RETROSPECTIVE COMMENTS ON THE ELASTIC WAVE SCATTERING PROBLEM

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ABSTRACT

Developments over the past several years have led to renewed study of theoretical methods for treating scattering of ultrasound by defects in elastic solids. Characteristically, many theories of scattering, until a few years ago, dealt with scalar waves and simple obstacles. Within that context two distinct regimes were apparent -- the long wave length, and the short wave or imaging regime. The treatment of vector fields in elastic solids is considered the more cumbersome, but we now have made progress -- still for idealized configurations. Specifically, the talk will deal with hypotheses on the types of problems which face us in the next generation of situations.

As the abstract indicates, the remarks to be given in this talk are retrospective rather than constituting a report of a particular current piece of research. Perhaps the history and the flow of emphasis may be of use in giving us some idea of what has been accomplished and what might be looked for in the next phase of this very interesting area.

Before going further, however, I have to acknowledge my colleagues, Gubernatis, Domany, Rose, Teitel and several other students at Cornell have contributed significantly to this work. Stimulating interactions with Kahn, Budiansky, Rice, and Kino, among others at the Materials Research Council, have also been extremely useful.

In 1973 at the first workshop meeting for this group, the point was made that much of the experimental work in wave scattering was being done was interpreted using acoustic wave results. Examples of scattering which had been studied were given which showed that, in fact, these scalar wave interpretations gave exactly wrong results for some important directions of scattering. This experience really set the stage for what I am reviewing here.

The Cornell group began its effort by attempting to place these results in context with some of the other approaches that have been carried out in the past. Clearly, elastic wave scattering has been one of the most venerable problems in mathematical physics. A partial inventory of some useful techniques include: partial wave expansions, the collection of groups of these partial waves to form a T-matrix (i.e., transition matrix, using quantum mechanical terminology), integral equation techniques, reciprocity methods, variational methods, and geometric diffraction methods. There are two regimes that must be considered in choosing a method: the short wave regime, which is the imaging regime, and the long wave length regime, in which the differences between the acoustic and the elastic cases are most pronounced. The question is "which method should you use in which regime?"

Our feelings are quite definite as regards the purposes at hand. We are concerned with applied physics and engineering applications. There is another closely related enterprise, namely mathematics and mathematical physics. In engineering, a useful theoretical framework should be one in which physical intuition or engineering data can be entered as conveniently as possible. On the other hand, it may well be that the method is approximate and the limits of that approximation have to be known. For that reason it is important that one have, as a resource from mathematical physics, some exact solution to the problem.

For that reason, we examined this set of possible methods and chose to concentrate, at least in the initial phase, on the integral equation method. The partial wave expansion, including work by Ying and Truell, and the matrix work by Pao and Varadan, all furnish an important reference for the theoretician and, eventually, for the experimentalist. However, the point about the integral equation methods and, more recently, the reciprocity methods, to which I will just refer briefly because Kino and Auld will develop these in detail, is that engineering data or physical intuition can be entered relatively easily.

Given that motivation as a background, we now work out the key equation and indicate the strategy which has been formulated, particularly by Gubernatis, in going about solving it at two levels of approximation. At the frequency \( \omega \), the scattering process is described by the integral equation

\[
U_i(\mathbf{r}) = U_i^0(\mathbf{r}) + \delta \rho \delta \left[ \int_{\mathbb{R}}^{\text{domain}} d^2 \mathbf{g}_{im}(\mathbf{r},\mathbf{r}') U_m(\mathbf{r}') \right] + \\
\delta c_{ijkl} \int_{\mathbb{R}}^{\text{domain}} d^2 \mathbf{g}_{ij,k}(\mathbf{r},\mathbf{r}') U_m(\mathbf{r}') \tag{1}
\]

where \( G_{ij} \) is the Green's function (response function) satisfying

\[
C_{ijkl} \frac{\partial}{\partial x_k} G_{im,jl} + \rho \frac{\partial^2}{\partial x_k^2} G_{im} + \delta c_{ijkl} G_{im} = 0 \tag{2}
\]

