1964

Some diffuse reflection problems in radiation aerodynamics

Stephen Nathaniel Falken

Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Aerospace Engineering Commons

Recommended Citation
https://lib.dr.iastate.edu/rtd/3848

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
This dissertation has been microfilmed exactly as received

FALKEN, Stephen Nathaniel, 1937—SOME DIFFUSE REFLECTION PROBLEMS IN RADIATION AERODYNAMICS.

Iowa State University of Science and Technology Ph.D., 1964 Engineering, aeronautical

University Microfilms, Inc., Ann Arbor, Michigan
SOME DIFFUSE REFLECTION PROBLEMS
IN RADIATION AERODYNAMICS

by

Stephen Nathaniel Falken

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subjects: Aerospace Engineering
Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Heads of Major Departments

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

1964
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEDICATION</td>
<td>iii</td>
</tr>
<tr>
<td><strong>I. LIST OF SYMBOLS</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>II. INTRODUCTION</strong></td>
<td>5</td>
</tr>
<tr>
<td><strong>III. GENERAL AERODYNAMIC FORCE ANALYSIS</strong></td>
<td>11</td>
</tr>
<tr>
<td><strong>IV. THE THEORY OF RADIATION AERODYNAMICS</strong></td>
<td>14</td>
</tr>
<tr>
<td>A. Fundamental Concepts</td>
<td>14</td>
</tr>
<tr>
<td>B. Diffuse Reflection and the Principle of Invariance</td>
<td>29</td>
</tr>
<tr>
<td>C. Isotropic Reflection</td>
<td>38</td>
</tr>
<tr>
<td>D. Anisotropic Reflection</td>
<td>44</td>
</tr>
<tr>
<td>E. Comparison between Radiation Aerodynamics and Rarefied Gas Dynamics</td>
<td>51</td>
</tr>
<tr>
<td><strong>V. APPLICATIONS OF RADIATION AERODYNAMICS</strong></td>
<td>58</td>
</tr>
<tr>
<td>A. Flat Plate</td>
<td>62</td>
</tr>
<tr>
<td>B. Sphere</td>
<td>74</td>
</tr>
<tr>
<td>C. Cylinder</td>
<td>82</td>
</tr>
<tr>
<td><strong>VI. DISCUSSION</strong></td>
<td>88</td>
</tr>
<tr>
<td>A. Flat Plate</td>
<td>88</td>
</tr>
<tr>
<td>B. Sphere</td>
<td>93</td>
</tr>
<tr>
<td>C. Cylinder</td>
<td>94</td>
</tr>
<tr>
<td><strong>VII. BIBLIOGRAPHY</strong></td>
<td>96</td>
</tr>
<tr>
<td><strong>VIII. ACKNOWLEDGEMENTS</strong></td>
<td>98</td>
</tr>
<tr>
<td><strong>IX. APPENDIX</strong></td>
<td>99</td>
</tr>
</tbody>
</table>
DEDICATION

It is a joy to dedicate this thesis to my wife Laurel, who provided the serene home environment so perfectly complementary to my research, and who has been a pillar of strength and an inspiration throughout.
# I. LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Albedo</td>
</tr>
<tr>
<td>b</td>
<td>Defined by Equation 28</td>
</tr>
<tr>
<td>c</td>
<td>Velocity of light</td>
</tr>
<tr>
<td>(\vec{c})</td>
<td>Velocity of a molecule</td>
</tr>
<tr>
<td>(C_D)</td>
<td>Dimensionless aerodynamic drag coefficient</td>
</tr>
<tr>
<td>(C_L)</td>
<td>Dimensionless aerodynamic lift coefficient</td>
</tr>
<tr>
<td>e</td>
<td>Defined by Equation 92</td>
</tr>
<tr>
<td>E</td>
<td>Electromagnetic energy</td>
</tr>
<tr>
<td>(\vec{E})</td>
<td>Electric field vector</td>
</tr>
<tr>
<td>F</td>
<td>Force</td>
</tr>
<tr>
<td>(\vec{F})</td>
<td>Flux of radiant energy</td>
</tr>
<tr>
<td>(\vec{f})</td>
<td>Force per unit area</td>
</tr>
<tr>
<td>f</td>
<td>Boltzmann distribution function</td>
</tr>
<tr>
<td>H</td>
<td>Defined by Equation 68</td>
</tr>
<tr>
<td>(\vec{H})</td>
<td>Magnetic field vector</td>
</tr>
<tr>
<td>I</td>
<td>Intensity of radiation energy</td>
</tr>
<tr>
<td>j</td>
<td>Emission coefficient</td>
</tr>
<tr>
<td>J</td>
<td>Defined by Equation 58</td>
</tr>
<tr>
<td>(\hat{k})</td>
<td>Unit vector</td>
</tr>
<tr>
<td>m</td>
<td>Mass of a molecule</td>
</tr>
<tr>
<td>(M_n)</td>
<td>Defined by Equation 70</td>
</tr>
<tr>
<td>(\vec{m})</td>
<td>Electromagnetic momentum</td>
</tr>
<tr>
<td>n</td>
<td>Number density of molecules</td>
</tr>
<tr>
<td>(\hat{n})</td>
<td>Unit vector</td>
</tr>
</tbody>
</table>
\( N_n \) Defined by Equation 126
\( P \) Pressure
\( \bar{F} \) Probability
\( \vec{r} \) Position vector
\( R \) Reflection function
\( R \) Gas constant
\( S \) Defined by Equation 53
\( \text{d}S \) Differential element of length
\( t \) Time
\( T \) Temperature
\( T \) Defined by Equation 122
\( u \) Component of \( \vec{c} \) in x-direction
\( U \) Defined by Equation 119
\( v \) Component of \( \vec{c} \) in y-direction
\( v \) Velocity of space vehicle
\( \text{dv} \) Differential element of volume
\( V \) Defined by Equation 120
\( w \) Component of \( \vec{c} \) in z-direction
\( W \) Defined by Equation 125
\( x \) Position coordinate on surface
\( X \) Defined in Equation 99
\( y \) Position coordinate on surface
\( Y \) Defined in Equation 100
\( z \) Position coordinate on surface
\( \alpha \) Angle of attack
\[ \gamma \quad \text{Angle defined by Equation 41} \]
\[ \delta \quad \text{Dirac } \delta\text{-function} \]
\[ \epsilon \quad \text{An angle in the sphere problem} \]
\[ \theta \quad \text{Polar angle in spherical coordinates} \]
\[ \kappa \quad \text{Absorption coefficient} \]
\[ \lambda \quad \text{Albedo for single scattering, or "degree of reflection"} \]
\[ \mu \quad \text{Cosine of the polar angle} \]
\[ \nu \quad \text{Frequency of radiation} \]
\[ \Pi \quad \text{Defined by Equation 110} \]
\[ \rho \quad \text{Density of reflecting material} \]
\[ \sigma \quad \text{Accommodation coefficient} \]
\[ d\sigma \quad \text{Element of surface area} \]
\[ d\Sigma \quad \text{Element of surface area} \]
\[ \tau \quad \text{Shear stress} \]
\[ \tau \quad \text{Optical thickness} \]
\[ \phi \quad \text{Azimuth angle in spherical coordinates} \]
\[ \chi \quad \text{Phase function} \]
\[ \psi \quad \text{Defined by Equation 68} \]
\[ d\omega \quad \text{Element of solid angle} \]
\[ d\Omega \quad \text{Truncated, semi-infinite cone} \]

**Subscripts**

\[ D \quad \text{Drag} \]
\[ i \quad \text{Incident to surface} \]
\[ L \quad \text{Lift} \]
\[ n \quad \text{Normal to surface} \]
o  Refers to incident parallel radiation
r  Reflected from surface
t  Tangential to surface
T  Total
x  x component
y  y component
z  z component
w  Maxwell distribution at the surface temperature
II. INTRODUCTION

In the year 1873, James Clerk Maxwell deduced theoretically that electromagnetic waves should exert pressure on matter. This idea was most revolutionary, and had a startling impact on the world of physics in the late nineteenth century, for it was advanced at a time when the wave vs. particle nature of light debate was active. Proponents of the corpuscular theory had proposed the possibility that pressure would be exerted by light if the light had a particle nature, and hope was held that experimental proof of the existence of light pressure would decisively rule out the wave theory. Yet Maxwell was predicting that waves themselves could exert pressure. It was then put squarely up to the experimentalists to verify the idea. Eventually the Russian physicist Lebedev in 1901 and the Americans Nichols and Hull in 1901-5 were able to perform laboratory experiments of sufficient accuracy to give unmistakable proof of the phenomenon of light pressure. An excellent discussion of these experiments and of the impact of Maxwell's prediction on the physics of the times is given by Henry (16). It is today a well established fact that electromagnetic waves transport not only energy but also momentum, and the exchange of this momentum with a material body by the processes of scattering and absorption results in a definite radiation force acting on the body. Although the magnitudes of radiation forces are small, there are cases where they are important and produce significant effects. A few such cases will be briefly discussed.

The rapid advance of space technology is making it possible for space vehicles to penetrate ever more deeply into the vast reaches of outer space.
In regions that are essentially free of the gravitational and geomagnetic fields of celestial bodies, the only external forces that exist to act on a vehicle are those due to streams of corpuscular radiation, such as the solar wind and cosmic showers, and to electromagnetic radiation. Electromagnetic radiation results from many sources, such as insolation (direct solar radiation), thermal radiation from stars, solar radiation reflected by planetary atmospheres, etc. It is of course important to the aerodynamicist to know these forces, since they are the only forces acting on a vehicle in field-free space, and precise control of the trajectory of the vehicle is desired. Another space application requiring accurate knowledge of radiation forces is in solar sailing. Here, navigation in space is accomplished by harnessing radiation forces by means of large sails, or control surfaces. The idea has emerged from science fiction status to distinct feasibility (11). The intense source of concentrated radiation afforded by the development of lasers opens up the possibility of a laser-carrying vehicle able to alter the trajectory of another vehicle by means of radiation forces. Interest in radiation forces has also been spurred by new concepts in propulsion systems which seek to produce propulsive thrust by the ejection of collimated beams of radiation. Particularly promising as a future means of providing a propulsion system for the exploration of inter-galactic space is a photon-laser system (14).

Thus, although Maxwell's discovery is over ninety years old, and the experimental verification of it over sixty, new interest has recently been aroused in radiation forces due to applications that the early pioneers in the field could never have dreamed of.
This paper will be concerned with the aerodynamic aspects of forces which result from light radiation. This comprises the specialty called radiation aerodynamics. As mentioned previously, the only external forces acting on a body in gravitational and geomagnetic field-free space are those due to high energy particle impingements and electromagnetic radiation. Before a complete specification of the forces acting on a vehicle can be given, more work needs to be done in both of these areas. In the particle impingement case, more needs to be known about the sources, constitution, density, and energy of particle streams in space, as well as the nature of the interaction of these streams with a body. Particularly important to know is the velocity distribution of the stream particles, because in principle the stream quantities of aerodynamic interest, such as pressure and shear, can be obtained as integral moments of the distribution function. There is no reason to assume that the high energy streams of space are in Maxwellian equilibrium. For electromagnetic radiation, the intensity and the sources of the radiation must be more fully known, as well as the characteristics of the interaction of the radiation with matter. Only then can aerodynamic analysis yield accurate information about radiation force effects.

Some excellent work has been done recently in expanding knowledge of the sources and characteristics of environmental radiation in space. Katzoff (19) provides a good review of this work. Yet little has been done with regard to developing a useful theory of the aerodynamic effects of radiation. The bulk of recent work in radiation transfer is concerned with the radiant heat transfer to a vehicle, but not with aerodynamic
effects. Cotter (11) has surveyed the dynamics of solar sails, but under the assumption that incident radiation either passes through the sail or undergoes specular (mirror-like) reflection from it. Dugan (13) has calculated the accelerations produced by radiation forces on a flat plate, a sphere, and a cone, but also under the assumption of specular reflection. Cunningham (12) has considered various specialized problems concerning the amount of various types of radiation incident on some bodies of aerodynamic interest, such as spinning flat plates. Fedor (15) has investigated the effect of solar radiation pressure on the spin of Explorer XH, a satellite which had four large solar cell paddles for obtaining power from the sun. It was found that the spin of the satellite increased by 20% after four months, and Fedor showed this effect to be attributable to solar radiation pressure. His analysis used the concept of reflectivity coefficient $\rho$, so that the incident and reflected pressure forces are related by $F_r = \rho F_i$, and he assumed completely specular reflection.

The work done to date on radiation pressure force effects has been based on the premise of specular reflection, in which the angles of incidence and reflection are equal. However, many materials are diffuse reflectors, so that radiation incident on them is reflected diffusely, i.e., in all directions. Some analyses of radiant energy interchange between surfaces have assumed diffuse reflection, but only the Lambert cosine law case, which is a simple, crude approximation to the actual laws of diffuse reflection. The actual reflection process at a surface is a complicated phenomenon depending upon the optical properties of the incident radiation and the reflecting material. Real surfaces deviate
from Lambert and specular reflection due to local irregularities in the surface profile. Because of this, and in view of the limitations of what has been done to date, a much more thorough analysis of radiation aerodynamics is requisite, and must take into account the more general case of diffuse reflection. This paper is intended to provide that analysis.

Aerodynamic analysis relates the lift and drag forces acting on a body to the pressure and shear due to the transmission of momentum from a "fluid" to the body. In radiation aerodynamics, in which the "fluid" is a stream of electromagnetic radiation travelling at the speed of light, the calculation of pressure and shear involves concepts which are probably not widely known at present among aerodynamicists. Therefore, in line with what has happened many times in recent years, the contemporary aerodynamicist is called upon to envelop a domain of knowledge which has previously belonged to the physicist. This thesis hopes to aid the aerodynamicist in this process. It is intended to do several things. First, the basic concepts of radiation transport theory are presented, these concepts being based largely on the foundational work of Ambarzumian (1) and Chandrasekhar (8). The methods obtained therefrom will then be turned toward the problem of determining pressure and shear. Finally, in a manner familiar to the aerodynamicist, the theory will be applied to the calculation of radiation lift and drag coefficients on three representative aerodynamic bodies: the flat plate, the sphere, and the cylinder. Further, the calculations will be done for several types of reflection processes. A further goal of this thesis is the determination of the influence of the reflection process (characterized by the
phase function) on the lift and drag coefficients. It is desired to see exactly how critical for aerodynamic purposes exact knowledge of the reflection process is. For example, does the assumption of specular reflection give a fair approximation to the actual aerodynamic coefficients in diffuse cases? Two types of diffuse reflection will be considered— isotropic (of which Lambert's law is a special case), and a case of anisotropic reflection.

A final goal will be to compare radiation aerodynamics with rarefied gas dynamics.
III. GENERAL AERODYNAMIC FORCE ANALYSIS

This section will present a method of aerodynamic force analysis which is widely used and quite general, in that the nature of the "fluid" is not specified. The method has been used with great success in analyzing problems in the free molecule flow regime of aerodynamics (25). The method presented here will serve to give a sense of direction for the development of the theory of radiation aerodynamics in the next chapter.

Attention is focused upon a differential element of area $d\sigma$. The incident stream makes an angle $\alpha$ with the positive direction of the $x$-axis. The angle $\alpha$ is known as the angle of attack of the area $d\sigma$ with respect to the incident stream. The incident stream is reflected from the body at $d\sigma$, and perhaps it is partly absorbed. As a result of this interaction, there is a net transfer of momentum from the stream to the body at $d\sigma$, and this produces a force on $d\sigma$. This force can be resolved into components normal and tangential to $d\sigma$, and these normal and tangential forces per unit area are the pressure and shear stress respectively which act on the surface element $d\sigma$. Two forces of particular aerodynamic interest are the lift and drag forces. The drag force acts on $d\sigma$ in the direction of the incident stream. The lift force acts at right angles to the drag force. The method of aerodynamic analysis seeks to relate the lift and drag forces to the pressure and the shear stress.

