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Fixed point theorems for non-continuous functions

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Fixed point theorems for non-continuous functions

by

Jack Emile Girolo

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I. INTRODUCTION

Let $f : X \to X$ be a function from a topological space $X$ to $X$. Then $f$ is said to have a **fixed point** if there exists an $x \in X$ such that $f(x) = x$ and $X$ is said to have the **continuous fixed point property** or **fixed point property** if every continuous function from $X$ to $X$ has a fixed point. The most fundamental theorem concerning fixed points is:

**Theorem 1.** [Brouwer] The $n$-cell $I^n$ has the fixed point property.

A subset $R$ of $X$ is said to be a **retract** of $X$ if there exists a continuous function $r : X \to R$ such that for each $y \in R$ $r(y) = y$. One of the most successful methods of showing that a space has the fixed point property is by means of the following theorem:

**Theorem 2.** If $X$ has the fixed point property and $R$ is a retract of $X$ then $R$ has the fixed point property.

**Proof:** Let $f : R \to R$ be continuous, $r : X \to R$ be the retraction and $i : R \to X$ be the identity function. Then $ifr : X \to X^1$ is continuous and by hypothesis $ifr$ has a fixed point. So there exists $x \in X$ such that $ifr(x) = x$

$ifr$ means the composition of the functions $r$, $f$ and $i$. 
which implies that $f(x) = x$. Thus $f$ has a fixed point.

**Example 1.** A dendrite $D$ is an acyclic (contains no simple closed curve), locally connected continuum. \[15\] and \[2; Corollary 13.5\] imply that $D$ is a retract of $I^2$. By Theorems 1 and 2 $D$ has the fixed point property.

**Example 2.** Every contractible polyhedron $^2$ $P$ has the fixed point property. \[10; Theorem V.3, p.60\] and \[9; Theorem 11.1, p.175\] imply, for some $n$, $P$ is a retract of $I^n$. By Theorems 1 and 2 $P$ has the fixed point property.

**Example 3.** \[1; p.120\] For each natural number $n$, let $B(n)$, a book with $n$ pages, be the subset of Euclidean 3-space defined by:

$$B(n) = \{(\theta, r, z) : \theta = \frac{2\pi k}{n}, \text{ } k \text{ an integer, } 0 \leq k \leq n-1, \} \quad 0 \leq z \leq 1 \text{ an } 0 \leq r \leq 1$$

Since $B(n)$ is a contractible polyhedron, it has the fixed point property.

Theorem 1 is used to prove the next theorem.

---

$^2$ In this dissertation all polyhedra are finite and connected.
Theorem 3. [Tychonoff] Let $L$ be a locally convex linear topological space and let $C$ be a compact convex subset of $L$. Then $C$ has the fixed point property.

Thus we see that spaces with the fixed point property are not necessarily finite dimensional.

If $G = \{u_1, \ldots, u_n\}$ is an open covering of a continuum $X$, then with $G$ and $X$ we associate a geometric complex $N(G, X)$ consisting of vertices $v_1, \ldots, v_n$ such that $\langle v_{i_1}, \ldots, v_{i_r} \rangle$ is a simplex of $N(G, X)$ if and only if $u_{i_1} \cap \cdots \cap u_{i_r} \neq \emptyset$. If for every $\varepsilon > 0$ there exists an open covering $G$ of $X$ such that the mesh of $G < \varepsilon$ and $N(G, X)$ is homeomorphic to $I$, then $X$ is a snake-like continuum. If for every $\varepsilon > 0$ there exists an open covering $G$ of $X$ such that the mesh of $G < \varepsilon$ and $N(G, X)$ is a tree (contains no simple closed curve), then $X$ is a tree-like continuum. A tree is a dendrite with only a finite number of non-separating points.

Figure 1. A snake-like continuum
Example 4. Let $X$ be the subset of the plane defined by

$$\{(0,t) : -1 \leq t \leq 1\} \cup \{(t,\sin \frac{1}{t}) : 0 < t \leq 1\}$$

See Figure 1. $X$ is a snake-like continuum.

![Figure 2. A tree-like continuum](image)

Example 5. Let $Y$ be the subset of the plane defined by $\{(t,0) : -1 \leq t \leq 0\} \cup X$, where $X$ is defined in Example 4. See Figure 2. $X$ is a tree-like continuum that is not snake-like.

Theorem 4. [Hamilton; 6] Every snake-like continuum has the fixed point property.

Example 4 together with Theorem 4 show that there are non-locally connected spaces with the fixed point property.

There are many spaces $X$ for which it is not known whether or not $X$ has the fixed point property.
Question 1. Does every tree-like continuum have the fixed point property?

Question 2. Does each compact convex subset of a linear topological space have the fixed point property?

Other unsolved problems concerning fixed points can be found in [1].

Definition 1. If \( f : X \rightarrow Y \) is a function from a topological space \( X \) to a topological space \( Y \), then the graph of \( f \), which we denote by \( \Gamma_f \), is \( \{(x, f(x)) : x \in X\} \) considered as a subset of the topological product space \( X \times Y \). Also the graph of \( f \) will be referred to as the graph over \( X \).

Definition 2. The statement that \( f \) is a connectivity function means that for each connected subset \( C \) of \( X \), \( \{(x, f(x)) : x \in C\} \) is a connected subset of \( X \times Y \).

Thus \( f \) is a connectivity function if the graph over each connected set is connected.

The following are examples of connectivity functions:

Example 6. Define \( f : I^n \rightarrow I \) by \( f(x) = \begin{cases} \sin \frac{1}{\|x\|}, & \|x\| \neq 0 \\ 0, & \|x\| = 0 \end{cases} \).

\[ \|x\| \] is the norm of \( x \); i.e. if \( x = (x_1, \ldots, x_n) \) then \( \|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \).
Example 7. Every continuous function.

Example 8. Any function $f : I \rightarrow X$ such that $f$ is connected.

O. H. Hamilton has extended the study of fixed points to non-continuous functions:

Theorem 5. [Hamilton; 7] If $f$ is a connectivity function of the n-cell into itself, then $f$ has a fixed point.

Theorem 5 gives motivation to the following:

Definition 3. If every connectivity function $f : X \rightarrow X$ has a fixed point, then $X$ is said to have the connectivity fixed point property.

Thus the n-cell has the connectivity fixed point property.

The significance of Theorem 5 to the study of fixed points may be that it is the connectivity structure of continuous functions which yields fixed points and approaching a fixed point problem with this in mind might be helpful.

Theorem 2 suggests:

Question 3. If $X$ has the connectivity fixed point property and $R$ is a retract of $X$, then does $R$ have the connectivity fixed point property?

An approach similar to the proof of Theorem 2 does not answer this question. The difficulty arises from the fact
that \( fr \) need not be a connectivity function.

**Example 9.** Let \( r : I^2 \to I \times 0 \) be defined by
\[
(r(x, y)) = (x, 0)
\]
and let \( f : I \times 0 \to I \times 0 \) be defined by
\[
f(x, 0) = \begin{cases} 
\left( \frac{1}{2} \left( \sin \frac{1}{x} \right) + \frac{1}{2} \right), & x \neq 0 \\
(0, 0), & x = 0 
\end{cases}
\]

Then \( r \) is a retraction and \( f \) is a connectivity function but \( fr \) is not a connectivity function.

**Definition 4.** A space \( X \) is said to be rim connected provided that for each \( x \in X \) and each open set \( U \) containing \( x \), there is a region \( V \) such that \( x \in V \subset \overline{V} \subset U \) and \( \text{bd}(V) \) is connected.

J. Stallings gives a partial answer to Question 3:

**Theorem 6.** [13] If \( P \) is a rim connected polyhedron with the fixed point property, then \( P \) has the connectivity fixed point property.

In this dissertation we continue the study of fixed points for continuous and non-continuous functions. In Chapter II we show that an infinite dimensional space, the

---

4. A region is a connected open set.
5. \( \text{bd}(V) \) = boundary of \( V \).
Hilbert Cube, has the connectivity fixed point property.

In Chapter III we answer Question 3 in the affirmative for 
\( X = P \) a polyhedron and \( R \) a rim connected retract of \( P \).

Let \( f : X \rightarrow Y \) be a function from \( X \) to \( Y \). Then 
\( f \times i : X \times I \rightarrow Y \times I \) is the function defined by 
\( f \times i((x,t)) = (f(x), t) \). In Chapter IV we study \( f \times i \) and 
give among other things a new characterization of continuous functions. In Chapter V we use connectivity methods to 
show that certain spaces have the fixed point property. In 
Chapter VI we will be concerned with relaxing the condition 
inherent in the definition of connectivity function in such 
a way as to get fixed point theorems for the n-cell.
II. AN INFINITE DIMENSIONAL FIXED POINT THEOREM

As was stated in the introduction, the object of this chapter is to prove that the Hilbert Cube has the connectivity fixed point property.

