Applications and extensions of the weighted integral

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APPLICATIONS AND EXTENSIONS
OF THE WEIGHTED INTEGRAL

by

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I. INTRODUCTION

In this paper we deal with the weighted refinement integral

\[ \left[ F, (w_1, w_2, w_3) \right] \int_a^b f(x) dg(x) \]  

of [1] and [2] as well as with an extension

\[ \left[ F, (w_1, w_2, w_3) \right] s \int_a^b f(x) dg(x) \]  

analogous to the extension of the Stieltjes mean sigma integral considered by P. Porcelli [3]. Throughout all functions are real-valued functions on the entire real axis, all intervals are finite and non-degenerate, and \((w_1, w_2, w_3)\) refers to an ordered triple of real numbers such that \(w_1 + w_2 + w_3 = 1\).

In Chapter II we define a Lebesgue-Stieltjes integral

\[ \text{(LS)} \int_{[a,b]} f(x) dg(x) \]  

in terms of the weighted refinement integral (1.1), where \(g\) is not even required to be quasi-continuous. This definition is consistent with the case where \(g\) is of
bounded variation. We then use a substitution theorem in [2] for the weighted refinement integral to establish the substitution formula

\[
(\text{LS}) \int_{[a,b]} h(x)f(x)dg(x) = (\text{LS}) \int_{[a,b]} h(x)dg(\text{LS}) \int_{[a,x]} f(u)dg(u)
+ h(a)f(a)[g(a^+) - g(a^-)]
\]

for this Lebesgue-Stieltjes integral.

In Chapter III we generalize the step function approach used by Porcelli [3] to define our extension (1.2) of the weighted refinement integral (1.1). Here \( g \) is assumed to be a function of bounded variation on every closed interval, and the function \( f \) is bounded on the closed interval \([a,b]\). If \( g_s \) is a saltus function and \( g_c \) a continuous function such that \( g = g_s + g_c \), then we show that (1.2) exists iff the weighted refinement integral

\[
(1.4) \quad [F, (w_1,w_2,w_3)] \int_a^b f(x)dg_s(x)
\]

and the Lebesgue-Stieltjes integral

\[
(1.5) \quad (\text{LS}) \int_{[a,b]} f(x)dg_c(x)
\]
exist. Furthermore, we show that when the extended integral (1.2) exists it is the sum of (1.4) and (1.5). A result for (1.2) is established which is analogous to the Lebesgue Dominated Convergence Theorem for the Lebesgue-Stieltjes integral. Then substitution for (1.2) is discussed and related to the case where the integrator of (1.2) is the product of two functions.

Next we describe how the work of H. Scharf [4] on integration-by-parts for the Lebesgue-Stieltjes integral leads rather naturally to a Lebesgue-Stieltjes integral (1.3) defined in terms of the extended integral

$$[F, (1,-1,1)] \int_{a}^{b} g(x) df(x).$$

This extension of the usual Lebesgue-Stieltjes integral extends the one of Chapter II for the case where $f$ is a function of bounded variation on every closed interval, $g$ is a function bounded on $[a,b]$, $g(a^-)$ and $g(b^+)$ exist and are finite, $g(x^+)$ exists for all $x$ in $[a,b]$ such that $f(x^+) \neq f(x)$, and $g(x^-)$ exists for all $x$ in $(a,b]$ such that $f(x^-) \neq f(x)$.

Finally, we extend the Gronwall inequality as studied by W. Schmaedeke and G. Sell [5]. We correct an error in [5] and show that under looser conditions than those of [5]
if \( k \) is a non-negative function, \( \epsilon \geq 0 \), and

\[
f(t) \leq \epsilon + \left[p, (w_1, w_2, w_3)\right] \int_a^t f(s)k(s)dg(s)
\]

for \( 0 \leq t \leq T \), then there exist constants \( T' \) and \( K \), depending on \( g \) and \( k \) but not \( f \), such that \( 0 < T' \leq T \), \( 0 < K \), and

\[
f(t) \leq K\epsilon, \quad 0 \leq t < T'.
\]

Also, under appropriate conditions we show that the inequality holds for the extended integral.

For the sake of the reader we conclude this chapter with the definition of and five basic theorems on the weighted refinement integral.

**Definition 1.1.** Let \( f \) and \( g \) be functions. Let

\[
P = \{a = x_0 < x_1 < \cdots < x_n = b\}
\]

be a partition of the interval \([a, b]\). If the refinement limit of the sums

\[
\sum_{i=1}^n [w_1f(x_{i-1}) + w_2f(\zeta_i) + w_3f(x_i)][g(x_i) - g(x_{i-1})],
\]
\[ x_{i-1} < \xi_i < x_i \]

exists and is finite, it is denoted by

\[ \left[ F, (w_1, w_2, w_3) \right] \int_a^b f(x) dg(x) \]

and is called the weighted refinement integral of \( f \) with respect to \( g \) over \([a, b]\).

The first three theorems can be found in [1].

**Theorem 1.1.** Let \( s \) be a saltus function and \( f \) a function bounded on the interval \([a, b]\). If either \( w_2 \) or \( w_3 \) is different from 0, suppose \( f(x^+) \) exists for all points \( x \) in \([a, b]\) such that \( s(x^+) \neq s(x) \); if either \( w_1 \) or \( w_2 \) is different from 0, suppose \( f(x^-) \) exists for all points \( x \) in \([a, b]\) such that \( s(x^-) \neq s(x) \). Then the weighted refinement integral

\[ \left[ F, (w_1, w_2, w_3) \right] \int_a^b f(x) ds(x) \]

exists and equals

\[ \sum_{x \in [a, b]} \left[ (1 - w_1) f(x^+) + w_1 f(x) \right] [s(x^+) - s(x)] \]
\[ \sum_{x \in (a, b]} \left[ w_3 \cdot f(x) + (1 - w_3) \cdot f(x^-) \right] [s(x) - s(x^-)]. \]

**Theorem 1.2.** Let \( g \) be a function of bounded variation on \([a, b]\). Let \( g_s \) be a saltus function and \( g_c \) a continuous function such that \( g(x) = g_s(x) + g_c(x) \) for all \( x \) in \([a, b]\). Let \( f \) be a function bounded on \([a, b]\), and let \( D \) be the set consisting of all points of \([a, b]\) where \( f \) is discontinuous. Let \( g_c^* \) be the non-decreasing function such that

\[
\begin{align*}
g_c^*(x) & = 0, & x & = a \\
& = \nu(g_c, [a, x]), & a & < x \leq b \\
& = g_c^*(b), & x & > b.
\end{align*}
\]

Then, the weighted refinement integral (1.1) exists iff:

i) if either \( w_2 \) or \( w_3 \) is different from 0, \( f(x^+) \) exists for all \( x \) in \([a, b]\) such that \( g(x^+) \neq g(x) \);

ii) if either \( w_1 \) or \( w_2 \) is different from 0, \( f(x^-) \) exists for all \( x \) in \((a, b]\) such that \( g(x^-) \neq g(x) \);

iii) the outer \( g_c^* \)-measure of \( D \) is 0.

**Theorem 1.3.** Let \( g \) be a function of bounded variation on every closed interval. Let \( f \) be a bounded
function on the interval \([a, b]\) such that the weighted refinement integral

\[
[F, (w_1, w_2, w_3)] \int_a^b f(x)dg(x)
\]

exists. Then, the Lebesgue-Stieltjes integral

\[
\int_{[a,b]} f(x)dg(x)
\]

exists, and the weighted refinement integral is equal to

\[
\int_{[a,b]} f(x)dg(x)
+ \sum_{x \in [a,b]} \frac{[f(x^+) - f(x)][g(x^+) - g(x)]}{(1 - w_1)}
- \sum_{x \in (a,b]} \frac{[f(x) - f(x^-)][g(x) - g(x^-)]}{(1 - w_3)}
- f(b)[g(b^+) - g(b)] - f(a)[g(a) - g(a^-)]
\]

The next theorem is in the thesis of J. Baker [2].

**Theorem 1.4.** Let \(g\) be a function of bounded variation on \([a, b]\), and suppose \(f\) and \(h\) are functions
bounded on \([a,b]\). Suppose (1.1) exists. If either \(w_2\) or \(w_3\) is different from 0, suppose \(h(x^+)\) exists for all \(x\) in \([a,b]\) such that \(g(x^+) \neq g(x)\); if either \(w_1\) or \(w_2\) is different from 0, suppose \(h(x^-)\) exists for all \(x\) in \((a,b]\) such that \(g(x^-) \neq g(x)\). Let \(p\) be a function such that

\[
p(x) = \left[F, (w_1, w_2, w_3)\right] \int_a^x f(u)\,dg(u), \quad a < x \leq b
\]

\[
eq 0, \quad x = a.
\]

Then, the weighted refinement integral

\[
\left[F, (w_1, w_2, w_3)\right] \int_a^b h(x)\,dp(x)
\]

exists iff the weighted refinement integral

\[
\left[F, (w_1, w_2, w_3)\right] \int_a^b h(x)f(x)\,dg(x)
\]

exists, and in this case

\[
\left[F, (w_1, w_2, w_3)\right] \int_a^b h(x)dp(x) = \left[F, (w_1, w_2, w_3)\right] \int_a^b h(x)f(x)\,dg(x)
\]
\[- w_1(1-w_1) \sum_{x \in [a,b)} [h(x^+) - h(x)][f(x^+) - f(x)][g(x^+) - g(x)] \]
\[- w_2(1-w_3) \sum_{x \in (a,b]} [h(x) - h(x^-)][f(x) - f(x^-)][g(x) - g(x^-)]. \]

The last theorem is a variation of Theorem 1.4 and can be found in [2].

Theorem 1.5. Let \( f \) and \( h \) be functions of bounded variation on \([a,b]\). Suppose \( g \) is bounded on \([a,b]\), \( g(x^+) \) exists for all \( x \) in \([a,b)\) such that either \( f(x^+) \neq f(x) \) or \( h(x^+) \neq h(x) \), and \( g(x^-) \) exists for all \( x \) in \((a,b]\) such that \( f(x^-) \neq f(x) \) or \( h(x^-) \neq h(x) \). Let \( p \) be defined as in Theorem 1.4. Finally, suppose (1.1) exists. Then, the weighted refinement integral

\[[F, (w_1,w_2,w_3)] \int_a^b h(x)dp(x)\]

exists iff the weighted refinement integral

\[[F, (w_1,w_2,w_3)] \int_a^b h(x)f(x)dg(x)\]

exists, and the formula of Theorem 1.4 holds.
II. A LEBESGUE-STIELTJES INTEGRAL

In this chapter we define a Lebesgue-Stieltjes integral in terms of the weighted refinement integral by using the formula in Theorem 1.3. As we will see from the definition, the integrator function is not even required to be quasi-continuous. Then we show that a substitution formula holds for this Lebesgue-Stieltjes integral.

