii) $|f(t_1) - f(t_2)| < \varepsilon, \forall t_1, t_2 \in E_i(e), i = 1, 2, ..., n$.

Proof: Suppose $f$ is in $K(\mu, P, X)$ and let $\varepsilon$ be a positive real number. Theorem 2.16 implies the existence of $\Delta_1 = \left\{ G_j(e) \right\}_{j=1}^{m}$ and

$\Delta_2' = \left\{ E_i(e) \right\}_{i=1}^{q}$ in $D(\mu, P)$ such that

a) $\sum_{j=1}^{m} \mu(G_j(e)) < \frac{\varepsilon}{2} < \varepsilon$,

b) $|f(t_1) - f(t_2)| < \frac{\varepsilon}{2} < \varepsilon, \forall t_1, t_2 \in E_i(e), i = 1, 2, ..., q,$
c) \(|f(t)| < \frac{\varepsilon}{2}, \forall t \in T - (\bigcup_{i=1}^{q} E_i(\varepsilon))\).

Since \(P\) is a proto-algebra, there exists \(\{A_i\}_{i=1}^{k}\) in \(D(\mu, P)\) such that
\[
T = \bigcup_{i=1}^{k} A_i.
\]
Therefore, the difference \(T - \bigcup_{j=1}^{m} (\bigcup_{i=1}^{q} G_j(\varepsilon_i) \cup (\bigcup_{i=1}^{q} E_i(\varepsilon_i)))\) is given by
\[
\bigcup_{i=1}^{m} (A_i - \bigcup_{j=1}^{m} (G_j(\varepsilon_i) \cup (E_i(\varepsilon_i))))
\]
and
\[
A_i - \bigcup_{j=1}^{m} (G_j(\varepsilon_i) \cup (E_i(\varepsilon_i)))
\]
is the union of a sequence \(\{B_{ij}\}_{j=1}^{p(i)}\) in \(D(\mu, P)\) for \(i = 1, 2, \ldots, k\). Since the sequence \(\{A_i\}_{i=1}^{k}\) is in \(D(\mu, P)\), the collection \(\mathcal{D}\) defined by
\[
\mathcal{D} = \{B_{ij}: j = 1, 2, \ldots, p(i); i = 1, 2, \ldots, k\},
\]
is a pairwise disjoint collection. Let \(\Delta_2'\) be the collection \(\mathcal{D}\) ordered as a sequence and let \(E_1(\varepsilon), E_2(\varepsilon), \ldots, E_n(\varepsilon)\) be the elements of \(\Delta_2'\). If \(\Delta_2\) is the sequence \(\{E_i(\varepsilon)\}_{i=1}^{n}\) in \(D(\mu, P)\), then it is clear that the non-empty elements of \(\Delta_1 \cup \Delta_2\) form a \(P\)-partition of \(T\). If \(t_1\) and \(t_2\) are in \(E_i(\varepsilon)\) for \(i = q + 1, q + 2, \ldots, n\), then by c), it follows that
\[
|f(t_1) - f(t_2)| \leq |f(t_1)| + |f(t_2)| < \varepsilon.
\]

Suppose now that for every positive real number \(\varepsilon\), there exists \(\Delta_1 = \{G_j(\varepsilon)\}_{j=1}^{m}\) and \(\Delta_2 = \{E_i(\varepsilon)\}_{i=1}^{n}\) such that the non-empty elements of \(\Delta_1 \cup \Delta_2\) form a \(P\)-partition of \(T\) and such that statements i) and ii) are satisfied. For each positive integer \(k\), let \(e_k\) be \(2^{-k}\) and let \(\Delta_1(k) = \{G_j(\varepsilon_k)\}_{j=1}^{m(k)}\) and \(\Delta_2(k) = \{E_i(\varepsilon_k)\}_{i=1}^{n(k)}\) be the sequences associated with \(e_k\). Let \(\Gamma'(k)\) be the \(P\)-partition of \(T\) determined by the non-empty elements of \(\Delta_1(k) \cup \Delta_2(k)\). Since \(T\) is non-empty, at least one of
\( \Delta_1(k) \) and \( \Delta_2(k) \) must contain a non-empty set. However, one of \( \Delta_1(k) \) and \( \Delta_2(k) \) may contain only the empty set for some values of \( k \). Therefore, if \( \Gamma'(k) \) contains only elements of \( \Delta_1(k) \) or only elements of \( \Delta_2(k) \), then let \( \Gamma(k) = \Gamma'(k) \cup \{ \emptyset \} \); otherwise, \( \Gamma(k) \) is defined to be \( \Gamma'(k) \). Let \( \Gamma(k) \) be the sequence \( \{ H_i(k) \}_{i=1}^{p(k)} \) and let the elements of \( \Gamma(k) \) be indexed in a manner such that

\[
H_i(k) \in \Delta_1(k) \text{ for } i = 1, 2, \ldots, r(k), \text{ and } \\
H_i(k) \in \Delta_2(k) \text{ for } i = r(k) + 1, \ldots, p(k).
\]

Suppose now that \( \Omega' \) is a \( P \)-partition of \( T \) that refines \( \Gamma'(k) \) for some fixed integer \( k \). If \( \Gamma(k) \) is \( \Gamma'(k) \), then let \( \Omega \) be \( \Omega' \); otherwise, \( \Omega \) is defined to be \( \Omega' \cup \{ \emptyset \} \). Let \( \Omega \) be the sequence \( \{ F_j \}_{j=1}^{s} \) and let \( \{ F_{ij} \}_{j=1}^{s(i)} \) be the subcollection of \( \Omega \) such that \( H_i(k) \) is \( \bigcup_{j=1}^{s(i)} F_{ij} \). Then it follows that

\[
\sum_{i=1}^{r(k)} \sum_{j=1}^{s(i)} \mu(F_{ij}) = \sum_{i=1}^{r(k)} \mu(H_i(k)) = \sum_{j=1}^{m(k)} \mu(G_j^\varepsilon(k)) < 2^{-k}, \text{ and}
\]

\[
|f(t_1) - f(t_2)| < 2^{-k}, \quad \forall t_1, t_2 \in F_{ij}, j = 1, 2, \ldots, s(i); i = r(k) + 1, \ldots, p(k),
\]

where the last inequality follows because \( F_{ij} \) is a subset of some element of \( \Delta_2(k) \) for the values of \( i \) and \( j \) indicated. In general, suppose \( \Omega' \) is a refinement of the \( P \)-partition determined by \( \Delta_1 \) and \( \Delta_2 \) sequences satisfying i) and ii). If \( \Omega \) is \( \Omega' \) in case neither \( \Delta_1 \) nor \( \Delta_2 \) contains only the empty set and \( \Omega \) is \( \Omega' \cup \{ \emptyset \} \) otherwise, then there
exist subcollections $\Omega_1$ and $\Omega_2$ of $\Omega$ such that $\Omega$ is $\Omega_1 \cup \Omega_2$, and $\Omega_1$ and $\Omega_2$ ordered as sequences satisfy i) and ii), respectively.

Let $\xi'(1)$ be $\Gamma'(1)$ and $\xi(1)$ be $\mathcal{I}(1)$. For each positive integer $k$ greater than 1, let $\xi'(k)$ be a $\Pi$-partition of $\mathcal{T}$ which refines $\xi'(k - 1)$ and $\Gamma'(k)$. If $\mathcal{T}(k)$ is $\Gamma'(k)$, then let $\xi(k)$ be $\xi'(k)$; otherwise, $\xi(k)$ is defined to be $\xi'(k) \cup \{\varnothing\}$. Let $\xi(k)$ be the sequence $\{F_j(k)\}_{j=1}^{w(k)}$ and assume this sequence is indexed in such a manner that for a fixed integer $e(k)$ with $1 \leq e(k) \leq w(k)$, the subcollection $\xi_1(k) = \{F_j(k)\}_{j=1}^{e(k)}$ of $\xi(k)$ satisfies condition i) and the subcollection $\xi_2(k) = \{F_j(k)\}_{j=e(k)+1}^{w(k)}$ of $\xi(k)$ satisfies condition ii). Let $\zeta$ be any choice function on $\mathcal{P} - \{\varnothing\}$; that is, if $E$ is in $\mathcal{P} - \{\varnothing\}$, then $\zeta(E)$ is an element of $E$. Define the function $g_k$ on $\mathcal{T}$ to $X$ by

$$g_k = \sum_{j=1}^{w(k)} f(\zeta(F_j(k))) \chi_{F_j}(k), k = 1, 2, 3, \ldots,$$

so that $g_k$ is clearly in $S_1(\mathcal{P}, X)$. If $f$ is $\mu$-equivalent to the bounded function $g$ on $\mathcal{T}$ to $X$, then $\overline{f}$ is $\overline{g}$. Hence, it is sufficient to prove the theorem under the assumption that $f$ is bounded. This condition on $f$ will be assumed. The proof is completed by showing that the sequence $\{g_k\}_{k=1}^{\infty}$ is norm Cauchy and that it $\mu$-converges to $\overline{f}$.

If $\{E_i\}_{i=1}^{n}$ and $\{F_j\}_{j=1}^{m}$ are $\Pi$-partitions of $\mathcal{T}$, then $\sum_{i=1}^{n} \mu(E_i)$ and $\sum_{j=1}^{m} \mu(F_j)$ are equal, so there exists a real number $S_1$ such that $S_1$ exceeds $\sum_{i=1}^{n} \mu(E_i)$ for all $\Pi$-partitions $\{E_i\}_{i=1}^{n}$. Since $f$ is bounded, there exists a real number $S$, greater than $S_1$, such that

$$|\overline{f}(t)| < S, \forall t \in \mathcal{T}.$$
It should be kept in mind throughout the remainder of this proof that for each \( g \), if \( t \) is in \( T \), then \( g(t) \) is the function \( f \) evaluated at some point \( t \) in \( T \). Also, since \( \mu \) is only finitely additive, all sequences in \( D(\mu, P) \) are finite. Let \( \left\{ \left( \sum_{j=1}^{m} p_{j}(n,m) \right) \right\} \) be a sequence in \( D(\mu, P) \) such that

\[
\bigcup_{j=1}^{p(n,m)} C_{j}(n,m) = \left( \bigcup_{j=1}^{e(n)} F_{j} \right) \bigcup \left( \bigcup_{j=1}^{e(m)} F_{j} \right),
\]

and let \( T - \left( \bigcup_{j=1}^{p(n,m)} C_{j}(n,m) \right) \) be given by \( \bigcup_{i=1}^{s(n,m)} H_{i}(n,m) \), where \( \{ H_{i}(n,m) \} s(n,m) \) is in \( D(\mu, P) \). If \( t \) is in \( \bigcup_{j=1}^{p(n,m)} C_{j}(n,m) \), then

\[
|g_{n}(t) - g_{m}(t)| < 2S, \forall n, m.
\]

Let \( \varepsilon \) be a positive real number and pick \( N \) to be a positive integer such that \( 2^{-N} < \varepsilon/7S \). Suppose that \( n > m > N \) and that \( t \) is in \( \bigcup_{i=1}^{s(n,m)} H_{i}(n,m) \).

Then for some integers \( k_{1} \) and \( k_{2} \) with \( e(n) + 1 \leq k_{1} \leq w(n) \) and \( e(m) + 1 \leq k_{2} \leq w(m) \), \( t \) is in \( F_{k_{1}}(n) \cap F_{k_{2}}(m) \). Since \( \phi'(n) \) refines \( \phi'(m) \), it follows that \( F_{k_{2}}(m) \) contains \( F_{k_{1}}(n) \). Hence, the inequalities follow:

\[
|g_{n}(t) - g_{m}(t)| \leq |g_{n}(t) - f(\phi_{k_{1}}^{(n)})| + |f(\phi_{k_{1}}^{(n)}) - f(\phi_{k_{2}}^{(m)})| + |f(\phi_{k_{2}}^{(m)}) - g_{m}(t)|
\]

\[
< 2^{-n} + 2^{-m} + 2^{-m} < 3.2^{-m}.
\]

Therefore, pointwise on \( T \),

\[
|g_{n}(\cdot) - g_{m}(\cdot)| \leq 2S \cdot \sum_{j=1}^{p(n,m)} \chi_{C_{j}(n,m)} + 3 \cdot 2^{-m} \sum_{i=1}^{s(n,m)} \chi_{H_{i}(n,m)}.
\]
This implies that
\[ \| \tilde{g}_n - \tilde{g}_m \| \leq 2S \left( 2^{-n} + 2^{-m} \right) + 3 \cdot 2^{-m} \cdot S < 7S \cdot 2^{-m} < 7S \cdot 2^{-N} < \epsilon. \]

Therefore, the sequence \( \{ \tilde{g}_n \}_{n=1}^\infty \) is norm Cauchy.

It remains to show that \( \{ \tilde{g}_n \}_{n=1}^\infty \) \( \mu \)-converges to \( \overline{f} \). Let \( \epsilon \) be a positive real number and pick the positive integer \( N \) such that \( 2^{-N} < \epsilon \).

If \( n \geq N \), then
\[ \sum_{j=1}^{e(n)} \mu(F_j^{(n)}) < 2^{-n} < \epsilon, \text{ and if } t \text{ is in } T - \left( \bigcup_{j=1}^{e(n)} F_j^{(n)} \right), \]
then \( t \) is in \( F_{k_0}^{(n)} \) for some integer \( k_0 \) with \( e(n) + 1 \leq k_0 \leq v(n) \). Hence, it follows that for \( n \geq N \),
\[ |g_n(t) - f(t)| = |f(\tilde{\phi}(F_{k_0}^{(n)})) - f(t)| < 2^{-n} < \epsilon, \forall t \in T - \left( \bigcup_{j=1}^{e(n)} F_j^{(n)} \right). \]

Therefore, the sequence \( \{ \tilde{g}_n \}_{n=1}^\infty \) \( \mu \)-converges to \( \overline{f} \), and this completes the proof.

It should be noted that in each of the two preceding theorems, the restriction that \( \mu \) is only finitely additive is necessary for the proofs given, since a difference \( A - (\bigcup_{i=1}^\infty B_i) \), where \( A, B_1, B_2, \ldots \) are in \( P \), is not necessarily a countable union of elements of \( P \). As an example of where this fails, let \( P \) be the proto-algebra of Example 1.1.

Let \( \{ q_i \}_{i=1}^\infty \) be the set of all rational numbers in \([0, 1]\). Then \((0, 1)\) is in \( P \) and \( \{ q_i \}_{i=1}^\infty \) is in \( P \) for \( i = 1, 2, 3, \ldots \), but \((0, 1) - (\bigcup_{i=1}^\infty \{ q_i \}) \) is not a countable union of elements of \( P \). Suppose it is true that \( P \) is a
proto-ring of sets having the property that if $A, B_1, B_2, B_3, \ldots$ are in $P$, then $A - \bigcup_{i=1}^{\infty} B_i$ is a countable union of elements of $P$. The proof given for Theorem 2.16 is then valid without the restriction that $\mu$ be only finitely additive. In particular, it may be noted that Theorem 2.16 is true without the restriction on $\mu$ if $P$ is a $\sigma$-algebra of sets. This added restriction on $P$, even if one allows countably infinite $P$-partitions, does not seem to be enough to permit the removal of the restriction on $\mu$ in Theorem 2.17. One runs into difficulty in trying to define the sequence $\{\delta_k\}_{k=1}^{\infty}$ in this case.

If $P$ is a proto-algebra and if $\eta$ is a choice function on $P - \{\emptyset\}$, then the $\eta$-integral of $f$ with respect to the $p$-volume $\mu$ is the limit under refinement, when it exists, of sums of the form $\sum_{j=1}^{s} f(\eta(F_j))\mu(F_j)$ where $\{F_j\}_{j=1}^{s}$ is a partition of $T$. This integral will be denoted $\int_T f d\mu$.

Corollary 2.17.1. If $\mu$ is only finitely additive, $P$ is a proto-algebra, $\bar{f}$ is in $K(\mu, P, X)$ and $f$ is $\mu$-equivalent to a bounded function, then $f$ is $\eta$-integrable for all choice functions $\eta$ on $P - \{\emptyset\}$ and $\int_T \bar{f} d\mu = \int_T f d\mu$.

Proof: Assume $f$ is $\mu$-equivalent to the bounded function $f^1$ on $T$ to $X$ and let $S$ be as in the proof of Theorem 2.17. That is, $\sum_{i=1}^{n} \mu(E_i) < S$ for all $P$-partitions $\{E_i\}_{i=1}^{n}$ and $|f^1(t)| < S$, $\forall t \in T$.

Let $\varepsilon$ be a positive real number and $\bar{\eta} = \{F_j\}_{j=1}^{P}$ the $P$-partition in the
statement of Theorem 2.17 corresponding to $\epsilon/4S$ with $\{F_j\}_{j=1}^n$ the $\Lambda_1$ sequence and $\{F_j\}_{j=n+1}^m$ the $\Lambda_2$ sequence. Suppose that $\Gamma = \{G_i\}_{i=1}^m$ is a $P$-partition that refines $\xi$. Let $\{G_{ji}\}_{i=1}^{s(j)}$ be the subcollection of $\Gamma$ such that $F_j = \bigcup_{i=1}^{s(j)} G_{ji}$, for $j = 1, 2, \ldots, p$. Then $|f_1(t) - f_1(\psi(G_{ji}))| < 2S$ for all $t$ in $T$ and $G_{ji}$ for $i = 1, 2, \ldots, s(j)$ and $j = 1, 2, \ldots, p$. Also, $|f_1(t) - f_1(\psi(G_{ji}))| < \epsilon/2S$ for $t$ in $G_{ji}$, $i = 1, 2, \ldots, s(j)$ and $j = n + 1, \ldots, p$. Let $g$ be the $P$-simple function

$$g = \sum_{i=1}^m f_1(\psi(G_{ji}))\chi_{G_{ji}}.$$

Then it follows that

$$\left| \int_T f_0 d\mu - \sum_{i=1}^m f_1(\psi(G_{ji}))\mu(G_{ji}) \right| = \left| \int_T f_0 d\mu - \int_T g d\mu \right|$$

$$= \left| \int_T (f_0 - g) d\mu \right|$$

$$\leq \int_T |f_0(\cdot) - g(\cdot)| \, d\mu$$

$$\leq 2S \cdot \sum_{j=1}^n \sum_{i=1}^{s(j)} \mu(G_{ji}) + \frac{\epsilon}{2S} \cdot \sum_{j=n+1}^p \sum_{i=1}^{s(j)} \mu(G_{ji})$$

$$< 2S \cdot S + \frac{\epsilon}{2S} \cdot S = \epsilon.$$

Therefore, $\left(\int_T f_0 d\mu\right)$ exists and is equal to $\int_T f_0 d\mu$. This completes the proof.

It should be noted that the choice of the $P$-partition $\xi$ in the proof of the corollary was independent of the choice function $\psi$. In this sense, the function $\tilde{f}$ in the statement of the corollary is
uniformly integrable with respect to the set of all choice functions on $P - \{\emptyset\}$. 
III. THE SPACE $H(T, P)$ AND THE INTEGRAL FOR P-VOLUMES OF BOUNDED VARIATION

It is assumed throughout this chapter that $T$ is a non-void set and that $P$ is a proto-ring of subsets of $T$ with the property that $\bigcup \{E:E \in P\}$ is $T$. The set functions considered in this chapter are $p$-volumes defined on $P$ and if the supremum of the set

$$\left\{ \sum_{i=1}^{n} |\mu(E_i)| : \left\{ F_i \right\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\}$$

is finite, then $\mu$ is said to be of bounded variation. The set of all $p$-volumes on $P$ that are of bounded variation is denoted by $H(T, P)$. The space $H(T, P)$ is of course a linear space under pointwise operations.

All of the $p$-volumes considered in this chapter are assumed to be of bounded variation. Finally, as in the previous chapter, $(X, | |)$ is a Banach space with norm $| |$, and $\mathbb{R}$ denotes the real numbers with the usual norm.