**Definition 5.** A connected topological space $X$ is semi-locally connected if for each point $x \in X$ and each open set $U$ containing $x$ there exists an open set $V$ such that $x \in V \subseteq U$ and $X - V$ has at most a finite number of components.

**Definition 6.** A monotone upper-semi-continuous decomposition of a space $X$ is a collection $G$ of mutually exclusive closed and connected subsets of $X$ whose union is $X$ such that if $g \in G$ and $U$ is an open set containing $g$ there is an open set $V$ containing $g$ such that every element of $G$ that intersects $V$ is a subset of $U$.

The lemma that follows is a modification of a lemma due to Stallings and the proof of the revised lemma is essentially the same as the proof of the original lemma as given by Stallings. Stallings Lemma has played an important role in the study of connectivity functions: see [4], [5], [12] and [13].

**Lemma 1.** [13; p.253], [4; p.184] Suppose that $X$ is a compact metric semi-locally connected space and $f : X \to Y$
is a connectivity function where \( Y \) is a \( T_1 \) space. Then if \( C \) is a closed subset of \( Y \), \( K \) a closed subset of \( X \), and \( G' \) denotes the collection of components of \( f^{-1}(C) \cap K \), the set \( G \) consisting of \( G' \) together with all of the degenerate subsets of \( X - (f^{-1}(C) \cap K) \) is a monotone upper-semi-continuous decomposition of \( X \) and \( G' \) as a subset of the decomposition space is totally disconnected.

**Theorem 7.** [7; p.750] If \( f : X \rightarrow Y \) is a connectivity function of a \( T_1 \) space \( X \) into a \( T_1 \) space \( Y \), \( p \) a point of \( X \), \( V \) and \( U \) open sets containing \( p \) and \( f(p) \) respectively, then every non-degenerate connected subset of \( X \) containing \( p \) contains a point \( q \) of \( V \) distinct from \( p \) such that \( f(q) \in U \).

In that which follows we use Lemma 1 and Theorem 7 to show that, on a certain class of "locally cohesive" spaces, connectivity functions are "peripherally" continuous. This result was originally proven by Whyburn [17]; we give a new proof using techniques developed by Stallings [Theorem 4; 13].

**Definition 7.** [7] If \( f : X \rightarrow Y \) is a function, then \( f \) is peripherally continuous means that for each \( x \in X \), each open \( V \subseteq X \) for which \( x \in V \), each open \( U \subseteq Y \) for which \( f(x) \in U \), there exists a neighborhood \( N \) of \( x \)
such that $\overline{N} \subseteq V$ and $f(\partial(N)) \subseteq U$.

**Definition 8.** [17] A space $X$ is locally cohesive at $p$ if for each neighborhood $U$ of $p$, there is a region $V$, $V \subseteq U$, $p \in V$ with $\partial(V)$ connected, and for each representation $V = A \cup B$ ($A$ and $B$ are closed and connected) with $\partial(V) \subseteq A$ and $p \in B$, $A \cap B$ is connected. Such a $V$ is a canonical region for $p$. If $X$ is locally cohesive at each of its points then $X$ is locally cohesive.

**Lemma 2.** Let $p$ be an element of a canonical region $V$ in a Peano Continuum $X$. Suppose $K \subseteq V$ separates $p$ and $\partial(V)$. Then there is a continuum $C$, $C \subseteq K$, $C$ is minimal with respect to being closed and separating $p$ and $\partial(V)$.

Proof: Since $\overline{V}$ is metric there is a closed subset $K'$ of $K$ such that $K'$ separates $p$ and $\partial(V)$ [16; p.43]. By Zorn's Lemma $K'$ may be assumed to be minimal. Suppose $K'$ is not connected. Then there is a representation $K' = A \cup B$ where $A$ and $B$ are closed. Let $C_{\alpha}$, where $\alpha$ ranges over some index set, be the components of $\overline{V} - (A \cup B)$. Let $p \in C_{\alpha_1}$ and $\partial(V) \subseteq C_{\alpha_2}$ where $\alpha_1 \neq \alpha_2$. Since $K'$ is minimal $K' \subseteq \overline{C_{\alpha_1}}$ and $K' \subseteq \overline{C_{\alpha_2}}$. So we construct the following representation of $\overline{V}$:
A' = \overline{C_{\alpha_1}} \quad \text{and} \quad B' = \overline{V} - \overline{C_{\alpha_1}}.

Then A' \cup B' = \overline{V}, A' \text{ and } B' \text{ are continua and } A' \cap B' = K' \text{ which is a contradiction. Therefore set } C = K'.

**Theorem 8.** If \( f : X \to Y \) is a connectivity function of a locally cohesive Peano continuum \( X \) into a regular Hausdorff space \( Y \), then \( f \) is peripherally continuous.

**Proof:** Given \( x \in X \), neighborhoods \( V \) of \( x \) and \( U \) of \( f(x) \), we must find a neighborhood \( N \) of \( X \) such that \( \overline{N} \subset V \) and \( f(\text{bd}(N)) \subset U \). We assume that \( V \) is a canonical region. Let \( V^* \) be a neighborhood of \( x \) such that \( \overline{V^*} \subset V \) and let \( U^* \) be a neighborhood of \( f(x) \) such that \( \overline{U^*} \subset U \). Then consider the set \( D = f^{-1}(U^*) \cap \overline{V^*} \) and the family of its components \( \{D_\alpha\} \).

**Statement 1.** There is one \( D_\alpha \) which contains the boundary of an open set \( M \) such that \( x \in M \subset V \).

Assuming Statement 1 is true then the conclusion of the theorem is immediate: \( \overline{M} \subset V \) and \( f(\text{bd}(M)) \subset U \). So set \( N = M \).

So the final objective of that which follows will be to prove Statement 1.
By Lemma 1 \( \{D_\alpha\} \) together with the degenerate sets of \( \overline{V} - D \) form a monotone upper-semi-continuous decomposition of \( \overline{V} \). Let \( D' \) be the decomposition space and \( T: \overline{V} \rightarrow D' \) the continuous mapping associated with the decomposition [16; p.126]. Let \( d_\alpha = T(D_\alpha) \). Define \( D_\alpha < D_\beta(d_\alpha < d_\beta) \) if \( D_\beta(d_\beta) \) separates \( D_\alpha(d_\alpha) \) and \( \text{bd}(V) \) in \( \overline{V}(T(\text{bd}(V)) \text{ in } D') \). This is a partial ordering.

**Statement 2.** Each \( D_\alpha \) is less than or equal to, with respect to \( < \), some maximal element \( D_\beta \). Certain parts of the verification of Statement 2 will use the original space \( \overline{V} \) and other parts will use the decomposition space \( D' \).

Let \( \{D_{\alpha_\tau}\} \) be a linearly ordered subset, where the indices \( \{\tau\} \) are themselves linearly ordered and \( u < \tau \) if and only if \( D_{\alpha_u} < D_{\alpha_\tau} \). Let

\[
L = \cap_{\tau \geq \tau} \bigcup_{\alpha \geq \tau} D_{\alpha_u} \quad \text{and} \quad \ell = \cap_{\tau \geq \tau} \bigcup_{\alpha \geq \tau} d_{\alpha_u}.
\]

It will be shown that \( L \) is an upper bound for the linearly ordered subset and that \( L \) is some \( D_{\alpha_\tau} \).

\( L \) is compact and nonempty.

\( L \) separates \( D_{\alpha_\tau} \) from \( \text{bd}(V) \) in \( \overline{V} \). This follows from the fact that if \( L \) did not separate \( D_{\alpha_\tau} \) and \( \text{bd}(V) \),...
one could get an arc from $D_\alpha$ to $bd(v)$. Then it can be shown that the arc would meet $L$.

Since $V$ is rim connected at each point, it follows that $L$ is non-degenerate.

It will be shown that $l$ is degenerate. Suppose $p$ and $q$ are elements of $l$. There exist disjoint connected open sets $M$ and $N$ about $p$ and $q$, respectively. Since we have assumed that $l$ is non-degenerate, $d_{\alpha u}$ has no upper bound. Then we can find $d_{\alpha u_1}, d_{\alpha u_2}$ and $d_{\alpha u_3}$ such that:

1. $d_{\alpha u_1} < d_{\alpha u_2} < d_{\alpha u_3}$
2. $d_{\alpha u_1} \in M$ and $d_{\alpha u_3} \in N$

But then $d_{\alpha u_3}$ would not separate $d_{\alpha u_1}$ and $T(bd(v))$, which is a contradiction. Thus $l$ is degenerate.

Since $T$ is continuous, $L \subseteq T^{-1}(l)$. From the definition of $T$, $T^{-1}(l)$ is a point or some $D_\alpha$. Since $L \subseteq T^{-1}(l)$ and $L$ is non-degenerate, $T^{-1}(l)$ is some $D_\beta$. But then $L = D_\beta$ and $D_\beta = D_{\alpha u}$ for some $u$. Thus $\{D_{\alpha u}\}$
has an upper bound $D_{\alpha}$ and hence each $D_{\alpha}$ is less than or equal to some maximal $D_{\beta}$. This completes the proof of Statement 2.

