A quantitatively accurate theory of stable crack growth in single phase ductile metal alloys under the influence of cyclic loading

Peter Huffman

Iowa State University
A quantitatively accurate theory of stable crack growth in single phase ductile metal alloys under the influence of cyclic loading

by

Peter Joel Huffman

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Major: Materials Science and Engineering

Program of Study Committee:
Scott P. Beckman, Major Professor
Scott Chumbley
Wei Hong
Ralph Napolitano
Alan Russell

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DEDICATION

I would like to dedicate this dissertation to the people who encouraged me when I first made the decision to attend college, namely Bob Benda and family, and Jess Lorentzen and family. I would also like to dedicate this to Dan Palan, who first introduced me to materials engineering as a discipline, and without whom, I may never have started down this path.
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ABSTRACT

Although fatigue has been a well studied phenomenon over the past century and a half, there has yet to be found a quantitative link between fatigue crack growth rates and materials properties. This work serves to establish that link, in the case of well behaved, single phase, ductile metals. The primary mechanisms of fatigue crack growth are identified in general terms, followed by a description of the dependence of the stress intensity factor range on those mechanisms. A method is presented for calculating the crack growth rate for an ideal, linear elastic, non-brittle material, which is assumed to be similar to the crack growth rate for a real material at very small crack growth rate values. The threshold stress intensity factor is discussed as a consequence of “crack tip healing”. Residual stresses are accounted for in the form of an approximated residual stress intensity factor. The results of these calculations are compared to data available in the literature. It is concluded that this work presents a new way to consider crack growth with respect to cyclic loading which is quantitatively accurate, and introduces a new way to consider fracture mechanics with respect to the relatively small, cyclic loads, normally associated with fatigue.
CHAPTER 1. INTRODUCTION

In 1849, Robert Stephenson, chairman of the British Institution of Mechanical Engineers, commented about fatigue failures

"I am only desirous to put the members on their guard against being satisfied with less than incontestable evidence as to a molecular change in iron, for the subject is one of serious importance, and the breaking of an axle has on one occasion rendered it questionable whether or not the engineer and superintendent would have a verdict of manslaughter returned against them". Carlson and Kardomateas (1995)

Fatigue persists in being one of the most common causes of equipment and structure failure in the modern world. Suresh (2004); Anderson (2005); Mikheevskiy et al. (2012); Lados and Apelian (2004); Courtney (2005) In spite of an enormous body of work on this general topic, there has yet to be a quantitative link between materials properties, applied load, and crack growth rates, that enables an accurate prediction of fatigue crack growth behavior. A number of possible mechanisms have been identified that are likely contributors to fatigue crack growth; however, calculations based on these models fail to yield quantitatively accurate predictions. Recent work has identified multi-parameter crack growth driving forces that have improved the description of crack growth rates. Noroozi et al. (2005); Xiong and et al (2008); Kujawski (2001); Kwofie and Rahbar (2011); Lu and Liu (2012) However, these multi-parameter driving force methods require fatigue data, in the form of crack growth rates, strain-life, or both. This work will establish an expression with a functional form similar to those of the multi-parameter driving force models, with constants calculated from basic materials properties. The goal of this work is to identify the causal link between crack growth rate and applied load, as a function of those measurable materials properties.

This work will be divided into several sections, each detailing a piece of the crack growth
rate puzzle, which will finally be assembled to form a clear and quantifiable description of fatigue crack growth rates for well behaved, single phase, ductile metals. As a starting point, the two primary candidates for fatigue crack growth mechanisms are identified and discussed. Following that is an examination of how the various forms of energy, namely plastic work and elastic energy storage, drive these mechanisms. The relationships between applied load and the forms of energy that drive the crack growth mechanisms are identified and calculated. An idealized material behavior is formulated to calculate the crack growth rate at low values, near a few atomic diameters per cycle. A concept of “crack tip healing” is then introduced as a partial explanation of the crack growth rate behavior near the threshold stress intensity factor range. A method of approximating a residual stress intensity factor is also discussed. Finally, calculations based on these concepts are compared to measured data for a variety of materials, from numerous sources.

1.1 Methods in fatigue

In the last few decades, the most common approaches to fatigue have included the stress-life approach, the strain-life approach, and the fracture mechanics approach. Each will be discussed in more detail, in the following sections.

1.1.1 Stress life

The stress-life approach was originally developed in 1860, by Wohler. Wohler (1860) To make use of the stress-life approach, a stress-life relationship must first be established. In uniaxial stress-life, specimens are cyclicly loaded in uniaxial tension and compression until they fail, from some predetermined minimum stress to a maximum stress. It is a common practice to perform these tests at a stress ratio of $R = -1$. The stress ratio is the ratio of minimum stress to maximum stress, $R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}}$. A stress ratio of $-1$ is often referred to as fully reversed loading. Suresh (2004); Courtney (2005) The stress amplitude, $\sigma_{\text{max}} - \sigma_{\text{min}}$, or stress range, $\sigma_{\text{max}} - \sigma_{\text{min}}$, is plotted against the number of cycles, $N_f$, or the number of stress reversals, $2N_f$, as is shown schematically in figure 1.1.
Figure 1.1 schematic of a stress life diagram showing the stress ratio effect, where the Y axis is stress range. The notable features are that at a higher stress ratio, although the curve shape is similar, the stress values are much lower for any particular life. Or, to put it another way, for a particular stress range, the life time is considerably shorter for a higher stress ratio. Stress ratio can also be quantified in terms of mean stress, as it is in this figure.
If a part is repeatedly loaded between two load levels, the stresses can be calculated, and the stress range and stress ratio can be used to estimate how many loadings a part will take before it fails. One weakness of the stress-life method is inconsistent failure criteria. Because of differences between test specimens and actual parts in service, or between different parts, it could take a different number of cycles for a crack to initiate and propagate until final fracture occurs. It is also well known that this method works well for high cycle fatigue, when the stress range is relatively small, but not as well for low cycle fatigue, when yielding and plastic deformation occur in a larger volume, and are not highly localized to the crack tip. Suressh (2004) Because of these weaknesses, the stress-life approach is often used only to establish a fatigue limit, or an endurance limit. For many steels, and some other materials with a body centered cubic crystal structure, it has been thought that there is a stress below which fatigue will not lead to part failure. Courtney (2005) Although there is some disagreement about this, it is generally agreed that the traditional fatigue limit is fictitious and a lower limit related to dislocation irreversibility exists at a lower stress range. Bathias (1999); Pyttel et al. (2011); Zettl et al. (2006); Muller-Bollenhagen et al. (2010); Krupp et al. (2010) The traditional fatigue limit is chosen as the stress at which the stress-life is $10^7$ cycles, and parts intended for long life are designed to stay below that stress level during service. For materials that have been known not to show traditional fatigue limit behavior, an endurance limit is used, which is often chosen as the stress at which the stress-life is $10^6$ cycles. When parts must be designed above the fatigue limit or endurance limit, the strain life can often give a more accurate gauge of part life. Suressh (2004); Courtney (2005)

1.1.2 Strain life

The strain-life bears some similarity to the stress-life, although its development and use are nearly a century more recent. Similar to a stress-life, test specimens are cyclically loaded until failure. The difference is that for a strain-life, a strain measurement device, such as an extentometer, is attached and a feedback loop is used to limit the maximum and minimum loads such that a maximum and minimum strain are reached, respectively. In 1910, Basquin found that there was a power law relationship between strain and fatigue life, for low strains.
This should have been fairly obvious, considering the known relationship between stress and life, and the linear relationship between stress and strain at small stresses. In 1954, Manson and Coffin independently showed a power law relationship between plastic strains and fatigue life. Manson (1953); Coffin (1954) The Manson Coffin equation,

\[ \frac{\Delta \varepsilon_p}{2} = \varepsilon_f (2N_f)^c \]  

(1.1)

where \( \varepsilon_p \) is the plastic strain, \( \varepsilon'_f \) is the fatigue ductility coefficient, and \( c \) is the fatigue ductility exponent, describes the relationship between plastic strain and fatigue life. Both the fatigue ductility coefficient and exponent are fitting parameters. Combined with the relationship described by Basquin, the elastic and plastic portions of strain can be combined to make the Basquin Manson Coffin equation,

\[ \frac{\Delta \varepsilon}{2} = \frac{\Delta \varepsilon_e}{2} + \frac{\Delta \varepsilon_p}{2} = \frac{\sigma'_f}{E} (2N_f)^b + \varepsilon'_f (2N_f)^c \]  

(1.2)

where \( b \) is the fatigue strength exponent, \( \sigma'_f \) is the fatigue strength coefficient, and \( E \) is the Young’s modulus. Frequently this is simply referred to as the Manson Coffin equation. A schematic strain-life is shown in figure 1.2. Like the stress-life relationship, the strain-life relationship is sensitive to the stress ratio. There are a number of techniques for accounting for the stress ratio, such as the Smith-Watson-Topper energy based approach, and the Morrow mean stress correction. Smith et al. (1970); Socie and Morrow (1980)

Even if the stress ratio effect can be accounted for, realistic service loads are never so regular. These irregular loads lead to what is known as variable amplitude fatigue. A schematic of a variable amplitude load history is shown in figure 1.3. Another schematic showing the case of an occasional overload is shown in figure 1.4.

Obviously, it can become difficult to count cycles because of the nature of variable amplitude loading. This might not be as obvious in figure 1.4, but it is clear from figure 1.3. For this reason, fatigue life is most often discussed in terms of reversals, instead of cycles. Suresh (2004) The most often used method for estimating variable amplitude fatigue life is the Palmgren-Miner rule. Wulpi (2000); Suresh (2004); Courtney (2005) The Palmgren-Miner rule assumes that damage from each strain reversal that could be considered a tensile peak is equivalent to \( \frac{1}{N_f} \),
Figure 1.2  Schematic of a strain life diagram showing the elastic, plastic, and total strain life. Typical strain life curves for metals will range in life from 10, 100, or 1000 reversals, to 1,000,000 or 10,000,000 reversals. Strain ranges will generally range between 0.01 and 0.001, give or take an order of magnitude on either side. For other materials, this can widely vary.
Figure 1.3  Schematic of variable amplitude loading. This is a case of random loading. Aspects of a load history might be randomized in order to simulate unknown service loads, when performing variable amplitude fatigue calculations.
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where $N_f$ is the number of such reversals expected to cause failure if each reversal were at that strain range and load ratio. Miner (1945) The Miner rule can be expressed as

$$\sum_{i=1}^{n} \frac{n_i}{N_{fi}} = C, \quad (1.3)$$

where $n_i$ is the number of times that a particular strain range is occurs at a particular stress ratio, $N_{fi}$ is the number of reversals until failure for the strain range and stress ratio $i$, and $C$ is a constant, representing a state of damage. Ideally, $C$ should always be 1. However, it has been found that $C$ often varies between 0.7 and 2.2. This is considered a shortcoming of the Palmgren-Miner rule, and since there has been a considerable amount of work on the topic of variable amplitude fatigue. Fatemi and Yang. (1998); Huffman and Beckman (2013); Walther and Eifler (2007); Rushton et al. (2007); Varvani-Farahani et al. (2007); George et al. (2007); Pinto et al. (2010); Sumsel and Tayolor (2011) Many of these techniques for calculating variable amplitude fatigue life use variations on the linear damage rule proposed by Miner, while some use more intrinsically accurate non-linear damage rules. Fatemi and Yang. (1998) While many of these methods consider the strain history in terms of the order in which the strains are applied, they often do not consider that a particular load will cause a lesser or greater amount of damage if the state of damage when that load is applied is lesser or greater. Lemaitre and Chaboche (1985) Many of these use a rain flow cycle counting algorithm, which has the purpose of counting cycles that are considered to be the ones that are responsible for causing damage, or crack growth. ASTM (2011) A weakness of the rain flow algorithm is that while it has a way of accounting for the order in which strains are applied, the use of rain flow counted cycles does not directly calculate damage at each cycle. This has the potential to lead to inaccuracies in lifetime calculations. Huffman and Beckman (2013)

Strain-life methods have some of the same shortcomings as stress-life calculations. It can be difficult to account for the complicated state of stress for a part in service, and express that in terms of the types of axial and torsional loads that are used in testing. The criteria for failure of a test specimen might not match the criteria for failure for a part in service. For example, both strain-life and stress-life often assume that once a crack has initiated, the part has failed. Many parts can remain in service without failing long after a crack is initiated. Suresh (2004);
Courtney (2005) Because of this, the field of damage tolerant approaches based on fracture mechanics has grown substantially in the past few decades.

