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Multivariate and generalized polykays in statistical structures

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John James Kinney

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The importance of the formulae lies in their generality; they are applicable to all distributions for which the expressions have a meaning. In the present state of our knowledge any information, however incomplete, as to sampling distributions is likely to be of frequent use, irrespective of the fact that moment functions only provide statistical estimates of high efficiency for a special type of distribution.

R. A. Fisher
I. CHAPTER ONE: PROLEGOMENA

A. Early Historical Development

When the determination of the distribution of a general sample moment appeared intractable, many statisticians confined their interest in the problem to the determination of the moments of sample moments, most of them considering sampling from an infinite population, but some, notably Pearson (1899), selection from a finite population. Among very early writers the works of Pearson (1902, 1906), Student (1908), Isserlis (1918), Tschouprow (1923) and Neyman (1925) were important contributions and resulted in various methods and a large number of formulas for moments of moments. Student's (1908) method in particular became popular and was used by many later authors, while Tschouprow (1923) worked out a large number of formulas.

Thiele (1903) made what was much later to prove an important contribution -- he introduced new symmetric functions, adopted possibly from some ideas of Laplace, which he called half-invariants. His half-invariants (or semi-invariants) of the sample values were functionally identical however to those of the population which produced them and here an essential point, later to be found by Fisher, was missed. Thiele worked out the half-invariants of the distributions of his half-invariants, but with limited success as the higher orders proved intractable, and he saw no pattern in the results.
St. Georgescu (1932) and Craig (1928) probably carried the work in this direction as far as it is practicable to go. St. Georgescu's work is as much noted for its method as its results since the method shows similarity to that used by Fisher. Both he and Craig found formulas for the semi-invariants of the simultaneous distribution of two moments about the mean thus abandoning the semi-invariants of this distribution and making considerable advances.

But the solution of the original problem was still elusive; despite the fact that he obtained extensive results, Craig (1928) wrote:

It rather seems that the best hopes of effectively further simplifying the problem of sampling for statistical characteristics lie either in the discovery of a new kind of symmetric function of all the observations which may be used to characterize frequency functions and which will be more amenable than either moments or semi-invariants for use in sampling problems, or in, what may very well prove to be much better and more feasible, the abandonment of the method of characterizing frequency functions by symmetric functions of the observations altogether.

Coincidentally, the first of these possibilities was involved in the solution to the problem contained in Fisher's important 1929 paper introducing the $k$ statistics. These, which apply to both univariate and multivariate distributions, have the property that their expectation, over all possible random samples of a fixed size, is the corresponding population cumulant moment function, or semi-invariant of Thiele. The introduction of unbiased estimates of population cumulants,
rather than identical functions of sample values, proved to be
the needed key, for Fisher, using remarkable insight, was
able to provide a scheme for finding the cumulants of the
simultaneous distribution of the $k$ statistics. He in
fact provided a table of univariate formulae up to the 10th
degree and gave some results of the 12th degree, thus, with
the $k$ statistics, going far beyond results attempted with
moments of moments or moments of semi-invariants.

Later Wishart (1933) showed how Fisher's work can be
used to obtain formulas for moments of moments and related
the work of Fisher, St. Georgescu, and Craig. But the problem
of moments of moments has never seriously been returned to,
primarily because of the new direction given the subject by
Fisher, and partially because of the varied new applications
which have been found.

A general understanding of the $k$ statistics (particu-
larly Fisher's contributions to the subject) is necessary
since the work that follows is directly dependent upon it.
The next section is given then to the $k$ statistics; section
C concerns the cumulants of the $k$ statistics, while sections
E and F treat the multivariate case and generalized polykays
respectively.
The semi-invariants of Thiele are defined by the generating relation

\[ 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \ldots = \exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\} \quad (1) \]

provided that all the quantities above exist. The terms "half invariants" and "seminvariants" have often been used for them but Dressel (1940) has pointed out that other functions have equal claim to such description and so the term "cumulants" will be used here for the specific quantities denoted by \( \kappa \)'s in (1). The cumulants are independent in general since any dependency among them would imply a dependency between the moments.

Kendall (1940a) has shown that the cumulants, except \( \kappa_1 \), are independent of the origin of measurement, and also that the \( \kappa \)'s are similarly origin independent.

Specific relations between the moments and cumulants can be found. Kendall and Stuart (1969) show that

\[ \mu_r' = \sum_{m=0}^{r} \sum_{l} \frac{\kappa_{p_1}}{p_1!} \frac{\kappa_{p_2}}{p_2!} \ldots \frac{\kappa_{p_m}}{p_m!} \frac{\pi_l}{\pi_1! \pi_2! \ldots \pi_m!} \]

where \( \Sigma \) extends over all non-negative integral values of \( l \) the \( \pi \)'s such that \( \sum_{i=1}^{m} p_i \pi_i = r \). Kendall and Stuart give the result of applying the formula up to and including \( r=10 \).

The cumulants can also be expressed in terms of the
moments:

\[ \kappa_r = r! \sum_{m=0}^{\infty} \frac{\mu_1^{\pi_1} \mu_2^{\pi_2} \cdots \mu_m^{\pi_m}}{\pi_1! \pi_2! \cdots \pi_m!} \prod_{i=1}^{m} \left( \frac{p_i^{\mu_i}}{p_i!} \right) \left( \frac{-1}{(r-\rho)} \right)^r \]

where \( \Sigma \) is now over all non-negative integral \( \pi_i \)'s such that

\[ \sum_{i=1}^{m} p_i \pi_i = r \quad \text{and} \quad \sum_{i=1}^{m} \pi_i = \rho. \]

Expressions for \( \kappa_r \) up to and including \( r=10 \) can be found in Kendall and Stuart (1969).

If a sample of size \( n \) is available it is convenient to denote sample sums such as

\[ \sum_{x_1 \neq x_2 \cdots x_s} p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \]

where the summation extends over all distinct values of the subscripts, by \( [p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}] \). If the \( x_i \)'s are selected by some independent process from an infinite population and the expectation is taken over all possible random samples of size \( n \) then

\[ E[p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}] = n(n-1) \cdots (n-\rho+1) \mu_1^{\pi_1} \mu_2^{\pi_2} \cdots \mu_s^{\pi_s} \]

where \( \rho = \sum_{i=1}^{s} \pi_i. \)

For example,

\[ [21] = \sum_{i,j=1}^{3} x_i^2 x_j = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2 \]

and
\[ E \sum_{i,j=1}^{3} x_i^2 x_j = 3.2.1 \mu_2^2 \mu_1^4 = 6\mu_2^4 \mu_1^4. \]

\([\pi_1 \pi_2 \ldots \pi_s]\) is sometimes called an augmented monomial symmetric function (Kendall and Stuart, 1969, David, Kendall, and Barton 1966). These functions behave quite simply if the sampling has been done from a finite population, say of size \(N\). In that case

\[ E[\pi_1 \pi_2 \ldots \pi_s] = E \sum_{i=1}^{N} x_1 \ldots x_{\pi_1} x_{\pi_1+1} \ldots \pi_s \alpha \]

where \(\alpha\) is Cornfield's (1944) variable:

\[
\alpha = \begin{cases} 
1 \text{ if } x_{1}, \ldots, x_{\pi_1+\pi_2+\ldots+\pi_s} \text{ are in the sample } \\
0 \text{ otherwise. } 
\end{cases}
\]

Then

\[ E[\pi_1 \pi_2 \ldots \pi_s] = \frac{n(n-1)\ldots(n-\rho+1)}{N(N-1)\ldots(N-\rho+1)} \sum_{i=1}^{N} \alpha \]

So if \(\langle p_1 p_2 p_2 \ldots p_s \rangle\) is defined as \(\frac{[\pi_1 \pi_2 \ldots \pi_s]}{n(\rho)}\), then

\[ E\langle p_1 p_2 p_2 \ldots p_s \rangle = \langle p_1 p_2 p_2 \ldots p_s \rangle'. \]

The prime indicates that the bracket is evaluated over the whole population. Tukey (1950) refers to this as "inheritance on the average". This property was known to Fisher who
indicated it to Anderson (1935).

The use of monomial symmetric functions makes the expression of the $k$'s particularly simple. It is evident that if

$$ k_r = r! \sum_{m=0}^{r} \frac{(-1)^{r-m}(r-1)! \langle p_1 \cdots p_r \rangle}{\pi_1! \cdots \pi_m! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_m!)^{\pi_m}} $$

where the bracket contains $\pi_i p_i$'s, $\sum_{i=1}^{m} p_i \pi_i = r$ and $\sum_{i=1}^{m} \pi_i = r^2$ then $E(k_r) = k_r$.

The assumption of the uniqueness of the $k$ statistics is inherent in all the writing on the subject. A proof more detailed than that in Kendall and Stuart (1969) is now given.

First, it may be noted that a symmetric function, that is a rational, integral, algebraic function of variables $x_i$, $i = 1, 2, \ldots, n$ say, which is invariant under the permutation group of the subscripts of the $x$'s, can be expressed uniquely in terms of the brackets (O'Toole, 1931). It follows that the power sums or $k$ statistics are also sufficient to describe uniquely any symmetric function of the observations.

Now suppose that $E(k_r) = E(l_r) = \kappa_r$, where $k_r$ and $l_r$ are not identical functions and $l_r$ is symmetric. Then $E(k_r - l_r) = 0$. But $k_r - l_r$ is a symmetric function and hence can be expressed in terms of the brackets. Consequently,
the expectation of this function is a function of the moments and thus the equation \( E(k_p - \lambda_p) = 0 \) establishes an impossible relation among the moments (Kendall, 1940a). So the \( k \) statistics are unique.