It may be noted that if $P$ is a proto-algebra and if $\mu$ is a non-negative $p$-volume, then $\mu$ is in $H(T, P)$. This observation follows from the fact that if $\left\{ E_i \right\}_{i=1}^{n}$ and $\left\{ F_j \right\}_{j=1}^{m}$ are finite sequences in $D(\mu, P)$ with $\bigcup_{i=1}^{n} E_i$ contained in $\bigcup_{j=1}^{m} F_j$, then

$$\sum_{i=1}^{n} \mu(E_i) \leq \sum_{j=1}^{m} \mu(F_j).$$

This implies that if $\left\{ E_i \right\}_{i=1}^{n}$ and $\left\{ F_j \right\}_{j=1}^{m}$ are $P$-partitions of $T$, then

$$\sum_{i=1}^{n} \mu(E_i) = \sum_{j=1}^{m} \mu(F_j),$$

and that the supremum of the set
\[ \left\{ \sum_{k=1}^{q} \mu(G_k) : \{G_k\}_{k=1}^{q} \text{ is a finite sequence in } D(\mu, P) \right\} \]

is less than or equal to \( \sum_{i=1}^{n} \mu(E_i) \) where \( \{E_i\}_{i=1}^{n} \) is a \( P \)-partition of \( T \).

If \( P \) is only a proto-ring, then it need not be true that every non-negative \( p \)-volume belongs to \( H(T, P) \). An example of this is provided by Lebesgue measure defined on the Lebesgue measurable subsets of \( R \).

Suppose now that \( \mu \) belongs to \( H(T, P) \) and that \( \{E_i\}_{i=1}^{n} \) is a finite sequence in \( D(\mu, P) \). Since \( \mu(E_i), i = 1, 2, \ldots, n \), satisfies the inequalities

\[ -|\mu(E_i)| \leq \mu(E_i) \leq |\mu(E_i)|, \]

it follows that

\[ -\sum_{i=1}^{n} |\mu(E_i)| \leq \sum_{i=1}^{n} \mu(E_i) \leq \sum_{i=1}^{n} |\mu(E_i)|. \]

Therefore, if \( E \) is in \( P \), then both the infimum and the supremum of the set

\[ \left\{ \sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \text{ and} \right\} \]

\[ \bigcup_{i=1}^{n} E_i \subseteq E \right\} \]

are finite real numbers since \( \mu \) is in \( H(T, P) \).

**Definition 3.1.** If \( \mu \) is in \( H(T, P) \) and if \( E \) is in \( P \), then the functions \( \mu^+ \) and \( \mu^- \) are defined on \( P \) to \( R \) by the equations

\[ \mu^+(E) = \sup \left\{ \sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\} \]

\[ \mu^-(E) = \inf \left\{ \sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\} \]
\[ \mu^-(E) = \inf \left\{ \sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\} \]

and \[ \mathcal{E} \in \bigcup_{i=1}^{n} E_i \subseteq E \}

\[ \mu^+(E) = \inf \left\{ \sum_{i=1}^{n} \mu(E_i) : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\} \]

and \[ \mathcal{E} \in \bigcup_{i=1}^{n} E_i \subseteq E \}

It should be noted that since \( \{0\} \) belongs to \( D(\mu, P) \), it follows that \( \mu^+(E) \geq 0 \) and that \( \mu^-(E) \leq 0 \) for all \( E \) in \( P \).

Suppose now that \( \mu \) is in \( H(T, P) \) and that \( \{F_j\}_{j=1}^{n} \) is in \( D(\mu, P) \). Assume also that \( \bigcup_{j=1}^{n} F_j \) belongs to \( P \). Let \( \varepsilon \) be a positive real number and for \( j = 1, 2, \ldots, n \), choose a finite sequence \( \{E_{ji}\}_{i=1}^{k(j)} \) in \( D(\mu, P) \) such that \( \bigcup_{i=1}^{k(j)} E_{ji} \subseteq F_j \) and

\[ \mu^+(F_j) - \frac{\varepsilon}{2^j} < \sum_{i=1}^{k(j)} \mu(E_{ji}) \leq \mu^+(F_j). \]

Since for each positive integer \( m, m \leq n \), the collection

\[ \{E_{ji} : i = 1, 2, \ldots, k(j); j = 1, 2, \ldots, m\} \]

ordered as a sequence is a finite sequence in \( D(\mu, P) \) and \( \bigcup_{j=1}^{m} \bigcup_{i=1}^{k(j)} E_{ji} \subseteq \bigcup_{j=1}^{m} F_j \), it follows from Definition 3.1 that

\[ (a) \sum_{j=1}^{m} \sum_{i=1}^{k(i)} \mu(E_{ji}) \leq \mu^+ \left( \bigcup_{j=1}^{n} F_j \right), \text{ for all positive integers } m, m \leq n. \]

Hence, it follows from this that

\[ \sum_{j=1}^{n} \sum_{i=1}^{k(i)} \mu(E_{ji}) \leq \mu^+ \left( \bigcup_{j=1}^{n} F_j \right) \]

where this inequality clearly follows from (a) if \( n \) is a positive integer,
and if \( n = \infty \), it follows from (a) by taking the limit of the left-hand side of (a) as \( m \to \infty \). Therefore, it is true that
\[
\sum_{j=1}^{n} \mu^+(F_j) - \varepsilon < \sum_{j=1}^{n} \sum_{i=1}^{k(j)} \mu(E_{j,i}) \leq \mu^+(\bigcup_{j=1}^{n} F_j).
\]

Since \( \varepsilon \) was an arbitrary positive real number, this implies that
\[
(1) \quad \sum_{j=1}^{n} \mu^+(F_j) \leq \mu^+(\bigcup_{j=1}^{n} F_j)
\]

for all \( \{F_j\}_{j=1}^{n} \) in \( D(\mu, P) \) for which \( \bigcup_{j=1}^{n} F_j \) is in \( P \). Suppose now that \( \{F_j\}_{j=1}^{n} \) is in \( D(\mu, P) \), \( \bigcup_{j=1}^{n} F_j \) is in \( P \) and \( \varepsilon \) is a positive real number.

Choose a finite sequence \( \{G_k\}_{k=1}^{q} \) in \( D(\mu, P) \) such that \( \bigcup_{k=1}^{q} G_k \subseteq \bigcup_{j=1}^{n} F_j \) and
\[
(2) \quad \mu^+(\bigcup_{j=1}^{n} F_j) - \varepsilon < \sum_{k=1}^{q} \mu(G_k) \leq \mu^+(\bigcup_{j=1}^{n} F_j).
\]

For \( k = 1, 2, \ldots, q \) and \( j = 1, 2, \ldots, n \), let \( \{E_{kji}\}_{i=1}^{p(k,j)} \) be a finite sequence in \( D(\mu, P) \) such that
\[
G_k \cap F_j = \bigcup_{i=1}^{p(k,j)} E_{kji}.
\]

It follows from this that
\[
(3) \quad G_k = \bigcup_{j=1}^{n} (G_k \cap F_j) = \bigcup_{j=1}^{n} \bigcup_{i=1}^{p(k,j)} E_{kji}, \quad k = 1, 2, \ldots, q,
\]

and
\[
F_j \supseteq \bigcup_{k=1}^{q} (F_j \cap G_k) = \bigcup_{k=1}^{q} \bigcup_{i=1}^{p(k,j)} E_{kji}, \quad j = 1, 2, \ldots, n.
\]

Furthermore, for \( j = 1, 2, \ldots, n \), the collection
\[
\{E_{kji} : i = 1, 2, \ldots, p(k, j); k = 1, 2, \ldots, q\}
\]
ordered as a sequence is a finite sequence in \( D(\mu, F) \), so it follows
that

\[
(d) \quad \sum_{k=1}^{q} \sum_{i=1}^{p(k,j)} \mu(E_{kji}) \leq \mu^+(F_j), \quad j = 1, 2, \ldots, n.
\]

Also, since \( \mu \) is in \( H(T, P) \), it follows in the case that \( n = \infty \) that the
monotone increasing sequence

\[
\left( \sum_{j=1}^{m} \sum_{k=1}^{q} \sum_{i=1}^{p(k,j)} |\mu(E_{kji})| \right)_{m=1}^{\infty}
\]

is bounded above. Therefore, in the case \( n = \infty \), the series

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{q} \sum_{i=1}^{p(k,j)} \mu(E_{kji})
\]

is absolutely convergent. Hence, in any case
it is true that

\[
(e) \quad \sum_{j=1}^{n} \sum_{k=1}^{q} \sum_{i=1}^{p(k,j)} \mu(E_{kji}) = \sum_{k=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{p(k,j)} \mu(E_{kji}).
\]

Therefore, statements (b), (c), (e) and (d), in that order, yield the
relations

\[
\mu^+ \left( \bigcup_{j=1}^{n} F_j \right) - \varepsilon < \sum_{k=1}^{q} \mu(E_k) = \sum_{k=1}^{q} \sum_{j=1}^{n} \sum_{i=1}^{p(k,j)} \mu(E_{kji}) = \sum_{j=1}^{n} \sum_{k=1}^{q} \sum_{i=1}^{p(k,j)} \mu(E_{kji}) \leq \sum_{j=1}^{n} \mu^+(F_j).
\]

Since \( \varepsilon \) was an arbitrary positive real number, it follows that

\[
(2) \quad \mu^+ \left( \bigcup_{j=1}^{n} F_j \right) \leq \sum_{j=1}^{n} \mu^+(F_j).
\]
for all \( \{ F_j \}_{j=1}^n \) in \( D(\mu, P) \) for which \( \bigcup_{j=1}^n F_j \) is in \( P \). Finally, statements (1) and (2) give the result that if \( \mu \) is in \( H(T, P) \) and \( \{ F_j \}_{j=1}^n \) is in

\[
D(\mu, P) \text{ with } \bigcup_{j=1}^n F_j \text{ in } P, \text{ then } \\
\sum_{j=1}^n \mu^+(F_j) = \mu^+\left( \bigcup_{j=1}^n F_j \right).
\]

The result that \( \sum_{j=1}^n \mu^-(F_j) = \mu^-(\bigcup_{j=1}^n F_j) \) for \( \mu \) in \( H(T, P) \) and \( \{ F_j \}_{j=1}^n \) in \( D(\mu, P) \) with \( \bigcup_{j=1}^n F_j \) in \( P \), may be obtained in a similar manner, or by noting that \( \mu^- = \mu^+ - [\mu^+] \).

Suppose now that \( \mu \) is in \( H(T, P) \), \( E \) is in \( P \) and \( \varepsilon \) is a positive real number. Choose a finite sequence \( \{ E_i \}_{i=1}^n \) in \( D(\mu, P) \) such that

\[
\bigcup_{i=1}^n E_i \subseteq E
\]

and

\[
\mu^+(E) - \varepsilon < \sum_{i=1}^n \mu(E_i) \leq \mu^+(E).
\]

Let \( \{ E_i \}_{i=n+1}^q \) be a finite sequence in \( D(\mu, P) \) such that \( E - \left( \bigcup_{i=1}^n E_i \right) \) is

\[
\bigcup_{i=n+1}^q E_i,
\]

so that

\[
E = \bigcup_{i=1}^q E_i.
\]

It follows that \( \mu(E) = \sum_{i=1}^q \mu(E_i) \) and

\[
\mu^-(E) \leq \sum_{i=n+1}^q \mu(E_i).
\]

Therefore, it is true that

\[
\mu^+(E) + \mu^-(E) - \varepsilon < \sum_{i=1}^n \mu(E_i) + \sum_{i=n+1}^q \mu(E_i) = \mu(E),
\]
and since $\varepsilon$ was arbitrary, this shows that

$$\mu^+(E) + \mu^-(E) \leq \mu(E), \forall E \in \mathcal{P}.$$ 

Now suppose $E$ is in $\mathcal{P}$ and $\varepsilon$ is a positive real number. Choose a finite sequence $\{F_j\}_{j=1}^m$ in $D(\mu, \mathcal{P})$ such that $\bigcup_{j=1}^m F_j \subset E$ and

$$\sum_{j=1}^m \mu(F_j) < \mu^-(E) + \varepsilon.$$

Let $\{F_j\}_{j=m+1}^r$ be a finite sequence in $D(\mu, \mathcal{P})$ such that $E - \bigcup_{j=1}^m F_j$ is

$$\bigcup_{j=m+1}^r F_j,$$ so that

$$E = \bigcup_{j=1}^r F_j$$

and

$$\sum_{j=m+1}^r \mu(F_j) \leq \mu^+(E).$$

Then it follows that

$$\mu(E) = \sum_{j=1}^r \mu(F_j)$$

$$= \sum_{j=1}^m \mu(F_j) + \sum_{j=m+1}^r \mu(F_j)$$

$$< \mu^+(E) + \mu^-(E) + \varepsilon.$$ 

Hence, it is true that

$$\mu(E) \leq \mu^+(E) + \mu^-(E), \forall E \in \mathcal{P},$$

and since the reverse inequality is also true,

$$\mu(E) = \mu^+(E) + \mu^-(E), \forall E \in \mathcal{P}.$$
So far it has been shown that if $\mu$ is in $H(T, P)$, then $\mu^+$ and $\mu^-$ are $p$-volumes on $P$ and $\mu$ is $\mu^+ + \mu^-$. Also, $\mu^+$ and $\mu^-$ are countably additive if $\mu$ is countably additive. Suppose $\mu$ is in $H(T, P)$ and let $M_\mu$ be the supremum of the set

$$\left\{ \sum_{i=1}^{n} |\mu(E_i)| : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\}.$$

Let $\epsilon$ be a positive real number. Let $\{F_j\}_{j=1}^{m}$ be any finite sequence in $D(\mu, P)$ and choose a finite sequence $\{E_j\}_{j=1}^{q(j)}$ in $D(\mu, P)$ for $j = 1, 2, \ldots, m$ such that $\bigcup_{i=1}^{q(j)} E_{ji} \subseteq F_j$ and

$$\mu^+(F_j) - \frac{\epsilon}{2^j} < \sum_{i=1}^{q(j)} \mu(E_{ji}).$$

Then it follows that

$$0 \leq \mu^+(F_j) < \sum_{i=1}^{q(j)} \mu(E_{ji}) + \frac{\epsilon}{2^j},$$

so that

$$|\mu^+(F_j)| < \sum_{i=1}^{q(j)} |\mu(E_{ji})| + \frac{\epsilon}{2^j}, \quad j = 1, 2, \ldots, m.$$

This implies that

$$\sum_{j=1}^{m} |\mu^+(F_j)| < \sum_{j=1}^{m} \sum_{i=1}^{q(j)} |\mu(E_{ji})| + \epsilon$$

$$\leq M_\mu + \epsilon.$$

Since $\{F_j\}_{j=1}^{m}$ was an arbitrary finite sequence in $D(\mu, P)$, this proves that $\mu^+$ is in $H(T, P)$. Finally, since $H(T, P)$ is a linear space, if $\mu$ is in $H(T, P)$, then $\mu^- = \mu - \mu^+$ is in $H(T, P)$, also.
These results constitute a proof of the next theorem.

**Theorem 3.1.** If \( \mu \) is in \( H(T, P) \), then \( \mu^+ \) and \( \mu^- \) are in \( H(T, P) \) and \( \mu = \mu^+ + \mu^- \). Also, if \( \mu \) is in \( H(T, P) \) and \( \mu \) is countably additive, then \( \mu^+ \) and \( \mu^- \) are also countably additive.

**Definition 3.2.** If \( \mu \) is in \( H(T, P) \), then the variation function of \( \mu \), written \( \overline{\mu} \), is defined to be the difference \( \mu^+ - (\mu^-) \), or in symbols \( \overline{\mu} = \mu^+ - (\mu^-) \).

Theorem 3.1 implies that if \( \mu \) is in \( H(T, P) \), then \( \overline{\mu} \) is in \( H(T, P) \). Also, it should be noted that if \( E \) is in \( P \), then

\[
|\mu(E)| \leq |\mu^+(E)| + |\mu^-(E)|
\]

\[
= \mu^+(E) + [- (\mu^-)(E)]
\]

\[
= \overline{\mu}(E).
\]

For each \( \mu \) in \( H(T, P) \), let the real number \( M_\mu \) be defined by

\[
M_\mu = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\}.
\]

It is clear that \( M_\mu \) is non-negative and that \( M_\mu \) is zero if and only if \( \mu(E) \) is zero for all \( E \) in \( P \). Also, it is easy to see that if \( \alpha \) is in \( \mathbb{R} \), then \( M_{\alpha \mu} = |\alpha| \cdot M_\mu \). Suppose that \( \mu \) and \( \nu \) are in \( H(T, P) \). It follows that

\[
M_{\mu+\nu} = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i) + \nu(E_i)| : \{E_i\}_{i=1}^{n} \text{ is a finite sequence in } D(\mu, P) \right\}
\]

\[
\leq \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| + \sum_{i=1}^{n} |\nu(E_i)| : \{E_i\}_{i=1}^{n} \text{ is a finite sequence} \right\}
\]
Therefore, if $\delta$ is the function defined on $H(T, P)$ by

$$\delta(\mu) = M, \forall \mu \in H(T, P),$$

then $\delta$ has all of the properties of a norm function and the space $(H(T, P), \delta)$ is a normed linear space. Whenever $H(T, P)$ is referred to as a normed linear space it is to be understood that the norm is the one induced by $\delta$. It should be noted that the discussion preceding Theorem 3.1 shows that if $\mu$ is in $H(T, P)$, then $\delta(\mu^+) \leq \delta(\mu)$.

Suppose now that $\{\mu_n\}_{n=1}^{\infty}$ is a norm Cauchy sequence of elements of $H(T, P)$. Let $E$ be an element of $P$. Since

$$|\mu_n(E) - \mu_m(E)| \leq \delta(\mu_n - \mu_m),$$

it follows that the sequence $\{\mu_n(E)\}_{n=1}^{\infty}$ of real numbers is norm Cauchy and therefore has a limit in $\mathbb{R}$. Let $\mu$ be the function on $P$ to $\mathbb{R}$ defined by the equation

$$\mu(E) = \lim_{n \to \infty} \mu_n(E), \forall E \in P.$$
\[
\sum_{i=1}^{P} \lim_{n \to \infty} \mu_n(E_i)
\]

\[= \sum_{i=1}^{P} \mu(E_i). \]

Therefore, it follows that \( \mu \) is a \( p \)-volume. Suppose now that \( \{E_i\}_{i=1}^{P} \) is a finite sequence in \( D(\mu, P) \). Since \( \{\mu_n\}_{n=1}^{\infty} \) is norm Cauchy and since

\[|\delta(\mu_n) - \delta(\mu_m)| \leq \delta(\mu_n - \mu_m),\]

it follows that \( \{\delta(\mu_n)\}_{n=1}^{\infty} \) is a norm Cauchy sequence of real numbers. Hence, there is a real number \( L \) such that \( \delta(\mu_n) \leq L \) for \( n = 1, 2, 3, \ldots \). For \( i = 1, 2, 3, \ldots, p \), there exists a positive integer \( n_i \) such that

\[|\mu(E_i) - \mu(E_i)| < 2^{-i}, \forall n \geq n_i.\]

Let \( N \) be the maximum of \( n_1, n_2, \ldots, n_p \), so that if \( n \geq N \), then

\[\sum_{i=1}^{P} |\mu(E_i)| \leq \sum_{i=1}^{P} |\mu(E_i) - \mu(E_i)| + \sum_{i=1}^{P} |\mu_n(E_i)|\]

\[< \sum_{i=1}^{P} 2^{-i} + L\]

\[< 1 + L.\]

Therefore, \( \mu \) belongs to \( H(T, P) \).

Let \( \varepsilon \) be a positive real number and pick a positive integer \( N \) such that

\[\delta(\mu_n - \mu_m) < \frac{\varepsilon}{3}, \forall n, m \geq N.\]

Choose a finite sequence \( \{E_i\}_{i=1}^{P} \) in \( D(\mu, P) \) such that
For $i = 1, 2, \ldots, p$, there exists a positive integer $n_i$, $n_i > N$, such that

$$|\mu(E_i) - \mu_n(E_i)| < \frac{\varepsilon}{3} \cdot 2^{-i}, \forall n \geq n_i.$$
the norm topology, it is a complete space, and this proves the next theorem.