Let $\{D_{\beta}\}$ be the set of maximal elements of the set $\{D_{\alpha}, <\}$. Let $E_{\beta} = \{x \in V : D_{\beta}$ separates $x$ from $bd(V)\}$, $\tilde{D}_{\beta} = D_{\beta} \cup E_{\beta}$ and $\tilde{D} = \bigcup \tilde{D}_{\beta}$. Then $E_{\beta}$ is open and $D \subset \tilde{D}$.

Statement 3. The components of $\tilde{D}$ are just the sets $\tilde{D}_{\beta}$.

Since $\tilde{D}_{\beta}$ is connected, each component of $\tilde{D}$ is of the form $\bigcup \tilde{D}_{\beta\tau}$. Now we know that $\bigcup D_{\beta\tau}$ is not connected unless there is only one $D_{\beta\tau}$ involved. Suppose there is more than one $D_{\beta\tau}$ involved. There is a partition $[\tau] = \{\tau_1\} \cup \{\tau_2\}$ such that $\bigcup D_{\beta\tau_1}$ and $\bigcup D_{\beta\tau_2}$ are disjoint open sets in $\bigcup D_{\beta\tau}$ which cover $\bigcup D_{\beta\tau}$. Let $F$ and $G$ be disjoint open sets such that $\bigcup D_{\beta\tau_1} \subset F$ and $\bigcup D_{\beta\tau_2} \subset G$. Now we show that $\bigcup \tilde{D}_{\beta\tau_1}$ and $\bigcup \tilde{D}_{\beta\tau_2}$ is a disjoint open cover of $\bigcup \tilde{D}_{\beta\tau}$. Disjoint since $\tilde{D}_{\beta\tau_1} \cap \tilde{D}_{\beta\tau_2} = \emptyset$ if $\tau_1 \neq \tau_2$. Next we show that $\bigcup \tilde{D}_{\beta\tau_1}$ and $\bigcup \tilde{D}_{\beta\tau_2}$ are open in $\bigcup \tilde{D}_{\beta\tau}$. This is immediate from the
fact that every \( E_{\beta_1} \) is open and \( D_{\beta_1} \subset F, \ D_{\beta_2} \subset G. \)

Hence the components of \( \tilde{D} \) are precisely the sets \( \tilde{D}_{\beta} \) and this completes the proof of Statement 3.

**Statement 4.** \( \overline{V} - \tilde{D} \) is connected.

Suppose that \( \overline{V} - \tilde{D} \) is not connected. By Lemma 2, since \( V \) is a canonical region, some component, say \( \tilde{D}_{\beta} \), separates \( \overline{V} \). Now let \( \{ C_{\alpha} \} \) be the components of the set \( \overline{V} - D_{\beta} \). Suppose \( \text{bd}(V) \subset C_{\beta} \). Then \( E_{\beta} = \bigcup_{\alpha \neq \beta} C_{\alpha} \) and \( \overline{V} - \tilde{D}_{\beta} = C_{\beta} \) which is connected. This completes the proof of Statement 4.

Now we show that \( x \in \text{int} \tilde{D} \). If not then, by Statement 4, \( (\overline{V} - \tilde{D}) \cup \{ x \} \) is connected and \( f((\overline{V} - \tilde{D}) \cap V^*) \cup \{ x \}) \cap V^* = \{ f(x) \} \) which is contrary to Theorem 7.

Since \( V \) is locally connected, there is a component \( \tilde{D}_{\beta} \) of \( \tilde{D} \) such that \( x \in \text{int} \tilde{D}_{\beta} \) and this completes the proof of Statement 1 and, consequently, of Theorem 8.

The Hilbert Cube \( H \) is the set of all sequences \( x(x_1, x_2, \ldots) \), each \( x_i \) is a real number \( 0 \leq x_i \leq 1/i. \)

\[ \|x\| = \left( \sum_{i=1}^{\frac{1}{2}} x_i^2 \right)^{\frac{1}{2}}. \]

It is known that the Hilbert Cube is homeomorphic to \( I^w \). We will work with \( I^w \) and the metric \( p \) which we will use in the one induced by the standard
homeomorphism $h$ of $H$ onto $I^w$ defined by
$$\begin{align*}
h((x_1, x_2, \ldots, x_i, \ldots)) &= (1 \cdot x_1, 2 \cdot x_2, \ldots, i \cdot x_i, \ldots).
\end{align*}$$
Thus if $x(x_1, x_2, \ldots), y(y_1, y_2, \ldots) \in I^w$ then
$$p(x,y) = \|h^{-1}(x) - h^{-1}(y)\| = \left(\sum (x_i - y_i)^2 / i^2\right)^{1/2}.$$ 

Lemma 3. The Hilbert Cube is locally cohesive.

Proof: Let $p \in U$, $U$ an open set in $I^w$. There exists a collection of intervals $J_n$ such that:

1. $J_n \subseteq [0,1]
2. J_n = [0,1]$ for all but a finite number of $n$'s.
3. At least two $J_n$'s are half open or open intervals.
4. $p \in \bigcap_{n=1}^{\infty} J_n$
5. $\bigcap_{n=1}^{\infty} J_n$ is open and $\bigcup_{n=1}^{\infty} J_n \subseteq U$.

From the definition of the $J_n$'s it follows that $\text{bd}(\bigcap_{n=1}^{\infty} J_n)$ is connected. Also $\bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} \overline{J_n}$. Thus $\bigcap_{n=1}^{\infty} J_n$ is a Hilbert Cube and since the Hilbert Cube is contractible it follows, by [16; Theorem 7.4], that it is unicoherent. Thus $\bigcap_{n=1}^{\infty} J_n$ is a canonical region.
The next lemma strengthens the usefulness of peripheral continuity.

**Lemma 4.** Let $f$ be a peripherally continuous function from $X$ to $Y$ where $X$ is locally cohesive. Then for each $x \in X$, each open $V \subseteq X$ for which $x \in V$, each open $U \subseteq Y$ for which $f(x) \in U$ there exists a connected neighborhood $R$ of $x$ such that $\bar{R} \subseteq V$, $\partial(R)$ is connected, and $f(\partial(R)) \subseteq U$.

**Proof:** We may assume that $V$ is a canonical region. From the definition of peripheral continuity, there is an open set $N$ such that $\bar{N} \subseteq V$ and $f(\partial(N)) \subseteq U$. $\partial(N)$ separates $x$ and $\partial(V)$. By Lemma 2 there is a continuum $K$, $K \subseteq \partial(N)$, which separates $x$ and $\partial(V)$. Let $R$ be the component of $V - K$ that contains $x$. $\partial(R) \subseteq K$ and since $K$ is minimal, $K \subseteq \partial(R)$. Therefore $\partial(R) = K$ and since $R$ is open in $X$ the proof is complete.

The next two theorems are concerned with $I^\omega$. Hamilton [7] proved analogous theorems for $I^n$ and his proof carries over to $I^\omega$.

**Theorem 9.** Let $f$ be a peripherally continuous function of $I^\omega$ into itself. Let the faces $x_i = 0$ and $x_i = 1$ be designated by $A_i$ and $B_i$, respectively. If $x$ is the point $x(x_1, x_2, \ldots)$ let $f(x)$ be the point $(x_1', x_2', \ldots)$. 
For each $i$, $i$ a natural number, let $L_i$, $E_i$, $G_i$ designate subsets of $I^w$ for which $x'_i \leq x_i$, $x'_i = x_i$, $x'_i \geq x_i$, respectively. Then the components of $L_i$, $E_i$ and $G_i$ are closed and if $q(q_1, q_2, \ldots)$ is a point in the common boundary between a non-degenerate component $E$ of $L_i$ or $G_i$ and a connected subset of $I^w - E$, then $q \in E_i$.

Proof: Let $q(q_1, q_2, \ldots)$ be a limit point of a component $E$ of $L_i$ and suppose $q \notin L_i$. By definition of $L_i$, $q'_i = q_i + d$, $d > 0$. Since $f$ is peripherally continuous there is a connected open set $D$ of diameter $< d/3 \cdot i$ containing $q$ such that $E - (D \cap E) \neq \emptyset$ and if $x \in \text{bd}(D)$, then $p(f(x), f(q)) < d/3 \cdot i$. Then the connected set $E$, since it contains points outside of $\overline{D}$ and within $D$, must contain a point $x \in \text{bd}(D)$. That is $p(f(x), f(q)) < d/3 \cdot i$. Hence $|x'_i - q'_i| < d/3$ and $|x_i - q_i| < d/3$. With $q'_i = q_i + d$ these inequalities give $x'_i > x_i + d/3$ and this contradicts the fact that $x$ is in $L_i$. Hence the assumption that $q$ does not belong to $L_i$ is false. Therefore $E_i$ is closed.

