Optimal synergetic maneuvers through dynamic programming

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INTRODUCTION

The importance of optimization in the study and planning of orbital or interplanetary missions has long been recognized by those individuals and organizations concerned with such missions.

Optimization in this sense refers to maximizing a "return" in terms of such things as reliability, useful payload, scientific measurements and data, or versatility of the vehicle, while incurring a minimum "expense" which may refer to such things as actual economic costs, time, fuel, or simplicity of design, often admitting a compromise with physical constraints.

As early as 1925, Hohmann (1) investigated the optimization of the transfer between two coplanar circular orbits. The resultant two-impulse Hohmann transfer requires a lower total velocity increment, hence, less fuel, than any other two-impulse transfer.

A few of the subsequent analyses of optimal orbit transfers are summarized briefly below.

Barrar (2) and Horner (3) proved analytically that the two-impulse Hohmann transfer is always superior to the optimum single-impulse transfer between intersecting, coplanar, common axis elliptical orbits or between a circular orbit and an intersecting, coplanar elliptical orbit. Altman and Pistiner (4) optimized the two-impulse transfer for arbitrary coplanar
orbits.

Lawden (5), using an elementary mathematical approach, studied optimal transfers between coplanar elliptical orbits with the number of impulses specified.

McCue (6) numerically optimized two-impulse transfers between inclined elliptical orbits. Analytical results for this case were later developed by Lee (7).

During this same period, older optimization techniques have been refined, new techniques have been developed, and the range of applications has increased greatly. The two best known and most widely used optimization theories for trajectory analysis are the calculus of variations, probably best defined by Forsyth (8), and the method of gradients, used extensively by Kelley (9) and Bryson and Denham (10), (11).

A more recent optimization technique, developed by Bellman (12), is dynamic programming. Dynamic programming and its relation to the classical calculus of variations are described in detail by Dreyfus (13). To this date, dynamic programming has been applied sparingly to trajectory analysis because of the excessive computer storage requirements.

Recently, however, a successive sweep method with dynamic programming has been described by McReynolds (14) and improved by Jacobson (15) under the heading of differential dynamic programming.

The differential dynamic programming approach promises to eliminate the major difficulty associated with dynamic pro-
gramming, that of excessive memory requirements.

The problem considered in this study is that of minimizing the velocity loss incurred by a winged reentry vehicle during the initial phase of atmospheric entry. Several cases are investigated for constraints on terminal time and final altitude or final flight path angle.

Possible applications of this type of maneuver could include such missions as:

1. Reconnaissance. A lifting vehicle could descend to a desired altitude at a desired time, for photographic purposes, while minimizing velocity loss. This would minimize the fuel required for reorbiting.

2. Intercept and/or rendezvous. If two vehicles are separated but moving in the same orbit plane, the lead vehicle might descend into the atmosphere while the trailing vehicle continued in its original orbit. The result would be a decrease in the angular separation of the two spacecraft. The lead vehicle might then reorbit for an intercept. If the velocity loss in the atmospheric descent had been minimized, the overall velocity increment required for the total maneuver should be lowered. For large initial angular separations and near-Earth orbits, this type of maneuver might require less time, for the same amount of fuel, than an exo-atmospheric intercept maneuver.

3. Initial phase of an orbital plane change. Shaver (16) has shown that for large orbital plane changes, such a
synergetic maneuver may require less fuel than an exo-atmospheric propulsive plane change.

4. Delivery of intercontinental weapons.

To simplify the analysis of this study, the following assumptions are made:

1. The Earth is flat and has a constant gravitational attraction in the region under consideration. For altitude changes encountered in this study, the actual gravitational attraction varies by less than one percent of the assumed value of 0.006 mile/(second)^2. The maximum range error due to the flat Earth assumption is less than two and one-half percent because of the relatively short time involved.

2. The upper limit of the sensible atmosphere is at 50 statute miles. This limit is arbitrary and represents a compromise between other suggested limits. For example, Shaver (16) uses approximately 40 miles and Bleick and Faulkner (17) use approximately 57 miles.

3. The atmospheric density model has an exponential decay rate. This assumption is a common first approximation for atmospheric entry problems. See Shi, Pottsepp and Eckstein (18).

4. The vehicle is represented by a point mass.

Initial conditions at atmospheric penetration are:

1. \( h_0 = 50 \) statute miles
2. \( \gamma_0 = -1.0 \) degree
3. \( V_0 = 5 \) mps (26400 fps).
These conditions represent an "average" entry path for a deorbiting vehicle. Final time was fixed at 180 seconds for most cases.

Forces acting on the vehicle during the maneuver are assumed to be lift, drag, and a constant gravitational force. The lift and drag relationship, which is a simplification of that suggested by Bleick and Faulkner (17), is

\[ C_L = 1.7\alpha |\alpha|, \]
\[ C_D = 0.042 + 1.46|\alpha^3| \]

which is the familiar hypersonic 3/2-power relationship and is reasonably valid to an angle of attack of approximately 20 degrees.

Comparisons of results with those obtained using a simple parabolic drag polar are also presented to emphasize a major contribution of this study which is to demonstrate the feasibility of the dynamic programming approach to optimizing dissipative trajectory problems.

In addition, an aerodynamic heating analysis is presented in Appendix B to illustrate maximum heating rates and total heat loads encountered during the optimal maneuvers.
LIST OF SYMBOLS

a  Predicted improvement of return function
A  Vehicle reference area
Cl  Lift coefficient
Cd  Drag coefficient
D  Aerodynamic drag
E  Total energy of the vehicle
g  Acceleration due to gravity
h  Altitude
H  Hamiltonian
J  General return function
L  Aerodynamic lift
m  Vehicle mass
Ph  Percentage of vehicle energy loss which is absorbed as heat
Q  Heat absorbed by the vehicle
S  Optimal return function
t  Time
V  velocity of the vehicle
W  Vehicle weight
x  Horizontal range
y  State vector for unconstrained problem
z  State vector for constrained problem
α  Vehicle angle of attack
p  Atmospheric density decay factor
\( \gamma \) Vehicle flight path angle measured with respect to the local horizon

\( \lambda \) Lagrange multiplier

\( \varphi \) Atmospheric density

\( \phi \) Constraint function

Subscripts

\( f \) Final condition

\( i,j,k,m \) Component of state vector

\( 0 \) Initial condition

\( l \) General time

Special Notation

\( \dot{y} \) The dot denotes differentiation with respect to time

\( S_z \) The subscript denotes partial differentiation with respect to \( z \)

\( \bar{y} \) The bar denotes a reference value

\( \alpha^* \) The asterisk denotes a maximizing value
Problem definition

The problem of optimizing a synergetic maneuver, subject to the assumptions listed in the introduction, may be formally stated as follows:

In the class of functions $x(t)$, $h(t)$, $V(t)$, $\gamma(t)$ and $\alpha(t)$, determine that particular set which maximizes the return function

$$J(y_f, t_f) = V(t_f)$$

subject to the differential equations of motion

$$\dot{x} = V \cos \gamma$$

$$\dot{h} = V \sin \gamma$$

$$\dot{V} = -\frac{D}{m} - g \sin \gamma$$

$$\dot{\gamma} = \frac{L}{mV} - \frac{g \cos \gamma}{V}$$

where

$x$ is the horizontal distance in miles

$h$ is the altitude in miles

$V$ is the velocity in miles per second

$\gamma$ is the flight path angle in radians

$y$ is the state variable vector $(x, h, V, \gamma)$

$L$ is the lift in pounds

$D$ is the drag in pounds
$g$ is the acceleration due to gravity in mile/(second)$^2$

$m$ is the mass of the vehicle in slugs

$\alpha$ is the vehicle angle of attack in radians

The dot above a variable designates the derivative with respect to time $t$.