1.1.3 Fracture mechanics

The earliest formulation of modern fracture mechanics came from Griffith, who proposed the idea that thermodynamics would apply to the fracture of materials. Anderson (2005); Griffith (1921) Using the stress analysis of Inglis in ref Inglis (1913), he proposed an energy balance relationship between the formation of a crack in terms of new surfaces, the input strain energy, and the strain energy relieved by the forming crack,

\[
\frac{dE}{dA} = \frac{d\Pi}{dA} + \frac{dW_s}{dA} = 0
\]  

where \( A \) is the cross-sectional area of some material, \( \Pi \) is the potential energy from sources such as the externally applied load and the internal strain energy, and \( W_s \) is the energy required to create new surfaces.

For a through-thickness crack in an infinitely wide plate, such as shown in figure 1.5, Griffith showed that the potential energy could be expressed as

\[
\Pi = \Pi_0 - \frac{\pi \sigma^2 a^2 B}{E}
\]  

where \( \Pi_0 \) is the potential energy of an uncracked plate, \( 2a \) is the size of the crack, \( B \) is the thickness of the plate, and \( E \) is the elastic modulus. The energy required to form the surfaces of the crack is, of course, related to the surface energy of the material by

\[
W_s = 4aB\gamma_s
\]  

where \( \gamma_s \) is the solid-air surface energy for the material. Keeping in mind equation 1.5, it can be shown that

\[
\frac{d\Pi}{dA} = \frac{\pi \sigma^2 a}{E}
\]  

and the energy to create the surfaces, per area of surface created is

\[
\frac{dW_s}{dA} = 2\gamma_s
\]
Figure 1.5  Schematic of a through crack in an infinite plate, where the load is applied perpendicular to the direction of the crack. The crack will also grow perpendicular to the applied load.
from which it can be shown that the fracture stress of a brittle, linear elastic material, can be expressed as

$$\sigma_f = \left( \frac{2E\gamma_s}{\pi a} \right)^{1/2}. \quad (1.9)$$

Griffith showed that this method worked very well for brittle materials, such as window glass. Unfortunately, the case is not so simple for ductile metals. Fortunately, that is because ductile metals are tougher than glass by roughly 3 orders of magnitude. Anderson (2005) The Griffith equation need not be abandoned, though, because it can be generalized. The equation for fracture stress can be generalized in terms of fracture energy,

$$\sigma_f = \left( \frac{2Ew_f}{\pi a} \right)^{1/2} \quad (1.10)$$

where $w_f$ is the fracture energy. Anderson (2005)

A related method proposed by Irwin, introduced a concept of the energy release rate,

$$G = -\frac{d\Pi}{dA} \quad (1.11)$$

where $G$ is called the energy release rate. In this case, it is not a time sensitive rate, but an amount of energy released per unit area created by the crack. Irwin (1956) This is the earliest conception of a crack growth driving force. Anderson (2005) The energy release rate relates to the ideal Griffith problem in a fairly obvious way,

$$G = \frac{\pi \sigma^2 a}{E}. \quad (1.12)$$

By energy balance, the crack will grow whenever the elastic energy released by elastic relaxation from the loss of traction in the area of a crack is greater than the energy needed to create new surfaces for the crack, or

$$G_c = \frac{dW_x}{dA}, \quad (1.13)$$

where $G_c$ is an energy based fracture toughness and $W_x$ is the work necessary to create new surfaces. This is, perhaps, best described by examining the potential energy of a body under a load. For an elastic body, the potential energy can be defined by

$$\Pi = U - F \quad (1.14)$$
where $U$ is the stored strain energy resulting from an external load, $F$ is the work done by the external load, and $\Pi$ is the potential energy of the body. If there is a difference between the stored elastic energy in a body from the application of a load, and the work done by the load, that energy must have been dissipated somehow. The concept of load control, as discussed in regard to a stress-life test, will be used here to help describe the concept of an energy release rate, $G$. The work done by an applied load, $F$ from equation 1.14, is equivalent to a load times displacement, or $P\Delta$, where $P$ is the applied load, and $\Delta$ is the displacement of that load. The stored elastic energy, $U$, can then be described by

$$U = \int_0^\Delta P\Delta = \frac{P\Delta}{2}.$$  

(1.15)

Figure 1.6 shows a schematic of a cracked plate, loaded by a hanging body of fixed mass.

The elastic energy difference for a change in crack size caused by a hanging load can be related to the load displacement curve, as shown in figure 1.7. For the same load, a growing crack would lead to a greater displacement. In terms of physical work, more work has been performed on the system. However, instead of the work transforming into potential energy, as in strain energy stored in a spring, the work drives a crack to grow. The area indicated in the load displacement schematic represents the difference in strain energy for the same load, for two different crack sizes. Taking equation 1.15, the expression for work done by an applied load, and recalling equation 1.14,

$$\Pi = \frac{P\Delta}{2} - P\Delta$$  

(1.16)

and so

$$\Pi = -U.$$  

(1.17)

Recalling that the plate thickness is $B$, and the change in area is $Bda$, where $a$ is the crack size, the energy release rate can be described as

$$G = \frac{1}{B} \left( \frac{dU}{da} \right)_P$$  

(1.18)

which, directly in terms of the load and geometry, is

$$G = -\frac{P}{2B} \left( \frac{d\Delta}{da} \right)_P.$$  

(1.19)
Figure 1.6  Schematic of a cracked plate with the top fixed, and a mass hanging from the bottom. If the mass were to eat a bunch of tacos for lunch, the load would change, and the crack might grow. Fracture mechanics can be used to calculate how many tacos the mass could eat, before causing unstable crack growth, which would lead to the mass falling.
Figure 1.7  Load vs. displacement curve for two different crack sizes in the same cracked plate, at one load level. The area between the curves indicates the difference in strain energy for the same load. As shown, the larger the crack, the greater the displacement.
Considering a great deal of modern fatigue testing is done in strain control, such as strain-life tests, it is pertinent to examine the difference in the energy release rate for a crack growing in the strain controlled condition. For strain control, the change in work done by the applied load is by definition 0. A force that does not move an object does no mechanical work. A load displacement curve for a constant displacement, at two different crack sizes, is shown in figure 1.8.

![Load vs. Displacement Curve](image)

**Figure 1.8** Load vs. displacement curve for two different crack sizes in the same cracked plate, at one displacement level. The area between the curves indicates the difference in strain energy for the same displacement. As shown, the same displacement requires a greater load, for a smaller crack.

The energy release rate for the constant displacement crack growth is

\[
G = -\frac{1}{B} \left( \frac{dU}{da} \right)_{\Delta}
\]  

(1.20)
which, in terms of load, is

$$G = -\frac{\Delta}{2B} \left( \frac{dP}{da} \right)_{\Delta}.$$  

(1.21)

The stiffness of a material is often described using the elastic modulus, $E = \frac{\sigma}{\epsilon}$, however, sometimes it is convenient to consider the stiffness of a structure, $S = \frac{P}{\Delta}$, where $S$ is the body stiffness. Likewise, the compliance of a structure, which is the inverse of stiffness, is

$$C = \frac{\Delta}{P}.$$  

(1.22)

where $C$ is the compliance. Using this equation for compliance, it can be seen from equations 1.19 and 1.21 that for both constant load, and constant displacement,

$$G = \frac{P^2}{2B} \frac{dC}{da}$$  

(1.23)

for a linear elastic material. From this, the change in strain energy in a body for a change in crack size is,

$$\left( \frac{dU}{da} \right)_P = \left( \frac{dU}{da} \right)_{\Delta}.$$  

(1.24)

There is an insignificant difference in $(dU)_P$ and $(dU)_{\Delta}$ for the same change in crack size, of the magnitude $dPd\Delta/2$, which is neglected. Anderson (2005) In fracture mechanics, the question arises of whether or not a crack growth will be stable. Unstable crack growth refers to a condition in which a crack will freely propagate through an entire body. Stable crack growth refers to when a crack will extend, but cease to grow until the load, or displacement, have changed. The concept of crack stability led to the development of the resistance curve, or “R curve”.

For a brittle material, the R curve will have a very “square” shape to it. This indicates that virtually any crack that will grow, will be unstable and continue to grow until fracture, for an applied load. For an example of how an applied load curve is displayed on an R curve, see figure 1.9. The crack will grow from $a_i$ to the point on the R curve where the load curve first crosses it. In the case of figure 1.9, there would be no crack advance. For a higher load, it is clear that there would be an instability that would cause catastrophic failure, for any condition in which the crack would extend.
Figure 1.9  R curve showing a non growing crack, in a cracked plate of finite size made of brittle material, at one load level.
For a constant displacement, the curve for which can be seen in figure 1.10, a crack could advance until the load is too low to satisfy the condition for crack growth, which is the energy release rate. The energy release rate can be plotted on an R curve as a straight line starting at the origin, with a positive slope. Recall from earlier, that at a constant displacement, as the crack grows, the load will decrease. It is important to know that the energy release rate, load, and displacement based curves are all dependent on how the load is applied to the geometry. Anderson (2005) The R curve for a tough material, such as most structural metals, is a rising curve. The slope tends to be much higher, and a crack is much more likely to be stable at higher load levels or displacements, than would be expected for a brittle material. A crack in

Figure 1.10 R curve showing an advancing crack, in the same cracked plate made of brittle material, at one displacement level.
a tough material can grow for a considerable distance, and still stop before extending all the way through the piece. Figure 1.11 shows a schematic R curve for a tough material, with a load curve.