It can be shown in the same way that each component of $E_i$ or $G_i$ is closed.

Let $q(q_1, q_2, \ldots)$ be a point in the common boundary between a component $E$ of $L_i$ and some connected subset $R$ of $I^w - E$ and suppose $q$ does not belong to $E_i$. Since
Let \( q \in L_i \), \( q_i = q_i^l + d \) for some \( d > 0 \). Let \( \delta \) be a positive number \(< \frac{d}{3} \cdot i \) such that a spherical region with center \( q \) and diameter \(< \delta \) does not contain all of \( E \) and \( R \).

Since \( f \) is peripherally continuous, it follows by Lemma 4 that there is a connected domain \( D \) with respect to \( \Gamma^w \) of diameter \(< \delta \) containing \( q \) with connected boundary \( F \) such that

1. \( D \) contains a point \( z \in R \),
2. \( R - (\overline{D} \cap R) \neq \emptyset \),
3. if \( x(x_1^l, x_2, \ldots) \in F \), then \( p(T(x), T(q)) < \frac{d}{3} \cdot i \).

Then since \( |x_i^l - q_i^l| < \frac{d}{3} \), \( |x_i - q_i| < \frac{d}{3} \), and since \( q_i = q_i^l + d \), it follows that \( x_i > x_i^l \). Hence \( x \in L_i \) and therefore \( F \subset E \). But the connected set \( R \) contains a point of \( F \) and hence a point of \( E \). This contradicts \( R \subset \Gamma^w - E \). Hence the assumption that \( q \) does not belong to \( E_i \) is false. Similarly it can be shown that each point common to the boundaries of a component \( E \) of \( G_i \) and a connected set of \( \Gamma^w - E \) is in \( E_i \).

**Theorem 10.** Every peripherally continuous function \( f \) of \( \Gamma^w \) into itself has a fixed point.

**Proof:** Let \( N_i \) be the component of \( G_i \) which contains \( A_i \). Then by Theorem 9, \( N_i \) is closed. Let \( \{G^i_\alpha\} \) be the collection of all components of \( \Gamma^w - N_i \) which contain
points of $B_i$. Let $H_i = \left[ \bigcup_{\alpha} G_i^\alpha \right] \cup B_i$. Then $H_i$ is connected since $B_i$ is connected and each $G_i^\alpha$ is connected and contains a point of $B_i$. Let $F_i = N_i \cap \overline{H_i}$. Then by Theorem 9, $F_i$ is a subset of $E_i$ and $F_i$ is closed.

We note that no component $C$ of $I^w - F_i$ contains points of both $A_i$ and $B_i$. For suppose $C$ contains a point $a$ of $A_i$ and a point $b$ of $B_i$. Then $a \in N_i$ and $b \in H_i$. Hence $C$ contains a point of $F_i$, the common boundary between $H_i$ and $N_i$. This contradicts the fact that $C \subset I^w - F_i$.

Next let $g$ be the function of $I^w$ into itself defined as follows: for each point $x(x_1, x_2, \ldots)$ of $I^w$, let $g(x)$ be designated by $x''(x''_1, x''_2, \ldots)$. Let $d_i(x) = p(x, F_i)$. Then if $x$ belongs to a component of $I^w - F_i$ which contains a point of $B_i$ and hence by the paragraph above no point of $A_i$, let

$$x''_i = x_i - \frac{1}{2}(d_i(x) \cdot x_i).$$

Then since $x_i \neq 0$, $x''_i \neq x_i$. If $x \in F_i$, let $x''_i = x_i$. If $x$ belongs to a component of $I^w - F_i$, which contains no point of $B_i$, let

$$x''_i = x_i + \frac{1}{2} d_i(x)(1 - x_i).$$

Since $1 - x_i \neq 0$, we have that $x''_i \neq x_i$.

Since $d_i/2 < 1$, $0 \leq x''_i \leq 1$. Also $x_i = x''_i$ if and only if $x \in F_i \subset E_i$; that is only if $x_i = x'_i$. The function $g$ is, by its definition, a continuous function.
of $I^w$ into itself and hence, by the fact that $I^w$ has
the continuous fixed point property, leaves some point $z$
of $I^w$ fixed. That is for each $i$, $i$ a natural number,
$z_i = z_i^\prime$. $z \in \cap E_i$ and hence by definition of $E_i$, $z_i = z_i^\prime$,
for each natural number $i$. That is $f(z) = z$ and $f$, as
required, leaves a point of $I^w$ fixed.

**Theorem 11.** The Hilbert Cube has the connectivity fixed
point property.

**Proof:** This follows immediately from Lemma 3, Theorem 8
and Theorem 10.
III. A FIXED POINT THEOREM FOR RETRACTS

In this chapter we will give a partial answer to Question 3. We will show that a large class of retracts have the connectivity fixed point property. The statement and proof of the main theorem follows preliminary definitions and theorems.

Let \( X \) be a continuum and \( A, B \) be subcontinua such that \( A \cup B = X \). Then \( N(A,B) \) is the number of components of \( A \cap B \).

**Definition Q.** A continuum \( X \) is \( m \)-coherent, \( m \) a natural number or \( \infty \), if and only if

\[
m = \sup \{ N(A,B) : A \text{ and } B \text{ are subcontinua such that } A \cup B = X \}.
\]

W. C. Chewning has established the following:

**Theorem 12.** Let \( r \) be a connectivity retraction from \( X \) to \( R \) where \( X \) is an \( m \)-coherent Peano continuum and \( R \) is rim connected, then \( R \) is locally cohesive.

**Proof:** The result follows from [5; Theorem 1] and [3; 1.15 and 1.20].
Lemma 5. There is an \( \epsilon > 0 \) associated with each rim connected Peano continuum \( X \) such that if two non-disjoint regions \( M \) and \( N \) with connected boundaries have diameter \( < \epsilon \), \( N \subseteq M \) and \( M \subseteq N \), then \( \text{bd}(M) \cap \text{bd}(N) \neq \emptyset \).

Proof: Suppose \( M \) and \( N \) are rim connected regions in \( X \) which satisfy \( N \subseteq M \), \( M \subseteq N \), \( N \cap M \neq \emptyset \). Suppose \( \text{bd}(M) \cap \text{bd}(N) = \emptyset \). We show \( M \cup N = X \).

\( \text{bd}(M) \cap N \neq \emptyset \). Let \( p \in M \cap N \), \( q \in N - (M \cap N) \). There is an arc \( a \subset N \) with end points \( p \) and \( q \). Since \( \text{bd}(M) \) separates \( p \) and \( q \), \( \text{bd}(M) \) must meet \( a \). Since \( a \subset N \), \( \text{bd}(M) \) meets \( N \). Likewise \( \text{bd}(N) \cap M \neq \emptyset \).

\( \text{bd}(M) \subset N \). \( \text{bd}(N) \) separates \( X \). Since \( \text{bd}(M) \cap \text{bd}(N) = \emptyset \) and \( \text{bd}(M) \) is connected, \( \text{bd}(M) \) is in one component of the separation. Since \( \text{bd}(M) \cap N \neq \emptyset \), \( \text{bd}(M) \subset N \). Likewise \( \text{bd}(N) \subset M \).

Thus \( M \cup N = \overline{M} \cup \overline{N} \). \( M \cup N \) is open and closed and hence must be \( X \).

So we may choose \( \epsilon \) to be \( \frac{\text{diameter } X}{2} \).

Definition 10. [13] That a metric space \( X \) is uniformly locally \( n \)-connected means: For any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for any \( x \in X \), any integer \( k \), \( 0 \leq k \leq n \), and any continuous \( \psi : S^k \to N(x, \delta) \), there exists an extension of \( \psi \) to a continuous \( \psi' : E^{k+1} \to N(x, \epsilon) \). 

\( S^k \) is the \( k \)-dimensional sphere. \( E^{k+1} \) is the \( k+1 \)-dimensional cell.
Definition 11. [13] In such a space we shall use the following notation:

\[ \delta(\varepsilon) = \sup \delta > 0 \text{ for all } x \in X, \text{ all integers } k, 0 \leq k \leq n, \text{ and for all continuous } \psi : S^i \to N(x, \delta) \text{ there exists an extension of } \psi, \psi' : E^{k+1} \to N(x, \varepsilon) \]

Set:

\[ \delta_0(\varepsilon) = \varepsilon \]

\[ \delta_1(\varepsilon) = \delta\left(\frac{\varepsilon}{2}\right) \]

\[ \delta_2(\varepsilon) = \delta\left(\frac{\delta_1(\varepsilon)}{5}\right) \]

\[ \vdots \]

\[ \delta_{p+1}(\varepsilon) = \delta\left(\frac{\delta_p(\varepsilon)}{5}\right) \]

and the following properties hold:

(A) For any \( \varepsilon > 0, \delta_p(\varepsilon) > 0 \)

(B) \( \delta_{p+1}(\varepsilon) < \delta_p(\varepsilon) \)

(C) \( \varepsilon < \varepsilon' \) implies \( \delta_p(\varepsilon) < \delta_p(\varepsilon') \)

Definition 12. [13] If \( f : X \to Y \) is a function from a topological space \( X \) to a topological space \( Y \), then \( f \) is almost continuous provided any open set of \( X \times Y \) which
contains \( \Gamma_f \) also contains the graph of a continuous function with domain \( X \) and range a subset of \( Y \).

