Potential flow and drawdown for a well in a partially confined porous medium

Charles Warren Boast
Iowa State University

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THE FLOW MEDIUM

We solve the problem of saturated water flow to a well fully penetrating a semi-confined porous medium. We seek an explicit solution which is in the form of a hydraulic head function \( \phi \), a function which for any point in space gives the height to which water would stand in a tube placed at the point. The problem has been studied previously using sand tanks, electric analogs and numerical methods. Kirkham's (1964) analytic method lacks convergence when approximations of higher order than those given by Kirkham are used.

Figure 1 shows a well of radius \( a \) fully penetrating a porous medium which is bounded below by a horizontal impermeable barrier. We place a system of cylindrical coordinates \((r, z)\) at the center of the bottom of the well. Initially there is a horizontal water table a distance \( h \) above

![Figure 1. The flow medium with boundaries numbered (1) through (5) for the condition "i = 0"](image_url)
the impermeable barrier and the flow medium is a cylinder of radius $b$ and height $h$. We assume that at radius $r = b$ there exists one of two conditions. There is either a constant head source of water with head equal to the initial height of the water table (this condition is denoted $i = 0$ and is shown in Figure 1) or there is a vertical impermeable barrier at $r = b$ (this condition is denoted $i = 1$, and is not shown in Figure 1).

At time $t = 0$ the water level in the well is lowered a distance $y$ and maintained at this level. We choose the level of water in the well as the reference level for hydraulic head measurements. As flow begins the water table draws down and approaches an equilibrium shape as time increases. We assume that there is an exact boundary between water-saturated soil below the water table and "dry" soil above the water table, i.e. we assume that there is no capillary fringe above the water table and that when the water table draws down all of the water of the "drainable porosity" $f$ ("fraction" of space) is removed instantaneously. There is no slowly draining porosity.

For $i = 1$, if there is no source of water above the flow medium, the equilibrium shape of the water table is a horizontal plane at height $z = w = h - y$. Therefore, for $i = 1$ we will (rather arbitrarily) assume that rainfall, distributed evenly over the soil surface and varying with time, keeps the height of the water table at radius $r = b$ at the initial height of the water table, $z = h$. For simplicity we assume that the rain percolates instantly through the "dry" porous medium.

In neither case $i = 0$ nor $i = 1$ do we attempt to represent reality in the dry soil. We believe that the solution we give for the
saturated flow medium can be used as one half of a "two medium" solution to the more complicated saturated-unsaturated mixed problem. Numerical methods have been applied to this problem by Hall (1955) and Taylor and Luthin (1969).

Henceforth by "flow medium" we mean only that part of the porous medium which is water saturated.

We assume that the rate of water flow through an infinitesimally small area \( dA \) in the flow medium is proportional to the size of the area and to the rate of change of the hydraulic head in a direction normal to the area. This relationship, known as Darcy's Law, is \( dQ = -k_n(\partial \phi / \partial n) dA \) where the proportionality constant \( k_n \), the hydraulic conductivity in the normal direction, has dimensions length/time. We assume that throughout the flow medium the hydraulic conductivity in all horizontal directions is a constant \( k_n \), and in the vertical direction is a constant \( k_v \). We also assume that \( k_n = k_v \) (denoted simply \( k \)) but the removal of this assumption can be accomplished by a simple transformation of coordinates as in Maasland (1957, especially Theorems II and III).

With the above assumptions the hydraulic head function \( \phi \) satisfies Laplace's equation, which in cylindrical coordinate form is

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 .
\]

Here the middle term is identically zero because the axial symmetry of the problem makes \( \partial \phi / \partial \theta \) zero. Here \( \theta \) is the longitudinal angle.
THE WATER TABLE

We represent the shape of the water table by a dimensionless function \( \sigma(r) \). At any radius \( r \) the height of the water table is given by \( h\sigma(r) \). The height of the upper boundary of the surface of seepage in the well \( h\sigma(a) \) and the height of the water table at any radius \( r \), \( h\sigma(r) \), are shown in Figure 1. We choose \( \sigma(r) \) so that \( h\sigma(b) = h \). We want the function \( \sigma(r) \) to have as simple a form as possible so we "build into" \( \sigma(r) \) known facts about the shape of the water table.

According to Muskat (1937, p. 291), Casagrande (1937, Figure 6e, p. 139), and Shaw and Southwell (1941, pp. 6-7 and pp. 15-16) the water table approaches the well face vertically, that is, tangentially at equilibrium. Hence, we choose as one term in our representation of the water table shape a function with infinite slope at \( r = a \). One such function viewed in the \( (r,z) \) plane is an ellipse with horizontal semi-axis of length \( b - a \), with vertical semi-axis of length \( h\sigma_1 \), where \( \sigma_1 \) is a constant which we have not yet determined, and with center at the point \((b, h - h\sigma_1)\).

Letting \((r,z_e)\) be a point on the ellipse, the equation for the ellipse is

\[
\frac{(b - r)^2}{(b - a)^2} + \frac{(z_e - (h - h\sigma_1))^2}{(h - (h - h\sigma_1))^2} = 1 .
\]

We are considering the upper left quarter of the ellipse only. We solve the last equation for \( z_e/h \) obtaining

\[
\frac{z_e}{h} = 1 - \sigma_1 \left[1 - \sqrt{1 - \left(\frac{b - r}{b - a}\right)^2}\right]. 
\]

(2)
The representation of $\sigma(r)$ which we use includes this term as well as other terms given below.