The maximization of $J(y_f, t_f)$ is also subject to a set of initial conditions

\begin{align*}
t_0 &= 0 \quad (6) \\
y(0) &= y_0 \quad (7)
\end{align*}

and terminal constraints of the form

\[ \phi(y_f) = 0 \quad (8) \]

where the subscript 0 refers to initial values and the subscript f refers to final values.

The coordinate system is shown in Figure 1 and a detailed derivation of the equations of motion is given in Appendix A.

The terminal constraints, Equation 8, have the following forms in this study:

\begin{align*}
\gamma(t_f) &= 0 \quad (9) \\
h(t_f) - h_{\text{specified}} &= 0 \quad (10)
\end{align*}

Also, the vehicle angle of attack $\alpha$ is considered to be the control variable and it appears in both the lift $L$ and drag $D$ terms of Equations 4 and 5. Lift and drag are defined by

\begin{align*}
L &= \frac{\rho v^2}{2} A C_L(\alpha) \quad (11) \\
D &= \frac{\rho v^2}{2} A C_D(\alpha) \quad (12)
\end{align*}
Figure 1. Coordinate system used for synergetic maneuvers.
where

\[ \rho \text{ is the atmospheric density in slugs/(mile)}^3 \]

\[ A \text{ is the reference area of the vehicle in (mile)}^2 \]

Bleich and Faulkner (17) use a current reentry glider drag polar given by

\[ C_L = 1.82 \sin \alpha |\sin \alpha| \cos \alpha \tag{13} \]

\[ C_D = 0.042 + 1.46 |\sin^3 \alpha| \tag{14} \]

In the present investigation, Equations 13 and 14 are approximated by

\[ C_L = 1.7 \alpha |\alpha| \tag{15} \]

\[ C_D = 0.042 + 1.46 |\alpha^3| \tag{16} \]

where \( \alpha \) is in radians. Graphical comparisons of Equations 13-16 are given in Figures 2 and 3.

An arbitrary parabolic drag polar

\[ C_D = 0.04 + 1.5 C_L^2 \tag{17} \]

where

\[ C_L = 0.2 \sin 6 \alpha \tag{18} \]

is also used in this study for several general comparisons although it is recognized that a parabolic polar is not valid for known hypersonic vehicles.

**Method of solution**

The use of dynamic programming for trajectory analysis has not been popular because of its similarity to a "brute force" approach. Even with modern computers, the memory
Figure 2. Lift coefficient versus angle of attack.
Figure 3. Lift coefficient versus drag coefficient
requirements quickly exceed memory availability when a systematic comparison of all possible paths between two points is undertaken. The application of constraints may significantly reduce storage requirements but dynamic programming is still inefficient when applied to a multi-dimensional problem.

However, a differential dynamic programming approach outlined by Jacobson (15) eliminates the need for an extensive comparison of trajectories while still retaining the simplicity of the basic dynamic programming formulation. A second-order algorithm, suggested by Jacobson (15), with differential dynamic programming is used in this study. The flow chart in Figure 4 illustrates the computational scheme and the logic of the algorithm.

**Unconstrained problem** Equations 1-7 define what is commonly known as a problem of the Mayer type. Since a maximum of the performance index

$$J(y_f, t_f)$$

is desired, the optimal return function is defined as

$$S(y(t), t) = \max_{\alpha(t)} [J(y_f, t_f)], \quad t_0 \leq t \leq t_f$$

(20)

where \(\alpha(t)\) is the time history of the vehicle angle of attack. Note that once the optimal control function \(\alpha_{opt}(t)\) is defined, \(S(y, t)\) has the same value at any point on the optimal trajectory.

In accordance with the theory, two necessary conditions for an optimum must be satisfied. The first of these necessary
1. Specify: Initial conditions $y(0)$
   Nominal multipliers $\lambda$
   Nominal control $a(t)$

2. Integrate equations of motion from $t_0$ to $t_f$ to define the reference trajectory $\bar{y}(t)$

3. Compute nominal return $J^*(t_f)$
   Apply boundary conditions to optimal return function:
   $S_z(t_f)$, $S_{zz}(t_f)$, $a(t_f)$

4. Solve $H_a(\bar{y}, \lambda, a^*, S_z, t) = 0$ for $a^*(t)$

Q1. Is $t \leq 0$?

Q2. Is $\frac{\Delta S}{a(\bar{y}, t_f)} < C$? (See page 23.)

5. Compute $S_z(t)$, $\dot{S}_{zz}(t)$, $\dot{a}(t)$

6. Compute $S_z(t - \Delta t)$, $S_{zz}(t - \Delta t)$, $a(t - \Delta t)$

7. Set $t = t - \Delta t$

Q3. Is $C \leq 0$?

8. Set $C = C - \Delta C$

9. Compute $\delta \lambda = -S_{\lambda\lambda}^{-1}\delta \lambda$

10. $H_a(\bar{y} + \delta y, \lambda + \delta \lambda, a^* + \delta a, S_z + S_{zz}\delta z, t) = 0$ for $a(t)$

11. Compute $\dot{y}(\bar{y} + \delta y, a, t)$

12. Compute $y(t + \Delta t)$, $\delta y = y(t + \Delta t) - \bar{y}(t + \Delta t)$

Q4. Is $t + \Delta t \geq t_f$?

13. Set $t = t + \Delta t$

14. Replace $\bar{y} = \bar{y} + \delta y$
conditions is
\[ \dot{S} = S_t + \max_{\alpha} S_j \dot{y}_j = 0, \quad j = 1, \ldots, 4 \] (21)
where the repeated index indicates summation. This may be recognized as the Hamilton-Jacobi equation. Also, the last term of Equation 21
\[ \max_{\alpha} S_j \dot{y}_j = H \] (22)
is the first-order Hamiltonian function for the system. The second necessary condition is then
\[ H_{\alpha_{\text{opt}}} = S_j (\dot{y}_j)_{\alpha_{\text{opt}}} = 0 \] (23)
The boundary condition corresponding to Equations 22 and 23 is
\[ S(y_f, t_f) = J(y_f, t_f) \] (24)
Again, according to the theory, if a "nominal" control law \( \bar{a}(t) \) is specified, with known initial conditions and the equations of motion, a nominal trajectory \( \bar{y}(t) \) and a nominal return
\[ J(\bar{y}_f, t_f; \bar{a}) \] (25)
which is probably non-optimal, are generated. The next step is to determine an improved control expression
\[ a(t) = \bar{a}(t) + \delta a(t) \] (26)
There are several methods available for determining this improved control. Some of these methods are described by McReynolds (14). Unfortunately, most of the suggested methods require a nominal trajectory which is reasonably close to the
optimum. If the nominal trajectory is not close enough to the optimum, severe instability in the numerical process may and usually does occur. The allowable deviation of the nominal trajectory from the optimum, for numerical stability, depends on the non-linearity of the system. This allowable deviation is sometimes referred to as the radius of convergence of the algorithm.

A second-order algorithm selected for this study is one which enlarges the radius of convergence. It is described in detail below and is outlined in the flow chart of Figure 4.