The first few \( k \) statistics are then

\[
\begin{align*}
k_1 &= \langle 1 \rangle \\
k_2 &= \langle 2 \rangle - \langle 11 \rangle \\
k_3 &= \langle 3 \rangle - 3\langle 21 \rangle + 2\langle 111 \rangle \\
k_4 &= \langle 4 \rangle - 3\langle 22 \rangle - 4\langle 31 \rangle + 12\langle 112 \rangle - 6\langle 1111 \rangle.
\end{align*}
\]

The weight or order or degree of an augmented monomial symmetric function \( [p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}] \) is \( \sum_{i=1}^{s} \pi_i p_i \). A similar definition applies to the bracket function \( \langle p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \rangle \). It may be noted that \( k_r \) is a linear function of brackets, each of weight \( r \) and so \( k_r \) is said to be of weight \( r \). The weight of a product of \( k \) statistics is the sum of the weights of the factors comprising it. David, Kendall and Barton (1966) give tables of the \( k \) statistics in terms of monomial symmetric functions for weight 12 only. This was first given by Abdel-Aty (1954) and can be specialized to lower weights, although not easily. Expressions for the \( k \)'s can also be obtained.

The expression of the \( k \) statistics in terms of power sums is often convenient. Fisher stated his original results in this fashion and later (Cornish and Fisher, 1937) gave a
derivation of the $k$ statistics in these terms. David, Kendall and Barton (1966) give tables of the $k$ statistics in terms of the power sums $\sum_{i=1}^{n} x_i^r$ up to and including weight 11. The largest $k$ statistic known in terms of power sums, $k_{12}$, has been found by Zia-Ud-Din and Ahmad (1960), its expression occupying two and one-half pages of print. The sheer bulk of the formulas probably precludes much further extension in this direction although the $k$ statistics can be expressed in terms of multiple index $k$ statistics, or polykays, without as much difficulty.

Interest in the power sums undoubtedly stems from computational considerations and the fact that they are sufficient to describe any symmetric function of the observations, that is, any symmetric function has a unique expression in terms of power sums (Kendall, 1940a). It follows that the brackets, and hence the $k$ statistics, are also sufficient to describe uniquely any symmetric function of the observations. In the case of finite populations inheritance on the average renders unbiased estimation of symmetric functions essentially trivial. For infinite populations the definition of the $k$ statistics provides the solution for the unbiased estimation of a linear function of population cumulants.
The computation of the \( k \) statistics themselves has not received great attention. Tukey (1950) does discuss it briefly, as does Keeping (1962). Schaeffer and Dwyer (1963) have given the most extensive treatment of this matter but they deal only with desk calculation. It is apparent that if the \( k \) statistics are to be used with greater frequency, further computational methods must be devised and be readily accessible.

Although formulas involving moments about the sample mean are common, very little literature exists concerning polykays of deviates. Dwyer (1964) does treat the subject, however. He lets \( k_p^p (x) \), \( x \) being a vector of sample observations and \( d_p^p (x-k_1) \). Then allowing \( p \) to denote a partition without unit parts, Dwyer obtains formulas for \( d \) in terms of the usual polykays and also states some recursion results for polykays of deviates. Further investigation of this line of inquiry might well prove fruitful.

Kendall (1942) investigated the possibility that a system of seminvariant statistics (rational, integral, algebraic and symmetric functions which are invariant under a change of origin) could be found whose sampling properties were simpler than the \( k \) statistics. That is, can a system of seminvariant statistics \( \lambda \) and seminvariant parameters \( \lambda \) be found so that the sampling \( \lambda \)'s of the \( \lambda \)'s can be expressed more simply in terms of population \( \lambda \)'s than can
the cumulants for the \( k \) statistics? Kendall shows that all such systems have some similar characteristics and concludes that the \( k \) statistics are very probably, but not certainly, the simplest system.

C. The Cumulants of the \( k \) Statistics

Fisher (1929) defined quantities which he called cumulants for the simultaneous distribution of the \( k \) statistics and he was able to give a series of rules for their determination.

Let the expected value of \( k_{p_1} k_{p_2} \ldots k_{p_s} \) be denoted by \( \mu(p_1,p_2 \ldots p_s) \). The cumulants of the simultaneous distribution of \( k_{p_1}, k_{p_2}, \ldots, k_{p_s} \) defined by

\[
\text{Ee}_{i=1}^s k_{p_i} t_i = \sum_t \mu(p_1,p_2 \ldots p_s) \frac{t_{p_1} t_{p_2} \ldots t_{p_s}}{t_{\pi_1}! t_{\pi_2}! \ldots t_{\pi_s}!} \\
= \exp\{\sum (p_1,p_2 \ldots p_s) \frac{t_{p_1} t_{p_2} \ldots t_{p_s}}{t_{\pi_1}! t_{\pi_2}! \ldots t_{\pi_s}!}\} \tag{2}
\]

where \( p_1, p_2, \ldots, p_s \) and \( \pi_1, \pi_2, \ldots, \pi_s \) are integers and
the summations are understood to extend over all terms of
the indicated form.

In order to show the magnitude of Fisher's accomplishment
and also to illustrate the algebraic method, a specific example
will be considered, that of the joint distribution of \( k_2 \) and
\( k_3 \) and in particular the determination of \( \kappa(23) \).

From (2) the generating relation is

\[
1 + \mu(2)t_2 + \mu(3)t_3 + \mu(23)t_2t_3 + \ldots
\]

\[
= \exp\{\kappa(2)t_2 + \kappa(3)t_3 + \kappa(23)t_2t_3 + \ldots\}. \tag{3}
\]

Now \( \kappa(r) = \mu(r) = k_r \) and \( \mu(23) = \kappa(23) + \kappa(2)\kappa(3) \)
from (3). Also

\[
k_2k_3 = \frac{1}{n(3)n(2)} \left\{ n^3s_2s_3^n - n^2s_1^2s_3^2 - 3n^2s_1s_2^2
\right. \\
\left. + 5ns_1^3s_2^2 - 2s_1^5 \right\}
\]

where

\[
s_r = \sum x_i^r.
\]

Using tables of monomial symmetric functions in terms of the
power sums such as Kendall, David and Barton (1966) the
following relations may be written:

\[
E(s_2s_3) = n\mu_5 + n(n-1)\mu_3\mu_2
\]
So

\[ E(s_1^2 s_2) = n \mu_5 + 2n(n-1)\mu_4 \mu_1 + n(n-1)\mu_3 \mu_2^2 + n(n-1)(n-2)\mu_4^2 \mu_1^2 \]

\[ E(s_1^3 s_2) = n \mu_5 + n(n-1)\mu_4 \mu_1^2 + 2n(n-1)\mu_3^2 \mu_1 + n(n-1)(n-2)\mu_4 \mu_1^2 \]

\[ E(s_3) = n \mu_5 + 3n(n-1)\mu_4 \mu_1 + 4n(n-1)\mu_3 \mu_2^2 + n(n-1)(n-2)\mu_4^2 \mu_1^2 \]

\[ E(s_1^5) = n \mu_5 + 5n(n-1)\mu_4 \mu_1^2 + 10n(n-1)\mu_3 \mu_2^3 + n(n-1)(n-2)\mu_4 \mu_1^3 \]

\[ E(k_3k_2) = \mu(32) = \frac{1}{n} \mu_5 + \frac{n^2-5n+10}{n(n-1)} \mu_4 \mu_1^2 - \frac{5}{n} \mu_4 \mu_1^2 \]

\[ - \frac{n^2-15n+20}{n(n-1)} \mu_3 \mu_1^2 - \frac{(n^2-5n+10)}{n(n-1)} \mu_2 \mu_1^2 \]

\[ + \frac{5(n-3)(n-4)}{n(n-1)} \mu_2 \mu_1^3 - \frac{2(n-3)(n-4)}{n(n-1)} \mu_1^5 \]

Now one makes use of the relationships between the moments about the origin and the cumulants such as
\[ u_5 = \kappa_5 + 5\kappa_4 \kappa_1 + 10\kappa_3 \kappa_2 + 10\kappa_3 \kappa_1^2 + 15\kappa_2^2 \kappa_1 \]

\[ + 10\kappa_2 \kappa_1^3 + \kappa_1^5 \]  

in order to express \( \mu(32) \) in terms of the cumulants. It is very useful to note at this point that \( \mu(32) \) is independent of the origin and hence its expression in terms of cumulants cannot involve \( \kappa_1 \). In fact \( \kappa_1 \) can occur only in the formula \( \kappa(1) = \kappa_1 \). Thus the expression of \( \mu(32) \) in terms of cumulants is greatly simplified and one obtains

\[ \mu(32) = \frac{1}{n} \kappa_5 + \frac{n+5}{n-1} \kappa_3 \kappa_2 \]

Now \( \kappa(32) = \mu(32) - \kappa(3) \kappa(2) \) so

\[ \kappa(32) = \frac{1}{n} \kappa_5 + \frac{6}{n-1} \kappa_3 \kappa_2 \]

Fisher (1929) describes the expression of \( \kappa(3\cdot2) \) in a similar fashion. He was able to break through this algebra, whose complexity increases sharply with increasing total weight of the cumulant, and proceed to the final result. His rules for writing any \( \kappa(p_1 \ldots p_s) \) may be summarized as follows