Definition 2.1. Let $f$ be a function of bounded variation on every closed interval. Let $g$ be a function bounded on every closed interval. Let $a$ be a real number. Suppose $g(x^+)$ exists for every real number $x > a$, suppose $g(x^-)$ exists for every real number $x > a$ such that $f(x^-) \neq f(x)$, and suppose $g(a^-)$ exists. If $b$ is a real number greater than $a$ such that the weighted refinement integral

\[
(2.1) \quad [F, (1,-1,1)] \int_a^b f(x)dg(x)
\]

exists, let

\[
(2.2) \quad (LS) \int_{[a,b]} f(x)dg(x)
\]

denote the real number
\[ [F, (1,-1,1)] \int_a^b f(x)dg(x) + f(a)[g(a) - g(a^-)] \]
\[ + f(b)[g(b^+) - g(b)]. \]

Let
\[ (LS) \int_{[a,a]} f(x)dg(x) = f(a)[g(a^+) - g(a^-)]. \]

Note that in this definition since \( g(x^-) \) is not assumed to exist for a real number \( x > a \) such that \( f(x^-) = f(x) \), \( g \) may not be quasi-continuous on a closed interval \( [a,b] \) over which (2.2) exists. According to Theorem 1.3, (2.2) agrees with the usual Lebesgue-Stieltjes integral when \( g \) is of bounded variation on every closed interval.

**Theorem 2.1.** Let \( f \) and \( h \) be functions of bounded variation on every closed interval. Let \( g \) be a function bounded on every closed interval. Let \( a \) be a real number. Suppose \( g(x^+) \) exists for every real number \( x \geq a \), suppose \( g(x^-) \) exists for every real number \( x > a \) such that either \( f(x^-) \neq f(x) \) or \( h(x^-) \neq h(x) \), and suppose \( g(a^-) \) exists. Finally, suppose (2.2) exists for every real number \( b > a \).

Let
\[ p(x) = (\text{LS}) \int_{[a,x]} f(u)dg(u), \quad x \geq a \]
\[ = p(a), \quad x < a. \]

Then, for any real number \( b > a \), the integral

\[ (\text{LS}) \int_{[a,b]} h(x)f(x)dg(x) \]

exists iff the integral

\[ (\text{LS}) \int_{[a,b]} h(x)dp(x) \]

exists, and in this case

\[ (\text{LS}) \int_{[a,b]} h(x)f(x)dg(x) = (\text{LS}) \int_{[a,b]} h(x)dp(x) + h(a)f(a)[g(a^+) - g(a^-)]. \]

Proof. (a) First, suppose \( b \) is a real number greater than \( a \) such that \((2.4)\) exists.

Let \( q \) be the function such that
\[ q(x) = [F, (1,-1,1)] \int_{a}^{x} f(u)dg(u), \quad x > a \]

\[ = 0, \quad x \leq a. \]

Let

\[ r(x) = p(x) - q(x) \]

(2.8)

for all real numbers \( x \). We note that

\[ r(x) = f(a)[g(a^{+}) - g(a^{-})], \quad x \leq a \]

\[ = f(a)[g(a) - g(a^{-})] + f(x)[g(x^{+}) - g(x)], \quad x > a. \]

By hypothesis, the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x)f(x)dg(x) \]

exists. Then, by Theorem 1.5 the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x)dg(x) \]

(2.10)
exists and equals (2.9). It follows from Theorem 3.5 of [1] that, for every real number $c > a$, the weighted refinement integral

$$[F, (0,1,0)] \int_{a}^{c} g(x)df(x)$$

exists and

$$q(c) = f(c)g(c) - f(a)g(a) - [F, (0,1,0)] \int_{a}^{c} g(x)df(x).$$

Thus, if $c$ is a real number greater than $a$ and if $K$ is a positive real number such that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x$ in $[a, c]$, then we have

$$|q(x)| \leq 2K^2 + K \cdot V(f, [a, c])$$

for all $x$ in $[a, c]$. Thus, $q$ is bounded on every closed interval. Also, $q(x^+)$ exists for every real number $x > a$, and $q(x^-)$ exists for every real number $x > a$ such that $h(x^-) \neq h(x)$. $q(a^-)$ exists and is 0. Thus, by definition, the integral

$$(LS) \int_{[a,b]}^{} h(x)dq(x)$$
exists. Moreover,

\[(LS) \int_{[a,b]} h(x)f(x)dg(x) = (LS) \int_{[a,b]} h(x)dg(x) + h(a)f(a)[g(a) - g(a^-)].\]

The function \(r\) is bounded on every closed interval. \(r(x^+)\) exists and is \(f(a)[g(a) - g(a^-)]\) for every real number \(x > a\), \(r(a^-)\) exists and is \(f(a)[g(a^+) - g(a^-)]\), and \(r(x^-)\) exists and is \(f(a)[g(a) - g(a^-)]\) for every real number \(x > a\) such that \(h(x^-) \neq h(x)\). Let \(h_s\) be a saltus function and \(h_c\) a continuous function such that \(h = h_s + h_c\). By Theorem 1.1 the weighted refinement integral

\[ [F, (0,1,0)] \int_a^b r(x)dh_s(x) \]

exists and equals

\[ f(a)[g(a) - g(a^-)][h_s(b) - h_s(a)]. \]

Since \(r(x^+)\) exists for all points \(x\) of \([a,b]\), the set of discontinuities of \(r\) on \([a,b]\) is countable, and so
the weighted refinement integral

\[ [F, (0,1,0)] \int_{a}^{b} r(x) dh_c(x) \]

exists and equals

\[ f(a)[g(a) - g(a^-)][h_c(b) - h_c(a)]. \]

Thus, the weighted refinement integral

\[ [F, (0,1,0)] \int_{a}^{b} r(x) dh(x) \]

exists and equals

\[ f(a)[g(a) - g(a^-)][h(b) - h(a)]. \]

Now, the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x) dr(x) \]

exists and equals
Therefore, the integral

\[(LS) \int_{[a,b]} h(x)dr(x)\]

exists and equals

\[-h(a)f(a)[g(a^+) - g(a)].\]

Thus, (2.5) exists and (2.6) holds.

(b) Next, suppose \(b\) is a real number greater than \(a\) such that (2.5) exists.

Let \(g\) be the function given by (2.7) and \(r\) the function given by (2.8). By hypothesis, the weighted refinement integral

\[[F, (1, -1, 1)] \int_{a}^{b} h(x)dp(x)\]
exists. As shown previously, the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x)dx(x) \]

exists. Thus, the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x)dg(x) \]

exists. Hence, by Theorem 1.5 the weighted refinement integral

\[ [F, (1,-1,1)] \int_{a}^{b} h(x)f(x)dg(x) \]

exists. Therefore, (2.4) exists.

This completes our proof.
III. THE EXTENDED WEIGHTED INTEGRAL

In this chapter we extend the weighted refinement integral and correspondingly the class of integrable functions in a manner analogous to that of [3]. To say that $f$ is a step function we mean that if $[a,b]$ is a closed interval, there is a partition \(a = x_0 < x_1 < \cdots < x_n = b\) of \([a,b]\) such that, for each integer \(i = 1, 2, \cdots, n\), \(f(x)\) is constant on the open interval \((x_{i-1}, x_i)\).

Definition 3.1. Let \(g\) be a function of bounded variation on every closed interval. Let \(g^*\) be the non-decreasing function such that

\[
g^*(x) = \begin{cases} \neg V(g, [x,0]), & x < 0 \\ \ = 0, & x = 0 \\ \ = V(g, [0,x]), & x > 0. \end{cases}
\]

Let \(f\) be a function bounded on the closed interval \([a,b]\) of the real axis. Let \((w_1, w_2, w_3)\) be an ordered triple of real numbers such that \(w_1 + w_2 + w_3 = 1\). If either \(w_2\) or \(w_3\) is different from 0, suppose \(f(x^+)\) exists for all points \(x\) of \([a,b]\) such that \(g(x^+) \neq g(x)\); if
either $w_1$ or $w_2$ is different from 0, suppose $f(x^-)$ exists for all points $x$ of $(a,b]$ such that $g(x^-) \neq g(x)$.

Suppose there is a sequence $\{f_n\}_{n=1}^{\infty}$ of step functions uniformly bounded on $[a,b]$ such that the following statements hold:

i) $\lim_{n \to +\infty} f_n(x) = f(x)$ for all $x$ in $[a,b] - Z$ where $Z$ is a subset of $[a,b]$ of outer $g^*$-measure 0;

ii) if either $w_2$ or $w_3$ is different from 0, $\lim_{n \to +\infty} f_n(x^+) = f(x^+)$ for all $x$ in $[a,b)$ such that $g(x^+) \neq g(x)$;

iii) if either $w_1$ or $w_2$ is different from 0, $\lim_{n \to +\infty} f_n(x^-)$ for all $x$ in $(a,b]$ such that $g(x^-) \neq g(x)$.

Then, we say that $f$ is $(w_1,w_2,w_3)$ $g$-summable over $[a,b]$.

It follows immediately from the definition that if $f_1$ and $f_2$ are $(w_1,w_2,w_3)$ $g$-summable functions over $[a,b]$ and $\alpha$ is a number, then $\alpha f_1$, $f_1 + f_2$, and $f_1 \cdot f_2$ are $(w_1,w_2,w_3)$ $g$-summable over $[a,b]$. Also note that any quasi-continuous function is $(w_1,w_2,w_3)$ $g$-summable over $[a,b]$.

**Theorem 3.1.** Let $f$ be a function that is $(w_1,w_2,w_3)$ $g$-summable over $[a,b]$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of step functions uniformly bounded on $[a,b]$ such that the
three statements of Definition 3.1 hold. Then the Lebesgue-Stieltjes integral

\[
(\text{LS}) \int_{[a,b]} f(x) \, dg(x)
\]

exists. Moreover,

\[
\lim_{n \to +\infty} [F, (w_1, w_2, w_3)] \int_a^b f_n(x) \, dg(x)
\]

exists and equals

\[
(\text{LS}) \int_{[a,b]} f(x) \, dg(x)
\]

\[
+ \left\{ (1 - w_1) \sum_{ x \in [a,b) } \frac{[f(x^+) - f(x)][g(x^+) - g(x)]}{x^+ - x} \right\}
\]

\[
(3.1)
\]

\[
- (1 - w_3) \sum_{ x \in (a, b] } \frac{[f(x) - f(x^-)][g(x) - g(x^-)]}{x^+ - x}.
\]

- \( f(b)[g(b^+) - g(b)] - f(a)[g(a) - g(a^-)] \}.