The nominal, or reference, trajectory is defined by the nominal control \( \bar{a} \), the nominal state \( \bar{y} \), and the nominal return \( \bar{J} \). The optimal trajectory may then be defined by

\[
\alpha = \bar{a} + \delta \alpha \\
y = \bar{y} + \delta y \\
S(y,t) = \bar{J}(\bar{y}_f + \delta y_f, t_f)
\]

and

\[
- \frac{\partial S}{\partial t} (\bar{y} + \delta y , t) = \text{Max}_{\delta \alpha} \left[ S_y (\bar{y} + \delta y , t) \dot{y}_j (\bar{y} + \delta y , \bar{a} + \delta a , t) \right]
\]

In this study, it is assumed that \( S \) is smooth enough to allow a power series expansion about \( \bar{y} \) so that, with higher order terms neglected,

\[
S(\bar{y} + \delta y , t) = S(\bar{y} , t) + S_y \delta y_j + \frac{S_{yy} \delta y_j \delta y_k}{2}
\]
and

$$S_{y_j}(\bar{y} + \delta y, t) = S_{y_j}(\bar{y}, t) + S_{y_j\gamma_k}(\bar{y}, t)\delta y_k$$  \hspace{1cm} (32)$$

where

$$S(\bar{y}, t) = H + a(\bar{y}, t)$$  \hspace{1cm} (33)$$

Substitution of Equations 31-33 into Equation 30 yields

$$- \frac{\partial J}{\partial t} - \frac{\partial a}{\partial t} - \frac{\partial S_{y_j}}{\partial t} \delta y_j - \frac{1}{2} \frac{\partial S_{y_j\gamma_k}}{\partial t} \delta y_j \delta y_k$$

$$= \max_{\delta a} [(S_{y_j} + S_{y_j\gamma_k}\delta y_k) \dot{y}_j(\bar{y} + \delta y, \bar{a} + \delta a, t)]$$ \hspace{1cm} (34)$$

The higher order terms in Equations 31-34 may be neglected if $\delta y$ is kept sufficiently small by means of a technique to be explained later.

Next, consider Equation 34 for $\delta y = 0$

$$- \frac{\partial J}{\partial t} - \frac{a}{t} = \max_{\delta a^*} [S_{y_j}(\bar{y}, \bar{a} + \delta a^*, t)]$$

$$= \max_{\delta a^*} H(\bar{y}, \bar{a} + \delta a^*, S_y, t)$$ \hspace{1cm} (35)$$

where $a^*$ is the maximizing control for $\delta y = 0$.

The maximizing $\delta a^*$ may then be obtained by setting the first derivative of the Hamiltonian function

$$H_{a}(\bar{y}, a^*, S_y, t) = 0$$ \hspace{1cm} (36)$$

and requiring the second derivative

$$H_{aa}(\bar{y}, a^*, S_y, t) \leq 0$$ \hspace{1cm} (37)$$

Equation 37 is equivalent to the Legendre-Clebsch condition of classical optimization theory.
The necessary condition for maintaining optimality is then

$$H_\alpha(y + \delta y, \alpha^* + \delta \alpha, S_y + S_{y*} \delta y_j, t) = 0$$  \hspace{1cm} (38)

where $\delta \alpha$ is now measured from $\alpha^*$. Hence, Equation 34 becomes

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial y} \frac{\partial y_j}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial y \partial y} \delta y_j \delta y_k$$

$$= \max_{\delta \alpha} \left[ (S_y + S_{y*} \delta y_j) \dot{y}_j (\ddot{y} + \delta y, \alpha^* + \delta \alpha, t) \right]$$

$$= \max_{\delta \alpha} H(\ddot{y} + \delta y, \alpha^* + \delta \alpha, S_y + S_{y*} \delta y_j, t)$$  \hspace{1cm} (39)

Expansion of the right-hand side of Equation 39 about $\bar{y}$, $\alpha^*$ yields, with higher order terms neglected.

$$\max_{\delta \alpha} \left[ H + H_\alpha \delta \alpha + H_y \delta y_j + S_{y*} \dot{y}_j \delta y_k + \frac{H_{aa}}{2}(\delta \alpha)^2 \right.$$  
$$+ \left( H_{ay_j} + S_{y*} \dot{y}_j \delta y_k \right) \delta y_j \delta \alpha + \frac{1}{2} \left( H_{y_j y_k} \right.$$  
$$+ S_{y*} \dot{y}_j \dot{y}_k ) \delta y_j \delta y_k \left\} \right.$$  \hspace{1cm} (40)

where

$$H_\alpha = S_{y*} \dot{y}_j \delta \alpha$$  \hspace{1cm} (41)

$$H_y = S_{y*} \dot{y}_j \delta \alpha$$  \hspace{1cm} (42)

$$H_{ay_j} = S_{y*} \dot{y}_j \delta \alpha$$  \hspace{1cm} (43)

$$H_{y_j y_k} = S_{y*} \dot{y}_j \dot{y}_k$$  \hspace{1cm} (44)

$$H_{aa} = S_{y*} \dot{y}_j \delta \alpha$$  \hspace{1cm} (45)

All quantities in Equations 40-45 are evaluated at $(\bar{y}, \alpha^*, t)$.

Note that if
the terms involving $\delta a$ in Equation 40 are

\[
\left( H_{ay_j} + s_{y_jy_k}(\dot{y}_k) \right) \delta y_j \delta a + \frac{H_{aa}}{2} (\delta a)^2
\]  

(46)

Maximizing Equation 40, with respect to $\delta a$, yields

\[
H_{aa} \delta a + \left( H_{ay_j} + s_{y_jy_k}(\dot{y}_k) \right) \delta y_j = 0
\]  

(47)

Therefore, the correction to $a^*$ is given by

\[
\delta a = \beta_j \delta y_j
\]  

(48)

where

\[
\beta_j = -\frac{H_{ay_j} + s_{y_jy_k}(\dot{y}_k)\alpha}{H_{aa}}
\]  

(49)

Substitution of Equations 40 and 48 into Equation 39 and equating coefficients now gives

\[
-\frac{\partial J}{\partial t} - \frac{\partial a}{\partial t} = H
\]  

(50)

\[
-\frac{\partial s_{y_j}}{\partial t} = H_{y_j} + s_{y_jy_k}\dot{y}_k
\]  

(51)

\[
-\frac{\partial s_{y_jy_k}}{\partial t} = H_{y_jy_k} + s_{y_jy_m}(\dot{y}_m)_{y_k} + s_{y_ky_m}(\dot{y}_m)_{y_j} - \beta_j \delta_{jk} H_{aa}
\]  

(52)

where all terms are evaluated at $(\bar{y}, a^*, t)$.

Along the nominal trajectory,
\[
\frac{dS(\bar{y}, t)}{dt} = \frac{d(\bar{J} + a)}{dt} = \frac{\partial(\bar{J} + a)}{\partial t} + S_{y_j} \dot{y}_j (\bar{y}, \bar{a}, t) \tag{53}
\]

\[
\frac{dS_{y_j}}{dt} = \dot{S}_{y_j} = \frac{\partial S_{y_j}}{\partial t} + S_{y_j} \dot{y}_k (\bar{y}, \bar{a}, t) \tag{54}
\]

\[
\frac{dS_{y_j y_k}}{dt} = \ddot{S}_{y_j y_k} = \frac{\partial S_{y_j y_k}}{\partial t} \tag{55}
\]

where higher order terms have been neglected.