This relation is arrived at with the aid of Figure 1, which shows the lift force, drag force, total pressure, and total shear acting on the element $d\sigma$. The incident and reflected streams are, for convenience, taken to be in the $x$-$y$ plane of a rectangular coordinate system with origin.
Figure 1. The lift force, drag force, total pressure, and total shear acting on a differential element of area $d\sigma$. 
at $d\sigma$. Then, all forces will also be in the x-y plane. Note that the x-axis is drawn "inward", so that the outward drawn unit normal vector $\hat{n}$ is in the negative x-direction. In the right-handed system, the positive z-direction is out of the paper. It is seen that the differential lift and drag forces acting on $d\sigma$ are given by

$$dF_L = (P_T \sin \alpha + \tau_T \cos \alpha) \, d\sigma$$  \hspace{1cm} (1)$$

$$dF_D = (P_T \cos \alpha + \tau_T \sin \alpha) \, d\sigma$$  \hspace{1cm} (2)$$

where, it is to be noted, $P_T$ and $\tau_T$ are functions of $\alpha$. The total pressure and shear are obtained from the incident and reflected pressures and shears by

$$P_T = P_I + P_r$$  \hspace{1cm} (3)$$

$$\tau_T = \tau_I - \tau_r$$  \hspace{1cm} (4)$$

In many aerodynamic problems (e.g., free molecule flow), the incident values of the pressure and shear are known from the known properties of the incident stream. However, the characteristics of the stream after reflection from the surface are distorted from those of the incident stream as a result of interaction with the surface material. The fundamental problem in aerodynamics then is the determination of the properties of the reflected stream. With the reflected pressure and shear known, the total pressure and shear can be obtained from Equations 3 and 4. Then $dF_L$ and $dF_D$ are found from Equations 1 and 2. Integration of Equations 1 and 2 over the surface area of a given body gives the total lift and drag forces on the body.

In radiation aerodynamics, the stream consists of beams of radiation travelling at the speed of light. The theory of radiation aerodynamics then must develop equations for obtaining the pressure and shear due to both the incident and reflected streams.
IV. THE THEORY OF RADIATION AERODYNAMICS

A. Fundamental Concepts

It is the purpose of this section to introduce the fundamental concepts and terminology of radiation transport theory, and to derive general expressions for radiation pressure and shear.

The space through which electromagnetic radiation is transported is characterized by an electric field vector $\vec{E}$, a magnetic field vector $\vec{H}$, and electromagnetic energy which is given (23) by

$$E = \frac{1}{2\pi} \int \left( E^2 + H^2 \right) \, dv$$  \hspace{1cm} (5)

where $dv$ denotes integration over all space available to the field. From Maxwell's equations, the momentum of the space is derived as (6)

$$\vec{M} = \frac{1}{4\pi c} \int (\vec{E} \times \vec{H}) \, dv$$  \hspace{1cm} (6)

and is in the direction of propagation of the radiation. For light waves in empty space, $\vec{E}$ and $\vec{H}$ are orthogonal and of equal magnitude (6), and so

$$\vec{M} = \left( \frac{1}{4\pi c} \int E^2 \, dv \right) \hat{k} = \left( \frac{1}{4\pi c} \int H^2 \, dv \right) \hat{k} = \left( \frac{1}{4\pi c} \int \frac{E^2 + H^2}{2} \, dv \right) \hat{k}$$  \hspace{1cm} (7)

where $\hat{k}$ is a unit vector in the direction of $\vec{M}$, and $\hat{k}$ is assumed to be a constant vector, independent of time.

From Equations 5 and 7 it is seen that the momentum and the energy of light radiation are related by

$$\vec{M} = \frac{E}{c} \hat{k}$$  \hspace{1cm} (8)

This highly important result underlies the theory of radiation aerodynamics.

Radiation aerodynamics is concerned with the aerodynamic effects of radiation forces on bodies. From Newton's Second Law of Motion and
Equation 8, the force exerted by a beam of light of energy $E$ is

$$\vec{F} = \frac{\partial \vec{m}}{\partial t} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \hat{k}$$

(9)

The force exerted per unit area of a differential element of area $d\sigma$ is given by

$$\vec{F} = \frac{\vec{F}}{d\sigma} = \frac{1}{d\sigma} \frac{d\vec{E}}{d\sigma} c \hat{t} \hat{k}$$

(10)

If the direction of the beam is not normal to the surface element $d\sigma$, Equation 10 will have both a normal (pressure) and a tangential (shear) component.

Thus, the important characteristic of the radiation field from the point of view of the aerodynamicist is its energy, or more specifically, the rate of energy transmitted across a surface element of area $d\sigma$. The next step will be a brief review of the manner in which energy is described in radiation transport theory.

Consider first the special case of parallel radiation, i.e., a radiation field characterized by parallel beams. Let the radiation be incident on an element of area $d\sigma$, making an angle $\theta$ with the normal to $d\sigma$. It is desired to investigate the amount of radiant energy that crosses $d\sigma$ per time interval $dt$, and in the frequency interval $(\nu, \nu + d\nu)$. For this purpose, the concept of intensity is introduced as follows

$$I_\nu = \lim_{d\sigma, dt, d\nu \to 0} \frac{d\nu}{d\sigma \cos \theta \, dt \, d\nu}$$

(11)

The $\cos \theta$ factor appears since the energy that crosses $d\sigma$ is that which crosses the projection of $d\sigma$ normal to the beam, i.e., $d\sigma \cos \theta$.

It is now necessary to generalize the concept of intensity for application to diffuse radiation fields. In this case, radiation from all
directions is incident on $d\sigma$. With reference to Figure 2, the following construction proves useful in defining the intensity $(l)$.

1) At an arbitrary point $P$ of the element $d\sigma$, a unit vector $\hat{n}$ normal to $d\sigma$ is drawn.
2) At an angle $\theta$ with respect to $\hat{n}$, a line $L$ is drawn from $P$. Let $L$ be the axis of a cone of solid angle $d\omega$.
3) Through every point on the boundary of $d\sigma$, a line is drawn parallel to the nearest generator of the cone $d\omega$. These lines are then the generators of a truncated, semi-infinite cone $d\Omega$ which is similar to $d\omega$.

Thus, at every angle $\theta$, there is a truncated cone $d\Omega$ whose cross-sectional area perpendicular to $L$ at the point $P$ is $d\sigma \cos \theta$. Again, consideration is given to the rate at which energy flows across $d\sigma$ in the frequency interval $(\nu, \nu + d\nu)$, only now, this energy is completely confined within $d\Omega$. The intensity of the radiation is defined by

$$I_0 = \lim_{d\sigma, dt, d\nu, d\omega \to 0} \frac{dE}{d\sigma \cos \theta \ dt \ d\omega}$$  \hspace{1cm} (12)

If the intensity of radiation is known, the energy can be calculated from

$$dE_0 = I_0 \ d\sigma \cos \theta \ dt \ d\omega$$  \hspace{1cm} (13)

Equation 13 can be integrated over the frequency spectrum of the radiation, giving

$$dE = I d\sigma \cos \theta \ dt \ d\omega$$  \hspace{1cm} (14)

Equation 13 gives the energy for radiation passing through $d\sigma$ confined to a direction characterized by $d\omega$. The amount of energy passing through $d\sigma$ in all directions is given by

$$dE_0 = \int_0^{\infty} dE_0 = d\sigma \ dt \ d\omega \int_0^{\infty} I_0 \cos \theta \ d\omega$$  \hspace{1cm} (15)
Figure 2. Construction of the truncated cone $d\Omega$ for defining the diffuse intensity
This energy, per unit area, time, and frequency, is called the flux of radiation. Hence,

$$\mathcal{F}_v = \int I_v \cos \theta \, d\omega$$  \hfill (16)$$

Finally, the total flux is defined as the integral of Equation 16 over the full frequency spectrum.

$$\mathcal{F} = \int \mathcal{F}_v \, d_o = \int \int I_v \cos \theta \, d\omega \, do$$  \hfill (17)$$

This completes the necessary statements of the terminology and basic concepts of radiation transport. The determination of the intensity of a radiation field is a fundamental problem in radiation transport, and in radiation aerodynamics as well.

There now remains the task of introducing an expression for the solid angle $d\omega$. A system of spherical coordinates with origin at the center of $d\sigma$, and x-axis along the inward normal to $d\sigma$, will be employed. The polar angle $\theta$ of a beam will be measured from the outward normal $\hat{n}$, and the azimuth angle $\phi$ will be measured from the z-axis, as shown in Figure 5. The element of solid angle is the area intercepted on the surface of a sphere of unit radius by the rays drawn from the origin, and is given by

$$d\omega = \sin \theta \, d\theta \, d\phi$$  \hfill (18)$$

The ranges of $\theta$ and $\phi$ depend on the physical problem. For problems in which radiation traverses $d\sigma$ through both upper and lower faces, the ranges are

$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi.$$
Figure 3. Spherical coordinate system in radiation field
In radiation aerodynamics, however, $d\sigma$ is an element of area on the surface of a solid body. Therefore, radiation can traverse $d\sigma$ through one face only, and the appropriate range of angles is

$$0 \leq \theta \leq \pi/2$$

$$0 \leq \phi \leq 2\pi$$

Equation 10,

$$\vec{f} = \frac{1}{c} \frac{dE}{d\sigma dt} \hat{k}$$

will now be considered in the context of the discussion of intensity.

First, for radiation fields of the parallel type, the force per unit area is, using Equation 11 integrated over all frequencies,

$$\vec{f} = \frac{1}{c} \int I \cos \theta \hat{k}$$

(19)

From the equations which relate spherical and cartesian coordinate systems, the force per unit area $\vec{f}$ has components along the $x, y,$ and $z$ axes given by

$$f_x = \frac{I}{c} \cos \theta \cos \theta$$

$$f_y = \frac{I}{c} \cos \theta \sin \theta \sin \phi$$

$$f_z = \frac{I}{c} \cos \theta \sin \theta \cos \phi$$

(20)

Writing the pressure and shear as $P = f_x$, $\tau_y = f_y$, and $\tau_z = f_z$, there results

$$P = \frac{I}{c} \cos^2 \theta$$

$$\tau_y = \frac{I}{c} \cos \theta \sin \theta \sin \phi$$

$$\tau_z = \frac{I}{c} \cos \theta \sin \theta \cos \phi$$

(21)

(22)

(23)

Analogous equations will next be found for the more general case of a diffuse radiation field. Integration of Equation 13 over all frequencies, and substitution of the result, Equation 14, into Equation 10 gives
\[ \mathbf{f}(\omega) = \frac{1}{c} \mathbf{I}(\omega) \cos \theta \, d\omega \hat{k}(\omega) \] (24)

for the force per unit area of \( d\sigma \) in a solid angle \( d\omega \). The components of \( \mathbf{f}(\omega) \) along the \( x, y, \) and \( z \) axes are then

\[ f_x(\omega) = \frac{1}{c} \mathbf{I}(\omega) \cos^2 \theta \, d\omega \]
\[ f_y(\omega) = \frac{1}{c} \mathbf{I}(\omega) \cos \theta \sin \theta \sin \phi \, d\omega \]
\[ f_z(\omega) = \frac{1}{c} \mathbf{I}(\omega) \cos \theta \sin \theta \cos \phi \, d\omega \] (25)

The pressure and shear components for a diffuse radiation field are thus obtained by integrating these equations over all solid angles. Using the expression for solid angle given by Equation 18 with the ranges of \( \theta \) and \( \phi \) appropriate to the problem gives

\[ P = \frac{1}{c} \int_\omega \int_\sigma \mathbf{I}(\theta, \phi) \cos^2 \theta \sin \theta \, d\theta \, d\phi \] (26)
\[ \tau_y = \frac{1}{c} \int_\omega \int_\sigma \mathbf{I}(\theta, \phi) \cos \theta \sin^2 \theta \sin \phi \, d\theta \, d\phi \] (27)
\[ \tau_z = \frac{1}{c} \int_\omega \int_\sigma \mathbf{I}(\theta, \phi) \cos \theta \sin^2 \theta \cos \phi \, d\theta \, d\phi \] (28)

The resultant shear stress is obtained from \( \tau_y \) and \( \tau_z \) by

\[ \tau = (\tau_y^2 + \tau_z^2)^{1/2} \] (29)

and acts in a direction given by

\[ \phi = \tan^{-1} \left( \frac{\tau_y}{\tau_z} \right) \] (30)

The importance of the concept of intensity in radiation aerodynamics may now be fully appreciated, inasmuch as the pressure and shear components, which are the main parameters of aerodynamic interest, are given as integral moments of the intensity. It is thus emphasized that the intensity is the basic unknown in radiation aerodynamics.
We shall now introduce what will be taken to be the "standard problem" of radiation aerodynamics. The radiation incident on do is characterized by a parallel radiation field, the waves being monochromatic of frequency \( v_o \). The radiation is in the direction \((\alpha, \phi)\) and has a constant intensity \( I_o \). The incident pressure and shear components are, from Equations 21, 22, and 23,

\[
P_i = \frac{I_o}{c} \cos^2 \alpha \tag{31}
\]

\[
\tau_{yi} = \frac{I_o}{c} \cos \alpha \sin \alpha \sin \phi \tag{32}
\]

\[
\tau_{zi} = \frac{I_o}{c} \cos \alpha \sin \alpha \cos \phi \tag{33}
\]

As a result of interaction with the surface material, the incident radiation is diffusely reflected. The reflected radiation field is characterized by a reflected intensity \( I_r(\theta, \phi) \), where \( \theta \) and \( \phi \) are the polar and azimuth angles of a reflected ray. The reflected pressure and shear components are, from Equation 26, 27, and 28,

\[
P_r = \frac{1}{c} \oint_0^{2\pi} \oint_0^{\pi/2} I_r(\theta, \phi) \cos \theta \sin \theta \ d\phi \ d\theta \tag{34}
\]

\[
\tau_{yr} = \frac{1}{c} \oint_0^{2\pi} \oint_0^{\pi/2} I_r(\theta, \phi) \cos \theta \sin^2 \theta \sin \phi \ d\phi \ d\theta \tag{35}
\]

\[
\tau_{zr} = \frac{1}{c} \oint_0^{2\pi} \oint_0^{\pi/2} I_r(\theta, \phi) \cos \theta \sin^2 \theta \cos \phi \ d\phi \ d\theta \tag{36}
\]

It is assumed that the frequency of the diffusely reflected radiation is the same as that of the incident radiation, \( v_o \).

A key point which underlies this analysis is that the incident and reflected streams can be treated independently of each other. If the
intensity of the incident beam were modified by interaction with the reflected beam, there would be no such independence, and a more complicated situation would result. It is therefore important to emphasize that radiation aerodynamics is predicated on the "two-independent streams" assumption. The basis for this assumption is that the cross section for the scattering of light by light is extremely small, in the absence of external fields (20). Jauch and Rohrlich (17) estimate the differential cross section for photon-photon scattering to be of the order of $10^{-31}$ cm$^2$/sterad, which is beyond experimental measurements. However, the intensities afforded by laser beams offer hope of being able to measure this very small cross section (20).

For all practical purposes then, beams of light pass through each other essentially unaffected, so that the incident and reflected streams may indeed be treated independently.

A brief discussion of the reflection process will now be given. The reflection of radiation by a medium is a complicated phenomenon which depends on the optical properties of both the radiation and the medium. Specular and diffuse reflection are the interaction processes which represent the two extremes, or limiting cases, of all possible types of interactions. Specular reflection is simple, mirror-like reflection, in which no tangential momentum is exchanged between the stream and the surface and the normal component of the momentum is reversed in direction. This type of reflection corresponds to no interaction between stream and surface. Diffuse reflection corresponds to complete interaction, in which the incident beam penetrates below the surface of the body, undergoes scattering within the body, and is emitted from the surface in all
directions, the intensity characterized by spatial randomness. The specular case is by far the easier to handle, and is the only one treated in aerodynamic applications thus far. The diffuse case requires knowledge of the angular properties of the reflected intensity. Irregularities in the surface profile of materials cause them to be diffuse reflectors. Diffuse reflection can be thought of as the more general case. The bulk of this thesis will deal with diffuse reflection.

Returning now to Equations 34, 35, and 36, it is seen that the immediate task of radiation aerodynamics is the determination of $I_r(\theta, \phi)$ for diffuse reflection. This leads to a consideration of the equation of transfer.

The equation of transfer is the fundamental equation of radiation transfer theory. It is an integro-differential equation which governs the variation of intensity with depth and direction in a medium. Every problem in radiation transport is essentially one of solving the equation of transfer subject to appropriate boundary conditions (7). Since excellent derivations of this equation are given by Ambarzumian (1) and Chandrasekhar (8), the equation will merely be stated here and briefly discussed.

The equation of transfer is derived by considering the changes affected, i.e., gains and losses, in the intensity of radiation traversing an element of volume in a medium. The medium is characterized by an absorption coefficient $\kappa_\nu$ and an emission coefficient $j_\nu$, which are both functions of frequency. The volume element is conveniently taken to be a small cylinder of cross section $d\sigma$ and height $ds$, oriented so that radiation passes through the element normally, and enters it through the lower base $d\sigma$. The density of the medium is $\rho$. The equation of transfer simply states that
\[
\frac{dI_b}{ds} = - \kappa_b^o I_b + j_b^o
\]  
(37)

In words, the change in intensity of radiation traversing a distance \(ds\) is due to both "absorption" and "emission". Here, the terms absorption and emission are used in the most general sense to denote all processes which respectively diminish and augment the intensity of the incident radiation. Two such processes exist for each case. For absorption:

1. The volume element may actually absorb (remove) radiation from the field.
2. The volume element may scatter radiation out of the element and into adjacent elements.