![R curve showing an advancing crack, in a finite cracked plate made of tough material, at one load level.](image)

A tough material with a crack driven by a constant displacement has an R curve with a crack growth driving force curve as shown in figure 1.12. A relatively high displacement is necessary in order for the crack to propagate all the way through the piece, which is what it means to have a high fracture toughness, such as is common in metals. It is also common for a metal to be deformable to relatively high plastic strains, often much greater than 10%, before catastrophic failure occurs, which is directly related.
Figure 1.12  R curve showing an advancing crack, in the same cracked plate made of tough material, at one displacement level.
The stresses in a material are orders of magnitude too low to break atomic bonds in the conventional sense, which known as ideal fracture. Courtney (2005). For a number of reasons, failure occurs at stresses well below the ideal fracture strength. For metals, this is a complicated matter involving dislocations and rearrangements of atoms from sliding motions, which requires substantially lower stress.

A factor that impacts both fracture of metals and brittle materials, is stress concentration from the effects of the geometry. In particular, a crack causes a significant, localized, increase in actual stress. For certain geometries and applied loads, and for isotropic, linear elastic materials, closed form solutions have been found to describe the stress distributions relative to a crack tip. A great number of publications have been produced for various geometries and load conditions. Irwin (1956); Sneddon (1946); Westergaard (1939); Williams (1957). It was shown that a general description of the stress field in a cracked body, for a linear elastic, isotropic material, is

\[
\sigma_{ij} = \left( \frac{k}{\sqrt{r}} \right) f_{ij}(\theta) + \sum_{m=0}^{\infty} A_m r^m g_{ij}(\theta) \tag{1.25}
\]

where \(\sigma_{ij}\) is the stress tensor, \(r\) is the distance from the crack tip, \(\theta\) is the angle from horizontal with respect to the direction of the crack, \(k\) is a constant, \(A_m\) is an amplitude and \(f_{ij}\) and \(g_{ij}\) are dimensionless functions of the angle \(\theta\). The most significant result of this is that as \(r\) approaches 0, the first term increases to infinity, and the following terms will diminish. Anderson (2005)

This proves that for a linear elastic material, the stress goes as \(1/\sqrt{r}\) regardless of the particular geometry and load. It should also be noted that this is a singularity; the stress goes to \(\infty\) as \(r \to 0\). The singularity will be critical for analysis for non-linear-elastic materials as well. However, that will be discussed later. This leads to an in-disposable concept in both fracture and fatigue engineering: the stress intensity factor, \(K\). The stress intensity factor replaces \(k\) by the relationship \(K = k\sqrt{2\pi}\). For this discussion, the focus will be on mode I loading, where the load on a cracked plate will be in the plane of the plate, perpendicular to the direction of the crack. For this discussion, \(K\) will be taken to mean \(K_I\), the mode I stress intensity factor.

A schematic of mode I loading is shown in figure 1.13.
Figure 1.13  Schematic of Mode I loading. The crack growth is on the plane perpendicular to the direction of the applied load. Figures 1.5 and 1.6 are also examples of mode I loading.
The stress field for mode I loading can be expressed as
\[
\lim_{r \to 0} \sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta),
\]
and the crack tip displacement field in the direction of loading is
\[
uy = \frac{K_I}{2\mu} \sqrt{\frac{r}{2\pi}} \sin \left( \frac{\theta}{2} \right) \left( \kappa + 1 - 2\cos^2 \left( \frac{\theta}{2} \right) \right)
\]
where \(\mu\) is the shear modulus, \(\kappa = 3 - 4\nu\) for plain strain, or \(\kappa = \frac{3-\nu}{1+\nu}\) for plain stress, and \(\nu\) is Poisson’s ratio. Although it is convenient to use only the leading term of equation 1.25, it has limitations that one should keep in mind. The stress intensity factor is generally expressed

![Figure 1.14](image-url)  
Schematic of the stress distribution in front of a crack tip, for mode I loading, at \(\theta = 0\), showing the difference between the actual stress distribution, and the result of using only the leading term of the stress distribution. A circle indicates the region ahead of the crack tip for which the solutions are virtually identical, known as the singularity dominated zone. Anderson (2005)
where \( Y \) is a geometrical factor. For this work, the ideal Griffith cracked plate will be used, where \( Y = 1 \), and so \( K = \sigma \sqrt{\pi a} \). For this ideal situation, the energy release rate and the stress intensity factor are directly related by

\[
G = \frac{K^2}{E}
\]  

(1.29)

where \( E' = E \) for plain stress, and \( E' = \frac{E}{1 - \nu^2} \) for plain strain.

For a non-linear elastic material, the energy release rate must be reevaluated. The non-linear energy release rate is

\[
J = -\frac{d\Pi}{dA}
\]  

(1.30)

where \( J \) is the energy release rate, and the other terms are defined the same as they were above. Rice (1968) The potential energy expression is now

\[
\Pi = U - P\Delta = -U^*
\]  

(1.31)

where \( U^* \) is the complimentary strain energy. For the linear elastic case, the complimentary strain energy was the same as the strain energy, \( U \), which was equal in magnitude and opposite in sign to the potential energy, \( \Pi \). A non-linear load displacement curve for two different crack sizes is shown schematically in figure 1.15.

The complimentary strain energy is

\[
U^* = \int_0^P \Delta dP.
\]  

(1.32)

The energy release rate can be related to the complimentary strain energy by

\[
J = \left( \frac{dU^*}{da} \right)_P
\]  

(1.33)

for a constant load and changing displacement, or the strain energy can be related to the energy release rate by

\[
J = \left( \frac{dU}{da} \right)\Delta
\]  

(1.34)

for a constant displacement and changing load. The difference between the constant load and constant displacement conditions is \( \frac{1}{2}dPd\Delta \), and much like in the linear elasticity case, it is
Figure 1.15  Load displacement curve for two different crack sizes in the same cracked plate for a non-linear elastic material, at one displacement level. The shaded area shows the difference in strain energy if two plates with different crack sizes, that are otherwise identical, are strained to the same displacement. A plate with a smaller crack will require a larger load to produce the same displacement as the plate with the larger crack.
small enough to be neglected. Anderson (2005) As in the linear elastic case, the energy release rate can be expressed as a function of the loads and displacements by

\[
J = \left( \frac{\partial}{\partial a} \int_0^P \Delta dP \right)_P = \int_0^P \left( \frac{\partial \Delta}{\partial a} \right)_P dP \tag{1.35}
\]

or

\[
J = -\left( \frac{\partial}{\partial a} \int_0^\Delta P d\Delta \right)_\Delta = \int_0^\Delta \left( \frac{\partial P}{\partial a} \right)_\Delta d\Delta. \tag{1.36}
\]

For this formulation of \( J \), it can be shown that in the linear elastic case the relationship between \( J \) and the stress intensity factor is

\[
J = K_I^2 E \tag{1.37}
\]

demonstrating that \( J \) is a generalized form of the energy release rate. It is critical to note that \( J \) cannot be readily applied to non-linear, non-elastic materials, in the same way that \( G \) was applied to linear elastic materials. However, it can be used to compare two similar crack sizes in terms of the energy required to cause them, supposing that they are both the product of an initial load, and not a crack growing in one plate from the application of that load. This is one of the reasons that prevent \( J \) from being a complete solution to calculating crack growth rates in fatigue. It is sometimes used as a crack growth rate driving force, however, instead of \( K \).

Rice proved mathematically that \( J \) is a path independent line integral. Rice (1968) Figure 1.16 shows a closed path, \( \Gamma^* \), as an example of the path for \( J \). The \( J \) integral can be evaluated along \( \Gamma^* \) by

\[
J^* = \int_{\Gamma^*} \left( w dy - T_i \frac{\partial u_i}{\partial x} ds \right) \tag{1.38}
\]

where \( w \) is the strain energy density, \( T_i \) is a directionally dependent component of the traction vector, \( u_i \) is the displacement along the direction \( i \), and \( ds \) is in incremental length of the line, along the path \( \Gamma^* \). The strain energy density is found by

\[
w = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \tag{1.39}
\]

and the traction vector is

\[
T_i = \sigma_{ij} n_j \tag{1.40}
\]
Figure 1.16  An arbitrary path that forms a closed loop, as per Rice in reference Rice (1968). The J integral along this entire line is always 0.
where \( n_j \) is the unit vector normal to \( \Gamma^* \). Rice used the divergence theorem to show that the equivalent area integral is

\[
J^* = \int_{A^*} \left( \frac{\partial w}{\partial x} - \frac{\partial}{\partial x_j} \left( \sigma_{ij} \frac{\partial u_i}{\partial x} \right) \right) dxdy \quad (1.41)
\]

where \( A^* \) is the area within the closed loop \( \Gamma^* \). With the strain energy density, assuming elastic potential behavior,

\[
\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \varepsilon_{ij}} \frac{\partial \varepsilon_{ij}}{\partial x} = \sigma_{ij} \frac{\partial w}{\partial x} \quad (1.42)
\]

and keeping with the small strain assumption, which is consistent with all of the work presented here,

\[
\frac{\partial w}{\partial x} = \frac{1}{2} \sigma_{ij} \left( \frac{\partial}{\partial x} \left( \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_j} \left( \frac{\partial u_j}{\partial x_i} \right) \right) = \sigma_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x} \right). \quad (1.43)
\]

Taking into account the equilibrium condition,

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad (1.44)
\]

and from the symmetry of stress-strain theory, \( \sigma_{ij} = \sigma_{ji} \),

\[
\sigma_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x} \right) = \frac{\partial}{\partial x_j} \left( \sigma_{ij} \frac{\partial u_i}{\partial x} \right). \quad (1.45)
\]

Equation 1.45 along with equation 1.42 shows that, according to equation 1.41, for any closed contour, \( \Gamma^* \), \( J = 0 \). If a closed contour is broken into multiple paths, such as shown in figure 1.17, then

\[
J = \sum_{i=1}^{h} J_i = 0 \quad (1.46)
\]

where \( h \) is the number of sections of the enclosed contour. In the example in figure 1.17, \( \Gamma_2 \) and \( \Gamma_4 \) are on the crack face, which has the property \( T_i = dy = 0 \), which makes \( J_2 = J_4 = 0 \), and considering equation 1.46, \( J_1 = -J_3 \). Therefore, even paths around the crack tip that are not closed contours will sum to 0, as long as they are in opposite directions. Or, equivalently, any two paths around the crack in the same direction will have the same value, proving that \( J \) is path independent.