When Equations 53-55 are substituted into Equations 50-52 and it is noted that

\[
\frac{d\bar{y}}{dt} = 0 \tag{56}
\]

the following characteristic equations are obtained

\[
- \dot{\alpha} = \bar{H} - H(\bar{y}, \bar{a}, S_y, t) \tag{57}
\]

\[
- \dot{S}_{y_j} = \bar{H}_{y_j} + S_{y_j} \dot{y}_k (\bar{y}_k - \bar{y}_k (\bar{y}, \bar{a}, t)) \tag{58}
\]

\[
- \dot{S}_{y_j y_k} = \bar{H}_{y_j y_k} + S_{y_j y_k} (\bar{y}_m - \bar{y}_m (\bar{y}, \bar{a}, t)) + S_{y_k y_m} (\bar{y}_j - \bar{y}_j (\bar{y}, \bar{a}, t)) - \bar{p}_{y_j} H \tag{59}
\]

The boundary conditions for Equations 57-59 are

\[
S(\bar{y}, t_f) = \bar{J} \tag{56}
\]

\[
a(t_f) = 0 \tag{60}
\]

\[
S_{y_j} (t_f) = \bar{J}_{y_j} \tag{56}
\]

\[
S_{y_j y_k} (t_f) = \bar{J}_{y_j y_k} \tag{56}
\]

To this point a reference trajectory \(\bar{y}(t)\) has been
generated based on a nominal control \( \bar{a}(t) \) and a nominal return \( \bar{J}(y_f, t_f) \) has been determined. The characteristic equations for the optimal return function \( S \) have been developed based on a maximizing control \( a^* \). The boundary conditions of Equation 60 are now applied and the characteristic equations are integrated backwards from \( t_f \) to \( t_0 \).

When the backward sweep from \( t_f \) to \( t_0 \) is completed, a forward integration of the equations of motion is again performed. The control \( a(t) \) applied on the forward sweep is determined on the basis of stored values of \( a^* \) and \( \delta y_j \)

\[
a(t) = a^*(t) + \delta a(t) = a^*(t) + \delta y_j(t)
\]

(61)

The iteration through successive backward and forward sweeps is continued until the predicted gain in return \( a(t_0) \) is less than a specified value.

Quite often, for nonlinear problems, it is found that \( \delta y_j \) is too large and the truncated expansions are inadequate. This situation occurred in the present study. Therefore, an alternate method was employed.

Recall that a necessary condition for maintaining optimality is

\[
\mathcal{H}_a(\bar{y} + \delta y, a^* + \delta a, S_y + \delta y_j, t) = 0
\]

(38)

Thus \( a = a^* + \delta a \) may be determined directly from Equation 38 on the forward sweep. This method also increases the radius of convergence and is therefore less sensitive to the choice of a nominal control.
The above technique still does not guarantee convergence to an optimum. The variation of the state \( \delta y \) may still become too large for Equation 38 to be valid over the entire corrected trajectory. This difficulty may be resolved in the following manner.

Note that during the backward integration of Equations 57-59, the quantity

\[
a(\bar{y}, t_1) = \int_{t_f}^{t_1} (\mathcal{H} - \mathcal{H}(\bar{y}, \bar{a}, S_y, t)) \, dt
\]

(62)

is obtained and represents the predicted gain in return when starting from \( \bar{y}(t_1), t_1 \) using the corrected control given by Equation 38 for \( t \in [t_1, t_f] \). The actual gain in return is obtained by integrating the equations of motion

\[
\frac{\dot{(\bar{y} + \delta y)}}{\delta c} = \bar{y}(\bar{y} + \delta y, \bar{a}, t^*) + \delta a, t
\]

(63)

from \( t_1 \) to \( t_f \) and computing

\[
\Delta S = S(\bar{y}, t_1) - \bar{J}
\]

(64)

Thus, \( \delta y \) is considered to be too large if

\[
\frac{\Delta S}{a(\bar{y}, t_1)} < C, \quad C \geq 0
\]

(65)

where the value of \( C \) is chosen on the basis of numerical stability criteria. A typical value of \( C \) is 0.5.

If \( t_1 > t_0 \), the nominal trajectory is followed from \( t_0 \) to \( t_1 \) and the corrected control is applied from \( t_1 \) to \( t_f \). The backward sweep is then started again at \( t_f \). If \( t_1 = t_0 \), the value of \( C \) may be decreased and the total algorithm repeated.
until $S$ is near zero.

**Constrained problem** The problem considered in this study is subject to terminal constraints of the form

$$\phi(y_x, t_x) = 0$$

(66)

This type of constraint may be included in the problem in two ways.

One common technique uses penalty functions. The return function is modified to take the form

$$J^* = J + K_j \phi_j^2$$

(67)

Since a maximum return is desired, the constants $K_j$ are given initial negative values and are allowed to approach $-\infty$ on successive iterations.

After considerable experimentation with penalty functions, the second technique, which involves the use of Lagrange multipliers, was selected for this study.

The augmented return function then takes the form

$$J^* = J + \lambda_j \phi_j$$

(68)

where the $\lambda_j$'s are Lagrange multipliers to be determined. The second order algorithm of differential dynamic programming offers a unique method for determining the Lagrange multipliers based on initial guesses and constraint violations on the reference trajectory.

Since the constraints are on terminal conditions, the multipliers are constant and may be treated as parameters to be
chosen optimally. A power series expansion of \( S(y_0, \bar{\lambda} + \lambda, t_0) \) about \( S(y_0, \bar{\lambda}, t_0) \) yields

\[
S(y_0, \bar{\lambda} + \delta \lambda, t_0) = S(y_0, \bar{\lambda}, t_0) + S_{\lambda} (y_0, \bar{\lambda}, t_0) \delta \lambda_j
\]

\[
+ \frac{1}{2} S_{\lambda \lambda} (y_0, \bar{\lambda}, t_0) \delta \lambda_j \delta \lambda_k
\]

(69)

with higher order terms neglected. Thus,

\[
\frac{\partial S}{\partial (\delta \lambda_j)} = S_{\lambda_j} + S_{\lambda j \lambda k} \delta \lambda_k = 0
\]

(70)

for optimal choices of \( \delta \lambda \). Hence,

\[
\delta \lambda_j = - S_{\lambda j \lambda k}^{-1} S_{\lambda k}
\]

(71)

where \( S_{\lambda j \lambda k}^{-1} \) is the inverse of the matrix

\[
S_{\lambda j \lambda k} = \begin{bmatrix}
S_{\lambda_1 \lambda_1} & \cdots & S_{\lambda_1 \lambda_n} \\
\vdots & \ddots & \vdots \\
S_{\lambda_n \lambda_1} & \cdots & S_{\lambda_n \lambda_n}
\end{bmatrix}
\]

(72)

The characteristic equations for \( S_\lambda \) and \( S_{\lambda \lambda} \) are obtained by returning to Equation 39 and writing

\[
- \frac{\partial \bar{y}}{\partial t} - \frac{\lambda}{\partial t} - \frac{\partial S y_j}{\partial t} \delta y_j - \frac{\partial S \lambda_j}{\partial t} \delta \lambda_j - \frac{1}{2} \frac{\partial S y_j y_k}{\partial t} \delta y_j \delta y_k
\]

\[
- \frac{\partial S y_j \lambda_k}{\partial t} \delta y_j \delta \lambda_k - \frac{1}{2} \frac{\partial S \lambda_j \lambda k}{\partial t} \delta \lambda_j \delta \lambda k
\]

\[
= \text{Max} \ H(\bar{y} + \delta y, \bar{\lambda} + \delta \lambda, \alpha^* + \delta \alpha, S_z + S_z z_j, t)
\]

(73)
where
\[ z = (x, h, v, \gamma, \lambda) \]  

A power series expansion, to second order, of the right-hand side of Equation 73 about \((\bar{y}, \bar{\lambda}, \alpha^*)\) gives
\[
\max_{\alpha} \left[ H + H_\alpha \delta \alpha + H_y \delta y_j + S_{y_j y_k} \delta y_j \delta y_k + S_{y_j \lambda_k} \delta y_j \delta \lambda_k \right. \\
+ \left. \left( H_{\alpha y} + S_{y_j y_k} (\dot{y}_k) \alpha \right) \delta y_j \delta \alpha + \frac{H_{\alpha \alpha}}{2} (\delta \alpha)^2 + S_{y_j \lambda_k} (\dot{y}_j) \alpha \delta \lambda_k \delta \alpha \right] \\
+ \frac{1}{2} \left[ H_y \delta y_j + S_{y_j y_m} (\dot{y}_m) \lambda_k \right. \\
+ \left. S_{y_j \lambda_k} (\dot{y}_j) \lambda_m \right] \delta y_j \delta \lambda_k \\
+ S_{y_j \lambda_k} \delta y_j \lambda_m 
\]  

(75)

where all quantities are evaluated at \((\bar{y}, \bar{\lambda}, \alpha^*, t)\).