1. If \( p = \sum_{i=1}^{s} p_i \pi_i \) then \( \kappa(p_1 \ldots p_s) \) is a linear function of cumulants and products of cumulants of order \( p \), none of which involve \( \kappa_1 \) except that \( \kappa(1) = \kappa_1 \).
2. The coefficient of $\kappa_{q_1}^{a_1} \kappa_{q_2}^{a_2} \cdots \kappa_{q_m}^{a_m}$ in $\kappa(p_1 \cdots p_s)$ is dependent on the two-way partition of the number

$$p = \sum_{i=1}^{s} p_i \pi_i = \sum_{i=1}^{m} a_i q_i$$

such that the columns total $P_1, P_1, \ldots, P_s$ and the rows total $q_1, q_1, \ldots, q_m$ as in the table below:

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>$P_1$</th>
<th>$\cdots$</th>
<th>$P_s$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_1$</td>
<td>$\ldots$</td>
<td>$q_m$</td>
<td></td>
</tr>
</tbody>
</table>

Many such partitions generally exist but those having the following properties may be ignored.

a. The non-zero entries fall into two or more mutually exclusive sets, called blocks, the elements of a block being rows and columns.

b. Some rows consists entirely of a single non-zero entry.

c. The non-zero entries fall into exactly two blocks, the intersection of the blocks consisting of exactly one column.

3. Each partition surviving the conditions of 2 is given a numerical and $n$-coefficient. The coefficient of $\kappa_{q_1}^{a_1} \cdots \kappa_{q_m}^{a_m}$ is the sum of the products of the numerical
and n-coefficients. The numerical coefficient is the number of distinct ways the column totals, considered as individuals, can be allocated to form the given pattern, divided by $a_1!a_2!...a_m!$

The n-coefficient depends entirely on the configuration of non-zero entries in the partition. The pattern is first divided into separates, a separate being formed for each distinct way in which the rows of the pattern can be formed into groups, each group containing at least one row. For example the pattern

\[
\begin{array}{c}
X X X \\
X - X \\
- X X
\end{array}
\]

where the X's stand for non-zero entries and - denotes a zero entry, forms five separates:

\[
\begin{array}{ccccccc}
X X X & X X X & X X X & X X X & X X X \\
X - X & \{X - X\} & X - X & X - X & X - X \\
- X X & - X X & - X X & - X X & - X X
\end{array}
\]

The third separate denotes the separation of the second row from the first and third and the last separate indicates that the rows have been divided into three groups, each consisting of one row. To form the n-coefficient for a particular separate, the factor $n(n-1)...(n-q+1)$ where $q$ is the number of groups in the separate, is multiplied by $R$. $R$ is formed by
considering the columns one at a time and assigning the factor \( \frac{(-1)^{p-1}(n-1)!}{n(n-1)\ldots(n-p+1)} \) to that column if the column has non-zero entries in \( \rho \) groups of the separate. \( R \) is then the product of the factors.

These rules may be applied to \( \kappa(32) \). \( \kappa(32) = a\kappa_5 + b\kappa_2 \) by rule one, there being no other partitions of five other than those containing unit parts.

Only one separate exists for finding \( a \), namely

\[
\begin{array}{ccc}
3 & 2 & 2 \\
\hline
3 & 2 & 2 \\
\end{array}
\]

Its numerical coefficient is 1 and \( n \)-coefficient is \( n\left(\frac{1}{n}\right)^2 = \frac{1}{n} \). Thus \( a = \frac{1}{n} \).

For the coefficient of \( \kappa_2 \kappa_2 \) there are three patterns,

\[
\begin{array}{ccc}
3 & - & 2 \\
\hline
3 & 2 & 2 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
2 & - & 2 \\
\end{array}
\quad
\begin{array}{ccc}
2 & 1 & 3 \\
\hline
1 & 1 & 2 \\
\end{array}
\]

the first two having zero coefficients by conditions 2a and 2b. The remaining pattern has numerical coefficient \( \frac{3!}{2!1!} \cdot \frac{2!}{1!1!} = 6 \) and the \( n \)-coefficient is formed from the separates

\[
\begin{array}{ccc}
X & X \\
\hline
X & X \\
\end{array}
\quad
\begin{array}{ccc}
X & X \\
\hline
........ \\
X & X \\
\end{array}
\]

and is accordingly \( n\left(\frac{1}{n}\right)^2 + n(n-1) \left(\frac{(-1)}{n(n-1)}\right)^2 = \frac{1}{n-1} \). So \( b = \frac{6}{n-1} \). Fisher (1929) gives a table of the more commonly occurring patterns and their \( n \)-coefficients.
Two additional rules which have frequent applications are:

4. Any pattern containing a column with only one non-zero entry has a pattern function equal to $1/n$ times the pattern function formed from the configuration with that column missing.

5. If $\kappa(p_1^{\pi_1} \ldots p_s^{\pi_s})$ is known, $\kappa(p_1^{\pi_1+1} \ldots p_s^{\pi_s+1})$ is formed by adding 1 to each subscript in the expression in each possible way and multiplying by $1/n$.

Rule 4 follows from the previous rules and rule 5 is an application of rule 4. The rules are difficult to apply in many cases and it is easy to omit separates. Wishart (1930) discusses $\kappa(3^2_2 3^1)$ for example, and finds that the pattern

$$
\begin{array}{cccc}
1 & 1 & - & - \\
1 & 1 & - & - \\
1 & - & 1 & - \\
- & 1 & - & 1 \\
- & - & 1 & 1 \\
- & - & - & 1 \\
\end{array}
$$

has 203 separates. Fisher (1929) considers $\kappa(4^2 2)$ and determines the term in $\kappa_6 \kappa_2^2$. The same example is discussed in detail by Kendall and Stuart (1969) and by Sukhatme (1939). No one has considered determining the $n$-coefficient by any simpler method than that advanced by Fisher, so all of the separates of a particular pattern
must be found. Fisher and Wishart (1931) and Wishart (1930, 1952a), however, do discuss limited alternatives in calculating the numerical coefficients.

Wishart (1930) proposed the use for a given partition having entries of ones, of a diagram having a vertex for each column and with as many lines stemming from that vertex as there are entries in the column. Lines from different vertices are joined if they occur in the same row of the diagram. For the pattern above the diagram is

```
D
\[ \begin{array}{c}
  \text{E} \\
  \text{A} \\
  \text{B}
\end{array} \]
```

The numerical coefficient is determined by counting the number of arrangements of the lines at each vertex which result in the same pattern. In this case, the vertices C, D and E can be arranged in 3! ways. The lines at A and B can be arranged in 3² ways and the lines at C, D and E arranged in 2³ ways once the order of the vertices is fixed. Finally, A and B can be arranged in 2 ways. The numerical coefficient is then \(3! \cdot 3^2 \cdot 2^3 \cdot 2 = 864\).

From the pattern itself this would be calculated as

\[
\frac{3!}{3!} \cdot \frac{2!}{2!} \cdot \frac{2!}{2!} \cdot \frac{2!}{2!} = 864
\]
as the number of distinct arrangements of the columns and the permutations of the rows.
and columns in the figure.

Wishart also discusses the calculation of coefficients for certain patterns from those of simpler patterns, such as the development of two patterns of \( \kappa(3^4) \) from that for \( \kappa(3^2) \).

Fisher and Wishart (1931) consider the calculation of a pattern coefficient from a simpler pattern by removing columns. Wishart (1952b) uses \( \kappa(4^2) \) and \( \kappa(4^3) \) as examples to show how all the patterns may be formed and how the numerical coefficients of some of the derived patterns are related.

This work provides interesting checks in a difficult example. It is not entirely clear, however, how the geometric diagrams are to be constructed when the entries are greater than 1; the diagrams also give no aid in the calculation of the n-coefficient.

Hsu and Lawley (1939) illustrate the calculation of \( \kappa(4^5) \) and \( \kappa(4^6) \) for normal populations using the diagrams for some of the patterns which occur. They also make use of Wishart's rules for finding coefficients for some patterns formed by inserting corners into simpler patterns. They use these results to work out the 5th and 6th moments of \( b_2 \) for normal samples.
D. Proofs of Fisher's Rules

Fisher (1929) presented some theoretical justification for his rules but this section of his paper is widely regarded as obscure (Kendall, 1963, Sukhatme, 1939).

Sukhatme (1939) works through in great detail the coefficient of $\kappa_6\kappa_2^2$ in $\kappa(4^2_2)$, this also being the example chosen by Fisher (1929). The example reveals that the algebraic procedure leads to considerations similar to those given by Fisher. For example, the term in $\mu(4^2_2)$ is seen to arise from pattern functions similar to Fisher's. Sukhatme concludes "The whole analysis...should materially help to produce a rigorous proof of [Fisher's] combinatorial methods."