Muskat (1937, p. 291), Casagrande (1937, Figure 6b), Shaw and Southwell (1941, p. 6), and Nahrgang (1951, p. 24) argue that at equilibrium the water table leaves a water source (e.g. for our problem the surface $r = b$ for $i = 0$) horizontally. They maintain that since boundary (1) is an equipotential and the water table is a streamline, the water table must intersect the equipotential orthogonally because of "general properties of a flownet" (Casagrande, 1937, pp. 140 and 161). It is well understood that the "general properties of a flownet" are not obeyed at points where streamlines and equipotentials are forced to be non-orthogonal, e.g. at the entrance and discharge points at the base of a dam resting on an impermeable barrier when the sides of the dam are not perpendicular to the impermeable barrier. What occurs at these points is either no flow or infinite flow depending on whether the streamlines are forced to diverge or converge.

Most examples of divergence or convergence of streamlines are, like the dam situation just mentioned, due to equipotentials and streamlines being forced non-orthogonal by "impermeable" and "equipotential" boundary conditions meeting non-orthogonally. There is, however, one common example of streamline convergence which is not caused by an impermeable and equipotential boundary condition meeting non-orthogonally. The example is the infinite rate of discharge into a well at the level of water in the well, the point $(a,w)$ in Figure 1, (and the corresponding situation in dams (see Muskat 1937, Figure 103 and p. 291)). We contend that irregular behavior can exist anywhere that boundary conditions change abruptly,
including the point \((b, h)\) of our problem. The water table streamline might leave this point slanting downward, in which case the streamlines would diverge and there would be no flow in the vicinity of the point.

Experimental data, e.g., sand tank work of Ehrenberger (1928, Figure 9) and electric analog work of Babbitt and Caldwell (1948, Figure 11), confirm this possibility. In work done at the hydraulic laboratory at Zurich and mentioned by Jaeger (1947, p. 164 and Figure 8) the water table seems to point straight down rather than move out horizontally into the flow medium at the entrance face of a dam. Relaxation solutions to the problem by Shaw and Southwell (1941, Figure 5), Boulton (1951, especially Figure 2), and Hall (1955, dashed lines on Figure 21) also show a downward effect. The effect if present is not easily seen in the work of Yang (1949).

In order to allow for the possibility of the water table having non-zero slope at \(r = b\) and also to allow for water tables of "any" shape we add a linear term and a sum of functions to Equation 2 to complete the representation of \(\sigma(r)\) as

\[
\frac{Z}{h} = \sigma(r) = 1 - \sigma_1 \left[1 - \sqrt{1 - \left(\frac{b - r}{a}\right)^2}\right] - \sum_{j=2}^{J} \sigma_j \left(\frac{b - r}{a}\right)^{j-1}
\]

where \(\sigma_1, \sigma_2, \ldots, \sigma_J\) are as yet undetermined constants. The linear term is the one with \(j = 2\).
SOLUTION TO THE POTENTIAL PROBLEM

At a given time the water table has a certain shape represented by the function \( \sigma(r) \). We wish to know the pattern of flow at this time. We assume that at the instant when we solve the potential problem a steady state condition exists in the flow medium. This steady state condition can be visualized by imagining a steady rainfall distributed over the soil surface in such a way as to hold the water table at the height \( h_0(r) \).

The mathematical problem is to find a hydraulic head function \( \phi(r,z) \) which satisfies Laplace's Equation and boundary conditions (1) through (5). We write the boundary conditions remembering that the reference level for hydraulic head is the height of water in the well, \( z = w \).

\begin{align*}
(1) \quad & \begin{cases} i = 0 & \phi(b,z) = y \\
 i = 1 & \frac{\partial \phi}{\partial r} \bigg|_{r = b} = 0 \end{cases} \quad 0 \leq z \leq h \\
(2) \quad & \frac{\partial \phi}{\partial z} \bigg|_{z = 0} = 0 \quad a \leq r \leq b \\
(3) \quad & \phi(a,z) = 0 \quad 0 \leq z \leq w \\
(4) \quad & \phi(a,z) = z - w \quad w \leq z \leq h_0(a) \\
(5) \quad & \phi(r,h_0(r)) = h_0(r) - w \quad a \leq r \leq b
\end{align*}

To avoid an excessive number of equations we now introduce some very compressed notation. We try to represent the function \( \phi(r,z) \) which satisfies Laplace's Equation and boundary conditions (1) through (5) in the
form

$$\phi(r,z) = y[l - \sum_{m=1}^{M} A_m R_m^{p_m}(r,z)]$$  \hfill (4)$$

Here the $A_m$ are constants and each superscript $p_m$ denotes either the superscript "B" or "C" where $R_m^{B}(r,z)$ and $R_m^{C}(r,z)$ are defined by

$$R_m^{B}(r,z) = [I_i(s_m b/h)K_0(s_m r/h) - (-1)^i K_i(s_m b/h)I_0(s_m r/h)] \cos(s_m z/h)$$

$$R_m^{C}(r,z) = [J_i(s_m b/h)Y_0(s_m r/h) - Y_i(s_m b/h)J_0(s_m r/h)] \cosh(s_m z/h)$$

The need for the notation $p_m = B$ or $p_m = C$ will become evident as we proceed. In these relations $I_i$ and $K_i$ are the ith order modified Bessel functions of first and second kind, $J_i$ and $Y_i$ are the ith order Bessel functions of the first and second kind, the $s_m$ are constants, and $i = 0$ or $i = 1$ is the same $i$ as we have introduced before to denote a constant head source or an impermeable barrier at $r = b$.