The following theorem follows from [13; Theorem 3].

**Theorem 13.** If \( f : X \to X \) is an almost continuous function and \( X \) is a Hausdorff space with the fixed point property, then \( f \) has a fixed point.

**Definition 13.** If \( r : P \to Y \), \( f : Y \to X \) and \( r \) is a continuous retraction, then \( fr \) is retraction almost continuous if every open set \( W \) which contains \( \Gamma_{fr} | Y \) also contains the graph of a continuous function defined on \( Y \).

**Remark 1.** \( fr \) is retraction almost continuous if and only if \( f \) is almost continuous.

The proof of the next theorem, the main theorem, uses the technique that Stallings used to prove Theorem 6.

**Theorem 14.** Suppose \( P \) is an \( n \)-dimensional polyhedron, \( Y \) a continuous rim connected retract of \( P \) and \( X \) a uniformly locally \((n-1)\)-connected metric space; let \( r \) be the retraction from \( P \) to \( Y \) and \( f \) a peripherally continuous function from \( Y \) to \( X \), then \( fr \) is retraction almost continuous.
Proof: Let \( \varepsilon \) be the number associated with Lemma 5 with respect to \( Y \). Let \( W \) be an open set containing \( \text{fr} \, | \, Y \). For each \( p \in Y \), let \( e_1(p) \) and \( e_2(p) \) be two positive numbers such that

1. \( e_1(p) < \varepsilon \)
2. \( N(p, e_1(p)) \times N(\text{fr}(p), e_2(p)) \subseteq W \)

By Theorem 12 and Lemma 4 there is a rim connected neighborhood \( V_{r(p)} \) about \( r(p) = p \) such that the diameter of \( V_{r(p)} < e_1(p) \) and

\[
\tilde{f}(\text{bd}(V_{r(p)})) \subseteq N(\text{fr}(p) \, : \, \frac{5}{\varepsilon} N(e_2(p)) \}
\]

Set \( U_p \) equal to the component of \( r^{-1}(V_{r(p)}) \) that contains \( p \). The \( U_p \)'s cover \( Y \), as a subset of \( P \), and hence there is a finite number, say \( U_{p_1}, \ldots, U_{p_m} \), which cover \( Y \) and is minimal with respect to covering \( Y \).

Suppose \( U_{p_i} \cap U_{p_j} \neq \emptyset \). Then

\[
\text{bd}(V_{r(p_i)}) \cap \text{bd}(V_{r(p_j)}) \neq \emptyset \quad \text{since} \quad U_{p_i} \cap U_{p_j} \neq \emptyset,
\]

\[
r(U_{p_i}) \cap r(U_{p_j}) = V_{r(p_i)} \cap V_{r(p_j)} \neq \emptyset. \quad V_{r(p_i)} \nsubseteq V_{r(p_j)}\]

if it was then \( U_{p_1}, \ldots, U_{p_m} \) would not be a minimal cover of \( Y \). Likewise \( V_{r(p_j)} \nsubseteq V_{r(p_i)} \). Therefore, by Lemma 5
bd(V_r(p_i)) \cap bd(V_r(p_j)) \neq \emptyset.

Set \( U = U \cup \bigcup_{p_i} \). There is a compact set \( Z \) such that \( Y \subset Z^0 \subset \overline{Z} \subset U \). Let \( n > 0 \) be less than the Lebesgue number of the covering \( U_{p_1}, \ldots, U_{p_m} \) with respect to \( Z \), 2. less than \( d(Y, \overline{Z}) \), and 3. less than \( d(p_i, \overline{P - U_{p_i}}) \), \( i = 1, \ldots, m \). Let \( P \) be triangulated with mesh \( < n/2 \) and so that each \( p_i \) is a vertex of the triangulation. Let \( P_x \) be the subpolyhedron of \( P \) consisting of all those simplexes which meet \( Y \). Let \( a_1, \ldots, a_r \) be the vertices of \( P_x \). A continuous function "g" will be constructed on \( P_x \).

To each vertex \( a_j \) assign one of the points \( p_i \), called \( p(a_j) \), such that the closed star of \( a_j \) (in this triangulation) is contained in \( U_{p(a_j)} \). This is permissible by 1 and 2. We do this in such a way that \( p(p_i) = p_i \). This is permissible by 3.

On the 0-skeleton of \( P_x \), \( P_x^0 = \{a_0, \ldots, a_r\} \) we define:

\[ g(a_j) = fr(p(a_j)). \]

\( Z^0 = \text{interior of } Z \). \( Z' = \text{complement of } Z \).
Suppose that a continuous function has thus been defined on the $k$-skeleton $P^k_*$ satisfying

**Condition-$k$.** If $\Delta$ is a $(k+1)$-simplex whose boundary is $\Delta' \subset P^k_*$, then for any vertex $a_i$ of $\Delta$ such that $e_2(p(a_i)) = \max\{e_2(p(a_j)) \mid a_j$ is a vertex of $\Delta\}$ it is true that

$$g(\Delta) \subset N(fr(p(a_i)); \delta_{n-k}(e_2(p(a_i)))).$$

Then $g$ will be extended to the $(k+1)$-skeleton $P^{k+1}_*$ so that Condition-$(k+1)$ is satisfied.

First check that Condition-0 is satisfied by the definition of $g$ for $k = 0$. It must be shown that if $\Delta = (a_i, a_j)$ is a 1-simplex and $e_2(p(a_j)) \geq e_2(p(a_i))$ then $g(\Delta') \subset N(fr(p(a_i)); \delta_n(e_2(p(a_j)))).$ This is true if $p(a_i) = p(a_j)$; Suppose $p(a_i) \neq p(a_j)$ and $e_2(p(a_j)) > e_2(p(a_i))$. We have $\Delta \subset U_{p(a_i)} \cap U_{p(a_j)}$. Thus there is a $c$ such that

$$c \in bd V_{r}(p(a_i)) \cap bd V_{r}(p(a_j)),$$

from which we get the following inequalities:
\[ d(g(a^*_1), f(c)) = d(f \circ r(p(a^*_1)), f(c)) < \frac{1}{5} \delta_n(e_2(p(a^*_1))) \]

\[ \leq \frac{1}{5} \delta_n(e_2(p(a^*_j))) > d(r(p(a^*_j)), f(c)) = d(g(a^*_j), f(c)) \]

Hence \[ d(g(a^*_1), g(a^*_j)) < \frac{2}{5} \delta_n(e_2(p(a^*_j))) < \delta_n(e_2(p(a^*_j))). \]

This is the proof that Condition-0 is satisfied.

Assume \( g \) defined on \( P^k_x \) satisfying the given Condition-k. It follows from the definition of \( \delta_{n-k} \) that if \( \Delta \) is a \( k+1 \) simplex and \( a^*_i \) is a vertex of \( \Delta \) for which \( e_2(p(a^*_i)) \) is maximal among the vertices of \( \Delta \), then there exists an extension of \( g|\Delta^* \) to \( \Delta \) such that \( g(\Delta) \subseteq N(g(a^*_i); \frac{1}{5} \delta_{n-k-1}(e_2(p(a^*_i)))) \). Let \( a^*_j \) be another vertex of \( \Delta \) for which \( e_2(p(a^*_j)) \) is maximal. If \( p(a^*_i) = p(a^*_j) \) we obviously have

\[ g(\Delta) \subseteq N(g(a^*_j); \frac{1}{5} \delta_{n-k-1}(\frac{e_2(p(a^*_i))}{5})) \]

If \( p(a^*_i) \neq p(a^*_j) \), then \( \Delta \subseteq U_{p(a^*_i)} \cap U_{p(a^*_j)} \). Hence there is a point \( c \in \text{bd } V_r(p(a^*_i)) \cap \text{bd } V_r(p(a^*_j)) \), and
\[ d(g(a_i), f(c)) = d(f \circ r(p(a_i)), f(c)) < \frac{1}{5} \delta_n(e_2(p(a_i))) \]

\[ < \frac{1}{5} \delta_{n-k-1}(e_2(p(a_i))) = \frac{1}{5} \delta_{n-k-1}(e_2(p(a_j))) < \frac{1}{5} \delta_n(e_2(p(a_j))) > \]

\[ d(fr(p(a_j)), f(c)) = d(g(a_j), f(c)). \]