For emission:

1. The material of the volume element may be a source of radiation, so that the volume element is self-emitting.
2. Radiation may be scattered into the element from other such elements.

At this point it is useful to make two assumptions regarding the nature of the medium.

1. The medium is not self-emitting, so that the emission term in Equation 37 need account only for radiation scattered into the volume element.
2. The medium will be assumed to be constituted of plane-parallel layers.

These assumptions will be assumed to hold for the aerodynamic applications to be considered.

With these assumptions, Equation 37 can be recast into "standard
In Equation 38, \( \tau \) is a quantity known as the optical thickness, defined as a dimensionless "depth" below the surface of the medium along the inward normal by

\[
\tau = \int_0^x k \rho \, dx \tag{39}
\]

where \( x \) is an element of distance in the direction of the inward normal, related to \( ds \) by \( dx = -\cos \theta \, ds \). In Equation 38,

\( (\theta, \phi) \) is the final direction of radiation emerging from a layer at a depth \( \tau \);

\( (\theta', \phi') \) is the initial direction of radiation before being scattered in \( (\theta, \phi) \);

\( \mu = \cos \theta \), i.e., the cosine of the polar angle between the final direction of a ray and the outward normal from the layer at \( \tau \);

\( \chi \) is the phase function, which will be discussed in detail.

The phase function introduces a very important concept. The angular distribution of diffusely emitted radiation depends on the phase function. To define it, it is useful to resort to the corpuscular theory of light, in which light is considered to be constituted of energy quanta, or photons. A quantum that undergoes scattering has different probabilities of being scattered in different directions. The question to be answered is, given a quantum in the direction \( (\theta', \phi') \) before scattering, what is the probability of it being scattered in the direction \( (\theta, \phi) \)?
element of solid angle $d\omega$ is taken about the direction $(\theta, \phi)$, what is the probability of the photon being scattered in directions lying within $d\omega$? This probability is proportional to $d\omega$, and is also a function of the angle $\gamma$ between the directions $(\theta, \phi)$ and $(\theta', \phi')$. The probability is written as

$$dP = x(\cos \gamma) \frac{d\omega}{4\pi}$$

where the function $x(\cos \gamma)$ is known as the phase function $(8)$, or the indicatrix of scattering $(l)$. It is this phase function that characterizes the scattering process, as will soon be apparent. The relation between the angles $\gamma, \theta, \phi, \theta', \text{and} \phi'$ may be shown to be

$$\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')$$

Since the total probability in all directions is unity,

$$\int_0^\pi x(\cos \gamma) \frac{d\omega}{4\pi} = 1$$

However, this assumes that all of the incident radiation is scattered, and none is absorbed. To account for the possibility that part of the incident radiation may be absorbed a parameter $\lambda$ is introduced and defined to be the fraction of incident energy in a given frequency that is scattered immediately, and in the same frequency, after being absorbed in a single process. Then, $(1-\lambda)$ is the fraction of incident energy that is converted into other forms of energy (i.e., undergoes true absorption), or into radiation of other frequencies. The parameter $\lambda$ is known as the "albedo for single scattering". When $\lambda = 1$, the scattering process is said to be conservative (perfect scattering). Equation 42, generalized to include the possibility of absorption (non-conservative scattering) becomes
The simplest phase function is $X(\cos \gamma)$ a constant. This corresponds to a scattering situation in which the probability of scattering in a given direction is independent of the direction before scattering. From Equation 43 it is seen that the constant must be $\lambda$. For

$$X(\cos \gamma) = \lambda$$

(44)

the scattering is said to be isotropic.

An important case of anisotropic scattering is

$$X(\cos \gamma) = \lambda(1 + b \cos \gamma)$$

(45)

where $b$ is a constant. The plus sign implies that the probability of scattering is greater at smaller angles of deviation. The minus sign applies to the case where the probability is greater at larger angles of deviation. The case of the Rayleigh phase function

$$X(\cos \gamma) = \frac{3}{4 \pi} (1 + \cos^2 \gamma)$$

(46)

is discussed thoroughly by Stone (27). Here, the incident wave is entirely scattered by an atom, molecule, or electron. The net scattered wave is just that which would be radiated by a single dipole oscillator, and this property of Rayleigh scattering is guaranteed when the scattering particle is small compared with the wavelength of the incident radiation. Thus, Rayleigh scattering is essentially a low frequency phenomenon.

The general practice is to suppose that the phase function can be expanded as a series in Legendre polynomials (8), i.e.,

$$X(\cos \gamma) = \sum_{i=0}^{\infty} \lambda_i P_i(\cos \gamma)$$

(47)

where the $\lambda_i$ are constants. In the applications of radiation aerodynamics
to be considered in this paper, only the phase functions $\chi(\cos \gamma) = \lambda$ and $\chi(\cos \gamma) = \lambda(1 + \cos \gamma)$ will be dealt with. With the equation of transfer and the phase function now presented, the problem of diffuse reflection can now be fully considered.

B. Diffuse Reflection and the Principle of Invariance

The physical picture of diffuse radiation is shown in Figure 4a. An incident radiation field, which is taken to be parallel, strikes a slab of material of optical thickness $\tau_1$. The incident radiation penetrates beneath the surface of the slab and undergoes scattering processes. The external effect of the internal scattering processes is that, at every point on the surface of the slab, radiation emerges in all directions.

In the analysis of diffuse radiation, the following sign convention will be employed. At any depth $\tau$, $\mu$ will denote the cosine of the polar angle which the radiation makes with the outward-drawn normal from $\tau$. The azimuth angle of this direction is $\phi$. The intensity of radiation at $\tau$ will be written $I(\tau_1 + \mu_1 \phi)$ and $I(\tau_1 - \mu_1 \phi)$ ($0 \leq \mu \leq 1$) to emphasize that the direction of radiation makes an acute angle $\cos^{-1} \mu$ with the outward-drawn normal and the inward-drawn normal, respectively. Thus, the intensity of the incident parallel field will be written as $I_o (-\mu_o, \phi_o)$.

For the analysis of diffuse reflection, the standard form of the equation of transfer requires an important modification. The meaning of the intensity in Equation 38 is limited only to the intensity of radiation that reaches depth $\tau$ as a result of scattering processes. However, it is
Incident radiation

Reflected Radiation

Figure 4a. Sketch of diffuse reflection showing radiation reflected from a given point on the surface
also possible for part of the parallel radiation which is incident on the
surface to penetrate to a depth \( \tau \) without undergoing any scattering at all.
The probability of this is \( e^{-\tau/\mu_0} \), and this effect requires writing the
equation of transfer as

\[
\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - \frac{1}{4\pi} \int_{0}^{2\pi} \int_{-1}^{1} x(\mu, \phi; \mu', \phi') I(\tau, \mu', \phi') d\mu' d\phi'
\]

\[- \frac{1}{4\pi} I_0 e^{-\tau/\mu_0} x(\mu, \phi; -\mu_0, \phi_0) \]  

Solutions of this equation must be found subject to the boundary condition

\[ I(0, -\mu, \phi) = 0 \quad (0 < \mu < 1) \]  

Equation 49 expresses the fact that no diffuse radiation is incident at
the surface of the slab. A further condition to be imposed is the bounded-
ness of the solution as \( \tau \to \infty \) (8).

Attention is drawn to the fact that it is assumed that none of the
radiation incident at \( \tau = 0 \) emerges at \( \tau = \tau_1 \), i.e., that all incident
radiation is reflected at \( \tau = 0 \). This is equivalent to assuming a "semi-
infinite medium".

Consideration is now given to the possibility of obtaining the
angular distribution of the reflected intensity from the equation of
transfer.

The theory of integro-differential equations has not advanced to the
point where general analytic methods of solution exist. Thus, numerical
methods must be resorted to. Two distinct paths have developed (8).
Numerical methods, aiming at high accuracy, have been developed in the
context of a particular problem. Also, approximate methods have been
developed which, though perhaps not as accurate in a given case as a more
specialized method, have sufficient generality to be applicable to several problems. The equation of transfer has been solved in certain special cases by replacing the integro-differential equation by a system of linear algebraic equations (8). However, in parallel with these efforts, another approach has developed which has been found to have far-reaching significance. This technique of solution is based on certain invariance principles. Because of the relative simplicity of the method, as well as the current interest in new applications of it, it will be the approach followed in this paper.

First, it is to be noted that there is a certain undesirability in attempting to solve Equation 48. The reflected intensity required in Equations 34, 35, and 36 for aerodynamic purposes is a function of the two variables \( \theta \) and \( \phi \). It is not a function of \( \tau \), because it is evaluated at the surface, where \( \tau = 0 \). Yet, solution of the equation of transfer yields the intensity as a function of the three variables \( \theta \), \( \phi \), and \( \tau \). From this solution, the required intensity at \( \tau = 0 \) could be obtained. It is natural to wonder whether the intensity at the surface could be obtained directly, without first having to find the intensity at all depths \( \tau \) below the surface.

In 1943, the Russian astrophysicist V. A. Ambarzumian made a major breakthrough in his field by introducing a new technique by which the diffusely reflected intensity at the surface could be obtained directly (2). Chandrasekhar extended the technique to other problems, and gave it the name "principle of invariance". In recent years Wing (28) has successfully applied the method to problems in neutron transport theory. Bellman and his collaborators at the Rand Corporation have not only used
it to develop the new area of mathematics known as invariant imbedding, but have applied it to the solution of a wide variety of contemporary problems in control theory (26), systems analysis (18), dynamics (5), and kinetic theory (3).

The Principle of Invariance may be stated as follows:

**Principle of Invariance:** the law of diffuse reflection by a semi-infinite plane-parallel medium is invariant to the addition (or subtraction) of layers of arbitrary optical thickness to (or from) the medium. In particular, the phase function and the value of \( \lambda \) are the same in the medium and in the additional layers.

The Principle of Invariant Imbedding, quoted directly from Bellman (5), puts this in more mathematical terms.

**Principle of Invariant Imbedding:** "Given a physical system, \( S \), whose state at any time \( t \) is specified by a state vector, \( x \), we consider a process which consists of a family of transformations applied to this state vector. Suitably enlarging the dimension of the original vector by means of additional components, the state vectors are made elements of a space which is mapped into itself by the family of transformations. In this way we obtain an invariant process, by imbedding the original process within the new family of processes. The functional equations governing the new process are the analytic expression of this invariance."

In other words, the functional equations for a process in a system characterized by a dimension \( x \) are obtained by imbedding the system within a larger one characterized by a dimension \( x + \Delta x \), calculating the changes in the process due to the addition of \( \Delta x \), and equating these
changes to zero.

As used in the problem of diffuse reflection, the principle of invariance becomes the basis for solving \( I(0, \mu, \phi) \), and the equation of transfer becomes an auxiliary tool in the solution.

Before the principle is employed, the concept of the reflection function will be introduced. Let \( I_1(\mu', \phi') \) represent the intensity incident on the surface \( \tau = 0 \) in the direction \((-\mu', \phi')\). Then the angular distribution of the reflected intensity can be related to the incident intensity by means of the reflection function \( R(\tau_1; \mu, \phi; \mu', \phi') \) as follows:

\[
I_\tau(\phi, +\mu, \phi) = \frac{1}{4\pi\mu} \int_0^{2\pi} \int_0^1 R(\tau_1; \mu, \phi; \mu', \phi') I_1(\mu', \phi') d\mu' d\phi'
\]  

(50)

An incident parallel field in the direction \((-\mu_0, \phi_0)\) is a special case of \( I_1(\mu', \phi') \), because

\[
I_1(\mu', \phi') = I_0 \delta(\mu' - \mu_0) \delta(\phi' - \phi_0)
\]

(51)

where \( \delta \) is the Dirac delta function. Then Equation 50 becomes

\[
I_\tau(\phi, +\mu, \phi) = \frac{1}{4\pi\mu} I_0 R(\tau_1; \mu, \phi; \mu_0, \phi_0)
\]

(52)

It will prove convenient to write the incident intensity in the form

\[
I_0 = \pi S
\]

(53)

so that Equation 52 becomes

\[
I_\tau(\phi, +\mu, \phi) = \frac{S}{4\pi} R(\mu, \phi; \mu_0, \phi_0)
\]

(54)

where the dependence on \( \tau_1 \) is no longer explicitly shown.

Equation 54 is known as the law of diffuse reflection. It will now be shown that this law is invariant to the subtraction of a thin layer.
Consider a thin layer within the slab of material shown in Figure 4, bounded by level surfaces at \( \tau = 0 \) and \( \tau = \tau_L \), where \( \tau \) is arbitrarily close to \( \tau = 0 \). Part of the parallel radiation field \( I_0 \) that is incident on the surface at \( \tau = 0 \) undergoes scattering upon penetration of the medium. However, an intensity \( I_0 e^{-\tau/\mu_0} \) arrives at \( \tau \) unaltered by scattering. Of this, a fraction \( R(\tau, \mu; \phi, \mu_0, \phi_0) \) is reflected at \( \tau \) in the direction \( (+\mu, \phi) \), so that the reflected intensity from this source is, from Equation 54,

\[
\frac{S}{4\pi} R(\mu, \phi; \mu_0, \phi_0) e^{-\tau/\mu_0}
\]

Also, part of the incident parallel field is scattered in the medium, and arrives at \( \tau \) as a diffuse field of intensity \( I(\tau, -\mu', \phi') \). Part of this field is reflected at \( \tau \), and the contribution to the reflected intensity in \( (+\mu, \phi) \) from this source is, using Equation 50,

\[
\frac{1}{4\pi\mu} \int_0^{2\pi} \int_0^1 R(\mu, \phi; \mu', \phi') I(\tau, -\mu', \phi') \, d\mu' \, d\phi'
\]

The total intensity of outward-directed, diffusely reflected radiation at \( \tau \) is due to both of these contributions, so that

\[
I(\tau, +\mu, \phi) = \frac{S}{4\pi} e^{-\tau/\mu_0} R(\mu, \phi; \mu_0, \phi_0) + \frac{1}{4\pi\mu} \int_0^{2\pi} \int_0^1 R(\mu, \phi; \mu', \phi') \, d\mu' \, d\phi'
\]

\[
I(\tau, -\mu', \phi') \, d\mu' \, d\phi'
\]

(55)

This important equation is the mathematical statement of the principle of invariance. The layer of thickness \( \tau \) can be regarded as having been subtracted from the slab, and thus unable to affect a net change in intensity. Equation 55 is the equating of the algebraic sum of all
changes to zero.

The principle of invariance, with the aid of the equation of transfer, may be used to derive an integral equation for the reflected function.

Differentiating Equation 55 and evaluating the resulting equation at \( \tau = 0 \) gives

\[
\left[ \frac{dI(\tau, +\mu, \phi)}{d\tau} \right]_{\tau=0} = -\frac{8}{4\mu_0} R(\mu, \Phi; \mu_0, \phi_0) + \frac{1}{4\mu} \int_0^2 \int_0^1 R(\mu, \Phi; \mu', \phi') \times
\]

\[
\left[ \frac{dI(\tau, -\mu', \phi')}{d\tau} \right]_{\tau=0} = d\mu' d\phi'
\]  

(56)

The required derivatives can be evaluated from the equation of transfer, Equation 48, which is conveniently written in the form

\[
\mu \frac{dI(\tau, \mu, \phi)}{d\tau} = I(\tau, \mu, \phi) - J(\tau, \mu, \phi)
\]  

(57)

where

\[
J(\tau, \mu, \phi) = \frac{S}{h} e^{-\tau/\mu_0} \chi(\mu, \phi; \mu_0, \phi_0) + \frac{1}{4\pi} \int_0^2 \int_{-1}^1 \chi(\mu, \phi; \mu'', \phi'') \times
\]

\[
d\mu'' d\phi''
\]  

(58)

From Equation 57,

\[
\left[ \frac{dI(\tau, +\mu, \phi)}{d\tau} \right]_{\tau=0} = \frac{1}{\mu} [I(0, +\mu, \phi) - J(0, +\mu, \phi)]
\]  

(59)

and

\[
\left[ \frac{dI(\tau, -\mu', \phi')}{d\tau} \right]_{\tau=0} = -\frac{1}{\mu'} [I(0, -\mu', \phi') - J(0, -\mu', \phi')] = \frac{1}{\mu'} J(0, -\mu', \phi')
\]  

(60)

In Equation 60, the boundary condition given by Equation 49 has been involved.
It is now a matter of substituting Equations 58, 59, and 60 into Equation 56, and carrying out the straightforward but tedious mathematics. The final result is the following integral equation for the reflection function

\[ \left( \frac{1}{\mu} + \frac{1}{\mu_0} \right) R(\mu, \phi; \mu_0, \phi_0) = X(\mu, \phi; -\mu_0, \phi_0) + \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \int_0^1 X(-\mu^t, \phi^t; -\mu_0, \phi_0) R(\mu, \phi; \mu^t, \phi^t) \frac{d\mu^t}{\mu^t} d\phi^t \]

\[ + \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^1 \int_0^1 \int_0^1 X(-\mu^t, \phi^t; \mu^t, \phi^t) R(\mu, \phi; \mu^t, \phi^t) R(\mu^t, \phi^t; \mu_0, \phi_0) \]

\[ \frac{d\mu^t}{d\phi^t} \frac{d\mu}{d\phi} \]

(61)

In arriving at Equation 61, the following two points should be noted:

1. The factor \( I(0, \mu'', \phi'') \) which appears in two integrands is replaced by \( \frac{S}{4\pi} R(\mu'', \phi''; \mu_0, \phi_0) \) according to Equation 54.

