OPTIMIZATION OF SCATTERING MATRIX FOR ELASTIC WAVES FROM VOIDS AND OBSTACLES

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ABSTRACT

It is shown that a family of scattering matrices for elastic and acoustic waves may be directly deduced from the boundary conditions at the surface of the defect, which in the present work is constrained to be a void or rigid immovable obstacle in a homogeneous isotropic medium, although the method can be readily generalized. From this family of matrices (which includes those derived and used by Waterman, and Pao and Varatharajulu) an optimum one may be chosen, and a criterion is given for doing so. This optimum T-matrix is numerically calculated for a variety of axially symmetric voids and obstacles and results are given for direct and mode-converted differential and total cross-sections.

INTRODUCTION

The scattering of elastic waves by flaws in a homogeneous and isotropic elastic medium is one of the simplest and most basic problems whose solution is of vital interest in nondestructive testing. Although it is simple on a relative scale, it can still be mathematically and computationally very involved, which is the principle reason numerical results have been sparse up to now, except for scattering from flaws with shapes of very high symmetry. The present work is an attempt to add a weapon to the arsenal which can be used to attack this problem.

In this report we shall present the principle of the method, its relation to other methods, and a sample of results which have been obtained. Details will be given elsewhere.

PRELIMINARIES

We will illustrate the procedure using equations for scalar waves, because the formalism is simpler. The development is parallel, but the equations are more complicated, for elastic waves.

The Helmholtz equation is

\[ (\nabla^2 + k^2) \psi = 0 \]

and we wish to find a solution

\[ \psi = \psi_{in} + \sum_{s} a_s \psi_s \]

which is a scattering wavefunction: i.e., the sum of an incident plane wave \( \psi_{\text{in}} \) and outgoing partial waves \( \psi_s \). \( \psi_{\text{in}} \) and \( \psi_s \) are solutions of (1); we need only to find the amplitudes \( a_s \) for which \( \psi \) satisfies the boundary conditions imposed by the presence of the defect, which we take to be either a void or an immovable obstacle in an otherwise homogeneous, isotropic, infinite medium.

Specifically we take the eigenfunction \( \psi_s \) to be

\[ \psi_s = \varphi_{k_m} = h_n^{(1)}(kr)Y_n^m(\theta, \phi) \]

where \( h_n^{(1)} = j_n + iy_n^\ast \) and

\[ \psi_{\text{in}} = e^{ik \cdot r} = \sum_{s, m} a_s \psi_s = 4\pi \sum_{l, m} \hat{\psi}_{l,m}(\theta, \phi) \]

with

\[ \hat{\psi}_{l,m} = j_n^{(1)}(kr)Y_n^m(\theta, \phi) \]

i.e., the part of \( \psi \) that is regular at the origin, which is always inside the defect.

BOUNDARY CONDITIONS

Figure 1 shows the geometry we assume; a defect with surface \( \Sigma \) of so far arbitrary shape, an incoming plane wave with wavevector \( \vec{k}_o \) at spherical polar angles \( \theta_o, \phi_o \), and a direction of observation \( \theta, \phi \).

The wavefunction \( \psi \) may be taken to be a velocity potential, in which case it is proportional to the pressure in an irrotational compressible fluid. The boundary conditions are then clearly

\[ \psi = 0 \quad \text{on} \quad \Sigma : \quad \text{void} \]

\[ \hat{n} \cdot \psi = 0 \quad \text{on} \quad \Sigma : \quad \text{obstacle} \]

(where \( \hat{n} \) is a unit outward normal on \( \Sigma \) for the two types of defects we are considering. If we combine (2), (4), and (6) we get

\[ \psi = 0 \quad \text{on} \quad \Sigma : \quad \text{void} \]

\[ \hat{n} \cdot \psi = 0 \quad \text{on} \quad \Sigma : \quad \text{obstacle} \]

(6a)

(6b)

Work supported by the U.S. Department of Energy.
\[ \Sigma(\mathbf{d}_s \mathbf{v}_s + a \mathbf{v}_s) = 0 \quad \text{on } \Sigma: \text{ void} \quad (7a) \]
\[ \Sigma(\mathbf{d}_s \mathbf{\hat{v}}_s + a \mathbf{\hat{v}}_s) = 0 \quad \text{on } \Sigma: \text{ obstacle} \quad (7b) \]

**MATRIX EQUATIONS**

Suppose now we introduce a set of functions \( \{f_j\} \) which is complete on \( \Sigma \), but otherwise arbitrary, multiply (7a), (7b) by one of them, and integrate over the surface \( \Sigma \). Then

\[ \Sigma[\mathbf{d}_s (f_j \mathbf{v}_s) + a (f_j \mathbf{v}_s)] = 0 \quad : \text{ void} \quad (8a) \]
\[ \Sigma[\mathbf{d}_s (f_j \mathbf{\hat{v}}_s) + a (f_j \mathbf{\hat{v}}_s)] = 0 \quad : \text{ obstacle} \quad (8b) \]

are, because \( \{f_j\} \) is complete, equivalent to (7a), (7b). We have introduced the following notation for the surface integral.

