APPLICATION OF MOOT TO SCATTERING OF ELASTIC WAVES FROM INCLUSIONS AND CRACKS

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ABSTRACT

The method of optimal truncation (MOOT), a convergent T-matrix scheme, has been applied to the computation of scattering of elastic waves from axially symmetric fluid and elastic inclusions imbedded in an isotropic homogeneous medium. Cones, pillboxes, and spheroids have been considered; an example of frequency and angular dependence of scattering from an oblate spheroid is given. Cracks may be considered as special cases of inclusions wherein the included material is identical to the host. A circular crack, for example, may be simulated by imposing free boundary conditions on the top surface of a pillbox and requiring continuity of displacements and surface tractions elsewhere. Alternatively, it may be feigned by an equatorially cloven spherical inclusion, wherein free boundary conditions are imposed on the bisecting plane and the spherical surfaces are welded.

INTRODUCTION

The problem we wish to solve is illustrated in Fig. 1. A flaw, which is a uniform inclusion in a homogeneous isotropic elastic medium, scatters an incident elastic wave \( \psi_{\text{inc}} \). The shape, orientation, density, and elastic constants of the inclusion are assumed to be known, and we want to calculate the scattered wave.

The method whereby we proceed can be briefly explained as follows. We expand the displacements in a series of partial waves:

\[
\hat{\psi}_{\text{ext}} = \sum p_{\text{pm}} \cdot \hat{p}_{\text{pm}} \quad (1)
\]

\[
\hat{\tau}_{\text{in}} = b_{\text{pm}} \cdot \hat{p}_{\text{pm}} \cdot \hat{p}_{\text{pm}} \quad (2)
\]

where each of the \( \hat{S} \)'s solves the elastic wave equation in the appropriate domain.

\[
\frac{\partial \hat{\psi}_{\text{pm}}}{\partial t} = -\omega^2 \hat{p}_{\text{pm}} = (\lambda+2\mu) \frac{\partial}{\partial x} \hat{p}_{\text{pm}} - \omega \hat{p}_{\text{pm}} \quad (3)
\]

We have assumed a steady-state situation wherein \( \hat{\psi}_{\text{in}} = \hat{\psi}_{\text{inc}} \). The eigenfunctions \( \hat{p}_{\text{pm}} \) are specified by the three integer indices \( p, \ell, m \), which arise naturally when Eq.(3) is written in spherical coordinates. Details, including explicit expressions for the basis functions, can be found elsewhere.\( \hat{p}_{\text{pm}} \)

The incident wave in Eq.(1) is expanded in the exterior regular functions \( \hat{S}_{\text{inc}} \); it is

\[
\hat{S}_{\text{inc}} = \sum_{p_{\text{pm}}} \hat{p}_{\text{pm}} \cdot \hat{p}_{\text{pm}} \cdot \hat{p}_{\text{pm}} \quad (4)
\]

for a plane wave with polarization \( p \) incident from \( (\theta_0, \phi_0) \). The wavenumber \( k \) depends on the polarization \( p \); its magnitude is \( k = \frac{\omega}{c_p} \sqrt{\lambda+2\mu} \) and \( k_t = \frac{\omega}{c_p} \sqrt{\mu} \) for longitudinal and transverse polarizations respectively. An incident longitudinal wave scatters into both polarizations; this is called mode conversion. \( \hat{p}_{\text{pm}} \) is a regular solution of Eq.(3) inside the flaw; it is the same as \( \hat{p}_{\text{pm}} \) except for the wavenumber, which is \( k' \) rather than \( k \).