Kendall (1940a,b) introduced certain symbolic operations and gave a complete set of proofs for Fisher's rules. He also established some properties of the $k$ statistics which are of some interest. Despite this evidence of interest and effort, purely combinatorial proofs of the rules Fisher first published in 1929 were not available until 1958 when James provided them. He also discussed for an arbitrary set of homogeneous symmetric functions of the observations the expression of their moments in terms of population moments, the expression of their moments in terms of population cumulants and the expression of their sampling cumulants in terms of population cumulants.
E. Multivariate Cumulants

Cumulants can be defined for multivariate distributions in a manner similar to that for univariate distributions. The cumulants for a bivariate distribution, which has been the case of primary interest, are defined by

\[ 1 + \frac{\mu_{1,0} t_1}{1!0!1!} + \frac{\mu_{0,1} t_2}{0!1!1!} + \frac{\mu_{1,1} t_1 t_2}{1!1!1!} + \ldots + \frac{\mu_{p,q} t_1^p t_2^q}{p!q!} \]

Then \( \mu_{pq} \) can be expressed in terms of the cumulants, and conversely (Fisher, 1929). Cook (1951a) first gave tables of these up to and including the 6th order. Tables through weight 8 can be found in David, Kendall and Barton (1966).

In particular,

\[ \kappa_{r,s} = \sum_{\pi_1! \pi_2! \ldots} \frac{(-1)^{\rho-1} (\rho-1)! r!}{(p_1!) \pi_1! (p_2!) \pi_2! \ldots} \]

\[ \times \frac{\mu_{1,1} \ldots \mu_{p,q} \pi_1^{p_1} \pi_2^{p_2} \ldots}{(q_1!) \pi_1^{q_1} (q_2!) \pi_2^{q_2} \ldots} \]

where \( (p_1 q_1)^{\pi_1} (p_2 q_2)^{\pi_2} \ldots \) is a partition of the bipartite number \( r, s \), that is, \( \sum_i p_i \pi_i = r \), \( \sum_i q_i \pi_i = s \), \( \sum_i \pi_i = \rho \), and the summation is over all such partitions.

Now if \( (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \) represents a sample
from a bivariate population a statistic \( k_{r,s} \), where

\[ E(k_{r,s}) = \kappa_{r,s} \]

can be defined, in exact analogy to the univariate case, as

\[
k_{r,s} = \sum (-1)^{p-1}(q-1)!r!s! \pi_1 \pi_2 \ldots \pi_r \pi_s \ldots \pi_q \pi_r \ldots
\]

where

\[
\langle(p_1 q_1)(p_1 q_1)\ldots\rangle = \frac{\prod_i p_1^\pi_i q_1^\pi_i}{n^\rho}
\]

and the summation is over all partitions of \( r, s \) and where

\[
E\frac{\prod_i p_1^\pi_i q_1^\pi_i}{n^\rho} = \mu_1^\pi_1 \mu_2^\pi_2 \ldots .
\]

Fisher gives a few values of \( k_{r,s} \), namely \( k_{1,1}, k_{2,1}, k_{3,1} \), and \( k_{2,2} \), in terms of power sums but no tables of these quantities, which Kendall called polybikays, exist. Kendall (1940c) gives a symbolic process whereby these multivariate \( k \) statistics may be obtained from the univariate. David, Kendall and Barton (1966) give tables of the bivariate augmented symmetric functions above in terms of bivariate sums \( s_{pq} = \sum_{i=1}^{n} x_i^p y_i^q \). Letting \( k_{(a_1^{\alpha_1} \ldots \alpha_2^{\beta_2} \ldots)} \) where there are \( r \alpha \)'s and \( s \beta \)'s, denote the \( \kappa_{rs} \) of the joint distribution of \( k_{\alpha\alpha} \) and \( k_{\beta\beta} \), Cook (1951) shows how Fisher's univariate rules may be applied to evaluate the
cumulant in terms of the bivariate cumulants. She also describes an operational method due to Kendall (1940c) by which the cumulants may be obtained from corresponding univariate results. Extensive results through weight 9 are given in this paper.

Cook (1951b) then applies these results to find

\[ \mu_2'(r), \mu_3'(r) \text{ and } \mu_4'(r) \]

to order \( n^{-2} \) where

\[
 r = \frac{\sum (x-x)(y-y)}{[\sum (x-x)^2 \sum (y-y)^2]^{1/2}}
\]

and specializes the results to bivariate normal surfaces.

Robson (1957) has applied multivariate polykays to unbiased ratio-type estimation and used them to find the variance of the Hartley-Ross estimator. He also constructed other ratio-type estimators and found unbiased estimates of them and unbiased estimators of their variance.

F. Polykays and Generalized Polykays

In 1940 Dressel introduced functions which he denoted by \( L_{ab...c} \) whose expectations, over all possible random samples of a fixed size, were products of Thiele's semi-invariants. So \( E(L_{ab...c}) = \kappa_a \kappa_b ... \kappa_c \). Dressel gave a complete table of these functions of weight at most 8, in terms of power sums. He apparently was motivated in this by
the fact that a moment of a given weight can be expressed in terms of cumulants and their products of that weight and if an unbiased estimate of this moment is desired, then the products of cumulants must be estimated unbiasedly as well.

However, it is also true that such functions arise naturally when unbiased estimation from finite populations is considered, and this fact was noticed, independently of Dressel's work, by Tukey in 1950. Tukey called the functions polykays and established many of their properties. Since polykays are the subject of Chapter II, further discussion of them will not be included here.

The generalized polykays represent another important aspect of the work which was stimulated by the discovery of the k statistics. Bipolykays were introduced by Hooke in 1956 when he considered sampling from a matrix, \(|x_{IJ}|, I = 1,2,\ldots,R\) and \(J = 1,2,\ldots,C\). A sample matrix is formed from the elements at the intersections of \(r\) chosen rows and \(c\) chosen columns and polynomial functions of these elements, which are symmetric in the sense that they are invariant under permutation of the rows and columns, are considered. A generalized symmetric mean is defined to be a function of the form

\[
\frac{1}{M} \sum_{p,q,\ldots,s,t} x_{pq}^{a_pq} \cdots x_{st}^{a_st}
\]

where \(M\) is the number of terms and the symbol \(\neq\) means
that unequal row (or column) subscripts must remain unequal in the summation. Certain linear functions of these generalized symmetric means, called bipolykays, are defined as symbolic products of polykays and are shown to be inherited on the average, as are the generalized symmetric means. Specific definitions, which are not needed here, can be found in Hooke (1956a,b).

It should be pointed out that the bipolykays have not been defined with any sort of cumulant property in mind, unlike the k statistics and the polykays. One consequence of this is that their pairing formulas are not simple. They are however important functions in studying two way crossed populations, and represented the first effort to associate a population structure with the symmetric functions arising from it.

More general work in this direction was done by Dayhoff (1964a) when he introduced generalized polykays, which are a generalization of Hooke's work to an n-way crossed population. Dayhoff was also able to specialize the generalized polykays to other populations involving both nested and crossed factors, thus greatly increasing the range of applicability of these polykays. He also demonstrated that the \( \Sigma \) functions which were introduced by Wilk (1955) and given general, easily specifiable definition by Zyskind (1958) were equivalent to the generalized polykays of the second degree,
a conjecture that had been made by Zyskind (1958). Dayhoff gave a complete set of formulas for generalized polykays of degrees 2, 3 and 4 for two and three factor structures. Subsequently, Carney (1967) considered the computation of the generalized symmetric means and polykays and their application to the estimation of variance components and to the computation of the variances and covariances of these estimates.

Due to the equivalence of the Σ's and generalized polykays of degree 2, the Σ's can be expressed in terms of components of variation, or, as did Dayhoff for two and three factor structures, in terms of generalized symmetric means of the second degree. This line of investigation is developed further in the present work where the expression of the Σ's for structures involving any number of factors is considered and rules are given whereby the Σ's can be formed, in terms of components of variation, by means of certain matrix products.

G. Overview

All of the problems dealt with here are concerned with the k statistics, polykays, and generalized polykays, a general context for which has been given in this chapter.

In Chapter II, polykays are described and a new method for their formation is proposed. New proofs of the properties of randomized sums are also given there. These proofs serve to
connect these properties of the \( k \) statistics with their definitions as originally given by Fisher.

Chapter III contains a study of the multivariate case -- a subject on which the literature is very scant. Multivariate \( k \) statistics are derived from univariate \( k \) statistics by a new symbolic method and a complete list of multivariate \( k \) statistics of weight 5 or less is presented. For the first time, multivariate polykays are expressed in terms of multivariate \( k \) statistics and a complete catalog of multivariate polykays of weight 8 or less is included.

The expression of the generalized polykays of the second degree, or \( \Sigma \)'s, in terms of components of variations is considered in Chapter IV. Statistical structures are classified as being unitary or non-unitary and the \( \Sigma \)'s are formed by means of matrix products in each case. This work for unitary structures is also applicable to structures involving any number of factors. All of the non-unitary structures with four or five factors are shown.

The initial part of Chapter V is a study of the problem of testing several samples, drawn from homoscedastic populations with possibly different means, for normality. The properties of the \( k \) statistics are brought into play here, using a technique which is applicable to both the normal and non-normal cases. The multivariate polykays are then applied
to a measure of bivariate dispersion, the generalized variance, an unbiased estimate of this quantity is found and its variance computed.
II. CHAPTER TWO: POLYKAYS

A. Early Research and Basic Results

Fisher in his 1929 paper and subsequent writings about the \( k \) statistics confined his discussion to infinite populations. Finite populations had however been of interest to writers on moments of moments from the beginning. Pearson (1899) for example gave formulas for the first four moments of the mean when sampling from a finite population which follows a simple frequency law. Neyman (1925) discussed moments of moments from more general finite populations. More recently, important contributions to the application of the \( k \) statistics in finite populations have been made by Irwin and Kendall, Tukey, and Wishart.