In the above expressions, \( U_i(\mathbf{r}) \) is the total displacement field and \( U_i^0(\mathbf{r}) \) is the so-called incident displacement field, which would equal the above total field if no scatterer were present. In the scatterer domain \( \mathbb{R} \) we assume that the property deviations are uniform, i.e., the density deviation \( \delta \rho \) and the elastic tensor deviation \( \delta c_{ijkl} \) are independent of \( \mathbf{r} \) in \( \mathbb{R} \). The host material \( H \) is of course characterized by the unperturbed density \( \rho \) and elastic tensor \( c_{ijkl} \). Where convenient we have used indicial notation in which the subscripts \( i, j, k, \ldots \) represent the Cartesian directions and in which repeated indices imply summation. A comma preceding a subscript implies differentiation with respect to the coordinate corresponding to the subscript (e.g., \( U_{i,m} = \partial U_i / \partial x_m \)). In order to
evaluate that integral one has to specify the displacement, which is unknown. This is where the approximations begin. In contrast to the partial wave method, it is possible to enter into the integrals of (1) the best guess for the displacement field \( \psi \) and strain \( \varepsilon \) in the vicinity of the flaw. This turns out to be very important from the point of view of the most recent developments as regards cracks and fracture parameters.

The systemization of the perturbation scheme is illustrated by the first Born approximation. If one assumes that the fields in the integrand are equal to the incident fields, then the scattering is found to be linear in \( \delta \psi \) and \( \delta \varepsilon \). This is the simplest case of the full systemization of the perturbation scheme. One can regard the eventual solution by perturbation expansions of equation (1) as a procedure in which, for example, one can order the terms according to powers of \( \delta \omega \) or, in another way, in terms of powers of \( \delta \varepsilon \). If one thinks of the derivative in the second integrand (eq. 1) as being a wave number, then the terms containing \( C_{ijkl} \) could be viewed as a gradient expansion in first order strain, second order strain and so on. There are many ways in which one could conceive of developing these terms as part of an approximate solution. The Born approximation is that one in which only first order terms in \( \delta \psi \) and \( \delta \varepsilon \) are included.

There is something else to note about this equation. The long wave length limit does not necessarily make the Born approximation a good one -- a very important point. The term proportional to \( \delta \psi \) vanishes at low or small \( \omega \) in the exact case, as predicted by the Born approximation. However, the term proportional to \( \delta \varepsilon \) must give the solution to the static elastic problem, i.e., for \( \omega = 0 \). This can be understood by imagining an applied, asymptotically uniform stress. If the flaw happens to be ellipsoidal in shape, one uses the Eshelby solution with a uniform internal strain. This exact solution must be the sum of all of the zero frequency terms in the expansion with respect to \( \delta \varepsilon \). This is not the case for the Born approximation.

This leads to a second approximation, in which the incident displacement and the strain fields appropriate to the exact solution around the flaw in the static limit are used to evaluate the integral. This gives a result, the quasi-static approximation, which is rigorously valid at low frequency.

This formulation provides a nice basis for going up higher and higher in the order of approximation. That, basically, is the motivation for why we felt that this approach is an engineering approach as contrasted with a sole emphasis on exact results. It is therefore complementary to the exact results which can be obtained by the partial wave expansion techniques which will be discussed elsewhere at this symposium.

An observation was made by Gubernatis\(^5\) in 1975 that the scattering can be, in general, represented in terms of a so-called f-vector, which again involves the basic parameters. The exact results in the long wave length limit were then developed by Kohn and Rice\(^1\) to systematize the interpretation of scattering in this regime. A very important point was the observation that there is richness in the elastic wave scattering which simply does not exist in the acoustic wave scattering and that this richness has to do with the structure of the term involving \( \delta \varepsilon \). Using an orderly procedure involving the manipulation of tensor quantities, Kohn and Rice were able to suggest a systematic technique for inverting data appropriate for this low frequency, long wavelength regime. In addition, other work has been stimulated from the long wave length scattering point of view. Richardson\(^11\) has developed a probabilistic inversion technique for this regime in which an exact low frequency theory is used in the modeling of the scattering measurements.

The appropriate philosophy is that there is not a single best way. For \( k \alpha \ll 1 \), the Born approximation\(^9\) has compared well with some exact solutions. The quasi-static scattering model which uses the Eshelby solution, is an improvement. This has enabled one to write down the recipe which has later developed into an experimental flow diagram for looking for the major properties of flaws. More remarkably, Budiansky and Rice\(^1\) also in the long wave limit, are able to relate the scattering by a crack directly to the critical fracture parameters.