Hence \( d(g(a_i), g(a_j)) < \frac{2}{5} \delta_{n-k-1}(e_2(p(a_i))) = \frac{2}{5} \delta_{n-k-1}(e_2(p(a_j))) \)
and since \( g(\Delta) \subset N(g(a_i); \frac{1}{5} \delta_{n-k-1}(e_2(p(a_i)))) \), it follows that \( g(\Delta) \subset N(g(a_j); \frac{3}{5} \delta_{n-k-1}(e_2(p(a_j)))) \). Suppose then that \( g \) is extended to each \((k+1)\)-simplex \( \Delta \) in this manner; then for each \((k+1)\)-simplex \( \Delta \) and vertex \( a_i \) of \( \Delta \) for which \( e_2(p(a_i)) \) is maximal among the vertices of \( \Delta \),

\[ g(\Delta) \subset N(g(a_i); \frac{3}{5} \delta_{n-k-1}(e_2(p(a_i)))) \]

Let \( \Delta^* \) be a \((k+2)\)-simplex and \( a_i \) a vertex for which \( e_2(p(a_i)) \) is maximal among the vertices of \( \Delta^* \). Then for each of the \((k+1)\)-simplexes in the boundary of \( \Delta^*, \Delta_1, \ldots, \Delta_{k+2} \), of which \( a_i \) is a vertex

\[ g(\Delta_j) \subset N(g(a_i); \frac{3}{5} \delta_{n-k-1}(e_2(p(a_i)))) \]
Let $a_s$ be a vertex of the remaining $(k+1)$-simplex $\Delta_0$, for which $e_2(p(a_s))$ is maximal among the vertices of $\Delta_0$; if $p(a_i) = p(a_s)$, it follows that

$$g(\Delta_0) \subset N(g(a_i); \frac{3}{5} \delta_{n-k-1}(e_2(p(a_i)))).$$

But if $p(a_i) \neq p(a_s)$, then $\Delta^* \subset Up(a_i) \cap Up(a_s)$; hence there is a $c \in bd Vr(p(a_i)) \cap bd Vr(p(a_s))$. Then

$$d(g(a_i), f(c)) \frac{1}{5} \delta_{n}(e_2(p(a_i))) < \frac{1}{5} \delta_{n-k-1}(e_2(p(a_i))) \geq \frac{1}{5} \delta_{n-k-1}(e_2(p(a_s))) > \frac{1}{5} \delta_{n}(e_2(p(a_s))) > d(g(a_s), f(c))$$

hence $g(\Delta_0) \subset N(g(a_i); \delta_{n-k-1}(e_2(p(a_i))))$. So for any $(k+2)$-simplex $\Delta^*$

$$g(\Delta^*) \subset N(g(a_i); \delta_{n-k-1}(e_2(p(a_i))))$$

for any vertex for which $e_2(p(a_i))$ is maximal. Thus Condition-$(k+1)$ is satisfied.

This procedure can be carried right up to and including the $n$-skeleton of $P_*$, $P_*^n = P_*$. Set

$$g^* = g|Y \circ r$$

Then $\Gamma g^* \subset \mathcal{W}$ and the theorem is proved.
Theorem 15. If $P$ is a finite polyhedron with the fixed point property and $X$ is a rim connected retract of $P$, then $X$ has the connectivity fixed point property.

Proof: This result follows from Remark 1 and Theorem's 8, 12, 13 and 14.

Example 10. Let $D$ be a dendrite and $I = [0,1]$. By $[3; 1.4]$, $D \times I$ is rim connected. Since $D \times I$ is a retract of $I^3$, $D \times I$ has the connectivity fixed point property. Since there exist dendrites $D$ such that $D \times I$ is not a finite polyhedron, Theorem 15 is indeed a generalization of Theorem 6.
IV. RESULTS CONCERNING $f \times _Z$

It is well known that every dendrite has the fixed point property, but the answer to the following question is not known: Does every dendrite have the connectivity fixed point property? Whyburn in a paper as yet not published, answers the above question in the affirmative if the dendrite has a countable number of end points.

One possible approach to the problem which is motivated by the results in Chapter III, might be:

1. Let $f : D \rightarrow D$ be a connectivity function defined on a dendrite $D$.
2. $f \times i : D \times I \rightarrow D \times I$ is a connectivity function.
3. By Theorem 15 and Example 10 there is an $(x,t)$ such that $f \times i((x,t)) = (x,t)$.
4. $f \times i((x,t)) = (f(x),t) = (x,t)$ implies that $f(x) = x$.

This attempt falls apart at step 2; that is to say there exists a connectivity function $f : D \rightarrow D$, where $D$ is a dendrite, such that $f \times i$ is not a connectivity function:

Example 11. Let $f : [0,1] \rightarrow [-1,1]$ be defined by

$$f(x) = \begin{cases} \sin \frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$
Then \( f \) has the desired properties.

We may ask under what conditions is \( f \times i \) a connectivity function. The next theorem shows that either \( f \times i \) does not have the desired connectivity properties or it, and consequently also \( f \), is continuous.

**Definition 14.** A function \( f: X \to Y \) from a topological space \( X \) to a topological space \( Y \) is a connected function if for each connected set \( C \subseteq X \), \( f(C) \) is a connected subset of \( Y \).

It follows from the definition of a connectivity function that every connectivity function is a connected function.

**Theorem 16.** Let \( f: X \to Y \) be a function from a connected space \( X \) to a \( T_1 \) space \( Y \). Then \( f \) is continuous if and only if \( f \times i \) is a connected function.

Proof: Suppose \( f \) is continuous. Then \( f \times i \) is continuous and hence a connected function.

Suppose \( f \) is not continuous. Then there is a net \( \{x_\alpha\} \) converging to \( x \in X \), but the net \( \{f(x_\alpha)\} \) does not converge to \( f(x) \). Since \( \{f(x_\alpha)\} \) does not converge to \( f(x) \), there exists an open set \( N, f(x) \notin N \), and \( \{f(x_\alpha)\} \) is not eventually in \( N \). Let \( E = \{x_\alpha : f(x_\alpha) \notin N\} \). Then \( E \) is a net converging to \( x \).
\[ C = \{(x_{\alpha}, t) : t \in I, x_{\alpha} \in E\} \cup \{(z, 1) : z \in X\} \cup \{(x, 0)\} \]

Then \( C \) is a connected set but \( f(C) \) is not connected since \( f \times i((x, 0)) = (f(x), 0) \) is an isolated point.

In contrast to Theorem 16 the next theorem shows, among other things, that if \( f \) is almost continuous then \( f \times i \) is almost continuous.

**Definition 15.** Let \( X, Y \) and \( Z \) be topological spaces and \( f : X \to Y \) a function from \( X \) to \( Y \). Then 
\( f \times i_Z : X \times Z \to Y \times Z \) is the function defined by

\[ f \times i_Z(x, z) = (f(x), z). \]

**Theorem 17.** Let \( f : X \to Y \) be almost continuous, then \( f \times i_Z \) is almost continuous.

**Proof:** Let \( W \) be an open set of \( X \times Z \times Y \times Z \) such that 
\[ \Gamma(f \times i_Z) = \{(x, z, f(x), z) : x \in X, z \in Z\} \subset W \]

For each \( (x, z, f(x), z) \in \Gamma(f \times i_Z) \) there exists 
\[ U_x \times V_z \times R_f(x) \times V_z, \text{ where } U_x, V_z \text{ and } R_f(x) \text{ are open such that} \]

$(x, z, f(x), z) \in U_x \times V_z \times R_f(x) \times V_z \subset W.$

Set $S = \bigcup_{x, z} U_x \times V_z \times R_f(x) \times V_z$ and define $P : S \to U_x \times R_f(x)$ by $P((x, z, y, z')) = (x, y)$. $P(S)$ is an open set and $\Gamma f \subset P(S) \subset X \times Y$. Since $f$ is almost continuous there exists $g : X \to Y$ such that $g$ is continuous and $\Gamma g \subset P(S)$. Then $g \times i_z$ is continuous and $\Gamma(g \times i_z) \subset S$. This completes the proof.

This theorem yields a wide range of almost continuous functions which are not connectivity functions.
V. A FIXED POINT THEOREM FOR THE GRAPH
OF A CONNECTIVITY FUNCTION

All previous connectivity fixed point results have been concerned with spaces that are compact and locally connected. The main result in this section is a connectivity fixed point theorem for spaces which are not necessarily compact or locally connected. The proof of the main theorem differs from the proofs of other connectivity fixed point theorems in that it does not rely upon the fixed point property of the respective spaces.

Definition 16. [8] If $\sigma$ and $\tau$ are collections of sets then $f^{-1}(\tau) = \{f^{-1}(V) : V \in \tau\}$ and $\sigma \times \tau = \{U \times V : U \in \sigma, V \in \tau\}$. If $\sigma$ is a subbase for a topology on $X$, then the resulting space will be denoted by $(X, \sigma)$ or by $X$ if the topology is apparent.

Definition 17. [8] The connectivity structure of $(X, \sigma)$ denoted by $C(\sigma)$ is the class of all connected subsets of $(X, \sigma)$.