2. The limits of \( \mu \) in the integral of Equation 58 are -1 to +1, but the value of the integral is zero between -1 and 0 due to Equation 49, when the integrand is evaluated at \( \tau = 0 \).

It can be shown by more elaborate considerations (8) that the reflection function is symmetric in the variables \( (\mu, \phi) \) and \( (\mu_0, \phi_0) \).

Equation 61 is an integral equation for the reflection function. For a given phase function the reflection function can in principle be found, and the diffusely reflected intensity can then be found from Equation 54 if the incident radiation field is parallel. In the next two sections, this procedure will be carried out for isotropic reflection,
and for a particular case of anisotropic reflection. For each case, the pressure and shear on an element of area of the surface of the slab due to diffusely reflected radiation will be determined. The medium is assumed homogeneous, so that $\lambda$ is a constant throughout.

C. Isotropic Reflection

As discussed previously, isotropic reflection is characterized by the phase function $\chi(\cos \gamma) = \lambda$, where $\lambda$ is a constant. As a consequence of this phase function, the diffuse radiation field is axially symmetric (1), and does not depend on the azimuth angle $\phi$. Thus, the intensity and reflection function may be written $I = I(\mu, \mu_o)$ and $R = R(\mu, \mu_o)$.

Equation 61 becomes,

$$\left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)R(\mu, \mu_o) = \lambda + \frac{\lambda}{2} \int_0^1 R(\mu'', \mu_o) \frac{d\mu''}{\mu''} + \frac{\lambda}{2} \int_0^1 R(\mu, \mu') \frac{d\mu''}{\mu''} + \frac{\lambda}{4} \int_0^1 \int_0^1 R(\mu, \mu'') \chi(\cos \gamma) R(\mu', \mu_o) \frac{d\mu'}{\mu'} \frac{d\mu''}{\mu''}$$

$$= \lambda \left[1 + \frac{1}{2} \int_0^1 R(\mu'', \mu_o) \frac{d\mu''}{\mu''} + \frac{1}{2} \int_0^1 R(\mu, \mu') \frac{d\mu'}{\mu'} + \frac{1}{4} \int_0^1 \int_0^1 R(\mu, \mu'') \chi(\cos \gamma) R(\mu', \mu_o) \frac{d\mu'}{\mu'} \frac{d\mu''}{\mu''}\right]$$

where integration over azimuth angle gives $2\pi$.

Equation 62 may be separated as follows:

$$\left(\frac{1}{\mu} + \frac{1}{\mu_o}\right)R(\mu, \mu_o) = \lambda \left[1 + \frac{1}{2} \int_0^1 R(\mu, \mu') \frac{d\mu'}{\mu'}\right] \left[1 + \frac{1}{2} \int_0^1 R(\mu'', \mu_o) \frac{d\mu''}{\mu''}\right]$$

Making use of the symmetry of $R(\mu, \mu')$ in $\mu$ and $\mu'$, it is seen that both
bracketed factors in Equation 63 must be the values for \( \mu \) and \( \mu_o \) of the same function. Defining,

\[
H(\mu_o) = 1 + \frac{1}{2} \int_0^1 R(\mu, \mu') \frac{d\mu'}{\mu'} = 1 + \frac{1}{2} \int_0^1 R(\mu', \mu) \frac{d\mu'}{\mu'},
\]

(64)

the reflection function may be written

\[
\left( \frac{1}{\mu} + \frac{1}{\mu_o} \right) R (\mu, \mu_o) = \lambda H(\mu) H(\mu_o)
\]

(65)

Substitution of Equation 65 into Equation 64 yields the following non-linear integral equation for \( H(\mu) \):

\[
H(\mu) = 1 + \frac{1}{2} \lambda \mu H(\mu) \int_0^1 \frac{H(\mu')}{\mu + \mu'} d\mu'
\]

(66)

Note that \( H \) is a function of two parameters, \( \mu \) and \( \lambda \). The dependence on \( \lambda \) will not be explicitly written. Finally, combining Equations 65 and 54, the reflected intensity in the isotropic diffuse case is

\[
I_r (0; \mu, \mu_o) = \frac{\lambda S}{4} \frac{\mu_o}{\mu + \mu_o} H(\mu) H(\mu_o)
\]

(67)

where the \( H \) function is the solution of Equation 66.

A variety of problems in the field of radiant transport have revealed the importance of non-linear integral equations of the form

\[
H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\psi(\mu')}{\mu + \mu'} d\mu'
\]

(68)

where \( \psi(\mu') \) is an even polynomial in \( \mu' \) satisfying the condition

\[
\int_0^1 \psi(\mu') d\mu' \leq \frac{1}{2}
\]

(69)

\( \psi(\mu') \) is called the characteristic function. Equation 66 is a special case of Equation 68 with \( \psi(\mu') = \lambda^2/2 \).
Excellent discussions of the solution of Equation 66 are given by Busbridge (7) and Chandrasekhar (8), and numerical solutions for various phase functions have been worked out by Chandrasekhar using an iteration procedure. Tables for the $H$ functions used in this thesis are given in Chandrasekhar (8).

Other functions which will prove to be of importance are the moments of $H$, defined by

$$M_n(\lambda) = \int_0^1 \mu^n H(\mu, \lambda) \, d\mu \quad (n \geq 0) \quad (70)$$

At this point, the expressions for the reflected pressure and shear on a small element area $d\sigma$ for isotropic reflection can be determined.

Equation 34 becomes, for axially symmetric fields,

$$P_r = \frac{2\pi}{c} \int_0^{\pi/2} I_r(\theta) \cos^2 \theta \sin \theta \, d\theta$$

In terms of $\mu = \cos \theta$, this may be written

$$P_r = \frac{2\pi}{c} \int_0^1 I_r(\mu, \mu_0) \mu^2 \, d\mu \quad (71)$$

For the isotropic case, Equation 67 is substituted into Equation 71, resulting in

$$P_r = \frac{\pi S \lambda}{2c} \mu_0 H(\mu_0) \int_0^1 \frac{\mu^2}{\mu + \mu_0} H(\mu) \, d\mu$$

or

$$P_r(\mu_0, \lambda) = \frac{I_0 \lambda}{2c} \mu_0 H(\mu_0) \int_0^1 \frac{\mu^2}{\mu + \mu_0} H(\mu) \, d\mu \quad (72)$$

Here the $H$-functions are presumed known from the solution of Equation 66.

Although Equation 72 can be solved directly by numerical methods, it is
informative to express the equation in terms of some meaningful parameters. To do this, consider the factor $\frac{\mu^2}{\mu + \mu_0}$ in the integrand. Since $\frac{\mu^2}{\mu + \mu_0} = \mu(1 - \frac{\mu}{\mu + \mu_0})$, Equation 72 can be written as

$$P_r = \frac{I_o}{2c} \mu_0 H(\mu_0) \left[ \frac{1}{6} \int \mu H(\mu) \, d\mu - \mu_0 \frac{1}{6} \int \frac{\mu H(\mu)}{\mu + \mu_0} \, d\mu \right]$$  \hspace{1cm} (73)

The first integral on the right side is, from Equation 70, the first moment of $H$, which is a function of $\lambda$, i.e., $M_1(\lambda)$. The second integral can be expressed in terms of a most important physical parameter, the "albedo", by means of the flux, introduced in Equations 16 and 17. The albedo is defined as the ratio of the flux of reflected radiation to the flux of incident radiation on a unit area of the surface element $dc$, i.e.,

$$A = \frac{\mathcal{F}_r}{\mathcal{F}_i}$$  \hspace{1cm} (74)

From Equation 11, and the definition of integrated flux as energy per area per time, $\mathcal{F}_i = I_o \cos \theta_o = I_o \mu_0$. From Equation 17,

$$\mathcal{F}_r = 2\pi \int_{0}^{1} I_r(\mu, \mu_0) \mu \, d\mu$$  \hspace{1cm} (75)

Substituting Equation 67 into Equation 75, the reflected flux becomes

$$\mathcal{F}_r = 2\pi \frac{\lambda S}{\mu} \mu_0 H(\mu_0) \frac{1}{6} \int \frac{\mu H(\mu)}{\mu + \mu_0} \, d\mu$$

$$= \frac{I_o}{2} \mu_0 H(\mu_0) \frac{1}{6} \int \frac{\mu H(\mu) \, d\mu}{\mu + \mu_0}$$  \hspace{1cm} (76)

The albedo is then

$$A = \frac{\lambda}{2} H(\mu_0) \int_{0}^{1} \frac{\mu H(\mu) \, d\mu}{\mu + \mu_0}$$  \hspace{1cm} (77)

Equation 73 can thus be written
or
\[ P_r (\mu_0, \lambda) = \frac{I_0 \mu_0}{c} \left[ \frac{\lambda}{2} \mathcal{A}(\mu_0) \mathcal{M}_1(\lambda) - A(\mu_0, \lambda) \mu_0 \right] \quad (78) \]

The albedo can be further reduced, for Equation 77 can be written
\[
A = \frac{\lambda}{2} \mathcal{H}(\mu_0) \int_0^1 \mathcal{H}(\mu) \left( 1 - \frac{\mu_0}{\mu + \mu_0} \right) d\mu
\]
\[
= \frac{\lambda}{2} \mathcal{H}(\mu_0) \left[ \int_0^1 \mathcal{H}(\mu) d\mu - \mu_0 \int_0^1 \frac{\mathcal{H}(\mu)}{\mu + \mu_0} d\mu \right] \quad (79)
\]
The second integral on the right hand side may be substituted for from Equation 66, which gives
\[
\mu_0 \int_0^1 \frac{\mathcal{H}(\mu)}{\mu + \mu_0} d\mu = \frac{\mathcal{H}(\mu_0) - 1}{\frac{\lambda}{2} \mathcal{H}(\mu_0)}
\]
The first integral on the right side is, from Equation 70, the zeroth moment of \( \mathcal{H}(\mu) \). Thus, Equation 79 becomes
\[
A = \frac{\lambda}{2} \mathcal{H}(\mu_0) \left[ \mathcal{M}_0(\lambda) - \frac{2}{\lambda} \frac{\mathcal{H}(\mu_0) - 1}{\mathcal{H}(\mu_0)} \right]
\]
\[
= \frac{\lambda}{2} \mathcal{H}(\mu_0) \mathcal{M}_0(\lambda) - \mathcal{H}(\mu_0) + 1
\]
Now, a fundamental property of the \( \mathcal{H} \) functions as defined by the integral equation, Equation 68 is that (8)
\[
\int_0^1 \mathcal{H}(\mu) \mathcal{P}(\mu) d\mu = 1 - \left[ 1 - 2 \left( \int_0^1 \mathcal{P}(\mu) d\mu \right) \right]^{1/2} \quad (80)
\]
For isotropic scattering, where \( \mathcal{P}(\mu) = \lambda/2 \), this reduces to
\[
\mathcal{M}_0(\lambda) = \frac{2}{\lambda} \left[ 1 - (1 - \lambda)^{1/2} \right] \quad (81)
\]
Using this result, the albedo becomes

\[ A = \frac{\lambda}{2} H(\mu_o) \frac{2}{\lambda} \left[ 1 - (1 - \lambda)^{\frac{1}{2}} \right] - H(\mu_o) + 1 \]

\[ = H(\mu_o) - H(\mu_o) (1 - \lambda)^{\frac{1}{2}} - H(\mu_o) + 1 \]

Thus,

\[ A (\mu_o, \lambda) = 1 - H(\mu_o, \lambda) (1 - \lambda)^{\frac{1}{2}} \tag{82} \]

To summarize, the reflected pressure has been expressed in Equation 78 in terms of the albedo, a parameter of special physical significance, and in terms of the first moment of \(H(\mu)\). The albedo has been shown to bear the radiation to the \(H\)-function of Equation 82.

Attention is now turned to the reflected shear. From symmetry considerations, the reflected shear is zero since the reflected radiation field is axially symmetric. At every angle \(0 \leq \theta \leq \pi/2\), the net tangential force acting on an element of surface area is zero, since at every azimuth angle \(\phi\) the tangential force is balanced by an oppositely directed tangential force at \(\phi + \pi\). This result can also be obtained directly from Equations 35 and 36. Thus, for isotropic scattering,

\[ \tau_r (\mu_o) = 0 \tag{83} \]

Before leaving the subject of isotropic scattering, the special case of Lambert's Law will be considered. Equation 67 gives the exact expression for the intensity of isotropically reflected radiation. Lambert's Law provides a simple, first approximation to this exact law. The distribution of the isotropically reflected radiation is taken to be independent of the angle of reflection, and is a constant depending on the angle of incidence and the intensity of the incident radiation. Lambert's Law is

\[ I_r (0, \mu, \phi; \mu_o, \phi_o) = \lambda \mu_o S \tag{84} \]
Substituting this into Equation 71 gives the reflected pressure for isotropic Lambert reflection as

$$P_r(\mu_o, \lambda) = \frac{2\pi}{c} \lambda \mu_o S \int_0^1 \mu^2 \, d\mu$$

$$= \frac{2}{3} \mu_o \lambda I_o / c$$  \hspace{1cm} (85)

**D. Anisotropic Reflection**

The goal now is to repeat the analysis of the previous section for a particular case of anisotropic reflection. The case corresponding to the phase function given in Equation 44 will be considered, with $b = 1$. Thus,

$$\chi (\cos \gamma) = \lambda (1 + \cos \gamma)$$  \hspace{1cm} (86)

where again $\lambda$ is a constant of the medium.

As mentioned before, Equation 86 corresponds to a moderate elongation of the phase function in the forward direction, i.e., there is a greater probability of scattering at small angles of deviation from the normal.

Since from Equation 41, $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')$, Equation 86 can be written as

$$\chi (\mu, \phi; \mu_o, \phi_o) = \lambda \left[ 1 + \mu \mu_o + (1 - \mu^2)^{3/2} (1 - \mu_o^2)^{3/2} \cos (\phi' - \phi) \right]$$  \hspace{1cm} (87)

This form of the phase function suggests that the reflection function be expressed in the form

$$R(\mu, \phi; \mu_o, \phi_o) = \lambda \left[ R^{(0)}(\mu, \mu_o) + (1 - \mu^2)^{3/2} (1 - \mu_o^2)^{3/2} R^{(1)}(\mu, \mu_o) \cos (\phi' - \phi) \right]$$  \hspace{1cm} (88)

where $R^{(0)}$ and $R^{(1)}$ are functions of $\mu$ and $\mu_o$ only.

Thus, the diffusely reflected intensity will be given by

$$I_r(0, \mu, \phi) = \frac{\lambda S}{4\mu} \left[ R^{(0)}(\mu, \mu_o) + (1 - \mu^2)^{3/2} (1 - \mu_o^2)^{3/2} R^{(1)}(\mu, \mu_o) \cos (\phi' - \phi) \right]$$  \hspace{1cm} (89)
which results from combining Equations 88 and 54.