Another critical analysis of stress fields for non-linear materials was independently performed by Hutchinson, and Rice and Rosengren. Rice and Rosengren (1968); Hutchinson
Figure 1.17  A set of paths that can form a closed loop, near a crack tip. Because paths $\Gamma_2$ and $\Gamma_4$ have $J$ integrals of 0, it must be that the integrals of $\Gamma_1$ and $\Gamma_3$ sum to 0, as long as all 4 paths together form a closed path.
(1968) Hutchinson proved mathematically, and Rice and Rosengren showed, that the stress times strain goes as $1/\sqrt{r}$ near a crack tip for any power-law hardening material. According to Hutchinson’s analysis,

$$\nabla^4 \phi + \gamma (\phi, \sigma_e, n, r, \alpha) = 0$$ (1.47)

where $\phi$ is a stress function, which Hutchinson expressed in the form

$$\phi = r^s \phi_1 (\theta) + r^t \phi_2 (\theta) + \ldots$$ (1.48)

where if the first term is the dominant term, $s < t$, and each subsequent term is of an order greater than $t$. Hutchinson then restricted the analysis to this dominant term, expressed as

$$\phi = C r^s \phi_1 (\theta)$$ (1.49)

where $C$ is an amplitude. An eigenvalue equation for $s$ was derived as

$$\left( n (s - 2) - \frac{\partial^2}{\partial \theta^2} \right) \left( \tilde{\sigma}_e^{n-1} \left( s (s - 3) \phi - 2 \phi'' \right) \right) + (n (s - 2) + 1) (n (s - 2)) \tilde{\sigma}_e^{n-1} \left( s (2s - 3) \phi - \phi'' \right) + 6 (n (s - 2) + 1) (s - 1) \left( \tilde{\sigma}_e^{n-1} \phi' \right)' = 0$$ (1.50)

where

$$\sigma_e = Cr^{s-2} \left( \tilde{\sigma}_r^2 + \tilde{\theta}_r^2 - \tilde{\sigma}_r \tilde{\sigma}_\theta + 3 \tilde{\sigma}_{r\theta}^2 \right)^{\frac{1}{2}}$$ (1.51)

and

$$\sigma_r = Cr^{s-2} \left( s \phi + \phi'' \right)$$ (1.52)

and

$$\sigma_\theta = Cr^{s-2} (s - 1) \phi$$ (1.53)

and

$$\sigma_{r\theta} = Cr^{s-2} (1 - 2) \phi'$$ (1.54)

For a number of $n$ values, given appropriate boundary conditions, $s$ was solved numerically, and the relationship

$$s = \frac{2n + 1}{n + 1}$$ (1.55)
which works well for both plain strain and plain stress, was found. Hutchinson (1968) The result of equation 1.55 implies that for all power-law hardening materials, the strain energy density goes as \(1/r\) near the crack tip, which plays a critical role in this work.

To solve for the magnitude of \(C\) which is expressed at times as \(\kappa\), consider the \(J\) integral for two paths around a crack tip, \(J_1\), far from the crack tip and in the region that can be well described by linear elasticity, and \(J_2\), where the stress can be described by equations 1.51, 1.52, 1.53 and 1.54. The integrand for \(J_2\) is

\[
w = \alpha \sigma_0 \varepsilon_0 \kappa^{n+1} \frac{n}{n+1} r^{(n+1)(s-2)} \sigma_0^{n+1}
\]

where \(\sigma_0\), \(n\), \(\alpha\), and \(\varepsilon_0\) are from the form of the Ramberg Osgood relationship when expressed in the form

\[
\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left( \frac{\sigma}{\sigma_0} \right)^n,
\]

and

\[
T_i \frac{\partial u_i}{\partial x} = \alpha \sigma_0 \varepsilon_0 \kappa^{n+1} r^{(n+1)(s-2)} \times 
\left( \sin \theta \left( \tilde{\sigma}_{rr} (\tilde{u}_\theta - \tilde{u}_r') - \tilde{\sigma}_{r\theta} (\tilde{u}_r + \tilde{u}_\theta') \right) + \cos \theta (n(s-2)+1) (\tilde{\sigma}_{rr}\tilde{u}_r + \tilde{\sigma}_{r\theta}\tilde{u}_\theta) \right)
\]

where \(\tilde{u}_r\) and \(\tilde{u}_\theta\) are the displacements

\[
u_r = \alpha \varepsilon_0 \kappa^{n+1} r^{(s-2)+1} \tilde{u}_r (\theta)
\]

and

\[
u_\theta = \alpha \varepsilon_0 \kappa^{n+1} r^{(s-2)+1} \tilde{u}_\theta (\theta).
\]

For the path \(r_2\),

\[
J_2 = \alpha \varepsilon_0 \kappa^{n+1} I_n
\]

where \(I_n\) is the result of the integral

\[
I_n = \int_{-\pi}^{\pi} \frac{n}{n+1} \tilde{\sigma}_e^{n+1} \cos \theta 
- \sin \theta \left( \tilde{\sigma}_{rr} (\tilde{u}_\theta - \tilde{u}_r') - \tilde{\sigma}_{r\theta} (\tilde{u}_r + \tilde{u}_\theta') \right) + \cos \theta (n(s-2)+1) (\tilde{\sigma}_{rr}\tilde{u}_r + \tilde{\sigma}_{r\theta}\tilde{u}_\theta) 
\]

\(d\theta\).

In order for equation 1.61 to be path independent, as has been established, the result must be independent of \(r\). This gives the same result as the numerical solution, equation 1.55, because
the \( r \) term only disappears when

\[
(n + 1) (s - 2) + 1 = 0
\]  

(1.63)

which can be solved for \( s \) to find

\[
s = \frac{2n + 1}{n + 1}.
\]  

(1.64)

With these results, the amplitude, \( \kappa \) or \( C \), is

\[
\kappa = \left( \frac{J}{\alpha \sigma_0 \varepsilon_0 I_n} \right)^{\frac{1}{n+1}}
\]  

(1.65)

which leads to

\[
\sigma_{ij} = \sigma_0 \left( \frac{E J}{\alpha \sigma_0 \varepsilon_0 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(n, \theta).
\]  

(1.66)

At very small values of \( r \), where the loading might not be proportional, and geometry changes at the crack tip violate the small strain assumption, the HRR singularity gives incorrect values for stress or strain. This should be obvious, because the singularity goes to \( \infty \) as \( r = 0 \) is approached.

Fracture mechanics based fatigue calculations arose from the need for damage tolerant approaches. Machine designs that stem from initiation based approaches such as stress-life or strain-life can be over engineered, such that structural components are much larger and heavier than they need to be. This is particularly important in the aerospace industry, where weight considerations are of critical importance. Suresh (2004) As demands for fuel efficiency increase, and weight reduction becomes a greater concern, other industries will seek to avoid over engineering components.

Fatigue methods based on fracture mechanics are also known as damage tolerant approaches, because of an assumption that within or on any engineering structure, there will be some flaw, and that the flaw will act as a crack that can grow by fatigue processes until final fracture, or until the part no longer fulfills its intended function. The interest in fatigue crack growth and crack growth processes has led to a great deal of literature on the topic. Paris et al. (1961); Paris and Erdogan (1961); Noroozi et al. (2005); Mikheevskiy et al. (2012); Alaoui et al. (2009); Xiaoping et al. (2008); Glancey and Stephens (2006); Lee et al. (2009); Romeiro et al. (2009); El-Zeghayar et al. (2011); Yakushiji et al. (1997); Bichler and Pippan (2007); Noroozi et al.
(2008) This work could be said to have started in 1921 with Griffith, and the early work in fracture mechanics. Griffith (1921) Fatigue crack growth, however, was not well described until Paris and Erdogan came up with a power law that describes the majority of crack growth, in terms of the stress intensity factor range, $\Delta K$. This relationship, frequently referred to as the Paris law, is

$$\frac{da}{dN} = C (\Delta K)^m$$  \hspace{1cm} (1.67)

where $da$ the incremental change in crack size, $dN$ the incremental change in the number of cycles, $\Delta K$ is the change in stress intensity factor, $C$ is the Paris law coefficient, and $m$ is the Paris law exponent. Both $C$ and $m$ are fitting parameters. A schematic of a crack growth rate curve can be seen in figure 1.18. The Paris law describes the region that appears to be linear in a log-log plot. On the left end of the Paris law portion of the curve, is a notable decrease in crack growth rate. The threshold stress intensity factor, $\Delta K_{th}$, is the point where this curve alone would predict no crack growth. It is worth note, that cracks can, in fact, grow at stress intensity factor ranges below $\Delta K_{th}$. Murtaza and Akid (1995); McDowell (1996) Another feature of a measured crack growth rate curve is on the far right, where there is a drastic increase in crack growth rate. This is where the fracture toughness is approached. It is obvious that when the stress intensity factor range nears the fracture toughness of the material, the crack growth rate will approach infinity. Anderson (2005); Suresh (2004) The point of fracture is marked by a red “X”. Efforts to develop a fatigue crack growth theory that successfully describes multiple materials have had very limited success, in spite of many of them using fatigue data, such as strain-life parameters. Suresh (2004) Some of the theories involve geometrical models, which relate crack tip displacement to empirical knowledge gained from measuring spacing of fatigue striations. Lardner (1967); Laird (1967) The crack growth rate expression developed from these ideas is

$$\frac{da}{dN} \approx \Delta \delta_t = \beta \frac{(\Delta K)^2}{\sigma'_y E'}$$  \hspace{1cm} (1.68)

where $\sigma'_y$ is the cyclic yield strength, $E'$ is the plane strain Young’s modulus, $\Delta \delta_t$ is a crack growth extension, and $\beta$ is related to things such as strain hardening, crack tip blunting behavior, and several other factors. This always results in a Paris law exponent of 2, which is
reasonable, but slightly low for most ductile metals. Suresh (2004) Dislocation and damage parameter based calculations have led to the relationship

\[
\frac{da}{dN} \propto (\Delta K)^4 \frac{G}{\sigma^2 U^*}
\]

(1.69)

where \( G \) is the shear modulus, and \( U^* \) is related to some critical value of energy absorbed, as calculated by the stress strain hysteresis loop. McCartney (1976); Weertman (1973) This type of damage accumulation model always leads to a Paris law exponent of 4, which is slightly higher than values tend to be for most ductile metals. These fatigue crack growth theories have serious shortcomings, in that they always have the same Paris law exponent, and they require often complex analysis or fitting. They also fall short when it comes to accounting for stress ratio effects.

The stress ratio effect is manifested in a left or right shift in the fatigue crack growth rate curve. The Paris' law slope tends to remain approximately the same at different stress ratios, for single phase ductile metals. Courtney (2005). Early efforts at characterizing the stress ratio effect have led to fatigue crack growth descriptions in the form of

\[
\frac{da}{dN} = C_a \left( \frac{\Delta K^{m_a}}{(1-R) K_c - \Delta K} \right)
\]

and

\[
\frac{da}{dN} = C_b \left( \frac{\Delta K^{m_b}}{(1-R)^{c_1}} \right)
\]

(1.70)

(1.71)

where \( C_a, C_b, m_a, m_b, \) and \( c_1 \) are fitting parameters. Forman et al. (1967); Walker (1970) The strength of these equations is that they can do a reasonably good job of accounting for the stress ratio effect. The obvious weakness is that in order to take advantage of these methods, a great deal of additional testing and fitting must be done.