**Theorem 3.2.** If $H(T, P)$ is given the norm topology induced by the norm function $\delta$ on $H(T, P)$, then $H(T, P)$ is a Banach space.

The following observation gives an alternate characterization for the topology of $H(T, P)$ when $P$ is a proto-algebra. Suppose that $\delta_1$ is defined on $H(T, P)$ by the equation

$$\delta_1(\mu) = \sum_{i=1}^{m} \mu(E_i), \forall \mu \in H(T, P),$$

where $\{E_i\}_{i=1}^{m}$ is a $P$-partition of $T$. Since $\mu$ is a $p$-volume, it follows that $\delta_1$ is independent of the $P$-partition, and hence a well defined function into the non-negative reals. If $\alpha$ is in $R$ and $\mu$ is $H(T, P)$, then $\alpha \mu$ is $|\alpha|\mu$, and if $\mu$ and $\nu$ are in $H(T, P)$, then $\mu + \nu \leq \overline{\mu + \nu}$.

It follows from this that $\delta_1$ is a norm for $H(T, P)$. Also, since

$$|\mu(E)| \leq \mu(E), \forall E \in P,$$

it follows that

$$\delta(\mu) \leq \delta_1(\mu), \forall \mu \in H(T, P).$$

If $\mu$ is a non-negative $p$-volume, then the monotone property for $\mu$ implies that

$$\delta(\mu) = \sum_{i=1}^{n} \mu(E_i)$$

where $\{E_i\}_{i=1}^{n}$ is a $P$-partition on $T$. This implies that $\delta(\mu)$ is $\delta_1(\mu)$ for non-negative $\mu$ in $H(T, P)$. Also, it should be noted that $\delta_1(\mu)$ is
\( \delta_1(\mu) = \delta(\mu) \) and that \( \delta(\mu^+) \leq \delta(\mu) \) for all \( \mu \) in \( H(T, P) \). It follows that

\[
\delta_1(\mu) = \delta_1(\mu) \\
= \delta(\mu) \\
\leq \delta(\mu + \mu) + \delta(-\mu) \\
= 2\delta(\mu^+) + \delta(\mu) \\
\leq 3\delta(\mu), \forall \mu \in H(T, P).
\]

Hence, it has been shown that

\[
\delta(\mu) \leq \delta_1(\mu) \leq 3\delta(\mu), \forall \mu \in H(T, P).
\]

In particular, the topology induced on \( H(T, P) \) by \( \delta_1 \) is equivalent to the topology determined by \( \delta \).

The assumption that \( P \) is a proto-algebra is now dropped. Suppose now that \( \{\mu_n\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( H(T, P) \), each of which is countably additive. Suppose also that \( \mu \) in \( H(T, P) \) is the limit of the sequence \( \{\mu_n\}_{n=1}^{\infty} \), so that

\[
\mu(E) = \lim_{n \to \infty} \mu_n(E), \forall E \in P.
\]

Let \( \{E_i\}_{i=1}^{\infty} \) be a sequence of pairwise disjoint elements of \( P \) such that

\[
E = \bigcup_{i=1}^{\infty} E_i \text{ is in } P. \text{ Let } \varepsilon \text{ be a positive real number and choose the positive integer } N \text{ such that}
\]

\[
\delta(\mu - \mu_n) < \frac{\varepsilon}{3}, \forall n \geq N.
\]
Since $\mu_N$ is countably additive, it follows that
\[ \mu_N \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu_N(E_i), \]
so the series $\sum_{i=1}^{\infty} \mu_N(E_i)$ converges. Let $N_1$ be an integer such that
\[ \sum_{i=m+1}^{\infty} \mu_N(E_i) < \frac{\varepsilon}{3}, \forall m \geq N_1. \]

It follows then that
\[
\begin{align*}
\left| \mu(E) - \sum_{i=1}^{m} \mu(E_i) \right| &\leq \left| \mu(E) - \mu_N(E) \right| + \left| \mu_N(E) - \sum_{i=1}^{m} \mu(E_i) \right| \\
&= \left| \mu(E) - \mu_N(E) \right| + \left| \sum_{i=1}^{\infty} \mu_N(E_i) - \sum_{i=1}^{m} \mu(E_i) \right| \\
&\leq \left| \mu(E) - \mu_N(E) \right| + \sum_{i=1}^{m} \left| \mu_N(E_i) - \mu(E_i) \right| \\
&\quad + \left| \sum_{i=m+1}^{\infty} \mu_N(E_i) \right| \\
&< \delta(\mu - \mu_N) + \delta(\mu - \mu_N) + \left| \sum_{i=m+1}^{\infty} \mu_N(E_i) \right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \forall m \geq N_1.
\end{align*}
\]

Therefore, it is true that
\[ \mu \left( \bigcup_{i=1}^{\infty} E_i \right) = \sum_{i=1}^{\infty} \mu(E_i), \]
and $\mu$ is countably additive. Since the difference of two countably additive p-volumes is a countably additive p-volume, it follows that the set of all countably additive elements in $H(T, \mathcal{P})$ is a closed subspace of $H(T, \mathcal{P})$ relative to the norm topology. This subspace of countably
additive $p$-volumes in $H(T, P)$ is denoted by $H_c(T, P)$.

**Definition 3.3.** If $v$ and $\mu$ are in $H(T, P)$, then $v$ is said to be absolutely continuous with respect to $\mu$, written $v \ll \mu$, if and only if for every positive real number $\varepsilon$, there exists a positive real number $\delta$ such that if $\{E_i\}_{i=1}^n$ is in $D(\mu, P)$ with $\sum_{i=1}^n \mu(E_i) < \delta$, then $\sum_{i=1}^n \nu(E_i) < \varepsilon$.

**Theorem 3.3.** Suppose that $v$ and $\mu$ are in $H(T, P)$. Then $v \ll \mu$ if and only if $v^+ \ll \mu$ and $v^- \ll \mu$.

**Proof:** If $\pi$ is in $H(T, P)$ and if $\{E_i\}_{i=1}^n$ is in $D(\mu, P)$, then

$$\sum_{i=1}^n \nu^+(E_i) = \sum_{i=1}^n \nu^-(E_i) \leq \sum_{i=1}^n \nu(E_i),$$

and

$$\sum_{i=1}^n \nu^-(E_i) = -\sum_{i=1}^n \nu^+(E_i) \leq \sum_{i=1}^n \nu(E_i).$$

It is clear from this that if $v \ll \mu$, then $v^+ \ll \mu$ and $v^- \ll \mu$.

Suppose now that $v^+ \ll \mu$ and $v^- \ll \mu$. Let $\varepsilon$ be a positive real number and pick $\delta$, a positive real number, such that if $\{E_i\}_{i=1}^n$ is in $D(\mu, P)$ with $\sum_{i=1}^n \mu(E_i) < \delta$, then

$$\sum_{i=1}^n \nu^+(E_i) < \frac{\varepsilon}{2},$$

and

$$\sum_{i=1}^n \nu^-(E_i) < \frac{\varepsilon}{2}.$$

It follows from this that if $\{E_i\}_{i=1}^n$ is in $D(\mu, P)$ with $\sum_{i=1}^n \mu(E_i) < \delta$, then

$$\sum_{i=1}^m \nu(E_i) = \sum_{i=1}^m \nu^+(E_i) - \sum_{i=1}^m \nu^-(E_i).$$
\[= \sum_{i=1}^{n} \overline{\nu}(E_i) + \sum_{i=1}^{n} \overline{\nu}(E_i)\]
\[< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\]

Therefore, it follows that \(\nu \ll \mu\), and this completes the proof.

**Theorem 3.4.** Suppose that \(\mu\) and \(\nu\) are in \(H(T, P)\) and \(\mu\) is countably additive. If \(\nu \ll \mu\), then \(\nu\) is countably additive.

**Proof:** Note first that \(\nu\) is countably additive if each of \(\nu^+\) and \(\nu^-\) are countably additive. Also, if \(\pi\) is in \(H(T, P)\), then \(\overline{\pi}\) and \(-\pi\) are the same, so that \(\nu^- \ll \mu\) if and only if \(-\nu^- \ll \mu\). Therefore, in view of Theorem 3.3, it is sufficient to prove the theorem under the assumption that \(\nu\) is non-negative. This assumption on \(\nu\) is made for the rest of the proof. In particular, this implies that \(\nu\) and \(\overline{\nu}\) are the same \(p\)-volume.

Suppose that \(\{E_i\}_{i=1}^{\infty}\) is in \(D(\mu, P)\) and that \(E = \bigcup_{i=1}^{\infty} E_i\) is in \(P\). Let \(\varepsilon\) be a positive real number and choose a positive real number \(\delta\) such that if \(\{F_j\}_{j=1}^{m}\) is in \(D(\mu, P)\) and \(\sum_{j=1}^{m} \overline{\nu}(F_j) < \delta\), then \(\sum_{j=1}^{m} \nu(F_j) < \varepsilon\).

Since \(\overline{\mu}(E) = \sum_{i=1}^{\infty} \overline{\mu}(E_i)\), the series \(\sum_{i=1}^{\infty} \overline{\mu}(E_i)\) converges and so there is a positive integer \(N\) such that
\[\sum_{i=n+1}^{\infty} \overline{\mu}(E_i) < \delta, \ \forall n \geq N.\]
This implies that
\[\overline{\mu}(E) - \sum_{i=1}^{n} \overline{\mu}(E_i) < \delta, \ \forall n \geq N.\]
For each positive integer \(n\), let \(\{E_{nj}\}_{j=1}^{q(n)}\) be a finite sequence in \(D(\mu, P)\).
such that

\[ E - \left( \bigcup_{i=1}^{n} E_i \right) = \bigcup_{j=1}^{q(n)} E_{nj}. \]

It follows from this that

\[ E = \left( \bigcup_{i=1}^{n} E_i \right) \bigcup \left( \bigcup_{j=1}^{q(n)} E_{nj} \right), \]

and the right hand side is a finite union of pairwise disjoint elements of \( P \) for \( n = 1, 2, 3, \ldots \). Therefore, it follows that

\[ \mu(E) = \sum_{j=1}^{q(n)} \mu(E_{nj}) = \mu(E) - \sum_{i=1}^{n} \mu(E_i) \]

\[ < \delta, \forall n \geq N, \]

and this implies that

\[ |\psi(E) - \sum_{i=1}^{n} \psi(E_i)| = \psi(E) - \sum_{i=1}^{n} \psi(E_i) \]

\[ = \sum_{j=1}^{q(n)} \psi(E_{nj}) \]

\[ < \epsilon, \forall n \geq N. \]

Hence, \( \psi(E) \) is \( \sum_{i=1}^{\infty} \psi(E_i) \) and \( \psi \) is countably additive. This completes the proof.

The integral will now be defined for \( \mu \) in \( H(T, P) \) and then relationships between indefinite integrals and absolute continuity will be studied in some detail. The following observation will be needed in these considerations.

Suppose that \( \mu \) and \( \nu \) are non-negative \( p \)-volumes on \( P \). It should be
observed that if \( \mu + \nu \) is countably additive, then each of \( \mu \) and \( \nu \) is also countably additive. This can be seen as follows. First, the monotone property of Theorem 1.4 shows that a non-negative \( p \)-volume \( \mu \) is always super-additive in the sense that if \( \{E_i\}_{i=1}^n \) is a sequence in \( D(\pi, \mathcal{P}) \) for some countably additive \( p \)-volume \( \pi \) and \( \bigcup_{i=1}^n E_i \) is in \( \mathcal{P} \), then

\[
\mu(\bigcup_{i=1}^n E_i) \geq \sum_{i=1}^n \mu(E_i).
\]

Suppose now that \( \mu + \nu \) is countably additive and that \( \mu \) is not countably additive. Then there exists a sequence in \( D(\mu + \nu, \mathcal{P}) \) such that

\[
\bigcup_{i=1}^\infty E_i \text{ is in } \mathcal{P} \quad \text{and} \quad \sum_{i=1}^\infty \mu(E_i) < \mu\left(\bigcup_{i=1}^\infty E_i\right).
\]

This implies that

\[
\sum_{i=1}^\infty (\mu + \nu)(E_i) = \sum_{i=1}^\infty \mu(E_i) + \sum_{i=1}^\infty \nu(E_i) < \mu\left(\bigcup_{i=1}^\infty E_i\right) + \nu\left(\bigcup_{i=1}^\infty E_i\right)
\]

\[
= (\mu + \nu)(\bigcup_{i=1}^\infty E_i),
\]

so that \( \mu + \nu \) is not countably additive. This is a contradiction, so it follows that if \( \mu \) and \( \nu \) are non-negative \( p \)-volumes and if \( \mu + \nu \) is countably additive, then \( \mu \) and \( \nu \) are both countably additive. This in turn implies that \( D(\mu + \nu, \mathcal{P}) \) is always a subset of \( D(\mu, \mathcal{P}) \) and \( D(\nu, \mathcal{P}) \).

Suppose now that \( f \) is a function on \( T \) to \( X \) for which \( f \) is in \( K(\mu + \nu, \mathcal{P}, X) \). Since \( f \) is in \( K(\mu + \nu, \mathcal{P}, X) \), there exists a sequence \( \{f_n\}_{n=1}^\infty \) in \( S(\mu + \nu, \mathcal{P}, X) \) that is norm Cauchy in \( K(\mu + \nu, \mathcal{P}, X) \) and \( (\mu + \nu) \)-convergent.
to \( f \). Since \( D(\mu + \nu, P) \) is a subset of \( D(\mu, P) \) and \( D(\nu, P) \) and since \( \nu(E) \) and \( \mu(E) \) are less than or equal to \((\nu + \mu)(E)\) for all \( E \) in \( P \), it follows that \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges and \( \nu \)-converges to \( f \). Also, since each \( f_n \) is a \( P \)-simple function, it is clear that \( \{f_n\}_{n=1}^{\infty} \) is in \( S(\mu, P, X) \) and \( S(\nu, P, X) \). Finally, if \( g \) is a \( P \)-simple function, then

\[
\int_T \overline{gd}(\mu + \nu) = \int_T \overline{gd}\mu + \int_T \overline{gd}\nu.
\]

In particular, it follows that if \( \|g\|_{\mu+\nu} \) and \( \|g\|_\nu \) are the norms for \( K(\mu + \nu, P, X) \), \( K(\mu, P, X) \) and \( K(\nu, P, X) \), respectively, then

\[
\| \overline{f_n} - \overline{f_m} \|_{\mu+\nu} = \int_T |\overline{f_n}(\cdot) - \overline{f_m}(\cdot)|d(\mu + \nu) = \int_T |\overline{f_n}(\cdot) - \overline{f_m}(\cdot)|d\mu + \int_T |\overline{f_n}(\cdot) - \overline{f_m}(\cdot)|d\nu = \| \overline{f_n} - \overline{f_m} \|_\mu + \| \overline{f_n} - \overline{f_m} \|_\nu.
\]

Therefore, the sequence \( \{f_n\}_{n=1}^{\infty} \) is norm Cauchy in each of the spaces \( K(\mu, P, X) \) and \( K(\nu, P, X) \). Then it follows that \( f \) belongs to \( K(\mu, P, X) \) and \( K(\nu, P, X) \), and

\[
\int_T \overline{fd}(\mu + \nu) = \int_T \overline{fd}\mu + \int_T \overline{fd}\nu.
\]

Suppose now that \( \mu \) is a non-negative \( p \)-volume. The space \( K_1(\mu, P, X) \) is defined to be the set of all functions \( f \) on \( T \) to \( X \) for which \( \overline{f} \) is in \( K(\mu, P, X) \). The integral and the function \( \| \|_\mu \) are defined on \( K_1(\mu, P, X) \) by the equations
\[ \int_T f d\mu = \int_T \overline{f} d\mu, \ \forall \mu \in \mathcal{K}_1(\mu, P, X), \ \text{and} \]

\[ \|f\|_\mu = \|\overline{f}\|_\mu, \ \forall \mu \in \mathcal{K}_1(\mu, P, X). \]

In view of the properties already derived for \( \mathcal{K}(\mu, P, X) \), it is clear that the function \( \|\|_\mu \) on \( \mathcal{K}_1(\mu, P, X) \) is a seminorm and that the integral is a continuous linear transformation into \( X \) relative to the seminorm topology. Also, it is clear that \( \mathcal{K}(\mu, P, X) \) is the quotient space obtained from \( \mathcal{K}_1(\mu, P, X) \) by identifying functions \( f \) and \( g \) in \( \mathcal{K}_1(\mu, P, X) \) for which \( \|f - g\|_\mu \) is zero.

If \( \mu \) is in \( \mathcal{H}(T, P) \), the space \( \mathcal{K}_1(\mu, P, X) \) is defined to be the space \( \mathcal{K}_1(||\mu||, P, X) \). Since \( \mu \) is the sum of two non-negative \( p \)-volumes \( \mu^+ \) and \( -(\mu^-) \), if \( f \) is in \( \mathcal{K}_1(\mu, P, X) \), then \( f \) is in \( \mathcal{K}_1(\mu^+, P, X) \) and \( \mathcal{K}_1(-(\mu^-), P, X) \). Therefore, the integral is defined on \( \mathcal{K}_1(\mu, P, X) \) by the equation

\[ \int_T f d\mu = \int_T f d\mu^+ - \int_T f d(\mu^-), \ \forall \mu \in \mathcal{K}_1(\mu, P, X), \]

and the seminorm \( \|\|_\mu \) is defined by

\[ \|f\|_\mu = \|f\|_\mu^-, \ \forall \mu \in \mathcal{K}_1(\mu, P, X). \]

Since \( \mu^- \) is non-positive, it follows that \( (\mu^-)^+(E) \) is zero for all \( E \) in \( P \), and therefore that

\[ \mu^- = (\mu^-)^+ + (\mu^-)^- \]

\[ = 0 + (\mu^-)^- \]

\[ = (\mu^-)^-. \]
This implies that
\[ \int_\mathcal{T} f d\mu (\mu^-) = \int_\mathcal{T} f d[\mu^-] + \int_\mathcal{T} f d[- \mu^-] \]
\[ = 0 - \int_\mathcal{T} f d(- \mu^-) \]
\[ = - \int_\mathcal{T} f d(- \mu^-), \forall \mu \in K_1 (\mu, P, X). \]

Therefore, the integral on \( K_1 (\mu, P, X) \) is given by
\[ \int_\mathcal{T} f d\mu = \int_\mathcal{T} f d\mu^+ + \int_\mathcal{T} f d\mu^-, \forall \mu \in K_1 (\mu, P, X). \]

The seminorm on \( K_1 (\mu, P, X) \) is given by
\[ \| f \|_\mu = \| f \|_{\mu^+} \]
\[ = \int_\mathcal{T} |f(\cdot)| d\mu, \forall \mu \in K_1 (\mu, P, X). \]

It follows readily from this that if \( f \) and \( g \) are in \( K_1 (\mu, P, X) \), then \( \| f - g \|_\mu \) is zero if and only if \( f \sim g \). Hence, if \( K(\mu, P, X) \) is the quotient space of \( K_1 (\mu, P, X) \) determined by identifying functions \( f \) and \( g \) in \( K_1 (\mu, P, X) \) for which \( \| f - g \|_\mu \) is zero, then \( K(\mu, P, X) \) is the space \( K(\mu, P, X) \). Finally, the integral and norm are defined on \( K(\mu, P, X) \) by defining the integral over \( \mathcal{T} \) of \( \bar{f} \) in \( K(\mu, P, X) \) to be the integral over \( \mathcal{T} \) of any \( g \) in \( K_1 (\mu, P, X) \) for which \( g \) is in \( \bar{f} \) and the norm of \( \bar{f} \) is defined to be the seminorm of any \( g \) in \( K_1 (\mu, P, X) \) for which \( g \) is in \( \bar{f} \); in symbols,
\[
\int_T \bar{f} d\mu = \int_T f d\mu, \forall \bar{f} \in \mathcal{K}(\mu, P, X), \text{ and}
\]

\[
\|\bar{f}\|_\mu = \|f\|_\mu, \forall \bar{f} \in \mathcal{K}(\mu, P, X).
\]

It may be noted that, if \(\bar{f}\) is in \(\mathcal{K}(\mu, P, X)\), then

\[
\int_T \bar{f} d\mu = \int_T \bar{f}^+ d\mu + \int_T \bar{f}^- d\mu, \text{ and}
\]

\[
\|\bar{f}\|_\mu = \|f\|_\mu
\]

\[
= \int_T |f(\cdot)| \, d\mu
\]

\[
= \int_T |f(\cdot)| \, d\mu^+ + \int_T |f(\cdot)| \, d\mu^- (\mu^-)
\]

\[
= \|\bar{f}\|_{\mu^+} + \|\bar{f}\|_{\mu^-}.
\]

Also, it follows that

\[
|\int_T \bar{f} d\mu| = |\int_T \bar{f}^+ d\mu + \int_T \bar{f}^- d\mu|
\]

\[
\leq |\int_T \bar{f}^+ d\mu| + |\int_T \bar{f}^- d\mu|
\]

\[
\leq \|\bar{f}\|_{\mu^+} + \|\bar{f}\|_{\mu^-}
\]

\[
= \|\bar{f}\|_\mu, \forall \bar{f} \in \mathcal{K}(\mu, P, X),
\]

so the integral on \(\mathcal{K}(\mu, P, X)\) is a continuous linear transformation into \(X\).