Proof. By the Lebesgue Dominated Convergence Theorem,

\[
(\text{LS}) \int_{[a,b]} f(x) \, dg(x)
\]
exists, and

\[ \lim_{n \to +\infty} \int_{[a,b]} f_n(x)dg(x) = \int_{[a,b]} f(x)dg(x). \]

From Theorem 1.3, we see that for each positive integer \( n \)

\[ [F, (w_1, w_2, w_3)] \int_{[a,b]} f_n(x)dg(x) = \int_{[a,b]} f(x)dg(x) \]

\[ + \left( 1 - w_1 \right) \sum_{x \in [a,b]} \left[ f_n(x) - f_n(x^-) \right] [g(x^+) - g(x)] \]

\[ - \left( 1 - w_3 \right) \sum_{x \in (a,b)} \left[ f_n(x) - f_n(x^-) \right] [g(x) - g(x^-)] \]

\[ - f_n(b) [g(b^+) - g(b)] - f_n(a) [g(a) - g(a^-)]. \]

To see that the first sum behaves properly, let \( \epsilon > 0 \) be given. Let \( M \) be a positive real number such that \( |f(x)| < M \) on \([a,b]\) and such that \( |f_n(x)| < M \) on \([a,b]\) for every positive integer \( n \). Let \( \{y_i\}_{i=1}^{+\infty} \) be a one-to-one sequence of points of \([a,b]\) containing all points of this interval where \( g \) is not continuous from the right. Let \( p \) be a positive integer such that
\[
\sum_{i=p+1}^{+\infty} \left| g(y_i^+) - g(y_i) \right| < \frac{\varepsilon}{12M}.
\]

If either \( w_2 \) or \( w_3 \) is different from 0, let \( N \) be a positive integer such that, for each integer \( i = 1, 2, \ldots, p \) for which \( g(y_i^+) \neq g(y_i) \),

\[
\left| f_n(y_i) - f(y_i) \right| < \frac{\varepsilon}{3[g^*(b) - g^*(a)] + 1}
\]

and

\[
\left| f_n(y_i^+) - f(y_i^+) \right| < \frac{\varepsilon}{3[g^*(b) - g^*(a)] + 1}
\]

for every positive integer \( n > N \). If either \( w_2 \) or \( w_3 \) is different from 0, we have for every integer \( n > N \) that

\[
\left| \sum_{x \in [a,b)} \left[ f_n(x^+) - f_n(x) \right] [g(x^+) - g(x)] \right|
\]

\[
- \sum_{x \in [a,b)} \left[ f(x^+) - f(x) \right] [g(x^+) - g(x)]\right|
\]

\[
\leq \sum_{i=1}^{p} \left| f_n(y_i^+) - f(y_i^+) \right| \left| g(y_i^+) - g(y_i) \right|
\]

\[
+ \sum_{i=1}^{p} \left| f_n(y_i) - f(y_i) \right| \left| g(y_i^+) - g(y_i) \right|
\]
\[ + 4M \sum_{i=p+1}^{+\infty} |g(y_i^+) - g(y_i^-)| \frac{\varepsilon}{\beta} + \frac{\varepsilon}{\alpha} + \frac{\varepsilon}{\gamma} = \varepsilon. \]

If either \( w_1 \) or \( w_2 \) is different from 0, the convergence of the sum containing left-hand limits follows similarly. QED

The following definition extends the weighted refinement integral to the linear space of \((w_1, w_2, w_3)\) \(g\)-summable functions.

**Definition 3.2.** Let \( f \) be a function which is \((w_1, w_2, w_3)\) \(g\)-summable over \([a, b]\). Let \( \{f_n\}_{n=1}^{+\infty} \) be a sequence of step functions uniformly bounded on \([a, b]\) such that the three statements of Definition 3.1 hold. Then, we let

\[(3.2) \quad [F, (w_1, w_2, w_3)]_s \int_a^b f(x)dg(x)\]

denote the limit

\[ \lim_{n \to +\infty} [F, (w_1, w_2, w_3)]_s \int_a^b f_n(x)dg(x). \]
It is obvious from Theorem 3.1 that this extended integral is independent of the particular sequence of step functions used and is equal to (3.1).

**Theorem 3.2.** Let $f$ be a function which is $(w_1, w_2, w_3)$ g-summable over $[a, b]$. Then, the weighted refinement integral

$$\int_a^b f(x)dg_s(x)$$

exists and the Lebesgue-Stieltjes integral

$$\int_{[a,b]} f(x)dg_c(x)$$

exists, where $g_s$ and $g_c$ are respectively a saltus function and a continuous function such that $g = g_s + g_c$. Moreover, the extended integral (3.2) is equal to the sum of (3.3) and (3.4).

**Proof.** The existence of the weighted refinement integral (3.3) follows from Theorem 1.1, and the existence of the Lebesgue-Stieltjes integral (3.4) follows from the Lebesgue Dominated Convergence Theorem. From Theorem 1.3 we have that the Lebesgue-Stieltjes integral
exists and that \((3.3)\) equals

\[
\begin{align*}
& (LS) \int_{[a,b]} f(x)dg_s(x) \\
& + \left\{(1-w_1) \sum_{x \in [a,b]} [f(x^+) - f(x)][g(x^+) - g(x)] \right\} \\
& - \left\{(1-w_2) \sum_{x \in (a,b]} [f(x) - f(x^-)][g(x) - g(x^-)] \right\} \\
& - f(b)[g(b^+) - g(b)] - f(a)[g(a) - g(a^-)]
\end{align*}
\]

Adding \((3.4)\) to this we obtain formula \((3.1)\). QED

The next theorem is the converse of Theorem 3.2 and is the main result on the extended integral.

**Theorem 3.3.** Let \( f \) be a function bounded on the closed interval \([a,b]\). Let \( g \) be a function of bounded variation on every closed interval, and let \( g_s \) and \( g_c \) be respectively a saltus function and a continuous function such that \( g = g_s + g_c \). Suppose that the weighted refinement
integral

(3.3) \[ [F, (w_1, w_2, w_3)] \int_a^b f(x) dg_s(x) \]

and the Lebesgue-Stieltjes integral

(3.4) \[ (LS) \int_{[a,b]} f(x) dg_c(x) \]

exist. Then, \( f \) is \( (w_1, w_2, w_3) \)-summable over \( [a,b] \).

Proof. In view of Theorems 1.2 and 1.3, the existence of the weighted refinement integral (3.3) implies that:

i) if either \( w_2 \) or \( w_3 \) is different from 0, \( f(x^+) \)
exists for all \( x \) in \( [a,b] \) such that \( g(x^+) \neq g(x) \);

ii) if either \( w_1 \) or \( w_2 \) is different from 0, \( f(x^-) \)
exists for all \( x \) in \( (a,b] \) such that \( g(x^-) \neq g(x) \);

iii) the Lebesgue-Stieltjes integral

\[ (LS) \int_{[a,b]} f(x) dg_s(x) \]

exists. Thus, the Lebesgue-Stieltjes integral

\[ (LS) \int_{[a,b]} f(x) dg(x) \]
exists. Therefore, there is a sequence \( \{f_n\}_{n=1}^{\infty} \) of step functions uniformly bounded on \([a,b]\) such that

\[
\lim_{n \to +\infty} f_n(x) = f(x) \quad \text{for all } x \in [a,b] - Z, \quad \text{where } Z \text{ is a subset of } [a,b] \text{ of outer } g^*-\text{measure 0}.
\]

Suppose that either \( w_2 \) or \( w_3 \) is different from 0. We will now construct a new sequence of step functions uniformly bounded on \([a,b]\) and having the first two properties of Definition 3.1. First, suppose \( \{x_j\}_{j=1}^{\infty} \) is a one-to-one sequence of points of \([a,b]\) consisting of all points \( x \) of this interval such that \( g(x^+) \neq g(x) \). For each positive integer \( n \), let the points \( x_1, x_2, \ldots, x_n \) be ordered as \( \xi_1,n < \xi_2,n < \cdots < \xi_n,n \) and let \( s_n \) be a positive real number less than

\[
\min\{\xi_2,n - \xi_1,n, \ldots, \xi_n,n - \xi_{n-1},n, b - \xi_n,n\}
\]

such that, if \( 1 \leq j \leq n \), \( |f(x) - f(\xi_j,n^+)\| < \frac{1}{n} \) for all \( x \) in \((\xi_j,n, \xi_j,n + s_n]\). For each positive integer \( n \), let \( \bar{f}_n \) be the step function such that

\[
\bar{f}_n(x) = f(\xi_j,n), \quad x \in (\xi_j,n, \xi_j,n + s_n] \quad \text{and} \quad 1 \leq j \leq n
\]

\[
= f_n(x), \quad x \text{ elsewhere}.
\]
Now, we observe that if \( j \) is a positive integer, then
\[
\overline{f}_n(x_j^+) = f(x_j^+) \text{ for all integers } n \geq j, \text{ so } \lim_{n \to +\infty} \overline{f}_n(x_j^+) = f(x_j^+). \]
Next, we show that if \( x \) is a point of \([a,b) - \mathbb{Z}\),
then \( \lim_{n \to +\infty} \overline{f}_n(x) = f(x) \). Let \( \varepsilon > 0 \) be given. Let \( n' \)
be a positive integer such that \( \frac{1}{n'} < \varepsilon \) and such that
\[
|f_n(x) - f(x)| < \varepsilon \text{ for all integers } n > n'. \]
Let \( n \) be a particular integer such that \( n > n' \). If there is
a positive integer \( j \leq n \) such that \( x \) is in
\([\xi_j, n, \xi_j + \delta_n]\), then
\[
|\overline{f}_n(x) - f(x)| = |f(\xi_j, n) - f(x)| < \frac{1}{n} < \varepsilon.
\]
Otherwise,
\[
|\overline{f}_n(x) - f(x)| = |f_n(x) - f(x)| < \varepsilon.
\]
Thus, \( \lim_{n \to +\infty} \overline{f}_n(x) = f(x) \). Second, suppose \( \{x_j\}_{j=1}^N \) is
a one-to-one finite sequence of points of \([a,b)\) consisting
of all points \( x \) of this interval such that \( g(x^+) \neq g(x) \).
We may proceed in a manner similar to the above to determine
a sequence \( \{\overline{f}_n\}_{n=1}^{+\infty} \) of step functions uniformly bounded
on \([a,b] \) and having the first two properties of Definition
3.1.
Suppose now that either $w_1$ or $w_2$ is different from 0. Also, suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ is such that
\[
\lim_{n \to +\infty} f_n(x^-) = f(x^-) \quad \text{for all } x \in (a,b] \text{ such that } g(x^-) \neq g(x).
\]
This could be assured by proceeding in a manner analogous to that above for right-hand limits.