\[ (u,v) = \int \mathbf{d}S \, u \mathbf{v} \quad : \Sigma \quad (9) \]

Equations (8a), (8b) are of the form

\[ Q d + Q a = 0 \quad , \quad (10) \]

which may in principle be solved for

\[ a = - Q^{-1} Q d \equiv T d \quad , \quad (11) \]

defining the T-matrix, which linearly transforms the incident wave amplitudes \( d_s \) into the outgoing wave amplitude \( a_s \). If \( \{f_j\} \) is complete, \( T \) is symmetric. Then (11) is identical to the relation between \( T \) and \( Q \) given in ref. (1); namely \( T = -(Q^{-1} Q)^t \).

**CONSTRAINTS ON \( \{f_j\} \)**

In a computer calculation all of the sets \( \{f_j\}, \{\mathbf{v}_s\}, \{\mathbf{\hat{v}}_s\} \) are finite, and although (11) gives the exact answer for complete (infinite) sets \( T \) is a matrix of infinite rank), one must in practice always truncate the basis sets and matrices.

Then, if one keeps the same number of basis functions in all sets

\[ j, s = 1, 2, \ldots, N \]

we will be calculating \( T^{(N)} \) which is an NN approximation to the \( \infty \times \infty \) matrix, and the goodness of the approximation will be dependent on the truncated set \( \{f_j\}_{j \leq N} \) we choose. Our choice of \( \{f_j\}_{j \leq N} \) will affect

1. the rate of convergence of \( a_s^{(N)} \) to the exact \( a_s \) as \( N \) increases,
2. the conditioning of the \( Q^{(N)} \)-matrix, and
3. the convenience, speed, and accuracy of the numerical evaluation of the \( Q\)-matrix elements.

(2) above means that, for example, if a poor choice of \( \{f_j\}_{j \leq N} \) is made, then some of the equations (8) may be nearly linearly dependent on the others, causing numerical instabilities and/or inaccuracies in the inversion of \( Q^{(N)} \).

The choice of \( \{f_j\}_{j \leq N} \) which was made by Waterman (1), and which corresponds to that used in the elastic wave case by his (2) and by Varatharan and Pao (3), is, for the case of the obstacle, \( f_j = \mathbf{\hat{v}}_s \), giving

\[ Q_{js} = (\mathbf{\hat{v}}_j \cdot \mathbf{n} \mathbf{\hat{v}}_s) \quad , \quad (12) \]
\[ \bar{Q}_{js} = (\mathbf{\hat{v}}_j \cdot \mathbf{n} \mathbf{\hat{v}}_s) \quad . \quad (13) \]

They made this choice of \( f \) because it was natural and convenient in their development of the formalism, which was very different from ours. We will choose \( f \) differently.

**OPTIMIZATION**

In order to motivate a unique specification of \( \{f_j\}_{j \leq N} \), we ask what can be learned by considering the surface integrals

\[ I = \int \mathbf{d}S \, u \mathbf{v} \quad : \text{ void} \quad (13a) \]
\[ I = \int \mathbf{d}S \, u \mathbf{\hat{v}} \quad : \text{ obstacle} \quad (13b) \]

These vanish if and only if \( \mathbf{v} \) satisfies the boundary conditions exactly, which is possible for most surfaces \( \Sigma \) only if \( N = \infty \). For finite \( N \), \( I > 0 \), and we require that the \( N \) coefficients \( a_s \) in (2) be chosen so that \( I \) is a minimum, which implies

\[ \frac{\partial I}{\partial a_s} = 0 \quad s = 1, \ldots, N \quad . \quad (14) \]

Specifically, for the void,

\[ \frac{\partial I}{\partial a_s} = \int \mathbf{d}S \, \mathbf{v}_s \cdot \mathbf{v}_s + \sum_{j=1}^{N} \mathbf{a}_s \mathbf{\hat{v}}_j \mathbf{\hat{v}}_s = 0 \quad , \quad (15) \]

which is (10) with

\[ Q_{js} = (\mathbf{\hat{v}}_j \cdot \mathbf{n} \mathbf{\hat{v}}_s) \quad (16a) \]
\[ \bar{Q}_{js} = (\mathbf{\hat{v}}_j \cdot \mathbf{n} \mathbf{\hat{v}}_s) \quad . \quad (16b) \]

These are to be compared with (12). Corresponding differences appear between the \( Q \)-matrices our criterion prescribes and those which are used in refs. (2) and (3) for elastic wave scattering.

It is intuitively reasonable, because of our optimization criterion that the solution we obtain should maximize fidelity to the boundary conditions on \( \Sigma \), that the results of our calculation should converge better for displacements and stresses in the near zone (surface stress concentration, etc.). But it is not a priori obvious that our results for cross sections, which depend on far fields only, should be optimum. This can only be determined by comparing detailed calculations.
APPLICATION

We will not present the formal expressions for displacements, stresses, and matrix elements here, that being reserved for later publication. We will just show some results for direct and mode-converted scattering from spheres, prolate and oblate spheroids, pillboxes and cones. There is a good reason that the above set of shapes are all axially symmetric, as follows.