Hooke's law for the elastic solid is embodied in the expression for the stress tensor \( \sigma_{ij} \):

\[
\sigma_{ij} = \frac{1}{2} \sum_{k'k} c_{ijkl} (\hat{p}_{k',l} + \hat{p}_{k',k}) \quad (5)
\]

in cartesian coordinates. The index following the
comma signifies differentiation. For the isotropic case Eq. (5) simplifies to
\[ \sigma_{ij} = \mu (S_{ij} + S_{ji}) + \lambda \delta_{ij} \text{div} \Sigma \] (6)
in terms of the Lamé elastic moduli \( \lambda \) and \( \mu \). If \( \Sigma \) denotes the unit normal to \( \Sigma \) (see Fig. 2), the surface traction may be written as the contraction of \( \sigma_{ij} \) with it;
\[ t_i = \sum \sigma_{ij} \hat{n}_j. \] (7)

Fig. 2. Surface coordinate system with basis \((\hat{N}, \hat{P}, \hat{Q})\). A cartesian system; \( \hat{N} \) is the unit normal, \( \hat{P} \) and \( \hat{Q} \) are in the surface.

In the cartesian coordinate system shown in Fig. (2), this equation reduces to
\[ t_n = 2\mu S_{nn} + \lambda \text{div} \Sigma \] (8)
\[ t_p = \mu (S_{np} + S_{pn}) \] (9)
\[ t_\phi = \mu (S_{n\phi} + S_{\phi n}) \] (10)
which are convenient expressions because the boundary conditions which determine the displacements and stresses are most easily expressed in terms of normal and parallel components at the surface.

MINIMUM PRINCIPLE

If the \( a_{\text{plm}} \)'s of Eq. (1) are found, then it is easy to get the various observables like cross-sections and phases from them. They are determined by the boundary conditions on \( \Sigma \) and may be found in the following way. For the present, for illustrative purposes we consider the simplest flaws: fixed rigid obstacles or voids. For these cases the boundary conditions are
\[ \Sigma_{\text{ex}} = 0 \quad \text{on} \quad \Sigma \quad \text{(rigid obstacle)} \] (9)
\[ \Sigma_{\text{ex}} = 0 \quad \text{on} \quad \Sigma \quad \text{(void)}. \] (10)

If (9) or (10) hold, then surely the surface integrals
\[ I = \int \Sigma \sigma_{ij} \Sigma_{\text{ex}} \, d\Sigma \] (11)
and
\[ J = \int \Sigma \sigma_{ij} \Sigma_{\text{ex}} \, d\Sigma \] (12)
must vanish if \( \Sigma \) is an exact solution. But our computers are finite, and the sum over \( \lambda \) in (1) must be truncated, say at \( \lambda = \lambda_{\text{max}} \). We take the best choice of \( a_{\text{plm}} \lambda = 0,1, \ldots, \lambda_{\text{max}} \) to be that which minimizes I or J. Thus the equations from which the amplitudes may be determined are
\[ \frac{\partial I}{\partial a_{\text{plm}}} = 0 \quad \text{(obstacle)} \] (13)
\[ \frac{\partial J}{\partial a_{\text{plm}}} = 0 \quad \text{(void)}. \] (14)

It is easy to show that the sequence
\[ \ldots I_{\text{ex}} \geq I_{\text{ex}}_{\text{max}} \geq \ldots \] (15)
converges to \( I_{\text{ex}} = 0 \) if an exact solution exists which satisfies appropriate smoothness conditions.

MATRIX EQUATIONS

The fixed rigid obstacle provides a simple example for the derivation of the matrix equation from which the \( a_{\text{plm}} \)'s may be determined. Now \( I \) is, explicitly
\[ I = \int \Sigma \sum p_{\text{plm}} \Sigma_{\text{ex}} + \sum a_{\text{plm}} S_{\text{ex}}^{(+)} \, d\Sigma. \] (16)
The derivatives of this equation are
\[ \frac{\partial I}{\partial a_{\text{plm}}} = \int \Sigma q_{p_{\text{plm}},p_{\text{plm}}} \Sigma_{\text{ex}} + \sum a_{\text{plm}} S_{\text{ex}}^{(+)} \] (17)
where
\[ q_{p_{\text{plm}},p_{\text{plm}}} = \int \Sigma \Sigma^{(+)} \Sigma_{\text{ex}}^{(+)} \] (18)
and
\[ \Sigma_{\text{ex}}^{(+)} = \int \Sigma \Sigma^{(+)} \Sigma_{\text{ex}} \] (19)

Equation (17) can be abbreviated
\[ qa + \hat{Q}d = 0, \] (20)
or
\[ a = -\hat{Q}^{-1}Qd = Td. \] (21)

This defines the T-matrix, which is independent of the direction and polarization of the incident wave; this information is contained in \( Q_{\text{plm}} \).