Equations 87 and 88 must now be substituted into Equation 61, and the lengthy simplification process is then carried out. It is found that the equations for \( R^{(0)} \) and \( R^{(1)} \) separate, and both \( R^{(0)} \) and \( R^{(1)} \) are expressible in terms of H-functions. The following results are taken from Chandrasekhar (8). The first equation for \( R^{(0)} \) is

\[
\frac{1}{\mu + \mu_o} R^{(0)}(\mu, \mu_o) = H(\mu)H(\mu_o) \left[ 1 - e(\mu + \mu_o) - (1 - \lambda)\mu \mu_o \right], \tag{90}
\]

where \( H(\mu) \) is the solution of

\[
H(\mu) = 1 + \frac{1}{\mu} \int_0^1 \frac{1 + \frac{(1 - \lambda)\mu^2}{1 + \mu^2}}{H(\mu') \, d\mu'}; \tag{91}
\]

and \( e \) is a constant for a given \( \lambda \) related to the zeroth and first moments of \( H(\mu) \) by

\[
e(\lambda) = \frac{\lambda (1 - \lambda) M_1(\lambda)}{2 - \lambda M_0(\lambda)}; \tag{92}
\]

The second equation for \( R^{(1)} \) is

\[
\frac{1}{\mu + \mu_o} R^{(1)}(\mu, \mu_o) = H^{(1)}(\mu)H^{(1)}(\mu_o); \tag{93}
\]

where \( H^{(1)}(\mu) \) is the solution of

\[
H^{(1)}(\mu) = 1 + \frac{\lambda}{4} \mu H^{(1)}(\mu) \int_0^1 \frac{(1 - \mu^2)}{\mu + \mu^2} H^{(1)}(\mu') \, d\mu'; \tag{94}
\]

Comparing Equations 91 and 94 with Equation 68, the general H-function equation, it is seen that the characteristic function for Equation 91 is

\[
\psi(\mu') = \frac{\lambda}{2} \left[ 1 + (1 - \lambda) \mu'^2 \right]; \tag{95}
\]

and for Equation 94

\[
\psi(\mu^1) = \frac{\lambda}{4} (1 - \mu^1); \tag{96}
\]
The reflected intensity is found from Equations 89, 90, and 93 to be

\[
I_r(\theta, \phi) = \frac{\lambda S}{4} \frac{\mu_o}{\mu + \mu_o} \left\{ H(\mu)H(\mu_o) \left[ 1 - e(\lambda)(\mu + \mu_o) - (1 - \lambda)\mu_o \right] \right. \\
+ \left. (1 - \mu^2)^{1/2} (1 - \mu_o^2)^{1/2} \left[ H^1(\mu)H^1(\mu_o) \cos(\phi_0 - \phi) \right] \right\}
\]  

(97)

For each value of \(\lambda\), this is of the form

\[
I_r(\theta, \phi) = X(\mu, \mu_o) + Y(\mu, \mu_o) \cos(\phi_0 - \phi)
\]  

(98)

where

\[
X(\mu, \mu_o) = \frac{\lambda S}{4} \frac{\mu_o}{\mu + \mu_o} \left\{ H(\mu)H(\mu_o) \left[ 1 - e(\lambda)(\mu + \mu_o) - (1 - \lambda)\mu_o \right] \right\}
\]  

(99)

\[
Y(\mu, \mu_o) = \frac{\lambda S}{4} \frac{\mu_o}{\mu + \mu_o} \left[ (1 - \mu^2)^{1/2} (1 - \mu_o^2)^{1/2} \left[ H^1(\mu)H^1(\mu_o) \right] \right]
\]  

(100)

Equation 34 gives the general equation for the pressure on an element of area of a surface due to a diffuse radiation field as

\[
P_r = \frac{1}{c} \int_0^{2\pi} \int_0^{\pi/2} I_r(\theta, \phi) \cos^2 \theta \sin \theta \, d\theta \, d\phi
\]

In terms of \(\mu\), this is

\[
P_r = \frac{1}{c} \int_0^{2\pi} \int_0^1 I_r(\mu, \phi) \mu^2 \, d\mu \, d\phi
\]  

(101)

It is most informative to substitute Equation 98 into Equation 101, for this discloses that only the X term in the expression for intensity contributes to the pressure. Thus,

\[
P_r = \frac{1}{c} \int_0^1 \int_0^{2\pi} \left[ X(\mu, \mu_o) + Y(\mu, \mu_o) \cos(\phi_0 - \phi) \right] \mu^2 \, d\phi \, d\mu
\]

\[
= \frac{1}{c} \int_0^1 \left[ 2\pi X(\mu, \mu_o) + Y(\mu, \mu_o) \int_0^{2\pi} \cos(\phi_0 - \phi) \, d\phi \right] \mu^2 \, d\mu
\]
For the shear, write Equations 35 and 36 in terms of \( \mu \) as follows:

\[
\tau_{yr} = \frac{1}{c} \int_0^{2\pi} \int_0^1 \mathcal{I}_x(\mu, \phi) \mu (1 - \mu^2)^{\frac{1}{2}} \sin \phi \, d\phi \, d\mu
\]  

and

\[
\tau_{zr} = \frac{1}{c} \int_0^{2\pi} \int_0^1 \mathcal{I}_z(\mu, \phi) \mu (1 - \mu^2)^{\frac{3}{2}} \cos \phi \, d\phi \, d\mu
\]

Substitution of Equation 98 into Equation 103 gives

\[
\tau_{zr} = \frac{1}{c} \int_0^{2\pi} \int_0^1 \left[ \mathcal{I}_x(\mu, \phi) + \mathcal{I}_z(\mu, \phi) \cos (\phi_0 - \phi) \right] \mu (1 - \mu^2)^{\frac{1}{2}} \cos \phi \, d\phi \, d\mu
\]

\[
= \frac{1}{c} \left\{ \int_0^1 \mathcal{I}_x(\mu, \phi_0) \mu (1 - \mu^2)^{\frac{1}{2}} \left( \int_0^{2\pi} \sin \phi \, d\phi \right) \, d\mu + \int_0^1 \mathcal{I}_z(\mu, \phi_0) \mu (1 - \mu^2)^{\frac{3}{2}} \right\}
\]

\[
= \frac{\pi}{c} \cos \phi_0 \int_0^1 \mathcal{I}_z(\mu, \phi_0) \mu (1 - \mu^2)^{\frac{3}{2}} \, d\mu \tag{105}
\]

Similarly, substitution of Equation 98 into Equation 104 gives

\[
\tau_{yr} = \frac{\pi}{c} \sin \phi_0 \int_0^1 \mathcal{I}_z(\mu, \phi_0) \mu (1 - \mu^2)^{\frac{3}{2}} \, d\mu \tag{106}
\]

These equations show that only the \( Y \) term in the expression for intensity contributes to the shear, and they also show explicitly the relation between the shear components and the incident azimuth angle \( \phi_0 \). Equations 105 and 106 show clearly that the reflected shear is zero when \( Y \) is zero, i.e., when there is no \( \phi \) dependence in the reflected intensity. Thus,
only anisotropic phase functions produce shear. Another important general result is that the reflected shear acts along the same direction as the incident shear. For, from Equations 30, 105, and 106,
\[
\phi_r = \tan^{-1} \left( \frac{\tau_y}{\tau_z} \right) = \tan^{-1} \left( \frac{\sin \phi}{\cos \phi_o} \right) = \tan^{-1} \tan \phi_o,
\]
and thus,
\[
\phi_r = \phi_o \quad (107)
\]
This is a result that holds when the reflected intensity has the form of Equation 98. Lastly, it is seen from Equations 29, 105, and 106 that the magnitude of the reflected shear is given by
\[
\tau_r = \frac{\pi}{c} \int_0^1 Y(\mu, \mu_0, \mu(1-\mu^2)^\frac{1}{2}) \, d\mu \quad (108)
\]
Explicit expressions for the reflected pressure and shear will now be obtained. The pressure is found by combining Equations 99 and 102. Thus,
\[
P_r = \frac{2\pi}{c} \frac{\lambda^2}{4} \mu_0 H(\mu_0) \int_0^1 [1 - e(\lambda)(\mu+\mu_0)-(1-\lambda)\mu_0] \frac{\mu^2 H(\mu)}{\mu+\mu_0} \, d\mu \quad (109)
\]
An attempt will be made to simplify this equation. Define the parameter
\[
\frac{P_r}{\mu_0^2 T^2 c^3} \quad (110)
\]
Then,\[
\frac{P_r}{\mu_0^2 T^2 c^3} = H(\mu_0) \left[ \int_0^1 \frac{\mu^2 H(\mu)}{\mu+\mu_0} \, d\mu - e(\lambda) \int_0^1 \mu^2 H(\mu) \, d\mu - (1-\lambda)\mu_0 \int_0^1 \frac{\mu^3 H(\mu)}{\mu+\mu_0} \, d\mu \right] \quad (111)
\]
Since
\[
\frac{\mu}{\mu+\mu_0} = 1 - \frac{\mu_0}{\mu+\mu_0} \quad (112)
\]
the third integral can be written as
\[ \int_0^1 \mu^2 H(\mu) d\mu - \mu_0 \int_0^{\mu_0} H(\mu) d\mu. \]
Collecting terms in Equation 113,
\[ \int_1 = H(\mu_0) \left\{ [1 + \mu_0^2 (1-\lambda)] \int_0^1 \frac{\mu^2 H(\mu)}{\mu + \mu_0} d\mu - [e(\lambda) + \mu_0 (1-\lambda)] \int_0^1 \frac{\mu^2 H(\mu)}{\mu + \mu_0} d\mu \right\} \]
However, the second integral is just the second moment of \( H(\mu) \), i.e.,
\( M_2(\lambda) \). Also, repeated application of Equation 112 reduces the first integral to
\[ \int_1 = M_1(\lambda) - \mu_0 M_0(\lambda) + \mu_0^2 \int_0^1 \frac{H(\mu)}{\mu + \mu_0} d\mu \]
Therefore,
\[ \int_1 = H(\mu_0) \left\{ [1 + \mu_0^2 (1-\lambda)] [M_1 - \mu_0 M_0 + \mu_0^2 \int_0^1 \frac{H(\mu)}{\mu + \mu_0} d\mu] - M_2 [e(\lambda) + \mu_0 (1-\lambda)] \right\} \]
The next step is the evaluation of the integral in Equation 115. To do this, the integral equation for the \( H \)-function is resorted to. Equation 91 may, by a change of notation, be written as
\[ H(\mu_0) = 1 + \frac{\lambda}{2} \mu_0 H(\mu_0) \int_0^1 \frac{1+(1-\lambda)\mu_0^2}{\mu + \mu_0} H(\mu) d\mu \]
This may be expanded as
\[ H(\mu_0) = 1 + \frac{\lambda}{2} \mu_0 H(\mu_0) \left[ \int_0^1 \frac{H(\mu)}{\mu + \mu_0} d\mu + (1-\lambda) \int_0^1 \frac{\mu_0^2}{\mu + \mu_0} H(\mu) d\mu \right] \]
Equations 114 and 117 are simultaneous equations for the integrals
\[ \int_0^1 \frac{H(\mu) d\mu}{\mu + \mu_0} \quad \text{and} \quad \int_0^1 \frac{\mu_0^2 H(\mu) d\mu}{\mu + \mu_0}. \]
Solving for the second of these, using the result in Equation 113, simplifying and collecting terms, amounts to a
immediacy and planning horizon, all of which were scores developed for the household. Correlations were made for each of these measures with socioeconomic status. The relationships among the time variables are summarized in Table 10.

The mean duration scores showed an inverse relationship between socioeconomic status and duration of credit commitments. That is, as the socioeconomic level rose, couples tended to use credit contracts of shorter duration. The coefficient (r = -.3594) showed the duration score negatively correlated with socioeconomic status at the .001 level of significance.

Socioeconomic status was positively related to the mean scores for immediacy which progressed from 23.9 for the low status group to 27.5 for the upper group. A positive correlation (r = .3227) between immediacy and socioeconomic status was significant at the .01 level. These two data, the significant r value and the increasing mean scores indicated that wants of couples became more urgent as the socioeconomic level rose, or that the lower group was less willing or able to obtain credit.

The planning variable moved inconsistently among the socioeconomic groups. The highest mean score for planning (3.25) was achieved by the middle status group with the mean score for the high status group falling between that value and the 2.78 mean score obtained for the low group. However, the r value of .1182 was not significant.
straightforward but laborious exercise in algebra. The final result gives for the reflected pressure

\[ P_r(\mu_o,\lambda) = \frac{1}{c} \mu_o \left\{ H(\mu_o,\lambda) [U(\lambda)+\mu_o V(\lambda)] - \mu_o \right\} \]  

(118)

where

\[ U(\lambda) = \frac{\lambda}{2} [M_1(\lambda) - e(\lambda) M_2(\lambda)] \]  

(119)

and

\[ V(\lambda) = 1 - \frac{\lambda}{2} [M_o(\lambda) + M_2(\lambda)] + \frac{\lambda^2}{2} M_2(\lambda) \]  

(120)

In a similar manner, the reflected shear is obtained by combining Equations 108 and 100. Thus,

\[ \tau_r = \frac{\pi}{c} \frac{\lambda_0}{1} \mu_o (1-\mu_o)^{\frac{3}{2}} H^{(1)}(\mu_o) \int_0^1 \frac{\mu(1-\mu_o)}{\mu+\mu_o} H^{(1)}(\mu) d\mu \]  

(121)

Define the parameter

\[ T = \frac{\tau_r}{\mu_o (1-\mu_o)^{\frac{3}{2}} I_o \lambda/4c} \]  

(122)

Then,

\[ T = H^{(1)}(\mu_o) \left[ \int_0^1 \frac{\mu}{\mu+\mu_o} H^{(1)}(\mu) d\mu - \int_0^1 \frac{\mu^3}{\mu+\mu_o} H^{(1)}(\mu) d\mu \right] \]  

(123)

In the same manner as was followed for the pressure, the integrals in Equation 123 can be reduced to moments of \( H^{(1)}(\mu) \), and to the integral

\[ \int_0^1 \frac{H^{(1)}(\mu)}{\mu+\mu_o} d\mu \]. This last integral is evaluated by means of the \( H^{(1)} \) equation, Equation 94. Once again, the procedure is lengthy, and just the results will be stated.

\[ \tau_r(\mu_o,\lambda) = \frac{I_o}{c} \mu_o (1-\mu_o)^{\frac{3}{2}} [1 + W(\lambda) H^{(1)}(\mu_o,\lambda)] \]  

(124)

where

\[ W(\lambda) = \frac{\lambda}{4} [N_o(\lambda) - N_2(\lambda)] - 1 \]  

(125)
and the moments of $H^{(1)}$ are defined by

$$N_n(\lambda) = \int_0^1 \mu^n H^{(1)}(\mu, \lambda) \, d\mu \quad (n \geq 0) \quad (126)$$

From Equation 80, with $\psi(\mu) = \frac{\lambda}{4} (1-\mu^2)$, it is found that

$$\frac{\lambda}{4} \int_0^1 H^{(1)}(\mu, \lambda) (1-\mu^2) \, d\mu = 1 - \left[ 1 - 2 \int_0^1 \frac{\lambda}{4} (1-\mu^2) \, d\mu \right]^{1/2}$$

so that the zeroth and second moments of $H^{(1)}$ are related by

$$N_0(\lambda) - N_2(\lambda) = \frac{1}{4} \left[ 1 - (1-\lambda/3)^{1/2} \right] / \lambda \quad (127)$$

Therefore, Equation 125 becomes

$$W(\lambda) = - (1 - \lambda/3)^{1/2} \quad (128)$$

E. Comparison between Radiation Aerodynamics and Rarefied Gas Dynamics

A comparison of the primary features of radiation aerodynamics and rarefied gas dynamics reveals some most interesting similarities and differences between them. These will be brought out and discussed in this section.

Rarefied gas dynamics is that area of gas dynamics in which the gas does not behave as a continuum, but rather exhibits some properties of its molecular structure ($2^k$). A rarefied gas is characterized by a mean free path of the order of some pertinent body dimension. Free molecule flow is the regime of rarefied gas dynamics of extreme rarefaction, wherein collisions between molecules are so infrequent as to be considered negligible, and the mean free path is very much greater than a body dimension.
The basic unknown in rarefied gas dynamics is the distribution function \( f(\vec{r}, \vec{v}, t) \), which gives the number density of molecules in phase space, i.e., the number of molecules per unit volume at time \( t \) having position vectors in the range \((\vec{r}, \vec{r} + d\vec{r})\) and velocities in the range \((\vec{v}, \vec{v} + d\vec{v})\). Knowledge of the distribution function is requisite to the determination of the macroscopic gas dynamic parameters. The evolution of the distribution function in \( \vec{r}, \vec{v}, \) and \( t \) space is governed by the Boltzmann equation, given (9) in vector form as

\[
\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{v}} = \int \int (f'_1 f'_1 - f f') k \, d\omega \, d\vec{v} \tag{129}
\]

The Boltzmann equation represents the conservation of the distribution function in a unit volume of phase space due to the effects of both molecular collisions and drift (continuous particle motion). Equation 129 considers the effect of two-particle collisions on the distribution function. The subscript 1 denotes the distribution function of one of the species of molecules. The \( f \) without the subscript denotes the distribution function of the other. Primes denote values after a collision. Unprimed variables refer to values prior to a collision. Further, \( k \) is a scalar factor, \( \vec{F} \) is the external force per unit mass, and \( d\omega \) the element of solid angle.

The condition of equilibrium flow is \( \frac{\partial f}{\partial t} = 0 \). If the distribution function does not vary with \( r \), and there are no external forces, the equilibrium condition requires the right hand side of the Boltzmann equation to be zero, which leads to \( f f'_1 = f'_1 f \). It is well known (9) that this last requirement is uniquely satisfied by the Maxwell distribution function.
\[ f(u,v,w) = n(2\pi RT)^{-3/2} \exp \left(-\frac{c^2}{2RT}\right) \] (130)

where \( u, v, \) and \( w \) are the \( x, y, \) and \( z \) components of \( \vec{c} \), \( n \) is the number density of particles, \( R \) is the gas constant and \( T \) is the temperature.