The phrase "almost uniform with respect to \(\mu\)" means "almost uniform with respect to \(\bar{\mu}\)" as previously defined and "almost everywhere with
respect to \( \mu \) means "almost everywhere with respect to \( \tilde{\mu} \)" as previously defined. Also, a subset \( B \) of \( T \) is said to be \( \mu \)-null if and only if it is \( \tilde{\mu} \)-null. The terms "\( \mu \)-equivalent" and "\( \mu \)-convergent" are defined to mean "\( \tilde{\mu} \)-equivalent" and "\( \tilde{\mu} \)-convergent," respectively. It should be noted that this means, for example, two functions \( f \) and \( g \) on \( T \) to \( X \) are \( \mu \)-equivalent, denoted \( f \sim \tilde{\mu} g \), if and only if for every positive real number \( \epsilon \), there exists \( \left\{ E_i \right\}_{i=1}^{n} \) in \( D(\mu, P) \) such that \( \sum_{i=1}^{n} \tilde{\mu}(E_i) < \epsilon \) and

\[
|f(t) - g(t)| < \epsilon, \forall t \in T \setminus \left( \bigcup_{i=1}^{n} E_i \right).
\]

Similarly, the sequences for \( \mu \)-convergence and \( \mu \)-null are drawn from \( D(\mu, P) \) and not \( D(\tilde{\mu}, P) \). However, it turns out that \( D(\mu, P) \) and \( D(\tilde{\mu}, P) \) are always equal. This is clear if \( \mu \) is countably additive, for then \( \tilde{\mu} \) is also countably additive. It was observed earlier that if the sum of two non-negative \( \mu \)-volumes is countably additive, then each of the summands is countably additive. Therefore, if \( \tilde{\mu} \) is countably additive, then \( \mu^+ \) and \( (\mu^-) \) are also countably additive. This implies that \( \mu \) is countably additive if and only if \( \tilde{\mu} \) is countably additive. Hence, \( D(\mu, P) \) and \( D(\tilde{\mu}, P) \) are the same for all \( \mu \in H(T, P) \).

It is then easy to check that if \( \mu \) is replaced by \( \tilde{\mu} \) in the conclusion of the statement of Theorem 2.10, in a) of Theorem 2.16 and in i) of Theorem 2.17, then Theorems 2.9-2.17 and Corollary 2.17.1 are valid for \( K(\mu, P, X) \) with \( \mu \in H(T, P) \).

If \( \mu \) is in \( H(T, P) \), \( E \) is in \( P \) and \( \tilde{f} \) is in \( K(\mu, P, X) \), then the notation \( \int_{E} \tilde{f} d\mu \) stands for \( \int_{T} \chi_{E} \cdot \tilde{f} d\mu \). The integral \( \int_{T} \chi_{E} \tilde{f} d\mu \) will normally be written in the form \( \int_{T} \chi_{E} \tilde{f} d\mu \).
Definition 3.4. If $\mu$ is in $H(T, P)$ and $\bar{f}$ is in $K(\mu, P, R)$, then the statement that $\nu_{\bar{f}}$ is the indefinite integral of $\bar{f}$ means that $v_f$ is the set function defined on $P$ by the equation

$$v_f(E) = \int_E \bar{f}d\mu, \forall E \in P.$$ 

If $f$ and $g$ are $\mu$-equivalent functions on $T$ to $R$ and if $\bar{f}$ is in $K(\mu, P, R)$, then it is clear from Definition 3.4 that $\nu_{\bar{f}} = \nu_g$. Therefore, if $\mu$ is in $H(T, P)$ and $\bar{f}$ is in $K(\mu, P, R)$, then $\nu_{\bar{f}}$ will be written as $v_{\bar{f}}$.

Since $\chi_{E \cup F} = \chi_E + \chi_F$ whenever $E$ and $F$ are disjoint, it is clear that $v_{\bar{f}}$ is a $p$-volume for each $\bar{f}$ in $K(\mu, P, R)$. If $\bar{f}$ is in $K(\mu, P, R)$ and $\{E_i\}_{i=1}^n$ is a finite sequence in $D(v_{\bar{f}}, P)$, then

$$\sum_{i=1}^n |v_{\bar{f}}(E_i)| = \sum_{i=1}^n \left| \int_{E_i} \bar{f}d\mu \right|$$

$$\leq \sum_{i=1}^n \int_{E_i} |\bar{f}(\cdot)|d\mu$$

$$\leq \|\bar{f}\|_\mu,$$

so that $v_{\bar{f}}$ is in $H(T, P)$.

Theorem 3.5. If $\{\bar{f}_n\}_{n=1}^\infty$ is a norm Cauchy sequence of elements of $K(\mu, P, R)$, then the sequence $\{v_{\bar{f}_n}\}_{n=1}^\infty$ is uniformly absolutely continuous with respect to $\mu$ and for $n = 1, 2, 3, \ldots$,

$$\nu_{\bar{f}_n}(E) = \int_E |\bar{f}_n(\cdot)|d\bar{\mu}, \forall E \in P.$$ 

The proof of Theorem 3.5 will be accomplished by establishing
the next four theorems.

**Theorem 3.6.** If \( \overline{f} \) is in \( S(\mu, P, R) \), then \( \nu_f \ll \mu \) and

\[
\overline{\nu}_f(E) = \int_E |f(\cdot)| \, d\overline{\mu}, \quad \forall E \in P.
\]

Proof: Since \( \nu_f = \nu_g \) for all \( g \) in \( \overline{f} \), it is sufficient to assume that \( f \) is in \( S_1(P, R) \). Let \( f \) be given by

\[
f = \sum_{i=1}^{n} a_i \chi_{A_i},
\]

where \( \{A_i\}_{i=1}^{n} \) is a finite sequence in \( D(\mu, P) \). Let \( E \) be in \( P \) and for \( i = 1, 2, \ldots, n \), choose a finite sequence \( \{A_{ij}\}_{j=1}^{k(i)} \) in \( D(\mu, P) \) such that

\[
E \cap A_i = \bigcup_{j=1}^{k(i)} A_{ij}.
\]

Since each \( A_{ij} \) is a subset of \( E \), it follows that

\[
\chi_{A_{ij}} \cdot f = \chi_{A_{ij}} \cdot \chi_E \cdot f = a_i \chi_{A_{ij}}, \quad i = 1, 2, \ldots, n;
\]

\[
j = 1, 2, \ldots, k(i).
\]

This implies that \( \nu_f(A_{ij}) \) is \( a_i \mu(A_{ij}) \) for each \( A_{ij} \), so that

\[
\overline{\nu}_f(A_{ij}) = a_i \mu(A_{ij})
\]

\[
= |a_i| \cdot \overline{\mu}(A_{ij})
\]

\[
= \int_{A_{ij}} |\overline{f}(\cdot)| \, d\overline{\mu}, \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, k(i).
\]
Let \( E = ( \bigcup_{i=1}^{n} \bigcup_{j=1}^{k(i)} A_{ij} ) \) be the union of a finite sequence \( \{ E_p \}_{p=1}^{r} \) in \( D(\mu, P) \), so that \( f \) restricted to \( \bigcup_{p=1}^{r} E_p \) is zero. Then it follows that

\[
\nu_f(E) = \sum_{i=1}^{n} \sum_{j=1}^{k(i)} \nu_f(A_{ij}) + \sum_{p=1}^{r} \nu_f(B_p)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{k(i)} \int_{A_{ij}} |f(\cdot)|d\mu + 0
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{k(i)} \int_{A_{ij}} \chi_{A_{ij}} \cdot |f(\cdot)|d\mu + \sum_{p=1}^{r} \int_{E_p} \chi_{E_p} \cdot |f(\cdot)|d\mu
\]

\[
= \int_{E} \chi_{E} \cdot |f(\cdot)|d\mu
\]

Since \( f \) is in \( S_1(P, R) \), there is a positive real number \( c \) such that

\[
c > |f(t)|, \quad \forall t \in T.
\]

Let \( \varepsilon \) be a positive real number, and suppose the sequence \( \{ E_i \}_{i=1}^{q} \) is in \( D(\mu, P) \) with

\[
\sum_{i=1}^{q} \mu(E_i) < \frac{\varepsilon}{c}.
\]

Then it follows that
\[
\sum_{i=1}^{q} v_{f}(E_{i}) = \sum_{i=1}^{q} \int_{E_{i}} |\tilde{f}(\cdot)| d\mu \\
\leq \sum_{i=1}^{q} c \cdot \bar{\mu}(E_{i}) \\
\leq c \cdot \frac{\varepsilon}{c} = \varepsilon,
\]
so that \( v_{f} \ll \mu \). This completes the proof.

**Theorem 3.7.** If \( \{f_{n}\}_{n=1}^{\infty} \) is a norm Cauchy sequence of elements of \( S(\mu, P, R) \), then the sequence \( \{v_{f_{n}}\}_{n=1}^{\infty} \) is uniformly absolutely continuous with respect to \( \mu \).

**Proof:** Let \( \varepsilon \) be a positive real number and choose a positive integer \( n_0 \) such that

\[
\|\bar{f}_{n} - \bar{f}_{m}\|_{\mu} < \frac{\varepsilon}{2}, \quad \forall n, m \geq n_0.
\]

Since \( v_{f_{n}} \), \( n = 1, 2, \ldots, n_0 \), is absolutely continuous with respect to \( \mu \), there exists a positive real number \( \varepsilon_{1} \) such that if \( \{E_{i}\}_{i=1}^{q} \) is in \( D(\mu, P) \) and \( \sum_{i=1}^{q} \bar{\mu}(E_{i}) < \varepsilon_{1} \), then \( \sum_{i=1}^{q} v_{f_{n}}(E_{i}) < \varepsilon/2 \), \( n = 1, 2, \ldots, n_0 \).

If \( n > n_0 \) and \( \{E_{i}\}_{i=1}^{q} \) is in \( D(\mu, P) \) with \( \sum_{i=1}^{q} \bar{\mu}(E_{i}) < \varepsilon_{1} \), then

\[
\sum_{i=1}^{q} v_{f_{n}}(E_{i}) = \sum_{i=1}^{q} \int_{E_{i}} |\bar{f}_{n}(\cdot)| d\mu \\
\leq \sum_{i=1}^{q} \int_{E_{i}} |\bar{f}_{n}(\cdot) - \bar{f}_{n_0}(\cdot)| d\mu + \sum_{i=1}^{q} \int_{E_{i}} |\bar{f}_{n_0}(\cdot)| d\mu \\
\leq \|\bar{f}_{n} - \bar{f}_{n_0}\|_{\mu} + \sum_{i=1}^{q} v_{f_{n_0}}(E_{i}) \\
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This completes the proof.

**Theorem 3.8.** If $\mathcal{F}$ is in $K(\mu, P, R)$, then $\nu_{\mathcal{F}} \ll \mu$ and

$$
\nu_{\mathcal{F}}(E) = \int_E |\mathcal{F}(\cdot)|d\mu, \quad \forall E \in P.
$$

**Proof:** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $S(\mu, P, R)$ such that

$$
limit_{n \to \infty} \|f_n - f\|_\mu = 0.
$$

If $\{E_i\}_{i=1}^{P}$ is a finite sequence in $D(\mu, P)$, then it follows that

$$
\left| \sum_{i=1}^{P} \nu_{f_n}(E_i) - \nu(E_i) \right| = \sum_{i=1}^{P} \left| \int_{E_i} f_n d\mu - \int_{E_i} f d\mu \right|
$$

$$
\leq \sum_{i=1}^{P} \int_{E_i} |f_n(\cdot) - f(\cdot)|d\mu
$$

$$
\leq \|f_n - f\|_\mu,
$$

so that $limit_{n \to \infty} \delta(\nu_{f_n} - \nu) = 0$. Also, for every $E$ in $P$,

$$
\left| \nu(E) - \nu_{f_n}(E) \right| \leq |\nu^+(E) - \nu_{f_n}^+(E)| + |\nu^-(E) - \nu_{f_n}^-(E)|
$$

$$
\leq 2 \cdot \delta(\nu - \nu_{f_n}).
$$

This implies that

$$
\nu(E) = \lim_{n \to \infty} \nu_{f_n}(E)
$$

$$
= \lim_{n \to \infty} \int_E |\mathcal{F}_n(\cdot)|d\mu
$$

$$
= \int_E |\mathcal{F}(\cdot)|d\mu, \quad \forall E \in P.
$$
Also, if \( \{V_i\}_{i=1}^P \) is in \( D(\mu, P) \), then
\[
\sum_{i=1}^P \nu(E_i) = \sum_{i=1}^P \int_{E_i} |\tilde{f}(\cdot)| d\mu
\leq \sum_{i=1}^P \int_{E_i} |\tilde{f}(\cdot) - \tilde{f}_n(\cdot)| d\mu + \sum_{i=1}^P \int_{E_i} |\tilde{f}_n(\cdot)| d\mu
\leq \|\tilde{f} - \tilde{f}_n\|_{\mu} + \sum_{i=1}^P \nu_f(E_i)
\]
for all positive integers \( n \). Since \( \lim_{n \to \infty} \|\tilde{f} - \tilde{f}_n\|_{\mu} = 0 \) and \( \{\nu_f\}_{n=1}^\infty \) is uniformly absolutely continuous with respect to \( \mu \), this implies that \( \nu \ll \mu \). This completes the proof.

**Theorem 3.9.** If \( \{\tilde{f}_n\}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( K(\mu, P, \mathbb{R}) \), then the sequence \( \{\nu_{\tilde{f}_n}\}_{n=1}^\infty \) is uniformly absolutely continuous with respect to \( \mu \).

The argument of Theorem 3.7 may now be used for the proof of Theorem 3.9. This completes the proof of Theorem 3.5.

If \( \{\tilde{f}_n\}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( K(\mu, P, \mathbb{R}) \), then the inequality
\[
|\tilde{f}_n(t) - \tilde{f}_n(t)| \leq |f_n(t) - f_m(t)|, \forall t \in \mathbb{T},
\]
implies that \( \{\tilde{f}_n(\cdot)\}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( K(\mu, P, \mathbb{R}) \). Therefore, the following is an immediate corollary to Theorem 3.5.

**Corollary 3.5.1.** If \( \{\tilde{f}_n\}_{n=1}^\infty \) is a norm Cauchy sequence of elements of \( K(\mu, P, \mathbb{R}) \), then the sequence \( \{\nu_f(\cdot)\}_{n=1}^\infty \) is uniformly absolutely continuous with respect to \( \mu \) and for \( n = 1, 2, 3, ... \),
Theorem 3.10. Let \( f \) be a function on \( T \) to \( X \), and suppose \( \{ f_n \}_{n=1}^{\infty} \) is a sequence of elements of \( K(\mu, P, X) \). Then \( f \) belongs to \( K(\mu, P, X) \) and \( \| f_n \|_\mu \to 0 \) if and only if

i) \( f \to f \), and

ii) \( \{ v |f_n(\cdot)| \}_{n=1}^{\infty} \) is uniformly absolutely continuous with respect to \( \mu \).

Proof: If \( f \) is in \( K(\mu, P, X) \) and \( \| f_n - f \|_\mu \to 0 \), then i) follows from Theorem 2.14 and ii) follows from Corollary 3.5.1.

Assume that i) and ii) hold. In view of the paragraph following the proof of Theorem 2.14, it suffices to show that the sequence \( \{ f_n \}_{n=1}^{\infty} \) is norm Cauchy in order to complete the proof. For each positive integer \( n \), choose \( g_n \) in \( S_1(P, X) \) such that

\[
\| f_n - g_n \|_\mu < 2^{-n},
\]

and

\( g_n \to f \).

Clearly, \( \{ f_n \}_{n=1}^{\infty} \) is norm Cauchy if \( \{ g_n \}_{n=1}^{\infty} \) is norm Cauchy. Notice also that if \( \{ E_i \}_{i=1}^{m} \) is in \( D(\mu, P) \), then

\[
\sum_{i=1}^{m} \int_{E_i} |g_n(\cdot)| \mu \leq \sum_{i=1}^{m} \int_{E_i} |f_n(\cdot)| \mu \]

where

\[
\overline{v} |f_n(\cdot)| (E) = \int_{E} |f_n(\cdot)| \mu, \forall E \in \mathcal{F}.
\]
\[ \leq \sum_{i=1}^{m} \int_{E_i} |g_n(\cdot) - f_n(\cdot)| \, d\mu + \sum_{i=1}^{m} \int_{E_i} |f_n(\cdot)| \, d\mu \]

\[ \leq \|g_n - \frac{\sum_{i=1}^{m}}{\|f_n\| \mu} \| + \sum_{i=1}^{m} \|f_n(\cdot)| (E_i). \]

Therefore, since \( \{\int_{E_i} f_n(\cdot) \}_{i=1}^{\infty} \) is uniformly absolutely continuous with respect to \( \mu \), for every positive \( \varepsilon \), there exists a positive real number \( \delta \), such that if \( \{E_i\}_{i=1}^{m} \) is in \( \mathcal{D}(\mu, P) \) and \( \sum_{i=1}^{m} \nu(E_i) < \delta \), then

\[ \sum_{i=1}^{m} \int_{E_i} |g_n(\cdot)| \, d\mu < 2^{-n} + \varepsilon. \]

Let \( g_n \) be given by

\[ g_n = \sum_{i=1}^{p(n)} a_{ni} X_{ni}, \]

where \( \{X_{ni}\}_{i=1}^{p(n)} \) is a finite sequence in \( \mathcal{D}(\mu, P) \) for \( n = 1, 2, 3, \ldots \).