If \( s = p, l, m \) where \( p = 1 \) (longitudinal polarization), \( 2, 3 \) (transverse polarization), \( l = 0, 1, \ldots, t_{\text{max}} \) and \( m = -l, \ldots, +l \), then it can easily be shown that, if \( \Sigma \) is axially symmetric,

\[
0_{plm} p l \prime m \prime = 0_{nm} 0_{plm} p l \prime m \prime , \tag{17}
\]

for all boundary conditions and also for \( \tilde{C} \) and \( T \). This is the basis for a gross simplification and acceleration of the calculation, because \( T \) can be computed for each azimuthal eigenvalue \( m \) separately. For axially symmetric flaws, as will be seen from the figures, we can get good results for quite large \( k \alpha \); it being sufficient to take \( l_{\text{max}} \) of the order of 1 or 2 \( k \alpha \) to obtain visual convergence (i.e., the computer-generated plots for successively larger \( l_{\text{max}} \) are indistinguishable to the eye). The largest matrix (17) which must be inverted has rank 3 \( (l_{\text{max}} + 1) \), which is trivial for \( l_{\text{max}} = 14 \), the largest we have needed.

But if \( \Sigma \) has no symmetries, (17) has no \( \delta_{nm} \) factor, the rank of \( Q \) is \( 3(1 + 3 + 5 + \cdots + 2l_{\text{max}} + 1)^2 = 3(l_{\text{max}} + 1)^2 \) and the situation is quite different. Only relatively long wavelengths could be considered.

The figures which follow will be self-explanatory, with the following guide to notation.

**DIRECT**

**CONVERTED**

Differential cross-sections are plotted as functions of \( (\theta, \mu) \), the spherical polar angles in the coordinate system of the flaw, whose axis of symmetry is along the Z axis. The incoming wave is longitudinally polarized with wavevector \( k \). A zero cross-section is plotted at -100 dB.

- \( \theta_0 \) = angle of incoming wavevector \( (\theta_0 = 0) \)
- \( \sigma_{L} = 10 \log_{10} \) (longitudinal cross-section/\( \pi a^2 \))

- \( t_1, t_2, t \) = partial and total mode-converted contributions.

- \( \lambda, \mu = \text{Lamé elastic parameters} \)
- \( a = \text{largest radius of flaw} \)

**Type** = (0,1) = (void, rigid obstacle)

- \( AR = (\text{height/diameter}) \) of defect
  - \( H = 0 \) if defect is a sphere or spheroid
  - \( H = \text{height of cone}/\alpha \) if \( H < 0 \)

- \( l_{\text{max}} = \text{cutoff in } l \); varies from 2 to 14 in plots
- \( n_{\text{leg}} = \text{number of points in numerical integration of surface integrals} \); performed by Gauss-Legendre quadrature.

The time consumed by a CDC 6600 for a given flaw in computing 25 \( \times \) 25 values of \( \sigma_{L} \) and \( \sigma_{T} \) for each of 5 incident angles \( \theta_0 \) is approximately \( l_{\text{max}} n_{\text{leg}} / 5 \) sec.

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<th>( \sigma_{T1} )</th>
<th>( \sigma_{T2} )</th>
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<th>Type</th>
<th>( AR )</th>
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Some indication of the rate of convergence of the calculation with increase of the truncation parameter $k_{\text{max}}$ can be obtained by studying the results for the scattering from a sphere for $k_{\text{max}} = 10$, 12, and 14 of an incident wave with $k = 10$. The differences between cross-sections for $k_{\text{max}} = 12$ and $k_{\text{max}} = 14$ are typically about .1 db or less (too small to see on the graphs), which is about 2%. In contrast, the cross-section for $k_{\text{max}} = 10$ and $k_{\text{max}} = 12$ are sometimes as much as 2 db. Although we have not studied the rate of convergence in any systematic, quantitative way, these trends are not inconsistent with the results of Johnson and Truell (4) on scattering from spherical elastic inclusions. To obtain .6% accuracy for scattering with $k = 10$ from an elastic inclusion, they find it necessary to go to $k_{\text{max}}$ from 10 to 16, depending on the materials involved. Admittedly, the sphere is a poor test of the method, because the matrices are then diagonal in $k$ as well as $m$; consequently any expansion in $k,m$ eigenfunctions should give the same answers.

A better test is afforded by some less symmetric defect. It would be interesting to compare, for an unsymmetric shape like a cone, the rates of convergence of our method, the method used by Varadan and Pao, and the one proposed by Waterman (5) in which he would force the T-matrix to be symmetric at each stage of truncation $N$.

**ACKNOWLEDGMENT**

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**REFERENCES**