Again noting that
\[ H_\alpha(\bar{y}, \bar{\lambda}, \alpha^*, S_y, t) = 0 \]  

(76)

maximization of Equation 75 with respect to \(\delta \alpha\) gives
\[
H_{\alpha \alpha} \delta \alpha + \left[ H_{\alpha y} + S_{y_j y_k} (\dot{y}_k) \alpha \right] \delta y_j + S_{y_j \lambda_k} (\dot{y}_j) \lambda_k \delta \lambda_k = 0 
\]  

(77)

Thus,
\[
\delta \alpha = - \frac{\left[ H_{\alpha y} + S_{y_j y_k} (\dot{y}_k) \alpha \right] \delta y_j + S_{y_j \lambda_k} (\dot{y}_j) \lambda_k \delta \lambda_k}{H_{\alpha \alpha}} 
\]  

(78)

Substituting Equation 78 into Equation 75 and equating coefficients of Equations 73 and 75 provides the desired characteristic equations
\[
- \ddot{\xi}_\lambda = S_{y_j \lambda_k} (\dot{y}_j - \dot{y}_j(\bar{y}, \bar{\lambda}, t)) 
\]  

(79)
\[ \dot{S}_{y_j} = S_{y_j} (\dot{y}_j) - \beta_{y_j} \dot{y}_j \alpha \]  
\[ \dot{S}_{x_j} = -\beta_{x_j} \dot{x}_j \alpha \]  
where
\[ \beta_{y_j} = \frac{-S_{y_j} (\dot{y}_j)}{H_{\alpha\alpha}} \]  
As before, if \( \beta_{y_j}, \delta_{y_j} \) or \( \beta_{x_j} \delta_{x_j} \) are too large for the second order expansion to remain valid, the necessary condition for optimality
\[ H_{\alpha} (\ddot{y} + \delta_{y_j} + \delta_{x_j} \alpha + \delta_{\alpha} S_z + S_{zz_j} \delta_{z_j}, t) = 0 \]  
may be solved directly for \( \alpha \). If convergence is still unsatisfactory, the application of the partial iteration given by Equations 62-65 and the accompanying discussion should allow a solution to be reached.
RESULTS

Numerical solutions were obtained for several constrained and unconstrained trajectories. Table 1 summarizes the cases considered in this study.

All computations were performed on an IBM 360/65 digital computer using Fortran IV language with single precision accuracy.

The following initial conditions were used for all cases investigated

\[ h_0 = 50.0 \text{ miles} \]
\[ V_0 = 5.0 \text{ miles per second} \]
\[ \gamma_0 = -1.0 \text{ degree} \]

The nominal, or reference, trajectories are based on a constant angle of attack and the initial values of the Lagrange multipliers for the constrained trajectories were set equal to zero.

Hypersonic Drag Polar

The hypersonic drag polar is defined by

\[ C_L = 1.7a|a| \]
\[ C_D = 0.042 + 1.45|a|^3 \]

The optimal angle of attack histories for the unconstrained and constrained cases are shown in Figure 5. Note in particular the trajectory for \( \gamma_f = 0 \). An arbitrary upper limit of 25 degrees was placed on the angle of attack and this limit
### Table 1. Summary of cases investigated

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of drag polar</th>
<th>Final time</th>
<th>Final altitude</th>
<th>Final flight path angle</th>
<th>Wing loading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Hypersonic^a</td>
<td>180</td>
<td>Free</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
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<td>Hypersonic</td>
<td>180</td>
<td>50</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>Hypersonic</td>
<td>180</td>
<td>Free</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>Parabolic^b</td>
<td>180</td>
<td>Free</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>5</td>
<td>Parabolic</td>
<td>180</td>
<td>50</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
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<td>180</td>
<td>Free</td>
<td>0</td>
<td>30</td>
</tr>
<tr>
<td>7</td>
<td>Parabolic</td>
<td>160</td>
<td>Free</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>8</td>
<td>Parabolic</td>
<td>200</td>
<td>Free</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>9</td>
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<td>Free</td>
<td>50</td>
</tr>
<tr>
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<td>180</td>
<td>Free</td>
<td>Free</td>
<td>70</td>
</tr>
<tr>
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<td>Parabolic</td>
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<td>45</td>
<td>Free</td>
<td>30</td>
</tr>
<tr>
<td>12</td>
<td>Parabolic</td>
<td>180</td>
<td>55</td>
<td>Free</td>
<td>30</td>
</tr>
</tbody>
</table>

^a The hypersonic drag polar is defined by

\[
C_L = 1.7\alpha|\alpha|
\]
\[
C_D = 0.042 + 1.46|\alpha^3|
\]

^b The parabolic drag polar is defined by

\[
C_L = 0.2 \sin 6\alpha
\]
\[
C_D = 0.04 + 1.5C_L^2
\]
Figure 5. Angle of attack histories for reference and optimal trajectories
is reached near the end of the trajectory. The upper limit on angle of attack was considered necessary because of the approximate drag polar employed. When the limit was removed, the angle of attack near the end of the trajectory became unreasonably large and convergence was unsatisfactory.

The optimal velocity, altitude and flight path angle profiles are presented in Figures 6, 7 and 8. The optimal velocity-altitude relationships are shown in Figure 9. Of particular interest in this figure is the shape of the unconstrained trajectory. The small peak near the end of the trajectory indicates a trade-off of potential and kinetic energy in order to maximize the final velocity.

Figures 10 and 11 show the angle of attack variation with velocity and altitude respectively.

Parabolic Drag Polar

The parabolic drag polar is defined by

\[ C_L = 0.2 \sin \theta \]
\[ C_D = 0.04 + 1.5 C_L^2 \]

from which

\[ \alpha_{C_L_{\text{max}}} = 15^\circ \]

Comparisons of optimal angle of attack histories for the parabolic and hypersonic drag polars are given in Figures 12, 13 and 14. A significant feature of these comparisons is the similarity of the shapes of the curves for the two dissimilar drag polars. This similarity is more obvious when the lift-
Figure 6. Velocity histories for reference and optimal trajectories
Figure 7. Altitude histories for reference and optimal trajectories
Figure 8. Flight path angle histories for reference and optimal trajectories
Figure 9. Velocity-altitude profiles
Figure 10. Velocity vs angle of attack
Figure 11. Altitude vs angle of attack

Altitude $h$, miles

Angle of attack $\alpha$, degrees

- Nominal
- $h_f = 50$ miles
- $\gamma_f = 0^\circ$
- Unconstrained
Figure 12. A comparison of optimal angle of attack histories for different lift-drag relationships and no trajectory constraints.
Figure 13. A comparison of optimal angle of attack histories for different lift-drag relationships and a fixed final altitude.
Figure 14. A comparison of optimal angle of attack histories for different lift-drag relationships and a fixed final flight path angle.

