1970

A plane thermoelastic crack problem

Kenneth Edgar Richards Jr.

Iowa State University

Follow this and additional works at: https://lib.dr.iastate.edu/rtd

Part of the Applied Mechanics Commons

Recommended Citation

https://lib.dr.iastate.edu/rtd/4354

This Dissertation is brought to you for free and open access by the Iowa State University Capstones, Theses and Dissertations at Iowa State University Digital Repository. It has been accepted for inclusion in Retrospective Theses and Dissertations by an authorized administrator of Iowa State University Digital Repository. For more information, please contact digirep@iastate.edu.
RICHARDS, Jr., Kenneth Edgar, 1942-
A PLANE THERMOELASTIC CRACK PROBLEM.

Iowa State University, Ph.D., 1970
Engineering Mechanics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED
A PLANE THERMOELASTIC CRACK PROBLEM

by

Kenneth Edgar Richards, Jr.

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major Subject: Engineering Mechanics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Ames, Iowa

1970
# TABLE OF CONTENTS

I. INTRODUCTION 1

II. PROBLEM DESCRIPTION 6

III. EQUATIONS AND THERMOELASTIC EQUILIBRIUM 11

IV. SOLUTION 17
   A. Boundary Conditions 17
   B. Solid Strip 20
      1. Temperature problem 20
      2. Stress problem 21
      3. Example 27
   C. Cracked Strip 33
      1. Temperature problem 33
      2. Stress problem 41

V. NUMERICAL SOLUTION 52
   A. Preliminary Considerations 52
   B. Conditioning of Equations 54
   C. Example 68

VI. RESULTS 79

VII. LITERATURE CITED 108

VIII. ACKNOWLEDGMENTS 111

IX. APPENDIX A 112
   A. Remarks Regarding Integrals Used 112
   B. Series Representations 119

X. APPENDIX B 137
I. INTRODUCTION

Considerable attention has been devoted to the study of stresses in the vicinity of cracks of various geometries subjected to several types of loadings, and lying in regions of differing shapes. A complete review of the relevant literature in this general field would produce a sizeable volume. We are primarily interested in the study of cracks associated with media in which thermal gradients are present, an area which has received considerably less attention. We therefore restrict the discussion of previous studies of isothermal crack problems to the efforts of a few authors whose work is representative of the field in general.

A major contribution to the analytic study of crack problems is Ian N. Sneddon. At a series of lectures presented at North Carolina State College in 1961 (1), Sneddon outlined the major contributions (including his own) of numerous authors to the study of various crack problems, typical examples of which are briefly described here.

Five solutions for the classical Griffith crack problem (a crack occupying the segment x=0, -c<y<c, for example), where the crack is opened by prescribed internal pressure, are discussed for the case where the crack is located in an infinite thin plate. The methods of solution, for the most part, make use of the elastic solution for an elliptical hole
loaded by internal pressure. The Griffith crack is then obtained by examining the solution as the minor axis of the ellipse tends to zero.

Sneddon's solution involves the reduction of the problem to that of a pair of dual integral equations. The integral equations are derived from potential functions, and can be solved if the pressure applied to the surface of the crack can be expressed in terms of polynomials in the coordinate along the crack.

Several axially symmetric three dimensional problems are discussed also. Typically, the geometry associated with these problems is that of a thin, penny-shaped crack imbedded in an infinite medium. Sneddon considers two cases of particular interest where the internal pressure, and either the heat flux or the temperature on the crack surface, are prescribed functions of the radial distance from the center of the crack. He then finds the results necessary to determine stresses and displacements for zero internal pressure, and constant heat flux on crack surface temperature, but he does not calculate the resulting temperature distribution.

Lowengrub (2) considers the problem of a penny shaped crack of unit radius contained in an infinite slab of thickness $2\delta$ loaded by internal pressure. He reduces the problem to the solution of a Fredholm integral equation, for which he obtains an approximate solution for large values of $\delta$. 
Lowengrub (3) also considers the two-dimensional analogue of the slab by solving for the stresses in a strip containing a line crack loaded by internal pressure. His solution is restricted to wide strips.

Several papers of interest have appeared recently. Sneddon (4) considers the problem of determining the stress which must be applied to the surface of a Griffith crack in an infinite medium to cause the crack to open to a prescribed shape. Olesiak and Sneddon (5) consider the analogous problem for the penny-shaped crack in an infinite medium. In both cases, the problem is reduced to the solution of a pair of dual integral equations.

Srivastava and Palaiya (6) consider the problem of the distribution of thermal stress in a semi-infinite elastic solid containing a penny-shaped crack, the plane of the crack being parallel to the edge of the solid. They consider two different mechanical boundary conditions on the surface of the solid: a stress free boundary, and one fully clamped. The thermal conditions considered are either the temperature or heat flux being prescribed on the surface of the crack. In all cases, the problems are reduced to the solutions of dual integral equations. They then reduce the dual integral equations to a Fredholm equation, which is then solved by an iterative scheme which is valid if the crack is far from the boundary. They solve the Fredholm equations numerically for
the case where the plane of the crack is about one crack radius from the surface of the solid.

Das (7) considers the problem of a long circular cylinder containing a concentric penny-shaped crack perpendicular to the axis of the cylinder. The temperature on the surface of the crack is prescribed, and the cylinder is free to radiate heat from its surface. He reduces the problem to the solution of two Fredholm equations, which he solves numerically for a constant crack temperature.

Kassir (8) solves the problem of the distribution of thermal stresses around an elliptical crack in an infinite solid for the special cases where either the temperature or the heat flux on the crack is given in terms of a polynomial in \( x \) and \( y \), where the \( x-y \) plane is the plane of the crack. To obtain the solution, he looks for potential functions from which the temperature, stresses, and displacement may be derived, and which reduce to polynomials which are of the same degree as those prescribing the boundary conditions.

Svoboda (9) has conducted an experimental investigation of the influence of thermal stresses on the stability of a crack. He compares his experimental results for thin, short cracks in large plexiglass sheets with a solution he obtains by complex variable methods for the case of a uniform flow of heat disturbed by an insulated crack. He concludes that a real crack does offer a great resistance to heat flow, but he
suggests that assuming an insulated condition on the crack surface may cause as much as a ten percent difference in theoretical and experimental values. His data is somewhat scattered and does not support any stronger conclusions.

In this thesis, we consider a plane thermoelastic crack problem with a radiation boundary condition imposed on the crack surface. This choice of a more general boundary condition was actually made prior to the publication of Svoboda's paper mentioned previously, but it was made with certain reservations on the part of the author.
II. PROBLEM DESCRIPTION

The infinite elastic strip shown in Figure 1 contains a line crack \(-a \leq y \leq a\) parallel to, and equidistant from, the edges \(x = \pm w\). The strip, initially stress-free and at a constant temperature, is clamped along the edges to prevent movement perpendicular to the edges, but to allow free movement in a direction parallel to the edges. The crack itself will not support either normal or shear stresses.

The faces of the strip are thermally insulated. Thermal radiation is free to occur from the edges of the crack in the plane of the strip, and the temperature of the environment into which the crack is radiating is presumed known.

Known temperature distributions are applied and maintained on the edges of the strip until the system reaches thermal and mechanical equilibrium. We wish to determine the temperature, stress, and displacement fields at equilibrium.

The symmetry of the mechanical boundary conditions suggests the possibility of dividing the strip into subregions, the lines of division being the coordinate axes, and obtaining a solution for each subregion. This approach has the advantage that each subregion is simply-connected, but it has the disadvantage that it is not possible to specify the mechanical boundary conditions along the lines of division for arbitrary edge temperature.
Figure 1. The cracked strip
To overcome this difficulty, we may proceed as follows. Consider, for purposes of discussion, temperature distributions $T_L(y)$ and $T_R(y)$ which are even functions of $y$ and are applied at the left and right edges of the strip respectively. We need then consider only the portion of the strip $y > 0$.

Let

$$T_1(y) = \frac{1}{2}(T_L(y) + T_R(y)) \quad (2.1)$$

and

$$T_2(y) = \frac{1}{2}(T_L(y) - T_R(y)) \quad (2.2)$$

It is apparent that we may use superposition to obtain the solution of the original temperature problem by first solving the symmetric problem ($T_1$ applied at both edges), then the antisymmetric problem ($T_2(y)$ applied at the left edge and $-T_2(y)$ applied at the right edge), and then adding the two solutions, providing that we specify the thermal boundary conditions on the crack to be such that, when combined, their sum is equivalent to the original radiation condition.

There evidently is no unique decomposition for the radiation condition. One which appears reasonable to the author is to specify a radiation condition for both the symmetric and antisymmetric problems, with the temperature
of the environment into which the crack is radiating to be zero for the antisymmetric problem and to be equal to the known environmental temperature for the symmetric problem. Although this formulation may be more complicated than other possible alternatives, it does offer the advantage that it provides for the same type of thermal phenomenon in both the symmetric and antisymmetric problems.

In the case of the symmetric problem, we need consider only the portion \(-w < x < 0\). By symmetry, the appropriate boundary conditions on the line of division for \(y > a\) are zero temperature gradient, zero shear stress, and zero displacement in the x direction.

We are unable to fully specify the boundary conditions on the line of division for the antisymmetric problem. It is clear that the temperature must be zero for \(y > a\), and the displacement in the y direction must vanish. It is not possible, however, to specify the normal stress, normal displacement, or shear stress. It is possible, at least in principle, to express one of these quantities in terms of an unknown function, solve for the stresses and displacements in the two adjacent subregions in terms of the unknown function, and then use the physical requirement that the stresses and displacements in the two adjacent subregions must match at the common boundary to determine the unknown function, thereby completing the solution.
The problem becomes even more complicated if we remove the restriction that the applied temperatures be even in y, for the type of matching process just described would have to be performed along the x axis also.

The essential elements for the solution of any of the subproblems described above are identical, except for the matching process. With the goal of obtaining numerical results in addition to a formal solution, we restrict ourselves to the problem of symmetric boundary temperatures which are even functions of y. We consider the strip to be a representative cross-section of a solid whose dimension in the z direction perpendicular to the plane is large, and assume that all variables are independent of z. The plane strain formulation is then appropriate. We further assume that the equations of linear elasticity apply, and we are thus restricted to considering temperature ranges sufficiently small so that the elastic and thermal properties of the strip remain essentially constant.
III. EQUATIONS OF THERMOELASTIC EQUILIBRIUM

It is convenient to describe the steady-state temperature $\bar{T}(x, y, z)$ in a body by

$$\bar{T}(x, y, z) = T_0 \{1 + \theta(x, y, z)\}, \quad (3.1)$$

where $T_0$ is taken as a constant reference temperature of zero stress and strain. Such a temperature distribution must obey Laplace's equation

$$\nabla^2 \bar{T} = 0, \quad (3.2)$$

or, equivalently,

$$\nabla^2 \theta = 0. \quad (3.3)$$

In addition to Laplace's equation, the appropriate field equations include the equilibrium equations which, in the absence of body forces, are:

$$\frac{\partial \bar{T}_{xxx}}{\partial x} + \frac{\partial \bar{T}_{xy}}{\partial y} + \frac{\partial \bar{T}_{xz}}{\partial z} = 0 \quad (3.4a)$$

$$\frac{\partial \bar{T}_{xy}}{\partial x} + \frac{\partial \bar{T}_{y}}{\partial y} + \frac{\partial \bar{T}_{yz}}{\partial z} = 0 \quad (3.4b)$$

$$\frac{\partial \bar{T}_{xz}}{\partial x} + \frac{\partial \bar{T}_{yz}}{\partial y} + \frac{\partial \bar{T}_{z}}{\partial z} = 0, \quad (3.4c)$$
where \( \bar{\sigma}_{x} \), \( \bar{\sigma}_{y} \), \( \bar{\sigma}_{z} \), and \( \bar{\tau}_{xy} \), \( \bar{\tau}_{xz} \), \( \bar{\tau}_{yz} \) are the normal and shear stress components respectively.

The Duhamel-Neumann stress-strain relations (10, p. 359) complete the set of governing differential equations. They are:

\[
\frac{\partial \bar{\sigma}_{x}}{\partial x} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + \lambda \left( \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - \alpha (3\lambda + 2\mu) \frac{T_0}{\partial \theta} \tag{3.5a}
\]

\[
\frac{\partial \bar{\sigma}_{y}}{\partial y} = (\lambda + 2\mu) \frac{\partial^2 v}{\partial y^2} + \lambda \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) - \alpha (3\lambda + 2\mu) \frac{T_0}{\partial \theta} \tag{3.5b}
\]

\[
\frac{\partial \bar{\sigma}_{z}}{\partial z} = (\lambda + 2\mu) \frac{\partial^2 w}{\partial z^2} + \lambda \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \alpha (3\lambda + 2\mu) \frac{T_0}{\partial \theta} \tag{3.5c}
\]

\[
\frac{\partial \bar{\tau}_{xy}}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right) \tag{3.5d}
\]

\[
\frac{\partial \bar{\tau}_{xz}}{\partial x} = \mu \left( \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} \right) \tag{3.5e}
\]

\[
\frac{\partial \bar{\tau}_{yz}}{\partial y} = \mu \left( \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{3.5f}
\]

where \( \bar{u} \), \( \bar{v} \), and \( \bar{w} \) are the displacements in the \( \bar{x} \), \( \bar{y} \), and \( \bar{z} \) directions, \( \lambda \) and \( \mu \) are the Lamé constants, and \( \alpha \) is the coefficient of linear thermal expansion.

We confine our attention here to the case of plane strain. Thus, \( \bar{w} \) is identically zero, and all quantities are independent of \( \bar{z} \).

It is convenient to write the equations in dimensionless form. Take the shear modulus \( \mu \) as the unit of stress and a
characteristic length \( l \), appropriately chosen for the particular problem, as the unit of length, i.e.,

\[
\sigma_x = \frac{\bar{\sigma}_x}{\mu}
\]

and

\[
x = \frac{\bar{x}}{l},
\]

with similar expressions for the remaining variables.

The equations then become:

\[
\nabla^2 \theta = 0 \quad (3.6)
\]

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad (3.7a)
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad (3.7b)
\]

\[
\sigma_x = \beta^2 \frac{\partial u}{\partial x} + (\beta^2 - 2) \frac{\partial v}{\partial x} - b \theta, \quad (3.8a)
\]

\[
\sigma_y = \beta^2 \frac{\partial v}{\partial y} + (\beta^2 - 2) \frac{\partial u}{\partial y} - b \theta, \quad (3.8b)
\]

\[
\tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad (3.8c)
\]

where

\[
\beta^2 = \frac{\lambda + 2\mu}{\mu} = \frac{2(1-v)}{1-2v}, \quad (3.9)
\]

\[
b = \alpha (3\beta^2 - 4) T_0, \quad (3.10)
\]
and \( \nabla^2 \) becomes the two-dimensional Laplace operator, with respect to \( x \) and \( y \).

In Equation 3.9, \( \nu \) is Poisson’s ratio. The absence of the bars in Equations 3.6 through 3.8 indicates that the non-dimensionlization previously described has been performed.

We note that \( \tau_{xz} \) and \( \tau_{yz} \) are identically zero under the plane strain assumption, and \( \sigma_z \) must be calculated from Equation 3.5c after \( u \) and \( v \) are determined.

Sneddon (11) has introduced a particularly convenient method of solution for this system of equations by introducing three sufficiently continuous functions \( \chi(x,y) \), \( \phi(x,y) \) and \( \psi(x,y) \) and expressing the displacements and temperature by

\[
\begin{align*}
  u &= \chi_x - \beta^2 \phi_x + (\beta^2-1)\chi_{xx} + \psi_x \phi_x - \psi, \quad (3.11a) \\
  v &= \chi_y + \phi_y + (\beta^2-1)\chi_{xy} + \psi_y, \quad (3.11b) \\
  \theta &= (2/h)\chi_x \quad (3.11c)
\end{align*}
\]

where the subscripts on \( \chi \), \( \phi \), and \( \psi \) denote partial differentiation with respect to the indicated variable.

The stress-displacement equations become
\[ \sigma_x = \beta^2 \left[ x_{xx} - \phi_{xx} + (\beta^2 - 1) x \phi_{xxx} + x \psi_{xx} \right] \\
+ (\beta^2 - 2) \left[ x_{yy} + \phi_{yy} + (\beta^2 - 1) x \phi_{yyy} + x \psi_{yy} \right] - 2 \psi_x, \quad (3.12a) \]

\[ \sigma_y = \beta^2 \left[ x_{yy} + \phi_{yy} + (\beta^2 - 1) x \phi_{xyy} + x \psi_{yy} \right] \\
+ (\beta^2 - 2) \left[ x_{xx} - \phi_{xx} + (\beta^2 - 1) x \phi_{xxx} + x \psi_{xx} \right] - 2 \psi_x, \quad (3.12b) \]

and

\[ \sigma_{xy} = 2x_{xy} + 2x \left[ (\beta^2 - 1) \phi_{xyy} + \psi_{xy} \right]. \quad (3.12c) \]

The stresses must also satisfy the equilibrium Equations 3.7a,b.

In terms of \( \chi, \phi, \) and \( \psi, \) these are

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \beta^2 (\nabla^2 \chi)_x + (\beta^2 - 2) \nabla^2 \psi \]

\[ + \beta^2 (\beta^2 - 1) x (\nabla^2 \phi)_{xx} + \beta^2 (\beta^2 - 2) (\nabla^2 \phi)_x \]

\[ + \beta^2 x (\nabla^2 \psi)_x \quad (3.13a) \]

and

\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \beta^2 (\nabla^2 \chi)_y + \beta^2 (\beta^2 - 1) x (\nabla^2 \phi)_{xy} \]

\[ + \beta^2 (\nabla^2 \phi)_y + \beta^2 x (\nabla^2 \psi)_y. \quad (3.13b) \]

It is apparent from Equations 3.13 that the equilibrium equations will be satisfied identically if \( \chi, \phi, \) and \( \psi \) are harmonic functions. We therefore restrict ourselves to consider functions of this type and make use of this restriction to write the stress-strain laws in the following form.

\[
\sigma_x = 2\chi_{xx} - 2(\beta^2 - 1)\phi_{xx} + 2(\beta^2 - 1)x\phi_{xxx} \\
+ 2x\psi_{xx} - 2\psi_x \tag{3.14a}
\]

\[
\sigma_y = -2\chi_{xx} - 2(\beta^2 - 1)\phi_{xx} - 2(\beta^2 - 1)x\phi_{xxx} \\
- 2x\psi_{xx} - 2\psi_x \tag{3.14b}
\]

\[
\tau_{xy} = 2\chi_{xy} + 2x[(\beta^2 - 1)\phi_{xy} + \psi_{xy}] \tag{3.14c}
\]

In summary, the displacements and temperature are written in terms of derivatives of \( \chi, \psi, \) and \( \phi \). The stresses are then expressed in terms of these functions through the stress-strain laws. By restricting \( \chi, \psi, \) and \( \phi \) to be harmonic functions, we automatically satisfy the equilibrium equations. The boundary conditions determine the forms of the harmonic functions for a particular application of this formulation.
IV. SOLUTION
A. Boundary Conditions

We choose the strip half width w as the unit of length, and, having restricted the applied edge temperatures to be identical even functions of y, need only consider the portion of the strip shown in Figure 2, and defined by \(-1 \leq x \leq 0, 0 \leq y\).

Equations 3.6 through 3.10 are the governing differential equations, subject to the following boundary conditions,

\[
\begin{align*}
\theta_y(x,0) &= 0 \quad \text{\((-1 < x < 0\))} \quad (4.1) \\
\theta(-1,y) &= T(y) \quad \text{\((y \geq 0)\)} \quad (4.2) \\
\theta_x + h\theta(0,y) &= h\theta_c(y) \quad \text{\((0 < y < \delta)\)} \quad (4.3) \\
\theta_x(0,y) &= 0 \quad \text{\((y > \delta)\)} \quad (4.4) \\
u(-1,y) &= 0 \quad \text{\((y > 0)\)} \quad (4.5) \\
\tau_{xy}(-1,y) &= 0 \quad \text{\((y > 0)\)} \quad (4.6) \\
\tau_{xy}(0,y) &= 0 \quad \text{\((y > 0)\)} \quad (4.7) \\
v(x,0) &= 0 \quad \text{\((-1 < x < 0)\)} \quad (4.8) \\
\tau_{xy}(x,0) &= 0 \quad \text{\((-1 < x < 0)\)} \quad (4.9) \\
\sigma_x(0,y) &= 0 \quad \text{\((0 < y < \delta)\)} \quad (4.10)
\end{align*}
\]
Figure 2. Cracked strip quadrant

Figure 3. Uncracked strip quadrant
\[ u(0,y) = 0 \quad (y \geq \delta) \quad (4.11) \]

In the above expressions, \( \delta = \frac{a}{w} \), \( T(y) \) is the applied edge temperature, \( \theta_c(y) \) is the temperature of the environment into which heat is radiating from the crack, and \( h \) is given by

\[ h = \frac{4e\sigma T_o^3 w}{\kappa} \quad (4.12) \]

The strip width appears from the non-dimensionalization of the radiation equation, \( e \) is the coefficient of emissivity, \( \sigma \) is the Stefan-Boltzmann constant, \( T_o \) is the absolute reference temperature, and \( \kappa \) is the coefficient of thermal conductivity. These constants (except \( w \) and \( T_o \)) depend on the type of material being considered. The coefficient of emissivity also depends on the nature of the radiating surface.

To provide a means of evaluating the influence of the crack on the stresses, displacements and temperature in the strip, we first consider the problem of an uncracked strip of identical outer geometry and boundary conditions, as shown in Figure 3. The solution to this problem is quite straightforward, due to the absence of the mixed boundary conditions on \( x=0 \).

Equations 4.3, 4.4, 4.10, and 4.11 are replaced by

\[ \theta_x(0,y) = 0 \quad (y > 0) \quad (4.13) \]
and

$$u(0,y) = 0 . \quad (y \geq 0) \quad (4.14)$$

All other boundary conditions previously listed for the cracked strip also apply to the uncracked strip.

B. Solid Strip

1. Temperature problem

We take the Fourier cosine transform of Equation 3.6 and apply the boundary condition given by Equation 4.1 to obtain

$$\frac{d^2 \theta(x;\lambda)}{dx^2} - \lambda^2 \theta(x;\lambda) = 0, \quad (4.15)$$

where

$$\theta(x;\lambda) = \frac{1}{2} \int_{0}^{\infty} \theta(x,y) \cos \lambda y \, dy.$$  

Equation 4.15 has the solution

$$\theta(x;\lambda) = A(\lambda) \sinh \lambda x + B(\lambda) \cosh \lambda x, \quad (4.16)$$

where $A(\lambda)$ and $B(\lambda)$ must be determined from the remaining thermal boundary conditions.

Equation 4.2 then implies

$$\theta(x;\lambda) = \frac{1}{\cosh \lambda} \{ A(\lambda) \sinh \lambda (1+x) + T(\lambda) \cosh \lambda x \}, \quad (4.17)$$
where \( \overline{T}(\lambda) \) is the Fourier cosine transform of the applied boundary temperature.

We apply the remaining boundary condition, Equation 4.13, to show

\[
A(\lambda) = 0, \tag{4.18}
\]

and then take the inverse transform of Equation 4.17 to obtain

\[
\theta(x,y) = \frac{1}{2} \int_{-\infty}^{\infty} \overline{T}(\lambda) \frac{\cosh \lambda x}{\cosh \lambda} \cos \lambda y \, d\lambda. \tag{4.19}
\]

For a given \( T(y) \), Equation 4.19 will give the temperature distribution in the strip, provided that \( T(y) \) possesses a Fourier cosine transform.

2. Stress problem

We now apply the method outlined in Chapter III to solve for the stresses and displacements in the uncracked strip.

The harmonic function \( \psi(x,y) \) is related to the temperature by

\[
\psi_x = (b/2) \theta,
\]

from which it follows that

\[
\psi = (b/2) \int \theta \, dx + g(y), \tag{4.20}
\]
where \( g \) is an arbitrary function of \( y \).