For each positive integer \( n \), define the set \( B_n \) by the equation

\[ B_n = \bigcup_{j=1}^{n} \bigcup_{i=1}^{p(j)} A_{ji}. \]

Choose a finite sequence \( \{C_{ni}\}_{i=1}^{q(n)} \) in \( \mathcal{D}(\mu, P) \) such that

\[ B_n = \bigcup_{i=1}^{q(n)} C_{ni}, \quad n = 1, 2, 3, \ldots. \]

Since \( B_1 \subset B_2 \subset B_3 \subset \ldots \), it follows that the sequence \( \{\sum_{i=1}^{q(n)} \mu(C_{ni})\}_{n=1}^{\infty} \) is a monotone increasing sequence, and since \( \mu \) is in \( H(T, P) \) and \( q(n) \) is finite for each \( n \), the sequence is bounded above. Let \( \{D_{nmk}\}_{k=1}^{s(n,m)} \) be a finite sequence in \( \mathcal{D}(\mu, P) \) such that

\[ q(n) \bigcup_{k=1}^{s(n,m)} D_{nmk} = q(n) \bigcup_{i=1}^{C_{ni}} - \bigcup_{i=1}^{C_{mi}} \text{ for all pairs of positive integers } n \text{ and } m. \]

Let \( \epsilon \) be a positive real number. Since the sequence \( \{\sum_{i=1}^{q(n)} \mu(C_{ni})\}_{n=1}^{\infty} \) converges, it is Cauchy and there exists a positive integer \( N \) such that
\[ |\sum_{i=1}^{q(n)} \mu(C_{n_i}) - \sum_{i=1}^{q(m)} \mu(C_{m_i})| < \epsilon, \forall n, m \geq N. \]

Now for each \( n \), \( \chi_{B_n} \) is \( \sum_{i=1}^{q(n)} \chi_{C_{n_i}} \) and so is in \( S_1(P, X) \). Also, since \( B_1 \subset B_2 \subset \ldots \), if \( n > m \), then

\[ s(n,m) \sum_{k=1}^{s(n,m)} \mu(D_{nmk}) \]

and if \( n \leq m \), then \( \chi_{B_n} - \chi_{B_m} \) is zero. Therefore, it follows that if \( n > m \geq N \), then

\[ \left| \| \chi_{B_n} \|_\mu - \| \chi_{B_m} \|_\mu \right| \leq \| \chi_{B_n} - \chi_{B_m} \|_\mu \]

\[ = \int_{T} \chi_{B_n} - \chi_{B_m} \, d\mu \]

\[ = \sum_{k=1}^{s(n,m)} \mu(D_{nmk}) \]

\[ = \sum_{i=1}^{q(n)} \mu(C_{n_i}) - \sum_{i=1}^{q(m)} \mu(C_{m_i}) \]

\[ < \epsilon. \]

This implies there exists a positive real number \( M \) such that

\[ \| \chi_{B_n} \|_\mu \leq M, \text{ } n = 1, 2, 3, \ldots \]

Let \( \epsilon \) be a positive real number and fix the positive integer \( n_0 \) such that \( 2^{-n_0} < \epsilon/5 \). Then there exists a positive real number \( \delta \) such that

if \( \{E_i\}_{i=1}^t \) is in \( D(\mu, P) \) and \( \sum_{i=1}^t \mu(E_i) < \delta \), then \( \sum_{i=1}^t \nu_{g_n}(E_i) \)

\[ < 2^{-n_0} + 2^{-n}. \]

Since \( \{g_n\}_{n=1}^\infty \) is \( \mu \)-convergent to \( f \), it is \( \mu \)-Cauchy.

Therefore, there exists a positive integer \( N > n_0 \) such that if
n, m \geq N_j$, then there exists a sequence \( \{s(n,m)\}_{j=1}^{s(n,m)} \) in \( D(\mu, \mathcal{F}) \) such that

\[ \sum_{j=1}^{s(n,m)} \mu(F_j(n,m)) < \delta \text{ and} \]

\[ |g_n(t) - g_m(t)| < \frac{1}{2} n_0, \quad \forall t \in T - (\bigcup_{j=1}^{s(n,m)} F_j(n,m)). \]

Finally, if \( n > m \), then

\[ |g_n(\cdot) - g_m(\cdot)| = \chi_{B_n} \cdot |g_n(\cdot) - g_m(\cdot)|, \]

so that for \( n > m \geq N_n \), it follows that

\begin{align*}
\|g_n - g_m\|_{\mu} &= \int_T |g_n(\cdot) - g_m(\cdot)| \, d\mu \\
&= \int_T \chi_{B_n} \cdot |g_n(\cdot) - g_m(\cdot)| \, d\mu \\
&= \sum_{j=1}^{s(n,m)} \int_{F_j(n,m)} |g_n(\cdot) - g_m(\cdot)| \, d\mu \\
&\leq \sum_{j=1}^{s(n,m)} \int_{F_j(n,m)} |g_n(\cdot)| \, d\mu + \sum_{j=1}^{s(n,m)} \int_{F_j(n,m)} |g_m(\cdot)| \, d\mu \\
&\leq \sum_{j=1}^{s(n,m)} \left( \frac{1}{2} n_0 \cdot \mu(B_n) \right) \cdot \int_{F_j(n,m)} |g_n(\cdot)| \, d\mu \\
&\leq \sum_{j=1}^{s(n,m)} \left( \frac{1}{2} n_0 \cdot \mu(B_n) \right) \cdot \|g_n\|_{\mu} \\
&\leq 2^{-n_0} + 2^{-n} + 2^{-n_0} + 2^{-m} + 2^{-n_0} \\
&< 5 \cdot 2^{-n_0} < \varepsilon.
\end{align*}
Hence, the sequence \( \{ f_n \}_{n=1}^{\infty} \) is norm Cauchy in \( K(\mu, P, X) \), and this completes the proof.

The next theorem is the analogue to the classical dominated convergence theorem.

**Theorem 3.11.** If \( g \) is in \( K(\mu, P, X) \) and if \( \{ f_n \}_{n=1}^{\infty} \) is a sequence of elements of \( K(\mu, P, X) \) with the property that \( |g(\cdot)| \geq |f_n(\cdot)| \), \( n = 1, 2, 3, \ldots \), then \( f_n \overset{\mu}{\to} f \) if and only if \( f \) is in \( K(\mu, P, X) \) and \( \|f_n - f\| \overset{\mu}{\to} 0 \).

The relation \( \geq \) in the statement of the theorem is the partial order relation of Definition 2.9.

**Proof:** Since \( |g(\cdot)| \geq |f_n(\cdot)| \) for each \( n \), it follows that

\[
\sum_{i=1}^{q} \int_{E_i} |f_n(\cdot)| d\mu \leq \sum_{i=1}^{q} \int_{E_i} |g(\cdot)| d\mu
\]

for all \( n \) and for all sequences \( \{E_i\}_{i=1}^{q} \) in \( D(\mu, P) \). Therefore, condition ii) of Theorem 3.10 holds. Hence, if \( f_n \overset{\mu}{\to} f \), then \( f \) is in \( K(\mu, P, X) \) and \( \|f_n - f\| \overset{\mu}{\to} 0 \).

The converse is one-half of Theorem 3.10. This completes the proof.
IV. THE SPACE $Q(T, P)$, $L^p$ SPACES AND THE RADON-NIKODYM THEOREM

It is assumed throughout this chapter that $T$ is a non-void set and that $P$ is a proto-ring of subsets of $T$ with the property that $\bigcup\{E: E \in P\}$ is $T$. The space $(X, |\cdot|)$ is a real Banach space with norm $|\cdot|$, and $\mathbb{R}$ denotes the real numbers with the usual norm.

If $Y$ is a compact Hausdorff space and if $B$ is the Borel algebra of subsets of $Y$ (the smallest $\sigma$-algebra containing all of the closed sets), then it is well known that the space $H^1(Y, B)$ of real valued regular Borel measures defined on $B$, with the variation norm, is the adjoint space to the space of all real valued continuous functions on $Y$, with the supremum norm. (See, for example, [4, p. 262]). This would seem to suggest that the space $H(T, P)$ introduced in the previous chapter should be the adjoint space to some normed linear space. It turns out that this is the case.

**Definition 4.1.** A function $f$ on $T$ to $\mathbb{R}$ is said to be quasi-continuous with respect to $P$ if and only if $f$ satisfies the condition: for every positive real number $\varepsilon$, there exists a finite sequence $\{E_i\}_{i=1}^n$ of pairwise disjoint elements of $P$ such that $f$ restricted to $T - \left(\bigcup_{i=1}^n E_i\right)$ is zero and

$$|f(t_1) - f(t_2)| < \varepsilon, \forall t_1, t_2 \in E_i, i = 1, 2, \ldots, n.$$ 

The space of all functions on $T$ to $\mathbb{R}$ that are quasi-continuous with respect to $P$ will be denoted by $Q(T, P)$. When there is only one proto-ring under consideration, the elements of $Q(T, P)$ will be said to be quasi-continuous with no mention of $P$. 
The following example is the reason the term "quasi-continuous" was chosen for the elements of $Q(T, P)$.

**Example 4.1.** Let $T$ be the real number interval $[0, 1]$, and $P$ is the proto-algebra consisting of $\emptyset$, all singleton subsets of $T$ and all open subintervals of $T$. This is the proto-algebra of Example 1.1. Then Definition 4.1 is precisely the classical definition for quasi-continuity on $[0, 1]$.

The next definition extends to proto-rings the concept of refinement as previously defined for $P$-partitions by elements of a proto-algebra.

**Definition 4.2.** Suppose that $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^m$ are finite sequences of pairwise disjoint elements of $P$. The sequence $\{F_j\}_{j=1}^m$ is said to refine the sequence $\{E_i\}_{i=1}^n$ if and only if each $E_i$ is the union of the elements of some subcollection of $\{F_j\}_{j=1}^m$.

Suppose that $\{E_i\}_{i=1}^n$ and $\{F_j\}_{j=1}^m$ are finite sequences of pairwise disjoint elements of $P$. For each pair of positive integers $i$ and $j$, let $\{A_{ijk}\}_{k=1}^p(i,j)$, $\{B_{ijk}\}_{k=1}^q(i,j)$ and $\{C_{ijk}\}_{k=1}^r(i,j)$ be finite sequences of pairwise disjoint elements of $P$ such that

\[
E_i \cap F_j = \bigcup_{k=1}^{p(i,j)} A_{ijk},
\]

\[
E_i - F_j = \bigcup_{k=1}^{q(i,j)} B_{ijk}, \text{ and}
\]

\[
F_j - E_i = \bigcup_{k=1}^{r(i,j)} C_{ijk}.
\]

Let $\Gamma_{ij}$ be the collection defined by

\[
\Gamma_{ij} = \bigcup_{k=1}^{p(i,j)} A_{ijk} \cup \bigcup_{k=1}^{q(i,j)} B_{ijk} \cup \bigcup_{k=1}^{r(i,j)} C_{ijk}.
\]
and let \( \Gamma \) be \( \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} T_{ij} \). If \( \{G_k\}_{k=1}^{s} \) is the collection \( \Gamma \) ordered as a (finite) sequence, it follows that \( \{G_k\}_{k=1}^{s} \) refines each of the sequences \( \{E_i\}_{i=1}^{n} \) and \( \{F_j\}_{j=1}^{m} \). Therefore, when ordered by refinement, the collection of all finite sequences of pairwise disjoint elements of \( \mathcal{P} \) is a directed set. This directed set of sequences will be denoted by \( \Lambda \), and whenever a partial ordering on \( \Lambda \) is used, it is assumed to be the refinement ordering.

Suppose now that \( f \) belongs to \( Q(T, \mathcal{P}) \) and let \( \epsilon \) be a positive real number. Let \( \{E_i\}_{i=1}^{n} \) in \( \Lambda \) be the sequence in Definition 4.1 and suppose that \( \{F_j\}_{j=1}^{m} \) in \( \Lambda \) refines \( \{E_i\}_{i=1}^{n} \), so each \( E_i \) is the union of the elements of some subcollection \( \{F_{ij}\}_{j=1}^{q(i)} \) of \( \{F_j\}_{j=1}^{m} \). It is clear that

\[
\bigcup_{i=1}^{n} E_i \text{ is } \bigcup_{i=1}^{n} \bigcup_{j=1}^{q(i)} F_{ij} \text{ and that }
\]

\[
|f(t_1) - f(t_2)| < \epsilon, \forall t_1, t_2 \in F_{ij}, j = 1, 2, \ldots, q(i);
\]

\[
i = 1, 2, \ldots, n.
\]

Also, if \( F_j \) is contained in \( T - \bigcup_{i=1}^{n} E_i \) for some \( j \), then

\[
|f(t_1) - f(t_2)| = |0 - 0| < \epsilon, \forall t_1, t_2 \in F_j.
\]

It follows then that \( f \) restricted to \( T - \bigcup_{j=1}^{m} F_j \) is zero and

\[
|f(t_1) - f(t_2)| < \epsilon, \forall t_1, t_2 \in F_j, j = 1, 2, \ldots, m.
\]

Therefore, every refinement of a sequence satisfying the conditions of Definition 4.1 also satisfies those conditions.

Assume \( f \) is in \( Q(T, \mathcal{P}) \) and for each positive integer \( n \), choose
\(\{E_i^{(n)}\}_{i=1}^{q(n)}\) in \(\Delta\) such that \(f\) restricted to \(T - \bigcup_{i=1}^{q(n)} E_i^{(n)}\) is zero and

\[
|f(t_1) - f(t_2)| < 2^{-n}, \quad \forall t_1, t_2 \in E_i^{(n)}, \; i = 1, 2, \ldots, q(n).
\]

Define \(\{F_j^{(1)}\}_{j=1}^{p(1)}\) to be the sequence \(\{E_i^{(1)}\}_{i=1}^{q(1)}\) and for \(n = 2, 3, 4, \ldots\), choose \(\{F_j^{(n)}\}_{j=1}^{p(n)}\) in \(\Delta\) so that \(\{F_j^{(n)}\}_{j=1}^{p(n)}\) refines \(\{F_j^{(n-1)}\}_{j=1}^{p(n-1)}\) and \(\{E_i^{(n)}\}_{i=1}^{q(n)}\). Let \(\psi\) be any choice function on \(P - \{\phi\}\), and define the \(P\)-simple function \(g_n\) on \(T\) to \(X\) by

\[
g_n = \sum_{j=1}^{p(n)} f(\psi(F_j^{(n)})) \chi_{F_j^{(n)}}(n), \; n = 1, 2, 3, \ldots.
\]

If \(n\) is a positive integer and \(t\) is in \(T - \bigcup_{j=1}^{p(n)} F_j^{(n)}\) then \(f(t)\) and \(g(t)\) are both zero. If \(t\) is in \(F_j^{(n)}\) for some \(j\), then

\[
|f(t) - g_n(t)| = |f(t) - f(\psi(F_j^{(n)}))| < 2^{-n}.
\]

Therefore, it follows that

\[
|f(t) - g_n(t)| < 2^{-n}, \quad \forall t \in T,
\]

and \(f\) can be uniformly approximated by a \(P\)-simple function. Conversely, suppose \(f\) is a function on \(T\) to \(X\) that vanishes on the complement of the union of a sequence \(\{E_i\}_{i=1}^{n}\) in \(\Delta\) and that can be uniformly approximated on \(T\) by a \(P\)-simple function. Let \(\varepsilon\) be a positive real number and choose a \(P\)-simple function \(g\) on \(T\) to \(X\) such that

\[
|f(t) - g(t)| < \frac{\varepsilon}{2}, \quad \forall t \in T.
\]
Let $g$ be given by $\sum_{i=1}^{m} a_i \chi_{A_i}$ where $\{A_i\}_{i=1}^{m}$ is in $\Delta$. Then it follows that
\[
|f(t_1) - f(t_2)| \leq |f(t_1) - g(t_1)| + |g(t_1) - g(t_2)| \\
+ |g(t_2) - f(t_2)| \\
< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon, \forall t_1, t_2 \in A_i, i = 1, 2, \ldots, n.
\]

Therefore, it follows that $f$ is in $Q(T, P)$. Since $g$ is clearly bounded, it also follows from this that $f$ is bounded.

Let $\beta$ be the real valued function defined on $Q(T, P)$ to $R$ by the equation
\[
\beta(f) = \sup \{|f(t)| : t \in T\}, \forall f \in Q(T, P).
\]

It is clear that $\beta$ is a norm for $Q(T, P)$, and when $Q(T, P)$ is referred to as a normed linear space, it is to be understood that $\beta$ is the norm.

**Theorem 4.1.** If $\mu$ is in $H(T, P)$ and $f$ is in $Q(T, P)$, then $\overline{f}$ is in $K(\mu, P, R)$.

**Proof:** Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of $P$-simple functions such that
\[
|f(t) - g_n(t)| < 2^{-n}, \forall t \in T.
\]

Let $\{E_i\}_{i=1}^{s}$ in $\Delta$ be chosen so that $f$ vanishes on $T - (\bigcup_{i=1}^{s} E_i)$, and let $E$ be $\bigcup_{i=1}^{s} E_i$. Define the sequence $\{f_n\}_{n=1}^{\infty}$ of $P$-simple functions by
\[
f_n = \chi_E \cdot g_n, n = 1, 2, 3, \ldots,
\]
so that $\overline{f_n}$ is in $S(\mu, P, R)$ for all $n$. It is clear that
\[ |f(t) - f_n(t)| < 2^{-n}, \forall t \in T. \]

Therefore, it follows that if \( n > m \), then
\[
|f_n(t) - f_m(t)| \leq |f_n(t) - f(t)| + |f(t) - f_m(t)|
< 2^{-n} + 2^{-m}
< 2 \cdot 2^{-m}, \forall t \in T.
\]

Also, it is true that
\[
|f_n(t) - f_m(t)| = 0, \forall t \in T - E.
\]

Hence, it follows that if \( n > m \), then
\[
\|f_n - f_m\| = \int_{T} |f_n(\cdot) - f_m(\cdot)| d\mu
= \int_{T} \chi_E \cdot |f_n(\cdot) - f_m(\cdot)| d\mu
< 2 \cdot 2^{-m} \int_{T} \chi_E d\mu
= 2 \cdot 2^{-m} \cdot \sum_{i=1}^{\infty} \mu(E_i).
\]

Therefore, \( \{f_n\}_{n=1}^{\infty} \) is norm Cauchy and it obviously \( \mu \)-converges to \( f \), so that \( f \) is in \( K(\mu, P, R) \). This completes the proof.

The next theorem gives essentially the same result as an example of Wilansky, [10, p. 93]. It is included here for completeness.

**Theorem 4.2.** The Banach space \( H(T, P) \) is the adjoint to the normed linear space \( Q(T, P) \).
Proof: Let $\mu$ be in $H(T, P)$ and let $\mu(f)$ be $\int_T \bar{f} d\mu$ for every $f$ in $Q(T, P)$. Choose $\{E_i\}_{i=1}^n$ in $\Delta$ such that $f$ vanishes on $T - (\bigcup_{i=1}^n E_i)$ and let $E = \bigcup_{i=1}^n E_i$. It follows that

$$|\mu(f)| = |\int_T \bar{f} d\mu|$$

$$\leq \int_T |\bar{f}(\cdot)| d\mu$$

$$\leq \int_T \beta(f) \cdot \chi_E d\mu$$

$$= \beta(f) \cdot \sum_{i=1}^n \mu(E_i)$$

$$= \beta(f) \cdot (\sum_{i=1}^n \mu^+(E_i) + \sum_{i=1}^n \mu^-(E_i))$$

$$\leq \beta(f) (\delta(\mu^+) + \delta(\mu^-)), \forall \mu \in Q(T, P).$$

Therefore, $\mu$ is in $Q^*(T, P)$, the adjoint of $Q(T, P)$, and $H(T, P) \subset Q^*(T, P)$.