Methods by which scientists and engineers have attempted to characterize or predict the stress ratio effect have also involved multi-parameter crack growth driving forces, instead of extra fitting parameters. Noroozi et al. (2005); Lu and Liu (2012); Xiong and et al (2008); Kwofie and Rahbar (2011); Kujawski (2001) Noroozi, Glinka, and Lambert use a two parameter driving force for fatigue crack growth analysis that can be expressed as

\[
\frac{da}{dN} = C \left( \Delta K^{(1-p) \gamma} \right)^p
\]

(1.72)
Figure 1.18  Schematic of a fatigue crack growth rate. Notable features are the linear looking region along most of the log-log plot, the decrease on the left side, and the increase on the right side. The middle region is known as the Paris law region, where a power law described in Paris et al. (1961) generally describes the crack growth behavior. The left side is bounded by the threshold stress intensity factor range, and on the right is the drastic increase in crack growth rate as the maximum stress intensity factor, $K_{max}$, approaches the fracture toughness of the material, $K_{IC}$. 
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where $\Delta K_{tot}$ is the total stress intensity factor range, $K_{max,tot}$ is the total maximum stress intensity factor, $C$ is the fatigue crack growth coefficient, $p$ is the driving force constant, and $\gamma$ is the fatigue crack growth equation exponent. Noroozi et al. (2005) The total stress intensity factors differ from that which was applied by an amount related to the residual stresses that develop in advance of the crack tip as a result of strain hardening within the zone of plastic deformation. Their technique assumes that the material can be considered to be composed of identical material blocks, all of the size $\rho^*$, which is believed to be a materials constant. The crack growth in a material is considered as the fracturing of successive blocks of material, of size $\rho^*$. The fatigue crack growth rate, then, is related to the material block size by

$$\frac{da}{dN} = \frac{\rho^*}{N}$$

(1.73)

where, in this case, $\frac{da}{dN}$ is the average crack rate for a crack propagating through the material block in $N$ cycles. After a rather elegant stress and energy analysis, Noroozi et. al. show that

$$N_f = \frac{1}{2} \left( \frac{1}{\sigma_f \varepsilon_f} \times \frac{(\psi_{y,1})^2}{2(n'+3)/(n'+1) \pi E \rho^*} \times (K_{max,tot}^2)_{n'/1} \left( \Delta K_{tot}^2 \right)^{1/(n'+1)} \right)^{(1/(b+c))}$$

(1.74)

where $\sigma_f$, $\varepsilon_f$, $b$, and $c$, are from the Basquin-Manson-Coffin, $n'$ is from the Ramberg Osgood fit of the stress-strain relationship, and $\psi_{y,1}$ is from a weight function that describes the average state of stress within the first material block. The crack growth rate, with respect to the number of cycles necessary for failure of the initial material block, is

$$2\rho^* \left( \frac{1}{\sigma_f \varepsilon_f} \times \frac{(\psi_{y,1})^2}{2(n'+3)/(n'+1) \pi E \rho^*} \times (K_{max,tot}^2)_{n'/1} \left( \Delta K_{tot}^2 \right)^{1/(n'+1)} \right)^{-(1/(b+c))} = \frac{da}{dN} = \frac{\rho^*}{N_f}.$$  

(1.75)

Because everything in equation 1.75 are constants except for $\Delta K_{tot}$ and $K_{max,tot}$, it is possible to express it in the form of equation 1.72. This particular derivation is for the assumption of predominantly plastic behavior near the crack tip. The parameters $C$, $p$, and $\gamma$, are calculated as

$$C = 2\rho^* \left( \frac{(\psi_{y,1})^2}{2(n'+3)/(n'+1) \sigma_f \varepsilon_f \pi E \rho^*} \right)^{-(1/(b+c))},$$

(1.76)
\[ p = \frac{n'}{n' + 1}, \]  
(1.77)

and

\[ \gamma = -\frac{2}{b + c}, \]  
(1.78)

where the only unknown constant is \( \rho^* \). At the time of this writing, this is found by approximation, or fitting. The driving force is then expressed as

\[ \Delta \kappa = \Delta K_{tot}^{(1-p)} K_{max,tot}^p. \]  
(1.79)

The calculation for \( \Delta K_{tot} \) comes from calculating \( K_{max,tot} \) and \( K_{min,tot} \), which vary depending on the stress ratio. In other words, they depend on whether or not \( K_{min,tot} \) is effectively negative, or how much the effective stress near the crack tip is affected by the residual stress from previous applications of the cyclic load. The residual stress factor, \( K_r \), is calculated using \( \rho^* \). This makes the determination of \( \rho^* \) a critical factor in using the Noroozi-Glinka-Lambert method. Another difficulty with the method is that it requires the parameters from the Basquin-Manson-Coffin equation.

1.2 The missing piece

The multi-parameter fatigue crack growth driving force discussed above requires a substantial amount of fatigue related information in order to use it. At minimum, the Basquin-Manson-Coffin equation is required, along with the cyclic stress-strain relationship. In order to fit \( \rho^* \), fatigue crack growth information is also necessary. Other multi-parameter crack growth driving forces suffer a similar malady, requiring a great deal of information, as well as at least one fitting parameter. On the practical side of engineering, a goal is to get as much information as possible, from as little as possible. This decreases the cost of testing, as well as the time it takes to do the testing. This brings to mind the question of how little data can be used, to get the information needed. At the minimum, for design purposes, the relationship between stress and strain must be known for any structural material. Ideally, this information could be used to calculate a fatigue crack growth rate. The following work will consider this possibility. It starts from the standpoint that the fatigue crack growth rate, being a mechanical response
to a mechanical load, should be related to the cyclic stress strain relationship, which is also a mechanical response to a mechanical load. As indicated in the cartoon, figure 1.19, it could be expected that one could be used to find the other, and vise versa.

Figure 1.19  The cyclic stress-strain relationship, and the fatigue crack growth equation, both relate a mechanical response to a mechanical load. An assumption made by this work, is that these two properties must be closely related to each other.
CHAPTER 2. THE CRACK GROWTH RATE EXPRESSION

The functional purpose of this work is to create an expression for the fatigue crack growth rate, from simple mechanical properties. There are a number of mechanisms which may contribute to the behavior, which are considered here in very general terms.

2.1 Introduction

The theory and modeling will first introduce the crack growth mechanisms, in very general terms. These terms are then related to the strain energy density calculations that can be performed based on the stress—strain relationship of a ductile metal. The basic mechanisms are chosen based on observations of crack surfaces. The relationship between the energy and the loading is examined, and used to identify fatigue crack growth rate parameters. In order to identify another parameter, an idealized linear elastic material is considered, and how the stress distribution will change as a function of a changing load. Then, a concept of crack tip healing is introduced and explained. Part of the environmental dependence of the fatigue crack growth rate, and the threshold stress intensity factor, are explained in terms of crack tip healing.

2.2 Theory and modeling

2.2.1 Mechanisms of fatigue crack growth

In the past few decades, a number of mechanisms have been explored regarding the causes of fatigue crack growth. Laird (1979); Starke and Williams (1989); Tanaka (1989); Roach et al. (2013); Mughrabi (2013) While models based on these mechanisms may be qualitatively satisfactory, none of them are successfully quantitative without the use of arbitrary fitting parameters. Additionally, many of them lack a satisfactory description of the stress ratio
effect. Multiple parameter crack growth models that have recently been developed using a less mechanistic approach have been more successful than previous work in capturing the stress ratio effect. Noroozi et al. (2005); Xiong and et al (2008); Kujawski (2001); Kwofie and Rahbar (2011); Lu and Liu (2012) In some of these models, parameters are often combined to form a so called “crack growth driving force”. Noroozi et al. (2005); Xiong and et al (2008); Kujawski (2001); Kwofie and Rahbar (2011); Lu and Liu (2012) Although there are numerous ways to interpret multi-parameter crack growth driving forces, it is taken here that they imply multiple mechanism crack growth. The well known stress ratio effect is often accounted for in part by the use of two independent variables, $\Delta K$ and $K_{\text{max}}$, which can be related to different crack growth mechanisms.

It is well known that elastic energy and brittle fracture are directly related; this concept is the foundation of linear elastic fracture mechanics (LEFM). Anderson (2005) Plasticity and plastic work are fundamentals of elastic-plastic fracture mechanics (EPFM), and are related to the fracture of tough, ductile materials. Anderson (2005) Fracture from a fatigue surface, as shown in 2.1(c), is not definitively either brittle, as in fig. 2.1(a), or ductile, as in fig. 2.1(b). This work assumes that instead of one mechanism or the other driving crack growth, both mechanisms act in concert, and interact with each other. The generalized mechanisms will be referred to as elastic, and plastic, in this work.

2.2.2 Energy of fatigue crack growth mechanisms

These two well studied methods of crack propagation are separately described by LEFM, and EPFM. Anderson (2005) Although efforts in using these fracture mechanics ideas to quantifiably predict fatigue crack growth have been somewhat unsuccessful, relationships between these concepts and fatigue crack growth rates are obvious. A power law relationship, known as the Paris Law, indicates a link between the stress intensity factor range, $\Delta K$, and fatigue crack growth. This links LEFM to fatigue crack growth rates. Another concept, crack tip opening displacement (CTOD), shows a power law relationship with fatigue crack growth in terms of EPFM. A feature common to these methods in fracture mechanics is a $\frac{1}{x}$ singularity, where $x$ is the distance from the crack tip, at the crack tip for $\sigma \times \varepsilon$. This is the well known Hutchinson Rice
Figure 2.1 Fracture surface of A36 steel. On the same surface, 3 very different behaviors can be seen.
Rosengren (HRR) singularity. Independent analysis by Hutchinson, and Rice and Rosengren, shows that the strain energy density for a power law hardening material obeys the $\frac{1}{2}$ singularity. Using this, along with an appropriate stress–strain relationship such as the Ramberg Osgood (RO) equation,

$$\varepsilon = \frac{\sigma}{E} + \left( \frac{\sigma}{K} \right)^{1/n} \quad (2.1)$$

where $\varepsilon$ is strain, $E$ is the elastic modulus, $\sigma$ is stress, $K$ is the strain hardening coefficient, and $n$ is the strain hardening exponent, the elastic–plastic stress and strain distribution is calculated similar to Molski and Glinka’s work in ref Molski and Glinka (1981). Because the idealized linear elastic strain energy density distribution is related to the strain energy density distribution from the elastic–plastic solution, the elastic–plastic stress can be calculated by equating the areas under the stress–strain curves, as seen in fig. 2.2. The RO stress–strain relationship is very convenient, in that there are separable, non-interacting, terms for the elastic and plastic parts of the strain. This means that the strain energy density is easily segregated into elastic and plastic portions.