It follows from the harmonic nature of both \( \psi \) and \( \theta \) that

\[
\frac{d^2 g}{dy^2} = 0.
\]

Thus,

\[
g(y) = Ay + B, \quad (4.21)
\]

where \( A \) and \( B \) are constants.

It is apparent from Equation 3.11a that \( B \) represents a rigid body displacement, and may therefore be taken as zero. Furthermore, since \( u \) must be an even function of \( y \), it follows that \( A \) must be zero. Then, from Equations 4.19 and 4.20, \( \psi \) is given by

\[
\psi(x,y) = \frac{b}{\sqrt{2\pi}} \int_0^\infty \frac{\tilde{\psi}(\lambda) \sinh \lambda x}{\lambda \cosh \lambda} \cos y \, d\lambda. \quad (4.22)
\]

It follows from physical considerations that \( \sigma_x, \sigma_y' \), and \( u \) are even functions of \( y \), whereas \( \tau_{xy} \) and \( v \) are odd functions of \( y \). Equations 3.11 and 3.14 show that \( u, \sigma_x' \), and \( \sigma_y \) depend on \( x \) derivatives of the harmonic functions, while \( \tau_{xy} \) and \( v \) depend on derivatives which are first order in \( y \). We therefore choose the following integral representations for \( \chi(x,y) \) and \( \phi(x,y) \),

\[
\chi(x,y) = \int_0^\infty \{K_1(\lambda) \sinh \lambda x + K_2(\lambda) \cosh \lambda x\} \cos y \, d\lambda \quad (4.23)
\]
where $K_1$, $K_2$, $K_3$, and $K_4$ will be chosen to satisfy the mechanical boundary conditions. We further assume throughout the following that the order of differentiation and integration may be interchanged, and that integrals may be combined prior to integration.

The forms for $\psi(x,y)$, $\phi(x,y)$ and $\chi(x,y)$ given above satisfy the homogeneous conditions on $v$ displacement and shear stress on $y=0$, as given by Equations 4.8 and 4.9.

Equation 4.7 requires that

$$\int_0^\infty \lambda^2 K_1(\lambda) \sin \lambda y \, d\lambda = 0,$$

which we satisfy by taking $K_1(\lambda)$ to be identically zero.

Equation 4.6 is equivalent to

$$\int_0^\infty \lambda^2 K_2(\lambda) \sinh \lambda \sin \lambda y \, d\lambda$$

$$+ (\beta^2-1) \int_0^\infty \lambda^3 \{K_4(\lambda) \cosh \lambda - K_3(\lambda) \sinh \lambda\} \sin \lambda y \, d\lambda$$

$$+ (b/\sqrt{2\pi}) \int_0^\infty \lambda T(\lambda) \sin \lambda y \, d\lambda = 0,$$

which we satisfy by setting

$$\lambda \sinh \lambda \{K_2(\lambda) - \lambda (\beta^2-1) K_3(\lambda)\} + \lambda^2 (\beta^2-1) K_4(\lambda) \cosh \lambda$$

$$+ (b/\sqrt{2\pi}) T(\lambda) = 0.$$
The displacement condition given by Equation 4.5 further restricts the forms of \(K_2(\lambda), K_3(\lambda),\) and \(K_4(\lambda).\) This restriction is given by

\[
\lambda \sinh \lambda \{K_2(\lambda) - \lambda (\beta^2 - 1)K_3(\lambda)\} + \lambda^2 (\beta^2 - 1)K_4(\lambda) \cosh \lambda \\
+ \left(\frac{b}{\sqrt{2\pi}}\right) \overline{T}(\lambda) + \lambda \beta^2 \{K_3(\lambda) \cosh \lambda - K_4(\lambda) \sinh \lambda\} \\
- \left(\frac{b}{\sqrt{2\pi}}\right) \left(\frac{\overline{T}(\lambda) \sinh \lambda}{\lambda \cosh \lambda}\right) = 0, \quad (4.28)
\]

which follows from an argument similar to the one which led to Equation 4.27.

The first three groups of terms in Equation 4.28 are identical to the left hand side of Equation 4.27; thus Equation 4.28 may be replaced by

\[
\lambda \beta^2 \{K_3(\lambda) \cosh \lambda - K_4(\lambda) \sinh \lambda\} - \left(\frac{b}{\sqrt{2\pi}}\right) \left(\frac{\overline{T}(\lambda) \sinh \lambda}{\lambda \cosh \lambda}\right) = 0 \quad (4.29)
\]

We now solve Equations 4.27 and 4.29 for \(K_3(\lambda)\) and \(K_4(\lambda)\) in terms of \(K_2(\lambda)\) and \(\overline{T}(\lambda),\) to obtain

\[
K_3(\lambda) = -\left\{\frac{\sinh^2 \lambda}{\lambda (\beta^2 - 1)}\right\} K_2(\lambda) - \left\{\frac{b \sinh \lambda}{\sqrt{2\pi} \lambda^2 \beta^2 (\beta^2 - 1)}\right\} \overline{T}(\lambda) \quad (4.30)
\]

and

\[
K_4(\lambda) = -\left\{\frac{\sinh \lambda \cosh \lambda}{\lambda (\beta^2 - 1)}\right\} K_2(\lambda) - \left\{\frac{\beta \overline{T}(\lambda)}{\sqrt{2\pi} \lambda^2 \beta^2 (\beta^2 - 1)}\right\} \{\cosh \lambda \} + \frac{1}{\cosh \lambda} \quad (4.31)
\]

Thus,

\[
\chi(x,y) = \int_0^\infty K_2(\lambda) \cosh \lambda x \cos \lambda y \, d\lambda \quad (4.32)
\]
\begin{equation}
\phi(x,y) = -\int_0^\infty \frac{K_2(\lambda) \sinh \lambda \cosh \lambda (1+x)}{\lambda (\beta^2 - 1)}
+ \frac{b\bar{T}(\lambda)}{\sqrt{2\pi}} \frac{(\cosh \lambda (1+x)}{\beta^2 - 1} \cosh \lambda x \cos y \, d\lambda. \tag{4.33}
\end{equation}

We use Equations 3.11a, 4.22, 4.32, and 4.33 to write \( u \) in terms of \( K_2(\lambda) \) and \( \bar{T}(\lambda) \).

\begin{equation}
\begin{aligned}
\phi(x,y) &= \int_0^\infty \{K_2(\lambda) (\lambda \sinh \lambda x + \frac{\beta^2 \sinh \lambda \sinh \lambda (1+x)}{\beta^2 - 1})
- \lambda x \sinh \lambda \cosh \lambda (1+x)) + (b/\sqrt{2\pi}) \bar{T}(\lambda) \frac{\sinh \lambda (1+x)}{\lambda (\beta^2 - 1)}
- \frac{x \cosh \lambda (1+x)}{\beta^2} + \frac{x \cosh \lambda x}{\beta^2 \cosh \lambda} \} \cos y \, d\lambda \tag{4.34}
\end{aligned}
\end{equation}

The remaining boundary condition, Equation 4.14, requires that

\begin{equation}
\int_0^\infty \{K_2(\lambda) \frac{\beta^2 \sinh^2 \lambda}{\beta^2 - 1} + (b/\sqrt{2\pi}) \bar{T}(\lambda) \frac{\sinh \lambda}{\lambda (\beta^2 - 1)} \} \cos y \, d\lambda = 0, \tag{4.35}
\end{equation}

which we satisfy by taking

\begin{equation}
K_2(\lambda) = \frac{-b\bar{T}(\lambda)}{\sqrt{2\pi} \lambda \beta^2 \sinh \lambda}. \tag{4.36}
\end{equation}

Thus, by direct substitution of Equation 4.36 into Equation 4.34, it follows that
To determine the remaining displacement and the stresses, we first take the derivatives of Equations 4.22, 4.32, and 4.33 as indicated by Equations 3.11b and 3.12. Then, using Equation 4.36, we find

\[
\begin{align*}
    u(x, y) &= \frac{b}{\sqrt{2\pi} \beta^2} \int_0^\infty \bar{T}(\lambda) \{ \frac{xcosh\lambda x}{cosh\lambda} - \frac{sinh\lambda x}{sinh\lambda} \} \cos\lambda y \, d\lambda. \\
    \sigma_x(x, y) &= \frac{1}{2} \frac{b}{\beta^2} \int_0^\infty \bar{T}(\lambda) \frac{\lambda}{cosh\lambda} (\lambda c x sinh\lambda x - c x cosh\lambda) \\
    &\quad - \frac{\lambda c x c o s h \lambda}{s i n h \lambda} \cos\lambda y \, d\lambda, \\
    \sigma_y(x, y) &= \frac{1}{2} \frac{b}{\beta^2} \int_0^\infty \bar{T}(\lambda) \frac{\lambda c o s h \lambda}{s i n h \lambda} \\
    &\quad - \frac{1}{c o s h \lambda} (c o s h \lambda x + \lambda c x s i n h \lambda x) \cos\lambda y \, d\lambda, \\
    \tau_{xy}(x, y) &= \frac{1}{2} \frac{b}{\beta^2} \int_0^\infty \lambda \bar{T}(\lambda) \{ \frac{sinh\lambda x}{sinh\lambda} - \frac{xcosh\lambda x}{cosh\lambda} \} \sin\lambda y \, d\lambda.
\end{align*}
\]

For a specified \( T(y) \), the above expressions give the resulting stresses and displacements in terms of the Fourier cosine transform of \( T(y) \). We consider a particular case in the following section.
3. Example

For purpose of example, we consider the temperature applied at the boundary to be given by

\[
T(y) = \begin{cases} 
T & (0 < y < A) \\
0, & (y > A)
\end{cases}
\]  

(4.41)

where \( T \) is a constant.

Then

\[
\overline{T}(\lambda) = \left( \frac{2}{\pi} \right) \frac{1}{2} \int_0^\infty T(y) \cos \lambda y \, dy
\]

\[
= \left( \frac{2}{\pi} \right) \frac{1}{2} \int_0^A T \cos \lambda y \, dy
\]

\[
= \frac{1}{2} \frac{2}{\pi} T \frac{\sin \lambda A}{\lambda} .
\]

(4.42)

In the work that follows, many of the needed integrals have been found in the tables. The source of each integral used is given, and the reader is referred to Appendix A for a discussion of the details.

Substitute Equation 4.42 into Equations 4.36 through 4.40.

The temperature in the strip is given by
\[ \theta(x,y) = \frac{2T}{\pi} \int_0^\infty \frac{\sin \lambda \cosh \lambda}{\lambda \cosh \lambda} \cos y \, d\lambda \]

\[ = \frac{T}{\pi} \int_0^\infty \frac{\cosh \lambda \sin \lambda (\ell+y)}{\lambda \cosh \lambda} \, d\lambda \]

\[ + \frac{T}{\pi} \int_0^\infty \frac{\cosh \lambda \sin \lambda (\ell+y)}{\lambda \cosh \lambda} \, d\lambda \]

\[ = \frac{T}{\pi} \{ \tan^{-1} \left( \frac{\sinh \frac{\ell+y}{2}}{\cos \frac{\pi x}{2}} \right) + \tan^{-1} \left( \frac{\sinh \frac{\ell-y}{2}}{\cos \frac{\pi x}{2}} \right) \} . \]

(12, p. 517) (4.43)

The stresses are determined as follows

\[ \tau_{xy}(x,y) = \frac{2bT}{\pi \beta^2} \int_0^\infty \left\{ \frac{\sinh \lambda x}{\sinh \lambda} - \frac{x \cosh \lambda x}{\cosh \lambda} \sin \lambda \sin y \right\} d\lambda \]

\[ = \frac{bT}{\pi \beta^2} \left\{ \int_0^\infty \frac{\sinh \lambda x}{\sinh \lambda} \cos (\ell-y) \, d\lambda \right\} 
- x \int_0^\infty \frac{\cosh \lambda x}{\cosh \lambda} \cos (\ell-y) \, d\lambda 
- \int_0^\infty \frac{\sinh \lambda x}{\sinh \lambda} \cos (\ell+y) \, d\lambda 
+ x \int_0^\infty \frac{\cosh \lambda x}{\cosh \lambda} \cos (\ell+y) \, d\lambda \} \]
\[
\sigma_y(x, y) = \frac{2bT}{\pi \beta^2} \int_0^\infty \frac{\sin \lambda \{ \lambda \cosh \lambda x \}}{\lambda \sinh \lambda} \left( - \frac{1}{\cosh \lambda} (\cosh \lambda x + \lambda x \sinh \lambda x) \right) \cos \lambda y \, d\lambda
\]

\[
= \frac{bT}{\pi \beta^2} \left( \frac{\sinh \pi (l+y)}{2 \cosh \pi (l+y) + \cos \pi x} + \frac{\sinh \pi (l-y)}{\cosh \pi (l-y) + \cos \pi x} \right) + \sin \lambda (l-y) \, d\lambda
\]

\[
= \frac{bT}{\pi \beta^2} \left\{ \frac{\pi}{2} \left( \frac{\sinh \pi (l+y)}{2 \cosh \pi (l+y) + \cos \pi x} - \frac{\sinh \pi (l-y)}{\cosh \pi (l-y) + \cos \pi x} \right) - \tan^{-1} \left( \frac{\sinh \pi(l+y)}{\cos \pi x^2} \right) - \tan^{-1} \left( \frac{\sinh \pi (l-y)}{\cos \pi x^2} \right) \right\}
\]

\[
= \frac{\sin \pi x}{\cosh \pi (l+y) + \cos \pi x} \cdot \frac{\sin \pi (l+y)}{\cos \pi x^2} + \frac{\sin \pi x}{\cosh \pi (l-y) + \cos \pi x} \cdot \frac{\sin \pi (l-y)}{\cos \pi x^2} \}
\]

(12, p. 504) (4.44)
\[ \sigma_x = \frac{2b_T}{\pi \beta^2} \int_0^\infty \sin \lambda \left\{ \frac{1}{\cosh \lambda} (\lambda \sinh \lambda x - \cosh \lambda x) - \frac{\lambda \cosh \lambda x}{\sinh \lambda} \right\} \cos \lambda y \, d\lambda \]

\[ = \frac{b_T}{\pi \beta^2} \int_0^\infty \left\{ \frac{\lambda \sinh \lambda x}{\cosh \lambda} - \frac{\cosh \lambda x}{\lambda \cosh \lambda} - \frac{\cosh \lambda x}{\sinh \lambda} \right\} \sin \lambda (l-y) \, d\lambda \]

\[ + \sin \lambda (l+y) \, d\lambda \]

Making use of previous results, we find

\[ \sigma_x(x,y) = \frac{b_T}{\pi \beta^2} \left\{ \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi (l+y)}{2}}{\cosh \pi (l+y) + \cos \pi x} + \frac{\sin \frac{\pi x}{2} \sinh \frac{\pi (l-y)}{2}}{\cosh \pi (l-y) + \cos \pi x} \right\} \]

\[ - \tan^{-1}\left( \frac{\sin \frac{\pi x}{2}}{\cos \frac{\pi x}{2}} \right) \cdot \frac{\sinh \pi (l-y)}{2 \cosh \pi (l+y) + \cos \pi x} \]

\[ - \tan^{-1}\left( \frac{\sin \frac{\pi (l+y)}{2}}{\cos \frac{\pi (l+y)}{2}} \right) \cdot \frac{\sinh \pi (l-y)}{2 \cosh \pi (l-y) + \cos \pi x} \}

\[ = \frac{\pi}{2} \left\{ \frac{\sin \pi (l+y)}{\cosh \pi (l+y) + \cos \pi x} + \frac{\sin \pi (l-y)}{\cosh \pi (l-y) + \cos \pi x} \right\} \}

\[ (4.46) \]

We next find the displacements.

\[ u(x,y) = \frac{b_T}{\pi \beta^2} \int_0^\infty \sin \lambda \left\{ \frac{x \cosh \lambda x}{\cosh \lambda} - \frac{\sinh \lambda x}{\sinh \lambda} \right\} \cos \lambda y \, d\lambda \]

\[ = \frac{b_T}{2 \pi \beta^2} \int_0^\infty \left\{ \frac{x \cosh \lambda x}{\lambda \cosh \lambda} - \frac{\sinh \lambda x}{\lambda \sinh \lambda} \right\} \sin \lambda (l+y) \]

\[ + \sin \lambda (l-y) \, d\lambda \]
\[ v(x, y) = \frac{bT}{\pi \beta^2} \left\{ \int_0^\infty \sinh \lambda \left\{ \frac{1}{\lambda \cosh \lambda} + \frac{1}{\sinh \lambda} \right\} \right. \]
\[ \left. - \frac{x \sinh \lambda x}{\cosh \lambda} \sin y \, d\lambda \right\} \]
\[ = \frac{bT}{\pi \beta^2} \left\{ \int_0^\infty \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin y \sin \lambda \, d\lambda \right. \]
\[ + \int_0^\infty \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin y \sin \lambda \, d\lambda \]
\[ - \int_0^\infty \frac{x \sinh \lambda x}{\lambda \cosh \lambda} \sin y \sin \lambda \, d\lambda \} \]  

We were unable to obtain a closed-form expression for the first integral in Equation 4.48. A derivation of a series representation for this integral is given in Appendix A, as is also the evaluation of the second integral in
Equation 4.48. The third integral may be evaluated using known results.

\[
\int_0^\infty \frac{x \sinh \lambda x}{\lambda \cosh \lambda} \sin y \sin \lambda \, d\lambda = \frac{x}{2} \int_0^\infty \frac{\sinh \lambda x}{\lambda \cosh \lambda} \cos (\lambda y) \, d\lambda
\]

\[
- \frac{x}{2} \int_0^\infty \frac{\sinh \lambda x}{\lambda \cosh \lambda} \cos (\lambda y) \, d\lambda
\]

\[
= \frac{x}{2} \left\{ \frac{1}{2} \ln \left( \frac{\cosh \pi \frac{1}{2} (\lambda y) + \sin \pi \frac{1}{2} x}{\cosh \pi \frac{1}{2} (\lambda y) - \sin \pi \frac{1}{2} x} \right) - \frac{1}{2} \ln \left( \frac{\cosh \pi \frac{1}{2} (\lambda y) + \sin \pi \frac{1}{2} x}{\cosh \pi \frac{1}{2} (\lambda y) - \sin \pi \frac{1}{2} x} \right) \right\}
\]

\[
(4.49)
\]

Then, with the results indicated above and in Appendix A, we are able to write the following expressions for the remaining displacement:

\[
v(x, y) = \frac{b T}{\pi \beta^2} \left\{ \frac{1}{4} \ln \left( \frac{\cosh \pi \frac{1}{2} (\lambda y) + \cos \pi x}{\cosh \pi \frac{1}{2} (\lambda y) + \cos \pi x} \right) \right. \\
- \frac{x}{4} \ln \left( \frac{\cosh \pi \frac{1}{2} (\lambda y) + \sin \pi x}{\cosh \pi \frac{1}{2} (\lambda y) - \sin \pi x} \right) \left( \frac{\cosh \pi \frac{1}{2} (\lambda y) + \sin \pi x}{\cosh \pi \frac{1}{2} (\lambda y) - \sin \pi x} \right) \right\}
\]

\[
+ \frac{\pi y}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2} \frac{\sinh \frac{2n-1}{2} \pi y}{(2n-1)^2} \cos \left( \frac{2n-1}{2} \pi x \right)
\]

\[
(0 < y < l) \quad (4.50a)
\]
\[ v(x, y) = \frac{bT}{\pi \beta^2} \left( \frac{1}{4} \ln \left( \frac{\cosh \pi \left( l + y \right) + \cos \pi x}{\cosh \pi \left( l - y \right) + \cos \pi x} \right) \right. \]

\[ - \frac{x}{4} \ln \left\{ \left( \frac{\cosh \frac{\pi (l-y)}{2}}{\cosh \frac{\pi (l+y)}{2}} - \frac{\sin \frac{\pi x}{2}}{\sin \frac{\pi y}{2}} \right) \left( \frac{\cosh \frac{\pi (l+y)}{2}}{\cosh \frac{\pi (l+y)}{2}} + \frac{\sin \frac{\pi x}{2}}{\sin \frac{\pi y}{2}} \right) \right\} \]

\[ + \frac{\pi \beta}{2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{\left( \frac{2n-1}{2} \right) \pi y}}{\sinh \left( \frac{2n-1}{2} \right) \pi \lambda \cos \left( \frac{2n-1}{2} \right) \pi x} \frac{1}{(2n-1)^2} \]

\[ (y > \lambda) \quad (4.50b) \]

This completes the solution for the solid strip.

C. Cracked Strip

1. Temperature problem

   The details of the solution for the temperature distribution in the cracked strip are identical to those in the case of the uncracked strip through Equation 4.17; therefore we begin at the point where we apply the mixed thermal boundary condition along \( x=0 \).

   We have

   \[ \Theta(x; \lambda) = \frac{1}{\cosh \lambda} \{ C(\lambda) \sinh \lambda (1 + x) + \overline{T}(\lambda) \cosh \lambda x \}, \quad (4.51) \]

   where we have replaced \( A(\lambda) \) in Equation 4.17 by \( C(\lambda) \) to avoid possible confusion.

   We take the inverse transform of Equation 4.51 to obtain
\[ \theta(x,y) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{0}^{\infty} \frac{C(\lambda) \sinh \lambda (1+x) + F(\lambda) \cosh \lambda x}{\cosh \lambda} \cos \lambda y \, d\lambda, \]  

(4.52)

and then differentiate with respect to \( x \) to find

\[ \theta_x(x,y) = \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \int_{0}^{\infty} \lambda \left( \frac{C(\lambda) \cosh \lambda (1+x) + F(\lambda) \sinh \lambda x}{\cosh \lambda} \right) \cos \lambda y \, d\lambda. \]  

(4.53)

We now apply the mixed boundary condition given by Equations 4.3 and 4.4, and group terms to obtain the following set of dual integral equations:

\[ \int_{0}^{\infty} C(\lambda) \{ \lambda + h \tanh \lambda \} \cos \lambda y \, d\lambda = \left( \frac{1}{2} \right) h \theta_c(y) \]

\[ - h \int_{0}^{\infty} \frac{F(\xi)}{\cosh \xi} \cos \xi y \, d\xi \quad \text{(0 < y < \delta)} \]  

(4.54a)

\[ \int_{0}^{\infty} \lambda C(\lambda) \cos \lambda y \, d\lambda = 0 \quad (y>\delta) \]  

(4.54b)

Note that the right hand side of Equation 4.54a is presumed known.

We choose the method due to C. J. Tranter (13, pp. 111-114) to solve the dual integral pair for \( C(\lambda) \). Tranter's method consists of first choosing a series expansion for \( C(\lambda) \) which automatically satisfies the homogeneous integral equation, but which is general enough to allow us to attempt to satisfy the non-homogeneous equation by an appropriate
choice of the coefficients in the expansion. The problem then becomes one of determining the correct values for these coefficients.

We first make the substitution

\[ \cos \lambda y = \{\pi \lambda y/2\}^{1/2} J_{-1/2}(\lambda y), \]

where \( J_{-1/2}(\lambda y) \) is the Bessel function of the first kind, and introduce the notation

\[ F(\lambda) = \lambda^{3/2} C(\lambda), \quad (4.55) \]

\[ H(\lambda) = 1 + \frac{h \tanh \lambda}{\lambda}, \quad (4.56) \]

and

\[ \frac{\overline{F}(y)}{\sqrt{y}} = \frac{h \theta_c(y)}{\sqrt{y}} - h \int_0^\infty \xi^{1/2} \frac{\overline{T}(\xi)}{\cosh \xi} J_{-1/2}(\xi y) d\xi. \quad (4.57) \]

Then, Equations 4.54 become

\[ \int_0^\infty H(\lambda) F(\lambda) J_{-1/2}(\lambda y) d\lambda = \frac{\overline{F}(y)}{\sqrt{y}} \quad (0 \leq y \leq \delta) \quad (4.58a) \]

\[ \int_0^\infty F(\lambda) J_{-1/2}(\lambda y) d\lambda = 0. \quad (y > \delta) \quad (4.58b) \]

Represent \( F(\lambda) \) by the series

\[ F(\lambda) = \sum_{m=0}^{\infty} A_m J_{2m+1/2}(\delta \lambda), \quad (4.59) \]

with the \( A_m \) to be determined.