Suppose now that $F$ is in $Q^*(T, P)$ and define $\mu_F$ on $P$ by the equation

$$\mu_F(E) = F(\chi_E), \forall E \in P.$$

Since $F$ is additive and $\chi_{E \cup F}$ is $\chi_E + \chi_F$ for $E$ and $F$ disjoint, it follows that $\mu_F$ is a p-volume on $P$ and that $F(\chi_E)$ is $\int_T \chi_E d\mu_F$. Suppose that $\{E_i\}_{i=1}^n$ is a finite sequence in $D(\mu_F, P)$ and let $\{E_k(i)\}_{i=1}^n$ be a re-ordering of $\{E_i\}_{i=1}^n$ and $m$ an integer, $1 \leq m \leq n$, such that

$$\mu_F(E_k(i)) \geq 0, i = 1, 2, \ldots, m,$$
\[ \mu_F(P_{k(i)}) < 0, \quad i = m + 1, \ldots, n. \]

Then it follows that

\[
\sum_{i=1}^{n} |\mu_F(E_i)| = \left| \sum_{i=1}^{m} \mu_F(P_{k(i)}) - \sum_{i=m+1}^{n} \mu_F(P_{k(i)}) \right|
\]

\[
= \left| \sum_{i=1}^{m} F(\chi_{P_{k(i)}}) - \sum_{i=m+1}^{n} F(\chi_{P_{k(i)}}) \right|
\]

\[
= \left| F(\chi_{\bigcup_{i=1}^{m} P_{k(i)}}) - \chi_{\bigcup_{i=m+1}^{n} P_{k(i)}} \right|
\]

\[
\leq (Q^*(T, P) \text{ norm of } F) \cdot 1.
\]

Therefore, \( \mu_F \) is in \( H(T, P) \). Since \( \mu_F(\cdot) \) and \( F(\cdot) \) are equal on the set of all characteristic functions of sets in \( P \) and \( \mu_F(\cdot) \) and \( F(\cdot) \) are linear, they are equal on the span of the characteristic functions. But the span of the characteristic functions is dense in \( Q(T, P) \) and \( \mu_F(\cdot) \) and \( F(\cdot) \) are continuous, so that \( \mu_F(\cdot) \) and \( F(\cdot) \) are the same element of \( Q^*(T, P) \). Hence, it follows that \( Q^*(T, P) \subset H(T, P) \), and this completes the proof.

It should be observed that the space \( Q(T, P) \) need not be a complete space. For example, if \( T \) is \( R \) and \( P \) is the collection consisting of the empty set, all singleton subsets of \( T \) and all open intervals \((a, b)\), with \( a < b \) and \( a \) and \( b \) are finite real numbers, then \( P \) is a proto-ring. Then the function \( f \) defined by the equation

\[
f(t) = \begin{cases} 
\frac{1}{n}, & \text{if } t \in (-n, -n+1] \cup (n-1, n], \ n = 1, 2, 3, \ldots \\
0, & \text{otherwise}
\end{cases}
\]
is not in $Q(T, P)$, but $f$ is in the closure of $Q(T, P)$ since the sequence $\{g_n\}_{n=1}^{\infty}$ of elements of $Q(T, P)$ defined by

$$g_n = \chi_{(-n,n]} \cdot f, \ n = 1, 2, 3, \ldots$$

is norm Cauchy and converges uniformly to $f$. In fact, this behavior characterizes the closure of $Q(T, P)$ in the sense that, in general, the closure of $Q(T, P)$ consists of all functions on $T$ to $\mathbb{R}$ such that for every positive real number $\varepsilon$, there exists a sequence $\{E_i\}_{i=1}^{n}$ in $\mathcal{A}$ such that $f$ is uniformly approximated by a $\mathcal{P}$-simple function on $\bigcup_{i=1}^{n} E_i$, and

$$|f(t)| < \varepsilon, \ \forall t \in T - \bigcup_{i=1}^{n} E_i.$$  

It should be noted that $H(T, P)$ is the adjoint of the completion of $Q(T, P)$ since it is the adjoint of $Q(T, P)$. Also, it is easy to see that $Q(T, P)$ is complete if $\mathcal{P}$ is a proto-algebra.

Porcelli, [9], defines and discusses $L_p$ spaces for finitely additive measures on an algebra of sets. His definition yields complete $L_p$ spaces, and he is able to characterize the adjoint spaces to his $L_p$ spaces. Also, he obtains isometric isomorphic relations between his $L_p$ spaces and the $L_p$ spaces for countably additive measures on the Borel algebra of subsets of a compact Hausdorff space. The definition taken for the $L_p$ spaces in this dissertation will be the same as the Porcelli definition, except that the $L_p$ spaces will be defined for proto-rings rather than the algebra structure. Hence, the results included here on $L_p$ spaces are included mainly for completeness.

The compact Hausdorff space referred to earlier and the characterization
of the adjoint spaces will be obtained in a significantly different manner than that used by Porcelli. It should also be commented that Porcelli does not show that his $L_2$ space is a Hilbert space.

**Definition 4.3.** For a fixed $\mu$ in $H(T, P)$ and real number $p$, $1 \leq p < \infty$, define $L_p(T, \mu, P)$ to be the collection of all sequences \( \{f_n\}_{n=1}^\infty \) of $P$-simple functions on $T$ to $\mathbb{R}$ such that \( \lim_{n,m \to \infty} \int_T |f_n(\cdot) - f_m(\cdot)|^p d\mu = 0 \). Identify two sequences \( \{f_n\}_{n=1}^\infty \) and \( \{g_n\}_{n=1}^\infty \) as being the same if \( \lim_{n \to \infty} \int_T |f_n(\cdot) - g_n(\cdot)|^p d\mu = 0 \).

The function $\| \cdot \|_p$ defined on $L_p(T, \mu, P)$ by the equation

\[
\| \{f_n\}_{n=1}^\infty \|_p = \lim_{n \to \infty} \left( \int_T |f_n(\cdot)|^p d\mu \right)^{1/p}, \quad \forall \{f_n\}_{n=1}^\infty \in L_p(T, \mu, P)
\]

is a norm for $L_p(T, \mu, P)$, and in fact is a complete norm. These observations can perhaps be seen best by the following considerations. Suppose one defines $L_p^0(T, \mu, P)$ to be the set of all $f$ in $S(\mu, P, \mathbb{R})$ together with the real valued function $\alpha_p$ defined on $L_p^0(T, \mu, P)$ by

\[
\alpha_p(f) = \left( \int_T |f(\cdot)|^p d\mu \right)^{1/p}, \quad \forall f \in L_p^0(T, \mu, P).
\]

The arguments normally given for the Holder and Minkowski inequalities show that those inequalities are valid in the present setting and that $\alpha_p$ is a seminorm for $L_p^0(T, \mu, P)$. Hence, $L_p^0(T, \mu, P)$ is a seminormed linear space, and it is clear that $L_p(T, \mu, P)$ is obtained from $L_p^0(T, \mu, P)$ by first completing $L_p^0(T, \mu, P)$ and then identifying those elements of the completion for which the seminorm of their difference is zero.

This makes it clear that $L_p(T, \mu, P)$, $1 \leq p < \infty$, with the norm $\| \cdot \|_p$ is a complete normed linear space. The space $L_\infty(T, \mu, P)$ is defined as follows.
Definition 4.4. For a fixed $\mu$ in $H(T, P)$, the space $L^\infty(T, \mu, P)$ is defined to be the collection of all sequences $\{f_n\}_{n=1}^{\infty}$ of $P$-simple functions on $T$ to $\mathbb{R}$ such that $\left\{f_n\right\}_{n=1}^{\infty}$ is in $L_p(T, \mu, P)$ for all $p \geq 1$ and $\lim_{p \to \infty} \lim_{n \to \infty} \left( \int |f_n|^p d\mu \right)^{1/p}$ is finite.

It is easy to see that $L^\infty(T, \mu, P)$ is a linear space and that the function $\|\|_{\infty}$ defined on $L^\infty(T, \mu, P)$ by the equation

$$\|\{f_n\}_{n=1}^{\infty}\|_{\infty} = \lim_{p \to \infty} \|\{f_n\}_{n=1}^{\infty}\|_p$$

is a seminorm for $L^\infty(T, \mu, P)$ and a norm if sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are identified whenever $\|\{f_n - g_n\}_{n=1}^{\infty}\| = 0$. It will be seen later that $L^\infty(T, \mu, P)$ is the adjoint to $L_1(T, \mu, P)$ and so it will then follow that $L^\infty(T, \mu, P)$ is a Banach space.

If one defines $L^1_p(T, \mu, P)$, $1 \leq p < \infty$ to be the set of all $f$ in $K(\mu, P, \mathbb{R})$ for which $|f(\cdot)|^p$ is in $K(\mu, P, \mathbb{R})$ and defines a norm on $L^1_p(T, \mu, P)$ by

$$\alpha_p(f) = \left[ \int_T |f(\cdot)|^p d\mu \right]^{1/p},$$

then in the case that $\mu$ is countably additive, $L^1_p(T, \mu, P)$ is a complete space. This can be seen to be true by an argument much like the one in [4, p. 146] for classical $L_p$ spaces. Since $S(\mu, P, \mathbb{R})$ is dense in $K(\mu, P, \mathbb{R})$, it follows that $L_p(T, \mu, P)$ and $L^1_p(T, \mu, P)$ are the same when $\mu$ is countably additive.

The fact that $L_2(T, \mu, P)$ is a Hilbert space follows since it satisfies the parallelogram law. That is, if $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are in $L_2(T, \mu, P)$, then
It follows from this that the bilinear function \( \langle, \rangle \) on \( L^2(T, \mu, \mathbf{P}) \) defined by

\[
\langle \{ f_n \}_{n=1}^\infty, \{ g_n \}_{n=1}^\infty \rangle = \frac{1}{4} \left[ \| \{ f_n \}_{n=1}^\infty \|_2^2 + \| \{ g_n \}_{n=1}^\infty \|_2^2 - 2 \| \{ f_n \}_{n=1}^\infty \|_2 \right]
\]

is an inner product. It should be noted that if \( \mu \) is countably additive, so that \( \{ f_n \}_{n=1}^\infty \) and \( \{ g_n \}_{n=1}^\infty \) in \( L^2(T, \mu, \mathbf{P}) \) are identified with functions \( f \) and \( g \) respectively, then this gives the usual result for \( L^2 \) space that \( \langle f, g \rangle = \int_T fg \, d\mu \).
Suppose now that \( P \) is a proto-algebra of sets and that \( \mu \) is in \( H(T, P) \). Suppose also that \( Y \) is a compact Hausdorff space, \( B \) is the Borel algebra of subsets of \( Y \) and \( \nu \) is a regular Borel measure on \( B \).

If it could be shown that \( L_p(T, \mu, P) \) and \( L_p(Y, \nu, B) \) are isometrically isomorphic for all \( p, 1 \leq p \leq \infty \), then it would follow that the adjoint to \( L_p(T, \mu, P) \) is \( L_q(T, \mu, P) \) where \( p^{-1} + q^{-1} = 1 \). It turns out there is such a compact Hausdorff space. Porcelli, [9], in considering measures on an algebra of subsets \( A \) of a set \( T \), constructs a compact Hausdorff space \( Y \) by taking the collection of all ultrafilters [7, p. 116] in the algebra as the set \( Y \) and imposing a topology on \( Y \) determined by a particular map from the algebra into the set of all subsets of \( Y \).

Porcelli also suggests two other constructions that may be used to give essentially the same compact Hausdorff space. The result is that \( Q(T, A) \) and \( C(Y) \), the space of real valued continuous functions on \( Y \), are isomorphic and isometric. This in turn implies isomorphic isometric relations between \( H(T, P) \) and \( H_r(Y, B) \), the regular Borel measures on the Borel algebra of subsets of \( Y \), and then the isomorphic isometric relations between \( L_p(T, \mu, A) \) and \( L_p(Y, \mu', B) \) where \( \mu \) in \( H(T, A) \) and \( \mu' \) in \( H_r(Y, B) \) correspond. Porcelli does not present a complete proof of these isomorphic isometric relations in [9], so for completeness the corresponding results will be demonstrated in detail in this paper.

Actually, it should be pointed out that Porcelli obtains the adjoint characterizations of his \( L_p \) spaces independently of the isomorphic isometric relations for the \( L_p \) spaces.

There is a somewhat more direct method of determining an appropriate compact Hausdorff space \( Y \) than any of those suggested by Porcelli, and
that is the next thing to be considered here. It should be kept in mind that $P$ is assumed to be a proto-algebra, and this assumption holds throughout the rest of this chapter. In particular, recall that $Q(T, P)$ is then a complete space relative to the norm topology determined by the norm function $\beta$.

Let $Q(T, P)$ be given the partial order determined by $f \geq g$ if and only if $f(t) \geq g(t)$ for all $t$ in $T$ and $f$ and $g$ in $Q(T, P)$. Suppose that $f$ and $g$ are in $Q(T, P)$ and let $h$ be defined by

$$h(t) = \max(f(t), g(t)), \forall t \in T.$$  

Let $\varepsilon$ be a positive real number and \(\{E_i\}_{i=1}^{n}\) and \(\{F_j\}_{j=1}^{m}\) be $P$-partitions of $T$ such that

$$|f(t) - f(s)| < \varepsilon, \forall t, s \in E_i, i = 1, 2, \ldots, n,$$  

$$|g(t) - g(s)| < \varepsilon, \forall t, s \in F_j, j = 1, 2, \ldots, m.$$  

Let \(\{C_k\}_{k=1}^{q}\) be a $P$-partition that refines \(\{E_i\}_{i=1}^{n}\) and \(\{F_j\}_{j=1}^{m}\). Suppose $t$ and $s$ are in $C_k$ for some $k$, $1 \leq k \leq q$. If $h\{t, s\}$ is $f\{t, s\}$ or $g\{t, s\}$, then clearly

$$|h(t) - h(s)| < \varepsilon.$$  

Suppose that $h(t)$ is $f(t)$ and $h(s)$ is $g(s)$. Because of the definition of $h$ and the preceding statement, it may be assumed that $f(t) > g(t)$ and $g(s) > f(s)$. Then it follows that

$$g(s) - \varepsilon < g(t) < f(t) < f(s) + \varepsilon < g(s) + \varepsilon,$$
so that

$$- \varepsilon < f(t) - g(s) < \varepsilon.$$  

This implies that

$$|h(t) - h(s)| < \varepsilon,$$

so that $h$ is in $Q(T, P)$. Therefore, $Q(T, P)$ is a vector lattice. Also, it is clear that

$$\beta(h) = \max(\beta(f), \beta(g)),$$

so that $Q(T, P)$ is a lattice of type $M$. If $f$ and $g$ are in $Q(T, P)$ and $f \geq g$, then it follows that $\beta(f) \geq \beta(g)$, so that $Q(T, P)$ is an $M$-space, (see [7, p. 238] for definitions of a lattice of type $M$ and $M$-space).

Also, $Q(T, P)$ contains the function $i$ for which

$$i(t) = 1, \forall t \in T,$$

since $P$ is a proto-algebra. Therefore, $Q(T, P)$ is an $M$-space with unit.

Kelley and Namioka, [7, p. 242] show that each $M$-space with a unit is isomorphic and isometric, under evaluation, to the space of all continuous real valued functions on its spectrum. It may be noted that it is the requirement that $Q(T, P)$ have a unit for this result that imposes the restriction of a proto-algebra on $P$. Let $Y$ be the spectrum of $Q(T, P)$; that is, $\alpha$ is in $Y$ if and only if $\alpha$ is a continuous linear functional of norm one on $Q(T, P)$ and

$$\alpha(\max(f, g)) = \max(\alpha(f), \alpha(g)), \forall f, g \in Q(T, P).$$
Then $Y$ is known to be a compact Hausdorff space in the weak-$^\ast$ topology $w(Q^\ast(T, P), Q(T, P))$. Hence, the result that $Q(T, P)$ and $C(Y)$, the continuous real valued functions on $Y$ with the supremum norm, are isomorphic and isometric follows from the theorem of Kelley and Namioka. For every $f$ in $Q(T, P)$, $f'$ will denote the image of $f$ in $C(Y)$ determined by the isomorphism. Since $H(T, P)$ is the adjoint of $Q(T, P)$ and $H^r(Y, B)$ is the adjoint of $C(Y)$, where $B$ is the Borel algebra of $Y$ and $H^r(Y, B)$ is the regular Borel measures on $B$, it follows that $H(T, P)$ and $H^r(Y, B)$ are isometrically isomorphic. If $\mu$ is in $H(T, P)$, its image in $H^r(Y, B)$ will be denoted by $\mu'$. It may be noted that if $f$ is in $Q(T, P)$ and $\mu$ is in $H(T, P)$, then

$$
\mu(f) = \int_T f d\mu = \int_Y f' d\mu' = \mu'(f')
$$

where $\int_T f d\mu$ is the integral developed in this paper and $\int_Y f' d\mu'$ is the usual Borel integral.

Suppose now that $p$ is a real number, $1 < p < \infty$ and let $\{f_n\}_{n=1}^\infty$ be in $L_p(T, \mu, P)$. Since each $f_n$ is $P$-simple, $f_n$ is in $Q(T, P)$ for all $n$ and then $f_n'$ is in $C(Y)$ for all $n$. It follows that

$$
\lim_{n,m \to \infty} \|f_n' - f_m'\|_p = \lim_{n,m \to \infty} \left( \int_Y |f_n' - f_m'|^p d\mu \right)^{1/p}
$$

$$
= \lim_{n,m \to \infty} \left( \int_T |f_n - f_m|^p d\mu \right)^{1/p}
$$

$$
= 0,
$$

and since $L_p(Y, \mu', B)$ is complete, there exists $f'$ in $L_p(Y, \mu', B)$ such
that
\[ \lim_{n \to \infty} \| f_1^n - f_2^n \|_p = 0. \]

Define \( \sigma \) on \( L_p(T, \mu, \mathcal{P}) \) to \( L_p(Y, \mu', \mathcal{P}) \) by
\[ \sigma(\{f_n\}_{n=1}^{\infty}) = \lim_{n \to \infty} f_1^n, \quad \forall \{f_n\}_{n=1}^{\infty} \in L_p(T, \mu, \mathcal{P}), \]
where the limit is taken in \( L_p(Y, \mu', \mathcal{P}) \). It follows that
\[ \| \sigma(\{f_n\}_{n=1}^{\infty}) \|_p = \| \lim_{n \to \infty} f_1^n \|_p \]
\[ = \lim_{n \to \infty} \| f_1^n \|_p \]
\[ = \lim_{n \to \infty} \left( \int_Y |f_1^n|^p d\mu \right)^{1/p} \]
\[ = \lim_{n \to \infty} \left( \int_T |f_1^n|^p d\mu \right)^{1/p} \]
\[ = \left\| \{f_n\}_{n=1}^{\infty} \right\|_p, \]
so that \( \sigma \) is an isometry. Suppose that \( \sigma(\{f_n\}_{n=1}^{\infty}) \) and \( \sigma(\{g_n\}_{n=1}^{\infty}) \) are the same for two elements \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) of \( L_p(T, \mu, \mathcal{P}) \). Then
\[ \lim_{n \to \infty} f_1^n = \lim_{n \to \infty} g_1^n \text{ in } L_p(Y, \mu, \mathcal{P}), \] so that
\[ \lim_{n \to \infty} \int_T |f_1^n - g_1^n|^p d\mu = \lim_{n \to \infty} \int_T |f_1^n - g_1^n|^p d\mu = 0. \]
Therefore, \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) are the same element of \( L_p(T, \mu, \mathcal{P}) \) and \( \sigma \) is one-to-one. It is clear that \( \sigma \) is linear, so it is an isometric isomorphism into \( L_p(Y, \mu', \mathcal{P}) \).
Let $f'$ be in $L^p(Y, \mu', B)$. Then there exists a sequence $\left\{f'_{n}\right\}_{n=1}^{\infty}$ in $C(Y)$ such that $\lim_{n \to \infty} f'_{n} = f'$ in $L^p(Y, \mu', B)$. Let $f_n$ be in $Q(T, P)$ such that $f_n$ corresponds to $f'_n$ by the isomorphism between $Q(T, P)$ and $C(Y)$. Then it follows that

$$
\lim_{n,m \to \infty} \int_{T} |f_n - f_m|^p d\mu = \lim_{n,m \to \infty} \int_{Y} |f'_n - f'_m|^p d\mu'
$$

$$= 0,$$

so that $\left\{f_n\right\}_{n=1}^{\infty}$ is in $L_p(T, \mu, P)$ and $\sigma(\left\{f_n\right\}_{n=1}^{\infty})$ is $f'$. Therefore, $\sigma$ is an isometric isomorphism onto $L_p(Y, \mu', B)$.

These results are summarized in the following theorem.