The method of determining the pressure and shear for a stream that is characterized by a distribution function \( f(\vec{c}) \) will now be given. In every range of molecular velocities \( (\vec{c}, \vec{c} + d\vec{c}) \),

\[
\text{force per area} = \frac{\text{momentum per time}}{\text{area}} = \frac{\text{number of molecules}}{\text{volume}} \times \text{x component of velocity normal to area} \times \frac{\text{momentum}}{\text{molecule}}
\]

\[ = f(\vec{c}) \, d\vec{c} \times u \times m \] (131)

where the \( x \) component of \( \vec{c} \) is taken to be normal to the element of area. Equation 131 has normal (pressure) and tangential (shear) components, and so

\[ P(\vec{c}, \vec{c} + d\vec{c}) = m u^2 f(\vec{c}) \, d\vec{c} \] (132)
\[ \tau_y(\vec{c}, \vec{c} + d\vec{c}) = m u v f(\vec{c}) \, d\vec{c} \] (133)
\[ \tau_z(\vec{c}, \vec{c} + d\vec{c}) = m u w f(\vec{c}) \, d\vec{c} \] (134)

The integrated pressure and shear components are then

\[ P = \int m u^2 f(\vec{c}) \, d\vec{c} \] (135)
\[ \tau_y = \int m u v f(\vec{c}) \, d\vec{c} \] (136)
\[ \tau_z = \int m u w f(\vec{c}) \, d\vec{c} \] (137)

A similar analysis could be performed for the more general distribution function \( f(\vec{c}, \vec{r}, t) \). In free molecule flow, Equations 135, 136, and 137 are evaluated by means of the Maxwell distribution function.

With this brief sketch of the method of rarefied gas dynamic analysis now presented, similarities and differences with radiation aerodynamics
will now be discussed.

(1). In free molecule flow, the basic assumption is that the incident flow is completely undisturbed by the presence of a body, so that the effect of reflected molecules on the incident stream is negligible (24). This is the free molecule counterpart of the "two independent stream" assumption of radiation aerodynamics. Therefore, the theories of free molecule flow and radiation aerodynamics are founded on the same assumption. This assumption is valid only for the free molecule regime of rarefied gas dynamics, and breaks down as the fluid density increases.

(2). The distribution function plays the same role in rarefied gas dynamics as the intensity does in radiation aerodynamics, as far as the determination of macroscopic stream parameters is concerned. Equations 135, 136, and 137 show that these parameters in rarefied gas dynamics are obtained as integral moments of the distribution function. In radiation aerodynamics, we have shown that these parameters are integral moments of the intensity. For this reason, \( f \) and \( I \) are regarded as the basic unknowns in their respective fields.

(3). A comparison of Equations 135, 136, and 137 with Equations 26, 27, and 28 reveals the striking similarity in the structure of the equations. The following lists the correspondence between rarefied gas dynamics and radiation aerodynamic factors in the integrands of the equations:

\[
\begin{align*}
\text{m} & \sim \frac{1}{c} \text{ (a scalar factor)} \\
\text{u} & \sim \cos \theta \\
\text{v} & \sim \sin \theta \sin \phi \\
\text{w} & \sim \sin \theta \cos \phi \\
f(c) & \sim I(\alpha)
\end{align*}
\]
The transformation of the integrands in the corresponding equations can readily be accomplished. As an example, consider the integrands of Equations 135 and 26, which are the expressions for pressure. The transformation is as follows:

\[
muf^2 f(c) dc = \frac{\partial}{\partial c} \left[ muf^2 f(c) dc \right]
\]

\[
\sim \frac{1}{c} \left( \frac{\text{mass}}{\text{molecule}} \times \text{velocity}^2 \times \frac{\text{number of molecules}}{\text{unit volume}} \right) \frac{\text{length}}{\text{time}}
\]

\[
\sim \frac{1}{c} \left( \frac{\text{total mass}}{\text{area-time}} \times \text{velocity}^2 \right)
\]

\[
\sim \frac{1}{c} \left( \text{energy/area-time} \right)
\]

\[
= \frac{1}{c} \left( I \cos \theta \right) d\omega
\]

(4) The distribution function is the solution of the Boltzmann equation. The intensity is the solution of the equation of transfer. Both are integro-differential equations, and both are derived from conservation principles. In the Boltzmann equation, collisions may be both the gas-gas and gas-body types. Since there are no photon-photon interactions, the equation of transfer accounts for collisions of the photon-body type.

(5) In free molecule flow, the distribution function is Maxwellian. The incident flow parameters are calculated from Equations 135, 136, and 137 using the distribution function of Equation 130 corresponding to an incident stream temperature \( T_i \). The reflected flow parameters are similarly evaluated, but with a Maxwellian at a temperature characteristic of the reflected flow, \( T_r \). In general, \( T_r \) lies between \( T_i \) and the body temperature \( T_w \). Its value depends on the degree of interaction, or "accommodation", between the incident stream and the body. It is the practice in free molecule flow to treat the uncertainty in the value of
by introducing accommodation coefficients to represent the exchange of
normal momentum and tangential momentum between incident stream and body
as follows (25):

\[ \sigma_n = \frac{\tau_i - \tau_r}{\tau_i - \tau_w} \]  \hspace{1cm} (138)

\[ \sigma_t = \frac{P_i - P_r}{P_i - P_w} \]  \hspace{1cm} (139)

where the subscripts \( i \) and \( r \) refer to incident and reflected stream and
\( w \) denotes the parameters evaluated by means of the Maxwell distribution at
the temperature of the surface. In free molecule flow, parameters of the
reflected stream are related to those of the incident stream by means of
suitable accommodation coefficients. These coefficients are gross and
incomplete descriptions of average effects (10). Their introduction as
constants in free molecule calculations avoids solution of the difficult
Boltzmann equation, but the coefficients are responsible for the
uncertainty that currently plagues free molecule calculations. In
radiation aerodynamics, the incident pressure and shear are easily
calculated from the assumption of a parallel field. The reflected intensity
has been solved for in simple cases of diffuse reflection directly, by
means of invariance principles, and no need to resort to "accommodation
coefficients" exists in radiation aerodynamics.

(6) Specular and diffuse are most important concepts in both
radiation aerodynamics and rarefied gas dynamics, but slightly different
interpretations are involved in each discipline. In free molecule flow,
for example, a specular reflection process is one which there is no exchange
of tangential momentum between the incident molecules and the body, and
the normal momentum is reversed in direction. This is described by the accomodation coefficients being zero. The other extreme, $\bar{\sigma}_n = \bar{\sigma}_t = 1$ corresponds to "perfectly diffuse" reflection in which the reflected molecules leave the surface of the body completely "accomodated" to it. This condition is characterized by the reflected stream being in Maxwellian equilibrium at the surface temperature, and the spatial distribution follows the Lambert cosine law with respect to the polar angles of the reflected molecules. In the general case (the accomodation coefficients having values between zero and one), the reflected stream is in Maxwellian equilibrium at some temperature between the body and the incident stream, and the diffuse spatial distribution is anisotropic. In radiation aerodynamics, the terms specular and diffuse refer only to the spatial orientation of the reflected intensity, while in free molecule flow more than spatial randomness is implied. The degree of equilibrium between the incident stream and the body is also included in the concept. There is a correspondence between the phase function in radiation aerodynamics and the accomodation coefficients of free molecule flow, for they both serve as an indication of the deviation of the reflected stream from an isotropic condition.
V. APPLICATIONS OF RADIATION AERODYNAMICS

Everything required for the calculation of the differential lift and drag force on an element of surface area $d\sigma$ has been determined. The pressure and shear for the incident parallel radiation field are known. The pressure and shear for diffusely reflected radiation fields have been determined for isotropic and a type of anisotropic reflection. For both of these cases, the total pressure and shear can be obtained from Equations 3 and 4, and these total values are related to the differential lift and drag forces by Equations 1 and 2. Equations 1 and 2 may then be integrated over a given body to give the total lift and drag forces on the body due to light radiation.

This procedure will be carried out in this section for four types of reflection processes: isotropic, Lambert, anisotropic, and specular. The total lift and drag forces will be determined for three body shapes of particular aerodynamic importance: a flat plate, a sphere, and a cylinder.

It is interesting to note that the velocity of the body is assumed to be small compared with the velocity of light, so that the body may be considered at rest with respect to the incident radiation. This is a most valid assumption for actual space travel situations. For example, a vehicle travelling from earth to Mars using a Hohmann transfer ellipse for a 250 day trip, will approach Mars at a velocity of $23,322 \text{ km/sec}$ (21). For this case,

$$\frac{v}{c} = \frac{2.332 \times 10^4}{3 \times 10^8} \approx 10^{-4}$$

and the point is illustrated. However, since electromagnetic radiation is
incident on a body with the velocity of light, the magnitude and direction of the body velocity can always be neglected, from relativistic considerations.

The expressions for the differential lift and drag forces will now be calculated for each of the four types of reflection processes that have been mentioned. Figure 46 shows an element of a slab, inclined at an angle of attack $\alpha$ with respect to the incident stream, and having a surface area $d\sigma$.

Using the coordinate system of Figure 1, we take the azimuth angle of the incident stream to be $\pi/2$ with respect to the positive z axis, so that the incident shear has no z-component. The pressure and shear for the incident radiation are, from Equations 29, 31, 32, and 33 with $\phi_0 = \pi/2$

$$P_i = \frac{I_0}{c} \cos^2 \alpha$$

and

$$\tau_i = \frac{I_0}{c} \cos \alpha \sin \alpha$$

acting in the negative y direction. In these equations, $\cos \alpha$ has been substituted for $\mu_0$.

For isotropic reflection, Equations 78 and 83 give

$$P_r(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha \left[ \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right]$$

$$\tau_r(\alpha, \lambda) = 0$$

Therefore, since $P_T = P_i + P_r$, and $\tau_T = \tau_i - \tau_r$,

$$P_T(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha \left[ \cos \alpha + \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right]$$

$$\tau_T(\alpha) = \frac{I_0}{c} \cos \alpha \sin \alpha$$

Substitution of these equations into Equations 1 and 2 gives

$$dF_L = \left\{ \frac{I_0}{c} \cos \alpha \sin \alpha \left[ \cos \alpha + \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right] \right\} d\sigma$$

or

$$dF_L = \left\{ \frac{I_0}{c} \cos \alpha \sin \alpha \left[ \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right] \right\} d\sigma$$

(144)
Figure 4b. Element of area $d\sigma$ in a radiation field
\[ dF_D = \left\{ \frac{I_0}{c} \cos^2 \alpha \left[ \cos \alpha + \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right] + \frac{I_0}{c} \cos \alpha \sin^2 \alpha \right\} d\sigma \]

or

\[ dF_D = \frac{I_0}{c} \cos \alpha \left\{ 1 + \cos \alpha \left[ \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right] \right\} d\sigma \]

(145)

For the special case of Lambert reflection, Equation 85 gives

\[ P_r(\alpha, \lambda) = \frac{2}{3} \frac{I_0}{c} \lambda \cos \alpha \]

Thus,

\[ P_r(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha \left( \cos \alpha + \frac{2}{3} \lambda \right) \]

(146)

\[ \tau_r(\alpha) = \frac{I_0}{c} \cos \alpha \sin \alpha \]

(147)

\[ dF_L = \left[ \frac{I_0}{c} \cos \alpha \sin \alpha \left( \cos \alpha + \frac{2}{3} \lambda \right) - \frac{I_0}{c} \cos^2 \alpha \sin \alpha \right] d\sigma \]

or

\[ dF_L = \left( \frac{2}{3} \frac{I_0}{c} \lambda \cos \alpha \sin \alpha \right) d\sigma \]

(148)

\[ dF_D = \left[ \frac{I_0}{c} \cos^2 \alpha \left( \cos \alpha + \frac{2}{3} \lambda \right) + \frac{I_0}{c} \cos \alpha \sin^2 \alpha \right] d\sigma \]

or

\[ dF_D = \left[ \frac{I_0}{c} \cos \alpha \left( 1 + \frac{2}{3} \lambda \cos \alpha \right) \right] d\sigma \]

(149)

For the anisotropic case, Equations 118 and 124 give

\[ P_r(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha \left\{ H(\alpha, \lambda) [U(\lambda) + V(\lambda) \cos \alpha] - \cos \alpha \right\} \]

\[ \tau_r(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha \sin \alpha \left[ 1 + W(\lambda) H^{(1)}(\alpha, \lambda) \right] \]

Therefore,

\[ P_T(\alpha, \lambda) = \frac{I_0}{c} \cos \alpha H(\alpha, \lambda) \left[ U(\lambda) + \cos \alpha V(\lambda) \right] \]

(150)

and

\[ \tau_T(\alpha, \lambda) = - \frac{I_0}{c} \cos \alpha \sin \alpha W(\lambda) H^{(1)}(\alpha, \lambda) \]

(151)

The function \( W(\lambda) \) is negative for all \( \lambda \) from 0 to 1, so that the total shear in Equation 151 has a positive sign, and acts in the negative y direction. The differential lift and drag forces are

\[ dF_L = \left\{ \frac{I_0}{c} \cos \alpha \sin \alpha H(\alpha, \lambda) [U(\lambda)+V(\lambda) \cos \alpha] + \frac{I_0}{c} \cos^2 \alpha \sin \alpha \right\} W(\lambda) H^{(1)}(\alpha, \lambda) \} d\sigma \]

or

\[ dF_L = \frac{I_0}{c} \cos \alpha \sin \alpha \left\{ H(\alpha, \lambda) [U(\lambda)+V(\lambda) \cos \alpha] + \cos \alpha W(\lambda) H^{(1)}(\alpha, \lambda) \right\} d\sigma \]

(152)
Finally, the specular case will be treated. Allowing for possible absorption of some of the incident radiation, the following conditions define the case of specular reflection:

$$I_r(\theta_r, \phi_r) = \lambda I_o(\theta_o, \phi_o)$$  \hspace{1cm} (154)

$$\theta_r = \theta_o$$  \hspace{1cm} (155)

$$\phi_r = \phi_o$$  \hspace{1cm} (156)

and the normal component of momentum reverses direction. The reflected pressure and shear are found as follows:

$$P_r = \frac{1}{c} I_r \cos \theta_r \frac{1}{c} \lambda I_o \cos \theta_o$$

Therefore,

$$P_r = \lambda P_i$$  \hspace{1cm} (157)

also,

$$\tau_r = \frac{1}{c} I_r \cos \theta_r \sin \theta_r = \frac{1}{c} \lambda I_o \cos \theta_o \sin \theta_o$$

and so

$$\tau_r = \lambda \tau_i$$  \hspace{1cm} (158)

The total values of pressure and shear are, in terms of $\alpha$,

$$P_T(\alpha, \lambda) = (1 + \lambda) \frac{I_o}{c} \cos^2 \alpha$$  \hspace{1cm} (159)

and

$$\tau_T(\alpha, \lambda) = (1 - \lambda) \frac{I_o}{c} \cos \alpha \sin \alpha$$  \hspace{1cm} (160)

For the specular case, then,

$$dF_L = [(1 + \lambda) \frac{I_o}{c} \cos^2 \alpha \sin \alpha - (1-\lambda) \frac{I_o}{c} \cos^2 \alpha \sin \alpha] d\sigma$$

or

$$dF_L = (2\lambda \frac{I_o}{c} \cos^2 \alpha \sin \alpha) d\sigma$$  \hspace{1cm} (161)
\[ dF_D = \left[ (1+\lambda) \frac{I_0}{c} \cos^2 \alpha + (1-\lambda) \frac{I_0}{c} \cos \alpha \sin^2 \alpha \right] d\sigma \]

or
\[ dF_D = \left[ \frac{I_0}{c} \cos \alpha (1+\lambda \cos 2\alpha) \right] d\sigma \]  

(162)

The equations for the differential lift and drag forces will now be integrated over the surfaces of a flat plate, a sphere, and a cylinder. The results will be expressed in terms of dimensionless lift and drag coefficients. These coefficients will be plotted for each case, and graphs comparing these coefficients for the same body shape but different reflection laws will be given.

A brief note will be given regarding the numerical solution of the lift and drag equations. The \( H \) functions have been computed by Chandrasekhar (8) as function of \( \mu = \cos \alpha \) and \( \lambda \). These values are used in the numerical solutions. All moments of the \( H \) functions that are required have been computed numerically from Equations 70 and 126. Numerical integrations are required for evaluating the moments of the \( H \) functions, and also for other integrals involving the \( H \) functions. In each numerical integration, the trapezoidal rule was used. All numerical calculations were performed by the IBM 7074 Computer at the Iowa State Computation Center.