Suppose that either $w_2$ or $w_3$ is different from 0. Consider the case where there is a one-to-one sequence $\{x_j\}_{j=1}^{\infty}$ consisting of all points $x$ of $[a,b)$ such that $g(x^+) \neq g(x)$. Let the sequence $\{\bar{f}_n\}_{n=1}^{\infty}$ of step functions be as above. We now show that if $x$ is a point of $(a,b]$ such that $g(x^-) \neq g(x)$, then
\[
\lim_{n \to +\infty} \bar{f}_n(x^-) = f(x^-).
\]

Let $\epsilon > 0$ be given. Let $\bar{n}$ be a positive integer such that
\[
\frac{2}{\bar{n}} < \epsilon
\]
and such that $|f_n(x^-) - f(x^-)| < \epsilon$ for all integers $n \geq \bar{n}$. Let $n$ be a particular integer such that $n > \bar{n}$.

If there is a positive integer $j \leq n$ such that $x$ is in $(\zeta_j, n + \delta_n]$, there is a point $\bar{x}$ satisfying
\[
\zeta_j, n < \bar{x} < x \quad \text{such that} \quad |f(\bar{x}) - f(x^-)| < \frac{1}{n},
\]
and so
\[
|\bar{f}_n(x^-) - f(x^-)| \leq |f(\zeta_j, n) - f(\bar{x})| + |f(\bar{x}) - f(x^-)|
\]
\[
< \frac{2}{n} < \epsilon.
\]

Otherwise,
Thus, \( \lim_{n \to \infty} \frac{f_n(x^-)}{f_{n+1}(x^-)} = f(x^-) \). QED

**Corollary 3.1.** Suppose \( g \) is a function of bounded variation on every closed interval, and suppose \( f \) is a function bounded on the closed interval \([a, b]\) of the real axis. If the weighted refinement integral

\[
[F, (w_1, w_2, w_3)] {\int}_a^b f(x)dg(x)
\]

exists, then \( f \) is \((w_1, w_2, w_3)\) \( g \)-summable over \([a, b]\).

**Proof.** The existence of (3.5) implies the existence of the weighted refinement integral

\[
[F, (w_1, w_2, w_3)] {\int}_a^b f(x)dg_s(x)
\]

and by way of Theorem 1.3 the existence of the Lebesgue-Stieltjes integral

\[
(\text{LS}) \int_{[a, b]} f(x)dg_c(x),
\]

where \( g_s \) and \( g_c \) are a saltus function and a continuous
function, respectively, such that \( g = g_s + g_c \). The desired conclusion follows from Theorem 3.3. QED

The converse of Corollary 3.1 is not true as the following example shows.

**Example.** Let \( g(x) = x \) for all real numbers \( x \), and let \( f \) be the function such that

\[
\begin{align*}
f(x) &= 0, \quad x \text{ irrational} \\
&= 1, \quad x \text{ rational}.
\end{align*}
\]

The function \( f \) is discontinuous at every point of \([0,1]\). Hence by Theorem 1.2 the weighted refinement integral

\[
[F, (w_1, w_2, w_3)] \int_0^1 f(x)dg(x)
\]

does not exist. Let \( \{r_i\}_{i=1}^{\infty} \) be an enumeration of the rationals in \([0,1]\). Let \( \{f_n\}_{n=1}^{\infty} \) be the sequence of step functions such that, for each positive integer \( n \),

\[
f_n(x) = 1, \quad x = r_1, r_2, \ldots, r_n \\
= 0, \quad \text{otherwise}.
\]
It is clear that this sequence has all the properties in Definition 3.1. Hence, $f$ is $(w_1, w_2, w_3)$ $g$-summable over $[0,1]$.

Actually, it is easy to construct such examples. We see from Theorem 3.3 that any function $f$ bounded and Lebesgue integrable on $[a,b]$ such that the Lebesgue measure of the set of points of discontinuity of $f$ on $[a,b]$ is greater than 0 will be $(w_1, w_2, w_3)$ $x$-summable over $[a,b]$, but the weighted refinement integral of $f$ with respect to $x$ will not exist.

It is interesting to note at this point that we can use Theorems 3.2 and 3.3 to show that the linear space of functions $(w_1, w_2, w_3)$ $g$-summable over $[a,b]$ is a Banach space.

**Theorem 3.4.** With the supremum norm, the space of $(w_1, w_2, w_3)$ $g$-summable functions over a closed interval $[a,b]$ of the real axis is a Banach space.

**Proof.** Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $(w_1, w_2, w_3)$ $g$-summable functions over $[a,b]$ that is Cauchy in the supremum norm, i.e., given $\varepsilon > 0$ there exists a positive integer $N_\varepsilon$ such that for any integers $n, m > N_\varepsilon$

$$|f_n - f_m| = \sup\{|f_n(x) - f_m(x)| : x \in [a,b]\} < \varepsilon.$$
Thus, for each $x$ in $[a, b]$ the sequence $\{f_n(x)\}_{n=1}^{\infty}$ of real numbers is a Cauchy sequence. Let $f$ be a function such that $f(x) = \lim_{n \to \infty} f_n(x)$ for all $x$ in $[a, b]$. We note that this convergence is uniform since we are using the supremum norm. $f$ is bounded on $[a, b]$.

Suppose that either $w_2$ or $w_3$ is different from 0. Let $x$ be a point of $[a, b)$ such that $g(x^+) \neq g(x)$. We have that $\lim_{h \to 0^+} f_n(x+h)$ exists and is finite for every positive integer $n$, and $\lim_{n \to \infty} f_n(x+h) = f(x+h)$ uniformly in $h$ on $(0, b-x]$. Thus, the iterated limits

$$\lim_{n \to \infty} \lim_{h \to 0^+} f_n(x+h)$$

and

$$\lim_{h \to 0^+} \lim_{n \to \infty} f_n(x+h)$$

exist and are finite. Moreover,

$$\lim_{n \to \infty} \lim_{h \to 0^+} f_n(x+h) = \lim_{h \to 0^+} \lim_{n \to \infty} f_n(x+h)$$

$$= \lim_{h \to 0^+} f(x+h).$$

Thus, $f(x^+)$ exists, $\lim_{n \to \infty} f_n(x^+)$ exists, and these two are equal.
Similarly, we can show that if \( w_1 \) or \( w_2 \) is different from 0 and \( x \) is a point of \((a,b] \) such that \( g(x^-) \neq g(x) \), then \( f(x^-) \) exists. Thus, by Theorem 1.1 the weighted refinement integral

\[
\left[ F, (w_1,w_2,w_3) \right] \int_a^b f(x) \, dg_s(x)
\]

exists.

Since each \( f_n \) is \((w_1,w_2,w_3) \) \( g \)-summable over \([a,b] \), we have from Theorem 3.2 that for each positive integer \( n \) the Lebesgue-Stieltjes integral

\[
(\text{LS}) \int_{[a,b]} f_n(x) \, dg_c(x)
\]

exists. Now, since the sequence \( \{f_n\}_{n=1}^{\infty} \) is supremum norm Cauchy on \([a,b] \), it is uniformly bounded there. By the Lebesgue Dominated Convergence Theorem, the Lebesgue-Stieltjes integral

\[
(\text{LS}) \int_{[a,b]} f(x) \, dg_c(x)
\]

exists, and

\[
\lim_{n \to \infty} (\text{LS}) \int_{[a,b]} f_n(x) \, dg_c(x)
\]
exists and equals

\[
(\text{LS}) \int_{[a,b]} f(x) dg_c(x).
\]

Then, Theorem 3.3 implies that \( f \) is \( (w_1, w_2, w_3) \) \( g \)-summable over \([a, b]\). \quad QED

Theorem 3.3 can also be used to prove a limit theorem for the extended integral that is analogous to the Lebesgue Dominated Convergence Theorem.

**Theorem 3.5.** Let \( g \) be a function of bounded variation on every closed interval, and let \( f \) be a function bounded on the closed interval \([a, b]\). If either \( w_2 \) or \( w_3 \) is different from 0, suppose \( f(x^+) \) exists for all \( x \) in \([a, b]\) such that \( g(x^+) \neq g(x) \); if either \( w_1 \) or \( w_2 \) is different from 0, suppose \( f(x^-) \) exists for all \( x \) in \((a, b]\) such that \( g(x^-) \neq g(x) \). Suppose there is a sequence \( \{f_n\}_{n=1}^{\infty} \) of functions \( (w_1, w_2, w_3) \) \( g \)-summable over \([a, b]\) and uniformly bounded on \([a, b]\) such that:

i) \( \lim_{n \to +\infty} f_n(x) = f(x) \) for all \( x \) in \([a, b] - Z\), where \( Z \) is a subset of \([a, b]\) of outer \( g^* \)-measure 0.

ii) if either \( w_2 \) or \( w_3 \) is different from 0, \( \lim_{n \to +\infty} f_n(x^+) = f(x^+) \) for all \( x \) in \([a, b]\) such that \( g(x^+) \neq g(x) \);
iii) if either \( w_1 \) or \( w_2 \) is different from 0, 
\[
\lim_{n \to +\infty} f_n(x^-) = f(x^-) \text{ for all } x \text{ in } (a, b] \text{ such that } g(x^-) \neq g(x).
\]

Then, \( f \) is \((w_1, w_2, w_3)\)-g-summable over \([a, b]\), and
\[
\lim \int_a^b f_n(x) \, dg(x) = \int_a^b f(x) \, dg(x).
\]

Proof. By Theorem 1.1, the weighted refinement integral
\[
\int_a^b f(x) \, dg_s(x)
\]
exists. From the Lebesgue Dominated Convergence Theorem we see that the Lebesgue-Stieltjes integral
\[
\mathcal{L} \int_{[a, b]} f(x) \, dg_c(x)
\]
exists. Hence, it follows from Theorem 3.3 that \( f \) is \((w_1, w_2, w_3)\)-g-summable over \([a, b]\).