A special case of the Weber-Schafheitlin discontinuous
integral is

\[
\int_0^\infty J_{2m+1/2}(\delta \lambda) J_{-1/2}(y \lambda) \, d\lambda = \begin{cases} 
\frac{\Gamma(m+1/2) _2F_1(m+1/2,-m;1/2;\frac{y^2}{\delta^2})}{\sqrt{\pi y^2} \, m!} & (0 < y < \delta) \\
0 & (y > \delta)
\end{cases}
\]

(14, p. 487) (4.60)

where \( _2F_1 \) is Gauss' hypergeometric function.

Thus, it is clear that the representation for \( F(\lambda) \)
given in Equation 4.59 satisfies Equation 4.58b, assuming
that we may interchange orders of summation and integration.
We note that the representation given above for \( F(\lambda) \) is
not the only one which will satisfy the homogeneous equation.
In fact, any expansion of the form

\[
\lambda^{1-k} \sum_{n=0}^{\infty} A_n J_{2n+k-1/2}(\lambda)
\]

will meet this requirement. The parameter \( k \) may then be
chosen so as to make subsequent integrals converge. In
our case, the choice \( k=1 \) was the most appropriate one.

We use Equation 4.60 to derive a result which will be
needed in the work that follows. We write

\[
_2F_1(m+1/2,-m;1/2;\frac{y^2}{\delta^2}) = _2F_1(-m,m+1/2;1/2;\frac{y^2}{\delta^2})
\]

(15, p. 8)
\[ F_m(1/2,1/2,\frac{\nu^2}{\delta^2}) = \frac{\Gamma(m+1/2)}{\sqrt{\pi y \delta}} \frac{F_m(1/2,1/2,\frac{\nu^2}{\delta^2})}{m!} \]

(15, p. 83)

where \( F_m(1/2,1/2,\frac{\nu^2}{\delta^2}) \) is the Jacobi polynomial of degree \( m \).

Thus,

\[
\int_0^\infty J_{2m+1/2}(\delta \lambda) J_{-1/2}(y \lambda) d\lambda = \begin{cases} 
\frac{\Gamma(m+1/2)}{\sqrt{\pi y \delta}} F_m(1/2,1/2,\frac{\nu^2}{\delta^2}) & (0 < y < \delta) \\
0 & (y > \delta) 
\end{cases} 
\]

(4.61)

The right hand side of Equation 4.61 may be considered to be the Hankel transform of \( \lambda^{-1} J_{2m+1/2}(\delta \lambda) \), with kernel \( J_{-1/2}(\lambda y) \). Then, by the Hankel inversion theorem, it follows that

\[
\lambda^{-1} J_{2m+1/2}(\delta \lambda) = \frac{\Gamma(m+1/2)}{\sqrt{\pi \delta}} \int_0^\delta \frac{1}{\sqrt{\pi \delta}} F_m(1/2,1/2,\frac{\nu^2}{\delta^2}) J_{-1/2}(\lambda y) dy. 
\]

(4.62)

Substitute Equation 4.59 into Equation 4.58a, and interchange orders of summation and integration.

\[
\sum_{m=0}^\infty A_m \int_0^\infty H(\lambda) J_{2m+1/2}(\delta \lambda) J_{-1/2}(y \lambda) d\lambda = \frac{\bar{F}(y)}{\sqrt{y}} 
\]

(0 < y < \delta) (4.63)

Multiply both sides of Equation 4.63 by
\[ \sqrt{y} F_n\left(1/2, 1/2, \frac{y^2}{\delta^2}\right), \ n=0,1,2, \ldots, \] and integrate with respect to \( y \) from zero to \( \delta \).

\[
\int_0^\delta \sqrt{y} F_n\left(1/2, 1/2, \frac{y^2}{\delta^2}\right) dy \sum_{m=0}^\infty A_m \int_0^{\infty} H(\lambda) J_{2m+1/2}(\delta \lambda) J_{-1/2}(\lambda y) d\lambda
\]

\[= \int_0^\delta \bar{f}(y) F_n\left(1/2, 1/2, \frac{y^2}{\delta^2}\right) dy \quad n=0,1,2, \ldots \]  

(4.64)

Interchange orders of integration,

\[
\sum_{m=0}^\infty A_m \int_0^{\infty} H(\lambda) J_{2m+1/2}(\delta \lambda) d\lambda \int_0^\delta \sqrt{y} F_n\left(\frac{1}{2}, \frac{1}{2}, \frac{y^2}{\delta^2}\right) J_{-1/2}(\lambda y) dy
\]

\[= \int_0^\delta \bar{f}(y) F_n\left(1/2, 1/2, \frac{y^2}{\delta^2}\right) dy \quad n=0,1,2, \ldots \]  

(4.65)

The integration on \( y \) is performed by using Equation 4.62, which gives us

\[
\sum_{m=0}^\infty A_m \int_0^{\infty} \lambda^{-1} H(\lambda) J_{2m+1/2}(\delta \lambda) J_{2n+1/2}(\delta \lambda) d\lambda
\]

\[= \left(\frac{\delta}{\pi}\right)^{1/2} \frac{\Gamma(n+1/2)}{n!} \int_0^1 f(z) F_n\left(1/2, 1/2, z^2\right) dz \quad n=0,1,2, \ldots \]  

(4.66)

where we have also made the substitutions

\[ \frac{y}{\delta} = z \]
and
\[ \overline{f(y)} = f(z). \]

From Equation 4.57, \( f(z) \) is an even function of \( z \).

We can therefore express \( f(z) \) in terms of the series of Jacobi polynomials in \( z^2 \) given by
\[ f(z) = \sum_{k=0}^{\infty} B_k F_k(1/2,1/2,z^2). \quad (4.66) \]

The Jacobi polynomials have the orthogonality property
\[
\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} F_m(\alpha,\beta,x) F_n(\alpha,\beta,x) \, dx
\]
\[
= \begin{cases} 
0 & \text{for } m \neq n \\
\frac{\Gamma(\gamma)\Gamma(\alpha+1-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(\alpha+1-\gamma,n)}{(\alpha,n)(\beta,n)} \frac{n!}{\alpha+2n} & \text{for } m = n \text{ and } \text{Re}(\gamma) > 0; \text{Re}(\alpha-\gamma) > -1, \quad (15, p. 83). 
\end{cases} 
\quad (4.67)
\]

Appell's symbol \((\alpha,n)\) is defined by
\[
(\alpha,n) = \alpha(\alpha+1)...(\alpha+n-1),
\]

where \((\alpha,0) = 1.\)

Thus,
\[
\int_0^1 F_k(1/2,1/2,z^2) F_n(1/2,1/2,z^2) \, dz
\]
\[
= \frac{1}{2} \int_0^1 t^{-1/2} F_k(1/2,1/2,t) F_n(1/2,1/2,t) \, dt
\]
Multiply both sides of Equation 4.66 by $F_n(1/2,1/2,z^2)$ and integrate with respect to $z$ from zero to one. Interchange the order of summation and integration and use Equation 4.68 to obtain

$$B_n = \frac{1}{4n+1} \left( \frac{n!}{(1/2,n)^2} \right)^2 \int_0^1 \phi(z) F_n(1/2,1/2,z^2) \, dz. \quad (4.69)$$

We note that

$$\Gamma(n+1/2) = \sqrt{\pi} \left( \frac{1}{2},n \right) \quad (14, \text{ p. 256})$$

and then use Equation 4.69 in Equation 4.66 to write

$$\sum_{m=0}^{\infty} A_m \int_0^\infty \lambda^{-1} H(\lambda) J_{2m+1/2}(\delta \lambda) J_{2n+1/2}(\delta \lambda) \, d\lambda$$

$$= \sqrt{\delta} \frac{n!}{(4n+1)(1/2,n)} \cdot B_n. \quad n=0,1,2,\ldots \quad (4.70)$$

Equation 4.70 represents an infinite system of linear non-homogeneous algebraic equations with known right hand sides, which must be solved numerically for the $A_m$. Determining the $A_m$ is equivalent to finding $C(\lambda)$, and
knowing \( C(\lambda) \), we can use Equation 4.52 to determine the temperature throughout the strip.

We leave the details of the numerical solution to a later section and proceed to solve the stress problem.

2. **Stress problem**

The method of solution for the stresses and displacements in the cracked strip is identical to that used for the solution in the uncracked strip up to the point of application of the displacement boundary condition along \( x=0 \). We therefore outline only the more important details of the initial portion of the solution.

We find

\[
\psi(x,y) = \frac{b}{\sqrt{2\pi}} \int_{0}^{\infty} \frac{C(\lambda) \cosh \lambda (1+x) + T(\lambda) \sinh \lambda x}{\lambda \cosh \lambda} \cos \lambda y \, d\lambda
\]

by the same argument as before, and represent \( \chi(x,y) \) and \( \phi(x,y) \) by

\[
\chi(x,y) = \int_{0}^{\infty} [K_1(\lambda) \sinh \lambda x + K_2(\lambda) \cosh \lambda x] \cos \lambda y \, d\lambda \tag{4.72}
\]

and

\[
\phi(x,y) = \int_{0}^{\infty} [K_3(\lambda) \sinh \lambda x + K_4(\lambda) \cosh \lambda x] \cos \lambda y \, d\lambda. \tag{4.73}
\]

These forms satisfy Equations 4.8 and 4.9 as before.

To satisfy Equation 4.7, it is sufficient to take \( K_1(\lambda) \)
to be identically zero.

Equation 4.6 is satisfied if

\[ \lambda \sinh \lambda \{ K_2(\lambda) - \lambda (\beta^2 - 1) K_3(\lambda) \} + \lambda^2 (\beta^2 - 1) K_4(\lambda) \cosh \lambda \]

\[ + \frac{b \overline{\Phi}(\lambda)}{\sqrt{2\pi}} = 0, \]  \hspace{1cm} (4.74)

and Equation 4.5 requires

\[ \lambda \sinh \lambda \{ K_2(\lambda) - \lambda (\beta^2 - 1) K_3(\lambda) \} + \lambda^2 (\beta^2 - 1) K_4(\lambda) \cosh \lambda \]

\[ + \frac{b \overline{\Phi}(\lambda)}{\sqrt{2\pi}} + \lambda \beta^2 \{ K_3(\lambda) \cosh \lambda - K_4(\lambda) \sinh \lambda \} \]

\[ + \frac{b}{\sqrt{2\pi} \lambda \cosh \lambda} \{ C(\lambda) - \overline{\Phi}(\lambda) \sinh \lambda \} = 0. \] \hspace{1cm} (4.75)

If Equation 4.74 is satisfied, Equation 4.75 reduces to

\[ \lambda \beta^2 \{ K_3(\lambda) \cosh \lambda - K_4(\lambda) \sinh \lambda \} + \frac{b}{\sqrt{2\pi} \lambda \cosh \lambda} \{ C(\lambda) - \overline{\Phi}(\lambda) \sinh \lambda \} = 0. \]

\hspace{1cm} (4.76)

We solve Equations 4.74 and 4.76 for \( K_3(\lambda) \) and \( K_4(\lambda) \) in terms of \( K_2(\lambda) \) and \( \overline{\Phi}(\lambda) \).

\[ K_3(\lambda) = \frac{-K_2(\lambda) \sinh^2 \lambda}{\lambda (\beta^2 - 1)} - \frac{b}{\sqrt{2\pi} \lambda \beta^2} \{ \overline{\Phi}(\lambda) \sinh \lambda + C(\lambda) \} \] \hspace{1cm} (4.77)
\[ K_4(\lambda) = \frac{-K_2(\lambda) \sinh \lambda \cosh \lambda}{\lambda (\beta^2 - 1)} - \frac{b}{\sqrt{2\pi} \beta^2} (\overline{T}(\lambda) (\cosh \lambda \frac{\sinh \lambda}{\beta^2 - 1}) + \frac{1}{\cosh \lambda}) + C(\lambda) \tanh \lambda \]  (4.78)

From these results, we write the stresses and displacements in terms of \( K_2(\lambda) \) and \( \overline{T}(\lambda) \).

\[
u(x,y) = \int_0^\infty \{ K_2(\lambda) (\lambda \sinh \lambda x + e^\lambda \sinh \lambda (1+x) - \lambda x \sinh \lambda \cosh \lambda (1+x)) + \frac{b \overline{T}(\lambda)}{\sqrt{2\pi} \beta^2} (\frac{\sinh \lambda (1+x)}{\lambda (\beta^2 - 1)}) - \frac{x \cosh \lambda (1+x)}{\beta^2} + \frac{xcosh \lambda}{\beta^2 \cosh \lambda} 
+ \frac{x b C(\lambda) \sinh (1+x)}{\sqrt{2\pi} \beta^2 \cosh \lambda} \} \cos y \ d\lambda
\]  (4.79)

\[
u(x,y) = \int_0^\infty \{ K_2(\lambda) (-\lambda \cosh \lambda x + \sinh \lambda \{ \frac{\cosh \lambda (1+x)}{\beta^2 - 1}) + \frac{\overline{T}(\lambda)}{\sqrt{2\pi} \beta^2} (\frac{\cosh \lambda (1+x)}{\lambda (\beta^2 - 1)}) + \frac{\cosh \lambda x}{\lambda \cosh \lambda} 
+ \frac{x \sinh \lambda (1+x)}{\cosh \lambda} 
+ \frac{b C(\lambda)}{\sqrt{2\pi} \beta^2 \cosh \lambda} (\frac{\sinh \lambda (1+x)}{\lambda} - \frac{xcosh \lambda (1+x)}) \} \sin y \ d\lambda
\]  (4.80)
\[ \sigma_x(x,y) = 2 \int_0^\infty \{ \lambda K_2(\lambda) (\lambda \cosh \lambda x + \sinh \lambda \cosh (1+x)) \\
- \lambda x \sinh \lambda \sinh (1+x) + \frac{bT(\lambda)}{\sqrt{2\pi \beta}} (\cosh (1+x) - \frac{\cosh \lambda x}{\cosh \lambda}) \\
- \lambda x \sinh (1+x) + \frac{\lambda x \sinh \lambda x}{\cosh \lambda} + \frac{bC(\lambda)}{\sqrt{2\pi \beta}^2 \cosh \lambda} (\lambda x \cosh (1+x)) \\
- \sinh (1+x)) \} \cos \lambda y \, d\lambda \] (4.81)

\[ \sigma_y(x,y) = 2 \int_0^\infty \{ \lambda K_2(\lambda) (-\lambda \cosh \lambda x + \sinh \lambda \cosh (1+x)) \\
+ \lambda x \sinh \lambda \sinh (1+x) + \frac{bT(\lambda)}{\sqrt{2\pi \beta}} (\cosh (1+x) - \frac{\cosh \lambda x}{\cosh \lambda}) \\
+ \lambda x \sinh (1+x) - \frac{\lambda x \sinh \lambda x}{\cosh \lambda} - \frac{bC(\lambda)}{\sqrt{2\pi \beta}^2 \cosh \lambda} (\sinh (1+x)) \\
+ \lambda x \cosh (1+x) \} \cos \lambda y \, d\lambda \] (4.82)

\[ \tau_{xy}(x,y) = 2 \int_0^\infty \{ \lambda^2 K_2(\lambda) (x \sinh \lambda \cosh (1+x) - \sinh \lambda x) \\
+ \frac{b\lambda x T(\lambda)}{\sqrt{2\pi \beta}^2} (\cosh (1+x) - \frac{\cosh \lambda x}{\cosh \lambda}) \\
- \frac{b\lambda x C(\lambda) \sinh (1+x)}{\sqrt{2\pi \beta}^2 \cosh \lambda} \} \sin \lambda y \, d\lambda \] (4.83)

The mixed boundary condition on \( x=0 \), given by Equations 4.10 and 4.11, remains to be satisfied. Equations 4.79 and 4.81, evaluated at \( x=0 \), yield the following pair of dual
integral equations for $K_2(\lambda)$:

\[
\int_0^\infty \left\{ \lambda K_2(\lambda)(\lambda+\sinh\lambda \cosh\lambda) + \frac{bT(\lambda)}{\sqrt{2\pi}\beta} (\cosh\lambda - \frac{1}{\cosh\lambda}) \right. \\
- \left. \frac{bC(\lambda)\sinh\lambda}{\sqrt{2\pi}\beta^2 \cosh\lambda} \right\} \cos y \, d\lambda = 0 \quad (0 < y < \delta) \tag{4.84a}
\]

\[
\int_0^\infty \left\{ \frac{\beta^2 \sinh^2\lambda}{\beta^2-1} + \frac{bT(\lambda)}{\sqrt{2\pi}} \frac{\sinh\lambda}{\lambda(\beta^2-1)} \right\} \cos y \, d\lambda = 0 \quad (y > \delta) \tag{4.84b}
\]

We define

\[
\overline{G}(\lambda) = K_2(\lambda)\sinh^2\lambda + \frac{bT(\lambda)\sinh\lambda}{\sqrt{2\pi}\beta^2}
\]

and

\[
\overline{h}(\lambda) = \frac{\lambda}{\sinh^2\lambda} + \coth\lambda,
\]

regroup terms, and write the above pair of equations in the more familiar form

\[
\int_0^\infty \lambda \overline{G}(\lambda) \overline{h}(\lambda) \cos y \, d\lambda = \frac{b}{\sqrt{2\pi}\beta^2} \int_0^\infty \left\{ \frac{C(\xi)\sinh\xi + T(\xi)}{\cosh\xi} \right\} \cos \xi y \, d\xi \\
+ \frac{b}{\sqrt{2\pi}\beta^2} \int_0^\infty \frac{\xi T(\xi)}{\sinh\xi} \cos \xi y \, d\xi \quad (0 < y < \delta) \tag{4.85a}
\]

\[
\int_0^\infty \overline{G}(\lambda) \cos y \, d\lambda = 0. \quad (y > \delta) \tag{4.85b}
\]
It follows from Equation 4.52 that
\[
\int_0^\infty \frac{C(\xi) \sinh \xi + \bar{P}(\xi)}{\cosh \xi} \cos \gamma \, d\xi = \left( \frac{\pi}{2} \right) \theta(0, y). 
\] (4.86)

We express \( \cos \gamma y \) in terms of \( J_{-1/2}(\gamma y) \). Then let
\[
\lambda^{1/2} \bar{G}(\lambda) = G(\lambda),
\] (4.87)
\[
\lambda \bar{h}(\lambda) = h(\lambda),
\] (4.88)
\[
\bar{g}(\gamma) = \frac{b \delta(0, y)}{\sqrt{2\pi \beta}} + \frac{b}{\sqrt{2\pi \beta}^2} \int_0^\infty \frac{\xi^{3/2} \bar{P}(\xi)}{\sinh \xi} J_{-1/2}(\xi y) \, d\xi,
\] (4.89)
and rewrite Equations 4.85 in the form
\[
\int_0^\infty h(\lambda) G(\lambda) J_{-1/2}(\lambda y) \, d\lambda = \frac{\bar{g}(\gamma)}{\sqrt{y}} \quad (0 < y < \delta) 
\] (4.90a)
\[
\int_0^\infty G(\lambda) J_{-1/2}(\lambda y) \, d\lambda = 0. \quad (y > \delta) 
\] (4.90b)

Assuming the solution to the temperature problem has been completed, the right hand side of Equation 4.90a is known. We proceed to solve for \( G(\lambda) \) using Tranter's method.

Represent \( G(\lambda) \) by
\[
G(\lambda) = \lambda^{-1/2} \sum_{m=0}^{\infty} a_m J_{2m+1}(\lambda \delta),
\] (4.91)
where the $a_m$ are to be determined. Note that here an appropriate choice of $k$ is $k=3/2$ (see discussion following Equation 4.60).

We now develop some results which are needed to complete the solution.

Another specialization of the Weber-Schafheitlin discontinuous integral is

$$\int_0^\infty \lambda^{-1/2} J_{2m+1}(\delta \lambda) J_{-1/2}(y \lambda) \, d\lambda = \begin{cases} \frac{(2 \pi y)^{1/2}}{(2m+1)} \, \frac{1}{\sqrt{2}} & 2F_1 \left( \frac{m+1/2}{2}, -\frac{m-1/2}{2}; \frac{1}{2}; \frac{y^2}{\delta^2} \right) \\ 0 & (y > \delta) \end{cases}$$

(14, p. 487) (4.92)

We write Gauss' hypergeometric functions in terms of the Jacobi polynomials.

$$2F_1 \left( \frac{m+1/2}{2}, -\frac{m-1/2}{2}; \frac{1}{2}; \frac{y^2}{\delta^2} \right) = \left(1 - \frac{y^2}{\delta^2} \right)^{1/2} 2F_1 \left( -\frac{m}{2}, \frac{m+1}{2}; \frac{1}{2}; \frac{y^2}{\delta^2} \right)$$

(15, p. 8)

$$= \left(1 - \frac{y^2}{\delta^2} \right)^{1/2} F_m \left( 1, 1/2, \frac{y^2}{\delta^2} \right)$$

(15, p. 83) (4.93)

Then, Equation 4.92 becomes
We consider the right hand side of the above equation to be the Hankel transform of $\lambda^{-3/2}J_{2m+1}(\delta \lambda)$ and use the inversion theorem to write

$$\int_0^\infty \lambda^{-1/2} J_{2m+1}(\delta \lambda) J_{-1/2}(y \lambda) d\lambda = \begin{cases} \frac{1}{2} (1 - \frac{\lambda^2}{\delta^2})^{1/2} \frac{(2)}{\nu (2m+1)} F_m(1,1/2,\frac{\lambda^2}{\delta^2}) & (0 < y < \delta) \\ 0 & (y > \delta) \end{cases}$$

(4.94)

Equation 4.94 shows that the representation of $G(\lambda)$ given by Equation 4.91 satisfies Equation 4.90b, provided we may interchange the order of summation and integration. We assume this may be done, and substitute Equation 4.91 into Equation 4.90a to obtain

$$\sum_{m=0}^\infty a_m \int_0^\infty \lambda^{-1/2} h(\lambda) J_{2m+1}(\delta \lambda) J_{-1/2}(y \lambda) d\lambda = \frac{\bar{g}(y)}{\sqrt{y}}$$

(0 < y < \delta)

(4.96)

Multiply both sides of Equation 4.96 by

$$y^{1/2} (1 - \frac{\lambda^2}{\delta^2})^{1/2} F_n(1,1/2,\frac{\lambda^2}{\delta^2}), n=0,1,2,...$$

and integrate with respect to $y$ from zero to $\delta$. 
\[ \sum_{m=0}^{\infty} a_m \int_0^{\delta} \frac{1}{2} h(\lambda) J_{2m+1}(\delta \lambda) J_{-1/2}(y) \, dy \]

\[ = \int_0^{\delta} \overline{g}(y) (1 - \frac{y^2}{\delta^2})^{1/2} \frac{1}{2} F_n(1,1/2,\frac{y^2}{\delta^2}) \, dy \quad n=0,1,2,\ldots \]  

\[ (4.97) \]

Interchange orders of integration.

\[ \sum_{m=0}^{\infty} a_m \int_0^{\infty} \frac{1}{2} h(\lambda) J_{2m+1}(\delta \lambda) \, d\lambda \int_0^{\delta} \frac{1}{2} y^2 (1 - \frac{y^2}{\delta^2}) \frac{1}{2} F_n(1,1/2,\frac{y^2}{\delta^2}) J_{-1/2}(\lambda y) \, dy \]

\[ = \int_0^{\delta} \overline{g}(y) (1 - \frac{y^2}{\delta^2})^{1/2} F_n(1,1/2,\frac{y^2}{\delta^2}) \, dy \quad n=0,1,2,\ldots \]  

\[ (4.98) \]

Use Equation 4.95 in Equation 4.98, and let

\[ z = \frac{y}{\delta}, \quad g(z) = \overline{g}(y). \]

\[ \sum_{m=0}^{\infty} a_m \int_0^{\infty} \frac{1}{2} h(\lambda) J_{2m+1}(\delta \lambda) J_{2n+1}(\delta \lambda) \, d\lambda \]

\[ = \frac{\delta}{2n+1} \left( \frac{2}{\pi} \right) \int_0^{1} g(z) (1-z^2)^{1/2} F_n(1,1/2,z^2) \, dz \quad n=0,1,2,\ldots \]  

\[ (4.99) \]

It follows from Equation 4.89 that \( g(z) \) is an even function of \( z \), so we choose the representation.
\[ g(z) = \sum_{k=0}^{\infty} b_k F_k(1,1/2,z^2), \]  

where the \( F_k(1,1/2,z^2) \) are Jacobi polynomials which have the orthogonality property

\[
\int_0^1 (1-z^2)^{1/2} F_k(1,1/2,z^2) F_n(1,1/2,z^2) \, dz 
= \frac{1}{2} \int_0^1 (1-t)^{1/2} F_k(1,1/2,t) F_n(1,1/2,t) t^{-1/2} \, dt 
\]

\[
= \begin{cases} 
0 & \text{if } k \neq n \\
\frac{\pi}{4} & \text{if } k = n.
\end{cases}
\]

We use this property to find

\[
b_n = \frac{4}{\pi} \int_0^1 g(z) (1-z^2)^{1/2} F_n(1,1/2,z^2) \, dz \quad n=0,1,2,\ldots
\]

and

\[
\sum_{m=0}^{\infty} a_m \int_0^{\infty} \lambda^{-2} h(\lambda) J_{2m+1}(\delta \lambda) J_{2n+1}(\delta \lambda) \, d\lambda = \frac{\delta}{(4n+2)^{1/2}} \frac{\pi}{2} b_n.
\]

\[
n=0,1,2,\ldots
\]

We again have reduced the problem to that of solving an infinite system of linear non-homogeneous algebraic equations for the \( a_m \), which will be done numerically. Finding the \( a_m \) is equivalent to determining \( K_2(\lambda) \), which we then use...
in Equations 4.79 through 4.83 to calculate the displacements and stresses throughout the strip.