A. Flat Plate

One of the basic assumptions of the theory presented in this paper is that none of the radiation incident on the outer surface of a body emerges from the inner surface. The incident radiation is absorbed and reflected, but not transmitted clear through the body. In dealing with flat plates, this condition must be kept in mind. A very thin flat plate need not necessarily violate this condition, since the plate might be made of a
highly reflecting material. Thus, the reflectivity and the thickness
determine the validity of the assumption in a particular case.

Let the total surface area of the plate be \( \sigma \), so that \( \int \sigma \, d\sigma = \sigma \).

Integration of the differential lift and drag forces, which have the form
\( dF = f(\alpha, \lambda) \, d\sigma \) gives \( F = f(\alpha, \lambda) \sigma \). Further, since \( \frac{I_0}{c} \) and \( \frac{F}{\sigma} \) have the same
dimension, the ratio \( \frac{F}{\sigma \frac{I_0}{c}} \) is dimensionless. In general, for any body
of total surface area \( \sigma \), radiation lift and drag forces can be defined by

\[
C_L = \frac{F_L}{\sigma \frac{I_0}{c}} = \frac{F_L}{\sigma(P_1)_{\alpha=0}}
\]

and

\[
C_D = \frac{F_D}{\sigma \frac{I_0}{c}} = \frac{F_D}{\sigma(P_1)_{\alpha=0}}
\]

The following results appear for the flat plate upon integration of the
appropriate differential lift and drag equations for the appropriate case.

1. **Isotropic**

\[
C_L(\alpha, \lambda) = \sin \alpha \cos \alpha \left[ \frac{1}{2} M_1(\lambda) H(\alpha, \lambda) - A(\alpha, \lambda) \cos \alpha \right]
\]

\[
C_D(\alpha, \lambda) = \cos \alpha \left[ 1 + \cos \alpha \left[ \frac{1}{2} M_1(\lambda) H(\alpha, \lambda) - A(\alpha, \lambda) \cos \alpha \right] \right]
\]

These equations are plotted in Figures 7 and 13.

2. **Lambert**

\[
C_L(\alpha, \lambda) = \frac{1}{2} \sin 2\alpha
\]

\[
C_D(\alpha, \lambda) = \cos \alpha (1 + \frac{2}{3} \lambda \cos \alpha)
\]

These equations are plotted in Figures 8 and 14.

3. **Anisotropic**

\[
C_L(\alpha, \lambda) = \cos \alpha \sin \alpha \left[ H(\alpha, \lambda) [U(\lambda) + V(\lambda) \cos \alpha] + \cos \alpha \hat{H}(\lambda) H^{(1)}(\alpha, \lambda) \right]
\]

\[
C_D(\alpha, \lambda) = \cos \alpha \left\{ \cos \alpha \hat{H}(\alpha, \lambda) [U(\lambda) + V(\lambda) \cos \alpha] - \sin^2 \alpha \hat{H}(\lambda) H^{(1)}(\alpha, \lambda) \right\}
\]
Figure 5. Construction of a sphere from an element of surface area $d\sigma$
Figure 6a. Cylinder at a right angle to the incident radiation

Figure 6b. Construction of a cylinder from an element of surface area $d\sigma$
Figure 7. Lift coefficient vs. angle of attack for a flat plate with isotropic reflection for different degrees of reflection $\lambda$. 
Figure 8. Lift coefficient vs. angle of attack for flat plate with Lambert reflection for different degrees of reflection $\lambda$. 
ANGLE OF ATTACK (\( \alpha \))

<table>
<thead>
<tr>
<th>( \lambda = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90</td>
</tr>
<tr>
<td>0.70</td>
</tr>
<tr>
<td>0.50</td>
</tr>
<tr>
<td>0.30</td>
</tr>
<tr>
<td>0.10</td>
</tr>
</tbody>
</table>

COSINE OF ANGLE OF ATTACK (\( \mu \))

LIFT COEFFICIENT (\( C_L = \frac{F_L}{I_o/c} \))
Figure 9. Lift coefficient vs. angle of attack for a flat plate with anisotropic reflection for different degrees of reflection $\lambda$. 
Figure 10. Lift coefficient vs. angle of attack for a flat plate with specular reflection for different degrees of reflection $\lambda$. 
Figure 11. Influence of the reflection process on the lift coefficient of a flat plate for $\lambda = 1.0$
Figure 12. Influence of the reflection process on the lift coefficient of a flat plate $\lambda = 0.50$
Figure 13. Drag coefficient vs. angle of attack for a flat plate with isotropic reflection for different degrees of reflection \( \lambda \).
Figure 14. Drag coefficient vs. angle of attack for a flat plate with Lambert reflection for different degrees of reflection $\lambda$. 
These equations are plotted in Figure 9 and 15.

4. Specular

\[ C_L(\alpha, \lambda) = \lambda \sin 2\alpha \cos \alpha \]  
\[ C_D(\alpha, \lambda) = \cos \alpha (1 + \lambda \cos 2\alpha) \]

These equations are plotted in Figures 10 and 16.

It is of great interest to compare the effect of the reflection laws on the aerodynamic coefficients. Figures 11 and 12 compare the lift coefficient vs. angle of attack for \( \lambda = 1.0 \) and \( \lambda = 0.5 \) respectively, for the four laws of reflection. Figures 17 and 18 do the same thing for the drag coefficient.

E. Sphere

In applying the theory of radiation aerodynamics to spheres, note must be taken of the assumption of plane-parallel media, so that the theory is strictly applicable only to a body of sufficiently large radius of curvature. This condition is met for spherically shaped space vehicles.

The sphere presents an example of the need to integrate the differential lift and drag equations with respect to angle of attack. As shown in Figure 4b, the angle of attack of the slab element with respect to the incident stream is the angle between the direction of the incident stream and the normal to the surface. It is thus possible to " construct" a sphere from strips at constant angle of attack as shown in Figure 5. If the axis of the sphere is taken along the direction of the incident radiation, the angle of attack of a strip will be the angle between the axis and the radius drawn from the center of the sphere to any point on the surface of the strip. The method of finding the total lift and drag
Figure 15. Drag coefficient vs. angle of attack for a flat plate with anisotropic reflection for different degrees of reflection $\lambda$. 
Figure 16. Drag coefficient vs. angle of attack for a flat plate with specular reflection for different degrees of reflection $\lambda$. 
Figure 17. Influence of the reflection process on the drag coefficient of a flat plate, $\lambda = 1.0$
Figure 18. Influence of the reflection process on the drag coefficient of a flat plate for $\lambda = 0.50$.
forces on the sphere is the following.

Equations 1 and 2 have allowed the determination of the differential lift and drag forces on an element of area \( d\sigma \). We regard the slab element of Figure 4 to be an element on a strip. Integration of Equations 1 and 2 around the strip gives the lift and drag forces acting on the strip, and then integration of these forces over all strips give the total lift and drag forces on the sphere. Let the surface area of a strip be \( d\Sigma \).

Let

\[
\begin{align*}
dF^L_{\alpha} &= \left[ P_T(\alpha) \sin \alpha - \tau_T(\alpha) \cos \alpha \right] \, d\sigma \\
dF^D_{\alpha} &= \left[ P_T(\alpha) \cos \alpha + \tau_T(\alpha) \sin \alpha \right] \, d\sigma
\end{align*}
\]

be the forces on \( d\sigma \), and

\[
\begin{align*}
dF^L_{\Sigma} &= \left[ P_T(\alpha) \sin \alpha - \tau_T(\alpha) \cos \alpha \right] \, d\Sigma \\
dF^D_{\Sigma} &= \left[ P_T(\alpha) \cos \alpha + \tau_T(\alpha) \sin \alpha \right] \, d\Sigma
\end{align*}
\]

be the forces on a strip of surface area \( d\Sigma \).

As shown in Figure 5,

\[
d\sigma = (R \sin \alpha \, d\epsilon) \times (R \, d\alpha) = R^2 \sin \alpha \, d\alpha \, d\epsilon
\]

where \( R \) is the radius of the sphere.

Thus,

\[
d\Sigma = \int d\sigma = \int_{\epsilon=0}^{2\pi} R^2 \sin \alpha \, d\alpha \, d\epsilon
\]

so

\[
d\Sigma = 2\pi R^2 \sin \alpha \, d\alpha
\]

(173)

Therefore,

\[
\begin{align*}
dF^L_{\Sigma} &= 2\pi R^2 \left[ P_T(\alpha) \sin \alpha - \tau_T(\alpha) \cos \alpha \right] \sin \alpha \, d\alpha \\
dF^D_{\Sigma} &= 2\pi R^2 \left[ P_T(\alpha) \cos \alpha + \tau_T(\alpha) \sin \alpha \right] \sin \alpha \, d\alpha
\end{align*}
\]

(174)

and

(175)

The limits of integration on \( \alpha \) are 0 to \( \pi \). However, only half of the sphere is "wetted" by radiation, the other half being entirely
unexposed. This is due to the fact that the incident field is parallel. This, in effect the limits of $\alpha$ are from 0 to $\pi/2$.

From symmetry considerations it may be immediately concluded that the lift force on the sphere is zero. For, on any given strip, the lift acting at a point defined by the angles $(\alpha, \phi)$ is balanced by an equal but oppositely acting force at the point $(\alpha, \phi + \pi)$. Therefore, the subsequent analysis will deal only with drag. The task is to evaluate the equation

$$F_D = 2\pi R^2 \int_0^{\pi/2} [P_T(\alpha)\cos \alpha + T_T(\alpha) \sin \alpha] \sin \alpha \, d\alpha \quad (176)$$

for each of the four types of reflection considered.

1. **Isotropic**

Substitution of Equations 152 and 143 into Equation 176 gives

$$F_D = 2\pi R^2 \int_0^{\pi/2} \cos \alpha \sin \alpha \left\{1 + \cos \alpha \left[\frac{A}{2} H(\alpha,\lambda)M_1(\lambda) - A(\alpha,\lambda)\cos \alpha\right]\right\} d\alpha$$

It is convenient to introduce $\mu = \cos \alpha$ into this expression. Also, let the radiation drag coefficient be defined with respect to the projected area of the sphere $\pi R^2$. Thus, for a sphere,

$$C_D = \frac{F_D/\pi R^2}{I_0/c} \quad (177)$$

and so

$$C_D = 2 \int_0^1 \mu \left\{1 + \mu \left[\frac{A}{2} H(\mu,\lambda)M_1(\lambda) - A(\mu,\lambda)\right]\right\} d\mu$$

$$= 2 \left[\int_0^1 \mu \, d\mu + \frac{A}{2} M_1(\lambda) \int_0^1 \mu^2 H(\mu,\lambda) \, d\mu - \int_0^1 \mu^3 A(\mu,\lambda) \, d\mu\right]$$
The first integral is equal to $1/2$. The second is the second moment of $H$, $M_2(\lambda)$. To evaluate the third integral, it is recalled from Equation 82 that

$$A(\mu,\lambda) = 1 - H(\mu,\lambda) \sqrt{1 - \lambda}$$

Thus,

$$\int_0^1 \mu^3 A(\mu,\lambda) \, d\mu = \int_0^1 \mu^3 \, d\mu - \sqrt{1 - \lambda} \int_0^1 \mu^3 H(\mu,\lambda) \, d\mu = \frac{1}{4} - \sqrt{1 - \lambda} M_2(\lambda).$$

Therefore,

$$C_D(\lambda) = 2 \left[ \frac{1}{8} + \frac{\lambda}{2} M_1(\lambda) M_2(\lambda) - \frac{1}{4} + \sqrt{1 - \lambda} M_3(\lambda) \right]$$

and thus

$$C_D(\lambda) = \frac{1}{8} + \lambda M_1(\lambda) M_2(\lambda) + 2 \sqrt{1 - \lambda} M_3(\lambda) \quad \text{(178)}$$

2. Lambert

Substitution of Equation 146 and 147 into Equation 176 gives

$$F_D = 2\pi \lambda^2 \frac{I_0}{c} \int_0^{\pi/2} \cos \alpha \left( 1 + \frac{2}{3} \lambda \cos \alpha \right) \sin \alpha \, d\alpha$$

Thus,

$$C_D = 2 \int_0^1 (\mu + \frac{2}{3} \lambda \mu^2) \, d\mu$$

$$= 2 \left[ \frac{1}{8} + \frac{\lambda}{9} \right],$$

and so

$$C_D(\lambda) = 1 + \frac{1}{9} \lambda \quad \text{(179)}$$

3. Anisotropic

Substitution of Equations 150 and 151 into Equation 176 gives

$$F_D = 2\pi \lambda^2 \frac{I_0}{c} \int_0^{\pi/2} \cos \alpha \left\{ \cos \alpha H(\alpha,\lambda) \left[ U(\lambda) + V(\lambda) \cos \alpha \right] - \sin^2 \alpha W(\lambda) H^{(1)}(\alpha,\lambda) \right\} \sin \alpha \, d\alpha$$

Thus,

$$C_D = 2 \int_0^1 \left\{ \mu^2 H(\mu,\lambda) [U(\lambda) + V(\lambda)] - (1 - \mu^2) W(\lambda) H^{(1)}(\alpha,\lambda) \right\} \, d\mu$$
and so
\[ C_D = 2 \left\{ U(\lambda) \int_0^1 \mu^2 H(\mu,\lambda) \, d\mu + V(\lambda) \int_0^1 \mu^3 H(\mu,\lambda) \, d\mu - \bar{W}(\lambda) \int_0^1 \mu^3 H^{(1)}(\mu,\lambda) \, d\mu \right\} \]

Each of the integrals is a moment of \( H(\mu,\lambda) \) or \( H^{(1)}(\mu,\lambda) \). Thus, from Equations 70 and 126,
\[ C_D(\lambda) = 2 \left\{ U(\lambda) M_2(\lambda) + V(\lambda) M_3(\lambda) - \bar{W}(\lambda) \left[ N_1(\lambda) - N_3(\lambda) \right] \right\} \]  

4. Specular

Substitution of Equations 159 and 160 into Equation 176 gives
\[ F_D = \frac{2\pi R^2 I_0}{c} \int_0^{\pi/2} \cos \alpha (1 + \lambda \cos 2\alpha) \sin \alpha \, d\alpha \]

Thus,
\[ C_D = 2 \int_0^1 \mu \left[ 1 + \lambda (2\mu^2 - 1) \right] \, d\mu \]
\[ = 2 \left[ \frac{1}{3} + 2\lambda (\frac{1}{3}) - \lambda (\frac{1}{3}) \right] \]

Therefore,
\[ C_D(\lambda) = 1 \]  

Equations 176, 177, 178, and 179 are plotted in Figure 19.

C. Cylinder

An aerodynamic body of considerable importance is the right circular cylinder with longitudinal axis perpendicular to the direction of the incident stream, i.e., a cylinder transverse to the incident radiation, as shown in Figure 6 (a).
Figure 19. Drag coefficient vs. degree of reflection $\lambda$ for a sphere
The cylinder can be regarded as composed of circular discs, as shown in Figure 6 (b). The slab element of Figure 4 can be considered as an element of the disc. As the element of area dσ travels around the disc, the angle of attack between the normal from dσ and the direction of incident radiation varies from 0 to 2π. In the coordinate system of Figure 4, the z axis is parallel to the longitudinal axis of the cylinder.

The element of area dσ is equal to (Rdα)(dz), where R is the radius of the cylinder. The angle α varies from 0, where the normal is exactly parallel to the direction of the incident radiation, to 2π. However, only half of the cylinder is exposed to the radiation. Thus, only that part of the cylinder in the range of α (0, π/2) and (3π/2, 2π) are effective in contributing to lift and drag. The effective limits are (-π/2, π/2).

As in the case of the sphere, the lift on the cylinder is zero by a symmetry consideration. This is true only for the cylinder at an angle of attack of 90°.

The total drag force is obtained from Equation 1, and is for the cylinder

\[ F_D = \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ P_T(\alpha) \cos \alpha + \tau_T(\alpha) \sin \alpha \right] R d\alpha dz \]

where L is the total length of the cylinder. Integration over the length gives

\[ F_D = RL \int_{-\pi/2}^{\pi/2} \left[ P_T(\alpha) \cos \alpha + \tau_T(\alpha) \sin \alpha \right] d\alpha \quad (182) \]

This will now be evaluated for each of the four types of reflection considered.