Not much has been yet said about the short wave length limit. Here the integrals in the integral equation (1) are evaluated asymptotically. It could be evaluated by the standard method of constant phase, which leads to a series of approximations, the simplest of which is the Fraunhofer approximation. Keller's technique\(^1\) for treating this problem for other applications made use of certain canonical exact solutions -- for example, the Kirchhoff solution for an edge. These were used to correct the simple Fraunhofer picture by patching in these singular edge fields at non-analytical edge points. In the same sense, the treatment of the elastodynamics problem requires some exact solutions and this is the approach that Achenbach\(^12\) and his group have developed in detail in the last few years. In order to guide this approximate technique, i.e., geometric diffraction theory, one must again have available some exact solutions. Future interest will then focus on joining these two regimes, the long wave length regime and the short wave end of the elastodynamics regime.

As one proceeds further, the earlier methods perhaps seem primitive and indirect. In particular, it is clear that the reciprocity techniques which Kino\(^3\) and Auld\(^4\) have now added to the subject are operationally even more oriented toward engineering because they take the essence of the scattering matrix right out through the transducer to the electrical terminals. Indeed, this combines the dynamics, or the response, if you wish, of the circuitry with reciprocity relations for the elastic fields. This was again redeveloped on the basis of earlier results obtained for electromagnetic fields. Again, it is possible to go through a hierarchy of expansions. As before these can be guessed at using the incident field, static deformations, and so on. The whole machinery of working in the engineering data, or physical intuition applied to the reciprocity technique, with the added feature that the results can be used in the Fresnel as well as the Fraunhofer regime.
This, in a sense, is the history of where we have been. The program still has much richness in the matter of relating theory to more sophisticated interpretations of experiments on single defects. However, the next step certainly has to be that of treating systems with many defects. This case may not be so difficult in one sense. If the spacing between defects is very much greater than the wave length, that is the transit time between defects is very much greater than the transient time across the defect, it is clear that then one can use simply time windowing and reduce the problem to that of, essentially, a single defect. That is an obvious thing to say but it is not necessarily obvious from the point of view of the mathematical physicist.

What it means is that one operates in the high frequency limit as regards the interdefect scattering even though one may be operating in the low frequency limit with respect to the intradefect scattering.

However, there is another regime when the spacing between the defects is much less than the wave length. In that case one has to do an entirely different problem and that problem probably can get some guidance from the techniques which have been used to deal with the propagation of Schrödinger waves in disordered alloys (so called random alloys), in which case it is possible, I believe, to consider effective bulk elastic quantities because the defect is smaller than the wave length. This places one in the long wave length regime where one might take the Rice-Kohn results and do a further averaging to develop a correction to the propagation constant -- an average propagation constant of the system.

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REFERENCES


2. See, for example, V. V. Varadan and V. K. Varadan "Scattering of Elastic Waves by Oblate Spheroids and Cracks", paper in these proceedings.


4. B. Auld "Characterization of Surface Wave Scattering by Surface Breaking Cracks", paper in these proceedings.


12. See, for example, J. D. Achenbach, A. K. Gautesen, and H. McMaken "Application of Geometrical Diffraction Theory to QNDE Analysis", paper in these proceedings.
Cnen Tsai (Carnegie-Mellon University): Being a layman in this field, I would like to ask Professor Krumhansl and Bruce Thompson, as well, a question. You have developed and studied various analytical techniques for calculating scattering from defects or voids and I would imagine that these kinds of techniques can be also applied to anisotropic media in principle. My question is whether these kinds of theories have been used in specific applications involving anisotropy.

James Krumhansl (National Science Foundation): I can give you a partial answer. In the past, there have been calculations for the static case. I think one can see now how the static information can be plugged into the long wave length scattering problem. The answer to your question, therefore, is "Yes, the integral equation technique can be used in the long wave length limit for the static case. The Greens function is to be developed in Fourier representation and simply has to be calculated in integral form for an anisotropic system. Jim Gubernatis and I have done this to try to calculate the elastic limit. Now one has to solve the Greens function by brute force, but I believe one can do it."