Figure 2.2  The linear elastic strain energy density can be used to calculate the stress and strain for a power law strain hardening material.
2.2.3 How energy scales with load

Figure 2.3 shows the schematic separation of the components of strain energy density as a cross hatched red triangle, indicating the elastic portion of the elastic—plastic strain energy density, and the blue region below the curve, showing the plastic strain energy density or plastic work. In the case of cyclic stress—strain behavior, the amount of plastic work done, related to the plastic portion of the calculated strain energy density, is proportional to the stress range, $\Delta \sigma = \sigma_{\text{max}} - \sigma_{\text{min}}$. The plastic mechanism of crack growth is, therefore, taken to be proportional to $\Delta K$. The elastic crack growth mechanism is related to brittle fracture, and will be dependent on elastic energy. The elastic energy in the system that is available to do work, such as growing a crack, is proportional to the maximum tensile stress, $\sigma_{\text{max}}$, or $K_{\text{max}}$.

![Figure 2.3](image)

Figure 2.3 The strain energy density from the integrated RO stress—strain curve are separable into elastic and plastic components. For cyclic loading, the elastic component is proportional to $K_{\text{max}}$ and the plastic component is proportional to $\Delta K$.

With the two generalized mechanisms for crack growth identified as elastic and plastic,
the two separate parts of strain energy that drive those crack growth mechanisms are used to calculate how those parts of strain energy are each proportional to the applied load. A region of interest is chosen, ahead of the crack tip, over which to perform the calculations. This region is chosen as some arbitrary size and location well within the yield zone, but not including the singularity. A schematic of this region of interest can be seen in fig. 2.4, and the integral is expressed in eqn. 2.2.

\[ \text{Strain Energy} = \int_{x_0}^{x_1} \int_{\varepsilon_{\text{min}}}^{\varepsilon_{\text{max}}} \sigma(K, a, x) \, d\varepsilon \, dx \]  

(2.2)

It is found that both the elastic and plastic energy components have power law relationships with \( K \), which leads to a crack growth expression,

\[ \Delta a = C (\Delta K)^{m_p} (K_{\text{max}})^{m_e} \]  

(2.3)
where $C$ is a constant to be discussed in more detail later, $m_p$ is the plasticity exponent, and $m_e$ is the elasticity exponent. An immediate result of this is the Paris law exponent at a stress ratio of $R = 0$, which is equal to $m = m_p + m_e$.

### 2.2.4 The ideal material with a metastable crack

Identifying the scaling relationship between load and crack growth is a critical aspect of the solution; however, the magnitude must also be found. This requires the calculation of the particular crack growth rate at a particular load. It is assumed here that when the crack growth rate is on the order of $2 - 3$ atomic diameters, the total plastic deformation is very small. It is also assumed that the crack is metastable, in this regime. In this case, a metastable crack is considered to be a crack that only grows while the applied load is changing, and for any static load, the crack is static. The linear elastic (LE) solution is used to calculate how the stress distribution changes with an applied load. It should be noted that the authors are not stating that the stress distribution near the crack tip, on the atomic scale, will match those of the continuum solution in actuality.

The stress distribution for the ideal LE case, $\sigma_y(\sigma_{\text{applied}}, x, a)$ can be solved to find the position relative to the crack tip, $x_{th}$, where the stress is equal to the theoretical strength of the material, $\sigma = \sigma_{th}$. The crack growth at this stage can be expressed as the change in $x_{th}$ with the change in $\sigma_{\text{applied}}$, and the total change in crack size of the crack for the change in $\sigma_{\text{applied}}$ is then

$$\Delta a = \int_{\sigma_{\text{min}}}^{\sigma_{\text{max}}} \frac{dx_{th}}{d\sigma_{\text{applied}}} d\sigma_{\text{applied}}$$

as is shown schematically in fig. 2.5.

Although the crack growth at very low crack growth rates is often considered to be shear slip driven, the calculations from eqn. 2.4 are intended to identify a logical minimum for the mechanisms identified and discussed here. Under the assumption that shear slip driven mechanisms are responsible primarily for crack nucleation, and are not always active, a value of $\Delta K$ is calculated from choosing a value of $\Delta a$ in eq. 2.4 to be, for example, 3 atomic diameters. Suresh (2004); Cheng et al. (2012); Baker and Warner (2012); McDowell and Dunne (2011); Sugeta and et al (2004)
Figure 2.5  A schematic of how the LE stress distribution changes as a load is being changed, expressed as the distribution from two different loads.

In comparison to this work, Paris et.al. discuss in ref. Paris et al. (1999) the observation that the “knee” in FCG curves is approximately

\[ \frac{da}{dN} = b \]  \hspace{1cm} (2.5)

and

\[ \Delta K_{eff} = E\sqrt{b} \]  \hspace{1cm} (2.6)

where b is the burgers vector, and \( \Delta K_{eff} \) is taken as equivalent to \( \Delta K_{tot} \), which is discussed in section 3.1.1. This observation will be compared with the current work in section 3.2.

2.2.5 Crack tip healing

This work proposes a potential contributor to the threshold stress intensity factor phenomenon. Crack tip healing attempts to explain \( \Delta K_{th} \) behavior in different inert environments and at positive stress ratios. Explanations that have been offered in the past have included the idea that a crack can not grow distances of less than one or two atoms, crack closure in terms of
both surfaces, plasticity, and oxide formation, and a variety of other possible explanations for such behavior. Suresh et al. (1981); Steinbock and gudladt (2011); Petit and Sarrazin-Baudoux (2006); Kobayashi et al. (1997); Gall and et al (2005); Henaff et al. (1995); Ismarrubie and Sugano (2004); Kelestemur and Chaki (2001); Shimojo et al. (2000); Kelestemur and Chaki (2001); Yakushiji et al. (2001) The idea that a crack can’t grow less than one atom fails when the crack is considered in three dimensions. If only half of a crack front advances one atomic diameter, the overall growth of the crack could be considered to be 0.5 atomic diameters. Crack tip closure due solely to surface interaction fails to explain the existence of $\Delta K_{th}$ at positive stress ratios. Crack tip closure due solely to plasticity fails to explain the behavior of $K_{th}$ in different environments, which can be seen in fig. 2.6.

![Crack growth rate curves](figure26.png)

**Figure 2.6** Crack growth rate curves for AISI A542 Class 2 martensitic microstructure steel, taken from Suresh et al. (1981) The blue circles are cracks grown in air, the red squares are grown in $He$, and the black triangles are grown in $H_2$.

Crack tip healing considers the structure of the crack tip and the region nearby, the separation of atoms in that region, and the introduction of particles from the environment in which the testing is done. A schematic of an atomic arrangement in the vicinity of a crack tip is shown in fig. 2.7.
Figure 2.7  Schematics showing the atoms near a crack tip. In 2.7(a) the load is $K_{\text{max}}$, and the atoms near the crack tip are as far separated as they can be. In 2.7(b) the load is $K_{\text{min}}$, and the atoms near the crack tip are in close proximity. The atoms in some positions may get near enough to each other to form primary atomic bonds, in which case the crack effectively “heals”, or retreats some distance, dependent on the environment.
At the peak tensile load, the atoms nearest the crack tip are separated too far to form primary atomic bonds with each other. On unloading, however, elastic relaxations allow these atoms to approach each other, and if they get close enough to bond, and there is nothing interfering with bonding, they will. A caveat is that electronic restructuring of dangling bonds on the surface can lessen the energetic benefit of bonding across the top and bottom surfaces, thereby diminishing or slowing down the effect. As shown in fig. 2.7(b), the size of the particles in the local environment will directly impact the amount of bonding that can occur, and will thereby change $\Delta K_{th}$ accordingly. Smaller particles will get farther into the crack tip and prevent more bonding, thereby lowering $\Delta K_{th}$. In the case of high vacuum, much more bonding may occur and $\Delta K_{th}$ will increase. This phenomenon is testable in inert environments, where local stresses caused by things such as surface oxidation will not be present. The total crack growth rate equation can now be expressed as

$$\frac{da}{dN} = C(\Delta K)^{m_p}(K_{max})^{m_e} - n\epsilon_b$$ (2.7)

where $C$ can be found using the value calculated from eqn. 2.4 and the load relations from eqn. 2.2, and $n$ is on the order of 2 or 3 atomic diameters, but could depend on a variety of factors, such as environment or stress ratio. Although this concept is an alternative to crack tip closure as an explanation for the threshold stress intensity factor, it does not accommodate for the importance of residual stresses as plasticity induced crack tip closure attempts to do. The residual stress must be accounted for separately from crack tip healing.

### 2.3 The next step

The goal of this work was to create, from the cyclic stress–strain relationship, an expression for crack growth that could be used for damage tolerant fatigue calculations. The next chapter discusses how the theories can be put to use, for calculating the fatigue crack growth rate and threshold stress intensity factor.
CHAPTER 3. IMPLEMENTATION AND RESULTS

Before this work can be put to use, a few extra details are addressed. The residual stresses in front of the crack tip are accounted for, the stress-strain relationship is transformed into true stress–true strain, and the true stress–true strain is refit to new Ramberg Osgood parameters. Then the results of this work are compared to a variety of materials, including steel, aluminum, and titanium alloys.

3.1 Introduction

3.1.1 Residual stress

It is well known that due to strain hardening there are residual stresses near a crack tip after loading and unloading. The distribution of these stresses has a substantial effect on the development of the stress distribution from following loads. Noroozi et al. have developed calculations for defining a residual stress intensity factor, \( K_r \), which modifies the applied stress intensity factor, \( K_{appl} \), giving \( K_{tot} = K_{appl} + K_r \) at positive stress ratios, \( R > 0 \). Noroozi et al. (2005) The calculations are inconvenient for this work, however, due to the amount of information required. Here, an estimate is made based on the stress–strain relationship of the material that approximates \( K_r \) in a way that has dependence on \( R \) and \( K_{appl} \), and is in good agreement with the calculations by Noroozi et al.

The stress at \( \varepsilon = 0.05 \), \( \sigma_{0.05} \), along with the stress at first yield, \( \sigma_{fy} \), are used to formulate \( K_r \),

\[
K_r = K_{appl} \frac{\sigma_{0.05} - \sigma_{fy}}{\sigma_{0.05}}
\]

such that, including a linear stress ratio effect,

\[
K_{tot} = K_{appl} \left(1 - \frac{\sigma_{0.05} - \sigma_{fy}}{\sigma_{0.05}}\right) (1 - R)
\]
Figure 3.1 A schematic of the stress–strain curve with indications for the stress at first yield, $\sigma_{fy}$, and the stress at 5% strain.

where $K_{tot}$ is used in the calculations for crack growth rate, and crack growth rate is plotted against $\Delta K_{appl}$. Here, first yield is defined as when there is a 1% difference between the LE stress and the RO stress at the same strain. A comparison of these results with the results from the calculations done by Noroozi for 4340 steel can be seen in fig. 3.2.