The 1944 paper by Irwin and Kendall is of central importance in work concerning \( k \) statistics. Their main result, however, has never been fully explicated and for this reason a fairly detailed explanation follows. An extension of their result will also be made to the case of polykays.

Irwin and Kendall considered a finite population of size \( N \) and a random sample of size \( n \) drawn from it without replacement. Let \( \kappa_r \) and \( \kappa_r \) denote the \( r \)th \( k \) statistics from the population and the sample respectively. Suppose now that the finite population itself is regarded as a random sample from an infinite population with cumulants \( \kappa_r \). Then \( E(k_r) = E(K_r) = \kappa_r \) since \( k_r \) and \( K_r \) are the \( k \) statistics.
based on random samples of sizes $n$ and $N$ respectively. Let $E_N(k_r)$ denote the expectation of $k_r$ over the finite population. This expectation is necessarily a symmetric function of the finite population values since $k_r$ is expressible in terms of the bracket functions and these are symmetric and inherited on the average. Suppose now that $E_N(k_r) = L_r$ where $L_r$ is a symmetric function distinct from $K_r$. Then

$$E(L_r) = E[E_N(k_r)] = E(k_r) = \kappa_r$$

and so

$$E(K_r - L_r) = 0 \quad (1)$$

since $E(K_r) = \kappa_r$ also. But $K_r - L_r$ is a symmetric function and thus can be uniquely expressed in terms of symmetric sums or bracket functions (O'Toole, 1931). In turn, these quantities have expectation over the infinite population which are products of moments and (1) then illustrates a dependence among the moments or equates some non-trivial function of a single moment to zero, either of which is impossible for a general population (Kendall, 1940a). Thus

$$E_N(k_r) = K_r.$$  

Irwin and Kendall then state without proof that if a function $f$ is found such that $E(f) = \sum_r a_r \kappa_r$ then

$$E_N(f) = \sum_r a_r K_r. \quad (2)$$
The presumption that $f$ is symmetric is added in Kendall and Stuart (1969). To see that (2) is then true, suppose that $E_N(f) = \sum_{r} b_r L_r$ where the functions $L_r$ and $K_r$ are distinct with $L_r$ symmetric for every $r$ appearing in the sum. Then, as before,

$$E[\sum_{r} a_r K_r - \sum_{r} b_r L_r] = 0$$

and so, since the function $\sum_{r} a_r K_r - \sum_{r} b_r L_r$ is symmetric, it must itself be zero, establishing (2).

An analogous argument was used in Chapter One to establish the uniqueness of the $k$ statistics and can be used in an exactly similar way to establish the uniqueness of the polykays, soon to be introduced here.

Formula (2) has become known as the Irwin-Kendall principle and its chief importance lies in the fact that finite population results can be deduced from those for infinite populations, an interesting reversal of the usual situation. To illustrate the use of the principle, an example from the Irwin-Kendall paper (1944) will now be given. Suppose that it is desired to calculate the variance of

$$m_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

where $x_i$ denotes a sample value in a random sample of size $n$ drawn without replacement from a finite population of size $N$. Since $E_N(k_2) = k_2$, ...
\[ E_N(m_2) = \frac{n-1}{n} \frac{N}{N-1} M_2 \]

where \[ M_2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2 \] and \( \mu \) denotes the population mean.

Now

\[ E(k_2^2) = \frac{1}{n} \kappa_4 + \frac{n+1}{n-1} \kappa_2^2 \]

and

\[ E(K_2^2) = \frac{1}{N} \kappa_4 + \frac{N+1}{N-1} \kappa_2^2, \]

this relation being obtainable by application of Fisher's rules. Since

\[ E(k_2 K_2) = E[E_N(k_2 K_2 | N)] = E(k_2^2), \]

it follows that

\[ E(k_2 - K_2)^2 = E(k_2^2) - E(K_2^2). \]

Then, eliminating \( \kappa_2^2 \),

\[ E(k_2 - K_2)^2 = \left[ \frac{(N-n)(Nn-n-N-1)}{n(n-1)N(N+1)} \right] \kappa_4 + E(K_2^2) \cdot \frac{2(N-n)}{(n-1)(N+1)} \]

and so

\[ \text{var}_N(k_2) = E_N(k_2 - K_2)^2 = \frac{(N-n)(Nn-n-N-1)}{n(n-1)N(N+1)} \kappa_4 + \frac{2(N-n)}{(n-1)(N+1)} \kappa_2^2, \]

from which
\[
\text{var}_N(m_2) = \frac{(n-1)N(N-n)}{n^3(N-1)^2(N-2)(N-3)} \left[ (nN-n-1)(N-1)M_4 \right.
\]
\[
- (nN^2-3N^2+6N-3n-3)M_2^2 \right],
\]

where

\[
M_4 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2.
\]

Irwin and Kendall also provide formulas for \( E_N(k_1 - K_1)^r \) for \( r = 2, 3 \) and \( 4 \), \( E_N(k_r - K_r)(k_1 - K_1) \), and they discuss \( E(k_r - K_r)(k_2 - K_2) \).

This work can be simplified considerably if a linear function of the \( k \) statistics is known whose expectation is \( \kappa_2^2 \). In that case the Irwin-Kendall principle can be applied to the equality

\[
E(k_2^2) = \frac{1}{n} \kappa_4 + \frac{n+1}{n-1} \kappa_2^2
\]

to give \( E_N(k_2^2) \). For more general purposes, quantities are needed whose expectations are products of population cumulants.

Now

\[
\kappa_p = \sum_{m=1}^{p} \frac{(-1)^{p-1} \rho_1^{p-1}}{\rho_1! \rho_2^{p} \cdots \rho_m^{p}} \frac{\pi_1^{\rho_1} \pi_2^{\rho_2} \cdots \pi_m^{\rho_m}}{p_1! p_2! \cdots p_m!}
\]

and

\[
\kappa_q = \prod_{n=1}^{q} \frac{(-1)^{q-1} \rho_2^{q-1}}{\rho_2! \rho_2^{q} \cdots \rho_n^{q}} \frac{\alpha_1^{\rho_1} \alpha_2^{\rho_2} \cdots \alpha_n^{\rho_n}}{q_1! q_2! \cdots q_n! \alpha_1! \alpha_2! \cdots \alpha_n!}
\]
where

\[ \sum_{i=1}^{m} p_i \pi_i = p, \quad \sum_{i=1}^{m} \pi_i = \rho_1, \quad \sum_{i=1}^{n} q_i \alpha_i = q, \quad \sum_{i=1}^{n} \alpha_i = \rho_2. \]

Therefore

\[ \kappa_{pq} = \sum_{p, q} \frac{(-1)^{\rho_1 + \rho_2}(\rho_1 - 1)!(\rho_2 - 1)! p!q!\mu_1^{\pi_1} \mu_2^{\pi_2} \cdots \mu_m^{\pi_m} \alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}}{p_1! \cdots p_m! \pi_1! \cdots \pi_m! q_1! \cdots q_n! \alpha_1! \cdots \alpha_n!} \]

from which it is evident that if

\[ k_{pq} = \sum_{p, q} \frac{(-1)^{\rho_1 + \rho_2}(\rho_1 - 1)!(\rho_2 - 1)! p!q!\mu_1^{\pi_1} \mu_2^{\pi_2} \cdots \mu_m^{\pi_m} \alpha_1^{\alpha_1} \cdots \alpha_n^{\alpha_n}}{p_1! \cdots p_m! \pi_1! \cdots \pi_m! q_1! \cdots q_n! \alpha_1! \cdots \alpha_n!} \]

then \( E(k_{pq}) = \kappa_{pq} \), a similar procedure holding for products of more than two cumulants.

Dressel (1940) introduced functions \( L_{pq} \), expressed in terms of power sums, such that \( E(L_{pq}) = \kappa_{pq} \). It will soon be shown that the functions \( L_{pq} \) and \( k_{pq} \) are identical. Dressel gave a table of all \( L_{pq} \) through weight 8.

Tukey (1950) was interested in the properties of randomized sums, which are considered in section F of this chapter. He defined a symbolic multiplication of brackets as follows:
1) Except when brackets multiply brackets, $\circ$ is ordinary multiplication,

2) $\langle p \rangle \circ \langle q \rangle = \langle pq \rangle$,

3) $\circ$ multiplication is distributive.

Tukey then defined $k_{pq} = k_p \circ k_q$ where the $k$'s are first expressed in terms of brackets. Although Tukey's motivation for the $\circ$ multiplication is unclear, it is evident that this procedure also produces the functions given in (3). Tukey called the functions obtained by the $\circ$ multiplication polykays and concluded that their expectations are products of population cumulants. It seems more fruitful however to use this cumulant property in defining the function $k_{pqr}$...

Finally, to return to the example of the computation of $\text{var}_N(m_2)$, $K_{22}$ is the polykay whose expectation is $\kappa_2^2$. Since $E(k_2^2) = \frac{1}{n} \kappa_4 + \frac{n+1}{n-1} \kappa_2^2$, $E_N(k_2^2) = \frac{1}{n} K_4 + \frac{n+1}{n-1} K_{22}$ by the Irwin-Kendall principle. But $\text{var}_N(k_2) = E_N(k_2^2) - K_2^2$ and $K_2 = \frac{1}{N} K_4 + \frac{N+1}{N-1} K_{22}$, the determination of these kinds of relations being the subject of Sections B and C. Combining these results,

$$\text{var}_N(k_2) = (\frac{1}{n} - \frac{1}{N}) K_4 + (\frac{2}{n-1} - \frac{2}{N-1}) K_{22}$$

from which $\text{var}_N(m_2)$ can be obtained. This result was first given by Tukey (1950).
B. Determination of Polykays

Expression (3) for polykays in terms of brackets is useful but it is often desirable to have relations among the polykays themselves and expressions for them in terms of powers and products of the single $k$ statistics. There are several ways to do this, including a procedure which is new and which, like a method due to Wishart (1952a), is dependent on Fisher's rules. The new method will be illustrated in this section, while Wishart's method and its contrast with the method proposed, is the subject of Section C.