The details of the numerical solution are presented in the following section.
V. NUMERICAL SOLUTION

A. Preliminary Considerations

We have thus far presented a strictly formal solution. When we attempt to perform the numerical calculations, certain practical problems arise, which we now discuss.

The most obvious question we must consider concerns the size of the system of algebraic equations we should solve to obtain a good approximation to the solutions of both the temperature and stress problems. Formally, we have an infinite system of equations to solve. The inclination is therefore to solve a very large system.

On the other hand, the amount of machine time required grows rapidly when the size of the system of equations increases. Furthermore, the solution of the system of algebraic equations represents only a small portion of the actual calculations to be performed; the time required to perform all other calculations increases proportionately with the size of the system.

Fortunately, the case is often that the leading terms in a series make the largest contribution to the sum of the series and it is not therefore necessary to take a large number of terms in the series to obtain a reasonable approximation to its value.

This suggests a possible criterion for deciding on the number of terms to be considered. If the first few terms
contribute a high percentage of the value of the solution obtained from a somewhat larger system, then we consider the solution so obtained to be a good approximation to the exact solution. As we will later show, this appeared to be a good method in the case of the temperature problem.

Another possible criterion could be based on how closely the approximate solution satisfies the imposed boundary conditions. This proved to be a convenient criterion in the solution to the stress problem.

Another difficulty arises when we attempt to evaluate the integrals for the coefficients of the $A_m$ and $a_m$. These integrals are

$$C_{mn} \equiv \int_0^\infty \xi^{-1} H(\xi) J_{2m+1/2} (\delta \xi) J_{2n+1/2} (\delta \xi) d\xi, \quad m,n=0,1,2,\ldots$$

(5.1)

and

$$C_{mn} \equiv \int_0^\infty \xi^{-2} h(\xi) J_{2m+1} (\delta \xi) J_{2n+1} (\delta \xi) d\xi, \quad m,n=0,1,2,\ldots$$

(5.2)

for the coefficients of the $A_m$ and $a_m$ respectively, where

$$H(\xi) = 1 + \frac{htanh\xi}{\xi}$$

(5.3)

and

$$h(\xi) = \frac{\xi^2}{\sinh^2 \xi} + \xi \coth \xi$$

(5.4)
These integrals converge slowly, the coefficients of the Bessel functions approaching $1/\xi$ in both cases. This behavior, coupled with the oscillatory nature of the Bessel functions, makes evaluation of the integrals difficult. We first seek to write the integrals in a form more suitable for numerical evaluation.

B. Conditioning of Equations

To condition the integrals in Equations 5.1 and 5.2 for evaluation on the IBM 360/65 digital computer we use the formula of Sonine and Schafheitlin given by

$$
\int_0^\infty J_\mu(at)J_\nu(at)t^{-\lambda}dt = \frac{\frac{2}{\lambda}^\lambda \Gamma(\lambda) \Gamma\left(\frac{\mu+\nu+1}{2}\right)}{2\Gamma\left(\frac{-\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+\lambda+1}{2}\right)\Gamma\left(\frac{\mu-\nu+\lambda+1}{2}\right)}
$$

which is valid for

$$\text{Re}(\mu+\nu+1) > \text{Re}(\lambda) > 0, \quad a > 0 \quad (15, \text{p. 35}).$$

In particular, for $\lambda=1$, $\mu=2m+1/2$, and $\nu=2n+1/2$,

$$
\int_0^\infty J_{2m+1/2}(\delta \xi)J_{2n+1/2}(\delta \xi)^{-1}d\xi = \frac{\delta_{mn}}{4n+1}, \quad m, n = 0, 1, 2, \ldots,
$$

where $\delta_{mn}$ is the Kronecker delta.

Substitute Equation 5.3 into Equation 5.1 and use Equation 5.6 to obtain
\[c_{mn} = \frac{\delta_{mn}}{4n+1} + h \int_0^\infty \frac{\tanh \xi}{\xi^2} J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi) d\xi,\]

\[m, n = 0, 1, 2, \ldots,\]  

(5.7)

We now add and subtract

\[h \int_0^\infty \frac{J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi)}{\xi^2} d\xi\]

in Equation 5.7 to obtain

\[c_{mn} = \frac{\delta_{mn}}{4n+1} + h \int_0^\infty \frac{J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi)}{\xi^2} d\xi\]

\[\quad - h \int_0^\infty \frac{(1 - \tanh \xi)}{\xi^2} J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi) d\xi.\]  

(5.8)

For \(\lambda = 2,\) Equation 5.5 gives

\[\int_0^\infty \frac{J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi)}{\xi^2} d\xi\]

\[= \frac{\delta}{4} \frac{\Gamma(2) \Gamma(m+n)}{\Gamma(n-m+3/2) \Gamma(m+n+2) \Gamma(m-n+3/2)}\]

\[= \frac{\delta (-1)^{m-n}}{4\pi} \frac{1}{\{\frac{1}{4} (m-n)^2\}^2 \{m+n\} \{m+n+1\}},\]  

(5.9)

which is valid for
or
\[ n=0,1,2,\ldots \quad \text{and} \quad m=0,1,2,\ldots, \]

but not for both \( m \) and \( n \) equal to zero.

We note these manipulations result in a significant improvement in the rate of convergence of the remaining integral; the coefficient of the Bessel functions now approaches \((2/\xi)e^{-2\xi}\) for large values of \( \xi \).

We use these results in Equation 4.70, and separate the term corresponding to \( m=n=0 \), to obtain

\[
A_o \left\{ 1 + h \int_0^\infty \frac{\tanh \xi}{\xi^2} J_{1/2}^2(\delta \xi) d\xi \right\}
\]

\[
+ h \sum_{m=1}^\infty A_m \left\{ \frac{\delta(-1)^{m+1}}{\pi} \frac{1}{m(m+1)(4m^2-1)} \right\}
\]

\[
- \int_0^\infty \frac{(1-\tanh \xi)}{\xi^2} J_{2m+1/2}(\delta \xi) J_{1/2}(\delta \xi) d\xi = \delta^{1/2} B_0 \quad (5.10)
\]

and

\[
\frac{A_n}{4n+1} + h \sum_{m=0}^\infty A_m \left\{ \frac{(-1)^{m+n+1}}{\pi} \frac{1}{(4(m-n)^2-1)(m+n)(m+n+1)} \right\}
\]

\[
- \int_0^\infty \frac{(1-\tanh \xi)}{\xi^2} J_{2m+1/2}(\delta \xi) J_{2n+1/2}(\delta \xi) d\xi
\]
We use the series representation (Appendix A)

\[ \int_0^\infty \frac{\tanh \xi}{\xi^2} J_{1/2}(\delta \xi) d\xi = 1 + \frac{8}{\pi^3} \sum_{n=1}^\infty \frac{e^{-(2n-1)\pi\delta-1}}{(2n-1)^3}, \]

and replace Equation 5.10 with

\[ A_0 \{1 + h + \frac{8h}{\pi^3} \sum_{n=1}^\infty \frac{e^{-(2n-1)\pi\delta-1}}{(2n-1)^3}\} + h \sum_{m=1}^\infty A_m \frac{\delta(-1)^{m+1}}{\pi m(m+1)(4m^2-1)} \]

\[- \int_0^\infty \frac{(1-\tanh \xi)}{\xi^2} J_{2m+1/2}(\delta \xi) J_{1/2}(\delta \xi) d\xi = \delta^{1/2} B_0. \]

The temperature is then determined from

\[ \theta(x,y) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \frac{\Pi(\lambda) \cosh \lambda x}{\cosh \lambda} \cos y d\lambda \]

\[ + \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \frac{C(\lambda) \sinh \lambda (1+x)}{\cosh \lambda} \cos y d\lambda. \]

It follows from Equation 4.19 that the value of the first integral is the temperature in the solid strip. We indicate quantities which are known from the solution for
the solid strip by underlining.

Thus, substituting for $C(\lambda)$, we find

$$\theta(x,y) = \overline{\theta}(x,y) + \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} \left\{ \sum_{m=0}^{\infty} J_{2m+1/2}(\delta \lambda) \right\} \frac{\sinh\lambda(1+x)}{\lambda^{3/2} \cosh\lambda} \cos \lambda \ d\lambda.$$  

(5.14)

We anticipate that the first term in the series appearing in Equation 5.14 is the dominant one, and therefore choose to separate it from the remaining terms.

We write $J_{1/2}(\delta \lambda)$ in terms of $\sin(\lambda \delta)$ to obtain

$$\theta(x,y) = \overline{\theta}(x,y) + \frac{2A}{\pi \delta^{1/2}} \int_{0}^{\infty} \frac{\sin\lambda \delta \cos \lambda \sinh\lambda(1+x)}{\lambda^{2} \cosh\lambda} \ d\lambda$$

$$+ \left( \frac{2}{\pi} \right)^{1/2} \int_{0}^{\infty} \left\{ \sum_{m=1}^{\infty} J_{2m+1/2}(\delta \lambda) \right\} \frac{\sinh\lambda(1+x)}{\lambda^{3/2} \cosh\lambda} \cos \lambda \ d\lambda.$$  

(5.15)

It is shown in Appendix A that

$$\int_{0}^{\infty} \frac{\sin\lambda \delta \cos \lambda \sinh\lambda(1+x)}{\lambda^{2} \cosh\lambda} \ d\lambda = \Sigma_{0}(x,y; \delta),$$  

(5.16)

where
\[
\Sigma_\theta(x,y;\delta) = \begin{cases} 
\frac{\pi(1+x)}{2} 
& (y \leq \delta) \\
\frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{(2n-1)^2}{2} \pi \delta} \cosh\left(\frac{2n-1}{2} \pi y\sin\left(\frac{(2n-1)(1+x)\pi}{2}\right)\right) 
& (y > \delta) 
\end{cases}
\]

Thus,

\[
\theta(x,y) = \theta(x,y) + \frac{2A_0}{\pi \delta^{1/2}} \Sigma_\theta(x,y;\delta)
\]

\[
+ \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \left\{ \sum_{m=1}^{\infty} A_m J_{2m+1/2}(\delta \lambda) \right\} \frac{\sinh(1+x)}{\lambda^{3/2} \cosh \lambda} \cos \lambda y \, d\lambda.
\]

The coefficients of the \(a_m\) in the stress problem are

\[
c_{mn} = \int_0^\infty \left\{ \frac{1}{\sinh^2 \xi} + \frac{\coth \xi}{\xi} \right\} J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi) \, d\xi.
\]

We add and subtract
\[ \int_0^\infty \frac{J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi)}{\xi} \, d\xi \]
to obtain

\[ c_{mn} = \int_0^\infty \left( \frac{1}{\sinh^2 \xi} + \frac{\coth \xi - 1}{\xi} \right) J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi) \, d\xi \\
+ \int_0^\infty \frac{J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi)}{\xi} \, d\xi \quad m,n=0,1,2,\ldots \quad (5.20) \]

Use Equation 5.5 with \( \lambda = 1, \mu = 2m+1, \) and \( v = 2n+1 \) to obtain

\[ \int_0^\infty \frac{J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi)}{\xi} \, d\xi = \frac{\delta_{mn}}{4n+2} \quad m,n=0,1,2,\ldots \quad (5.21) \]

Thus, the system of equations is written as

\[ \frac{a_n}{4n+a} + \sum_{m=0}^\infty a_m \int_0^\infty \left\{ \frac{1}{\sinh^2 \xi} + \frac{\coth \xi - 1}{\xi} \right\} J_{2m+1}(\delta \xi) J_{2n+1}(\delta \xi) \, d\xi \]

\[ = \frac{\delta}{4n+2} \left( \frac{\pi}{2} \right)^{1/2} b_n, \quad n=0,1,2,\ldots \quad (5.22) \]

We note that \( \frac{\coth \xi - 1}{\xi} \) approaches \( 2e^{-2\xi} \) and \( \frac{1}{\sinh^2 \xi} \) approaches \( 4e^{-2\xi} \) for large \( \xi \).

It is possible to separate the stresses and displacements corresponding to the solid strip from the remaining
terms appearing in the cracked strip solution. We do so for convenience, again underlining the solid strip terms.

Many terms which appear in Equation 4.81 for $\sigma_x$ also appear in Equation 4.82 for $\sigma_y$. To avoid unnecessary repetition of numerical calculations we calculate the sum and difference of $\sigma_x$ and $\sigma_y$ first, and then separate them by addition and subtraction.

The algebraic manipulations are straightforward. We omit the details which lead to the following results:

\[
\begin{align*}
\text{u}(x,y) & = u(x,y) + \frac{1}{\delta} \int_0^\infty \left\{ \frac{\sinh \frac{\xi x}{\delta}}{\sinh \frac{\xi}{\delta}} + \frac{\delta^2 \sinh \frac{\xi (1+x)}{\delta}}{(\beta^2-1) \xi \sinh \frac{\xi}{\delta}} \right\} \\
& - \frac{\xi \cosh \frac{\xi (1+x)}{\delta}}{\sinh \frac{\xi}{\delta}} \left\{ \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \right\}
\end{align*}
\]

\[
\begin{align*}
\text{v}(x,y) & = v(x,y) + \frac{1}{\delta} \int_0^\infty \left\{ \frac{\cosh \frac{\xi (1+x)}{\delta}}{(\beta^2-1) \xi \sinh \frac{\xi}{\delta}} + \frac{\xi \sinh \frac{\xi (1+x)}{\delta}}{\sinh \frac{\xi}{\delta}} \right\} \\
& - \frac{\cosh \frac{\xi x}{\delta}}{\sinh \frac{\xi}{\delta}} \left\{ \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \right\} + \frac{b \delta^{3/2}}{\sqrt{2\pi} \beta^{3/2} \xi^{3/2}} \left[ \frac{\delta \sinh \frac{\xi (1+x)}{\delta}}{\xi \cosh \frac{\xi}{\delta}} \right]
\end{align*}
\]
\[ \tau_{xy}(x,y) = I_{xy}(x,y) + \frac{2}{\delta} \int_0^\infty \left\{ \frac{\xi \cosh\frac{\xi(1+x)}{\delta}}{\delta \sinh\frac{\xi}{\delta}} \right\} \left\{ \sum_{m=0}^\infty A_m J_{2m+1}(\xi) \right\} \sin\frac{\xi y}{\delta} \, d\xi, \quad (5.24) \]

\[ \frac{\delta^{1/2} b x \sinh\frac{\xi(1+x)}{\delta}}{(2\pi \xi)^{1/2} b^2 \cosh\frac{\xi}{\delta}} \left\{ \sum_{m=0}^\infty A_m J_{2m+1}(\xi) \right\} \sin\frac{\xi y}{\delta} \, d\xi, \]

\[ (5.25) \]

\[ \sigma_x(x,y) + \sigma_y(x,y) = g_x(x,y) + g_y(x,y) \]

\[ + \frac{4}{\delta} \int_0^\infty \left\{ \frac{\cosh\frac{\xi(1+x)}{\delta}}{\sinh\frac{\xi}{\delta}} \right\} \left\{ \sum_{m=0}^\infty A_m J_{2m+1}(\xi) \right\} \cos\frac{\xi y}{\delta} \, d\xi, \]

\[ (5.26) \]

\[ \sigma_x(x,y) - \sigma_y(x,y) = g_x(x,y) - g_y(x,y) \]

\[ + \frac{4}{\delta} \int_0^\infty \left\{ \frac{\frac{\xi \cosh\frac{\xi}{\delta}}{\delta \sinh\frac{\xi}{\delta}} - \frac{\xi x \sinh\frac{\xi(1+x)}{\delta}}{\delta \sinh\frac{\xi}{\delta}}}{\delta \sinh\frac{\xi}{\delta}} \right\} \left\{ \sum_{m=0}^\infty A_m J_{2m+1}(\xi) \right\} \]

\[ \delta \sinh\frac{\xi}{\delta} \]
\[
\delta^{1/2} b x \cosh \frac{\xi (1 + x)}{\delta} + \left\{ \frac{1}{2 \pi} \right\} \sum_{m=0}^{\infty} A_m J_{2m+1/2}(\xi) \cos \frac{\xi y}{\delta} \, d\xi. \]  
\]  
(5.27)

Note that the variable of integration \( \lambda \) has been replaced by \( \xi \), where

\[ \lambda = \frac{\xi}{\delta}. \]

In the above equations, \(-1 < x < 0\). On \( x = 0 \), several of the ratios of hyperbolic functions in the previous integrals approach one, which makes numerical evaluation difficult. We therefore consider evaluation of the stresses and displacements on \( x = 0 \) as a separate case.

It follows from Equations 5.24 and 4.40 that

\[ \tau_{xy}(0, y) = 0. \]

Equation 4.36 implies

\[ u(0, y) = 0. \]

These results satisfy the specified boundary conditions. Then,

\[ u(0, y) = \frac{\delta^2}{\beta^2 - 1} \int_0^\infty \frac{1}{3} \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \cos \frac{\xi y}{\delta} \, d\xi \]  
(5.28)

Integrals of the form
are specializations of the Weber-Schafheitlin type, having the values
\[ \frac{1}{2m+1} \cos \{ (2m+1) \sin^{-1} (y/\delta) \} \quad \text{for } 0 \leq y \leq \delta \]
and
\[ 0 \quad \text{for } y > \delta \]
(14, p. 487).

Interchange the order of summation and integration in Equation 5.28 and use the previous result to find
\[
\begin{align*}
\frac{\delta^2}{\beta^2 - 1} \sum_{m=0}^{\infty} \frac{a_m}{2m+1} \cos \{ (2m+1) \sin^{-1} (y/\delta) \} \\
0 \quad \text{(for } y > \delta) \end{align*}
\]
(5.29)

Note that Equation 5.29 describes the shape of the deformed crack for 0 \leq y \leq \delta and satisfies the displacement boundary condition for y > \delta.

For x = 0, Equation 5.23 gives
\[
v(0,y) = v(0,y) + \int_0^{\infty} \left[ \left\{ \frac{\coth \frac{\xi}{\delta}}{\xi (\beta^2 - 1)} - \frac{1}{\delta \sinh^2 \frac{\xi}{\delta}} \right\} \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \right] \sin \frac{\xi y}{\delta} \, d\xi.
\]
(5.30)
Add and subtract
\[
\frac{1}{\beta^2-1} \int_0^\infty \frac{1}{\xi} \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \sin \frac{\xi y}{\delta} \, d\xi
\]
to obtain
\[
v(0,y) = v(0,y) + I_v(y;\delta) + \frac{1}{\beta^2-1} \int_0^\infty \frac{1}{\xi} \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \sin \frac{\xi y}{\delta} \, d\xi,
\]
where
\[
I_v(y;\delta) = \int_0^\infty \left[ \frac{\coth \frac{\xi}{\delta} - 1}{\xi(\beta^2-1)} - \frac{1}{\delta\sinh^2 \frac{\xi}{\delta}} \right] \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \sin \frac{\xi y}{\delta} \, d\xi.
\]
\[
+ \left\{ \frac{b_0^{3/2}\tanh\frac{\xi}{\delta}}{\sqrt{2\pi} \beta \frac{5/2}{\xi}} \right\} \sum_{m=0}^{\infty} A_m J_{2m+1/2}(\xi) \sin \frac{\xi y}{\delta} \, d\xi.
\]
(5.32)

Following the method used previously for \(u(0,y)\), we use the result
\[
\int_0^\infty \frac{1}{\xi} J_{2m+1}(\xi) \sin \frac{\xi y}{\delta} \, d\xi = \begin{cases} \frac{1}{2m+1} \sin \{ (2m+1) \sin^{-1} (y/\delta) \} & (0 < y < \delta) \\ (-1)^m (2m+1) \{ y/\delta + (y^2/\delta^2 - 1)^{1/2} \}^{2m+1} & (y > \delta) \end{cases}
\]
(14, p. 487)
to write
\[
v(0,y) = v(0,y) + I_v(y;\delta) + \Sigma_v(y;\delta),
\]
(5.33)
where

\[
\Sigma_v(y;\delta) = \begin{cases} \\
\displaystyle \sum_{m=0}^{\infty} \frac{1}{\beta^2 - 1} \frac{(-1)^m a_m}{(2m+1)} \frac{\sin\left\{ (2m+1) \sin^{-1}(y/\delta) \right\}}{(2m+1)} \
\end{cases} \\
(y > \delta)
\]

\begin{align}
\sigma_x(0,y) + \sigma_y(0,y) &= \sigma_x(0,y) + \sigma_y(0,y) + I_\sigma(y;\delta) \\
&+ \frac{4}{\delta} \sum_{m=0}^{\infty} a_m \int_0^\infty J_{2m+1}(\xi) \cos\frac{\xi y}{\delta} \, d\xi \\
&- \frac{4b_\delta}{(2\pi)^{1/2}} A_\delta \int_\xi^{3/2} \frac{\tanh\frac{\xi}{\delta}}{\delta} J_{1/2}(\xi) \cos\frac{\xi y}{\delta} \, d\xi,
\end{align}

(5.35)

where

\[
I_\sigma(y;\delta) = \frac{4}{\delta} \int_0^\infty \left\{ \coth\frac{\xi}{\delta} - 1 \right\} \sum_{m=0}^{\infty} a_m J_{2m+1}(\xi) \\
- \frac{\delta^{3/2} b \tanh\frac{\xi}{\delta}}{(2\pi)^{1/2} \beta^{3/2}} \sum_{m=1}^{\infty} A_m J_{2m+1/2}(\xi) \cos\frac{\xi y}{\delta} \, d\xi.
\]

(5.36)
From (14, p. 487) we obtain another form of the Weber-Schafheitlin integral,

\[ \int_0^\infty J_{2m+1}(\xi) \cos \frac{\xi Y}{\delta} \, d\xi = \begin{cases} 
\frac{\cos((2m+1) \sin^{-1}(\frac{Y}{\delta}))}{(1-\frac{y^2}{\delta^2})^{1/2}} & (0 \leq y < \delta) \\
(-1)^{m+1} & (y > \delta) 
\end{cases} \] 

\[ \frac{y^2}{\delta^2} - 1 \right)^{1/2} \left\{ \frac{Y}{\delta} + \left( \frac{y^2}{\delta^2} - 1 \right)^{1/2} \right\}^{2m+1} \]

Now,

\[ \frac{4b \delta^{1/2}}{(2\pi)^{1/2} \delta^2} A_o \int_0^\infty \frac{\tanh \xi}{\xi^{1/2}} J_{1/2}(\xi) \cos \frac{\xi Y}{\delta} \, d\xi \]

\[ = \frac{4b}{\pi \delta^{1/2}} A_o \int_0^\infty \frac{\tanh \lambda}{\lambda^2} \sin \lambda \delta \cos \lambda y \, d\lambda \]

\[ = \frac{4b A_o}{\pi \delta^{1/2}} \Sigma (0, y; \delta), \] 

(5.38)

where \( \Sigma (x, y; \delta) \) is given by Equation 5.17.