The function $\phi$ defined by Equation 4 satisfies Laplace's Equation and boundary conditions (1) and (2) regardless of the values of the $A_m$, $p_m$, and $s_m$. Boundary conditions (3), (4), and (5) have yet to be satisfied.

We see no straight forward method by which the $A_m$, $p_m$, and $s_m$ can be chosen so as to guarantee that boundary conditions (3), (4), and (5) will be satisfied. We must choose the $p_m$ and $s_m$ more or less by intuition. After the choice of $p_m$ and $s_m$ is made, the values of the $A_m$ that will make $\phi$ of Equation 4 satisfy boundary conditions (3), (4), and (5) best can be found in a straight forward manner which we outline below. We mean "best" in the sense of a least square fit of the boundary
conditions. Whether boundary conditions (3), (4), and (5) are actually satisfied depends on how the $p_m$ and $s_m$ are chosen.

Choice of the $A_m$

When Equation 4 is substituted into boundary conditions (3), (4), (5) we obtain

\[
(3) \quad \frac{\partial}{\partial y} \bigg|_{r=a} = 0 = 1 - \sum_{m=1}^{M} p_m^m(a, z) \quad 0 \leq z \leq w
\]

\[
(4) \quad \frac{\partial}{\partial y} \bigg|_{r=a} = \frac{z - w}{y} = 1 - \sum_{m=1}^{M} p_m^m(a, z) \quad w \leq z \leq h\sigma(a)
\]

\[
(5) \quad \frac{\partial}{\partial y} \bigg|_{z=h\sigma(r)} = \frac{h\sigma(r) - w}{y} = 1 - \sum_{m=1}^{M} p_m^m(r, h\sigma(r)) \quad a \leq r \leq b
\]

With the purpose of condensing these three relations into one equation (so that we can utilize known methods for finding the values of the $A_m$ that give the best least square fit) we define a dummy variable $x$ on the range $-h\sigma(a) \leq x \leq (b - a)$ as

\[
x = \begin{cases} 
    z - h\sigma(a) & 0 \leq z \leq h\sigma(a) \\
    r - a & a \leq r \leq b
\end{cases}
\]

We can visualize the variable $x$ as moving from the point $(a,0)$ where $x$ has value $-h\sigma(a)$, via boundaries (3) and (4), to the point $(a,h\sigma(a))$ where $x$ has value 0. On this range $x$ has value $z - h\sigma(a)$. Then $x$ moves via boundary (5) to the point $(b,h)$ where $x$ has value $b - a$. On this range $x$ has value $r - a$. We substitute $x$ so defined into the three boundary conditions and rearrange obtaining
\[
M \sum_{m=1}^{M} A_m R_m (a, h^{\sigma}(a) + x) = 1
\quad - h^{\sigma}(a) \leq x \leq [w - h^{\sigma}(a)]
\]

\[
M \sum_{m=1}^{M} A_m R_m (a, h^{\sigma}(a) + x) = 1 - \frac{h^{\sigma}(a) + x - w}{y}
\quad [w - h^{\sigma}(a)] \leq x \leq 0
\]

\[
M \sum_{m=1}^{M} A_m R_m (a + x, h^{\sigma}(a + x)) = 1 - \frac{h^{\sigma}(a + x) - w}{y}
\quad 0 \leq x \leq b - a
\]

We condense these three equations into one equation by defining \( u_m(x) \) and \( f(x) \) as

\[
u_m(x) = \begin{cases} 
R_m (a, h^{\sigma}(a) + x) & - h^{\sigma}(a) < x < 0 \\
R_m (a + x, h^{\sigma}(a + x)) & 0 < x < b - a
\end{cases}
\]

\[
f(x) = \begin{cases} 
1 & - h^{\sigma}(a) \leq x \leq [w - h^{\sigma}(a)] \\
1 - \frac{h^{\sigma}(a) + x - w}{y} & [w - h^{\sigma}(a)] \leq x \leq 0 \\
1 - \frac{h^{\sigma}(a + x) - w}{y} & 0 \leq x \leq b - a
\end{cases}
\]

so that the three equations become

\[
M \sum_{m=1}^{M} u_m(x) = f(x)
\quad - h^{\sigma}(a) < x < b - a
\]

This equation is satisfied if and only if boundary conditions (3), (4), and (5) are satisfied by \( \varnothing \) of Equation 4.