The next theorem, which is due to Hildebrand and Sanderson, is a characterization of connectivity functions.

Theorem 18. [8] A function $f : (X, \sigma) \to (Y, \sigma)$ is a connectivity function if and only if $C(\sigma) = C(\sigma')$ where $\sigma' = \sigma \cup f^{-1}(\tau)$. 
Theorem 19. Let \((X,\sigma)\) be a space with the connectivity fixed point property. If \(\sigma'\) is a subbase for a topology on \(X\) such that \(\sigma \subseteq \sigma'\) and \(C(\sigma') = C(\sigma)\). Then \((X,\sigma')\) has the connectivity fixed point property.

Proof: Let \(f : (X,\sigma') \to (X,\sigma')\) be a connectivity function. The theorem will be established if we can show \(f : (X,\sigma) \to (X,\sigma)\) is a connectivity function. Let \(C\) be a connected subset of \((X,\sigma)\). Suppose there exist open sets \(U_1\) and \(U_2\) of \((X \times X, \sigma \times \sigma)\) such that

\[
U_1' = (\Gamma f \cap C) \cap U_1, \quad U_2' = (\Gamma f \cap C) \cap U_2
\]

and \(U_1' \cap U_2' = \emptyset\). Since \(\sigma \subseteq \sigma'\) \(U_1\) and \(U_2\) are open in \((X \times X, \sigma' \times \sigma')\). Thus \(U_1'\) and \(U_2'\) separate \(\Gamma f \cap C\).

But by hypothesis \(C \in C(\sigma)\): this together with \(C(\sigma) = C(\sigma')\) and \(f : (X,\sigma') \to (X,\sigma')\) a connectivity function lead to a contradiction. Therefore \(\Gamma f \cap C\) is connected and \(f : (X,\sigma) \to (X,\sigma)\) is a connectivity function. Thus there is an \(x\) such that \(f(x) = x\) and this completes the proof.

\(\Gamma f\) is the graph of \(f : (x,\sigma) \to (x,\sigma)\).
Theorem 20. Let \( f : (X, \sigma) \to (X, \tau) \) be a connectivity function and \( \sigma' = \sigma \cup f^{-1}(\tau) \). Then \( (X, \sigma') \) is homeomorphic to \( \Gamma_f \).

Proof: We may assume that \( f(X) = Y \). Let \( F : (X, \sigma') \to \Gamma_f \) be defined in the natural way: \( F(x) = (x, f(x)) \). Then \( F \) is one to one and onto; so we set \( F^{-1} = G : \Gamma_f \to (X, \sigma') \) defined by \( G((x, f(x))) = x \).

Let \( p_i : X \times X \to X \) be the projection to the \( i \)th coordinate. Then \( F \) is continuous since \( p_1 \circ F \) and \( p_2 \circ F \) are continuous.

We show that \( G \) is continuous. Let \( S \) be a subbase element of \( \sigma' = \sigma \cup f^{-1}(\tau) \). If \( S \in \sigma \) then
\[
G^{-1}(S) = (S \times Y) \cap \Gamma_f \text{ which is open in } \Gamma_f.
\]
If \( S \in f^{-1}(\tau) \) there exists \( U \subseteq Y \), \( U \) open in \( Y \) and \( f^{-1}(U) = S \).
\[
G^{-1}(S) = (X \times U) \cap \Gamma_f \text{ which is open. This completes the proof.}
\]

At this point a question arises: Given \( (X, \sigma) \) and a subbase for a topology on \( X \), call it \( \sigma' \), such that \( \sigma \subseteq \sigma' \) and \( C(\sigma) = C(\sigma') \), is there a space \( (Y, \tau) \) and a connectivity function \( f : (X, \sigma) \to (Y, \tau) \) such that \( \sigma' = \sigma \cup f^{-1}(\tau) \)? The answer is yes. Set \( Y = X \), \( \tau = \sigma' \) and \( f = \text{identity on } X \).

As a consequence of Theorems 19 and 20 we have the next theorem. Following the statement of the theorem a
different proof will be presented.

**Theorem 21.** Let $X$ be a space with the connectivity fixed point property, $f: X \to Y$ a connectivity function, then $\Gamma f$ has the connectivity fixed point property.

Alternate Proof: Let $g: \Gamma f \to \Gamma f$ be a connectivity function. Define $F: X \to \Gamma f$ by $F(x) = (x, f(x))$.

$g \circ F$ is a connectivity function. Let $C$ be a connected subset of $X$. We must show that

$$D = \{(x, y, f(y)) : x \in C \text{ and } g((x, f(x))) = (y, f(y))\}$$

is a connected subset of $X \times \Gamma f$. Since $f$ is a connectivity function $\{(x, f(x)) : x \in C\}$ is connected and since $g$ is a connectivity function,

$$E = \{(x, f(x), y, f(y)) : x \in C \text{ and } g((x, f(x))) = (y, f(y))\}$$

is a connected subset of $\Gamma f \times \Gamma f$. Define $P: E \to D$ by $P((x, x', y, y')) = (x, y, y')$. Since $P(E) = D$ and $P$ is continuous we see that $g \circ F$ is a connectivity function.

Then, by [8; 2.8], $p_1 \circ g \circ F: X \to X$, $p_1$ is the projection map to the first coordinate, is a connectivity function. Hence there is a point $x$ such that $p_1 \circ g \circ F(x) = x$
which implies that \( g((x,f(x))) = (x,f(x)) \). Thus \( g \) has a fixed point.

**Example 12.** The subset of \((n+1)\)-Euclidean space defined by

\[
\{(x,\sin \frac{1}{||x||}): 0 < ||x|| \leq 1\} \cup \{(0,0,\ldots,0)\}
\]

has the connectivity fixed point property.

The next theorem is due to Cornette.

**Theorem 22.** [4] If \( X \) is connected, separable and metric there is a connectivity function \( f \) with domain \( I \), range \( X \) and \( \overline{f} = I \times X \).

Combining Theorems 21 and 22 we get:

**Theorem 23.** If \( X \) is connected separable and metric then \( I \times X \) contains a dense subset with the connectivity fixed point property.

**Example 13.** Let \( X \) be a circle. Then \( I \times X \) is a cylinder and it does not have the fixed point property even though it contains a dense subset with the connectivity fixed point property.

At this point a reasonable question would be: If \( X \) has the fixed point property, does \( X \) have the connectivity
fixed point property? S. K. Hildebrand and D. E. Sanderson [8; 3.4] give a negative answer to this question.
VI. GENERALIZED FIXED POINTS

In this chapter we will be concerned with the following questions:

1. If \( f : I^n \rightarrow I^n \) is a connected function, then must \( f \) have a fixed point?

2. If \( f : I^n \rightarrow I^n \) has the property that the graph over connected and locally connected sets is connected, then must \( f \) have a fixed point?

3. If \( f : I^n \rightarrow I^n \) has the property that the graph over continua is connected, then must \( f \) have a fixed point?

4. If \( f : X \rightarrow Y \) is continuous, \( g : Y \rightarrow X \) is a connectivity function and \( X \) has the connectivity fixed point property, then must \( gf \) have a fixed point?

L. Vietoris [14; p.202], [11; p. 82] has constructed a connected function \( f : I \rightarrow I \) with no fixed points. The following is an example of a connected function \( f : I^n \rightarrow I^n \) with no fixed point.

Example 14. [7; Example 3] In \( n \)-dimensional Euclidean space, let \( B \) denote the unit ball, \( F \) the boundary of \( B \) and \( S_r \) the sphere centered at the origin and radius \( r \). For \( 0 \leq r \leq 1 \) the points of \( S_r \) will be denoted by \( (r, \theta) \)
with the understanding that \((r_1, \theta)\) and \((r_2, \theta)\) lie on a line passing through the origin. Let
\[
\{(1, \theta_i) : i \text{ a natural number}\}
\]
be a countable dense subset of \(F\). Let \(R\) be a half open ray with the following properties:

(i) \(R \subset B \setminus F\),

(ii) for \(i \geq 2\), \((1 - \frac{1}{i}, \theta_i) \in R\),

(iii) the end point of \(R\) is \((\frac{1}{2}, \theta_2)\),

(iv) for \(\frac{1}{2} < r < 1\), \(R \cap S_r\) consists of one and only one point.

We define a function \(f : B \cup F\) as follows
\[
f((0,0)) = (\frac{1}{2}, \theta_2); \quad \text{for } (r, \theta) \text{ where } 0 < r < 1, \quad f((r, \theta)) = R \cap S_{\frac{r+1}{2}}; \quad f|F\text{ is the antipodal map.}
\]

Clearly \(f\) has no fixed point and it's easy to see that \(f\) is a connected function. Thus for each \(n\), \(n\) a natural number, there is a connected function \(f : I^n \to I^n\) with no fixed point and the answer to question 1 is no.

If \(f : I \to I\) has the property that the graph over connected and locally connected sets is connected then \(f\) is a connectivity function and hence \(f\) has a fixed point.