First it may be noted that $k_{pq}$ is unique. This can be established by exactly the same form of argument used in establishing (1) earlier in this chapter. Furthermore, (2) may be extended as follows, where again a finite population of size $N$ is being regarded as a sample from an infinite population with cumulants $\kappa_r$ and a random sample of size $n$ is available from the finite population. Now suppose that for some symmetric function $f$,

$$E(f) = P$$

where $P$ is a polynomial function of the cumulants. Then

$$E_N(f) = P^*$$

(4)

where $P^*$ is the linear function corresponding to $P$ in which the powers and products of the infinite population
cumulants are replaced by the corresponding polykays of the finite population. Since (4) can be proved in a manner similar to that used to establish (2), the proof of (4) will be omitted.

Now as a first method for determining polykays, apparently the one Tukey had in mind in 1950, tables of symmetric functions, such as those of David, Kendall, and Barton (1966) may be used. To illustrate, consider \( k_2k_1 \).

\[
k_2k_1 = (\langle 2 \rangle - \langle 11 \rangle) \langle 1 \rangle = \langle 2 \rangle \langle 1 \rangle - \langle 11 \rangle \langle 1 \rangle .
\]

By tables of symmetric functions this can be written as

\[
\frac{s_2}{n} \frac{s_1}{n} - \frac{1}{n^2(n-1)} [s_1^2 - s_2] s_1
\]

where \( s_r = \sum_{i=1}^{n} x_i^r \). Then

\[
k_2k_1 = \frac{1}{n^2} \{[3] + [21]\} - \frac{1}{n^2(n-1)} \{[3] + 3[21] + [1^3] - [3] - [21]\}
\]

where the brackets are the augmented monomial symmetric functions defined in Chapter One.

Then, again from tables of symmetric functions,

\[
k_2k_1 = \frac{1}{n} \{k_{111} + 3k_{21} + k_3\} + \frac{n-3}{n} \{k_{111} + k_{21}\}
\]

\[- \frac{n-2}{n} k_{111}\]

So \( k_{21} = \frac{1}{n} k_3 - k_2k_1 \). Wishart (1952) gives a complete
set of formulas through weight 6 as well as a few of weights 7 and 8. Abdel-Aty (1954) gives a complete set of formulas of weight 12 in terms of augmented symmetric functions. However to use the method effectively for higher weights more extensive tables must be available and even then the algebra becomes very heavy. Wishart (1952a) has given another example of Tukey's procedure and has provided conversion tables of polykays and augmented symmetric functions through weight 6.

A direct way to evaluate polykays using Fisher's rules as they were originally stated can be found. This procedure can be justified without recourse to symbolic methods and a rule concerning unit parts can be established. Some general formulas and some new results will also be given.

The cumulants and means of the simultaneous distribution of the $k$ statistics are related by

$$1 + \mu(p)t + \mu(q)t + \mu(pq)t + \ldots + \frac{\mu(p^ab^rc^\ldots)t^a_t^b_t^c_t^\ldots}{abc!} + \ldots$$

$$= \exp\{\kappa(p)t + \kappa(q)t + \kappa(pq)t + \ldots + \frac{\kappa(p^ab^rc^\ldots)t^a_t^b_t^c_t^\ldots}{abc!} + \ldots \}$$

where $\mu(pqr\ldots) = E(k^k^k^r\ldots)$. So it follows that
\[1 + \mu(p) t_p + \mu(q) t_q + \mu(pq) t_p t_q + \ldots =
\]
\[1 + \{\kappa(p) t_p + \kappa(q) t_q + \kappa(pq) t_p t_q + \ldots}\]
\[+ \frac{1}{2!} \{\kappa(p) t_p + \kappa(q) t_q + \kappa(pq) t_p t_q + \ldots\}^2\]
\[+ \frac{1}{3!} \{\kappa(p) t_p + \kappa(q) t_q + \kappa(pq) t_p t_q + \ldots\}^3 + \ldots \quad (5)\]

Suppose an expression for \(k_{pq}\) is wanted.

On equating the coefficients of \(t_p t_q\) on both sides of (5) and noting that \(\kappa(p) = \kappa_p\), it follows that

\[\mu(pq) = \kappa(pq) + \kappa(p) \kappa(q)\]
or
\[\kappa_p \kappa_q = \mu(pq) - \kappa(pq)\]

Now \(\kappa(pq)\) can be evaluated by Fisher's rules in terms of products and powers of population cumulants and in general

\[\kappa(pq) = \frac{1}{n} \kappa_{p+q} + \Sigma C(r,s,...) \kappa_r \kappa_s \ldots\]

where the summation is over all partitions such that \(r+s+\ldots = p+q\), and where \(C(r,s,...)\) denotes the appropriate coefficient determined from the pattern.

Then \[\kappa_p \kappa_q = \mu(pq) - \frac{1}{n} \kappa_{p+q} - \Sigma C(r,s,...) \kappa_r \kappa_s \ldots\]

and so by the uniqueness property

\[k_{pq} = k_p k_q - \frac{1}{n} k_{p+q} - \Sigma C(r,s,...) k_{rs} \ldots\]

since its expectation is \(k_p k_q\). Also,
\[ \mu(pqr) = \kappa(pqr) + \kappa(p)\kappa(qr) + \kappa(q)\kappa(pr) \]

\[ + \kappa(r)\kappa(pq) + \kappa(p)\kappa(q)\kappa(r) \]

in a similar way and so since \( \kappa(pqr), \kappa(qr), \kappa(pr) \) and \( \kappa(pq) \)
can be expressed in terms of cumulants, \( k_{pqr} \) can be found. This procedure applies to a polykay of any order and with any number of parts.

As examples consider first \( k_{23} \). Since

\[ 1 + \mu(2)t_2 + \mu(3)t_3 + \mu(23)t_2t_3 + \ldots \]

\[ = 1 + \{ \kappa(2)t_2 + \kappa(3)t_3 + \kappa(23)t_2t_3 + \ldots \} \]

\[ + \frac{1}{2!} \{ \kappa(2)t_2 + \kappa(3)t_3 + \kappa(23)t_2t_3 + \ldots \}^2 + \ldots \]  \( \text{(6)} \)

it follows that

\[ \mu(23) = \kappa(23) + \kappa(2)\kappa(3) \cdot \]

Thus \( \kappa_2\kappa_3 = E(k_2k_3) - \kappa(23) \)
or

\[ \kappa_2\kappa_3 = E(k_2k_3) - \frac{1}{n} \kappa_5 - \frac{6}{n-1} \kappa_2\kappa_3, \]

the last term being found from the pattern

\[
\begin{array}{ccc|c}
1 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 \\
\end{array}
\]

So

\[ k_{23} = k_2k_3 - \frac{1}{n} \kappa_5 - \frac{6}{n-1} k_{23} \]
or

\[ k_{23} = \frac{n-1}{n+5} \{ k_2 k_3 - \frac{1}{n} k_5 \} \]

The bracket multiplication could be applied directly as an alternative. As a more difficult example, consider \( k_{223} \).

From (6),

\[ \mu(2^23) = \kappa(2^23) + 2\kappa(2)\kappa(23) + \kappa(2^2)\kappa(3) + [\kappa(2)]^2\kappa(3) \]

so

\[ \kappa_2^2\kappa_3 = \mu(2^23) - \kappa(2^23) - 2\kappa(2)\kappa(23) - \kappa(2^2)\kappa(3) \]

Now

\[ \kappa(2^23) = \frac{1}{n^2} \kappa_7 + \frac{24n-36}{n(n-1)} \kappa_4 \kappa_3 + \frac{16}{n(n-1)} \kappa_2 \kappa_5 \]

\[ \text{+ } \frac{48}{(n-1)^2} \kappa_2^2 \kappa_3 \]

where the coefficient of \( \kappa_4 \kappa_3 \) is determined from the patterns

\[ \begin{array}{ccc} 1 & 1 & 2 \ 1 & 1 & 1 \ 2 & 2 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & 2 & 1 \ 1 & -2 & 3 \ 2 & 2 & 3 \end{array} \]

(a) \quad \quad (b)

that of \( \kappa_2 \kappa_5 \) from the patterns

\[ \begin{array}{ccc} 1 & -1 & 2 \ 1 & 2 & 2 \ 2 & 2 & 3 \end{array} \quad \text{and} \quad \begin{array}{ccc} 1 & 1 & -2 \ 1 & 1 & 3 \ 2 & 2 & 3 \end{array} \]

(c) \quad \quad (d)
and that of $\kappa_2^2 \kappa_3$ from

\[
\begin{array}{c|ccc}
1 & -1 & 2 \\
1 & 1 & -2 \\
-1 & 2 & 3 \\
\hline
2 & 2 & 3
\end{array}
\quad \text{and} \quad
\begin{array}{c|ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\hline
2 & 2 & 3
\end{array}
\]