We define \( E \), the mean square error in satisfying the boundary conditions, as

\[
E = \frac{1}{h} \int_{-h^{\sigma}(a)}^{b-a} \left[ \sum_{m=1}^{M} u_m(x) - f(x) \right]^2 dx
\]
The values of the constants \( A_m \) that give the best least square fit, i.e. that make \( E \) smallest, may be found in two ways: (i) by solving the set of \( M \) simultaneous linear equations commonly called the normal equations

\[
\frac{\partial E}{\partial A_n} = 0 \quad n = 1, 2, \ldots, M
\]

or (ii) by using the Gram-Schmidt orthonormalization process as developed for easy calculation by Powers, Kirkham, and Snowden (1967) and computerized by Boast (1969, pp. 22-29 and 67-71). The two methods for finding the best values of \( A_m \) give identical results. The methods determine the best values of \( A_m \), that is, they minimize the error \( E \). But this is no guarantee that our solution satisfies boundary conditions (3), (4), and (5), that is, that \( E \) approaches zero as we increase the number of terms \( M \).

To see whether \( E \) approaches zero for large \( M \) we can use Bessel's inequality (which is easily calculated if the Gram-Schmidt orthonormalization method is used). Bessel's inequality in our notation is

\[
\frac{1}{M} \sum_{m=1}^{M} E_m^2 D_m \leq \frac{1}{h} \int_{-\sigma(a)}^{b-a} f^2(x) dx
\]

(see Boast, 1969, p. 32) where the constants \( D_m \) and \( E_m \) (not to be confused with the error \( E \)) are as in Powers, Kirkham, and Snowden (1967, Equations 25 and 45). The inequality becomes an equality for large \( M \) if and only if the error \( E \) approaches zero for large \( M \). In Equation 5, \( E_m \) and \( D_m \) depend on \( u_m(x) \) and if we do not choose \( p_m \) and \( s_m \) in \( u_m(x) \) correctly then \( E \) does not approach zero and boundary conditions (3), (4), and (5) are not satisfied.
Either the Gram-Schmidt orthonormalization method or the normal equations method requires evaluation of some integrals which are denoted $w_m$ and $u_{mn}$

\[
w_m = \frac{1}{h} \int_{-h\sigma(a)}^{b-a} f(x)u_m(x)dx \quad m = 1, 2, \ldots, M
\]

\[
u_{mn} = \frac{1}{h} \int_{-h\sigma(a)}^{b-a} u_m(x)u_n(x)dx \quad n = 1, 2, \ldots, m, m = 1, 2, \ldots, M
\]

For evaluation we break the integrals in the definitions of $w_m$ and $u_{mn}$ into two intervals $\int_{-h\sigma(a)}^{b-a} = \int_{-h\sigma(a)}^{0} + \int_{0}^{b-a}$. We evaluate over the first interval analytically and over the second interval numerically (with 33 equally spaced points) using Simpson's Rule.

Choice of the $p_m$ and $s_m$

In general we see no method for choosing the values of $p_m$ and $s_m$ which will guarantee that the function $\phi$ of Equation 4 satisfies boundary conditions (3), (4), and (5). However, in one case, that of a flat water table at $z = h\sigma(r) = h$, the solution is found simply (Kirkham and Van Bavel, 1948) by choosing $p_m = B$ and $s_m = (2m - 1)\pi/2$ for all $m$. This choice guarantees that boundary condition (5) is satisfied because each term $R_m^B(r, z)$ in the summation of Equation 4 contains the factor $\cos[(2m - 1)\pi z/2h]$ which evaluated at $z = h$ is zero. Hence, $\phi$ evaluated at $z = h$ becomes $\phi = y[1 - \sum_{m=1}^{M} R_m^B(r, h)] = y$ as demanded by boundary condition (5).
In order to satisfy boundary conditions (3) and (4) we define \( a_m \) by

\[
a_m = A_m \left[ I_1(s_m b/h)K_0(s_m a/h) - (-1)^i K_i(s_m b/h)I_0(s_m a/h) \right] \tag{6}
\]

Equation 4 evaluated at \( r = a \) becomes

\[
\frac{\partial \phi}{\partial y} \bigg|_{r=a} = 1 - \sum_{m=1}^{M} a_m \cos \left( \frac{(2m-1)\pi}{2h} \right) y
\]

so that boundary conditions (3) and (4) can be satisfied by choosing the \( a_m \) [using quarter range cosine theory (Kirkham, 1965, Equation 24)] as

\[
a_m = \frac{1}{(2m-1)\pi} \sin \left( \frac{(2m-1)\pi}{2} \right) - \frac{2}{h} \int_{0}^{h} \frac{\partial \phi}{\partial y} \bigg|_{r=a} \cos \left( \frac{(2m-1)\pi}{2h} \right) z \, dz
\]

\[
= \frac{8h}{(2m-1)^2\pi^2 y} \cos \left( \frac{(2m-1)\pi w}{2h} \right) \tag{7}
\]

These values of \( a_m \) are substituted in Equation 6 and the constants \( A_m \) are solved for algebraically. An equivalent method for finding the \( A_m \) would be to use the error minimization method of the previous section, but since in the special case \( \sigma(r) = 1 \) we can use Fourier series theory we can know that the error \( E \) is not only being minimized but that in fact \( E \) approaches zero as the number of terms increases, i.e. our solution is "converging" to the boundary conditions on boundaries (3) and (4).