3.2 Results

For accurate results, the cyclic RO equation,

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{1/n'}$$

(3.3)

where $K'$ is the cyclic strain hardening coefficient, and $n'$ is the cyclic strain hardening exponent, should be used, and not the monotonic RO curve. Some error may be introduced if the RO parameters are not measured for the same lot of material as was tested in fatigue crack growth rate testing. Additionally, the true stress–true strain curve will give the best description of the material behavior at the crack tip. Using the transformations $\varepsilon_t = ln(1+\varepsilon_e)$ and $\sigma_t = \sigma_e (1+\varepsilon_e)$ the RO curve can be refit so that the cyclic true stress–true strain curve
Figure 3.2  A comparison between the stress intensity factor used in calculations, $K_{tot}$, and the applied stress intensity factor, $K_{applied}$, as calculated by Noroozi et. al. Noroozi et al. (2005), and as estimated by this work.

parameters can be used in later calculations. The refitting is in good agreement to the critical, highly plastic, portion of the curve, as indicated in figure 3.3. It has been shown that a power law hardening relationship can describe large plastic strains reasonably well, when fit to true stress–true strain data. Tsuchida et al. (2012)

To validate this work, it is compared to a variety of materials at different stress ratios. The values used for the atomic diameters were based on the primary constituent, being iron, aluminum, or titanium, in the materials used for comparison here. A common estimate of the ideal strength is $\sigma_{th} = \sqrt{\frac{E\gamma}{\epsilon_b}}$, where $\gamma$ is surface energy and $\epsilon_b$ is bond length. Courtney (2005) This yields values $\approx 0.17E \leq \sigma_{th} \geq 0.23E$ for many metals. Courtney (2005) Here, a value of $\sigma_{th} = 0.20E$ is used in all cases. For this work, the number of atomic diameters for crack tip healing, $n$ from eqn. 2.7, was chosen to be 3 for all calculations. The value for $\Delta\sigma$ of interest in eqn. 2.4 used in this work is 3 atomic diameters. The results of the calculations can be seen
Figure 3.3  The RO curve can be re-fit to the calculated true stress–true strain curve. The fit works well at high plastic strains, which is the regime of interest for this work. The black curve represents the original RO curve, the blue line represents the true stress–true strain curve, and the red dashed line shows the power law refit using RO parameters.

in figs 3.4–3.12.

The observations of Paris et.al. in ref Paris et al. (1999) can be compared with this work using the assumptions that $\Delta K_{eff} = \Delta K_{tot}$ and that the burgers vector, $b$, is approximately one atomic diameter. From the current work, at the stress ratio $R = 0$, $da/dN = b = \left( \frac{\Delta K_{tot}}{E \epsilon_{th}} \right) \frac{1}{2\pi} - 3b$, which yields

$$E \sqrt{b} = \Delta K_{eff} = \Delta K_{tot} = E \epsilon_{th} \sqrt{\frac{2\pi}{4b}}.$$  \hspace{1cm} (3.4)

Using the estimate $\epsilon_{th} \approx 0.20$ discussed above, the first and fourth expressions from eqn.3.4 result in $1 \approx 1.003$, which the author considers to be satisfying.

As seen in fig.3.4, for most values of $\Delta K$ this work matches experiment very well. There is some disagreement as the threshold is approached, which is the likely result of a number of factors. Those factors include the incompleteness of the physics modeled, and the choice of using 3 atomic diameters as a crack tip healing distance instead of, for example, 2.
In fig. 3.5 there is good agreement with the exceptions of a widely dispersed experimental threshold stress intensity factor range, and a slight divergence between theory and experiment at high values of $\Delta K$. At a higher stress ratio, another good fit for the same material is seen in fig. 3.6, where there is less divergence at high $\Delta K$ than in the lower $R$ ratio, and measured threshold is in agreement with this work. Considering this, it is likely that the difference in $\Delta K_{th}$ at lower $R$ ratios is, at least in part, due to crack closure from surface contact.

Grade 2 CP titanium in the $\Delta K$ range examined in ref. Adib and Baptista (2007) at low and high $R$ ratios are shown in figs 3.7 and 3.8 respectively. Good fitting can be observed for the stress intensity factor ranges available, for all load ratios. For materials that have stress–strain relationships that are more exotic than can be described well with the RO curve, it is not uncommon to have an exotic fatigue crack growth rate curve, as well.

An example of a more complex fatigue crack growth rate curve can be seen in figs 3.9 and 3.10. An interesting feature is that at higher values of $\Delta K$ the experiment and theory are in good agreement. However, at lower values, in particular as the threshold is approached, the

Figure 3.4  A fatigue crack growth rate curve for D6ac steel after Jones et al. (2012).
Figure 3.5  Fatigue crack growth rate curves for 4340 steel at low stress ratios, after Dowling (1999); Taylor (1985); Swain et al. (1990); Wanhill (1972), adapted from Noroozi et al. (2007).
Figure 3.6 Fatigue crack growth rate curves for 4340 steel at high stress ratios, after Dowling (1999); Taylor (1985); Swain et al. (1990); Wanhill (1972), adapted from Noroozi et al. (2007).
Figure 3.7  Fatigue crack growth rate curves for commercially pure grade 2 titanium at low stress ratios, after Adib and Baptista (2007).

The experiment is substantially different from the results of this work. Because many materials are constituted of multiple phases, or have stress–strain behavior that is not well described by the RO curve, the integrated RO curve is not an accurate description of the strain energy distribution near a crack tip that results from the application of a load. Additional considerations will have to be taken in order to appropriately describe such phenomena as more complex stress–strain relationships, and the development of inhomogenously distributed stresses near a crack tip due to inhomogeneity in the material.

Figures 3.11 and 3.12 are comparable to figures 3.9 and 3.10 in that they are eccentric, compared to most fatigue crack growth rate curves. It is well known that titanium and titanium alloys show this behavior. Adib and Baptista (2007) In both cases, the curve may be attributed to the presence of multiple phases, the possibility of a stress–strain relationship that is not well described by the Ramberg Osgood equation, or both.
Figure 3.8 Fatigue crack growth rate curves for commercially pure grade 2 titanium at high stress ratios, after Adib and Baptista (2007).
Figure 3.9 Fatigue crack growth rate curves for 2024 T351 Aluminum at low stress ratios, after Wanhill (1994); Pang and Song (1994); Liu (1998) and adapted from Noroozi et al. (2005).
Figure 3.10  Fatigue crack growth rate curves for 2024 T351 Aluminum at high stress ratios, after Wanhill (1994); Pang and Song (1994); Liu (1998) and adapted from Noroozi et al. (2005).
Figure 3.11 Fatigue crack growth rate curves for Ti-6Al-4V at low stress ratios, after Yuen et al. (1974); Ritchie et al. (1999), adapted from Noroozi et al. (2005).

Figure 3.12 Fatigue crack growth rate curves for Ti-6Al-4V at high stress ratios, after Yuen et al. (1974); Ritchie et al. (1999), adapted from Noroozi et al. (2005).
CHAPTER 4. DISCUSSION, APPLICATION, AND CONCLUSION

Although a number of theories have been developed over the years, none have been entirely successful without the need for a number of fitting parameters. Anderson (2005); Courtney (2005); Suresh (2004) This work, although using a number of estimates and approximations, accurately predicts fatigue crack growth rates for a variety of materials. Particularly, single phase materials with stress–strain properties that are well described by the Ramberg Osgood relationship. This chapter will discuss the relationships found between the cyclic strain hardening exponent, \( n' \), coefficient, \( K' \), and elastic modulus, \( E \), with the Paris crack growth rate parameters, \( C \), and \( m \). An example of a strain–life curve for a cracked square specimen is calculated.

4.1 Discussion

Assuming the HRR singularity is approximately true, the strain energy density for a power law hardening material goes as \( 1/r \) away from a crack tip. Linear elasticity is a special case of power law hardening. This is taken as evidence that the strain energy density must be the same for the LE case, and the Ramberg Osgood relationship. If, for a constant crack size, \( K \propto \sigma_{\text{applied}} \), and strain energy density (SED) is

\[
SED = \int \varepsilon d\sigma, \tag{4.1}
\]

x which for the LE case is

\[
SED = \int \varepsilon d\sigma = \frac{\sigma^2}{2E}, \tag{4.2}
\]

then the total strain energy density at a point in front of the crack tip is proportional to \( \sigma^2 \). Because \( K \propto \sigma_{\text{applied}} \), then

\[
SED \propto K^2. \tag{4.3}
\]
Close to the crack tip, where the proportional loading assumption holds true, a condition should exist such that the plastic work portion of the strain energy density, $SED_{plastic}$ should be approximately equal to the total strain energy density from linear elasticity, $SED_{LE}$, or

$$SED_{plastic} \approx SED_{LE}. \quad (4.4)$$

This shows that the plastic work related exponent in this work, $m_p$, is always $\approx 2$. The elastic energy related exponent, $m_e$ is free from that constraint, and may vary as $0 \leq m_e \leq 2$.

Using the calculations presented, relatively simple relationships between the strain hardening properties and the fatigue crack growth properties can be found. Figures 4.1, 4.2, and 4.3 show that the Paris law exponent is nearly linearly dependent on the strain hardening exponent, $n'$, but independent of the strain hardening coefficient, $K'$, for steels, aluminum alloys, and titanium alloys respectively. The differentiation in the alloys being the elastic modulus, in these cases.

Figure 4.4 shows how this relationship is independent of the elastic modulus, meaning that for virtually all single phase ductile metals, the fatigue crack growth rate exponent can be related to the strain hardening exponent by the simple relationship

$$m \approx 3n' + 2.35 \quad (4.5)$$

where all terms are unitless.

This remarkably convenient result shows that in the cases where the ideas presented in this work are useful, the calculations are not necessary if the relationship described in equation 4.5 is known. It can be seen in figure 4.5 that for the Paris law coefficient, there is also little or no dependence on $K'$. It is certainly clear that there is, however, some dependence on composition. Although the most obvious relation is between $C$ and the modulus, the relationship found empirically is also dependent on the atomic radii. The relationship between the Paris law coefficient, $C$, and the strain hardening exponent, $n'$, was found to be

$$C \approx \alpha \times \exp \left( \beta \times n' \right) \quad (4.6)$$

where $\alpha$ was found to be

$$\alpha \approx -3.78 \times 10^{-15} \left( E \times \epsilon_b^{2.63} \right) + 1.45 \times 10^{-9} \quad (4.7)$$
Figure 4.1 The Paris law exponent, $m$, vs. cyclic strain hardening exponent, $n'$, at different values of the strain hardening coefficient, $K'$, for a material with an elastic modulus of 200,000 $MPa$. This work expects single phase steels to behave this way.
Figure 4.2  The paris law exponent, $m$, vs. cyclic strain hardening exponent, $n'$, at different values of the strain hardening coefficient, $K'$, for a material with an elastic modulus of 70,000 MPa. This work expects single phase aluminum alloys to behave this way.
Figure 4.3 The Paris law exponent, $m$, vs. cyclic strain hardening exponent, $n'$, at different values of the strain hardening coefficient, $K'$, for a material with an elastic modulus of 117,000 MPa. This work expects single phase titanium alloys to behave this way.
Figure 4.4 The Paris law exponent, $m$, vs. cyclic strain hardening exponent, $n'$, at different values of the strain hardening coefficient, $K'$, for a material with elastic moduli of 200,000 MPa, 70,000 MPa, and 117,000 MPa. This figure demonstrates the lack of dependence on elastic modulus, of the relationship between the strain hardening exponent and the fatigue crack growth exponent.
Figure 4.5  Paris law coefficient vs. strain hardening exponent. The relationship is independent of $K'$, but dependent on $n'$, as well as the primary constituent of the material. The relationship between $C$ and $n'$ is approximately exponential.
where \( E \) is the elastic modulus in MPa, and \( \epsilon_b \) is the atomic radius in angstroms, \( \AA \), and \( \beta \) was found to be

\[
\beta \approx -1.78 \times 10^{-5} (E \times \epsilon_b^{1.63}) - 5.77
\]  

(4.8)

with the same units for \( E \) and \( \epsilon_b \). The units of the term \( \alpha \) is meters, and \( \beta \) is unitless. Equations 4.5 and 4.6 are constructed based on the data presented here, and may not fit well for other materials, such as nickel based alloys.