Let

\[ \Sigma (y; \delta) = \begin{cases} 
\sum_{m=0}^{\infty} \frac{a_m \cos((2m+1) \sin^{-1}(\frac{Y}{\delta}))}{(\delta^2 - y^2)^{1/2}} & (0 \leq y < \delta) \\
\sum_{m=0}^{\infty} \frac{(-1)^{m+1} a_m}{(y^2 - \delta^2)^{1/2} \left\{ y/\delta + \left( \frac{y^2}{\delta^2} - 1 \right)^{1/2} \right\}^{2m+1}} & (y > \delta) 
\end{cases} \]

(5.39)
Then

\[ \sigma_x(0,y) + \sigma_y(0,y) = \sigma_x(0,y) + \sigma_y(0,y) + I_R(y;\delta) \]

\[ - \frac{4b A_0}{\pi B \delta^{1/2}} \sum \Theta(0,y;\delta) + \Sigma_\sigma(y;\delta). \]  

(5.39a)

We need not modify Equation 5.27. For \( x=0 \), we have

\[ \sigma_x(0,y) - \sigma_y(0,y) = \sigma_x(0,y) - \sigma_y(0,y) \]

\[ + \frac{4}{\delta^2} \int_0^\infty \frac{\xi}{\sinh^2 \frac{\xi}{\delta}} \left\{ \sum a_m J_{2m+1}(\xi) \right\} \cos \frac{\xi y}{\delta} \, d\xi. \]  

(5.40)

This completes the necessary conditioning of the equations for numerical solution. We emphasize that all results presented in this section are valid for any edge temperature satisfying the previously given symmetry requirements and possessing a Fourier cosine transform.

C. Example

We now complete the solution for the cracked strip with applied boundary temperatures as given by Equation 4.41. We here describe the numerical techniques used, and the results of the calculations are given in the following section. The coefficients calculated for the various
expansions are given in Appendix B, as are the coefficient matrices in the systems of equations for the $A_m$ and $a_m$.

From Equations 4.57 and 4.42 we obtain

$$f(y) = h\theta_C(y) - \frac{2hT}{\pi} \int_0^\infty \frac{\sin \xi y \cos \xi y}{\xi \cosh \xi} d\xi $$

$$= h\theta_C(y) - \frac{2hT}{\pi} \tan^{-1} \left( \frac{\sinh \frac{\pi y}{2}}{\cosh \frac{\pi y}{2}} \right).$$

$$(12, p. 517)$$

Thus,

$$f(z) = h\theta_C(\delta z) - \frac{2hT}{\pi} \tan^{-1} \left( \frac{\sinh \frac{\pi \delta z}{2}}{\cosh \frac{\pi \delta z}{2}} \right).$$

Then, it follows from Equations 4.69 and 5.41 that

$$B_n = h(4n+1) \left\{ \frac{(1/2,n)}{n!} \right\} \int_0^1 \theta_C(\delta z)$$

$$- \frac{2T}{\pi} \tan^{-1} \left( \frac{\sinh \frac{\pi \delta z}{2}}{\cosh \frac{\pi \delta z}{2}} \right) F_n(1/2,1/2,z^2) dz.$$  

$$(5.42)$$

At this point, it is necessary to specify $\theta_C(\delta z)$.

To eliminate the effects of the discontinuity in the applied temperature at $y=l$ from influencing the temperature distribution in the vicinity of the crack to any great extent, we take $\lambda \gg \delta$. Then, a reasonable assumption is that
\[ \theta_C(\delta z) = CT, \quad (5.43) \]

where \( C \) is a constant.

Since

\[ F_0(\alpha, \beta, t) = 1 \quad (15, \text{p. 83}) \]

for any \( \alpha, \beta, \) and \( t, \) it follows from the orthogonality relationship given by Equation 4.68 that

\[ B_0 = hT(C - \frac{2}{\pi} \int_0^1 \tan^{-1}(\frac{\sinh \frac{\pi \ell}{2}}{\cosh \frac{\pi \delta z}{2}}) \, dz) \quad (5.44) \]

and

\[ B_n = -\frac{2hT}{\pi^2} (4n+1) \left(\frac{1/2, n}{n!}\right)^2 \int_0^1 \tan^{-1}(\frac{\sinh \frac{\pi \ell}{2}}{\cosh \frac{\pi \delta z}{2}}) F_n(\frac{1}{2}, \frac{1}{2}, z^2) \, dz. \quad n=1, 2, 3, \ldots \quad (5.45) \]

We now choose numerical values for the necessary parameters.

For the radiation condition, we use the universal radiation constant

\[ \sigma = 5.685(10)^{-8} \text{ watt} \frac{\text{m}^2}{\text{K}^4}. \quad (16, \text{p. 36}) \]

For steel,

\[ \kappa = 0.108 \frac{\text{cal}}{\text{sec-cm} \cdot \text{K}}, \quad (17, \text{p. 2434}) \]

\[ \nu = 0.3, \]

\[ \alpha = 11.9(10)^{-6} \frac{\text{in/in}}{\text{K}}, \]
and
\[ e = 0.94. \]

(17, p. 2959)

The value of \( e \) given is for a rough surface. The values of \( \nu \) and \( \alpha \) are nominal values for steel.

We take
\[
C = 0.9, \\
\lambda = 5, \\
\delta = 0.1, \\
w = 12 \text{ in.}
\]

and
\[
T_o = 293^\circ K.
\]

The value of \( C \) is chosen under the assumption that \( \theta_c \) will be close to the temperature of the strip at that point if the crack were not present. We choose \( \lambda \) to be much larger than \( \delta \) to reduce the effect of the discontinuity of the edge temperature in the region of the crack.

Numerous quadrature techniques are available for the approximate evaluation of integrals, some of which are more suitable than others for a given integral.

For \( \lambda \gg \delta \), the inverse tangent appearing in Equations 5.44 and 5.45 is a slowly decreasing monotonic function in the interval from 0 to 1. Hence, the behavior of the arguments of the integrals closely approximates that of
a constant multiple of the Jacobi polynomials. For this reason, a suitable choice for the evaluation of these integrals appeared to be a subroutine designed for accurate integration of polynomials.

The subroutine selected was DQG32 (18, p. 303), a 32 point Gaussian quadrature formula capable in principle of integrating polynomials of up to degree 63 exactly.

The orthogonality relationship given in Equation 4.68 provided a check on the accuracy of machine calculations. Integrals of the Jacobi polynomials alone over z=0 to z=1 vanish for n=1, 2, 3,..., so these were calculated in addition to those appearing in Equation 5.42 as a test of the subroutine.

The Jacobi polynomial of degree n in Equation 5.45 has n distinct zeros, all of which lie between z=0 and z=1. As anticipated, this rapidly oscillating behavior introduced error in the numerical integration. The values of the test integrals should have been zero to sixteen digits (the number carried is double precision arithmetic), but this was not the case for orders higher than three.

However, f(z) is almost constant, and a comparison of the values of the test integrals with the actual integrals indicated that f(z) could be adequately represented by the first three terms in the expansion. In fact, the integral corresponding to n=0 is approximately ten orders of magnitude
larger than that corresponding to $n=2$, and thirteen orders of magnitude larger than that for $n=3$. Furthermore, the actual integral for $n=2$ was seven orders of magnitude larger than the test integral, an indication of significant computational difference.

We therefore set $B_n=0$ for $n>2$.

To calculate the coefficients of the $A_m$, it was first necessary to obtain values for the Bessel functions. The most practical method is to generate them in the machine.

A program was written which used the downward recursion process (19) to generate spherical Bessel functions of the first kind, which are given by

$$j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+1/2}(z)$$

The downward recursion process has the advantage that all orders from zero to a maximum specified order are generated for a given argument without the necessity of supplying any tabulated values. The process proved to be extremely efficient for this application; over 700 values for each of ten arguments are generated in less than 5.5 seconds.

The integrations were performed using subroutine QSF (18, p. 291), a quadrature method based on Simpson's rule which was first tested on functions with behavior similar to the product of spherical Bessel functions but which could be integrated analytically.
The infinite sum appearing in Equation 5.12 was evaluated using subroutine TEUL (18, p. 238), which relies on an Euler transformation to accelerate convergence.

The remaining calculations for the coefficients of the $A_m$ were straightforward, merely involving substitution of appropriate values of $m$ and $n$.

It was decided to try a 20 x 20 system. To determine the sensitivity of the system to the number of equations used, 5 x 5, 10 x 10, and 15 x 15 systems were also solved. Two different single precision methods were used to serve as a cross-check.

The first, GELG (18, p. 122) solves the system of linear equations by Gauss elimination. The second, MINV (18, p. 118) inverts the coefficient matrix. The right hand sides are then premultiplied by the inverse of the coefficient matrix.

The values of the $A_m$ obtained from each method for a given system size agreed in all cases to six out of seven digits. Furthermore, those values of the $A_m$ which could be compared for the different systems agreed closely; the worst difference was between $A_4$ in the 5 x 5 and 20 x 20 systems, this difference occurring in the sixth digit.

The summation appearing in the integral in Equation 5.18 was then calculated (having been approximated by 20 terms) for evenly spaced values of $\lambda$ and punched on cards.
by the computer. The temperature over a region of the plate given by \(-1 \leq x \leq 0, 0 \leq y \leq 3.5\delta\) was computed for a rectangular mesh of points with mesh size \(0.25\delta\). The solid strip temperature, the value of the sum, the value of the integral, and the cracked strip temperature were printed to allow comparison of the magnitudes of the various terms.

It was found that the contribution of the integral in Equation 5.18 was four to six orders of magnitude smaller than the contribution of the leading term in that equation. For this reason, it was not considered necessary to increase the number of terms, and the solution obtained from the 20 \(x\) 20 system was accepted as a good approximation to the temperature in the strip.

The first step in the solution for the stresses and displacements is the calculation of the \(b_n\).

From Equations 4.89 and 4.42, we obtain

\[
\bar{g}(y) = \frac{b\theta(0,y)}{(2\pi)^{1/2} \beta^2} + \frac{bT}{\pi \beta^2} \int_0^\infty \frac{\sin \xi \cos \xi y}{\sinh \xi} \, d\xi
\]

\[
= \frac{b\theta(0,y)}{(2\pi)^{1/2} \beta^2} + \frac{bT}{(2\pi)^{1/2} \beta^2} \frac{\sinh \pi \xi}{\cosh \pi \xi + \cos \pi y} . \quad (5.47)
\]

We write

\[
\theta(0,y) = \bar{\theta}(0,y) + \sum_{k=0}^{N} t_k y^{2k} . \quad (5.48)
\]
where the $t_k$ were determined from a least squares fit of seventeen data points calculated on $x=0$, $0 \leq y \leq \delta$. It was found that, for $N=2$, the root mean square error was $1.8(10)^{-6}$. This expansion was necessary because the integration subroutine used required a function, rather than a set of discrete values. The expansion was done in even powers of $y$ because of $y$ symmetry.

In attempting to integrate Equation 4.102 for the $b_n$, it was found that subroutine DQG32 gave inaccurate results for test integrations similar to those performed for the $B_n$ in the temperature problem. This difficulty was due to the presence of the square root.

Subroutine DNODE (20) proved to give satisfactory results for the test integrals, and was used for the calculation of the $b_n$. This subroutine uses a predictor-corrector method to reduce the stepsize used in the integration to maintain the accuracy specified by the user.

As in the case of the temperature problem, it was found that zero values could be taken for the $b_n$ for $n>2$ due to the smooth, nearly constant behavior of $g(z)$.

The Bessel functions appearing in Equation 5.22 were generated in the machine using a subroutine similar to that for the spherical Bessel functions. The subroutine was a modified version of subroutine BESSL (21), using the downward recursion process.
The coefficient matrix obtained was strongly diagonally dominant. The Jacobi iteration technique (22, p. 430) is particularly suitable for systems of equations of this type, giving rapidly converging solutions independent of the initial starting point for the iteration. A program using this technique was written and used to solve 5 \times 5, 10 \times 10, 15 \times 15, 20 \times 20, and 25 \times 25 systems. The resulting $a_m$ showed even less sensitivity to the size of the system than did the $A_m$ in the temperature problem.

The stresses and displacements were then calculated, except on $x=0$, by a method similar to that used for the temperature problem, using the 25 \times 25 system.

On $x=0$, the special equations previously developed were solved. $\Sigma_v(y;\delta)$ and $\Sigma_\sigma(y;\delta)$ were evaluated with subroutine TEUL. The integrations were performed using subroutine QSF.

We now consider the accuracy of the solution for the stresses and displacements.

The form of the solution we have chosen satisfies all mechanical boundary conditions exactly, independently of the values of the $A_m$ and $a_m$, with the exception of the homogeneous condition on $\sigma_x$ along the crack. We therefore confine our discussion to this boundary condition.

By extracting the stresses corresponding to those in the solid strip, we have effectively exhibited a "correction
term" which, in the case of $\sigma_x(0,y)$, $y<\delta$, should be of equal magnitude but of opposite sign to $\sigma_x(0,y)$. That is, if

$$\sigma_x(0,y) = \sigma_x(0,y) + \sigma_x^*(0,y),$$

then

$$\sigma_x^*(0,y) = -\sigma_x(0,y)$$

should be satisfied for $0<y<\delta$.

In our solution, this condition is closely approximated. In all cases, $\sigma_x^*(0,y)$ is to within less than 0.3% of the value necessary to satisfy this condition exactly.

We note (refer to Appendix B) that $a_{24}$ is equal to zero to our order of approximation, and that all $a_m$ are monotonically decreasing in magnitude. Furthermore, the coefficient matrix used to solve for the $a_m$ is such that the only non-zero terms in the lower right hand corner are on the diagonal. Thus, increasing the size of the system will not result in any more non-zero values for additional $a_m$.

It is possible that double precision arithmetic for all calculations might increase the accuracy of the solution. We did not feel, however, that this more expensive measure was justified on the basis of the results we obtained.
VI. RESULTS

The calculations previously described were performed for the temperature, stresses, and displacements at 615 evenly-spaced points in the region \(-1 \leq x \leq 0, 0 \leq y \leq 0.35\). The results of the calculations are shown on graphs in the following pages. The graphs have been drawn with the assumption that \(T\), the applied edge temperature, is negative (cooling), a condition necessary to assure that the crack opens.

It should be noted that the data has been plotted for \(x > 0\). Appropriate changes in the algebraic signs of the calculated data have been made where necessary based on conditions of symmetry or antisymmetry.

The temperatures in the cracked and uncracked strips are shown in Figure 4. It is apparent that the temperatures in the region considered are almost constant. The effect of the radiation condition imposed on the crack produces only a small change in the temperature distribution for the ambient temperature and value of \(\theta_c\) chosen. An increase in \(T_o\) or a decrease in the magnitude of \(\theta_c\) would increase the effect of the radiation condition.

The displacements in the \(x\) direction for the cracked and uncracked strips are shown in Figures 5 and 6 respectively. The large difference in the magnitude
Figure 4. Temperatures in cracked and uncracked strips
Figure 5. Displacements in x direction in the un-cracked strip
Figure 6. Displacements in the x direction in the cracked strip
Figure 6 (Continued)
of the displacements in the two cases is apparent. As can be seen from Figure 6, the value of $u$ is maximum at $x=y=0$, and decreases with increasing values of $x$ and $y$.

Figure 7 shows the displacements in the $y$ direction in the solid strip. The $y$ displacements in the cracked strip are shown in Figure 8. Notice that the opening of the crack permits larger displacements than are present in the uncracked strip along $x=0$. The rotation of material near the edges of the crack due to its opening reduce the magnitude of $v$ near $x=0$. The values of $v$ for $x$ near unity in the cracked strip approach those in the uncracked strip.

To further aid visualization of the displacement field in the vicinity of the crack, we have shown the distortion of an originally square grid placed on the uncracked and cracked strips and allowed to deform with the material upon cooling.

These grids are shown in Figures 9 and 10. The displacements are multiplied by a factor of 20 to make them more readily visible. The numbers on the axes refer to the original positions on the strip before deformation.

Figures 11 and 12 show $\sigma_x$ in the uncracked and cracked strips. This stress is almost constant throughout the region considered in the uncracked strip. The presence of the crack causes extremely high stresses to occur in the vicinity of the crack tip, the normal stress $\sigma_x$ being singular at the
Figure 7. Displacements in the y direction in the uncracked strip
Figure 8. Displacements in the y direction in the cracked strip
Figure 9. Distorted grid on the uncracked strip
Figure 10. Distorted grid on the cracked strip
Figure 11. \( \sigma_x \) in the uncracked strip
Figure 12. $\sigma_x$ in the cracked strip
Figure 12 (Continued)
Figure 12 (Continued)
tip of the crack. This condition would actually produce yielding of the material in the vicinity of the crack tip, a condition which cannot be described by the linear elastic model we have used. This effect is confined to the immediate vicinity of the crack tip, and should not seriously distort the values of the stresses we have obtained away from the crack tip.

Figure 13 is a plot of the stress surface $z=\sigma_x(x,y)$ for $0\leq x \leq 0.35, -0.35 \leq y \leq 0.35$. This plot is intended to provide an overall subjective view of the behavior of $\sigma_x$ in the area around the crack. Values of $z=0$ have been plotted around the edge of the surface to provide a reference plane, a practice we have followed for all such plots presented here.

Figure 14 is a plot of the portion of the $\sigma_x$ surface lying in the first quadrant from a point of view which places the crack in the lower left hand corner. The $y$ axis is nearly perpendicular to the plane of the page. This view was chosen to show the behavior of $\sigma_x$ in the area adjacent to the crack. The right portion of the surface is not vertical due to the linear interpolation used by the plotting subroutine for portions of the surface between data points.

The stress $\sigma_y$ occurring in the uncracked plate is shown in Figure 15. We observe that $\sigma_y$ is considerably smaller than $\sigma_x$. The change in sign is due to the temperature
Figure 13. Stress surface \( z = q(x,y) \) for the cracked strip, \(-0.35 \leq y \leq 0.35, 0 \leq x \leq 0.35\).
Figure 14. Stress surface $z = \sigma_x(x,y)$ for the cracked strip, $0 < x < 0.35$, $0 < y < 0.35$. 
Figure 15. $\sigma_y$ in the uncracked strip
gradient across the strip. The material near the edge is cooler than that in the center, a condition which induces compressive stresses in the warmer material and tensile stresses in the cooler material.

The distribution of $\sigma_y$ in the cracked strip is shown in Figure 16. We observe the singularity in $\sigma_y$ at the crack tip. We find a nearly hydrostatic state of stress exists in the material immediately adjacent to the crack tip.

Figure 17 provides an overall view of the distribution of $\sigma_y$ in the region near the crack. This graph is drawn to approximately the same scale as the corresponding graph for $\sigma_x$, and the edges of the surface are bordered by zero values. The presence of the border is not obvious due to the small values of $\sigma_y$ in the portions of the strip somewhat removed from the crack.

The shear stress $\tau_{xy}$ in the uncracked strip is shown in Figure 18. The shear is quite small throughout this region.

The shear stress in the cracked strip is shown in Figure 19. The large negative values of $\tau_{xy}$ indicate that this stress plays a more important role in maintaining equilibrium in the $x$ direction than in the case of the uncracked strip, a condition which is necessitated by the vanishing of $\sigma_x$ on the crack surface.
Figure 16. $\sigma_y$ in the cracked strip
Figure 16 (Continued)
Figure 16 (Continued)
Figure 17. Stress surface $z = \sigma_y(x, y)$ for the cracked strip, $0 \leq x \leq 0.35$, $-0.35 \leq y \leq 0.35$
Figure 18. $\tau_{xy}$ in the uncracked strip
Figure 19. $\tau_{xy}$ in the cracked strip
Figure 19 (Continued)
Figure 19 (Continued)
Figure 20 represents the shear stress surface. We observe the abrupt rise in shear stress in the immediate neighborhood of the crack tip and the rather rapid decrease to near-zero values. The antisymmetric behavior of $\tau_{xy}$ is apparent, although the downward peak is hidden from view. The zero values of $\tau_{xy}$ on $x=0$ are due to symmetry, and are not a result of bordering the surface by a reference plane.

The method of solution we have used is not restricted to small values of $\delta$, or, equivalently, to cracks far removed from the boundary. The rate of convergence of many of the integrals we have evaluated is dependent on the value of $\delta$, but these integrals can be evaluated for larger values of $\delta$ if sufficient computer time is available.
Figure 20. Stress surface $z = \tau_{xy}(x, y)$ for the cracked strip, $0 \leq x \leq 0.35$, $-0.35 \leq y \leq 0.35$
VII. LITERATURE CITED


11. Sneddon, Ian N. Solutions of the equations of thermo- 
elastic equilibrium. Archiwum Mechaniki Stosowanej 

12. Gradshteyn, I. S. and I. M. Ryzhik. Table of integrals, 
series and products. 4th edition. New York, N.Y., 

13. Tranter, C. J. Integral transforms in mathematical 
and Sons, Inc. 1956.

Printing Office. 1964.

15. Magnus, W. and F. Oberhettinger. Formulas and theorems 
for the functions of mathematical physics. Trans­ 
lated from the German by John Wermer. New York, 

Freeman and Company. 1962.

4th ed. Cleveland, Ohio, The Chemical Rubber 

18. International Business Machines Corporation. System/ 
360 Scientific Subroutine Package (360A-CM-03X) 

19. Corbató, F. J. and J. L. Uretsky. Generation of 
spherical Bessel functions in digital computers. 
Journal of the Association for Computing Machines 

20. DNODE: A package of fortran subroutines to solve 
ordinary differential equations. Unpublished type- 
written manuscript. Ames, Iowa, Computation Center 
Library, Iowa State University. ca. 1963.


VIII. ACKNOWLEDGMENTS

The author wishes to express his gratitude for the helpful guidance provided by Dr. Harry J. Weiss during the course of this work. He is also indebted to the Engineering Research Institute of Iowa State University for the support he received during a portion of this effort.
IX. APPENDIX A

A. Remarks Regarding Integrals Used

In the solution for the uncracked strip, we made use of integral tables to evaluate many of the needed integrals. Should the reader refer to the tables cited, he would find that the parameters appearing in certain of the integrals are subject to restrictions, some of which we have apparently violated. We list these integrals below (12, p. 504).

\[ \int_{0}^{\infty} \cos \alpha x \frac{\sinh \beta x}{\sinh \gamma x} \, dx = \frac{\pi}{2\gamma} \frac{\sin \frac{\pi \beta}{\gamma}}{\cosh \frac{\alpha \pi}{\gamma} + \cos \frac{\beta \pi}{\gamma}} \quad (A.1) \]

\[ \int_{0}^{\infty} \cos \alpha x \frac{\cosh \beta x}{\cosh \gamma x} \, dx = \frac{\pi}{\gamma} \frac{\cos \frac{\beta \pi}{2\gamma} \cosh \frac{\alpha \pi}{2\gamma}}{\cosh \frac{\alpha \pi}{\gamma} + \cos \frac{\beta \pi}{\gamma}} \quad (A.2) \]

\[ \int_{0}^{\infty} \sin \alpha x \frac{\cosh \beta x}{\sinh \gamma x} \, dx = \frac{\pi}{2\gamma} \frac{\sinh \frac{\alpha \pi}{\gamma}}{\cosh \frac{\alpha \pi}{\gamma} + \cos \frac{\beta \pi}{\gamma}} \quad (A.3) \]

\[ \int_{0}^{\infty} \sin \alpha x \frac{\sinh \beta x}{\cosh \gamma x} \, dx = \frac{\pi}{\gamma} \frac{\sin \frac{\beta \pi}{2\gamma} \sinh \frac{\alpha \pi}{2\gamma}}{\cosh \frac{\alpha \pi}{\gamma} + \cos \frac{\beta \pi}{\gamma}} \quad (A.4) \]

The restrictions on the parameters are

\[ |\text{Re} \beta| < \gamma, \, a > 0 \]
for all of the above integrals.