We return now to the problem of how to satisfy boundary conditions (3), (4), and (5) when the water table is not flat. For concreteness we consider a problem with specific dimensions: \( a/h = 0.1, b/h = 1.6 \) and \( w/h = 0.5 \), with a constant head source at \( r = b(i = 0) \), and with boundary (5) being represented by Equation 3 where we take two terms \( J = 2 \),
and take $\sigma_1 = 0.2862$ and $\sigma_2 = 0.0098$.

We present three methods for the choice of the $p_m$ and $s_m$ (from these the $A_m$ follows). The effectiveness of the three methods for solving the above problem is evaluated by calculating the left side of Bessel's inequality for $M = 1, M = 2, \ldots$ and using a technique described below to see whether the left side converges to the value of the right side as $M$ increases.

In the first method for choosing the constants $p_m$ and $s_m$ we let $p_m = B$ for all $m$, that is, we use functions of the same form as we used for the flat water table case. Now, however, we cannot satisfy boundary condition (5) simply by proper choice of the constants $s_m$. What is worse we have no way of deciding what values of $s_m$ we should use. Some choices for the values of the $s_m$ are obviously unwise, e.g. $s_m = d(2m-1)\pi/2$ where $d$ is any constant less than one, because this makes $\phi = 1$ at $z = dh$, an obvious physical impossibility. There remain, however, infinite possibilities for the method of choice of the $s_m$ which are not so obviously unwise as the above example.

A natural choice for the values of the $s_m$ is $s_m = (2m-1)\pi/2$, the same values that we used for the problem with a flat water table. We designate as METHOD I the choice $p_m = B, s_m = (2m-1)\pi/2$ for all $m$. Using these values for $p_m$ and $s_m$ we have calculated the best least square fit to boundary conditions (3), (4), and (5) for $M = 1, M = 2, \ldots M = 24$ with the Gram-Schmidt orthonormalization method. We have calculated the value of the right side of Bessel's inequality as $0.6856582$ and values of the left side for $M = 1, M = 2, \ldots M = 24$, as plotted in Figure 2. To see whether the left side of Bessel's inequality converges
to the right side we plot values of the left side versus $1/M$ and extrapolate values for $M$ finite to the vertical axis where $M$ is infinite ($1/M = 1/\infty = 0$).

Figure 2. Bessel's inequality evaluation of METHOD I and METHOD II

It is not certain whether the left side of Bessel's inequality is converging to the right side as $M$ increases, and hence whether the error $E$ in satisfying boundary conditions (3), (4), and (5) is going to zero. The value of METHOD I for solving the boundary value problem is in doubt.

In the second method for choosing the constants $p_m$ and $s_m$ we use both terms with $p_m = B$ and $p_m = C$. Let us denote the number of terms with $p_m = B$ by $M_B$ and the number of terms with $p_m = C$ by $M_C$, then the total number of terms $M$ is given by $M = M_B + M_C$. For the terms with $p_m = B$ we choose the $s_m$ as $s_m = \pi/2, 3\pi/2, 5\pi/2, \ldots$, $(2M_B - 1)\pi/2$. For the terms with $p_m = C$ we choose the $s_m$ to be the first $M_C$ positive zeroes of the expression

$$J_i(s_m b/h)Y_0(s_m a/h) - Y_i(s_m b/h)J_0(s_m a/h)$$
so that \( R_m^C(a, z) = 0 \). The motivation for choosing the \( p_m \) and \( s_m \) in this way may be more clear after we present METHOD III.

We have calculated the best least square fit to boundary conditions (3), (4), and (5) using 12 terms with \( p_m = B \) (\( M_B = 12 \)) and 12 terms with \( p_m = C \) (\( M_C = 12 \)) alternating terms of the two types (\( p_1 = B \), \( p_2 = C \), \( p_3 = B \), \( p_4 = C \), ... ) to give 24 terms as before. The values of the left side of Bessel's inequality shown in Figure 2 seem to converge to the value of the right side. The discrepancy between the two sides of Bessel's inequality is much less, that is the error in satisfying boundary conditions (3), (4), and (5) is much less using METHOD II than using METHOD I. Hence, we reject METHOD I.

To see if unbalancing the number of terms in \( M_B \) and \( M_C \) improves or worsens convergence we have calculated the best least square fit to boundary conditions (3), (4), and (5) with \( M_B = 8 \) and \( M_C = 16 \). The values of the left side of Bessel's inequality are somewhat worse (less) than for \( M_B = 12 \) and \( M_C = 12 \), though after the first few terms the difference is not enough to be seen in Figure 2. Despite the fact that letting \( M_B \) be equal to \( M_C \) appears to be a more efficient way to choose \( M_B \) and \( M_C \), most of the calculations we present have been done with \( M_B = 8 \) and \( M_C = 16 \). With these values of \( M_B \) and \( M_C \) equality in Bessel's relation is almost achieved.

The remainder of this chapter is a digression where we attempt to motivate the choice of the \( p_m \) and \( s_m \) in METHOD II. Since \( R_m^C(a, z) \) equals zero identically in METHOD II, the only terms which contribute to satisfying boundary conditions (3) and (4) are the ones with \( p_m = B \). If boundary condition (4) covered the range \( w \leq z \leq h \) rather than
$w \leq z \leq h\sigma(a)$ then the best values of the $A_m$ for a least square fit of boundary conditions (3) and (4) would be given in Equations 6 and 7 from Fourier series theory. Values of $A_1$ calculated by METHOD I and METHOD II for $M = 1, 2, \ldots, 24$ are shown in Figure 3.