Hypersonic polar
\[ C_L = 1.7a/|a| \]
\[ C_D = 0.042 + 1.46a^3 \]

Parabolic polar
\[ C_L = 0.2 \sin 6 \alpha \]
\[ C_D = 0.04 + 1.5 C_L^2 \]
to-drag ratios $L/D$ are compared as shown in Figures 15, 16 and 17. Thus, the $L/D$ ratio appears to be more important than the form of the drag polar for determining the optimal trajectories.

The unusual angle of attack history for $\gamma_f = 0$, shown in Figure 14, is partially explained by the optimal $L/D$ variation shown in Figure 17. The $L/D$ varies little from its maximum value over roughly the first half of the optimal trajectory. Then $L/D$ begins to decrease at about 80 seconds into the trajectory. This corresponds closely to the point at which $\gamma$ passes through zero. Thus, the remainder of the trajectory is a result of a modulation of $L/D$ in order to reach $\gamma = 0$ again at $t = t_f$.

The importance of the maximum $L/D$ to optimal maneuvers is by no means unique to the problems considered in this study. It has long been known that the maximum range of an aircraft is obtained by flying at maximum $L/D$. Bleick and Faulkner (17) also show that the initial portion of a maximum range trajectory for a reentry glider is flown at maximum $L/D$. Therefore, it should not be surprising to encounter maximum $L/D$ arcs on other optimal reentry trajectories.

The effects of varying final time, final altitude and vehicle wing loading on the optimal trajectories for a parabolic drag polar are shown in Figures 18, 19 and 20. Because of the similarity of trajectories for different drag polars the trends in these figures are considered to be quite general.
Figure 15. Lift-to-drag ratio variation along optimal unconstrained trajectories.
Figure 16. Lift-to-drag ratio variation along optimal trajectories with final altitude constrained

$h_f = 50$ miles
Figure 17. Lift-to-drag ratio variation along optimal trajectories with final flight path angle constrained
Figure 18. Effect of terminal time on optimal angle of attack history for a vehicle with a parabolic drag polar and no trajectory constraints
Figure 19. Effect of final altitude constraint on optimal angle of attack history for a vehicle with a parabolic drag polar.
Figure 20. Effect of wing loading on optimal angle of attack history for a vehicle with a parabolic drag polar and no trajectory constraints.
Aerodynamic Heating

The heating analysis used in this study is described in Appendix B. The resultant heating rates and total heat loads encountered on the optimal trajectories for the hypersonic drag polar are shown in Figures 21 and 22. For available materials, Shaver (16) indicates that heating rates of 100 Btu/ft$^2$sec and total heat loads of 10,000 Btu/ft$^2$ are readily acceptable without a loss of structural integrity or harmful effects on the crew of a manned vehicle. Since the maximum heating rate shown in Figure 21 is less than 40 Btu/ft$^2$sec and the maximum total heat load shown in Figure 22 is less than 2000 Btu/ft$^2$ the problem of aerodynamic heating is probably of minor importance for cases considered in this study.
Figure 21. Heat absorption rates on optimal trajectories for a hypersonic drag polar
Figure 22. Total heat absorbed on optimal trajectories for a hypersonic drag polar
SUMMARY

A successive sweep method with differential dynamic programming is used to develop the necessary conditions for minimization of the velocity loss of a lifting reentry vehicle during the initial phase of atmospheric entry. Final time is specified with final altitude and final flight path angle constraints being included by the use of Lagrange multipliers. The minimization of the velocity loss is particularly important if the vehicle is going to return to orbit since less fuel will be required to attain orbital velocity.

A second-order algorithm is used to determine the optimal angle of attack on both the backward and forward integrations. This algorithm also provides the characteristic equations necessary for determining the Lagrange multipliers of the constrained trajectories.

Numerical results are presented for several constrained and unconstrained trajectories. Results are compared for two lift-drag relationships:

1. An approximation of the hypersonic reentry glider drag polar given by Bleick and Faulkner (17)
   
   \[ C_L = 1.7 \alpha |\alpha| \]
   \[ C_D = 0.042 + 1.46 |\alpha|^3 \]

2. A parabolic drag polar
   
   \[ C_L = 0.2 \sin 6 \alpha \]
   \[ C_D = 0.04 + 1.5 C_L^2 \]
Although the optimal angle of attack histories vary considerably in magnitude for the two drag polars considered, the lift-to-drag ratio $L/D$ variations are quite similar. Thus, $L/D$ seems to be a better parameter for describing the optimal trajectories than angle of attack.

For the parabolic drag polar, the effects of varying final time, final altitude and vehicle wing loading on the optimal angle of attack are presented.

The results of a study of the aerodynamic heating effects are also presented for the optimal trajectories involving a hypersonic drag polar. The resulting heating rates and total heat loads seem to be well within acceptable limits for reentry vehicles.
RECOMMENDATIONS FOR FURTHER STUDY

Many extensions of the present study are apparent. For example, the entire class of minimum-time problems is relatively unexplored by the dynamic programming method.

Instead of maximum final velocity, final energy or heat load may be used as payoff functions. Final velocity may be constrained or horizontal range could be maximized.

There is also a full range of three-dimensional problems available. To this time, such multi-dimensional trajectory problems appear to have resisted solution by dynamic programming. The equations of motion applicable to one such problem, that of an orbital plane change, are developed in Appendix C.
LITERATURE CITED


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APPENDIX A

Equations of Motion

Consider a lifting body moving in the atmosphere over a flat Earth. If the vehicle is represented by a point mass,

\[ m\ddot{V} = \vec{D} + \vec{L} + mg \]  

(A.1)

Figure A.1. Force vector diagram for a lifting body over a flat Earth

Figure A.2. Body-referenced force diagram
Let

\[ \vec{V} = \mathbf{v} \hat{\mathbf{s}} \]  
\[ \vec{D} = -\mathbf{d} \hat{\mathbf{s}} \]  
\[ \vec{L} = \mathbf{l} \hat{\mathbf{n}} \]  
\[ m\vec{g} = -mg \hat{\mathbf{j}} \]  

where

\[ \hat{\mathbf{s}} = \cos \gamma \hat{\mathbf{i}} + \sin \gamma \hat{\mathbf{j}} \]  
\[ \hat{\mathbf{n}} = -\sin \gamma \hat{\mathbf{i}} + \cos \gamma \hat{\mathbf{j}} \]  

Substitution of Equation A.6 into Equation A.2 gives

\[ \vec{V} = \mathbf{v} \cos \gamma \hat{\mathbf{i}} + \mathbf{v} \sin \gamma \hat{\mathbf{j}} \]  

Differentiation of Equation A.8 yields

\[ \dot{\vec{V}} = (\dot{\mathbf{v}} \cos \gamma - \mathbf{v} \dot{\gamma} \sin \gamma) \hat{\mathbf{i}} + (\dot{\mathbf{v}} \sin \gamma + \mathbf{v} \dot{\gamma} \cos \gamma) \hat{\mathbf{j}} \]  