The crack growth rate parameters are shown in table 4.1, along with the results from Noroozi et al. (2007). The crack growth rate parameters are in good agreement with each other. The exponents are within 10% of each other, as is the log of the coefficient, which is within the difference in values that can be found in experiment, as can be seen in figures 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12.

Table 4.1 Effective Paris’ law crack growth rate parameters for this work at a stress ratio of \( R = 0 \), and the crack growth rate driving force presented in Noroozi et al. (2007)

<table>
<thead>
<tr>
<th>Material</th>
<th>( m )</th>
<th>( C )</th>
<th>( n \epsilon_b )</th>
<th>( p )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4340 steel</td>
<td>2.75</td>
<td>( 4.66 \times 10^{-11} )</td>
<td>( 7.56 \times 10^{-10} )</td>
<td>2.77</td>
<td>0.11</td>
</tr>
<tr>
<td>2024-T351</td>
<td>2.67</td>
<td>( 7.97 \times 10^{-10} )</td>
<td>( 8.58 \times 10^{-10} )</td>
<td>2.67</td>
<td>0.09</td>
</tr>
<tr>
<td>Ti6Al4V</td>
<td>2.70</td>
<td>( 1.97 \times 10^{-10} )</td>
<td>( 8.82 \times 10^{-10} )</td>
<td>2.53</td>
<td>0.096</td>
</tr>
</tbody>
</table>

4.2 Application

Fatigue data is often used in the form of a strain–life curve. Suresh (2004); Courtney (2005) The strain life curve is often developed by using smooth specimens and cyclically loading them between two displacement levels until they fail. Usually, this is done at a stress ratio of \( R = -1 \). Strain–life curves for some of the materials discussed here are shown in figure 4.6, and listed in table 4.2.

Although a common belief is that the strain–life curve for smooth tensile specimens represents fatigue crack initiation, and not growth, a similar curve can be developed for any geometry. Suresh (2004) For this work, as a demonstration of these methods in use, a square, pre cracked rod with cross sectional dimensions of \( 5mm \times 5mm \) is examined. The stress intensity factor is calculated from an equation in an appendix of calculations compiled in reference Suresh (2004),
Figure 4.6 Strain life of 4340 steel, 2024 – T351 aluminum, and Ti – 6Al – 4V titanium alloy, as reported in Noroozi et al. (2007).

Table 4.2 The materials properties and strain–life properties of 4340 steel, 2024 – T351 aluminum, and Ti – 6Al – 4V titanium alloy, as reported in Noroozi et al. (2007), originally from Leis (1972); Dowling (1999)

<table>
<thead>
<tr>
<th>Material</th>
<th>(E) (MPa)</th>
<th>(n)</th>
<th>(K) (MPa)</th>
<th>(\sigma_f) (MPa)</th>
<th>(b)</th>
<th>(\varepsilon_f)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4340 steel</td>
<td>200,000</td>
<td>0.123</td>
<td>1910</td>
<td>1879</td>
<td>-0.0859</td>
<td>0.64</td>
<td>-0.636</td>
</tr>
<tr>
<td>2024-T351</td>
<td>70,000</td>
<td>0.1</td>
<td>751.1</td>
<td>909.48</td>
<td>-0.1</td>
<td>0.36</td>
<td>-0.65</td>
</tr>
<tr>
<td>Ti6Al4V</td>
<td>117,000</td>
<td>0.106</td>
<td>1772</td>
<td>2030</td>
<td>-0.104</td>
<td>0.841</td>
<td>-0.688</td>
</tr>
</tbody>
</table>
as

\[ K_I = \sigma \sqrt{a} f \left( \frac{a}{W} \right) \]  \hspace{1cm} (4.9)

where \( W \) is the width of the plate, and

\[ f \left( \frac{a}{W} \right) = 1.99 - 0.41 \frac{a}{W} + 18.7 \left( \frac{a}{W} \right)^2 - 38.48 \left( \frac{a}{W} \right)^3 + 53.85 \left( \frac{a}{W} \right)^4. \]  \hspace{1cm} (4.10)

A schematic of the square specimen can be seen in figure 4.7. The strain life parameters from these calculations can be seen in table 4.3. Figure 4.8 shows the strain life for the pre cracked square specimen. As expected, the high cycle region, at lower strains, is considerably shorter in life than in the smooth specimens. The low cycle region, at higher strains, shows much longer life than the smooth specimens. In part, this could be related to how strains above \( \approx 0.004 \), the yield stress is exceeded, and the entire specimen would experience plastic deformation. This could also be from plastic deformation and crack tip blunting, which would considerably slow the crack growth. Suresh (2004) Differences between the strain-life of the smooth specimens from measurement, and from the results of these calculations, are most obvious with the titanium alloy. This is quite likely in part related to the offset in the measured threshold stress intensity factor, as seen in figure 3.11.

In order for this method to be successful, it must show a reasonable dependence on stress ratio. Figure 4.9 shows such a stress ratio dependence. The strain life curve at a stress ratio of \( R = 0.7 \) falls below the curve at \( R = 0 \), indicating a significantly reduced life for the same strain range, and it shows the same general shape of the curve.

### 4.3 Conclusion

This work presents a new, quantitatively accurate, theory for stable crack growth, with respect to fatigue crack growth. It is constructed from an examination of generalized mechanisms of crack growth and how the energy that drives those mechanisms relates to an applied load, assumptions about the behavior of those mechanisms at very small stress intensity factors, and the concept of crack tip healing. An estimation of a residual stress intensity factor is made, which is in reasonable agreement with previous work. There is very good agreement between this work and experiment, for simple, well behaved, materials. This work could lead to a new,
Table 4.3 Strain life properties for pre-cracked rectangular bars, made from 4340 steel, 2024-T351 aluminum, and Ti-6Al-4V titanium alloy, based on crack growth rate calculations from this work. Included is the initial crack size, \( a_i \). Materials properties are as reported in Noroozi et al. (2007), originally from Leis (1972); Dowling (1999).

<table>
<thead>
<tr>
<th>Material</th>
<th>( E ) (MPa)</th>
<th>( n' )</th>
<th>( K' ) (MPa)</th>
<th>( \sigma_f ) (MPa)</th>
<th>( b )</th>
<th>( \varepsilon_f )</th>
<th>( c )</th>
<th>( a_i ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4340 steel</td>
<td>200,000</td>
<td>0.123</td>
<td>1910</td>
<td>12,711</td>
<td>-0.30</td>
<td>4,923,710</td>
<td>-2.45</td>
<td>5.0 \times 10^{-6}</td>
</tr>
<tr>
<td>2024-T351</td>
<td>70,000</td>
<td>0.1</td>
<td>751.1</td>
<td>6359</td>
<td>-0.337</td>
<td>1.89 \times 10^9</td>
<td>-3.37</td>
<td>5.0 \times 10^{-6}</td>
</tr>
<tr>
<td>Ti6Al4V</td>
<td>117,000</td>
<td>0.106</td>
<td>1772</td>
<td>10,953</td>
<td>-0.340</td>
<td>29,031,488</td>
<td>-3.22</td>
<td>5.0 \times 10^{-6}</td>
</tr>
</tbody>
</table>
Figure 4.7  Schematic of pre-cracked square specimen used for strain life calculations in this work. This represents an example of how the strain life for a part in service might be calculated, when using a damage tolerant approach.
Figure 4.8 Strain life of 4340 steel, 2024-T351 aluminum, and Ti-6Al-4V titanium alloy, for a pre-cracked square rod specimen, as per calculations from this work.
Figure 4.9 Strain life of 4340 steel, for a pre-cracked square rod specimen, as per calculations from this work, at 2 different stress ratios.
better understanding of stable crack growth and fatigue. Additionally, for a known relationship between process parameters and cyclic stress—strain behavior, and a known load history, this work makes it possible to modify manufacturing processes for fatigue optimization in way that has never before been possible.

The criteria for success of a fatigue crack growth rate theory are, at minimum, that it describes the Paris’ law regime such that the power law parameters can be calculated and are comparable with the measured parameters. A higher degree of success is one that includes the stress ratio effect. Although the work of researches such as Noroozi et al. (2005) accomplishes both of those, it does so with at least one fitting parameter, and it often requires, at minimum, the strain-life curve parameters in terms of the Basquin Manson Coffin equation.

As evidenced in figures 3.4,3.5,3.6,3.7,3.8,3.9,3.10,3.11, and 3.12, this work can be used to calculate the crack growth rate parameters, including the stress ratio effect. This work can also estimate the threshold stress intensity factor, $K_{th}$, and how it can change with the stress ratio. This work requires very limited input, being only the cyclic stress strain properties, and the atomic radius of the primary elemental constituent.

As a simplification of the results of this work, equations 4.5 and 4.6 have been developed to aid in the use of this work for engineering purposes.
Table 4.4 Effective Paris’ law crack growth rate parameters for this work at a stress ratio of \( R = 0^* \), and the crack growth rate data for D6ac steel presented in Jones and Forth (2010), the mechanical properties of D6ac presented in Nachtigall (1977), the commercially pure titanium crack growth rate properties from Adib and Baptista (2007), and the commercially pure mechanical properties presented in Takao and Kusukawa (1996). *The crack growth rate properties from Adib and Baptista (2007) are for a stress ratio of \( R = 0.05 \).

<table>
<thead>
<tr>
<th>Material</th>
<th>( E ) (MPa)</th>
<th>( n' )</th>
<th>( K' ) (MPa)</th>
<th>( m )</th>
<th>( C )</th>
<th>( n_{\epsilon_b} )</th>
<th>( \gamma/m )</th>
<th>( p )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>D6ac steel</td>
<td>202,000</td>
<td>0.116</td>
<td>2841</td>
<td>2.73</td>
<td>4.62 × 10^{-11}</td>
<td>7.56 × 10^{-10}</td>
<td>2.625</td>
<td>0.05</td>
<td>2.47 × 10^{-11}</td>
</tr>
<tr>
<td>Grade 2 CP titanium</td>
<td>110,000</td>
<td>0.23</td>
<td>354</td>
<td>3.02</td>
<td>1.99 × 10^{-10}</td>
<td>8.82 × 10^{-10}</td>
<td>2.99</td>
<td>NA</td>
<td>4.00 × 10^{-11}</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