**Theorem 4.3.** If $P$ is a proto-algebra of subsets of a non-void set $T$ and $Y$ the spectrum of $Q(T, P)$ considered as a compact Hausdorff space in the weak-$\ast$ topology $w(Q^\ast(T, P), Q(T, P))$, then the following isometric isomorphic relations hold:

1. $Q(T, P) \cong C(Y)$,
2. $H(T, P) \cong \mathcal{H}_r(Y, B)$,
3. $L_p(T, \mu, P) \cong L_p(Y, \mu', B),$

where $B$ is the Borel algebra of $Y$, $\mathcal{H}_r(Y, B)$ is the regular Borel measures on $B$, $C(Y)$ is the real valued continuous functions on $Y$, $\mu'$ is the image of $\mu$ under (2) and $1 \leq p < \infty$.

The preceding theorem is also true for $p = \infty$, and this is most easily seen by showing that $L_\infty(T, \mu, P)$ is the adjoint of $L_1(T, \mu, P)$. Then $L_\infty(T, \mu, P)$ and $L_\infty(Y, \mu', B)$ are adjoints to the isometrically
isomorphic spaces $L^1(T, \mu, P)$ and $L^1(Y, \mu', B)$, respectively, and this implies the desired result. The next theorem will make it possible to use a result from the paper by Porcelli, [9], to conclude that the adjoint to $L^1(T, \mu, P)$ is $L^\infty(T, \mu, P)$.

Theorem 4.4. If $P$ is a proto-algebra and $\mu$ is in $H(T, P)$, then the algebra generated by $P$ is the set $A(P)$ defined by

$$A(P) = \left\{ E: E = \bigcup_{i=1}^{n} E_i \text{ for some } \{E_i\}_{i=1}^{n} \text{ in } \Delta \right\},$$

and $\mu$ has a unique extension to a finitely additive measure $\mu_1$ on $A(P)$. If $\mu$ is countably additive, then $\mu_1$ is also countably additive. Furthermore, $\widehat{(\mu)_1}$ is $\widehat{(\mu_1)}$, and if $f$ is in $S_1(P, X)$, then

$$\int_T f d\mu = \int_T f d\mu_1.$$

The integral $\int_T f d\mu$ is of course the integral of a $P$-simple function as defined in Chapter II, and the integral $\int_T f d\mu_1$ is the integral of a $P$-simple function, which is an $A(P)$-simple function, as defined in [4, p. 108]. The integral in [9] is the one in [4] for simple functions.

Proof: The union of any finite sequence of elements of $P$ can be written as the union of a sequence in $\Delta$, so $A(P)$ is closed under unions. Also, if $\{E_i\}_{i=1}^{n}$ is in $\Delta$, then $T - \left( \bigcup_{i=1}^{n} E_i \right)$ is the union of a sequence in $\Delta$ since $P$ is a proto-algebra. Therefore, $A(P)$ is an algebra that contains $P$, and any algebra that contains $P$ must contain $A(P)$. Hence, $A(P)$ is the algebra generated by $P$.

Suppose now that $\mu$ is in $H(T, P)$ and $\{E_i\}_{i=1}^{n}$ and $\{F_j\}_{j=1}^{m}$ are in $\Delta$ with
\[ \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j. \]

It was shown in Chapter I that this implies

\[ \sum_{i=1}^{n} \pi(E_i) = \sum_{j=1}^{m} \pi(F_j) \]

whenever \( \pi \) is a non-negative p-volume. However, since \( \mu \) is the difference of two non-negative p-volumes, this implies that

\[ \sum_{i=1}^{n} \mu(E_i) = \sum_{j=1}^{m} \mu(F_j). \]

The equation

\[ \mu_1(E) = \sum_{i=1}^{n} \mu(E_i), \quad \forall E \in A(P), \]

where \( E = \bigcup_{i=1}^{n} E_i \) and \( \{ E_i \}_{i=1}^{n} \) is in \( \Delta \), then gives a well defined function \( \mu_1 \) on \( A(P) \). Since \( \mu \) is finitely additive, \( \mu_1 \) is clearly an extension of \( \mu \) and is finitely additive on \( A(P) \). Suppose that \( \mu \) is countably additive and that \( \{ E_i \}_{i=1}^{\infty} \) is a pairwise disjoint sequence of elements of \( P \) with \( \bigcup_{i=1}^{\infty} E_i \) in \( P \). For \( i = 1, 2, 3, \ldots \), let \( \{ E_{ij} \}_{j=1}^{k(i)} \) be a sequence in \( \Delta \)

such that \( E_i = \bigcup_{j=1}^{\infty} E_{ij} \). Since \( \bigcup_{i=1}^{\infty} E_i \) is in \( P \), there is a sequence \( \{ F_{ij} \}_{j=1}^{m} \) in \( \Delta \) such that

\[ \bigcup_{j=1}^{m} F_j = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{k(i)} E_{ij}, \]

with both sides of this equation a union of pairwise disjoint elements of \( P \). It is an immediate result of the discussion following Theorem 1.4 that
\[
\sum_{j=1}^{m} \mu(F_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{k(i)} \mu(E_{i,j}),
\]

and this clearly implies that

\[
\mu_1(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu_1(E_i).
\]

Suppose now that \( \mu_2 \) is a finitely additive measure on \( A(P) \) such that \( \mu_2|_P \) is \( \mu \). Let \( E \) in \( A(P) \) be given by \( \bigcup_{i=1}^{n} E_i \) with \( \{E_i\}_{i=1}^{n} \) in \( \Delta \). Then

\[
\mu_2(E) = \mu_2(\bigcup_{i=1}^{n} E_i)
\]

\[
= \sum_{i=1}^{n} \mu_2(E_i)
\]

\[
= \sum_{i=1}^{n} \mu(E_i)
\]

\[
= \mu_1(E),
\]

so that \( \mu_2 \) and \( \mu_1 \) are the same on \( A(P) \), and the extension of \( \mu \) is unique.

In order to prove the next statement of the theorem it should first be observed that \( \overline{\mu_1} \) is the variation function as defined by Dunford and Schwartz, [4, p. 97]. They show, [4, pp. 98-99], that \( \overline{\mu_1} \) is given by \( (\mu_1)^+ + (\mu_1)^- \) where

\[
(\mu_1)^+(E) = \sup \left\{ \mu_1(F) : F \subseteq E \text{ and } F \in A(P) \right\}, \forall E \in A(P), \text{ and}
\]

\[
(\mu_1)^-(E) = - \inf \left\{ \mu_1(F) : F \subseteq E \text{ and } F \in A(P) \right\}, \forall E \in A(P).
\]

Suppose now that \( E \) is in \( P \). Then
\[(\mu_1)^+(E) = \sup \left\{ \mu_1(F) : F \subseteq E \text{ and } F \in A(P) \right\} \]
\[= \sup \left\{ \sum_{i=1}^{n} \mu(E_i) : \bigcup_{i=1}^{n} E_i \subseteq E \text{ and } \left\{ E_i \right\}_{i=1}^{n} \in A(P) \right\} \]
\[= \mu^+(E), \quad \text{and} \]
\[(\mu_1)^-(E) = -\inf \left\{ \mu_1(F) : F \subseteq E \text{ and } F \in A(P) \right\} \]
\[= -\inf \left\{ \sum_{i=1}^{n} \mu(E_i) : \bigcup_{i=1}^{n} E_i \subseteq E \text{ and } \left\{ E_i \right\}_{i=1}^{n} \in A(P) \right\} \]
\[= -\left[ \mu^-(E) \right]. \]

Therefore, it follows that if \(E\) is in \(P\), then
\[\bar{\mu}_1(E) = (\mu_1)^+(E) + (\mu_1)^-(E) \]
\[= \mu^+(E) - \mu^-(E) \]
\[= \bar{\mu}(E). \]

This implies that \(\bar{\mu}_1\) is an extension of \(\bar{\mu}\), and since the extensions are unique it follows that \(\bar{\mu}_1\) is \((\bar{\mu})_1\).

If \(f\) is in \(S_1(P, X)\) and \(f\) is given by \(\sum_{i=1}^{n} a_i X_{A_i}\) with \(\left\{ A_i \right\}_{i=1}^{n}\) a finite sequence in \(D(\mu, P)\), then in [4, p. 108] it is required that \(\left\{ A_i \right\}_{i=1}^{n}\) be a \(P\)-partition of \(T\) before the integral is defined. Since \(P\) is a proto-algebra, \(\left\{ A_i \right\}_{i=1}^{n}\) can be extended to a partition \(\left\{ A_i \right\}_{i=1}^{q}\) where \(1 \leq n \leq q\). Let \(a_i\) be zero for \(i = n + 1, \ldots, q\). Then \(f\) is given by \(\sum_{i=1}^{n} a_i X_{A_i}\) and the integral over \(T\) of \(f\) with respect to \(\mu_1\) is defined in [4] by the equation
\[\int_T f \, d\mu_1 = \sum_{i=1}^{q} a_i \mu_1\left( A_i \right),\]
and this implies
\[ \int_T f d\mu_1 = \sum_{i=1}^{n} a_i \chi_{A_i} \]
\[ = \sum_{i=1}^{n} a_i \mu(A_i) \]
\[ = \int_T f d\mu. \]

This completes the proof.

Since \( P \) is contained in \( A(P) \), it is clear that any \( P \)-simple function is an \( A(P) \)-simple function. Suppose \( f \) is an \( A(P) \)-simple function and let
\[ f = \sum_{i=1}^{n} a_i \chi_{A_i}, \]
where \( \{A_i\}_{i=1}^{n} \) is an \( A(P) \)-partition of \( T \). Then each \( A_i \), \( i = 1, 2, \ldots, n \), is the union of a sequence \( \{A_{ij}\}_{j=1}^{q(i)} \) in \( A \), so that
\[ f = \sum_{i=1}^{n} \sum_{j=1}^{q(i)} a_i \chi_{A_{ij}}, \]
and \( f \) is a \( P \)-simple function. Since for \( P \)-simple functions the integral as defined in this paper and the integral as defined in [9] agree, it follows that \( L_p(T, \mu, P) \) and \( L_p(T, \mu_1, A(P)) \) are the same for all \( p \), \( 1 \leq p \leq \infty \). In particular, it is shown by Porcelli, [9, p. 114], that \( L_1(T, \mu_1, A(P)) \) is \( L_1(T, \mu_1, A(P)) \). Therefore, it follows that \( L_1(T, \mu, F) \) is \( L_1(T, \mu, F) \). Finally, the adjoint of \( L_1(T, \mu_1, A(P)) \) is characterized by Porcelli as being isometrically isomorphic to the subspace \( L \) of \( H_T(Y, B) \) where \( \lambda \) is in \( L \) if and only if \( \lambda(E) \) is zero whenever \( \mu_1(E) \) is zero for each Borel set \( E \) of \( Y \). Since \( L_1(T, \mu, P) \) is \( L_1(T, \mu_1, A(P)) \),
this also characterizes the adjoint of $L_1(T, \mu, \mathcal{P})$.

Fefferman, [5], proves a form of the Radon-Nikodym theorem for finitely additive measures on an algebra of sets. The method of proof used in [5] could be adapted to p-volumes and proto-rings without much difficulty. However, the proof then would be essentially the same as Fefferman and would not introduce any new techniques. Hence, the following is included mainly for completeness.

Assume that $\mathcal{P}$ is a proto-algebra for the remainder of this chapter. Suppose that $\mu$ and $\nu$ are in $H(T, \mathcal{P})$ and that $\nu \ll \mu$. As before, $A(\mathcal{P})$ is the algebra generated by $\mathcal{P}$ and $\mu_1$ and $\nu_1$ are the extensions of $\mu$ and $\nu$, respectively, to $A(\mathcal{P})$. Suppose now that $\varepsilon$ is a positive real number and choose a positive real number $\delta$ such that if $\{E_i\}_{i=1}^n$ is in $D(\mu, \mathcal{P})$ and $\sum_{i=1}^n \mu(E_i) < \delta$, then $\sum_{i=1}^n \nu(E_i) < \varepsilon$. If $E$ is in $\mathcal{P}$ and $\bar{\mu}_1(E) < \delta$, then let $E$ be the union of the elements of the finite sequence $\{E_i\}_{i=1}^n$ in $\Delta$. It follows from Theorem 4.4 that

$$\sum_{i=1}^n \bar{\mu}(E_i) = \sum_{i=1}^n \bar{\mu}_1(E_i)$$

$$= \bar{\mu}_1(E)$$

$$< \delta,$$

and this implies that

$$\bar{\nu}_1(E) = \sum_{i=1}^n \bar{\nu}_1(E_i)$$

$$= \sum_{i=1}^n \bar{\nu}(E_i)$$

$$< \varepsilon.$$
Therefore, \( \nu \ll \mu \) and the Radon-Nikodym theorem of Fefferman, [5, Theorem 1], applies. Hence, there exists a sequence \( \{f_n\}_{n=1}^{\infty} \) of \( P \)-simple functions on \( T \) to \( \mathbb{R} \) such that

\[
\nu_1(E) = \lim_{n \to \infty} \int_{E} f_n \, d\mu_1
\]

uniformly for all \( E \) in \( A(P) \) and

\[
\lim_{n,m \to \infty} \int_{T} |f_n(\cdot) - f_m(\cdot)| \, d\mu_1 = 0.
\]

Therefore, \( \{f_n\}_{n=1}^{\infty} \) is a sequence of elements of \( S_1(P, \mathbb{R}) \) and Theorem 4.4 implies that

\[
\nu(E) = \lim_{n \to \infty} \int_{E} f \, d\mu
\]

uniformly for all \( E \) in \( P \) and

\[
\lim_{n,m \to \infty} \int_{T} |f_n(\cdot) - f_m(\cdot)| \, d\mu = 0.
\]

This proves the next theorem.

**Theorem 4.5. (Radon-Nikodym).** Suppose that \( \mu \) and \( \nu \) are in \( H(T, P) \) and that \( \nu \ll \mu \). Then there exists a sequence \( \{f_n\}_{n=1}^{\infty} \) of elements of \( S_1(P, \mathbb{R}) \) such that

\[
\nu(E) = \lim_{n \to \infty} \int_{E} f \, d\mu
\]

uniformly for all \( E \) in \( P \), and

\[
\lim_{n,m \to \infty} \int_{T} |f_n(\cdot) - f_m(\cdot)| \, d\mu = 0.
\]
A proof using a generalization of a Hahn decomposition, [6, p. 121], is given for a Radon-Nikodym theorem by Darst, [3], for finitely additive measures on an algebra of sets. By some fairly tedious modifications of Darst's results, that approach may also be used to prove Theorem 4.5.
V. COMPARISONS WITH THE INTEGRALS OF DUNFORD AND SCHWARTZ AND
OF BOGDANOVICH

The assumptions throughout this chapter relative to $T$, $(X, | |)$ and $R$ are the same as in the preceding chapters.

Suppose that $P$ is a proto-algebra of subsets of $T$ and that $\mu$ is in $H(T, P)$. Theorem 4.4 shows that there is a natural extension $\mu_1$ to a finitely additive measure on $A(P)$, the algebra generated by $P$. Dunford and Schwartz, [4, pp. 95-119], develop a space, $L(T, A(P), \mu_1, X)$ in their notation, of equivalence classes of functions on $T$ to $X$, integrable over $T$ with respect to $\mu_1$. The method of constructing $L(T, A(P), \mu_1, X)$ is somewhat similar to the method used in this paper to construct $K(\mu, P, X)$. It is natural to ask whether or not $L(T, A(P), \mu_1, X)$ and $K(\mu, P, X)$ are comparable, and if so, how they compare.

Suppose that $f$ and $g$ are functions on $T$ to $X$. The functions $f$ and $g$ are identified in [4] if and only if $f - g$ is $\mu_1$-null; or, equivalently, for every positive real number $\varepsilon$, the set $E(f - g, \varepsilon)$ defined by

$$E(f - g, \varepsilon) = \left\{ t \in T : |f(t) - g(t)| > \varepsilon \right\}$$

is $\mu_1$-null. Suppose $f - g$ is $\mu_1$-null and let $\varepsilon$ be a positive real number. Then $E(f - g, \varepsilon)$ is $\mu_1$-null, so there exists $G$ in $A(P)$ such that $\mu_1(G) < \varepsilon$ and $E(f - g, \varepsilon)$ is a subset of $G$. Let $\left\{G_k\right\}_{k=1}^{n}$ be a finite sequence in $D(\mu, P)$ such that $G = \bigcup_{k=1}^{n} G_k$. Then $\sum_{k=1}^{n} \mu(G_k) < \varepsilon$ and

$$|f(t) - g(t)| < \varepsilon, \forall t \in T - (\bigcup_{k=1}^{n} G_k),$$

so that $f$ and $g$ are $\mu$-equivalent. Suppose now that $f$ and $g$ are $\mu$-
equivalent and let $\epsilon$ be a positive real number. Let $\epsilon_1$ be any positive real number and define $\epsilon_2$ to be the minimum of $\epsilon$ and $\epsilon_1$. Then there exists \( \{G_k\}_{k=1}^n \) in $D(\mu, P)$ such that \( \sum_{k=1}^n \mu(G_k) < \epsilon_2 \leq \epsilon_1 \) and
\[
|f(t) - g(t)| < \epsilon_2 \leq \epsilon, \forall t \in T - \left( \bigcup_{k=1}^n G_k \right).
\]
Therefore, $E(f - g, \epsilon)$ is contained in $\bigcup_{k=1}^n G_k$. However, $\bigcup_{k=1}^n G_k$ is a cover of $E(f - g, \epsilon)$ by an element of $A(P)$ if and only if $n$ is finite.

In particular, if $\mu$ is only finitely additive, this shows that $E(f - g, \epsilon)$ is $\mu_1$-null and that $f - g$ is $\mu_1$-null. Hence, in the case that $\mu$ is only finitely additive and $P$ is a proto-algebra, the equivalence classes of functions in [4] and in this paper agree. However, if $\mu$ is countably additive, an equivalence class $\widetilde{f}$ in $X^{T}/\sim$ may be larger than the equivalence class containing $f$ as defined in [4]. Hence, $K(\mu, P, X)$ and $L(T, A(P), \mu_1, X)$ are comparable if $\mu$ is only finitely additive.

**Theorem 5.1.** Suppose that $P$ is a proto-algebra, $\mu$ is in $H(T, P)$, $A(P)$ is the algebra generated by $P$ and $\mu_1$ is the extension of $\mu$ to $A(P)$. If $\mu$ is only finitely additive, then $K(\mu, P, X)$ and $L(T, A(P), \mu_1, X)$ are the same normed linear spaces and the integrals on the two spaces are the same.

**Proof:** It has already been shown that $K(\mu, P, X)$ and $L(T, A(P), \mu_1, X)$ are both subsets of $X^{T}/\sim$ when $\mu$ is only finitely additive, and hence comparable. Also, if $f$ is $\mu$-equivalent to a $P$-simple function $g$, then in both developments the integral of $\widetilde{f}$ is defined to be the integral of $g$. Also, if $g$ is in $S_1(P, X) = S_1(A(P), X)$, then Theorem 4.4 shows that
\[
\int_T g d\mu = \int_T g d\mu_1,
\]
so that
\[
\int_T \overline{f} d\mu = \int_T \overline{f} d\mu_1, \forall f \in \mathcal{S}(\mu, P, X).
\]

Also, in both developments the norm of \(\overline{f}\) in \(\mathcal{S}(\mu, P, X)\) is given by the integral of \(|\overline{f}(\cdot)|\), so that a norm Cauchy sequence of simple functions is the same in both developments.