For our particular application, $\beta$ is real with absolute value less than one and $\gamma$ is equal to one. The parameter $a$, however, is not always positive.

The original proofs of Equations A.1 through A.4 were not available, so the reasons for the restrictions could not be determined. However, it does not appear that the restriction that $a$ be positive is necessary for our choice of $\beta$ and $\gamma$.

For a non-zero value of $a$, it is obvious from the even nature of the trigonometric and hyperbolic cosines that Equations A.1 and A.2 hold for positive and negative values of $a$. Furthermore, by replacing $a$ by minus $a$ and making use of the odd nature of the trigonometric and hyperbolic sines, we find that Equations A.3 and A.4 hold for negative values of $a$ as well.

We now show that Equations A.1 through A.4 are valid for $a=0$. We must show

$$\lim_{a \to 0} \int_0^\infty \phi(x; a; \beta) \, dx = \int_0^\infty \lim_{a \to 0} \phi(x; a; \beta) \, dx,$$

where $\phi(x; a; \beta)$ corresponds to the argument in each of the integrals in Equations A.1 through A.4.

This is trivial for Equations A.3 and A.4.

From (12, p. 344), we find
\[ \int_0^\infty \frac{\sinh \beta x}{\sinh x} \, dx = \frac{\pi}{2} \tan \frac{\beta \pi}{2} \text{ for } |\beta| < 1. \]

Now, for \( \gamma = 1 \)

\[ \lim_{a \to 0} \int_0^\infty \frac{\cos ax}{\sinh x} \frac{\sinh x}{\sinh x} \, dx = \lim_{a \to 0} \frac{\pi}{2} \frac{\sin \beta \pi}{\cosh \alpha \pi + \cos \beta \pi} \]

\[ = \frac{\pi}{2} \frac{\sin \beta \pi}{1 + \cos \beta \pi} \]

\[ = \frac{\pi}{2} \frac{2 \sin \frac{\beta \pi}{2} \cos \frac{\beta \pi}{2}}{1 + 2 \cos^2 \frac{\beta \pi}{2} - 1} \]

\[ = \frac{\pi}{2} \tan \frac{\beta \pi}{2}. \]

Thus, Equation A.5 holds for Equation A.1.

From (12, p. 344), we find

\[ \int_0^\infty \frac{\cosh \beta x}{\cosh x} \, dx = \frac{\pi}{2} \sec \frac{\beta \pi}{2} \text{ for } |\beta| < 1. \]

\[ \lim_{a \to 0} \int_0^\infty \frac{\cos ax}{\cosh x} \frac{\cosh \beta x}{\cosh x} \, dx = \lim_{a \to 0} \frac{\cos \frac{\beta \pi}{2} \cosh \alpha \pi}{\cosh \alpha \pi + \cos \beta \pi} \]

\[ = \frac{\pi}{2} \frac{\cos \frac{\beta \pi}{2}}{1 + \cos \beta \pi} \]

\[ = \frac{\pi}{2} \frac{\cos \frac{\beta \pi}{2}}{1 + 2 \cos^2 \frac{\beta \pi}{2} - 1} \]

\[ = \frac{\pi}{2} \sec \frac{\beta \pi}{2}. \]
Therefore, Equation A.2 is valid for \( a=0 \).

We now derive another result used to determine \( v(x,y) \)
in the solid strip.

We wish to evaluate

\[
I(x,y) = \int_0^\infty \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin \lambda \sin \lambda \ d\lambda \tag{A.6}
\]

for \( |x| < 1 \).

To prove \( I(x,y) \) converges, consider

\[
\int_0^\infty \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin \lambda y \sin \lambda \ d\lambda,
\]

where \( 0 < x < 1 \) and \( y > 0 \).

A necessary and sufficient condition for this integral to converge is that, given an \( \varepsilon \), we can find a \( \lambda_0 \) such that

\[
\left| \int_\lambda_1^{\lambda_2} \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin \lambda y \sin \lambda \ d\lambda \right| < \varepsilon
\]

for \( \lambda_2 > \lambda_1 \geq \lambda_0 \) (23, p. 22).

Now,

\[
\left| \int_\lambda_1^{\lambda_2} \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin \lambda y \sin \lambda \ d\lambda \right| \leq \int_\lambda_1^{\lambda_2} \left| \frac{\cosh \lambda x}{\lambda \sinh \lambda} \sin \lambda y \sin \lambda \right| d\lambda
\]

\[
\leq \int_\lambda_1^{\lambda_2} \frac{\cosh \lambda x}{\lambda \sinh \lambda} d\lambda = \int_\lambda_1^{\lambda_2} \frac{e^{\lambda x} + e^{-\lambda x}}{\lambda(e^\lambda - e^{-\lambda})} d\lambda
\]
\[ \int_{\lambda_1}^{\lambda_2} \frac{e^{-\lambda(1-x)} + e^{-\lambda(1+x)}}{\lambda(1-e^{-2\lambda})} \, d\lambda \]

\[ \leq \frac{1}{1-e^{-2\lambda_1}} \int_{\lambda_1}^{\lambda_2} \frac{e^{-\lambda(1-x)} + e^{-\lambda(1+x)}}{\lambda} \, d\lambda \]

\[ \leq \frac{1}{\lambda_1(1-e^{-2\lambda_1})} \int_{\lambda_1}^{\lambda_2} (e^{-\lambda(1-x)} + e^{-\lambda(1+x)}) \, d\lambda \]

\[ = \frac{1}{\lambda_1(1-e^{-2\lambda_1})} \left\{ \frac{1}{1-x} \left[ e^{-\lambda_1(1-x)} - e^{-\lambda_2(1-x)} \right] -1 \left[ e^{-\lambda_1(1+x)} - e^{-\lambda_2(1+x)} \right] \right\} \]

\[ \leq \frac{1}{\lambda_1(1-e^{-2\lambda_1})} \left\{ \frac{-\lambda_1(1-x) - \lambda_2(1-x)}{1-x} + \frac{-\lambda_1(1+x) - \lambda_2(1+x)}{1+x} \right\} . \]

It is clear that the above expression may be made as small as we choose by an appropriate choice of \( \lambda_0 \), so \( I(x,y) \) converges for \( x \lt l \).

We now prove

\[ \int_0^\infty \frac{\sinh \lambda x}{\sinh \lambda} \sin \lambda y \sin \lambda \lambda \, d\lambda \]

is uniformly convergent for \( 0 \lt x \lt y \lt 1 \) and all \( y \).
Clearly,\[
\frac{\sinh \lambda x \sin y \sin \lambda l}{\sinh \lambda} \leq \frac{\sinh \lambda x}{\sinh \lambda}.
\]

Furthermore\[
\int_0^\infty \frac{\sinh \lambda x}{\sinh \lambda} \, d\lambda = \frac{\pi}{2} \tan \frac{\pi x}{2} \quad (12, \text{p. 344}).
\]

Thus, the integral is uniformly convergent for \(0<x<\bar{x}<1\) (23, p. 22).

Next, consider\[
\int_0^\infty \frac{\cosh \lambda x}{\sinh \lambda} \cos y \sin \lambda l \, d\lambda \text{ for } 0<x<\bar{x}<1.
\]

We find\[
\frac{\cosh \lambda x}{\sinh \lambda} \cos y \sin \lambda l \leq \frac{\cosh \lambda x}{\sinh \lambda} \sin \lambda l
\]
\[
= \frac{\lambda \cosh \lambda x}{\sinh \lambda}.
\]

for \(\lambda>0\).

Now,\[
\int_0^\infty \frac{\lambda \cosh \lambda x}{\sinh \lambda} \, d\lambda = \left(\frac{\pi}{2} \sec \frac{\pi \bar{x}}{2}\right)^2. \quad (12, \text{p. 350}).
\]
Hence,

\[ \int_{0}^{\infty} \frac{\cosh \lambda x}{\sinh \lambda} \cos \lambda y \sin \lambda \, d\lambda \]

is uniformly convergent for \(0 < x < 1\).

The convergence of \(I(x,y)\) together with the uniform convergence of the two integrals considered above permits us to interchange orders of differentiation and integration when we evaluate

\[ \frac{\partial I(x,y)}{\partial x} \quad \text{and} \quad \frac{\partial I(x,y)}{\partial y}. \]

Thus,

\[
\frac{\partial I}{\partial x} = \int_{0}^{\infty} \frac{\sinh \lambda x}{\sinh \lambda} \sin \lambda y \sin \lambda \, d\lambda = \frac{1}{2} \int_{0}^{\infty} \left( \frac{\sinh \lambda x}{\sinh \lambda} \cos \lambda (\ell - y) - \frac{1}{2} \right) \frac{\sinh \lambda x}{\sinh \lambda} \cos \lambda (\ell + y) \, d\lambda = \frac{\pi}{4} \frac{\sin \pi x}{\cosh \pi (\ell - y) + \cos \pi x} - \frac{\pi}{4} \frac{\sin \pi x}{\cosh \pi (\ell + y) + \cos \pi x}.
\]

We integrate with respect to \(x\) to find

\[ I(x,y) = \frac{1}{4} \ln \frac{\cosh \pi (\ell + y) + \cos \pi x}{\cosh \pi (\ell - y) + \cos \pi x} + f(y), \quad (A.7) \]

where \(f(y)\) is an arbitrary function of \(y\).
\[ \frac{\partial I}{\partial y} = \int_0^\infty \frac{\cosh \lambda x}{\sinh \lambda} \cos \lambda y \sin \lambda \, d\lambda \]

\[ = \frac{1}{2} \int_0^\infty \frac{\cosh \lambda x}{\sinh \lambda} \sin (\ell + y) \, d\lambda + \frac{1}{2} \int_0^\infty \frac{\cosh \lambda x}{\sinh \lambda} \sin (\ell - y) \, d\lambda \]

\[ = \frac{\pi}{4} \frac{\sinh \pi (\ell + y)}{\cosh \pi (\ell + y) + \cos \pi x} + \frac{\pi}{4} \frac{\sinh \pi (\ell - y)}{\cosh \pi (\ell - y) + \cos \pi x} \]

Integrate with respect to \( y \).

\[ I(x, y) = \frac{1}{4} \ln \frac{\cosh \pi (\ell + y) + \cos \pi x}{\cosh \pi (\ell - y) + \cos \pi x} + g(x), \quad (A.8) \]

where \( g(x) \) is an arbitrary function of \( x \).

Comparison of Equations A.7 and A.8 shows that \( g(x) \) and \( f(y) \) are equal to a constant. It follows from Equation A.6 that

\[ I(x, 0) = 0. \]

Thus, the constant is zero, and

\[ I(x, y) = \frac{1}{4} \ln \frac{\cosh \pi (\ell + y) + \cos \pi x}{\cosh \pi (\ell - y) + \cos \pi x}. \quad (A.9) \]

**B. Series Representations**

To obtain the series representation of

\[ \int_0^\infty \frac{\cosh ax \sin bx \sin \lambda x}{x^2 \cosh x} \, dx \]
used to determine $v(x,y)$, we consider

$$\oint_C \frac{e^{az}(e^{i\beta z} - e^{i\gamma z})}{z^2 \cosh z} \, dz,$$

where

$$|a| < 1,$$
$$b > 0,$$
$$\ell > 0,$$
$$\beta = b - \ell > 0$$
$$\gamma = b + \ell,$$

and the contour C (Figure 21) consists of

(i) the semicircle $|z|=R$, $0<\arg z<\pi$;

(ii) the portions of the real axis $-R<x<-\varepsilon$ and $\varepsilon<x<R$,
where $\varepsilon$ is small;

(iii) the small semicircle $|z|=\varepsilon$, $0<\arg z<\pi$.

We choose $R=\pi n$, $n=1,2,3,\ldots$. The points $z=0, \frac{(2n-1)i\pi}{2}$ are simple poles.

We use the residue theorem to write

$$\oint_C \frac{e^{az}(e^{i\beta z} - e^{i\gamma z})}{z^2 \cosh z} \, dz = \sum_{n=1}^{N} \text{Res}_n,$$

where $\text{Res}_n$ is the residue of $\frac{e^{az}(e^{i\beta z} - e^{i\gamma z})}{z^2 \cosh z}$ at $z=\frac{(2n-1)i\pi}{2}$ for $n=1,2,\ldots,N$. 
Figure 21. The contour C
Now,
\[
\oint \frac{e^{az} (e^{i\beta z} - e^{i\gamma z})}{z^2 \cosh z} \, dz = \int_{-\infty}^{\infty} \frac{e^{ax} (e^{i\beta x} - e^{i\gamma x})}{x^2 \cosh x} \, dx
\]

\[
\text{and combine this integral with the first on the right hand side of Equation A.11 to find}
\]
\[
\int_{-\infty}^{\infty} \frac{e^{ax} (e^{i\beta x} - e^{i\gamma x})}{x^2 \cosh x} \, dx = \int_{-\infty}^{\infty} \frac{e^{-ax} (e^{-i\beta x} - e^{-i\gamma x})}{x^2 \cosh x} \, dx = \int_{\infty}^{\infty} \frac{e^{-ax} (e^{-i\beta x} - e^{-i\gamma x})}{x^2 \cosh x} \, dx,
\]

\[
\int_{\partial C} \frac{e^{az} (e^{i\beta z} - e^{i\gamma z})}{z^2 \cosh z} \, dz = \int_{R} \frac{e^{ax} (e^{i\beta x} - e^{i\gamma x})}{x^2 \cosh x} \, dx + \int_{-R}^{R} \frac{e^{ax} (e^{i\beta x} - e^{i\gamma x})}{x^2 \cosh x} \, dx
\]

\[
= 2 \int_{\epsilon}^{R} \cosh ax \left( \cos \beta x - \cos \gamma x \right) \frac{dx}{x^2 \cosh x} + 2i \int_{\epsilon}^{R} \sinh ax \left( \sin \beta x - \sin \gamma x \right) \frac{dx}{x^2 \cosh x}.
\]
To obtain a bound on the second integral on the right hand side of Equation A.11, we proceed as follows:

\[
\left| \frac{ieR\sin \theta (e^{i\beta R \cos \theta} - e^{-i\gamma R \cos \theta})}{e^{R \cos \theta} + e^{-R \cos \theta}} \right| = \frac{2}{R} \left| \frac{e^{R \cos \theta} (e^{i\beta R \cos \theta} - e^{-i\gamma R \cos \theta})}{e^{R \cos \theta} + e^{-R \cos \theta}} \right|
\]

\[
= \frac{2}{R} \left| \frac{e^{a(x+iy)} (e^{i\beta x} - e^{-i\gamma y})}{e^{x+y} - e^{-x-y}} \right|
\]

\[
= \frac{2}{R} \left| \frac{e^{ax} (e^{i\beta x} - e^{-i\gamma y})}{e^{x+y} - e^{-x-y}} \right|
\]

\[
= \frac{2}{R} \left| \frac{e^{-\beta y} e^{-\gamma y} \cos 2\sqrt{x} - i e^{-\gamma y} \sin 2\sqrt{x}}{e - e^{-2x} \cos 2y - e^{-4y}} \right|
\]

\[
= \frac{2e^{-\beta y}}{R} \left[ \frac{1 - 2e^{-2y} \cos 2\sqrt{x} + e^{-4y}}{e^{2(1-a)x} + 2e^{-2ax} \cos 2y + e^{-2x(1+a)}} \right]^{1/2}
\]

\[
\leq \frac{2}{R} \left[ \frac{4}{e^{-2ax} (e^{2x} + 2\cos 2y + e^{-2x})} \right]^{1/2}
\]

\[
= \frac{4}{R} \left[ \frac{1}{e^{-2ax} (2\cosh 2x + 2\cos 2y)} \right]^{1/2}
\]

for \( \beta > 0 \).

We must show that \( e^{-2ax}(2\cosh 2x + 2\cos 2y) \) can not vanish on \( z = R e^{i\theta} \) for \( 0 < \theta < \pi \). Since \( 0 < a < 1 \), the expression can not
vanish for \( x \neq 0 \). For \( x=0, y=R \) and \( 2\cosh x = 2 \).

Assume

\[
2 + 2\cos 2y = 0.
\]

Then,

\[
y = \frac{\pi(2n-1)}{2}, \quad n=1,2,3,\ldots
\]

This is a contradiction, since \( R \) has been restricted to avoid these values, which are the poles.

Thus,

\[
e^{-2ax}(2\cosh 2x + 2\cos 2y) \geq D^2,
\]
where \( D > 0 \).

Hence,

\[
\left|ie^{aRe^{i\theta}}(e^{i\beta e^{i\gamma e^{i\theta}}} - e^{-i\gamma e^{i\theta}})\right| \leq \frac{4}{DR}.
\] (A.13)

Thus,

\[
\left|\frac{ie^{aRe^{i\theta}}(e^{i\beta e^{i\gamma e^{i\theta}}} - e^{-i\gamma e^{i\theta}})}{e^{i\theta}}\right| \leq \frac{4\pi}{DR}.
\] (A.14)

The residue at \( z=0 \) is calculated to be

\[
\text{Res}_0 = \lim_{z \to 0} \frac{ze^{az}(e^{i\beta z} - e^{-i\gamma z})}{z^2 \cosh z}
\]
= \lim_{z \to 0} \frac{e^{az(i\beta e^{i\gamma z} - i\gamma e^{-i\gamma z})} + (e^{i\beta z} - e^{-i\gamma z})ae^{az}}{z \sinh z + \cosh z}

\text{where we have applied L'Hospital's rule.}

The residue at a typical pole on the positive y axis is calculated in the same manner.

\[ \text{Res}_{n} = \lim_{z \to \frac{(2n-1)i\pi}{2}} \frac{(z - \frac{(2n-1)i\pi}{2})e^{az(i\beta e^{i\gamma z} - i\gamma e^{-i\gamma z})}}{z^2 \cosh z} \]

\[ = \lim_{z \to \frac{(2n-1)i\pi}{2}} \frac{(z - \frac{(2n-1)i\pi}{2})e^{az(i\beta e^{i\gamma z} - i\gamma e^{-i\gamma z})}}{z^2 \sinh z + 2z \cosh z} \]

\[ = e^{a(2n-1)i\pi}(e^{i\beta \frac{(2n-1)i\pi}{2}} - e^{-i\gamma \frac{(2n-1)i\pi}{2}}) \frac{(2n-1)i\pi}{2} \sinh \frac{(2n-1)i\pi}{2} \]

\[ = 4(-1)^n \frac{-\beta \frac{(2n-1)i\pi}{2} - \gamma \frac{(2n-1)i\pi}{2} (2n-1)a\pi}{(2n-1)^2\pi^2} \]
Thus, letting $\epsilon \to 0$ and $R \to \infty$, it follows from Equations A.11, A.12, A.14, and A.15 that

\[
\int_{c} e^{az}(e^{i\beta z} - e^{i\gamma z})\frac{dz}{z^2 \cosh z} = 2\int_{0}^{\infty} \frac{\cosh x (\cos \beta x - \cos \gamma x)}{x^2 \cosh x} \, dx
\]

\[+ 2i \int_{0}^{\infty} \frac{\sinh x (\sin \beta x - \sin \gamma x)}{x^2 \cosh x} \, dx + \pi(\beta - \gamma), \quad (A.17)
\]

where we have used the result (24, p. 35)

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \int_{0}^{\pi} \frac{e^{i\epsilon e^{i\theta}}(e^{i\beta \epsilon e^{i\theta}} - e^{i\gamma \epsilon e^{i\theta}})}{\epsilon e^{i\theta} \cosh(\epsilon e^{i\theta})} \, d\theta = -\pi i \text{Res}_0.
\]

We use Equations A.10 (with $N=\infty$), A.16, and A.17 to find

\[2 \int_{0}^{\infty} \frac{\cosh x (\cos \beta x - \cos \gamma x)}{x^2 \cosh x} \, dx + 2i \int_{0}^{\infty} \frac{\sinh x (\sin \beta x - \sin \gamma x)}{x^2 \cosh x} \, dx \]

\[+ \pi(\beta - \gamma)
\]

\[= 2\pi i \sum_{n=1}^{\infty} \frac{4(-1)^n \{e^{-\frac{\beta(2n-1)\pi}{2}} - e^{-\frac{\gamma(2n-1)\pi}{2}}\} \{\sin \frac{(2n-1)\pi}{2} \cos \frac{(2n-1)\pi}{2}\}}{(2n-1)^2 \pi^2}.
\]

\[= 2\pi i \sum_{n=1}^{\infty} \frac{4(-1)^n \{e^{-\frac{\beta(2n-1)\pi}{2}} - e^{-\frac{\gamma(2n-1)\pi}{2}}\} \{\sin \frac{(2n-1)\pi}{2} \cos \frac{(2n-1)\pi}{2}\}}{(2n-1)^2 \pi^2}.
\]

\[= 2\pi i \sum_{n=1}^{\infty} \frac{4(-1)^n \{e^{-\frac{\beta(2n-1)\pi}{2}} - e^{-\frac{\gamma(2n-1)\pi}{2}}\} \{\sin \frac{(2n-1)\pi}{2} \cos \frac{(2n-1)\pi}{2}\}}{(2n-1)^2 \pi^2}.
\]

\[= 2\pi i \sum_{n=1}^{\infty} \frac{4(-1)^n \{e^{-\frac{\beta(2n-1)\pi}{2}} - e^{-\frac{\gamma(2n-1)\pi}{2}}\} \{\sin \frac{(2n-1)\pi}{2} \cos \frac{(2n-1)\pi}{2}\}}{(2n-1)^2 \pi^2}.
\]

Equate real parts to obtain

\[\int_{0}^{\infty} \frac{\cosh x (\cos \beta x - \cos \gamma x)}{x^2 \cosh x} \, dx = \frac{\pi(\beta - \gamma)}{2}.
\]
We recall the restriction that \( \beta = 0 \), and therefore, \( b > \ell \).
For \( b < \ell \), redefine \( \beta \) such that \( \beta = \ell - b \), and repeat the previous argument to find a result of the same form with the roles of \( b \) and \( \beta \) interchanged.

It then follows that

\[
\int_0^\infty \frac{\cosh \alpha x \sin \lambda y \sin \lambda k}{\lambda^2 \cosh \lambda} \, d\lambda
\]

\[
= \begin{cases} \\
\frac{\pi y}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\frac{(2n-1)\pi}{2} x}}{\sinh \frac{(2n-1)\pi y}{2} \cos \frac{2n-1}{2} x} \\ \frac{\pi \ell}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\frac{(2n-1)\pi}{2} y}}{\sinh \frac{(2n-1)\pi \ell}{2} \cos \frac{2n-1}{2} x} \\
\end{cases} 
\]

\( (y < \ell, |x| < 1) \), \( (y > \ell, |x| < 1) \), \( (A.20) \)

where we have used the previously defined values of \( \beta \) and \( \gamma \), replaced \( b \) with \( y \) and \( a \) with \( x \), and written the exponentials in terms of the hyperbolic function.

To determine the series representation of

\[
\int_0^\infty \frac{\tanh \xi}{\xi^2} J_{1/2}^2(\delta \xi) \, d\xi
\]
needed for the coefficient of $A_o$ in the temperature problem for the cracked strip, we first write

$$\int_0^\infty \frac{\tanh \xi}{\xi^2} J_1^2(\delta \xi) d\xi = \frac{2}{\pi \delta} \int_0^\infty \frac{\sinh \xi \sin^2 \delta \xi}{\xi^2 \cosh \xi} d\xi. \quad (A.21)$$

To evaluate the integral appearing on the right in Equation A.21, we evaluate

$$\oint \frac{\sinh(z(1-e^{2iaz}))}{z^3 \cosh z} dz,$$

where $a>0$ and $C$ is the contour used previously. The complex function has simple poles at $z=0, \frac{(2n-1)i\pi}{2}$, where $n=1, 2, 3, \ldots$.