For each positive integer \( n \),
\[ [F, (w_1, w_2, w_3)] s \int_a^b f_n(x)dg(x) = (LS) \int_{[a,b]} f_n(x)dg(x) \]

\[ + \{(1-w_1) \sum_{x \in [a,b]} [f_n(x^+) - f_n(x)][g(x^+) - g(x)] \]

\[ - (1-w_3) \sum_{x \in (a,b]} [f(x) - f_n(x^-)][g(x) - g(x^-)] \]

\[ - f_n(b)[g(b^+) - g(b)] - f_n(a)[g(a) - g(a^-)]. \]

Taking the limit as \( n \to \infty \) we have, as in the proof of Theorem 3.1, that

\[ \lim_{n \to \infty} (LS) \int_{[a,b]} f_n(x)dg(x) = (LS) \int_{[a,b]} f(x)dg(x), \]

that

\[ \lim_{n \to \infty} \sum_{x \in [a,b]} [f_n(x^+) - f_n(x)][g(x^+) - g(x)] \]

\[ = \sum_{x \in [a,b]} [f(x^+) - f(x)][g(x^+) - g(x)] \]

if either \( w_2 \) or \( w_3 \) is different from 0, and likewise for the sum involving left-hand limits. Thus,
Next we give a substitution formula for the extended integral.

**Theorem 3.6.** Let $f$ be a function that is $(w_1, w_2, w_3)$ $g$-summable over $[a, b]$. Let $p$ be the function such that

$$p(x) = 0, \quad x \leq a$$

$$= [F, (w_1, w_2, w_3)] s \int_a^b f(u)dg(u), \quad a < x \leq b$$

$$= p(b), \quad x > b.$$  \hspace{2cm} (3.6)

Let $h$ be a function that is $(w_1, w_2, w_3)$ $g$-summable over $[a, b]$. Then, $p$ is of bounded variation on every closed interval, $h$ is $(w_1, w_2, w_3)$ $p$-summable over $[a, b]$, and $h \cdot f$ is $(w_1, w_2, w_3)$ $g$-summable over $[a, b]$. Moreover,

$$= \lim_{n \to \infty} [F, (w_1, w_2, w_3)] s \int_a^b f_n(x)dg(x)$$

$$= [F, (w_1, w_2, w_3)] s \int_a^b f(x)dg(x). \quad \text{QED}$$
\[
= \left[ F, (w_1, w_2, w_3) \right] s \int_a^b h(x) f(x) g(x) \, dg(x)
\]

\[
-w_1 (1-w_1) \sum_{x \in [a,b)} [h(x^+) - h(x)] [f(x^+) - f(x)] [g(x^+) - g(x)]
\]

\[
-w_2 (1-w_2) \sum_{x \in (a,b]} [h(x) - h(x^-)] [f(x) - f(x^-)] [g(x) - g(x^-)].
\]

Proof. It has already been commented that \( h \cdot f \) is \((w_1, w_2, w_3)\) \( g \)-summable over \([a,b]\). \( h \) is \( g \)-measurable on \([a,b]\). Thus, \( h \) is also \( p \)-measurable on \([a,b]\). For each \( x \) in \([a,b]\),

\[
p(x^+) - p(x) = [w_1 f(x) + (1-w_1) f(x^+)] [g(x^+) - g(x)];
\]

for each \( x \) in \((a,b]\),

\[
p(x) - p(x^-) = [(1-w_2) f(x^-) + w_2 f(x)] [g(x) - g(x^-)].
\]

If either \( w_2 \) or \( w_3 \) is different from 0, then \( h(x^+) \) exists for each \( x \) in \([a,b]\) such that \( p(x^+) \neq p(x) \); if either \( w_1 \) or \( w_2 \) is different from 0, \( h(x^-) \) exists for each \( x \) in \((a,b]\) such that \( p(x^-) \neq p(x) \). It follows that \( h \) is \((w_1, w_2, w_3)\) \( p \)-summable over \([a,b]\).
Let $t$ be the saltus function such that

$$t(x) = 0, \quad x \leq a$$

$$= \left[ F, (w_1, w_2, w_3) \right] \int_a^x f(u)dg_s(x), \quad a < x \leq b$$

$$= t(b), \quad x > b.$$ 

Let $\phi$ be the continuous function such that

$$\phi(x) = 0, \quad x \leq a$$

$$= \left( LS \right) \int_{[a, x]} f(u)dg_s(x), \quad a < x \leq b$$

$$= \phi(b), \quad x > b.$$ 

We observe that

$$p(x) = t(x) + \phi(x)$$

for all real numbers $x$. Now, by Theorem 1.4
\[ [F, (w_1, w_2, w_3)] \int_a^b h(x) \, dt(x) \]

\[ = [F, (w_1, w_2, w_3)] \int_a^b h(x) \, dg(x) \]

\[ -w_1(1-w_1) \sum_{x \in [a,b]} [h(x^+) - h(x)][f(x^+) - f(x)][g(x^+) - g(x)] \]

\[ -w_2(1-w_2) \sum_{x \in (a,b]} [h(x) - h(x^-)][f(x) - f(x^-)][g(x) - g(x^-)]. \]

Also,

\[ (LS) \int_{[a,b]} h(x) \, d\phi(x) = (LS) \int_{[a,b]} h(x)f(x) \, dg(x). \]

In view of Theorem 3.2, the formula (3.7) holds. QED

The next theorem is a rather curious result on the extended integral relating the substitution formula to the case where the integrator is the product of two functions.

Theorem 3.7. Let \( f \) be a function of bounded variation on every closed interval. Let \( h \) be a function that is \((w_1, w_2, w_3)\) \( g \)-summable over \([a,b]\) and \((w_1, w_2, w_3)\) \( f \)-summable over \([a,b]\). Let \( p \) be the function defined as in Theorem 3.6. Then, \( h \) is \((w_1, w_2, w_3)\) \( f \cdot g \)-summable over \([a,b]\), and
\[ [F, (w_1, w_2, w_3)]^s \int_a^b h(x)df(x)g(x) \]

\[ -[F, (w_1, w_2, w_3)]^s \int_a^b h(x)f(x)dg(x) \]

\[ -[F, (w_1, w_2, w_3)]^s \int_a^b h(x)g(x)df(x) \]

\[ = [F, (w_1, w_2, w_3)]^s \int_a^b h(x)dp(x) \]

\[(3.8) \quad -[F, (w_1, w_2, w_3)]^s \int_a^b h(x)f(x)dg(x) \}

\[-(1-w_1)^2 \sum_{x \in [a, b]} h(x^+) [f(x^+)-f(x)] [g(x^+)-g(x)] \]

\[+w_1^2 \sum_{x \in [a, b]} h(x) [f(x^+)-f(x)] [g(x^+)-g(x)] \]

\[+(1-w_3)^2 \sum_{x \in (a, b]} h(x^-) [f(x)-f(x^-)] [g(x)-g(x^-)] \]

\[-w_3^2 \sum_{x \in (a, b]} h(x) [f(x)-f(x^-)] [g(x)-g(x^-)] \].
Proof. Let $k$ be the function such that

$$k(x) = f(x) \cdot g(x).$$

for all real numbers $x$. It is clear that $k$ is of bounded variation on every closed interval. If either $w_2$ or $w_3$ is different from 0, and if $x$ is a point of $[a, b)$ such that $k(x^+) \neq k(x)$, then either $f(x^+) \neq f(x)$ or $g(x^+) \neq g(x)$, and thus $h(x^+)$ exists. Similarly, we see that if $w_1$ or $w_2$ is different from 0, and if $x$ is a point of $(a, b]$ such that $k(x^-) \neq k(x)$, then $h(x^-)$ exists.

Let $f^*$ be the non-decreasing function such that

$$f^*(x) = -\mathcal{V}(f, [x, 0]), \quad x < 0$$

$$= 0, \quad x = 0$$

$$= \mathcal{V}(f, [0, x]), \quad x > 0.$$

Let $k^*$ be the non-decreasing function defined similarly for $k$. Let $r$ be the non-decreasing function such that, for every real number $x$,

$$r(x) = f^*(x) + g^*(x).$$
Both of the Lebesgue-Stieltjes integrals

\[(LS) \int_{[a,b]} h(x)df^*(x)\]

and

\[(LS) \int_{[a,b]} h(x)dg^*(x)\]

exist. Thus, the Lebesgue-Stieltjes integral

\[(LS) \int_{[a,b]} h(x)dr(x)\]

exists. Let \( \{h_n\}_{n=1}^{\infty} \) be a sequence of step functions uniformly bounded on \([a,b]\) such that \( \lim_{n \to \infty} h_n(x) = h(x) \) for all \( x \) in \([a,b] - Z\) where \( Z \) is a subset of \([a,b]\) of outer \( r \)-measure 0. The set \( Z \) is then of outer \( f^* \)-measure 0 and of outer \( g^* \)-measure 0. Also, \( Z \) is of outer \( k^* \)-measure 0. Thus, the Lebesgue Stieltjes integral

\[(LS) \int_{[a,b]} h(x)dk(x)\]

exists. In view of Theorem 3.3, the function \( h \) is \( (w_1, w_2, w_3) \) \( k \)-summable over \([a,b]\).
If either $w_2$ or $w_3$ is different from 0, and if $x$ is a point of $[a,b)$ such that $r(x^+) \neq r(x)$, then either $f(x^+) \neq f(x)$ or $g(x^+) \neq g(x)$, and thus $h(x^+)$ exists. If either $w_1$ or $w_2$ is different from 0, and if $x$ is a point of $(a,b]$ such that $r(x^-) \neq r(x)$, then either $f(x^-) \neq f(x)$ or $g(x^-) \neq g(x)$, and thus $h(x^-)$ exists. Then, let the sequence $\{h_n\}_{n=1}^{+\infty}$ have the following properties:

i) if either $w_2$ or $w_3$ is different from 0,
$$\lim_{n \to +\infty} h_n(x^+) = h(x^+) \text{ for all } x \in [a,b) \text{ such that } r(x^+) \neq r(x);$$

ii) if either $w_1$ or $w_2$ is different from 0,
$$\lim_{n \to +\infty} h_n(x^-) = h(x^-) \text{ for all } x \in (a,b] \text{ such that } r(x^-) \neq r(x).$$