The space \(L(T, A(P), \mu_1, X)\) is defined to be the collection of all \(\overline{f}\) in \(X^T/\hat{\mu}\) for which there is a norm Cauchy sequence \(\left\{\overline{f}_n\right\}_{n=1}^{\infty}\) in \(\mathcal{S}(\mu, P, X)\) converging to \(\overline{f}\) in \(\mu_1\)-measure. If \(\overline{f}\) is in \(L(T, A(P), \mu_1, X)\), then the integral of \(\overline{f}\) is defined by
\[
\int_T \overline{f} d\mu_1 = \lim_{n \to \infty} \int_T \overline{f}_n d\mu_1.
\]

Hence, if \(\mu_1\)-measure convergence and \(\mu\)-convergence are equivalent, then it is clear that \(K(\mu, P, X)\) and \(L(T, A(P), \mu_1, X)\) coincide and that
\[
\int_T \overline{f} d\mu = \int_T \overline{f} d\mu_1, \forall f \in K(\mu, P, X).
\]

Therefore, it must be shown that if \(\left\{f_n\right\}_{n=1}^{\infty}\) is a sequence of functions in \(S_1(P, X) = S_1(A(P), X)\) and if \(f\) is a function on \(T\) to \(X\), then \(f_n \to f\) if and only if \(\left\{f_n\right\}_{n=1}^{\infty}\) converges to \(f\) in \(\mu_1\)-measure as defined in [4].

The reader is referred to [4] for the notation which will be used when appropriate in the remainder of this proof.

Suppose that \(\left\{f_n\right\}_{n=1}^{\infty}\) is a sequence of elements of \(S_1(P, X)\) that converges in \(\mu_1\)-measure to a function \(f\) on \(T\) to \(X\). Let \(\varepsilon\) be a positive
real number. Then there exists a positive integer \( N \) such that if \( n \geq N \), then

\[
\inf_{\alpha \geq 0} \arctan(\alpha + \frac{1}{n} \{t \in T: |f_n(t) - f(t)| > \alpha\}) < \arctan \varepsilon.
\]

Therefore, for every \( n \geq N \), there is a positive real number \( \delta_n \) such that

\[
\arctan(\delta_n + \frac{1}{n} \{t \in T: |f_n(t) - f(t)| > \delta_n\}) < \arctan \varepsilon,
\]

and this implies that

\[
\delta_n + \frac{1}{n} \{t \in T: |f_n(t) - f(t)| > \delta_n\} < \varepsilon, \quad \forall n \geq N.
\]

It follows that

\[
0 \leq \frac{1}{n} \{t \in T: |f_n(t) - f(t)| > \delta_n\} < \varepsilon - \delta_n < \varepsilon, \quad \forall n \geq N.
\]

(Note that \( 0 < \varepsilon - \delta_n \), so \( 0 < \delta_n \).) Hence, if \( n \geq N \), there exists a finite sequence in \( D(\mu, \rho) \) such that

\[
\{G_k(n)\}_{k=1}^{q(n)},
\]

\[
\mu_1(\bigcup_{k=1}^{q(n)} G_k(n)) < \varepsilon. \quad \text{It follows that}
\]

\[
|f_n(t) - f(t)| \leq \delta_n < \varepsilon, \quad \forall t \in T - \bigcup_{k=1}^{q(n)} G_k(n),
\]

\[
\sum_{k=1}^{q(n)} \mu(G_k(n)) = \sum_{k=1}^{q(n)} \mu_1(G_k(n))
\]

\[
= \sum_{k=1}^{q(n)} \mu_1(G_k(n))
\]

\[
= \mu_1(\bigcup_{k=1}^{q(n)} G_k(n)) < \varepsilon.
\]
Therefore, the sequence \( \{f_n\}_{n=1}^{\infty} \) \( \mu \)-converges to \( f \).

Suppose now that \( \{f_n\}_{n=1}^{\infty} \) is a sequence of elements of \( S_1(P, X) \) that \( \mu \)-converges to a function \( f \) on \( T \) to \( X \). Let \( \varepsilon \) be a positive real number and choose \( \varepsilon_1 \) in \( \mathbb{R} \) such that \( \varepsilon_1 > 0 \) and

\[
\arctan(2\varepsilon_1) < \varepsilon.
\]

There exists a positive integer \( N \) such that if \( n \geq N \), then there exists a (finite) sequence \( \{G_k^{(n)}\}_{k=1}^{q(n)} \) in \( D(\mu, P) \) such that

\[
\frac{q(n)}{\mu_1(\bigcup_{k=1}^{q(n)} G_k^{(n)})} = \sum_{k=1}^{q(n)} \frac{1}{\mu_1(G_k^{(n)})} = \sum_{k=1}^{q(n)} \mu(G_k^{(n)}) < \varepsilon_1,
\]

and

\[
|f_n(t) - f(t)| \leq \varepsilon_1, \forall t \in T - \bigcup_{k=1}^{q(n)} (G_k^{(n)}).
\]

Therefore, if \( n \geq N \), then

\[
\mu_1(\{t \in T : |f_n(t) - f(t)| > \varepsilon_1\}) < \varepsilon_1, \text{ and}
\]

\[
\varepsilon_1 + \mu_1(\{t \in T : |f_n(t) - f(t)| > \varepsilon_1\}) < 2\varepsilon_1.
\]

This implies that

\[
\arctan \left[ \varepsilon_1 + \mu_1(\{t \in T : |f_n(t) - f(t)| > \varepsilon_1\}) \right] < \varepsilon, \forall n \geq N;
\]

therefore,

\[
\inf_{\alpha > 0} \arctan \left[ \alpha + \mu_1(\{t \in T : |f_n(t) - f(t)| > \alpha\}) \right] < \varepsilon, \forall n \geq N.
\]
Hence, the sequence \( \{f_n\}_{n=1}^{\infty} \) converges to \( f \) in \( \mu_1 \)-measure. This completes the proof.

The section of Dunford and Schwartz cited, [4, pp. 95-119], effectively treats only measures that are only finitely additive since integration with respect to a countably additive set function is treated separately. Hence, if \( \mu \) is countably additive, then the development in this paper more closely parallels the development by Bogdanowicz, [1]. Before proceeding with the comparison with the Bogdanowicz development there is one final observation concerning [4] that should be made. It is the note that while no measurability restrictions are placed on the functions when developing \( L(T, A(P), \mu_1, X) \) in [4], the functions in the \( L_p \) spaces of [4] are assumed to be measurable. Hence, even if \( \mu \) is only finitely additive, \( K(\mu, P, X) \) may be larger than the \( L_1 \) space in [4].

Suppose for the remainder of this chapter that \( P \) is a proto-ring of subsets of \( T \) and that \( \mu \) is a non-negative countably additive \( p \)-volume on \( P \). Bogdanowicz, [1], develops a complete seminormed space of functions on \( T \) to \( X \) starting from a countably additive non-negative set function defined on a pre-ring of subsets of \( T \). Such a set function is called a volume. If \( \nu \) is a volume on a pre-ring \( P_1 \) of subsets of \( T \), then the Bogdanowicz space will be denoted by \( L(\nu, P_1, X) \). The elements of \( L(\nu, P_1, X) \) are functions on \( T \) to \( X \) and not equivalence classes of functions. The analogue to \( L(\nu, P_1, X) \) would seem to be \( K_1(\mu, P, X) \), and the rest of this chapter shows that this is the case. The following theorem will be needed for this.
Theorem 5.2. Suppose that \( f \) and \( g \) are functions on \( T \) to \( X \). Then \( f \) and \( g \) are \( \mu \)-equivalent if and only if they are equal almost everywhere.

Proof: Assume \( f \) and \( g \) are \( \mu \)-equivalent. For each positive integer \( n \) let \( \{ E_i^{(n)} \}_{i=1}^{q(n)} \) be a sequence in \( D(\mu, F) \) such that \( \sum_{i=1}^{q(n)} \mu(E_i^{(n)}) < 2^{-n} \) and

\[
|f(t) - g(t)| < 2^{-n}, \quad \forall t \in T - (\bigcup_{i=1}^{q(n)} E_i^{(n)}).
\]

Define the sets \( E_n \), \( F_m \) and \( F \) by

\[
E_n = \bigcup_{i=1}^{q(n)} E_i^{(n)}, \quad n = 1, 2, 3, \ldots,
\]

\[
F_m = \bigcup_{n=m}^{\infty} E_n, \quad m = 1, 2, 3, \ldots,
\]

\[
F = \bigcap_{m=1}^{\infty} F_m.
\]

Since \( F \) is contained in \( F_m \) for \( m = 1, 2, 3, \ldots \) and since

\[
\sum_{n=m}^{\infty} \sum_{i=1}^{q(n)} \mu(E_i^{(n)}) < \sum_{n=m}^{\infty} 2^{-n} = 2^{-(m-1)}, \quad m = 1, 2, 3, \ldots,
\]

it follows that \( F \) is \( \mu \)-null. Also, if \( t \) is not in \( F \), then for some positive integer \( m \), \( t \) is not in \( E_n \) for all \( n \geq m \). This implies that

\[
|f(t) - g(t)| < 2^{-n}, \quad \forall n \geq m.
\]

Therefore,

\[
f(t) = g(t), \quad \forall t \notin F,
\]
and \( f \) and \( g \) are equal almost everywhere.

The definition of a \( \mu \)-null set makes the converse obvious.

This completes the proof.

Suppose now that \( P_1 \) is defined by the equation

\[
P_1 = \left\{ E : E = \bigcap_{i=1}^{n} E_i \text{ for some finite sequence } \{E_i\}_{i=1}^{n} \subseteq P \right\}.
\]

By choosing singleton sequences in the definition of \( P_1 \), it is easy to see that \( P_1 \) contains \( P \). Also, any pre-ring containing \( P \) must contain \( P_1 \) since a pre-ring is closed under finite intersections. Therefore, if \( P_1 \) is a pre-ring, it is the smallest pre-ring containing \( P \). It is clear that \( P_1 \) is closed under finite intersections, so to see that it is a pre-ring, one only needs to check the difference property for pre-rings. Suppose that \( E \) and \( F \) are in \( P_1 \). Then there exist finite sequences \( \{E_i\}_{i=1}^{n} \) and \( \{F_j\}_{j=1}^{m} \) of pairwise disjoint elements of \( P \) such that

\[
E = \bigcup_{i=1}^{n} E_i, \quad \text{and} \quad F = \bigcup_{j=1}^{m} F_j,
\]

since \( P \) is a proto-ring. Therefore,

\[
E - F = \left( \bigcup_{i=1}^{n} E_i \right) - \left( \bigcup_{j=1}^{m} F_j \right),
\]

and it has been shown before that a difference of the form \( \left( \bigcup_{i=1}^{n} E_i \right) - \left( \bigcup_{j=1}^{m} F_j \right) \) can be written as a finite union of pairwise disjoint elements of \( P \). Since \( P_1 \) contains \( P \), it follows that \( E - F \) is a finite union of elements of \( P_1 \). Therefore, \( P_1 \) is a pre-ring, and it will be
called the pre-ring generated by $P$. Suppose now that $E$ is in $P_1$.

Since $P$ is a proto-ring, there exists a finite sequence $\{E_i\}_{i=1}^n$ in $D(\mu, P)$ such that $E = \bigcup_{i=1}^n E_i$. Also, if $\{F_j\}_{j=1}^m$ is a second sequence in $D(\mu, P)$ with $E = \bigcup_{j=1}^m F_j$, then $\sum_{i=1}^n \mu(E_i)$ and $\sum_{j=1}^m \mu(F_j)$ are the same.

Define the function $\mu_1$ on $P_1$ by

$$\mu_1(E) = \sum_{i=1}^n \mu(E_i).$$

Suppose that $\{E_i\}_{i=1}^\infty$ is a pairwise disjoint sequence of elements of $P_1$ such that $E = \bigcup_{i=1}^\infty E_i$ is in $P_1$. Let $\{F_j\}_{j=1}^m$ be a finite sequence in $D(\mu, P)$ such that $E = \bigcup_{j=1}^m F_j$, and for $i = 1, 2, 3, \ldots$, let $\{G_{ik}\}_{k=1}^{p(i)}$ be a finite sequence in $D(\mu, P)$ such that $E_i = \bigcup_{k=1}^{p(i)} G_{ik}$. It follows that

$$\bigcup_{j=1}^m F_j = \bigcup_{i=1}^\infty \bigcup_{k=1}^{p(i)} G_{ik}$$

and the discussion following Theorem 1.4 shows that

$$\sum_{j=1}^m \mu(F_j) = \sum_{i=1}^\infty \sum_{k=1}^{p(i)} \mu(G_{ik})$$

since $\{G_{ik} : k = 1, 2, \ldots, p(i) ; i = 1, 2, \ldots\}$ is a pairwise disjoint collection. This implies that

$$\mu_1(E) = \sum_{j=1}^m \mu(F_j)$$

$$= \sum_{i=1}^\infty \sum_{k=1}^{p(i)} \mu(G_{ik})$$

$$= \sum_{i=1}^\infty \mu_1(E_i),$$
so that \( \mu_1 \) is a volume on \( P_1 \). It follows in a manner like that used in
the previous comparison with Dunford and Schwartz that \( \mu_1 \) is an
extension of \( \mu \) and it is easy to see that the extension is unique.

If \( E \) is in \( P_1 \), then \( E \) is \( \bigcup_{i=1}^{n} E_i \) for some finite sequence \( \{E_i\}_{i=1}^{n} \)
in \( D(\mu, P) \). Therefore, \( \chi_E \) is \( \sum_{i=1}^{n} \chi_{E_i} \) and this implies that any \( P_1 \)
simple function on \( T \) to \( X \) is a \( P \)-simple function on \( T \) to \( X \). Since
\( P_1 \) contains \( P \), the converse is also true and \( S_1(P, X) \) and \( S_1(P_1, X) \)
coincide.

The integral of a function \( f \) in \( S_1(P, X) \) is defined in [1] exactly
as it is in this paper. Also, the seminorm function \( \|
\) on \( S_1(P, X) \)
is defined in both developments by the equation
\[
\| h \| = \int_T |h(\cdot)|d\mu, \forall h \in S_1(P, X).
\]

In [1] a sequence \( \{ f_n \}_{n=1}^{\infty} \) of elements of \( S_1(P, X) \) is called a basic
sequence if there exists a sequence \( \{h_n\}_{n=1}^{\infty} \) of elements of \( S_1(P, X) \)
and a real number \( M > 0 \) such that
\[
f_n = \sum_{i=1}^{n} h_i, \ n = 1, 2, 3, \ldots, \text{ and}
\]
\[
\| h_n \| \leq M \cdot 4^{-n}, \ n = 1, 2, 3, \ldots.
\]

Suppose that \( \{ g_n \}_{n=1}^{\infty} \) is a seminorm Cauchy sequence of elements of \( S_1(P, X) \). Let \( n(1) \) be a positive integer such that
\[
\| g_n - g_m \| < 4^{-2}, \ \forall n, m \geq n(1).
\]

For \( k = 2, 3, 4, \ldots \), let \( n(k) \) be a positive integer, \( n(k) > n(k - 1) \),
such that
\[ \| g_n - g_m \| < 4^{-k+1}, \quad \forall n, m \geq n(k). \]

Let \( f_k \) be \( g_{n(k)} \), so that \( \{f_k\}_{k=1}^{\infty} \) is a subsequence of \( \{g_n\}_{n=1}^{\infty} \) and
\[ \| f_{k+1} - f_k \| = \| g_{n(k+1)} - g_{n(k)} \| < 4^{-k+1} \]
for \( k = 1, 2, 3, \ldots \).

Define \( h_1 \) to be \( f_1 \) and for \( k = 2, 3, 4, \ldots \), define \( h_k \) to be \( f_k - f_{k-1} \).

Then it follows that
\[ f_k = \sum_{i=1}^{k} h_i, \quad k = 1, 2, 3, \ldots \]
and
\[ \| h_k \| = \| f_k - f_{k-1} \| < 4^{-k}, \quad k = 2, 3, 4, \ldots \]

If \( M \) is the maximum of \( 1 \) and \( 4 \cdot \| h_1 \| \), then
\[ \| h_k \| \leq M \cdot 4^{-k}, \quad k = 1, 2, 3, \ldots \]

Therefore, \( \{f_k\}_{k=1}^{\infty} \) is a basic sequence, and every seminorm Cauchy sequence has a basic subsequence.

The space \( L(\mu_1, P_1, X) \) is defined in [1] to be the set of all functions \( f \) on \( T \) to \( X \) for which there exists a basic sequence \( \{f_n\}_{n=1}^{\infty} \) of elements of \( S_1(P, X) \) converging pointwise almost everywhere to \( f \).

Suppose now that \( f \) is in \( K_1(\mu, P, X) \) and let \( \{f_n\}_{n=1}^{\infty} \) be a seminorm Cauchy sequence of elements of \( S_1(P, X) \) \( \mu \)-converging to \( f \). Theorem 2.12 and the proof of Theorem 2.13 show that there is a subsequence
\[ \left\{ f_{n(k)} \right\}_{k=1}^{\infty} \] of \( \left\{ f_n \right\}_{n=1}^{\infty} \) and a function \( h \) on \( T \) to \( X \) such that \( \left\{ f_{n(k)} \right\}_{k=1}^{\infty} \) both \( \mu \)-converges and converges pointwise almost everywhere to \( h \). Since \( \left\{ f_{n(k)} \right\}_{k=1}^{\infty} \) is a subsequence of \( \left\{ f_n \right\}_{n=1}^{\infty} \), it is seminorm Cauchy and so there is a basic subsequence \( \left\{ g_m \right\}_{m=1}^{\infty} \) of \( \left\{ f_{n(k)} \right\}_{k=1}^{\infty} \). It follows that \( \left\{ g_m \right\}_{m=1}^{\infty} \) both \( \mu \)-converges and converges pointwise almost everywhere to \( h \).

Since \( \left\{ g_m \right\}_{m=1}^{\infty} \) is a subsequence of \( \left\{ f_n \right\}_{n=1}^{\infty} \), it also \( \mu \)-converges to \( f \).

Theorem 2.6 implies that \( f \) and \( h \) are \( \mu \)-equivalent, and then Theorem 5.2 implies that \( f \) and \( h \) are equal almost everywhere. Therefore, \( \left\{ g_n \right\}_{n=1}^{\infty} \) is a basic sequence of elements of \( S_1(P, X) \) that converges pointwise almost everywhere to \( f \), and \( f \) is in \( L(\mu_1, P_1, X) \). Also, the integral and seminorm are defined on \( L(\mu_1, P_1, X) \) by the equations

\[
\int_T f d\mu_1 = \lim_{n \to \infty} \int_T g_n d\mu_1 , \text{ and} \]

\[
\| f \| = \lim_{n \to \infty} \| g_n \|
\]

so it follows that the seminorm on \( K_1(\mu, P, X) \) is the seminorm of \( L(\mu_1, P_1, X) \) restricted to \( K_1(\mu, P, X) \) and

\[
\int_T f d\mu = \int_T f d\mu_1 , \forall f \in K_1(\mu, P, X).
\]

Since \( \mu \) is countably additive, \( K_1(\mu, P, X) \) is a complete subspace of \( L(\mu_1, P_1, X) \). However, \( S_1(P, X) \) is contained in \( K_1(\mu, P, X) \) and is a dense subspace of \( L(\mu_1, P_1, X) \), so it follows that \( K_1(\mu, P, X) \) is dense in \( L(\mu_1, P_1, X) \). Finally, \( L(\mu_1, P_1, X) \) is complete, so \( K_1(\mu, P, X) \) is a closed dense subspace of \( L(\mu_1, P_1, X) \) and it must be true that \( K_1(\mu, P, X) \) and \( L(\mu_1, P_1, X) \) coincide. This completes the proof of the following theorem.
Theorem 5.3. Suppose that \( \mu \) is a non-negative countably additive \( p \)-volume on the proto-ring \( P \) of subsets of \( T \), that \( P_1 \) is the pre-ring generated by \( P \) and \( \mu_1 \) is the extension of \( \mu \) to \( P_1 \). Then \( K_1(\mu, P, X) \) and \( L(\mu_1, P_1, X) \) are the same normed linear space and the integrals on the two spaces are the same.
VI. BIBLIOGRAPHY


VII. ACKNOWLEDGMENT

The author gratefully acknowledges the help and guidance of his major professor, Professor James Dyer, in selecting a topic and developing this dissertation. He also wishes to express his gratitude to an understanding wife, Mary Ann, who understands neither a definition nor a theorem in the entire dissertation.