We proceed as before.

$$\oint \frac{\sinh(z(1-e^{2iaz}))}{z^3 \cosh z} dz = 2\pi i \sum_{n=1}^N \text{Res}_n + \sum_{n=1}^N \text{Res}_n \quad (A.22)$$

$$\oint \frac{\sinh(z(1-e^{2iaz}))}{z^3 \cosh z} dz = \int_{-R}^{-\epsilon} \frac{\sinh(x(1-e^{2iax}))}{x^3 \cosh x} dx$$

$$+ i \int_{0}^{\pi} \frac{\sinh e^{i\theta}(1-e^{2iae^{i\theta}})}{\epsilon^2 e^{2i\theta} \cosh e^{i\theta}} d\theta + \int_{\epsilon}^{R} \frac{\sinh(x(1-e^{2iax}))}{x^3 \cosh x} dx$$
\[+i \int_0^\pi \frac{\sinh \text{Re}^{i\theta} (1-e^{-2i\text{Re}^{i\theta}})}{R^2 e^{2i\theta} \cosh \text{Re}^{i\theta}} \, d\theta. \tag{A.23}\]

\[= \int_{-\varepsilon}^{-\varepsilon} \frac{\sinh x (1-e^{-2iax})}{x^3 \cosh x} \, dx + \int_{-\varepsilon}^{-\varepsilon} \frac{\sinh x (1-e^{-2iax})}{x^3 \cosh x} \, dx\]

\[= -\int_{-\varepsilon}^{\varepsilon} \frac{\sinh x (1-e^{-2iax})}{R x^3 \cosh x} \, dx + \int_{-\varepsilon}^{\varepsilon} \frac{\sinh x (1-e^{-2iax})}{e x^3 \cosh x} \, dx\]

\[= 4 \int_{0}^{R} \frac{\sinh x \sin^2 ax}{x^3 \cosh x} \, dx \tag{A.24}\]

We now find a bound on the last integral in Equation A.23.

\[\left| \frac{\text{isinh\text{Re}^{i\theta} (1-e^{-2ia\text{Re}^{i\theta}})}}{R^2 e^{2i\theta} \cosh \text{Re}^{i\theta}} \right| = \frac{2}{R^2} \left| \frac{\sinh(x+iy)e^{ia(x+iy)} \sin a(x+iy)}{\cosh(x+iy)} \right|\]

\[= \frac{2e^{-ay}}{R^2} \left| \frac{(\sinh x \cos y + i \cosh x \sin y)(\sin ax \cosh ay + i \cos ax \sinh ay)}{\cosh x \cos y + i \sinh x \sin y} \right|\]

\[= \frac{2e^{-ay}}{R^2} \left( \sin^2 y + \sinh^2 x \right)^{1/2} \left( \sin^2 ax + \sinh^2 ay \right)^{1/2} \frac{1}{\cosh^2 x - \sin^2 y}\]

\[< \frac{2e^{-ay}}{R^2} \left( 1 + \sinh^2 x \right)^{1/2} \left( 1 + \sinh^2 ay \right)^{1/2} \frac{1}{\cosh^2 x - \sin^2 y}\]

\[= \frac{2e^{-ay}}{R^2} \left( \cosh^2 x \cosh^2 ay \right)^{1/2} \frac{1}{\cosh^2 x - \sin^2 y}\]
The only point where the denominator could possibly vanish is at $x=0$. But, on $z=Re^{i\theta}$, $x=0$ implies $y=n\pi$ and $\sin y=0$. Thus, the denominator is always positive.

Thus,

$$\left|\frac{i\sinh Re^{i\theta}(1-e^{2iaRe^{i\theta}})}{R^2 e^{2i\theta} \cosh Re^{i\theta}}\right| \leq \frac{2K}{R}, \quad (A.25)$$

where $K$ is a constant.

The residue at $z=0$ is given by

$$\text{Res}_0 = \lim_{z \to 0} \frac{z\sinh z(1-e^{2iaz})}{z^3 \cosh z}$$

$$= \lim_{z \to 0} \frac{(z + \frac{3}{2I} + \frac{5}{4I} + \ldots)(1-2iaz+2a^2z^2+\ldots)}{z^2(1 + \frac{3}{2I} + \frac{5}{4I} + \ldots)}$$

$$= -2ia. \quad (A.26)$$

A typical residue at $z=\left(\frac{2n-1}{2}\right)i\pi$ is given by

$$\text{Res}_n = \lim_{z \to \left(\frac{2n-1}{2}\right)i\pi} \frac{(z-\frac{2n-1}{2}i\pi)\sinh z(1-e^{2iaz})}{z^3 \cosh z}$$
\[
\lim_{z \to \frac{(2n-1)i\pi}{2}} \frac{(z-(\frac{2n-1}{2})i\pi)(-2i\sinh z + \frac{2\pi}{2}}{z^3 \sinh z + 3z^2 \cosh z}
\]

\[
\frac{(1-e^{-2iaz}) \cosh z + (1-e^{2iaz}) \sinh z}{z^3 \sinh z + 3z^2 \cosh z}
\]

\[
\sinh((\frac{2n-1}{2})i\pi)(1-e^{-i(2n-1)i\pi})
\]

\[
= \frac{8}{\pi^3 i} \frac{e^{-(2n-1)a\pi^3} - 1}{(2n-1)^3}
\]

We then let \(\epsilon \to 0\) and \(R \to \infty\) to find

\[
\int_0^\infty \frac{\sinh x \sin^2 ax}{x^3 \cosh x} \, dx - \frac{\pi a}{2} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)a\pi^3} - 1}{(2n-1)^3},
\]

where we note that the contribution from the large semicircle vanishes and the contribution from the small semicircle is \(-\pi i \text{ Res}_0\), as before.

Thus,

\[
\int_0^\infty \frac{\tanh \xi}{\xi^2} \left( \frac{j}{2} (\delta \xi) d\xi \right) = 1 + \frac{8}{\pi^3 \delta} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)\pi^3 \delta} - 1}{(2n-1)^3}
\]

The last series representation required is for
\[ \int_0^\infty \frac{\sin \lambda \cos \gamma \sinh \lambda (1+x)}{\lambda^2 \cosh \lambda} \, d\lambda, \text{ where } -1 < x < 0, \]

which was needed for the temperature in the cracked strip.

Now,
\[ \int_0^\infty \frac{\sin \lambda \cos \gamma \sinh \lambda (1+x)}{\lambda^2 \cosh \lambda} \, d\lambda = \frac{1}{2} \int_0^\infty \frac{\sin \lambda (\delta+y) \sinh \lambda (1+x)}{\lambda^2 \cosh \lambda} \, d\lambda \]
\[ + \frac{1}{2} \int_0^\infty \frac{\sin \lambda (\delta-y) \sinh \lambda (1+x)}{\lambda^2 \cosh \lambda} \, d\lambda. \tag{A.30} \]

We first evaluate
\[ \int_0^\infty \frac{\sin ax \sinh bx}{x^2 \cosh x} \, dx \text{ for } a > 0, \quad 0 < b < 1 \]

by considering
\[ \oint_C \frac{e^{iaz} \sinh bz}{z^2 \cosh z} \, dz, \]

where \( C \) is the same contour as before. Again, the points
\[ z = 0, \frac{(2n-1)i\pi}{2} \text{ for } n=1,2,3...,N \]
are simple poles.

\[ \oint_C \frac{e^{iaz} \sinh bz}{z^2 \cosh z} \, dz = 2\pi i \sum_{n=1}^{N} \text{Res}_n. \tag{A.31} \]
\[
\int_{C} \frac{e^{iaz} \sinh b z}{z^2 \cosh z} \, dz = \int_{-R}^{\infty} \frac{e^{iaz} \sinh b x}{x^2 \cosh x} \, dx + i \int_{0}^{\pi} \frac{e^{ia \theta} \sinh b e^{i \theta}}{e \cosh e^{i \theta}} \, d\theta
\]

\[
+ \int_{\varepsilon}^{R} \frac{e^{iaz} \sinh b x}{x^2 \cosh x} \, dx + i \int_{0}^{\pi} \frac{e^{ia \theta} \sinh b e^{i \theta}}{e \cosh e^{i \theta}} \, d\theta.
\]

(A.32)

We combine the first and third integrals on the right-hand side of Equation A.32.

\[
\int_{-R}^{\infty} \frac{e^{iaz} \sinh b x}{x^2 \cosh x} \, dx + \int_{\varepsilon}^{R} \frac{e^{iaz} \sinh b x}{x^2 \cosh x} \, dx
\]

\[
= \int_{R}^{\infty} \frac{e^{-iaz} \sinh b x}{x^2 \cosh x} \, dx + \int_{\varepsilon}^{R} \frac{e^{iaz} \sinh b x}{x^2 \cosh x} \, dx
\]

\[
= 2i \int_{\varepsilon}^{R} \frac{\sin a x \sin b x}{x^2 \cosh x} \, dx.
\]

(A.33)

We obtain a bound on the argument of the last integral on the right-hand side of Equation A.32 as follows:

\[
\left| \frac{ie^{ia \theta} \sinh b e^{i \theta}}{Re^{i \theta} \cosh e^{i \theta}} \right| = \frac{1}{R} \left| \frac{e^{i a(x+iy)} \sinh b(x+iy)}{\cosh(x+iy)} \right|
\]

\[
= \frac{e^{-ay}}{R} \left| \frac{\sinh b x \cos b y + i \cosh b x \sin b y}{\cosh x \cos y + i \sinh x \sin y} \right|
\]
\[ \frac{\sinh^2 bx \cos^2 by + \cosh^2 bx \sin^2 by}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}^{1/2} \]

\[ = \frac{1}{R} \left[ \frac{\sin^2 by + \sinh^2 bx}{\cosh^2 x - \sin^2 y} \right]^{1/2} \]

\[ = \frac{1}{R} \left[ \frac{\cosh^2 bx}{\cosh^2 x - \sin^2 y} \right]^{1/2} \]

\[ = \frac{1}{R} \left[ \frac{1}{(\cosh^2 x - \sin^2 y) / \cosh^2 bx} \right]^{1/2}. \quad (A.34) \]

For \( b < 1 \) on \( z = \text{Re}^i \theta, 0 \leq \theta \leq \pi, R = n\pi \), the denominator of the above expression cannot vanish.

Thus,

\[ \left| \frac{ie^{i\alpha} e^{-i\theta}}{\text{Re}^i \theta} \frac{\sinh \text{Re}^i \theta}{\cosh \text{Re}^i \theta} \right| \leq \frac{A}{R}, \quad (A.35) \]

where \( A \) is a positive constant.

As before,

\[ \lim_{\varepsilon \to 0} \int_0^1 \frac{e^{i\alpha e^{-i\theta}} \sinh \varepsilon e^{-i\theta}}{\varepsilon e^{i\theta} \cosh \varepsilon e^{-i\theta}} d\theta = -\pi i \text{Res}_0' \quad (A.36) \]

where

\[ \text{Res}_0 = \lim_{z \to 0} \frac{e^{iaz} \sinh bz}{z \cosh z} \]
\[
\lim_{z \to 0} \frac{(1+iaz - \frac{a}{2}z^2 + \ldots)(bz + \frac{b}{3!}z^3 + \ldots)}{z(1 + \frac{z^2}{2!} + \ldots)} = b. \quad (A.37)
\]

A typical residue at a pole on the positive y axis is given by

\[
\text{Res}_n = \lim_{z \to (2n-1)i\pi} \frac{(z-(2n-1)i\pi)e^{iaz}\sinhbz}{z^2 \cosh z}
\]

\[
= \lim_{z \to (2n-1)i\pi} \frac{(z-(2n-1)i\pi)(be^{iaz} \cosh bz + iae^{iaz} \sinh bz) + e^{iaz} \sinh bz}{z^2 \sinh z + 2z \cosh z}
\]

\[
= \frac{ia(2n-1)i\pi}{2} \cdot \frac{\sinh(2n-1)b\pi i}{\{(2n-1)i\pi\}^2 \sinh(2n-1)i\pi i}
\]

\[
= \frac{4(-1)^n e^{-\frac{(2n-1)a\pi}{2}} \sin(\frac{(2n-1)b\pi}{2})}{\pi^2 (2n-1)^2}. \quad (A.38)
\]

Then, letting \(\epsilon \to 0\) and \(R \to \infty\), we proceed as before to find

\[
\int_0^\infty \frac{\sin ax \sin bx}{x^2 \cosh x} \, dx = \frac{\pi b}{2} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\frac{(2n-1)a\pi}{2}} \sin(\frac{2n-1)b\pi}{2}}{(2n-1)^2}. \quad (A.39)
\]
for \( a > 0, \ 0 < b < 1 \).

For \( a < 0 \), we could consider

\[
\oint_C \frac{e^{-iaz} \sinh b z}{z^2 \cosh z} \, dz
\]

and follow the same argument as for \( a > 0 \). A more direct approach is to replace \( a \) by \(-a\) and use the odd behavior of the sine in Equation A.39 to obtain

\[
\int_0^\infty \frac{\sin ax \sin bx}{x^2 \cosh x} \, dx = -\frac{\pi b}{2} - 4 \sum_{n=1}^{\infty} \frac{(-1)^n e^{\frac{(2n-1)\pi a}{2}} \sin\left(\frac{(2n-1)\pi b}{2}\right)}{(2n-1)^2}
\]

for \( a < 0, \ 0 < b < 1 \).

We use Equations A.39 and A.40 in Equation A.30 and combine terms, which gives

\[
\int_0^\infty \frac{\sin \lambda \delta \cos \lambda y \sinh \lambda (1+x)}{\lambda^2 \cosh \lambda} \, d\lambda
\]

\[
= \left\{ \begin{array}{ll}
\frac{\pi (1+x)}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{(2n-1)\pi \delta}{2}} \cosh\left(\frac{(2n-1)\pi y}{2}\right) \sin\left(\frac{(2n-1)\pi (1+x)}{2}\right) \\
& (y \leq \delta)
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^n e^{-\frac{(2n-1)\pi y}{2}} \sin\left(\frac{(2n-1)\pi \delta}{2}\right) \sin\left(\frac{(2n-1)\pi (1+x)}{2}\right) \\
& (y > \delta)
\end{array} \right. 
\]
X. APPENDIX B

We list below the coefficient matrices and the coefficients used in the expansions needed for the temperature and stress problems.
Table 1. Coefficient matrix for temperature problem

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.007748E+00</td>
<td>7</td>
<td>3</td>
<td>-1.667476E-07</td>
<td>10</td>
<td>7</td>
<td>1.078955E-07</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1.919622E-04</td>
<td>7</td>
<td>4</td>
<td>2.501213E-07</td>
<td>10</td>
<td>8</td>
<td>-2.252553E-07</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2.001926E-01</td>
<td>7</td>
<td>5</td>
<td>-4.938293E-07</td>
<td>10</td>
<td>9</td>
<td>1.036498E-06</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1.283972E-05</td>
<td>7</td>
<td>6</td>
<td>2.116411E-06</td>
<td>10</td>
<td>10</td>
<td>2.439299E-02</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3.209888E-05</td>
<td>7</td>
<td>7</td>
<td>3.448826E-02</td>
<td>11</td>
<td>0</td>
<td>1.812473E-08</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.111689E-01</td>
<td>8</td>
<td>0</td>
<td>-6.293897E-08</td>
<td>11</td>
<td>1</td>
<td>-1.856500E-08</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2.751333E-06</td>
<td>8</td>
<td>1</td>
<td>6.584383E-08</td>
<td>11</td>
<td>2</td>
<td>1.965706E-08</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-3.851867E-06</td>
<td>8</td>
<td>2</td>
<td>-7.346216E-08</td>
<td>11</td>
<td>3</td>
<td>-2.157909E-08</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.283956E-05</td>
<td>8</td>
<td>3</td>
<td>8.842665E-08</td>
<td>11</td>
<td>4</td>
<td>2.469145E-08</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>7.695055E-02</td>
<td>8</td>
<td>4</td>
<td>-1.175783E-07</td>
<td>11</td>
<td>5</td>
<td>-2.970897E-08</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-9.171109E-07</td>
<td>8</td>
<td>5</td>
<td>1.814066E-07</td>
<td>11</td>
<td>6</td>
<td>3.814485E-08</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.105533E-06</td>
<td>8</td>
<td>6</td>
<td>-3.668443E-07</td>
<td>11</td>
<td>7</td>
<td>-5.363225E-08</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>-1.834222E-06</td>
<td>8</td>
<td>7</td>
<td>1.604944E-06</td>
<td>11</td>
<td>8</td>
<td>8.688420E-08</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>6.878336E-06</td>
<td>8</td>
<td>8</td>
<td>3.030727E-02</td>
<td>11</td>
<td>9</td>
<td>-1.834222E-07</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5.883957E-02</td>
<td>9</td>
<td>0</td>
<td>3.975095E-08</td>
<td>11</td>
<td>10</td>
<td>8.337375E-07</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>3.890773E-07</td>
<td>9</td>
<td>1</td>
<td>-4.119641E-08</td>
<td>11</td>
<td>11</td>
<td>2.222450E-02</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-4.367196E-07</td>
<td>9</td>
<td>2</td>
<td>4.483950E-08</td>
<td>12</td>
<td>0</td>
<td>-1.288250E-08</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5.895711E-07</td>
<td>9</td>
<td>3</td>
<td>-5.180025E-08</td>
<td>12</td>
<td>1</td>
<td>1.314514E-08</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-1.069962E-06</td>
<td>9</td>
<td>4</td>
<td>6.413359E-08</td>
<td>12</td>
<td>2</td>
<td>-1.379114E-08</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>4.279855E-06</td>
<td>9</td>
<td>5</td>
<td>-8.734389E-08</td>
<td>12</td>
<td>3</td>
<td>1.490660E-08</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4.762955E-02</td>
<td>9</td>
<td>6</td>
<td>1.375667E-07</td>
<td>12</td>
<td>4</td>
<td>-1.666032E-08</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>-1.924009E-07</td>
<td>9</td>
<td>7</td>
<td>-2.832255E-07</td>
<td>12</td>
<td>5</td>
<td>1.936585E-08</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2.084344E-07</td>
<td>9</td>
<td>8</td>
<td>1.258780E-06</td>
<td>12</td>
<td>6</td>
<td>-2.362819E-08</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>-2.547532E-07</td>
<td>9</td>
<td>9</td>
<td>2.703040E-02</td>
<td>12</td>
<td>7</td>
<td>3.071665E-08</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>3.668443E-07</td>
<td>10</td>
<td>0</td>
<td>-2.632855E-08</td>
<td>12</td>
<td>8</td>
<td>-4.367196E-08</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>-7.003394E-07</td>
<td>10</td>
<td>1</td>
<td>2.710292E-08</td>
<td>12</td>
<td>9</td>
<td>7.146320E-08</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>2.918081E-06</td>
<td>10</td>
<td>2</td>
<td>-2.904877E-08</td>
<td>12</td>
<td>10</td>
<td>-1.522477E-07</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>4.000740E-02</td>
<td>10</td>
<td>3</td>
<td>3.256016E-08</td>
<td>12</td>
<td>11</td>
<td>6.978021E-07</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1.058205E-07</td>
<td>10</td>
<td>4</td>
<td>-3.848021E-08</td>
<td>12</td>
<td>12</td>
<td>2.041008E-02</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>-1.122339E-07</td>
<td>10</td>
<td>5</td>
<td>4.863471E-08</td>
<td>13</td>
<td>0</td>
<td>9.406268E-09</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>1.296925E-07</td>
<td>10</td>
<td>6</td>
<td>-6.743460E-08</td>
<td>13</td>
<td>1</td>
<td>9.569852E-09</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>--------</td>
<td>---</td>
<td>---</td>
<td>--------</td>
<td>---</td>
<td>---</td>
<td>--------</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>9.968598E-09</td>
<td>15</td>
<td>4</td>
<td>6.295956E-09</td>
<td>17</td>
<td>2</td>
<td>3.382590E-09</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>-1.064757E-08</td>
<td>15</td>
<td>5</td>
<td>-6.895572E-09</td>
<td>17</td>
<td>3</td>
<td>-3.513836E-09</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>1.169145E-08</td>
<td>15</td>
<td>6</td>
<td>7.743690E-09</td>
<td>17</td>
<td>4</td>
<td>3.705502E-09</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>-1.325032E-08</td>
<td>15</td>
<td>7</td>
<td>-8.955748E-09</td>
<td>17</td>
<td>5</td>
<td>-3.971674E-09</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>1.559460E-08</td>
<td>15</td>
<td>8</td>
<td>1.073542E-08</td>
<td>17</td>
<td>6</td>
<td>4.334172E-09</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>-1.924009E-08</td>
<td>15</td>
<td>9</td>
<td>-1.346807E-08</td>
<td>17</td>
<td>7</td>
<td>-4.826898E-09</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>2.526477E-08</td>
<td>15</td>
<td>10</td>
<td>1.795742E-08</td>
<td>17</td>
<td>8</td>
<td>5.503974E-09</td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>-3.624945E-08</td>
<td>15</td>
<td>11</td>
<td>-2.612852E-08</td>
<td>17</td>
<td>9</td>
<td>-6.455281E-09</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>5.981155E-08</td>
<td>15</td>
<td>12</td>
<td>4.367196E-08</td>
<td>17</td>
<td>10</td>
<td>7.838558E-09</td>
</tr>
<tr>
<td>13</td>
<td>11</td>
<td>-1.283955E-07</td>
<td>15</td>
<td>13</td>
<td>-9.487354E-08</td>
<td>17</td>
<td>11</td>
<td>-9.951773E-09</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>5.925949E-07</td>
<td>15</td>
<td>14</td>
<td>4.427434E-07</td>
<td>17</td>
<td>12</td>
<td>1.341646E-08</td>
</tr>
<tr>
<td>13</td>
<td>13</td>
<td>1.886956E-02</td>
<td>15</td>
<td>15</td>
<td>1.639468E-02</td>
<td>17</td>
<td>13</td>
<td>1.972282E-08</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>-7.027669E-09</td>
<td>16</td>
<td>0</td>
<td>-4.152866E-09</td>
<td>17</td>
<td>14</td>
<td>3.328226E-08</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>7.133085E-09</td>
<td>16</td>
<td>1</td>
<td>4.200601E-09</td>
<td>17</td>
<td>15</td>
<td>-7.295199E-08</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>-7.388490E-09</td>
<td>16</td>
<td>2</td>
<td>-4.315232E-09</td>
<td>17</td>
<td>16</td>
<td>3.433037E-07</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>7.818510E-09</td>
<td>16</td>
<td>3</td>
<td>4.505107E-09</td>
<td>17</td>
<td>17</td>
<td>1.449372E-02</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>-9.468245E-09</td>
<td>16</td>
<td>4</td>
<td>-4.784926E-09</td>
<td>18</td>
<td>0</td>
<td>-2.609135E-09</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>9.414698E-09</td>
<td>16</td>
<td>5</td>
<td>5.178489E-09</td>
<td>18</td>
<td>1</td>
<td>2.632856E-09</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>-1.078954E-08</td>
<td>16</td>
<td>6</td>
<td>-5.723596E-09</td>
<td>18</td>
<td>2</td>
<td>-2.689476E-09</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>1.282673E-08</td>
<td>16</td>
<td>7</td>
<td>6.481134E-09</td>
<td>18</td>
<td>3</td>
<td>2.782216E-09</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>-1.597004E-08</td>
<td>16</td>
<td>8</td>
<td>-7.552678E-09</td>
<td>18</td>
<td>4</td>
<td>-2.916624E-09</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
<td>2.114551E-08</td>
<td>16</td>
<td>9</td>
<td>9.116842E-09</td>
<td>18</td>
<td>5</td>
<td>3.101341E-09</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>-3.057038E-08</td>
<td>16</td>
<td>10</td>
<td>-1.151175E-08</td>
<td>18</td>
<td>6</td>
<td>-3.349450E-09</td>
</tr>
<tr>
<td>14</td>
<td>11</td>
<td>5.079384E-08</td>
<td>16</td>
<td>11</td>
<td>1.543958E-08</td>
<td>18</td>
<td>7</td>
<td>3.680714E-09</td>
</tr>
<tr>
<td>14</td>
<td>12</td>
<td>-1.097393E-07</td>
<td>16</td>
<td>12</td>
<td>-2.258895E-08</td>
<td>18</td>
<td>8</td>
<td>-4.125553E-09</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>5.095064E-08</td>
<td>16</td>
<td>13</td>
<td>3.794942E-08</td>
<td>18</td>
<td>9</td>
<td>4.732253E-09</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>1.754528E-02</td>
<td>16</td>
<td>14</td>
<td>-8.283581E-08</td>
<td>18</td>
<td>10</td>
<td>-5.580798E-09</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>5.355766E-09</td>
<td>16</td>
<td>15</td>
<td>3.882932E-07</td>
<td>18</td>
<td>11</td>
<td>6.811437E-09</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>-5.425775E-09</td>
<td>16</td>
<td>16</td>
<td>1.538571E-02</td>
<td>18</td>
<td>12</td>
<td>-8.689074E-09</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>5.594579E-09</td>
<td>17</td>
<td>0</td>
<td>3.269559E-09</td>
<td>18</td>
<td>13</td>
<td>1.176645E-08</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>-5.876227E-09</td>
<td>17</td>
<td>1</td>
<td>-3.302865E-09</td>
<td>18</td>
<td>14</td>
<td>-1.736953E-08</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
<td>m</td>
<td>n</td>
<td>C_{mn}</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>---</td>
<td>---</td>
<td>-------</td>
</tr>
<tr>
<td>18</td>
<td>15</td>
<td>2.942603E-08</td>
<td>19</td>
<td>4</td>
<td>2.328594E-09</td>
<td>19</td>
<td>12</td>
<td>5.973735E-09</td>
</tr>
<tr>
<td>18</td>
<td>16</td>
<td>-6.473721E-08</td>
<td>19</td>
<td>5</td>
<td>-2.459686E-09</td>
<td>19</td>
<td>13</td>
<td>-7.652307E-09</td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>3.057037E-07</td>
<td>19</td>
<td>6</td>
<td>2.633755E-09</td>
<td>19</td>
<td>14</td>
<td>1.040314E-08</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>1.369949E-02</td>
<td>19</td>
<td>7</td>
<td>-2.862776E-09</td>
<td>19</td>
<td>15</td>
<td>-1.541363E-08</td>
</tr>
<tr>
<td>19</td>
<td>0</td>
<td>2.107379E-09</td>
<td>19</td>
<td>8</td>
<td>3.164636E-09</td>
<td>19</td>
<td>16</td>
<td>2.620318E-08</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>-2.124582E-09</td>
<td>19</td>
<td>9</td>
<td>-3.566676E-09</td>
<td>19</td>
<td>17</td>
<td>-5.783583E-08</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>2.165552E-09</td>
<td>19</td>
<td>10</td>
<td>4.112167E-09</td>
<td>19</td>
<td>18</td>
<td>2.739592E-07</td>
</tr>
<tr>
<td>19</td>
<td>3</td>
<td>-2.232371E-09</td>
<td>19</td>
<td>11</td>
<td>-4.872696E-09</td>
<td>19</td>
<td>19</td>
<td>1.298779E-02</td>
</tr>
</tbody>
</table>
Table 2. $B_n$ for temperature problem