1. Isotropic

Substitution of Equations 142 and 143 into Equation 182 gives
\[
\frac{F_D}{RL} = \frac{I_0}{c} \int_{-\pi/2}^{\pi/2} \cos \alpha \left\{ 1 + \cos \alpha \left[ \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - A(\alpha, \lambda) \cos \alpha \right] \right\} d\alpha
\]

It is conventional to refer the drag coefficient of a cylinder to the projected area of the cylinder, \(2RL\). Then, using Equation 164, and also using the expression for albedo given in Equation 82, we can write

\[
2C_D = \int_{-\pi/2}^{\pi/2} \cos \alpha \left\{ 1 + \cos \alpha \left[ \frac{\lambda}{2} H(\alpha, \lambda) M_1(\lambda) - \cos \alpha \cos \alpha H(\alpha, \lambda) \sqrt{1 - \lambda} \right] \right\} d\alpha
\]

We have been writing the \(H\)-function as \(H(\alpha, \lambda)\). Strictly speaking, this should be written \(H(\cos \alpha, \lambda)\), because of the way \(H\) is defined in Equation 68. Thus, the integrand of Equation 181 is an even function of \(\alpha\), and so

\[
2C_D = 2 \left\{ \int_0^{\pi/2} \cos \alpha d\alpha \frac{\lambda}{2} M_1(\lambda) \int_0^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) d\alpha - \int_0^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) d\alpha \right\}
\]

so that

\[
C_D(\lambda) = \frac{1}{3} + \frac{\lambda}{2} M_1(\lambda) \int_0^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) d\alpha + \sqrt{1 - \lambda} \int_0^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) d\alpha
\]

Note that the two integrals are not moments of \(H\) as defined by Equation 70. These integrals have to be evaluated numerically.

2. Lambert

Substitution of Equations 146 and 147 into Equation 182 gives

\[
\frac{F_D}{RL} = \frac{I_0}{c} \int_{-\pi/2}^{\pi/2} \cos \alpha \left( 1 + \frac{2}{3} \lambda \cos \alpha \right) d\alpha
\]

\[
= 2 \frac{I_0}{c} \left[ \int_0^{\pi/2} \cos \alpha d\alpha + \frac{2}{3} \lambda \int_0^{\pi/2} \cos^2 \alpha d\alpha \right]
\]
Thus, \( C_D(\lambda) = 1 + \frac{\pi}{6} \lambda \) \hspace{1cm} (185)

3. Anisotropic

Substitution of Equation 150 and 151 into Equation 182 gives

\[
\frac{F_D}{R_L} = \frac{I_0}{c} \int_{-\pi/2}^{\pi/2} \cos \alpha \left\{ \cos \alpha H(\alpha, \lambda) \left[ U(\lambda) + \cos \alpha V(\lambda) \right] - \sin^2 \alpha W(\lambda) \right\} \, d\alpha
\]

The integrand is an even function of \( \alpha \). Thus,

\[
C_D(\lambda) = U(\lambda) \int_{0}^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) \, d\alpha + V(\lambda) \int_{0}^{\pi/2} \cos^2 \alpha H(\alpha, \lambda) \, d\alpha - W(\lambda) \int_{0}^{\pi/2} \cos \alpha \sin \alpha H(1) (\alpha, \lambda) \, d\alpha
\]

Each integral in Equation 186 needs to be evaluated numerically.

4. Specular

Substitution of Equations 159 and 160 into Equation 182 gives

\[
\frac{F_D}{R_L} = \frac{I_0}{c} \int_{0}^{\pi/2} \cos \alpha (1 + \lambda \cos 2\alpha) \, d\alpha
\]

\[
= \frac{I_0}{c} \left[ \int_{0}^{\pi/2} \cos \alpha \, d\alpha + \lambda \int_{0}^{\pi/2} \cos \alpha \cos 2\alpha \, d\alpha \right]
\]

Therefore,

\[
C_D(\lambda) = 1 + \frac{\lambda}{3} \hspace{1cm} (187)
\]

Equations 184, 185, 186, and 187 are plotted in Figure 20.
Figure 20. Drag coefficient vs. degree of reflection $\lambda$ for a cylinder transverse to incident radiation.
VI. DISCUSSION

The results of the previous chapter are shown graphically in Figures 7-20. The significant features brought out by the curves will now be discussed. Since it is of particular aerodynamic interest to determine the influence that the reflection process has on the values of the aerodynamic coefficients, the format of this chapter will be to compare, for each body shape considered, the effect of the following reflection processes:

1. Isotropic and Specular
2. Isotropic and Lambert
3. Isotropic and Anisotropic

The first of these comparisons will determine the feasibility of approximating the aerodynamic coefficients of a diffuse reflector by assuming specular reflection, thereby greatly simplifying the calculations. If this is found not to be feasible, the second approximation will determine the effect of approximating isotropic reflection by Lambert reflection, which would simplify the diffuse calculations. Finally, the third approximation will serve to indicate the importance of precise knowledge of the diffuse phase function.

A. Flat Plate

Figures 7-10 show the variation of the lift coefficient of a flat plate with angle of attack for isotropic, Lambert, anisotropic, and specular reflection respectively, with $\lambda$ as a parameter. For each type of reflection, the curves show that reduction in the value of $\lambda$ reduces the lift, so that maximum lift at a given angle of attack is achieved for
total reflection of the incident radiation, and minimum lift results for
complete absorption. This is to be expected from Equation 1

\[ dF_L = (P_T \sin \alpha - \tau_T \cos \alpha) d\sigma \]

since the total pressure increases by the amount of the reflected pressure
which results when \( \lambda > 0 \), while the total shear is reduced by the amount
of the reflected shear which may result (for specular and anisotropic
reflection) when \( \lambda > 0 \). The curves also show that there is no lift at
angles of attack of 0° and 90°. This may also be deduced from Equation 1.

At 0°, the first term on the right-hand side is zero due to \( \sin \alpha \), and the
second term is zero since there is no shear component of the force per
unit area at 0°. At 90°, the first term is zero since there is no
pressure component at 90°, and the second term is zero due to the \( \cos \alpha \)
term. These results accord with those for a flat plate exposed to a
conventional "fluid" (e.g., free molecule flow), with one significant
difference. For a conventional fluid, both sides of the plate are exposed
to the flow, while it is clear that only one side of the plate will
experience incident radiation. If, for example, free molecule flow
calculations were carried out assuming only one side exposed, it would
be found that there would be lift at 90°. A further observation regarding
the lift curves is that maximum lift for the diffuse curves occurs at an
angle of attack of about 44° (\( \cos \alpha = 0.72 \)), while for specular reflection
this maximum occurs at an angle of attack of about 36° (\( \cos \alpha = 0.81 \)).

It is seen from Equations 169 and 170 that, for specular reflection with
\( \lambda = 1 \), lift is proportional to the square of the cosine of \( \alpha \), and to the
sine of \( \alpha \), while drag is proportional to the cube of the cosine of \( \alpha \).

This result has been reported by Cotter (11).
Figures 11 and 12 compare the lift coefficients for the four types of reflection processes, with a $\lambda$ of 1.0 (complete reflection) and 0.5 (half reflection and half absorption) respectively. The results presented in these figures will be discussed according to the three categories listed at the beginning of this chapter.

1. Specular reflection gives a markedly higher lift coefficient than any diffuse reflection process for angles of attack in the range of $0^\circ$ to about $66^\circ$. Specifically, between $18^\circ$ and $45^\circ$ specular reflection gives more than twice the lift. However, at large angles of attack in the range $72.5^\circ$ to $90^\circ$, specular reflection results in less lift than isotropic or Lambert reflection for $\lambda = 0.5$ and less lift than all three diffuse reflection processes for $\lambda = 1.0$. The important conclusion to be drawn is that for a wide range of angles of attack, specular reflection will not be an accurate approximation to an actual diffuse reflector in determining the lift coefficient of a flat plate.

2. The Lambert approximation to exact isotropic reflection is an excellent one for large values of $\lambda$, but deviates from the isotropic solution as $\lambda$ decreases. At $\lambda = 1.0$, the results for the lift coefficient are practically identical. At smaller values of $\lambda$, the Lambert approximation gives values for the lift coefficient of a flat plate that are as much as three times larger than the isotropic values.

3. The lift coefficients calculated for isotropic and anisotropic reflection are very close. Anisotropic reflection gives slightly
larger values of $C_L$ at large values of $\lambda$, but the reverse is true at smaller values of $\lambda$. It can be concluded that the type of anisotropic reflection considered is adequately approximated by isotropic reflection.

Figures 13-16 show the variation of the flat plate drag coefficient with angle of attack for the four reflection processes. While the behavior of the diffuse reflection curves are essentially similar, a unique feature appears in the specular case in the nature of a "crossover" point at an angle of attack of $45^\circ$. Below $45^\circ$, increasing $\lambda$ increases $C_D$, as in the case of diffuse reflection. However, above $45^\circ$ an increase in $\lambda$ results in a decrease in drag. These conclusions may be reached directly from an analysis of Equation 172. This interesting result has not, to the author's knowledge, been previously reported, due perhaps to the frequently made assumption with specular reflection that $\lambda = 1.0$.

For all cases, there is no drag at an angle of attack of $90^\circ$, and maximum drag occurs at $0^\circ$. Let us examine Equation 2,

$$dF_D = (F_T \cos \alpha + \tau_T \sin \alpha) \, d\alpha.$$  

At $90^\circ$, the first term on the right-hand side is zero due to $\cos \alpha$, and the second term is also zero since the total shear is zero at an angle of attack of $90^\circ$. This latter result follows directly from Equations 143, 147, 151, and 160, but there is unusual physical significance attached to it. For conventional fluids, shear is maximum at $90^\circ$, but the shear due to electromagnetic radiation is zero at $90^\circ$. This serves as an indication of the different mechanisms that produce shear. In conventional fluids, shear stress occurs as the result of an interchange of momentum between
the layer of fluid adjacent to the surface and the surface, the interchange
being accomplished by means of the molecules of the fluid. Since the
fluid molecules undergo random motion, it is possible for the molecules
to transfer momentum to the surface even though the stream direction is
exactly parallel to the surface. However, it was shown in Chapter III
that for radiation to exert a force on an element of area, radiative
energy has to pass through the area. Since no radiation incident on an
area with an angle of attack of $90^\circ$ can pass through the area, there will
be no shear stress. Therefore, there is no drag at an angle of attack of
$90^\circ$. This is unique to radiation aerodynamics. For free molecule flow,
for example, the drag coefficient of a flat plate at an angle of attack
of $90^\circ$ is related to the free stream Mach Number $M_\infty$, the gas specific
heat ratio $\gamma$, and the tangential accommodation coefficient $\tilde{\alpha}_t$ by (25)
\[
C_D = \frac{(8/\gamma \pi)^{\frac{1}{2}}}{\tilde{\alpha}_t/M_\infty}
\]
for both sides exposed to the flow, and
\[
C_D = \frac{(2/\gamma \pi)^{\frac{1}{2}}}{\tilde{\alpha}_t/M_\infty}
\]
for one side exposed. These show that, for $M_\infty \neq 0$, the drag coefficient
is zero only for specular reflection ($\tilde{\alpha}_t = 0$). For diffuse reflection
($\tilde{\alpha}_t = 1$), there is drag at an angle of attack of $90^\circ$. There is also
a drag effect for all values of $\lambda$.

Figures 17 and 18 compare the drag coefficients for the four types
of reflection processes with $\lambda = 1.0$ and 0.5, respectively. The
following conclusions are drawn:

1. Specular reflection gives larger drag coefficients than diffuse
reflection for angles of attack between $0^\circ$ and $25^\circ$, but smaller
ones between 25° and 90°, for λ = 1.0. As λ decreases, the specular drag coefficient becomes smaller than the diffuse drag coefficients at larger angles of attack. As in the lift case, the deviation between specular and diffuse values is appreciable.

2. The Lambert results are identical with the isotropic results at λ = 1.0, but at λ = 0.5 the Lambert approximation gives values of the drag coefficient that are too high. However, the Lambert approximation to isotropic reflection is more accurate at all values of λ for the drag coefficient than for the lift coefficient.

3. The anisotropic results for drag remain reasonably close to the isotropic results at all values of λ to conclude once again that isotropic reflection may adequately represent diffuse reflection under anisotropic phase functions.

B. Sphere

Figure 19 shows the drag coefficient of a sphere as a function of λ for the four types of reflection processes. The importance of the body shape to the conclusions drawn so far from the flat plate become evident for the sphere and cylinder.

1. The fact that the drag force on a sphere with specular reflection is a constant, independent of λ, is a most significant result. The deviation of the diffuse drag coefficients from the specular drag coefficient at large values of λ is apparent.

2. Lambert's Law is not a bad engineering approximation to the isotropic diffuse case, despite the markedly different shapes of the curves. For practical numerical calculations, Lambert
reflection is still a satisfactory approximation, although not as good as for the flat plate.

3. Similarly, isotropic reflection is an acceptable representation of anisotropic reflection.

It is interesting to note that specular reflection gives the smallest drag coefficient at every value of $\lambda$.

C. Cylinder

Figure 20 shows the drag coefficient of a cylinder transverse to the incident radiation as a function of $\lambda$ for the four types of reflection processes. Here, the anisotropic case gives the smallest drag for $\lambda$ between 0 and 0.963, and specular reflection gives the smallest drag at $\lambda = 1.0$. The same remarks apply as for the sphere. The drag coefficients for the sphere and the cylinder as functions of $\lambda$ are given in the Appendix for isotropic and anisotropic reflection.

It is important to note that the result for the sphere and cylinder have wider aerodynamic generality than may be evident. Since the parallel radiation field "wets" only half of each body, the shape of the body behind the wetted portion is immaterial, provided that the rear portion is unexposed to the radiation. Then the results would apply to a variety of shapes, such as sphere-cones, sphere-cylinder, cylinder-plates, etc.

To summarize then the results disclosed by the applications of radiation aerodynamics considered in this paper:

1. Specular reflection is not, in general, a suitable approximation to a diffusely reflecting material.
2. Lambert reflection is a satisfactory approximation to both kinds of diffuse reflection processes considered for large values of $\lambda$.

3. Isotropic reflection is an adequate representation of the anisotropic reflection process considered in this paper.
VII. BIBLIOGRAPHY


VIII. ACKNOWLEDGEMENTS

The author is deeply grateful to Dr. C. T. Hsu for hours of stimulating discussions. His helpful guidance will long be remembered and sincerely appreciated.

A profound feeling of thanks must also be expressed to Dr. E. W. Anderson for providing the environment of creative freedom which spawned this thesis, and for making possible the association with the Aerospace Engineering Department which the author prizes.

Finally, the author is most thankful to the Iowa Engineering Experiment Station for the support of this program; and to the Iowa State Computation Center for providing free time on the IBM 7074 Computer.
Table 1. Drag coefficients for a sphere and a transverse cylinder for isotropic and anisotropic reflection, as a function of the degree of reflection $\lambda$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Sphere</th>
<th></th>
<th>Cylinder</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Isotropic</td>
<td>Anisotropic</td>
<td>Isotropic</td>
<td>Anisotropic</td>
</tr>
<tr>
<td>0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>0.10</td>
<td>1.0095</td>
<td>1.0009</td>
<td>1.0089</td>
<td>1.0018</td>
</tr>
<tr>
<td>0.20</td>
<td>1.0189</td>
<td>1.0025</td>
<td>1.0192</td>
<td>1.0044</td>
</tr>
<tr>
<td>0.30</td>
<td>1.0299</td>
<td>1.0056</td>
<td>1.0313</td>
<td>1.0087</td>
</tr>
<tr>
<td>0.40</td>
<td>1.0428</td>
<td>1.0101</td>
<td>1.0456</td>
<td>1.0148</td>
</tr>
<tr>
<td>0.50</td>
<td>1.0585</td>
<td>1.0173</td>
<td>1.0631</td>
<td>1.0240</td>
</tr>
<tr>
<td>0.60</td>
<td>1.0780</td>
<td>1.0280</td>
<td>1.0850</td>
<td>1.0373</td>
</tr>
<tr>
<td>0.70</td>
<td>1.1037</td>
<td>1.0450</td>
<td>1.1140</td>
<td>1.0581</td>
</tr>
<tr>
<td>0.80</td>
<td>1.1399</td>
<td>1.0730</td>
<td>1.1554</td>
<td>1.0919</td>
</tr>
<tr>
<td>0.85</td>
<td>1.1652</td>
<td>1.0985</td>
<td>1.1848</td>
<td>1.1232</td>
</tr>
<tr>
<td>0.90</td>
<td>1.1997</td>
<td>1.1274</td>
<td>1.2251</td>
<td>1.1575</td>
</tr>
<tr>
<td>0.925</td>
<td>1.2230</td>
<td>1.1505</td>
<td>1.2525</td>
<td>1.1854</td>
</tr>
<tr>
<td>0.95</td>
<td>1.2538</td>
<td>1.1823</td>
<td>1.2891</td>
<td>1.2241</td>
</tr>
<tr>
<td>0.975</td>
<td>1.2999</td>
<td>1.2322</td>
<td>1.3445</td>
<td>1.2851</td>
</tr>
<tr>
<td>1.000</td>
<td>1.4497</td>
<td>1.4078</td>
<td>1.5295</td>
<td>1.5034</td>
</tr>
</tbody>
</table>