![Figure 3. Values of $A_1$ chosen by METHODS I and II](image)

The values of $A_1$ calculated in METHOD II are quite close to the value of $A_1$ given by Equations 6 and 7, 0.175, shown in Figure 3 by a dashed line. This could indicate that the terms with $p_m = B$ are not really contributing to the satisfying of boundary condition (5) and hence the coefficients $A_m$ of these terms might just as well be chosen directly by Equations 6 and 7, the quarter range Fourier cosine series, rather than through the Gram-Schmidt orthonomalization method.

In METHOD III we choose the constants $p_m$ and $s_m$ as in METHOD II but we choose the $A_m$ for terms with $p_m = B$ by Equations 6 and 7. We choose $M_B$ large enough that boundary conditions (3) and (4) are
satisfied as closely as we desire. All that remains is to choose the rest of the \( A_m \) (those with \( p_m = C \)) so boundary condition (5) is satisfied. Boundary conditions (3) and (4) remain satisfied regardless of the choice of these \( A_m \) since \( R_m^C(a,z) = 0 \). We choose the other \( A_m \) of METHOD III by basically the same technique as in METHODS I and II. By this procedure we place all of the \( p_m = C \) terms in Equation 4 first, write the sum in Equation 4 in two parts, and evaluate at \( z = \eta(r) \) obtaining

\[
\phi(r, \eta(r)) = y[1 - \sum_{m=1}^{M} A_m R_m^C(r, \eta(r)) - \sum_{m=M_C+1}^{M} A_m R_m^B(r, \eta(r))]
\]

\[a < r < b\]

We substitute for \( \phi(r, \eta(r)) \) using boundary condition (5), divide the equation by \( y \), and rearrange to find

\[
\sum_{m=1}^{M} A_m R_m^C(r, \eta(r)) = \frac{\eta(r) - w}{y} + 1 - \sum_{m=M_C+1}^{M} A_m R_m^B(r, \eta(r))
\]

\[a < r < b\]

Notice that the \( A_m \) on the right are different than the \( A_m \) on the left. We have chosen values of the \( A_m \) on the right using Equations 6 and 7 already. The only unknowns here are the \( A_m \) of the left side so we can define \( f(r) \), \( a < r < b \), as the right side of this equation and define \( u_m(r) = R_m^C(r, \eta(r)), a < r < b \). Then either the normal equations method or the Gram-Schmidt orthonormalization method of the last section can be used to find the \( A_m \). Here the integrals in the definition of \( E \), in the right side of Bessel's inequality, and in the definitions of \( w_m \) and \( u_{mn} \) have dummy variable \( r \) and end points \( a \) and \( b \).
For $M_B = 8$ we calculate the best least square fit to boundary condition (5) using 16 terms ($M_C - 16$). The values of the left side of Bessel's inequality shown in Figure 4 seem to converge to the value (star) of the right side. Hence, METHOD III seems to work.

![Graph showing the right side of Bessel's inequality](image)

Figure 4. Bessel's inequality evaluation of METHOD III with $M_B = 8$

We cannot directly compare METHOD III to the other methods as we were able to compare METHOD I and METHOD II in Figure 2 because the least square fit in METHOD III is over boundary (5) only, whereas with the other methods the least square fit is over all three boundaries (3), (4), and (5). METHOD III could be considered superior to METHOD II for two reasons: (i) choice of the $p_m$ and $s_m$ has some justification in METHOD III but very little justification in METHOD II (for example the choice of the $s_m$ (for terms with $p_m = C$) so that $R_m(a, z) = 0$ is necessary in METHOD III but is completely without justification in METHOD II) and (ii)
computation time for a given number of terms, $M_B$ and $M_C$, is lower using METHOD III than METHOD II. However, we see in the next chapter that METHOD II is in fact very much superior to METHOD III. The main purpose of presenting METHOD III at all is to motivate the choice of $p_m$ and $s_m$ in METHOD II. Indeed we conceived METHOD III first and found it to be a fairly poor method for solving the potential problem. We thought that METHOD II was only of academic interest and it was almost by accident that we discovered that it is greatly superior to METHOD III.

We remark that there may be many methods for choosing the $s_m$ in METHOD II. We arrived at the method we used to get the $s_m$ in such a circuitous manner that we wonder if almost any choice of the $s_m$ (so long as the choice is not "obviously unwise") might not work just as well as the choice we make.
The function \( h(r,t) \) which we use to represent the heights of points on the water table changes with time. We denote the height of the water table at time \( t \) by \( h(r,t) \). We remember that the depth of water in the well, \( y \), does not change with time.