(e) \hspace{1cm} (f)

From the previous example

\[\kappa(23) = \frac{1}{n} \kappa_5 + \frac{6}{n-1} \kappa_2 \kappa_3.\]

Also \[\kappa(2^2) = \frac{1}{n} \kappa_4 + \frac{2}{n-1} \kappa_2^2\]

so

\[\kappa_2^2 \kappa_3 = E(k_2^2 k_3) - \frac{1}{n^2} \kappa_7 - \frac{24n-36}{n(n-1)^2} \kappa_4^2 \kappa_3 - \frac{16}{n(n-1)} \kappa_2^2 \kappa_5\]

\[- \frac{48}{(n-1)^2} \kappa_2^2 \kappa_3 - 2 \kappa_2 \{\frac{1}{n} \kappa_5 + \frac{6}{n-1} \kappa_2 \kappa_3\}\]

\[\kappa_3 \{\frac{1}{n} \kappa_4 + \frac{2}{n-1} \kappa_2^2\}.\]

Thus

\[k_{223} = k_2^2 k_3 - \frac{1}{n} k_7 - \frac{24n-36}{n(n-1)^2} k_{43} - \frac{16}{n(n-1)} k_{25}\]

\[- \frac{48}{(n-1)^2} k_{223} - \frac{2}{n} k_{25} - \frac{12}{n-1} k_{223} - \frac{1}{n} k_{43}\]

\[- \frac{2}{n-1} k_{223},\]

or

\[k_{223} = \frac{(n-1)^2}{(n+5)(n+7)} \left[k_2^2 k_3 - \frac{1}{n^2} k_7 - \frac{2(n+7)}{n(n-1)} k_{25}\right.\]

\[- \frac{(n^2+22n-35)}{n(n-1)^2} k_{43}\].
C. Wishart's Method

The procedure just presented may be contrasted with that of Wishart (1952a). Wishart's method consists of applying Fisher's rules to the products of k statistics in exactly the same manner as they apply to cumulants except that a class of patterns which had zero coefficients before is now allowed and the coefficients are calculated using the other rules governing this. The class of patterns is that for which the columns fall into two or more classes, the non-zero entries of which fall into distinct sets of rows. For example in the expression for \( k(2^2) \) the pattern

\[
\begin{array}{ccc|c}
2 & - & 2 \\
- & 2 & 2 \\
2 & 2 & \\
\end{array}
\]

is not allowed. It is admitted in calculating \( k_2^2 \) and, by Wishart's procedure,

\[
k_2^2 = \frac{1}{n} k_4 + (1 + \frac{2}{n-1}) k_{22};
\]

the coefficient \( \frac{2}{n-1} \) is calculated from the pattern

\[
\begin{array}{ccc|c}
1 & 1 & 2 \\
1 & 1 & 2 \\
2 & 2 & \\
\end{array}
\]

In simple cases the methods do not differ greatly; in more complex instances Wishart's method demands that many more patterns be written and their coefficients evaluated
and this is rarely a trivial task. The method given here often allows many fewer patterns but does involve more algebra. For comparative purposes, $k_{223}$ will be found from Wishart's procedure.

Let $k_{2}^{2}k_{3} = ak_{7} + bk_{43} + ck_{52} + dk_{223}$, this being analogous to the expression of $\kappa(2^{2}3)$ in terms of cumulants. As before, $a = 1/n^2$, there being no new admissible patterns.

For $b$, the pattern

```
2 2 - 4
- - 3 3
```

with coefficient $1/n$ must be added to that of (a) and (b) given previously, so

$$b = \frac{1}{n} + \frac{2n^2 - 36}{n(n-1)^2} = \frac{n^2 + 2n - 35}{n(n-1)^2}.$$

The pattern

```
- 2 - 2
2 - 3 5
```

has coefficient $1/n$ and since the first two columns may be interchanged, $2/n$ must be added to $16/n(n-1)$ to give $c$.

So $c = \frac{2(n+7)}{n(n-1)}$. The patterns

```
2 - - 2
- 2 - 2
- - 3 3
2 2 3
```

```
2 - - 2
2 - - 2
- - 3 3
2 2 3
```

```
2 - - 2
- - 3 3
2 2 3
```
must be used in addition to (e) and (f) to calculate $d$. These patterns consist of the permutations of three patterns but they are all shown here for convenience. The total contribution will be found to be $1 + 14/n-1$, so

$$d = 1 + \frac{14}{n-1} + \frac{48}{(n-1)^2} = \frac{(n+5)(n+7)}{(n-1)^2}.$$  

All these results may be combined to give $k_{223}$ as before.

Wishart does not offer a proof of this procedure but Kendall (1952) does, establishing it in a manner similar to that used in his establishment of Fisher's rules, that is, by symbolic processes. Wishart also applies Fisher's rule about unit parts in this fashion: $k_{p,1}$ can be found by adding 1 in all possible ways to $P$ (a partition of $p$) and dividing by $n$ and attaching 1 without dividing by $n$. For example, since

$$k_2^2 = \frac{1}{n} k_4 + \frac{n+1}{n-1} k_{22},$$

it follows that
\[ k^2 k_1 = \frac{1}{n} k_5 + \frac{1}{n} k_{41} + \frac{2(n+1)}{n-1} k_{32} + \frac{n+1}{n-1} k_{221}. \]

**D. Proofs**

The procedure presented in Section B will now be shown in more general terms. Suppose \( k_{a b c \ldots} \) is wanted. The notation \( k_{a b c \ldots} \) means a \( k \) statistic with a subscript containing \( a \) \( p \)'s, \( b \) \( q \)'s, \( c \) \( r \)'s, and so on. The generating relation is

\[
1 + \mu(p)t_p + \mu(p^2)\frac{t_p^2}{2!} + \ldots + \mu(q)t_q + \ldots + \mu(pq)t_p t_q \ldots \\
+ \frac{\mu(p^a q^b r^c \ldots) t_p^a t_q^b t_r^c \ldots}{\alpha \beta \gamma \ldots} + \ldots = \\
= \exp\{\kappa(p)t_p + \kappa(p^2)\frac{t_p^2}{2!} + \ldots + \frac{\kappa(p^a q^b r^c \ldots) t_p^a t_q^b t_r^c \ldots}{\alpha \beta \gamma \ldots} + \ldots\} \\
= 1 + \{\kappa(p)t_p + \frac{\kappa(p^2)t_p^2}{2!} + \ldots\} \\
+ \frac{1}{2!} \{\kappa(p)t_p + \frac{\kappa(p^2)t_p^2}{2!} + \ldots\}^2 + \ldots (7)
\]

Upon equating the coefficients of \( t_p^a t_q^b t_r^c \ldots \) on both sides, it follows that

\[
\mu(p^a q^b r^c \ldots) = \sum_{m=1}^{z} \pi_1 \prod_{L=1}^{P} \kappa(p^L q^L r^L \ldots) \prod_{m=1}^{P} \kappa(p^m q^m r^m \ldots) \prod_{m}^{2} \frac{p_m q_m r_m}{\kappa(p^m q^m r^m \ldots)}\]
\[ x \frac{1}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_m!)^{\pi_m}} \frac{1}{(q_1!)^{\pi_1} \cdots (q_m!)^{\pi_m}} \ldots \]

\[ \times \frac{a!b!c! \ldots}{\pi_1! \cdots \pi_m!} \]

where the summation is over all partitions of
\[ z = p_1 a + p_2 b + p_3 c + \cdots \] where \( \sum_{i=1}^{m} p_i \pi_i = a, \sum_{i=1}^{m} q_i \pi_i = b, \ldots \)

The term on the right in \([\kappa(p)]^a [\kappa(q)]^b [\kappa(r)]^c \ldots\) has coefficient \( \frac{a!b!c! \ldots}{a!b!c! \ldots} = 1 \) so writing \( \Sigma^o \) to denote the summation without this partition, (8) becomes

\[ \kappa_{p}^{a} \kappa_{q}^{b} \kappa_{r}^{c} \ldots = \mu(p^{a} q^{b} r^{c} \ldots) \]

\[ x \frac{z}{\Sigma^o} [\kappa(p_1 q_1 r_1 \ldots)]^{\pi_1} \cdots [\kappa(p_m q_m r_m \ldots)]^{\pi_m} \]

\[ \times \frac{1}{(p_1!)^{\pi_1} \cdots (p_m!)^{\pi_m}} \frac{1}{(q_1!)^{\pi_1} \cdots (q_m!)^{\pi_m}} \ldots \]

\[ \frac{a!b!c! \ldots}{\pi_1! \cdots \pi_m!} \]

Now \( \mu(p^{a} q^{b} r^{c} \ldots) = E(k_{p}^{a} k_{q}^{b} k_{r}^{c} \ldots) \) and also each factor

\[ [\kappa(p_1^{i} q_1^{i} r_1^{i} \ldots)]^{\pi_i} \] may be expanded by Fisher's rules in cumulants and powers of cumulants of total weight

\( (p q_i + q q_i + \ldots) \pi_i \). Finally, since the polykays have expected values which are products of cumulants, the right side of (9) may be written as the expectation of a quantity.
which is \( k_{p_{a}q_{b}r_{c}...} \). This completes the general justification of the method.

Since Fisher's rules are applied to \( \kappa(p_{i}q_{i}... \), only the minimum number of patterns must be found, although the algebraic simplification remains.