Substitution of Equations A.3-A.7 and Equation A.9 into Equation A.1 now gives

\[ m\{(\dot{\mathbf{v}} \cos \gamma - \mathbf{v} \dot{\gamma} \sin \gamma) \hat{\mathbf{i}} + (\dot{\mathbf{v}} \sin \gamma + \mathbf{v} \dot{\gamma} \cos \gamma) \hat{\mathbf{j}}\} = -D(\cos \gamma \hat{\mathbf{i}} + \sin \gamma \hat{\mathbf{j}}) + L(-\sin \gamma \hat{\mathbf{i}} + \cos \gamma \hat{\mathbf{j}}) - mg \hat{\mathbf{j}} \]  
\[ = (-D \cos \gamma - L \sin \gamma) \hat{\mathbf{i}} + (-D \sin \gamma + L \cos \gamma - mg) \hat{\mathbf{j}} \]  

Separation of components of Equation A.10 yields

\[ m(\dot{\mathbf{v}} \cos \gamma - \mathbf{v} \dot{\gamma} \sin \gamma) = -D \cos \gamma - L \sin \gamma \]  
\[ m(\dot{\mathbf{v}} \sin \gamma + \mathbf{v} \dot{\gamma} \cos \gamma) = -D \sin \gamma + L \cos \gamma - mg \]  

Next multiply Equation A.11 by \cos \gamma and Equation A.12 by \sin \gamma and add to obtain
\( m \ddot{V} = -D - mg \sin \gamma \quad (A.13) \)

Then, multiply Equation A.11 by \(-\sin \gamma\) and Equation A.12 by \(\cos \gamma\) and add to get

\( m \ddot{V} = L - mg \cos \gamma \quad (A.14) \)

or

\( m \dot{\gamma} = \frac{L}{V} - \frac{mg \cos \gamma}{V} \quad (A.15) \)

Also,

\( \dot{V} = V \cos \gamma \hat{i} + V \sin \gamma \hat{j} = \dot{x} \hat{i} + \dot{h} \hat{j} \quad (A.16) \)

so

\( \dot{x} = V \cos \gamma \quad (A.17) \)

\( \dot{h} = V \sin \gamma \quad (A.18) \)

Equations A.13, A.15, A.17 and A.18 are the equations of motion used in the present study.
APPENDIX B

Aerodynamic Heating Analysis

Entry into the atmosphere at near-orbital velocities gives rise to significant aerodynamic heating. The mechanical energy of the vehicle is reduced by atmospheric drag and a percentage of the total mechanical energy loss is converted into heat which is transmitted to the vehicle. Most of the heat transmission is through convection in the boundary layer. A portion of the energy loss is also required to maintain the aerodynamic field around the vehicle. The resultant shock waves create a large temperature rise in the surrounding flow which causes some vehicle heating by radiation.

The total mechanical energy of the vehicle at any instant is considered to be the sum of the kinetic and potential energies

\[ E = \frac{m\dot{V}^2}{2} + mgh \]  \hspace{1cm} (B.1)

The rate of energy loss is obtained by differentiation of Equation B.1

\[ \dot{E} = m\ddot{V} + m\dot{h} \]  \hspace{1cm} (B.2)

for constant mass and a constant gravitational field.

Substitution of Equations 3 and 4 into Equation B.2 yields

\[ \dot{E} = -DV \]  \hspace{1cm} (B.3)

or

\[ \frac{\dot{E}}{A} = -\frac{DV}{A} \]  \hspace{1cm} (B.4)
where \( A \) is the same vehicle reference area used in
\[
D = C_D \frac{\rho v^2}{2} A \quad (B.5)
\]

According to Shaver (16), the approximate net percentage of total mechanical energy loss transmitted to the vehicle as heat may be given as a function of altitude as shown in Figure B.1. Shaver (16) reports these values as reliable within twenty percent.

The vehicle heating rate may thus be represented by
\[
\dot{Q} = - p_h \frac{E}{A} = p_h C_D \frac{\rho v^3}{2} \quad (B.6)
\]
where \( p_h \) is the percentage of the energy loss which is absorbed as heat by the vehicle. The total heat load on the vehicle is therefore
\[
Q_T = \int_0^{t_f} p_h C_D \frac{\rho v^3}{2} dt \quad (B.7)
\]

It must be emphasized that this analysis is intended only as a first approximation to heating phenomena.
Figure B.1. Conversion of vehicle energy into heat energy at the vehicle surface
APPENDIX C

Synergetic Orbital Plane Changes

The fundamental vector equations of motion for a lifting body moving within the atmosphere are

\[ \dot{V} = \frac{d\vec{R}}{dt} \]  \hspace{2cm} (C.1)

\[ \ddot{a} = \frac{d\dot{V}}{dt} = \frac{L}{m} + \frac{D}{m} + \vec{g} \]  \hspace{2cm} (C.2)

\[ \hat{R} = R_0 \cos \theta \]

\[ \hat{R} = R_0 \sin \theta \]

\[ \hat{V} = \dot{R}_0 \cos \theta \]

\[ \hat{V} = \dot{R}_0 \sin \theta \]

\[ \hat{\theta} = \theta \]

\[ \hat{\theta} = \theta \]

\[ \hat{\phi} = \phi \]

\[ \hat{\phi} = \phi \]

\[ \hat{\psi} = \psi \]

\[ \hat{\psi} = \psi \]

\[ \begin{bmatrix} \dot{\hat{u}}_K \\ \dot{\hat{u}} \\ \dot{\hat{u}}_\theta \end{bmatrix} = \begin{bmatrix} 0 & \dot{\psi} \cos \theta & \dot{\theta} \\ -\dot{\psi} \cos \theta & 0 & \dot{\phi} \\ -\dot{\theta} & -\dot{\psi} \sin \theta & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_K \\ \hat{u} \\ \hat{u}_\theta \end{bmatrix} \]  \hspace{2cm} (C.4)

Figure C.1. Basic coordinate systems

From Figure C.1,

\[ \hat{R} = R_0 \]

\[ \hat{V} = R_0 \]

\[ \hat{\theta} = \theta \]

\[ \hat{\phi} = \phi \]

\[ \hat{\psi} = \psi \]
where $\hat{u}_R$, $\hat{u}_\varphi$, and $\hat{u}_\theta$ are orthogonal unit vectors as indicated in the figure. Thus,

$$\vec{V} = \vec{R}$$

$$= \dot{R}\hat{u}_R + \hat{R}\dot{u}_R$$

$$= \dot{R}\hat{u}_R + R\dot{\varphi}\cos \theta \hat{u}_\varphi + R\dot{\theta}\hat{u}_\theta \quad (C.5)$$

But,

$$\vec{V} = V_R\hat{u}_R + V_\varphi\hat{u}_\varphi + V_\theta\hat{u}_\theta \quad (C.6)$$

so the velocity components are

$$V_R = \dot{R}$$

$$V_\varphi = R\dot{\varphi}\cos \theta$$

$$V_\theta = R\dot{\theta} \quad (C.7)$$

Acceleration is given by

$$\vec{a} = \vec{V}$$

$$= \ddot{R}\hat{u}_R + (\dot{R}\dot{\varphi}\cos \theta + R\ddot{\varphi}\cos \theta - \dot{R}\dot{\theta}\sin \theta)\hat{u}_\varphi$$

$$+ (\ddot{\varphi} + R\dot{\theta})\hat{u}_\theta + \dot{R}\dot{u}_R + R\dot{\varphi}\cos \theta \hat{\varphi} + R\dot{\theta}\hat{u}_\theta$$

$$= (\dddot{R} - R\dddot{\theta}\cos^2 \theta - R\dddot{\varphi} - \dot{R}\ddot{\theta})\hat{u}_R + (R\dot{\varphi}\cos \theta + 2R\dot{\varphi}\cos \theta$$