If either $w_2$ or $w_3$ is different from 0, then
$$\lim_{n \to +\infty} h_n(x^+) = h(x^+) \text{ for all } x \in [a,b) \text{ such that } f(x^+) \neq f(x), \text{ and } \lim_{n \to +\infty} h_n(x^+) = h(x^+) \text{ for all } x \in (a,b] \text{ such that } g(x^+) \neq g(x).$$

If either $w_1$ or $w_2$ is different from 0, then
$$\lim_{n \to +\infty} h_n(x^-) = h(x^-) \text{ for all } x \in (a,b] \text{ such that } f(x^-) \neq f(x), \text{ and } \lim_{n \to +\infty} h_n(x^-) = h(x^-) \text{ for all } x \in (a,b] \text{ such that } g(x^-) \neq g(x).$$

Let
\[ P = \{ a = x_0 < x_1 < \cdots < x_m = b \} \]

be a given partition of \([a, b]\), and for each integer \(i = 1, 2, \ldots, m\) let \(\zeta_i\) be a point of the open subinterval \((x_{i-1}, x_i)\) of \([a, b]\). For a positive integer \(n\),

\[
\sum_{i=1}^{m} \left[ w_1 h_n(x_{i-1}) + w_2 h_n(\zeta_i) + w_3 h_n(x_i) \right] \left[ f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) \right] \\
- \sum_{i=1}^{m} \left[ w_1 h_n(x_{i-1}) \frac{f(x_i) - f(x_{i-1})}{h_n(x_{i-1})} + w_2 h_n(\zeta_i) \frac{f(\zeta_i) - f(x_{i-1})}{h_n(\zeta_i)} + w_3 h_n(x_i) \frac{f(x_i) - f(x_{i-1})}{h_n(x_i)} \right] \left[ g(x_i) - g(x_{i-1}) \right] \\
- \sum_{i=1}^{m} \left[ w_1 h_n(x_{i-1}) \frac{g(x_i) - g(x_{i-1})}{h_n(x_{i-1})} + w_2 h_n(\zeta_i) \frac{g(\zeta_i) - g(x_{i-1})}{h_n(\zeta_i)} + w_3 h_n(x_i) \frac{g(x_i) - g(x_{i-1})}{h_n(x_i)} \right] \left[ f(x_i) - f(x_{i-1}) \right] \\
= -w_1 \sum_{i=1}^{m} \left[ h_n(x_i) - h_n(x_{i-1}) \right] \left[\frac{f(x_i) - f(x_{i-1})}{h_n(x_{i-1})} \right] \left[ g(x_i) - g(x_{i-1}) \right] \\
- w_2 \left\{ \sum_{i=1}^{m} \left[ h_n(\zeta_i) - h_n(x_{i-1}) \right] \left[\frac{f(\zeta_i) - f(x_{i-1})}{h_n(\zeta_i)} \right] \left[ g(\zeta_i) - g(x_{i-1}) \right] \right\} \\
+ \sum_{i=1}^{m} \left[ h_n(\zeta_i) - h_n(x_{i-1}) \right] \left[\frac{f(\zeta_i) - f(x_{i-1})}{h_n(\zeta_i)} \right] \left[ g(x_i) - g(x_{i-1}) \right] \\
+ \sum_{i=1}^{m} \left[ h_n(x_i) - h_n(x_{i-1}) \right] \left[\frac{f(x_i) - f(x_{i-1})}{h_n(x_i)} \right] \left[ g(\zeta_i) - g(x_{i-1}) \right] \\
\]
\[ - \sum_{i=1}^{m} h_n(x_i) [f(x_i) - f(x_i')] [g(x_i') - g(x_i')] \]

\[ + (w_1 - w_2) \sum_{i=1}^{m} h_n(x_i) [f(x_i) - f(x_{i-1})] [g(x_i) - g(x_{i-1})]. \]

Thus, for each positive integer \( n \),

\[ [F, (w_1, w_2, w_3)] \int_{a}^{b} h_n(x) df(x) g(x) \]

\[ - [F, (w_1, w_2, w_3)] \int_{a}^{b} h_n(x) f(x) dg(x) \]

\[ - [F, (w_1, w_2, w_3)] \int_{a}^{b} h_n(x) g(x) df(x) \]

\[ = - (w_1 - w_2) \left\{ \sum_{x \in [a, b]} \frac{[h_n(x)^+ - h_n(x^-)][f(x^+)^- - f(x^-)^+][g(x^+)^- - g(x^-)^+]}{x \in [a, b]} \right\} \]

\[ - w_2 \left\{ \sum_{x \in [a, c]} h_n(x) [f(x^+)^- - f(x^-)^+][g(x^+)^- - g(x^-)^+] \right\} \]

\[ - \sum_{x \in (a, b]} h_n(x) [f(x) - f(x^-)][g(x) - g(x^-)]. \]
\[ + (w_1 - w_3) \{ \sum_{x \in [a, b]} h_n(x^+) [f(x^+) - f(x)] [g(x^+) - g(x)] \]
\[ + \sum_{x \in (a, b)} h_n(x) [f(x) - f(x^-)] [g(x) - g(x^-)] \]
\[ = -(1 - w_3) \{ \sum_{x \in [a, b]} [h_n(x^+) - h_n(x^-)] [f(x^+) - f(x^-)] [g(x^+) - g(x^-)] \]
\[ + \sum_{x \in (a, b)} [h_n(x^-) - h_n(x)] [f(x^-) - f(x)] [g(x^-) - g(x)] \]
\[ - (1 - w_1 - w_3) \{ \sum_{x \in [a, b]} h_n(x) [f(x^+) - f(x)] [g(x^+) - g(x)] \]
\[ - \sum_{x \in (a, b)} h_n(x) [f(x) - f(x^-)] [g(x) - g(x^-)] \]
\[ + (w_1 - w_3) \{ \sum_{x \in [a, b]} h_n(x^+) [f(x^+) - f(x)] [g(x^+) - g(x)] \]
\[ + \sum_{x \in (a, b)} h_n(x) [f(x) - f(x^-)] [g(x) - g(x^-)] \]
\[ = - [w_1 (1 - w_1) \sum_{x \in [a, b]} h_n(x^+) - h_n(x)] [f(x^+) - f(x)] [g(x^+) - g(x)] \]
\[ + w_3 (1 - w_3) \sum_{x \in (a, b)} h_n(x^-) [f(x^-) - f(x^-)] [g(x^-) - g(x^-)] \]
Thus, we have for each positive integer \( n \) that

\[
\int_a^b [F, (w_1, w_2, w_3)] h_n(x) df(x) g(x) \]

\[
-[F, (w_1, w_2, w_3)] \int_a^b h_n(x) f(x) dg(x) \]

\[
-[F, (w_1, w_2, w_3)] \int_a^b h_n(x) g(x) df(x) \]

\[
= \{ [F, (w_1, w_2, w_3)] \int_a^b h_n(x) dp(x) \}
\]

\[
-[F, (w_1, w_2, w_3)] \int_a^b h_n(x) f(x) dg(x) \} \]
\[-(1-w_1)^2 \sum_{x \in [a, b)} h_n(x^+)[f(x^+)-f(x)][g(x^+)-g(x)] \]

\[+(w_1^2 \sum_{x \in [a, b)} h_n(x)[f(x^+)-f(x)][g(x^+)-g(x)] \]

\[+(1-w_3)^2 \sum_{x \in (a, b]} h_n(x^-)[f(x^-)-f(x)][g(x^-)-g(x^-)] \]

\[-w_3^2 \sum_{x \in (a, b]} h_n(x)[f(x^-)-f(x)][g(x)-g(x^-)]. \]

If either \( w_2 \) or \( w_3 \) is different from 0, and if \( x \) is a point of \( [a, b) \) such that \( k(x^+) \neq k(x) \), then either \( f(x^+) \neq f(x) \) or \( g(x^+) \neq g(x) \), and thus \( \lim_{n \to -\infty} h_n(x^+) = h(x^+) \). If either \( w_1 \) or \( w_2 \) is different from 0, and if \( x \) is a point of \( (a, b] \) such that \( k(x^-) \neq k(x) \), then either \( f(x^-) \neq f(x) \) or \( g(x^-) \neq g(x) \), and thus \( \lim_{n \to -\infty} h_n(x^-) = h(x^-) \). Hence,

\[ \lim_{n \to -\infty} [F, (w_1, w_2, w_3)] \int_a^b h_n(x)df(x)g(x) \]

\[= [F, (w_1, w_2, w_3)] \int_a^b h(x)df(x)g(x). \]
The function $p$ is of bounded variation on every closed interval. Let $p^*$ be the non-decreasing function such that

$$p^*(x) = -V(p, [x,0]), \quad x < 0$$

$$= 0, \quad x = 0$$

$$= V(p, [0,x]), \quad x > 0.$$ 

The set $Z$ is of outer $p^*$-measure 0. If either $w_2$ or $w_3^*$ is different from 0, and if $x$ is a point of $[a,b)$ such that $p(x^+) \neq p(x)$, then $g(x^+) \neq g(x)$, so $h(x^+)$ exists, and

$$\lim_{n \to +\infty} h_n(x^+) = h(x^+).$$

If either $w_1$ or $w_2$ is different from 0, and if $x$ is a point of $(a,b]$ such that $p(x^-) \neq p(x)$, then $g(x^-) \neq g(x)$, so $h(x^-)$ exists, and

$$\lim_{n \to +\infty} h_n(x^-) = h(x^-).$$

Thus, $h$ is $(w_1, w_2, w_3^*)$ p-summable over $[a,b]$. Moreover,

$$\lim_{n \to +\infty} \int_a^b h_n(x)dp(x) = \int_a^b h(x)dp(x).$$

Let $\ell$ be the function such that $\ell(x) = h(x)f(x)$ for all real $x$. For each positive integer $n$, let $\ell_n$ be
the function such that \( l_n(x) = h_n(x)f(x) \) for all real \( x \).

The function \( l \) is bounded on \([a,b]\). If either \( w_2 \) or \( w_3 \) is different from 0, \( l(x^+) \) exists for all \( x \) in \([a,b]\) such that \( g(x^+) \neq g(x) \). If either \( w_1 \) or \( w_2 \) is different from 0, \( l(x^-) \) exists for all \( x \) in \((a,b]\) such that \( g(x^-) \neq g(x) \). \( \{l_n\}_{n=1}^{\infty} \) is a sequence of functions uniformly bounded on \([a,b]\) such that, for each positive integer \( n \), \( l_n \) is \((w_1,w_2,w_3)\) \( g \)-summable over \([a,b]\).