As a practical matter it should be noted that the \( Q \) and T-matrices are diagonal in \( m \) if the flaw has axial symmetry. They can therefore be written in block-diagonal form (11), which usually reduces the amount of computation required by orders of magnitude. For this reason the shapes we calculate are always axially symmetric.

T-matrices which in the limit of \( k_{\text{max}} \rightarrow \infty \) become identical to that defined in (21) can be defined in an infinite number of ways (1, 3). This one is unique and optimal in the sense that it is generated by a convergent sequence of surface integrals.
BOUNDARY CONDITIONS

In Table I we bring together the boundary conditions for various kinds of defects, and show the integral functions of the surface fields which must be minimized in HOMOT.

**Table I**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Function</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Void</td>
<td>$S_{ex} = \text{unrestricted}$; $t_{ex} = 0$</td>
<td>$J = \int</td>
</tr>
<tr>
<td>Fixed Rigid Obstacle</td>
<td>$S_{ex} = 0$; $t_{ex} = \text{unrestricted}$</td>
<td>$I = \int</td>
</tr>
<tr>
<td>Elastic Inclusion (welded)</td>
<td>$S_{ex} = S_{in}$; $t_{ex} = t_{in}$</td>
<td>$J = \int</td>
</tr>
<tr>
<td>Elastic Inclusion (slippery)</td>
<td>$S_{ex} = S_{in}$; $t_{ex} = t_{in}$</td>
<td>$I = \int</td>
</tr>
<tr>
<td>Fluid Inclusion</td>
<td>$S_{ex} = S_{in}$; $t_{ex} = t_{in}$</td>
<td>$J = \int</td>
</tr>
</tbody>
</table>

Table I. Boundary conditions to be imposed on $\Sigma$ for various types of flaws. Conditions on displacements and surface tractions are given, as are the surface integral functions of the amplitudes which are to be minimized in several cases. Where more than one functional ($I$, $J$, and $K$) of the surface fields is to be minimized, we will minimize a positive linear combination of them, as discussed in the text. The Elastic Inclusion (unglued) can be made the basis for a treatment of a crack, as is also discussed in the text.

As an example of how the expressions given in Table I can be used to derive linear equations for the amplitudes we consider the case of the welded elastic inclusion. Here we need to worry about the internal displacements and stresses $S_{in}$ and $t_{in}$; thus after forming a positive linear combination of $I$ and $J$

$$K = \alpha I + \beta J$$

and using (1), (2), (4), and (8) we can write, supressing subscripts,

$$\frac{\partial K}{\partial \sigma} = 0 = Q_a - Q'$$

and

$$\frac{\partial K}{\partial \sigma} = 0 = -Q_a + Q_b - Q_c.$$
viewed as an identical bisected spherical inclusion. These 3 crack simulations are pictured in Fig. 4.

Fig. 3. Backscattering of an incident longitudinal elastic wave from an oblate (aspect ratio = 2) spheroidal inclusion of magnesium in stainless steel \( (\rho/\rho' = 4; \lambda/\lambda' = 2) \). The axis of the spheroid is at \( \theta = 0 \); the curves for successively larger angles of backscatter are displaced downward 10 dB. \( \theta_{max} = 8 \) was used for these calculations, which were done for 50 values of \( ka = 0.1 \) to 5.0. About 10 min. were consumed on a CDC 6600 to generate the scattered amplitudes which produced these data. Thirty Gauss-Legendre quadrature points were used in the numerical integrations which produced the matrix elements.

Fig. 4. Three ways to simulate a circular crack with M00T. (a) is an oblate spheroidal void, with aspect ratio 0. (b) is an identical inclusion, pillbox-shaped (or with any other rotationally symmetric shape) welded everywhere except over a free circular surface, where free boundary conditions are imposed, (c) comprises a pair of identical opposed hemispherical welded inclusions, each with its plane circular surface free.