$$- 2R\dot{\varphi}\sin \theta)\hat{u}_\varphi + (2\ddot{\varphi} + R\ddot{\theta} + R\dddot{\varphi}\sin \theta \cos \theta)\hat{u}_\theta$$

$$\quad (C.8)$$
From Figure C.3,
\[ \hat{\mathbf{L}} = L\hat{k} \quad \text{(C.9)} \]
\[ \hat{\mathbf{V}} = V\hat{i} \quad \text{(C.10)} \]

and, from Figure C.2, the following relationship between the
unit vectors \( \hat{i}, \hat{j}, \hat{k} \) and \( \hat{u}_R, \hat{u}_\nu, \hat{u}_\theta \) exists

\[
\begin{bmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{bmatrix} =
\begin{bmatrix}
(-\sin \phi) & (\cos \phi \cos \theta) \\
(-\sin \nu \cos \phi) & (-\sin \nu \cos \phi \sin \theta - \cos \nu \sin \phi) \\
(\cos \nu \cos \phi) & (\cos \nu \cos \phi \sin \theta - \sin \nu \sin \phi)
\end{bmatrix}
\begin{bmatrix}
\hat{u}_R \\
\hat{u}_\nu \\
\hat{u}_\theta
\end{bmatrix}
\]

where \( \phi, \theta, \nu \) are Euler angles.

Then,

\[
\begin{align*}
\vec{V} \cdot \hat{u}_R &= V_R = -V \sin \phi \\
\vec{V} \cdot \hat{u}_\nu &= V_\nu = V \cos \phi \cos \theta \\
\vec{V} \cdot \hat{u}_\theta &= V_\theta = V \cos \phi \sin \theta
\end{align*}
\]

Substitution of Equation C.11 into Equation C.8 and equating components gives

\[
\begin{align*}
\ddot{R} - R^2 \dot{\theta}^2 - R^2 \dot{\nu}^2 \cos^2 \theta &= \frac{L}{m} \cos \nu \cos \phi + \frac{D}{m} \sin \phi - g \\
2R \dot{\nu} \cos \theta - 2R \dot{\theta} \sin \theta + R \dot{\nu} \cos \theta &= \frac{L}{m} (\cos \nu \cos \theta \sin \phi - \sin \nu \sin \phi) - \frac{D}{m} \cos \phi \cos \nu \\
2R \ddot{\theta} + R \dot{\nu}^2 \sin \theta \cos \theta + R \ddot{\theta} &= \frac{L}{m} (\cos \nu \sin \phi \sin \theta + \sin \nu \cos \phi) - \frac{D}{m} \sin \phi \cos \nu
\end{align*}
\]

Substitution of Equations C.7 into Equations C.13 results in
\[
\begin{align*}
\dot{V}_R &= \ddot{R} = \frac{V_y^2 + V_\theta^2}{R} + \frac{L}{m} \cos \xi \cos \phi + \frac{D}{m} \sin \psi - g \\
\dot{V}_y &= R \ddot{\theta} \cos \theta + R \dot{\theta} \cos \theta - R \dot{\psi} \sin \theta \\
&= \frac{V_y (V_\theta \tan \theta - V_R)}{R} - \frac{L}{m} (\sin \xi \sin \gamma - \cos \xi \cos \gamma \sin \phi) \\
&- \frac{D}{m} \cos \gamma \cos \phi \\
\dot{V}_\theta &= R \ddot{\theta} + R \dot{\theta} = - \frac{(V_R V_\theta + V_y^2 \tan \theta)}{R} \\
&+ \frac{L}{m} (\sin \xi \cos \gamma + \cos \xi \sin \gamma \sin \phi) - \frac{D}{m} \sin \gamma \cos \phi \\
&\text{(C.14)}
\end{align*}
\]

Figures C.4 below are based on Equations C.12 and yield

Figure C.4. Velocity components

the relations

\[
\begin{align*}
\sin \phi &= - \frac{V_R}{V} \\
\cos \phi &= \frac{\sqrt{V_y^2 + V_\theta^2}}{V} \\
&\text{(C.15)}
\end{align*}
\]
\[
\sin \gamma = \frac{V_\theta}{\sqrt{V_\psi^2 + V_\theta^2}} \\
\cos \gamma = \frac{V_\psi}{\sqrt{V_\psi^2 + V_\theta^2}}
\]

(C.16)

Substitution of Equations C.15 and C.16 into Equations C.14 gives

\[
\dot{V}_R = \frac{(V_\psi^2 + V_\theta^2)}{R} + \frac{L}{m} \sqrt{\frac{V_\psi^2 + V_\theta^2}{V}} \cos \xi - \frac{D}{m} \frac{V_R}{V} - g
\]

\[
\dot{V}_\psi = V_\psi \frac{(V_\theta \tan \theta - V_R)}{R} - \frac{L}{m} \left( \frac{(V_\psi \sin \xi + V_R \cos \xi)}{\sqrt{V_\psi^2 + V_\theta^2}} \right) - \frac{D}{m} \frac{V_\psi}{V}
\]

\[
\dot{V}_\theta = -\frac{(V_\psi V_\theta + V_\theta^2 \tan \theta)}{R} + \frac{L}{m} \left( \frac{(V_\psi \sin \xi - V_R \cos \xi)}{\sqrt{V_\psi^2 + V_\theta^2}} \right) - \frac{D}{m} \frac{V_\theta}{V}
\]

(C.17)

Then, Equations C.7 and C.17 are the equations of motion.

Magnitude of plane change

The instantaneous plane of the trajectory is defined by the radius and velocity vectors, \( \vec{R} \) and \( \vec{V} \). To determine the orientation of the instantaneous plane with respect to the original orbit plane, consider Figures C.5 and C.6.
Application of the Law of sines to the spherical triangle in Figure C.6 gives

$$\frac{\sin i}{\sin \theta} = \frac{\cos \eta}{\sin (\nu - 1)} \frac{1}{\sin f} \quad \text{(C.18)}$$

Squaring Equation C.18 and rearranging,

$$\sin^2 i = \frac{\cos^2 \eta \sin^2 \theta}{\sin(\nu - 1)} \frac{\sin^2 \theta}{\sin^2 f} \quad \text{(C.19)}$$

Application of the Law of cosines to Figure C.6 gives

$$\cos f = \cos \theta \cos (\nu - 1) \quad \text{(C.20)}$$
Substitution of Equation C.20 into Equation C.19 and again rearranging gives
\[
\cos^2 \eta \sin^2 f = \cos^2 \eta (1 - \cos^2 f) = \sin^2 (\psi - 1)
\]
\[
= \cos^2 \eta \{1 - \cos^2 \theta \cos^2 (\psi - 1)\}
\]
\[
= \cos^2 \eta \{1 - \cos^2 \theta + \cos^2 \theta \sin^2 (\psi - 1)\}
\]
\[
= \cos^2 \eta \sin^2 \theta + \cos^2 \theta \sin^2 (\psi - 1)
\]
(C.21)

Substitution of Equation C.19 into Equation C.21 gives
\[
\sin \eta \cos \theta = 1.
\]
\[
= \sin^2 \theta + \cos^2 \psi \cos^2 \theta
\]
(C.22)

Thus,
\[
1 - \sin^2 \theta = \cos^2 \theta = \cos^2 \psi \cos^2 \theta
\]
(C.23)

Substitution of Equation C.16 into Equation C.23 gives
\[
\cos \theta = \frac{V_\psi \cos \theta}{\sqrt{V_\psi^2 + V_\theta^2}}
\]
(C.24)

which provides the terminal constraint for the synergetic plane change.