Moreover, we note that \( \lim_{n \to +\infty} l_n(x) = l(x) \) for all \( x \) in \([a,b] - Z \) and that \( Z \) is a subset of \([a,b]\) of outer \( g^* \)-measure 0. If either \( w_2 \) or \( w_3 \) is different from 0,

\[
\lim_{n \to +\infty} l_n(x^+) = l(x^+) \text{ for all } x \text{ in } [a,b) \text{ such that } g(x^+) \neq g(x).
\]

If either \( w_1 \) or \( w_2 \) is different from 0,

\[
\lim_{n \to +\infty} l_n(x^-) = l(x^-) \text{ for all } x \text{ in } (a,b] \text{ such that } g(x^-) \neq g(x).
\]

It now follows from Theorem 3.5 that \( l \) is \((w_1,w_2,w_3)\) \( g \)-summable over \([a,b]\) and that

\[
\lim_{n \to +\infty} \int_{a}^{b} l_n(x)dg(x) = \int_{a}^{b} l(x)dg(x).
\]

Similarly, we can show that \( h g \) is \((w_1,w_2,w_3)\) \( f \)-summable over \([a,b]\) and that
\[
\lim_{n \to +\infty} [F, (w_1, w_2, w_3)] \int_a^b h_n(x)g(x)df(x)
= [F, (w_1, w_2, w_3)] \int_a^b h(x)g(x)df(x).
\]

The desired formula follows. QED

Now we discuss another way to extend the Lebesgue-Stieltjes integral. Let \( f \) be a function of bounded variation on every closed interval, and let \( g \) be a function. If \( g \) is of bounded variation on every closed interval, then, for any closed interval \([a, b]\), both of the Lebesgue-Stieltjes integrals

\[
(3.9) \quad (\text{LS}) \int_{[a, b]} f(x)dg(x)
\]

and

\[
(3.10) \quad (\text{LS}) \int_{[a, b]} g(x)df(x)
\]

exist, and we have the integration-by-parts formula
\[ (LS) \int_{[a,b]} f(x)dg(x) \]

\[ = f(b^+g(b^+)-f(a^-)g(a^-)-(LS) \int_{[a,b]} g(x)df(x) \]

(3.11)

\[ -\sum_{x \in [a,b]} \frac{[f(x^+)-f(x)][g(x^+)-g(x)]}{x} \]

\[ +\sum_{x \in (a,b]} \frac{[f(x)-f(x^-)][g(x)-g(x^-)]}{x} \]

of Scharf [4]. Let \([a,b]\) be a given closed interval of the real axis. Suppose that \(g\) is bounded on \([a,b]\), that \(g(a^-)\) and \(g(b^+)\) exist and are finite, that \(g(x^+)\) exists for all \(x\) in \([a,b]\) such that \(f(x^+) \neq f(x)\), and that \(g(x^-)\) exists for all \(x\) in \((a,b]\) such that \(f(x^-) \neq f(x)\). If we suppose that the Lebesgue-Stieltjes integral (3.10) exists, we may use (3.11) to define (3.9). However, note that we are then supposing that \(g\) is \((1,1,1)\) \(f\)-summable over \([a,b]\). Substituting for the Lebesgue-Stieltjes integral (3.10) in equation (3.11), we have
\begin{align*}
(\text{LS}) \int_{[a,b]} f(x) dg(x) \\
= g(b^+) f(b) - g(a^-) f(a)
\end{align*}

\begin{equation}
(3.12) \quad -[\Phi, (1,-1,1)] s \int_a^b g(x) df(x)
\end{equation}

\begin{align*}
&- \sum_{x \in [a,b]} [f(x^+)-f(x)][g(x^+)-g(x)] \\
&+ \sum_{x \in (a,b]} [f(x)-f(x^-)][g(x)-g(x^-)].
\end{align*}

Using (3.12) we may define the Lebesgue-Stieltjes integral (3.9) in terms of the extended weighted refinement integral.

Again suppose that $f$ is a function of bounded variation on every closed interval and that $g$ is a function. Suppose that $g$ is bounded on the closed interval $[a,b]$, that $g(a^-)$ and $g(b^+)$ exist and are finite, that $g(x^+)$ exists for all $x$ in $[a,b)$ such that $f(x^+)$ $\neq f(x)$, and that $g(x^-)$ exists for all $x$ in $(a,b]$ such that $f(x^-) \neq f(x)$. Finally, suppose that the weighted refinement integral
exists. Then, according to Theorem 3.5 of [1],

$$[F, (1,-1,1)] \int_a^b f(x) dg(x)$$

exists, and

$$[F, (1,-1,1)] \int_a^b g(x) df(x)$$

Thus, (3.12) becomes

$$f(b)g(b) - f(a)g(a) - [F, (1,-1,1)] \int_a^b g(x) df(x)$$

$$+ \sum_{x \in (a, b]} [f(x^-) - f(x)] [g(x^-) - g(x)]$$

$$- \sum_{x \in [a, b]} [f(x^+) - f(x)] [g(x^+) - g(x)]$$

Thus, (3.12) becomes

$$\int_{[a,b]} f(x) dg(x)$$

$$= [F, (1,-1,1)] \int_a^b f(x) dg(x)$$
\[ +f(b)[g(b^+) - g(b)] + f(a)[g(a) - g(a^-)]. \]

This is the same as the formula of Chapter II.
IV. THE GRONWALL INEQUALITY

The last thing to be considered in this thesis is the Gronwall inequality. It is important in studying uniqueness of solutions of differential equations, and according to Schmaedeke and Sell [5] it is important in studying the existence of solutions in some optimal control problems.

We now give the main theorem of [5].

Theorem 4.1. Let $f$ and $g$ be functions of bounded variation on $[0,T]$ and let $\varepsilon > 0$. Suppose $f$ and $g$ are right continuous, that $f \geq 0$, and $g$ is increasing.

If

$$f(t) \leq \varepsilon + \int_0^T f(s)dg(x), \quad 0 < t < T,$$

then there exist constants $T'$ and $K$, depending on $g$ but not $f$, such that $0 < T' \leq T$, $K > 0$, and

$$f(t) \leq K\varepsilon, \quad 0 \leq t < T'.$$

After proving this theorem, Schmaedeke and Sell then state that if $f$ satisfies an inequality of the form

$$(4.1) \quad f(t) \leq \varepsilon + \int_0^T f(s)k(s)dg(s), \quad 0 \leq t \leq T,$$
this can be reduced to the case of the main theorem by replacing (4.1) with

\[ f(t) \leq \epsilon + [F, \left( \frac{1}{2}, 0, \frac{1}{2} \right)] \int_0^t f(s) dv(s), \]

where

\[ v(t) = [F, \left( \frac{1}{2}, 0, \frac{1}{2} \right)] \int_0^t k(s) dg(s). \]

However, a look at Theorem 1.4 tells us that this is not necessarily true. As we shall show, this last case may be proved in the same manner as the main theorem of [5] merely with the added assumptions that \( k \) is bounded and \( k \geq 0 \).

The next result generalizes Theorem 4.1. Our proof is patterned after the proof of Theorem 4.1 as given in [5]. But first we need a few definitions.

**Definition 4.1.** Let \( T \) be a positive real number. Let \( g \) be a non-decreasing function, and let \( k \) be a function bounded on \( [0,T] \) such that \( k(t) \geq 0 \) for all \( t \) in \( [0,T] \) and such that

\[ [F, (w_1, w_2, w_3)] \int_0^T k(t) dg(t) \]
exists. Let $\epsilon$ be a non-negative real number. Let $F([0,T], g, k, \epsilon)$ be the class of all functions $f$ bounded on $[0,T]$ such that $f(t) \geq 0$ for all $t$ in $[0,T]$, such that

$$[F, (w_1, w_2, w_3)] \int_0^T f(t)dg(t)$$

exists, and such that

$$f(0) \leq \epsilon$$

$$f(t) \leq \epsilon + [F, (w_1, w_2, w_3)] \int_0^t f(s)k(s)dg(s), \quad 0 < t \leq T.$$ 

**Definition 4.2.** Let $T$ be a positive real number. Let $g$ be a non-decreasing function, and let $k$ be a function bounded on $[0,T]$ such that $k(t) \geq 0$ for all $t$ in $[0,T]$, such that $A = \text{l.u.b.} \ k(t)$ is positive, and such that

$$[F, (w_1, w_2, w_3)] \int_0^T k(t)dg(t)$$

exists. If there is a point $t$ of the interval $(0,T)$ such that $|w_3| \cdot A \cdot [g(t) - g(t^-)] \geq 1$, let $T'$ be the smallest
such $t$; otherwise, let $T' = T$. If there is a point $t$ of the interval $(0, T')$ such that

$$[g(t) - g(t^-)] \geq \frac{1}{(|w_1| + |w_2| + |w_3|) \lambda A},$$

let these points be listed as

$$T_1 < T_2 < \cdots < T_n,$$

let $T_0 = 0$, and let $T_{n+1} = T'$; otherwise, let $n = 0$, let $T_0 = 0$, and let $T_{n+1} = T'$. Let $\alpha$ be the non-negative real number less than $1$ such that

$$\frac{\alpha}{(|w_1| + |w_2| + |w_3|) \lambda A} = \max \{[g(t) - g(t^-)]; t \in (T_{j-1}, T_j), j = 1, 2, \cdots, n+1\}.$$

Suppose there is a non-negative real number $\beta$ less than $1 - \alpha$ such that

$$[g(t^+ - g(t))] \leq \frac{\beta}{(|w_1| + |w_2| + |w_3|) \lambda A}.$$
for all points \( t \) of the interval \([0,T']\). Then, we say that \((g,k)\) is an "admissible function pair" for the interval \([0,T]\).

**Theorem 4.2.** Let \( T \) be a positive real number, and let \((g,k)\) be an admissible function pair for the interval \([0,T]\). Let \( A = \text{l.u.b. } k(t) \). If there is a point \( t \) of the interval \((0,T]\) such that \[w_2 \cdot A \cdot |g(t) - g(t^-)| \geq 1,\]

let \( T' \) be the smallest such \( t \); otherwise, let \( T' = T \).

Let \( \epsilon \) be a non-negative real number. Then, there is a non-negative real number \( K \) such that

\[ f(t) \leq K\epsilon, \quad 0 < t < T' \]

for all functions \( f \) in the class \( F([0,T],g,k,\epsilon) \).