Figure (4a) depicts the ingenious representation of a crack as the limit (aspect ratio \( \rightarrow 0 \)) of an oblate spheroidal void. Problems arise here in the evaluation of the surface integrals because the integrands often become singular when the surface approaches the origin (the dots in Fig. (4)). These difficulties can be partly mitigated in different formulations of T-matrix theory\(^4\), but we have not found a way to overcome them in M00T. The schemes illustrated in Fig. (4b & 4c) avoid singular integrands by keeping the origin away from the region of space where the fields are expanded in regular functions. Fig. (4c) has the apparent disadvantage that it requires an additional eigenfunction expansion; that is, space is divided into an external region and two internal regions. Because of the up-down symmetry when the crack is plane circular, though, this drawback can be circumvented, and for this case Fig. (4c) is the formulation of preference. The very asymmetry responsible for the inferiority of Fig. (4b) for the plane crack, though, makes it the preferable picture when the crack is not planar, e.g. a spherical cap or a Frisbee shape.

A difficulty common to all 3 crack simulations shown in Fig. (4) is caused by the fact that the displacements and stresses near a crack edge are singular; the displacement behaving like \( \rho^{1/2} \), the stress like \( \rho^{-1/2} \), where \( \rho \) is the distance to the edge. These singularities cannot be well represented by a partial wave expansion (1) and (2) with a reasonable number of terms, and an accurate description of the singularities appears to be critically important to a scattering calculation\(^5\).

The form of the singular displacements can be easily calculated asymptotically close to the crack edge in terms of just 3 independent amplitudes\(^5\). This fact can be exploited to solve the problem of the edge singularities in a number of ways. One way is to actually surround the crack edge with a torus and expand the displacement in the torus in terms of the amplitudes of the 3 singular modes. Then one has additional surface integrals over the torus and 3 additional amplitudes to determine for each \( m \). Another way is to require, in a least-squares sense, that the fields satisfy crack conditions at the quadrature points of the surface integrals nearest the crack edge. This is the simplest alternative and is presently being pursued.

REFERENCES

4. V. V. and V. K. Varadan, Elastic Wave Scattering by Rough Flaws and Cracks (these proceedings).

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SUMMARY DISCUSSION  
(William Visscher)

Don Yuhas (Sonoscan): You listed two types of elastic inclusions, one slippery and one with welded boundary conditions. Could you give a physical example of each?

William Visscher: Well, not really. The real reason that I listed the slippery inclusion is that it's a stepping stone from the welded inclusion to the fluid inclusion.

Don Yuhas: I'm just trying to get some idea so that if somebody goes into a lab should they measure in order to confirm what you're calculating.

William Visscher: I haven't calculated anything for the fluid inclusion yet, but the physically realistic ones are the welded inclusion and the fluid inclusion. Slippery elastic is sort of halfway between.

James Rice (Brown University): It's a very small point, but you said it was obvious that for the rigid inclusion, the displacements have to vanish from the surface, and I thought it was obvious that the displacements have to be those of a rigid body, which can move.

William Visscher: I should have said rigid immovable inclusion.

George Gruber (Southwest Research): I'm very dubious about extrapolating from an inclusion-like defect to a crack-like defect because I basically believe that inclusion-like defects behave like scatterers having smooth boundaries, and a crack-like defect has sharp boundaries and basically behaves like a diffracter. And, mathematically you might be able to come over, but physically the interaction mechanisms of the ultrasound with the defect types of these two basic types seem to be widely different and seem to come from a different direction.

James Krumhansl (Session Chairman): I guess that comment is an interesting comment. It has a variety of applications to discuss. Maybe you want to comment, Bill, but we will call it at that.

William Visscher: It was worries like that which motivated me to have a different expansion in the neighborhood of the crack edge. That is, you can put in exact forms for the displacement and stress in the immediate neighborhood of the crack edge and use separate expansions there.

James Krumhansl: So my comment isn't entirely opaque: certainly the crack properties, for example, by Rice and Budiansky have been mapped on the very long wavelength region where there is no diffraction.

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