We have from the previous chapter the solution to the potential problem for a flow medium with upper boundary, at any time \( t_0 \) (\( t_0 \) is not necessarily zero), of shape \( z = h(r,t_0) \). We wish to calculate the rate of fall of the water table at any radius \( r \). Kirkham and Gaskell (1950, Equation 6) give a formula for the rate of fall which, in our notation, is

\[
\frac{\partial h(r,t)}{\partial t} \bigg|_{t=t_0} = \frac{k}{f} \left[ \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial r} \frac{h}{r} \frac{\partial \sigma(r,t_0)}{\partial r} \right]
\]

where \( f \) is the drainable porosity. If the water table continues to fall at this rate for a time \( \Delta t \) (this is a valid assumption if \( \Delta t \) is small enough) then at time \( t = t_0 + \Delta t \) the height of the water table is given by

\[
z = h(r,t_0 + \Delta t) = \sigma(r,t_0) - \Delta t \frac{k}{f} \left[ \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial r} \frac{h}{r} \frac{\partial \sigma(r,t_0)}{\partial r} \right]
\]

If we divide this equation by \( h \) and define a variable \( \tau = \frac{tk}{hf} \) which we designate as "dimensionless time", then

\[
\sigma(r,t_0 + \Delta \tau) = \sigma(r,t_0) - \Delta \tau \left[ \frac{\partial \varphi}{\partial z} - \frac{\partial \varphi}{\partial r} \frac{h}{r} \frac{\partial \sigma(r,t_0)}{\partial r} \right]
\] (8)

We call the factor which is multiplied by \( \Delta \tau \) in Equation 8 the dimensionless rate of water table fall, \( \varphi(r,\tau)/\tau \). Equation 8 is completely
dimensionless, since \( \sigma \) is the dimensionless water table height.

We solve the potential problem with a constant head source at \( r = b \) (\( i=0 \)) and no rainfall. We wish to follow the shape of the water table \( h(r, \tau) \) as \( \tau \) goes to infinity. We do this by considering a finite number of evenly spaced time steps with \( \Delta \tau = 0.05 \). We first calculate the initial (\( \tau = 0 \)) pattern of flow and then calculate the dimensionless rate of water table fall at \( \tau = 0 \). We assume that this rate of water table fall continues unchanged for a dimensionless time \( \Delta \tau = 0.05 \) and calculate from Equation 8 the position of the water table at dimensionless time \( \tau = 0.05 \). We solve the potential problem for the pattern of flow at \( \tau = 0.05 \) and again compute the dimensionless rate of water table fall. We repeat this process 39 more times until \( \tau = 2.0 \). At \( \tau = 2.0 \) the water table shape is very near to equilibrium.

At any one of the time steps we can calculate the hydraulic head function \( \phi \) at any point in the flow medium. Using Table 1 of Zaslavsky and Kirkham (1965) we can find the stream function \( \psi \) which corresponds to our hydraulic head function \( \phi \) and calculate the stream function at any point in the flow medium. Flownets for \( \tau = 0.0, \tau = 0.5, \tau = 1.0, \) and \( \tau = 2.0 \) are shown in Figures 5, 6, 7, and 8.

The flownets show that the boundary conditions are satisfied. The boundary at \( r = b \) is an equipotential \( \phi = y \). The impermeable boundary at \( z = 0 \) is a streamline. The boundary below the level of water in the well is an equipotential \( \phi = 0 \). The hydraulic head on the surface of seepage in the well and on the water table is equal to the height above the reference level for head. Values of the stream function \( \psi \) (we choose the reference streamline \( \psi = 0 \) to be coincident with the bottom
Figure 5. Flownet for \( i = 0 \) at \( t = 0 \).
Figure 6. Flownet for \( i = 0 \) at \( \tau = 0.5 \)
Figure 7. Flownet for $i = 0$ at $\tau = 1.0$
Figure 8. Flownet for "i = 0" at $\tau = 2.0$
impermeable barrier) are expressed as fractions of the equilibrium rate of flow into the well $\psi_{eq}$. The equilibrium rate as given by Hantush (1962, Equation 2) is, in our notation,

$$\psi_{eq} = \pi k \frac{h^2 - w^2}{\ln(b/a)}$$

(9)

In Figure 8 we show with a dashed line the equilibrium water table shape calculated numerically by Hall (1955, Figure 21). Our solution is probably at fault where it differs with Hall's near the well. The streamline designated $\psi = \psi_{eq}$ flows into "dry" soil just outside the well which is physically impossible. The bad behavior of our solution near the well is a strong indication that it is our solution and not Hall's which is at fault near the well. Slow convergence of solutions to boundary value problems near points where the boundary conditions change abruptly is to be expected (cf. Powers, Kirkham, and Snowden, 1967, Figure 3). We suspect that by (i) improving the function we use to represent the shape of the water table, $\phi(r)$ of Equation 3, (ii) increasing the number of terms we take in Equation 4, (iii) increasing the number of points we take in the numerical integrations of the previous chapter, (iv) decreasing the length of the time steps $\Delta T$, or doing some combination of the above we could improve the behavior of our solution near the well.

We define a dimensionless stream function as $\psi/kya$. The equilibrium value $\psi_{eq}/kya$ of the dimensionless stream function for $a/h = 0.1$, $b/h = 1.6$, and $w/h = 0.5$ is 16.996 from Equation 9. Figure 9 shows values of $\psi/kya$ at points on the water table for $\tau = 0.0$, $\tau = 0.5$, $\tau = 1.0$, and $\tau = 2.0$. Initially ($\tau = 0$ curve in Figure 9) the stream...
function value near the well is greater than the equilibrium value and far from the well is less than the equilibrium value. With time the dimensionless stream function value approaches 17.0 all along the water table, that is, the water table becomes the equilibrium streamline.