**Proof.** (a) Let the partition

\[ \{0 = T_0 < T_1 < T_2 < \cdots < T_n < T_{n+1} = T'\} \]

of the interval \([0,T']\), the non-negative real number \( \alpha \), and the non-negative real number \( \beta \) be as in Definition 4.2.

Let \( \gamma \) be a positive real number less than \( 1 - (\alpha + \beta) \).

For each integer \( j = 1, 2, \cdots, n+1 \), let
\{T_{j-1} = t_0, j < t_1, j < t_2, j < \cdots < t_{m_j+1}, j = T_j \}

be a partition of the closed interval \([T_{j-1}, T_j]\), such that, for each \(i = 1, 2, \ldots, m_j + 1\),

\[|g(t'') - g(t')| \leq \left( \frac{1}{|w_1| + |w_2| + |w_3|} \right) \cdot \Lambda \]

for all \(t', t''\) in the open interval \((t_{i-1}, t_i, j)\).

(b) Let \(\bar{t}\) be a point of \([0, T]\) such that

\[|g(\bar{t}^-) - g(\bar{t})| < \left( \frac{1}{|w_1| + |w_2| + |w_3|} \right) \cdot \Lambda \cdot \]

Suppose there is a non-negative real number \(\bar{K}\) such that

\[f(t) \leq \bar{K} \epsilon, \quad 0 \leq t \leq \bar{t}\]

for all functions \(f\) in the class \(F([0, T], g, k, \epsilon)\). Let \(\hat{\epsilon}\) be a point of \((\bar{t}, T]\) such that

\[|g(\hat{\epsilon}) - g(\bar{t})| < \left( \frac{1}{|w_1| + |w_2| + |w_3|} \right) \cdot \Lambda \cdot \]

Let \(f\) any given function in \(F([0, T], g, k, \epsilon)\). Let

\[M = |g(\bar{t}) - g(0)|, \quad \text{and let} \quad \rho = |g(\hat{\epsilon}) - g(\bar{t})| \cdot \]

Let \(B(f)\) be
a non-negative real number such that \( f(t) \leq B(f) \) for all points \( t \) of \([\hat{t}, \hat{t}]\). For simplicity, let \( |w| = |w_1| + |w_2| + |w_3| \). For any \( t \) in \([\hat{t}, \hat{t}]\),

\[
f(t) \leq \varepsilon + \left[ F, (w_1, w_2, w_3) \right] \int_0^\hat{t} f(s) k(s) dg(s)
+ \left[ F, (w_1, w_2, w_3) \right] \int_\hat{t}^t f(s) k(s) dg(s)
\leq \varepsilon + [K \kappa M + B(f) \alpha \rho] \cdot |w|
= (1 + K \kappa M |w|) \cdot \varepsilon + B(f) \alpha \rho |w|.
\]

Then, for any \( t \) in \([\hat{t}, \hat{t}]\),

\[
f(t) \leq (1 + K \kappa M |w|) \cdot \varepsilon + [(1 + K \kappa M |w|) \cdot \varepsilon
+ B(f) \alpha \rho |w|] \cdot \alpha \rho |w|
= (1 + K \kappa M |w|) \cdot (1 + \alpha \rho |w|) \cdot \varepsilon + B(f) \alpha^2 \rho^2 |w|^2.
\]

Next, for any \( t \) in \([\hat{t}, \hat{t}]\),
\[ f(t) \leq (1 + \overline{KAM}|w|)(1 + A_\rho |w|) \epsilon \]

\[ + [(1 + \overline{KAM}|w|) \epsilon + B(f)A_\rho |w|] A_\rho^2 |w|^2 \]

\[ = (1 + \overline{KAM}|w|)(1 + A_\rho |w| + A_\rho^2 |w|^2) \epsilon + B(f)A_\rho^3 |w|^3. \]

Continuing by mathematical induction, we have for each positive integer \( j \) that

\[ f(t) \leq (1 + \overline{KAM}|w|)(1 + A_\rho |w| + A_\rho^2 |w|^2 + \cdots + A_\rho^j |w|^j) \epsilon \]

\[ + B(f)A_\rho^{j+1} |w|^{j+1} \]

for all \( t \) in \([\overline{t}, \hat{t}]\). Since the non-negative real number \( A_\rho |w| \) is less than 1, it follows that

\[ f(t) \leq (1 + \overline{KAM}|w|) \frac{1}{1 - A_\rho |w|} \epsilon \]

for all \( t \) in \([\overline{t}, \hat{t}]\).

Thus, there is a non-negative real number \( \hat{K} \) such that

\[ f(t) \leq \hat{K} \epsilon, \quad 0 \leq t \leq \hat{t} \]
for all functions $f$ in the class $F([0, T], g, k, \epsilon)$.

(c) Let $\bar{t}$ be a point of $[0, T)$ such that

$$\frac{g(\bar{t}^+)}{A} - g(\bar{t}) < \frac{1}{|w_1| + |w_2| + |w_3|}.$$

Suppose there is a non-negative real number $\bar{K}$ such that

$$f(t) \leq \bar{K}\epsilon, \quad 0 \leq t \leq \bar{t}$$

for all functions $f$ in the class $F([0, T], g, k, \epsilon)$. Let $t^*$ be a point of $(\bar{t}, T]$ such that

$$\frac{g(t^*) - g(\bar{t})}{A} < \frac{1}{|w_1| + |w_2| + |w_3|}.$$

From the proof in (b), it follows that there is a non-negative real number $K^*$ such that

$$f(t) \leq K^*\epsilon, \quad 0 \leq t < t^*$$

for all functions $f$ in the class $F([0, T], g, k, \epsilon)$.

(d) Let $\bar{\epsilon}$ be a point of $(0, T]$ such that

$$|w_3| \cdot A \cdot [g(\bar{\epsilon}) - g(\bar{\epsilon}^-)] < 1.$$
Suppose there is a non-negative real number $\tilde{K}$ such that

$$f(t) \leq \tilde{K}\varepsilon, \quad 0 \leq t < \varepsilon$$

for all functions $f$ in the class $F([0,T],g,k,\varepsilon)$.

Let $f$ be any given function in $F([0,T],g,k,\varepsilon)$. For any $t$ in $(0,\varepsilon)$,

$$f(\varepsilon) \leq \varepsilon + \left[ F, (w_1,w_2,w_3) \right]\int_0^t f(s)k(s)dg(s)$$

$$+ \left[ F, (w_1,w_2,w_3) \right]\int_t^\varepsilon f(s)k(s)dg(s)$$

$$\leq \{1+\tilde{K}A\omega\}w\{g(\varepsilon) - g(0)\} \cdot \varepsilon$$

$$+ \left[ F, (w_1,w_2,w_3) \right]\int_0^\varepsilon f(s)k(s)dg(s)$$

$$- \left[ F, (w_1,w_2,w_3) \right]\int_0^t f(s)k(s)dg(s) \}.$$ 

Now let this $t \to \varepsilon$. We obtain the inequality

$$f(\varepsilon) \leq \{1+\tilde{K}A\omega\}w\{g(\varepsilon) - g(0)\} \cdot \varepsilon$$
\[ + [w_2 f(\xi) g(\xi) + (1 - w_2) f(\xi') g(\xi') \cdot [g(\xi) - g(\xi')] \]

\[ \leq \{1 + \tilde{k}A |w| [g(\xi) - g(0)] \} \cdot \varepsilon \]

\[ + |1 - w_2| \tilde{k}A[g(\xi) - g(\xi')] \cdot \varepsilon \]

\[ + |w_2| A f(\xi') [g(\xi) - g(\xi')] \]

Therefore,

\[ f(\xi) \leq \frac{1 + \tilde{k}A |w| [g(\xi) - g(0)] + |1 - w_2| \tilde{k}A[g(\xi) - g(\xi')]}{1 - |w_2| A[g(\xi) - g(\xi')] \cdot \varepsilon} \]

Thus, there is a non-negative real number \( K' \) such that

\[ f(t) \leq K' \varepsilon, \quad 0 \leq t \leq \xi \]

for all functions \( f \) in the class \( F([0,T], g, k, \varepsilon) \).

(e) Let \( j \) be a particular positive integer not exceeding \( n + 1 \). For each integer \( i = 1, 2, \ldots, m_j \),
\[ [g(t_{i,j}) - g(t_{i-1,j})] \]

\[ = [g(t_{i,j}) - g(t_{i,j}^-)] + [g(t_{i,j}^-) - g(t_{i-1,j}^+)] \]

\[ + [g(t_{i-1,j}^+) - g(t_{i-1,j})] \]

\[ \leq \frac{\alpha + \beta + \gamma}{(|w_1| + |w_2| + |w_3|) \cdot A} \cdot \]

Moreover,

\[ [g(t_{m_j+1,j}^-) - g(t_{m_j,j})] \]

\[ = [g(t_{m_j+1,j}^-) - g(t_{m_j,j}^+)] + [g(t_{m_j,j}^+) - g(t_{m_j,j})] \]

\[ \leq \frac{\beta + \gamma}{(|w_1| + |w_2| + |w_3|) \cdot A} \cdot \]

Consider the case where the integer \( n \) is positive. We observe that \( f(0) \leq \epsilon \) for all \( f \) in the class \( F([0,T], g, k, \epsilon) \). We may then apply (b) successively to the intervals \( [t_{0,1}, t_{1,1}], [t_{1,1}, t_{2,1}], \ldots, [t_{m_1-1,1}, t_{m_1,1}] \) to obtain a non-negative real number \( K_1 \) such that
for all functions \( f \) in the class \( F([0,T],g,k,\varepsilon) \). Then, we use (c) to obtain a non-negative real number \( K_1' \) such that

\[
f(t) \leq K_1' \varepsilon, \quad 0 \leq t \leq T_1
\]

for all functions \( f \) in the class \( F([0,T],g,k,\varepsilon) \). Finally, we use (d) to obtain a non-negative real number \( K_1 \) such that

\[
f(t) \leq K_1 \varepsilon, \quad 0 \leq t \leq T_1
\]

for all functions \( f \) in the class \( F([0,T],g,k,\varepsilon) \).

We may repeat the process of the preceding paragraph to obtain a non-negative real number \( K \) such that

\[
f(t) \leq K \varepsilon, \quad 0 \leq t < T'
\]

for all functions \( f \) in the class \( F([0,T],g,k,\varepsilon) \).

Theorem 4.2 remains valid if we replace the weighted refinement integral by our extended weighted refinement integral.
V. BIBLIOGRAPHY


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