<table>
<thead>
<tr>
<th>n</th>
<th>$\frac{B_n}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.3612279E-02$</td>
</tr>
<tr>
<td>1</td>
<td>$-0.7392049E-07$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.3905875E-10$</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 3. $A_m$ for temperature problem

<table>
<thead>
<tr>
<th>m</th>
<th>$\frac{A_m}{T}$</th>
<th>m</th>
<th>$\frac{A_m}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-1.133521E-03$</td>
<td>10</td>
<td>$-1.225034E-09$</td>
</tr>
<tr>
<td>1</td>
<td>$1.040233E-06$</td>
<td>11</td>
<td>$9.255854E-10$</td>
</tr>
<tr>
<td>2</td>
<td>$-1.311911E-07$</td>
<td>12</td>
<td>$-7.163488E-10$</td>
</tr>
<tr>
<td>3</td>
<td>$4.060425E-08$</td>
<td>13</td>
<td>$5.657463E-10$</td>
</tr>
<tr>
<td>4</td>
<td>$-1.769681E-08$</td>
<td>14</td>
<td>$-4.545819E-10$</td>
</tr>
<tr>
<td>5</td>
<td>$9.273560E-09$</td>
<td>15</td>
<td>$3.707450E-10$</td>
</tr>
<tr>
<td>6</td>
<td>$-5.459068E-09$</td>
<td>16</td>
<td>$-3.063259E-10$</td>
</tr>
<tr>
<td>7</td>
<td>$3.482784E-09$</td>
<td>17</td>
<td>$2.560118E-10$</td>
</tr>
<tr>
<td>8</td>
<td>$-2.357129E-09$</td>
<td>18</td>
<td>$-3.161415E-10$</td>
</tr>
<tr>
<td>9</td>
<td>$1.669137E-09$</td>
<td>19</td>
<td>$1.841381E-10$</td>
</tr>
</tbody>
</table>

Table 4. Values of $t_k$ in Equation 5.48

<table>
<thead>
<tr>
<th>k</th>
<th>$t_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.807664E-03$</td>
</tr>
<tr>
<td>1</td>
<td>$0.927049E-02$</td>
</tr>
<tr>
<td>2</td>
<td>$0.600178E-00$</td>
</tr>
</tbody>
</table>
Table 5. Coefficient matrix for stress problem

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
<th>m</th>
<th>n</th>
<th>C_{mn}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>5.045486E-01</td>
<td>7</td>
<td>3</td>
<td>1.900918E-30</td>
<td>10</td>
<td>7</td>
<td>2.205356E-48</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4.158976E-06</td>
<td>7</td>
<td>4</td>
<td>9.063364E-33</td>
<td>10</td>
<td>8</td>
<td>7.089045E-51</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.666666E-01</td>
<td>7</td>
<td>5</td>
<td>3.327307E-35</td>
<td>10</td>
<td>9</td>
<td>2.009867E-53</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3.405522E-09</td>
<td>7</td>
<td>6</td>
<td>1.001193E-37</td>
<td>10</td>
<td>10</td>
<td>2.380952E-02</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.887253E-11</td>
<td>7</td>
<td>7</td>
<td>3.333333E-02</td>
<td>11</td>
<td>0</td>
<td>1.420363E-37</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>9.999996E-02</td>
<td>8</td>
<td>0</td>
<td>4.844725E-28</td>
<td>11</td>
<td>1</td>
<td>1.044871E-38</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>2.665481E-12</td>
<td>8</td>
<td>1</td>
<td>1.994476E-29</td>
<td>11</td>
<td>2</td>
<td>2.523861E-40</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2.444440E-14</td>
<td>8</td>
<td>2</td>
<td>2.900424E-31</td>
<td>11</td>
<td>3</td>
<td>3.292732E-42</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>9.01764E-17</td>
<td>8</td>
<td>3</td>
<td>2.381683E-33</td>
<td>11</td>
<td>4</td>
<td>2.841010E-44</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>7.142854E-02</td>
<td>8</td>
<td>4</td>
<td>1.339453E-35</td>
<td>11</td>
<td>5</td>
<td>1.809519E-46</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2.000487E-15</td>
<td>8</td>
<td>5</td>
<td>5.726728E-38</td>
<td>11</td>
<td>6</td>
<td>9.111498E-49</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>2.732515E-17</td>
<td>8</td>
<td>6</td>
<td>1.985118E-40</td>
<td>11</td>
<td>7</td>
<td>3.799022E-51</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.536325E-19</td>
<td>8</td>
<td>7</td>
<td>5.825457E-43</td>
<td>11</td>
<td>8</td>
<td>1.355871E-53</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5.424215E-22</td>
<td>8</td>
<td>8</td>
<td>2.941176E-02</td>
<td>11</td>
<td>9</td>
<td>4.245913E-56</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5.555555E-02</td>
<td>9</td>
<td>0</td>
<td>3.255507E-31</td>
<td>11</td>
<td>10</td>
<td>1.189051E-58</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1.451638E-18</td>
<td>9</td>
<td>1</td>
<td>1.651133E-32</td>
<td>11</td>
<td>11</td>
<td>2.173913E-02</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>2.765784E-20</td>
<td>9</td>
<td>2</td>
<td>2.882336E-34</td>
<td>12</td>
<td>0</td>
<td>9.245472E-41</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>2.061286E-22</td>
<td>9</td>
<td>3</td>
<td>2.794231E-36</td>
<td>12</td>
<td>1</td>
<td>8.050105E-42</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>9.308756E-25</td>
<td>9</td>
<td>4</td>
<td>1.830897E-38</td>
<td>12</td>
<td>2</td>
<td>2.250434E-43</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>3.043874E-27</td>
<td>9</td>
<td>5</td>
<td>9.016288E-41</td>
<td>12</td>
<td>3</td>
<td>3.357463E-45</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>4.545454E-02</td>
<td>9</td>
<td>6</td>
<td>3.568614E-43</td>
<td>12</td>
<td>4</td>
<td>3.282564E-47</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1.026395E-21</td>
<td>9</td>
<td>7</td>
<td>1.185456E-45</td>
<td>12</td>
<td>5</td>
<td>2.351018E-49</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>2.607011E-23</td>
<td>9</td>
<td>8</td>
<td>3.412496E-48</td>
<td>12</td>
<td>6</td>
<td>1.322297E-51</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2.488694E-25</td>
<td>9</td>
<td>9</td>
<td>2.631579E-02</td>
<td>12</td>
<td>7</td>
<td>6.122092E-54</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1.399595E-27</td>
<td>10</td>
<td>0</td>
<td>2.161635E-34</td>
<td>12</td>
<td>8</td>
<td>2.413512E-56</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>5.575029E-30</td>
<td>10</td>
<td>1</td>
<td>1.329064E-35</td>
<td>12</td>
<td>9</td>
<td>8.308981E-59</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>1.735702E-32</td>
<td>10</td>
<td>2</td>
<td>2.745668E-37</td>
<td>12</td>
<td>10</td>
<td>2.547127E-61</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3.846154E-02</td>
<td>10</td>
<td>3</td>
<td>3.104019E-39</td>
<td>12</td>
<td>11</td>
<td>7.060558E-64</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>7.110435E-25</td>
<td>10</td>
<td>4</td>
<td>2.344637E-41</td>
<td>12</td>
<td>12</td>
<td>2.000000E-02</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>2.328800E-26</td>
<td>10</td>
<td>5</td>
<td>1.318683E-43</td>
<td>13</td>
<td>0</td>
<td>5.966492E-44</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>2.772958E-28</td>
<td>10</td>
<td>6</td>
<td>5.906922E-46</td>
<td>13</td>
<td>1</td>
<td>6.094264E-45</td>
</tr>
<tr>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>----------</td>
<td>----</td>
<td>----</td>
<td>----------</td>
<td>----</td>
<td>----</td>
<td>----------</td>
</tr>
<tr>
<td>13</td>
<td>2</td>
<td>1.954397E-46</td>
<td>15</td>
<td>4</td>
<td>4.028228E-56</td>
<td>17</td>
<td>2</td>
<td>9.049867E-59</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>3.308209E-48</td>
<td>15</td>
<td>5</td>
<td>3.959497E-58</td>
<td>17</td>
<td>3</td>
<td>2.384818E-60</td>
</tr>
<tr>
<td>13</td>
<td>4</td>
<td>3.639205E-50</td>
<td>15</td>
<td>6</td>
<td>3.006185E-60</td>
<td>17</td>
<td>4</td>
<td>3.973699E-62</td>
</tr>
<tr>
<td>13</td>
<td>5</td>
<td>2.912341E-52</td>
<td>15</td>
<td>7</td>
<td>1.851457E-62</td>
<td>17</td>
<td>5</td>
<td>4.709951E-64</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>1.819274E-54</td>
<td>15</td>
<td>8</td>
<td>9.582146E-65</td>
<td>17</td>
<td>6</td>
<td>4.273693E-66</td>
</tr>
<tr>
<td>13</td>
<td>7</td>
<td>9.305297E-57</td>
<td>15</td>
<td>9</td>
<td>4.279284E-67</td>
<td>17</td>
<td>7</td>
<td>3.120822E-68</td>
</tr>
<tr>
<td>13</td>
<td>8</td>
<td>4.033317E-59</td>
<td>15</td>
<td>10</td>
<td>1.683259E-69</td>
<td>17</td>
<td>8</td>
<td>1.901389E-70</td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>1.520071E-61</td>
<td>15</td>
<td>11</td>
<td>5.927601E-72</td>
<td>17</td>
<td>9</td>
<td>9.931362E-73</td>
</tr>
<tr>
<td>14</td>
<td>0</td>
<td>3.819886E-47</td>
<td>16</td>
<td>0</td>
<td>1.531243E-53</td>
<td>17</td>
<td>13</td>
<td>0.0</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>4.543194E-48</td>
<td>16</td>
<td>1</td>
<td>2.426914E-54</td>
<td>17</td>
<td>14</td>
<td>0.0</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>1.658657E-49</td>
<td>16</td>
<td>2</td>
<td>1.126362E-55</td>
<td>17</td>
<td>15</td>
<td>0.0</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>3.163203E-51</td>
<td>16</td>
<td>3</td>
<td>2.678541E-57</td>
<td>17</td>
<td>17</td>
<td>1.428571E-02</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>3.890761E-53</td>
<td>16</td>
<td>4</td>
<td>4.052686E-59</td>
<td>18</td>
<td>0</td>
<td>5.968906E-60</td>
</tr>
<tr>
<td>14</td>
<td>5</td>
<td>3.459585E-55</td>
<td>16</td>
<td>5</td>
<td>4.383521E-61</td>
<td>18</td>
<td>1</td>
<td>1.240119E-60</td>
</tr>
<tr>
<td>14</td>
<td>6</td>
<td>2.388036E-57</td>
<td>16</td>
<td>6</td>
<td>3.645262E-63</td>
<td>18</td>
<td>2</td>
<td>7.165055E-62</td>
</tr>
<tr>
<td>14</td>
<td>7</td>
<td>1.343114E-59</td>
<td>16</td>
<td>7</td>
<td>2.448925E-65</td>
<td>18</td>
<td>3</td>
<td>2.083070E-63</td>
</tr>
<tr>
<td>14</td>
<td>8</td>
<td>6.373711E-62</td>
<td>16</td>
<td>8</td>
<td>1.377405E-67</td>
<td>18</td>
<td>4</td>
<td>3.806816E-65</td>
</tr>
<tr>
<td>14</td>
<td>9</td>
<td>2.619494E-64</td>
<td>16</td>
<td>9</td>
<td>6.662506E-70</td>
<td>18</td>
<td>5</td>
<td>4.925746E-67</td>
</tr>
<tr>
<td>14</td>
<td>10</td>
<td>9.514072E-67</td>
<td>16</td>
<td>10</td>
<td>2.829690E-72</td>
<td>18</td>
<td>6</td>
<td>4.859663E-69</td>
</tr>
<tr>
<td>14</td>
<td>11</td>
<td>3.103113E-69</td>
<td>16</td>
<td>11</td>
<td>1.072575E-74</td>
<td>18</td>
<td>7</td>
<td>3.844913E-71</td>
</tr>
<tr>
<td>14</td>
<td>12</td>
<td>9.207460E-72</td>
<td>16</td>
<td>12</td>
<td>3.167286E-77</td>
<td>18</td>
<td>8</td>
<td>2.529977E-73</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>2.511784E-74</td>
<td>16</td>
<td>13</td>
<td>0.0</td>
<td>18</td>
<td>9</td>
<td>1.419799E-75</td>
</tr>
<tr>
<td>14</td>
<td>14</td>
<td>1.724138E-02</td>
<td>16</td>
<td>14</td>
<td>0.0</td>
<td>18</td>
<td>10</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>2.427240E-50</td>
<td>16</td>
<td>15</td>
<td>0.0</td>
<td>18</td>
<td>11</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>3.548793E-51</td>
<td>16</td>
<td>16</td>
<td>1.515151E-02</td>
<td>18</td>
<td>12</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
<td>1.579294E-52</td>
<td>17</td>
<td>0</td>
<td>9.593011E-57</td>
<td>18</td>
<td>13</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>2.945427E-54</td>
<td>17</td>
<td>1</td>
<td>1.743597E-57</td>
<td>18</td>
<td>14</td>
<td>0.0</td>
</tr>
<tr>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
<td>m</td>
<td>n</td>
<td>$c_{mn}$</td>
</tr>
<tr>
<td>----</td>
<td>----</td>
<td>---------</td>
<td>----</td>
<td>----</td>
<td>---------</td>
<td>----</td>
<td>----</td>
<td>--------</td>
</tr>
<tr>
<td>18</td>
<td>15</td>
<td>0.0</td>
<td>20</td>
<td>7</td>
<td>4.8015e-77</td>
<td>21</td>
<td>17</td>
<td>0.0</td>
</tr>
<tr>
<td>18</td>
<td>16</td>
<td>0.0</td>
<td>20</td>
<td>8</td>
<td>0.0</td>
<td>21</td>
<td>18</td>
<td>0.0</td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>0.0</td>
<td>20</td>
<td>9</td>
<td>0.0</td>
<td>21</td>
<td>19</td>
<td>0.0</td>
</tr>
<tr>
<td>18</td>
<td>18</td>
<td>1.3513e-02</td>
<td>20</td>
<td>10</td>
<td>0.0</td>
<td>21</td>
<td>20</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>0</td>
<td>3.6887e-63</td>
<td>20</td>
<td>11</td>
<td>0.0</td>
<td>21</td>
<td>21</td>
<td>1.1627e-02</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>8.7394e-64</td>
<td>20</td>
<td>12</td>
<td>0.0</td>
<td>22</td>
<td>0</td>
<td>8.3514e-73</td>
</tr>
<tr>
<td>19</td>
<td>2</td>
<td>5.5977e-65</td>
<td>20</td>
<td>13</td>
<td>0.0</td>
<td>22</td>
<td>1</td>
<td>2.9133e-73</td>
</tr>
<tr>
<td>19</td>
<td>3</td>
<td>1.7881e-66</td>
<td>20</td>
<td>14</td>
<td>0.0</td>
<td>22</td>
<td>2</td>
<td>2.4901e-74</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>3.5707e-68</td>
<td>20</td>
<td>15</td>
<td>0.0</td>
<td>22</td>
<td>3</td>
<td>1.0313e-75</td>
</tr>
<tr>
<td>19</td>
<td>5</td>
<td>5.0262e-69</td>
<td>20</td>
<td>16</td>
<td>0.0</td>
<td>22</td>
<td>4</td>
<td>2.0977e-77</td>
</tr>
<tr>
<td>19</td>
<td>6</td>
<td>5.3746e-72</td>
<td>20</td>
<td>17</td>
<td>0.0</td>
<td>22</td>
<td>5</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>7</td>
<td>4.5931e-74</td>
<td>20</td>
<td>18</td>
<td>0.0</td>
<td>22</td>
<td>6</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>8</td>
<td>3.2172e-76</td>
<td>20</td>
<td>19</td>
<td>0.0</td>
<td>22</td>
<td>7</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>9</td>
<td>0.0</td>
<td>20</td>
<td>20</td>
<td>1.2195e-02</td>
<td>22</td>
<td>8</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>10</td>
<td>0.0</td>
<td>21</td>
<td>0</td>
<td>1.3800e-69</td>
<td>22</td>
<td>9</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>11</td>
<td>0.0</td>
<td>21</td>
<td>1</td>
<td>4.2334e-70</td>
<td>22</td>
<td>10</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>12</td>
<td>0.0</td>
<td>21</td>
<td>2</td>
<td>3.2970e-71</td>
<td>22</td>
<td>11</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>13</td>
<td>0.0</td>
<td>21</td>
<td>3</td>
<td>1.2582e-72</td>
<td>22</td>
<td>12</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>14</td>
<td>0.0</td>
<td>21</td>
<td>4</td>
<td>2.9698e-74</td>
<td>22</td>
<td>13</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>15</td>
<td>0.0</td>
<td>21</td>
<td>5</td>
<td>4.8715e-76</td>
<td>22</td>
<td>14</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>16</td>
<td>0.0</td>
<td>21</td>
<td>6</td>
<td>0.0</td>
<td>22</td>
<td>15</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>17</td>
<td>0.0</td>
<td>21</td>
<td>7</td>
<td>0.0</td>
<td>22</td>
<td>16</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>0.0</td>
<td>21</td>
<td>8</td>
<td>0.0</td>
<td>22</td>
<td>17</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
<td>19</td>
<td>1.2820e-02</td>
<td>21</td>
<td>9</td>
<td>0.0</td>
<td>22</td>
<td>18</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>2.2640e-66</td>
<td>21</td>
<td>10</td>
<td>0.0</td>
<td>22</td>
<td>19</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>1</td>
<td>6.1066e-67</td>
<td>21</td>
<td>11</td>
<td>0.0</td>
<td>22</td>
<td>20</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>4.3201e-68</td>
<td>21</td>
<td>12</td>
<td>0.0</td>
<td>22</td>
<td>21</td>
<td>0.0</td>
</tr>
<tr>
<td>20</td>
<td>3</td>
<td>1.5109e-69</td>
<td>21</td>
<td>13</td>
<td>0.0</td>
<td>22</td>
<td>22</td>
<td>1.1111e-02</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>3.2854e-71</td>
<td>21</td>
<td>14</td>
<td>0.0</td>
<td>23</td>
<td>0</td>
<td>4.9892e-76</td>
</tr>
<tr>
<td>20</td>
<td>5</td>
<td>5.0150e-72</td>
<td>21</td>
<td>15</td>
<td>0.0</td>
<td>23</td>
<td>1</td>
<td>1.9564e-76</td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>5.7920e-75</td>
<td>21</td>
<td>16</td>
<td>0.0</td>
<td>23</td>
<td>2</td>
<td>1.2347e-77</td>
</tr>
</tbody>
</table>
| Pressure (GPa) | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 | 2.3 | 2.4 | 2.5 | 2.6 | 2.7 | 2.8 | 2.9 | 3.0 | 3.1
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Temperature (°C) | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 | 2.3 | 2.4 | 2.5 | 2.6 | 2.7 | 2.8 | 2.9 | 3.0 | 3.1

Table 5 (continued)
Table 6. \( b_n \) for stress problem

<table>
<thead>
<tr>
<th>( n )</th>
<th>( b_n/(bT/\beta^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7973766E+00</td>
</tr>
<tr>
<td>1</td>
<td>-0.1312232E-04</td>
</tr>
<tr>
<td>2</td>
<td>0.1494674E-05</td>
</tr>
<tr>
<td>&gt;3</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 7. \( a_m \) for stress problem

<table>
<thead>
<tr>
<th>( m )</th>
<th>( a_m/(bT/\beta^2) )</th>
<th>( m )</th>
<th>( a_m/(bT/\beta^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.903540E-02</td>
<td>13</td>
<td>-3.190815E-43</td>
</tr>
<tr>
<td>1</td>
<td>-4.115956E-06</td>
<td>14</td>
<td>-2.194152E-46</td>
</tr>
<tr>
<td>2</td>
<td>1.839570E-07</td>
<td>15</td>
<td>-1.490364E-49</td>
</tr>
<tr>
<td>3</td>
<td>-3.695673E-12</td>
<td>16</td>
<td>-1.000866E-52</td>
</tr>
<tr>
<td>4</td>
<td>-3.566141E-15</td>
<td>17</td>
<td>-6.650286E-56</td>
</tr>
<tr>
<td>5</td>
<td>-3.162796E-18</td>
<td>18</td>
<td>-4.374348E-59</td>
</tr>
<tr>
<td>6</td>
<td>-2.642882E-21</td>
<td>19</td>
<td>-2.849424E-62</td>
</tr>
<tr>
<td>7</td>
<td>-2.112552E-24</td>
<td>20</td>
<td>-1.838632E-65</td>
</tr>
<tr>
<td>8</td>
<td>-1.631315E-27</td>
<td>21</td>
<td>-1.175366E-68</td>
</tr>
<tr>
<td>9</td>
<td>-1.225157E-30</td>
<td>22</td>
<td>-7.443677E-72</td>
</tr>
<tr>
<td>10</td>
<td>-8.992170E-34</td>
<td>23</td>
<td>-4.644688E-75</td>
</tr>
<tr>
<td>11</td>
<td>-6.470626E-37</td>
<td>24</td>
<td>-0.000000E-00</td>
</tr>
<tr>
<td>12</td>
<td>-4.578128E-40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>