For \(n > 1\) Example 14 has the property that the graph over connected and locally connected sets is connected [7; Example 3]. Consequently the answer to question 2 is yes.
for \( n = 1 \) and no for \( n > 1 \).

**Definition 18.** A function \( f : X \rightarrow Y \) is a weak-connectivity function if the graph over continua is connected.

If \( f : I \rightarrow I \) is a weak-connectivity function then \( f \) is a connectivity function and hence \( f \) has a fixed point.

The next example is an analytic description of Example 14 for \( n = 2 \). Using this we will show that \( f \) is not a weak-connectivity function.

**Figure 3.** A retraction of the disc

**Example 15.** Let \( B \) be those points \( (r, \theta) \), where \( r \) and \( \theta \) are polar coordinates, in the plane with \( 0 \leq r \leq 1 \).

Let \( R \) be a half-open ray:

\[
R = \left\{ (r, \theta) : \frac{1}{2} \leq r < 1 \text{ and } \theta = \frac{1}{1-r} - 2 \right\}.
\]

Let \( F = \{(r, \theta) : r = 1\} \). Then we define the function \( f : B \rightarrow R \cup F \subset B \), see Figure 3, as follows:
We will show that \( f \) is not a weak-connectivity function by constructing a continuum \( C \) such that \( \Gamma(f \mid C) \) is not connected. Let \( S \subset B \) be the spiral defined by:

\[
S = \{(r, \frac{2r}{1-r}) : 0 < r < 1\}
\]

and let \( C = S \cup F \).

Surely \( C \) is a continuum and we show that \( \Gamma(f \mid S) \cup \Gamma(f \mid F) \) is a separation of \( \Gamma(f \mid C) \). A typical element \( p \) of \( \Gamma(f \mid S) \) is

\[
p = (r, \theta, f(r, \theta)) = (r, \frac{2r}{r-1}, \frac{r+1}{2}, \frac{2r}{r-1}).
\]

A typical element \( q \in \Gamma(f \mid F) \) is

\[
q = (1, \phi, f(1, \phi)) = (1, \phi, 1, \phi + \pi).
\]

If \( \Gamma(f \mid S) \) and \( \Gamma(f \mid F) \) did not separate \( \Gamma(f \mid C) \) then one could find elements \( p \) and \( q \) which were arbitrarily
close to each other. But then there would exist natural numbers \( k \) and \( \ell \) such that

(i) \( \phi + 2\pi k \) is arbitrarily close to \( \frac{2r}{r-1} \),

and

(ii) \( \phi + \pi + 2\pi \ell \) is arbitrarily close to \( \frac{2r}{r-1} \).

But it is not possible to satisfy both (i) and (ii) and this completes the proof.

One can show by an analogous development, for \( n > 2 \), that the function \( f \), as defined in Example 14, is not a weak connectivity function.

Consequently the answer to question 3 for \( n \geq 2 \) remains open. As a consequence of [13; Proposition 2] every almost continuous function is a weak-connectivity function and if the converse of this statement was true then the converse plus Theorem 13 would give an affirmative answer to question 3 for \( n \geq 2 \).

Example 9 gives a retraction \( r : I^2 \to I \times 0 \) and a connectivity function \( f : I \times 0 \to I \times 0 \) such that \( fr \) is not a connectivity function, but \( fr \) does have a fixed point. Another example is if \( P \) is a polyhedron, \( r : P \to R \) is a retraction to a rim-connected retract \( R \) and \( f : R \to R \) is a connectivity function, then, by Theorem 15, \( fr \) has a fixed point. It is not known whether or not \( fr \) is a connectivity function. Given a space \( X \) with the
fixed point property, a continuous function \( f : X \rightarrow Y \) and a connectivity function \( g : Y \rightarrow X \), we investigate whether or not \( gf \) has a fixed point. We show for the case \( X = Y = I \) that \( gf \) is a connectivity function and hence \( gf \) has a fixed point.

**Definition 19.** Let \( f : X \rightarrow Y \). Then \( N \) is said to be a continuum open subset of \( X \times Y \) if \( X \times Y - N \) is a continuum and \( f \) is said to be a continuum almost continuous function (cac) if every continuum open set which contains the graph of \( f \) also contains the graph of a continuous function.

**Theorem 24.** Given \( Y \) and \( Z \) are Hausdorff spaces \( f : X \rightarrow Y \) is continuous and \( g : Y \rightarrow Z \) is cac, then \( gf \) is cac.

**Proof:** Let \( N \) be a continuum open set in \( X \times Z \) containing \( \Gamma(gf) \). Set \( C = (X \times Z) - N \), which is a continuum. Let \( f_* : X \times Z \rightarrow Y \times Z \) be defined as follows:

\[
f_* = f \times i.
\]

Then \( f_*(\Gamma(gf)) \subseteq \Gamma g \) and \( f_*^{-1}(\Gamma g) = \Gamma(gf) \). \( f_*(C) \) is a continuum in \( Y \times Z \) such that \( f_*(C) \cap \Gamma g = \emptyset \). Set \( M = (Y \times Z) - f_*(C) \) which is continuum open and contains
Therefore there is a continuous function $G: Y \to Z$ such that $\Gamma G \subset M$. Then $Gf: X \to Z$ is continuous, $f_*(\Gamma(Gf)) \subset \Gamma G \subset M$ and $f_*^{-1}(\Gamma G) = \Gamma(Gf)$; but $f_*^{-1}(\Gamma G) \subset N$. Thus $\Gamma(Gf) \subset N$ and this completes the proof.

**Theorem 25.** Let $f: I \to I$ be a connectivity function, then $f$ is cac.

**Proof:** $I \times I$ will be considered as a subset of an open disk $D$. See Figure 4. Suppose $N$ is a continuum open set in $I \times I$ that contains $\Gamma f$. Set $C = I \times I - N$, which is a continuum.

$C$ can not meet both $I \times 0$ and $I \times 1$. For if it did, since $\Gamma f$ is connected, a component of $I \times I - C$ would contain $\Gamma f$. This component would contain an arc $a$ with end points $(0,f(0))$ and $1,f(1))$. There is an arc $b \subset (D - I \times I) \cup \{(0,f(0)),(1,f(1))\}$ with end points...

![Figure 4. $I \times I \subset D$](image1)

![Figure 5. Arcs a and b](image2)
$(0, f(0))$ and $(1, f(1))$. Set $s = a \cup b$ which is a simple closed curve. See Figure 5. Then by the Jordan Curve Theorem $s$ separates $I \times 0$ and $I \times 1$ and consequently $s$ separates $C$ which is a contradiction. Thus $I \times 0$ or $I \times 1$ is the graph of a continuous function contained in $N$.

**Theorem 26.** Let $f : I \to I$ be a cac function, then $f$ is a connectivity function.

Proof: Suppose $f$ is not a connectivity function. Then there exists open sets $\alpha$ and $\beta$ that separate $\Gamma f$. Let $C'' = I^2 - (\alpha \cup \beta)$. $C''$ separates $\Gamma f$ and since $I^2$ is unicoherent there is a continuum $C'' \subset C''$ that separates $\Gamma f$. Let

$$A_1 = \{ (0,t) : (0,t) \in I^2 \text{ and } (0,t) \text{ is not in the component of } (I^2 - C'') \cap (0 \times I) \text{ that contains } (0, f(0)) \}.$$

$$A_2 = \{ (1,t) : (1,t) \in I^2 \text{ and } (1,t) \text{ is not in the component of } (I^2 - C'') \cap (1 \times I) \text{ that contains } (1, f(1)) \}.$$

$$B_1 = (0 \times I) - A_1 \text{ and } B_2 = (1 \times I) - A_2.$$

Then $B_1$ and $B_2$ are connected subsets of $I \times 0$ and $I \times 1$, respectively. Let $C'$ be a continuum defined by
C' = A_1 \cup A_2 \cup C". Since f is cac and C' \cap \Gamma f = \emptyset, there exists a continuous function g : I \rightarrow I such that \Gamma g \subset U \subset I^2 - C' where U is a component of I^2 - C'. We note that B_1 \subset U. There exists a point (p, f(p)) \notin U. Suppose the component of I^2 - C' that contains (p, f(p)) is V. Set B_2 equal to the component of V \cap (p \times I) that contains (p, f(p)). Let A_2 = (p \times I) - B_2 and C = C' \cup A_2. Then C is a continuum disjoint from \Gamma f; so there is a continuous function h : I \rightarrow I such that \Gamma h \subset I^2 - C. Since B_1 \subset U, \Gamma h \subset U which is not possible. Thus we conclude that \Gamma f is connected and f is a connectivity function.

Theorems 24, 25 and 26 imply

Theorem 27. Let f : I \rightarrow I be continuous and g : I \rightarrow I be a connectivity function, then gf is a connectivity function and hence gf has a fixed point.
VII. BIBLIOGRAPHY


VIII. ACKNOWLEDGEMENT

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