As \( r/h \) approaches \( a/h = 0.1 \) the dimensionless stream function value should not decrease as it does in Figure 9. This is another symptom of the bad behavior of our solution near the well.

![Figure 9](image-url)

**Figure 9.** Dimensionless stream function values at points \((r, h\sigma(r))\) on the water table for \(i = 0\) (dashed line \(\psi/k\nu = 16.996\) is the equilibrium value given by Equation 9)
Calculated values of the dimensionless rate of water table fall for $\tau = 0$ and $\tau = 0.5$ are shown in Figure 10. For $\tau = 0.5$ we show the rate of fall $\partial \sigma(r, \tau) / \partial \tau$ calculated both by METHOD II and METHOD III. METHOD II is seen to be much better.

The reason why the calculated values of the rate of water table fall (even when calculated by METHOD II) are oscillatory is that these values, $\partial \phi / \partial z - \partial \phi / \partial z - \partial \phi / \partial r \partial \sigma(r, \tau_0) / \partial r$, are derivatives of $\phi$. The values of $\phi$ which we calculate are very slightly oscillatory and these oscillations show up much more in their derivatives. If the values of the dimensionless rate of water table fall shown in Figure 10 are used in Equation 8 to find the next ($\tau = 0.55$) water table position, the new position calculated is oscillatory. Hence, we approximate this new water table position by a few (3 or 4) terms of the function given in Equation 3. We use a least squares fit (not to be confused with the least square approximation which is central to the previous chapter) for the approximation.

![Figure 10. Dimensionless rate of water table fall](image-url)
It is the water table position defined by this approximation that we use, not the water table position calculated directly from Equation 8.

Finally we give our solution to the problem with boundary (1) at \( r = b \) taken as an impermeable barrier (\( i = 1 \)) with \( a/h = 0.1 \), \( b/h = 1.6 \), and \( w/h = 0.5 \). We remember that for \( i = 1 \) we (rather arbitrarily) assume that rainfall, falling at an equal rate per unit area everywhere over the soil surface but varying with time, keeps the height of the water table at the radius \( r = b \) but not at other radii at the initial height of the water table, \( z = h \). This rainfall should not be confused with the rainfall presented at the beginning of the third chapter. That rainfall was merely a device for visualizing how a falling water table flow problem can be treated as a series of steady state flow problems. The rainfall we introduce now for the times \( \Delta \tau \) actually occur while the water table falls. At equilibrium the only source of water to the well is this rainfall and since this is distributed evenly over the soil surface the value of the stream function at any point \((r, h\sigma(r))\) on the water table should be proportional to the area of soil surface of radius larger than the radius of the point

\[
\psi \propto \pi b^2 - \pi r^2 \propto (b/h)^2 - (r/h)^2
\]

Figure 11 shows values of the dimensionless stream function \( \psi/kya \) at points on the water table versus \((b/h)^2 - (r/h)^2\) for \( \tau = 0.0 \), \( \tau = 0.5 \), \( \tau = 1.0 \), and \( \tau = 2.0 \). Note that the abscissa scale is not linear.

With time the fit to a straight line in Figure 11 is seen to improve. After \( \tau = 1.0 \) the desired fit to a straight line is not visibly improved, but behavior near the well similar to that seen in the case of a
constant head source at \( r = b \) \( (i = 0) \) begins to appear so we stop our solution at the dimensionless time \( \tau = 1.0 \). We show a flownet of \( \tau = 1.0 \) in Figure 12. We assume that the flow at \( \tau = 1.0 \) is the equilibrium flow \( \psi_{eq} \). Values of the stream function \( \psi \) are expressed as fractions of \( \psi_{eq} \). The flownet shows that the boundary conditions are
Figure 12. Flownet for "i = 0" at τ = 1.0
satisfied.

As one may quickly calculate, the annular areas between any two pairs of adjacent streamlines computed for the points where the streamlines intersect the water table are equal. The equipotential $\phi = y$ occurs only at points on the circle $r = b, z = h$. The streamline $\psi = \psi_{eq}$ occurs only at points on the circle $r = a, z = h\sigma(a) = 0.72h$. 
SUMMARY

We solve analytically the steady state boundary value problem of potential flow to a well where the upper boundary of the flow medium is a curved surface. Starting with the water table in a flat initial position we let the water table fall for short periods of time and solve the potential problem after each time step. With time the water table approaches an equilibrium position.

First we find the equilibrium position of the water table for a flow medium with a constant head source of water at the outer radial boundary of the flow medium. The equilibrium flow rate into the well agrees with known theory. The equilibrium water table shape agrees closely with the shape found by another using numerical techniques to solve the flow problem.

Second we find the equilibrium position and the equilibrium flow rate when the outer radial boundary is impermeable and rainfall is applied to the soil surface. This solution is probably of more value than the solution to the constant head problem because it is similar to the problem of flow in a well field where the only source of recharge is rainfall.


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