Polykays involving unit parts can be found from those not involving unit parts in a particularly simple way. In order to find \( \kappa(p_{a}q_{b}r_{c}...1) \), it is noted that the pattern

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\hline
p_{a}q_{b}r_{c}...1
\end{array}
\]

is used, where the unit entry in the last column can occur in any row. Moreover the pattern function for any such pattern is \( \frac{1}{n} \) times the pattern function for \( \kappa(p_{a}q_{b}r_{c}... \). It follows that \( \kappa(p_{a}q_{b}r_{c}...1) \) will involve each of the terms in \( \kappa(p_{a}q_{b}r_{c}...) \) but with 1 added in all possible ways (corresponding to its appearance in each possible column) to the subscripts in the cumulants \( \kappa(p_{a}q_{b}r_{c}...) \) with each term divided by \( n \).

The polykay \( k_{p_{a}q_{b}r_{c}...1} \) can be found from
\[ \mu(p^a q^b r^c \ldots l) = \kappa_p^{a \ k_1^{b \ k_r^{c \ \ldots \ k_l}} + \sum_{m=1}^{z+1} \Sigma^o [\kappa(p^m q^m \ldots l) \ \pi_m^m \times \frac{l}{(p^m_1 \ldots p^m_n)! \ldots \pi_m^m)} \times \ldots \times \frac{\alpha_1 \beta_1 \ldots}{\pi_l^l \ldots \pi_m^m}. \]

The summation is now over all partitions of \( pa + qb + rc + \ldots + l \), so for each such partition some one of the factors will be \( \kappa(p^i q^i \ldots l) \), say. As has been shown, this can be formed from \( \kappa(p^i q^i \ldots l) \) by use of the unit rule. Now the additional unit part can be inserted in any of the factors and produces a different partition for each such addition. When the result is expanded in terms of powers and products of single subscript \( \kappa \)'s the result differs from \( \kappa^a_{p^b q^c} \ldots \) by the addition of a \( l \) in all possible ways to the subscripts, each term being divided by \( n \), in addition to terms representing partitions of \( pa + qb + \ldots \) multiplied by \( \kappa(l) \), and the leading term is \( \kappa^a_{p^b q^c} \ldots k_1 \). The products of cumulants are now written as expectations of polykays and an expression for \( k^a_{p^b q^c} \ldots l \) is obtained.

The unit rule can be given then for the product of a \( k \) statistic and \( k_1 \): \( k^a_{p^b q^c} \) is formed from the expression for \( k^a_p \) in terms of polykays by adding \( l \) in all possible ways to each part of each subscript and dividing by \( n \) and attaching \( l \)
to each subscript without dividing by $n$.

E. Extensions

Some new general formulas for polykays can also be derived. In particular, formulas for $k_{pq}$ with $p$ equal to 2 or 3 will be shown and some specific new explicit results for various values of $q$ given.

The polykay

$$k_{pq} = k_p k_q - \frac{1}{n} k_{p+q} - \Sigma C(r,s,...)k_{rs}...$$

where the summation is over all partitions of $p,q$ with non-zero pattern functions and $C(r,s...)$ is the appropriate coefficient. All partitions containing unit parts may be ignored since they have zero coefficients.

In particular if $p = 2$, $k_{2q}$ can be written

$$k_{2q} = k_2 k_q - \frac{1}{n} k_{2+q} - \Sigma C(r,s)k_{rs}$$

since the partitions of $2,q$ having non-zero pattern functions can have at most two parts. One of these is $k_{2q}$ with coefficient $\frac{2q}{n-1}$ determined from the pattern

$$\begin{array}{c|c|c}
1 & 1 & 2 \\
1 & q-1 & q \\
2 & q & \\
\end{array}$$

Furthermore, any other $C(r,s)$ is determined from the pattern...
where \( r, s \neq 2 \), which has numerical coefficient \( \frac{2q!}{(r-1)!(s-1)!} \) if \( r \neq s \) and numerical coefficient \( \frac{q!}{(r-1)!(s-1)!} \) if \( r = s \), unless \( q = 2 \). If \( q = 2 \), the coefficient is 2. Since \( r + s - 2 = q \), and since the pattern function is \( \frac{1}{n-1} \), it follows that

\[
k_{2q} = k_2 k_q - \frac{1}{n} k_{2+q} - \frac{2q}{n-1} k_{2q} - \frac{(21)}{n-1} \delta(r,s) (q-1) k_{rs}
\]

where the summation is under the previous conditions and

\[
\delta(r,s) = \begin{cases} 
1 & \text{if } r \neq s \\
0 & \text{otherwise}
\end{cases}
\]

and \( r, s, q \neq 2 \).

For example,

\[
k_{23} = \frac{n-1}{n+5} \{ k_2 k_3 - \frac{1}{n} k_5 \}
\]

since there are no partitions of 2,3 satisfying the conditions.

Also

\[
k_{26} = \frac{n-1}{n+11} \{ k_2 k_6 - \frac{1}{n} k_8 - \frac{30}{n-1} k_{53} - \frac{20}{n-1} k_{44} \}
\]

from (10). Both of these formulas, as well as \( k_{24} \) and \( k_{25} \) which follow similarly, are given by Wishart. Since no formulas for weight greater than 8 are available, a few more
results are given here. One could go beyond weight 12 but no expressions for $k_r$, $r>12$, are presently known. The new formulas are:

\[
\begin{align*}
  k_{27} &= \frac{n-1}{n+13} \{ k_2 k_7 - \frac{1}{n} k_9 - \frac{42}{n-1} k_{36} - \frac{70}{n-1} k_{45} \}, \\
  k_{28} &= \frac{n-1}{n+15} \{ k_2 k_8 - \frac{1}{n} k_{10} - \frac{56}{n-1} k_{37} - \frac{112}{n-1} k_{46} - \frac{70}{n-1} k_{55} \}, \\
  k_{29} &= \frac{n-1}{n+17} \{ k_2 k_9 - \frac{1}{n} k_{11} - \frac{72}{n-1} k_{38} - \frac{168}{n-1} k_{47} - \frac{252}{n-1} k_{56} \},
\end{align*}
\]

and

\[
\begin{align*}
  k_{2,10} &= \frac{n-1}{n+19} \{ k_2 k_{10} - \frac{1}{n} k_{12} - \frac{90}{n-1} k_{39} - \frac{240}{n-1} k_{48} \\
  &\quad - \frac{252}{n-1} k_{66} - \frac{420}{n-1} k_{57} \}
\end{align*}
\]

From the generating relation (5),

\[
\mu(3r) = \kappa(3r) + \kappa(3)\kappa(r)
\]

and so

\[
\kappa_3^r = E(k_3 k_r) - \left\{ \frac{1}{n} \kappa_3^r + \sum_{s+t=r+3} \frac{3}{n-1} \kappa^s \kappa^t \right\}
\]

\[
+ \sum_{m+n+p=r+3} \frac{(3!) \delta(m,n,p)}{(n-1)(n-2)} \left( \kappa^r - \frac{1}{2^{(m,n,p)}} \kappa^m \kappa^n \kappa^p \right)
\]

where \( \delta(m,n,p) = \begin{cases} 0 & \text{if } m=n=p \\ 1 & \text{otherwise} \end{cases} \)
\[
\lambda(m,n,p) = \begin{cases} 
1 & \text{if exactly two of } m,n,p \text{ are equal}, \\
0 & \text{otherwise}
\end{cases}
\]

\[s \neq 2; \ s,t,r \neq 3, \text{ and } \binom{r}{m-1,n-1,p-1} = \frac{r!}{(m-1)!(n-1)!(p-1)!},\]

so

\[
k_{3r} = \frac{2(n-1)}{3r^2 + 3r + 2n - 2} \left\{ k_{3r} - \frac{1}{n} k_{3r+1} - \sum_{s+t=r+3} \frac{3}{n-1} \binom{r}{s-2} k_{st} \right\}
\]

\[
- \sum_{s+t=r+3} \frac{(3!)^2 \delta(m,n,p)_n}{(n-1)(n-2)} \binom{r}{m-1,n-1,p-1} \frac{1}{2\lambda(m,n,p)} k_{mnp} \}
\]

(12)

where \(s \neq 2\) and \(r \neq 3\).

In (11) the coefficient of \(\kappa_s \kappa_t\) comes from the pattern

\[
\begin{array}{c|c|c}
s \hline
2 & s-2 & s \\
1 & t-1 & t \\
3 & r & \\
\end{array}
\]

In applying the formula terms in \(\kappa_s \kappa_t\) and \(\kappa_t \kappa_s\) are calculated separately, where \(s \neq t\). No unit parts are involved on the right side of (11). The pattern

\[
\begin{array}{c|c|c|c}
m-1 \hline
1 & m \\
n-1 \hline
1 & n \\
p-1 \hline
1 & p \\
3 & r \\
\end{array}
\]

gives the terms in \(\kappa_m \kappa_n \kappa_p\). The leading coefficient in (12) is found by considering the patterns

\[
\begin{array}{c|c|c|c}
2 \hline
1 & 3 \\
1 & r-1 & r \\
3 & r & \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
2 \hline
r-2 & r \\
1 & 2 & 3 \\
3 \hline
r & \\
\end{array}
\]
with pattern functions \( \frac{3r}{n-1} \) and \( \frac{3r(r-1)}{2(n-1)} \). Writing these terms on the left gives 1 + \( \frac{3r}{n-1} + \frac{3r(r-1)}{2(n-1)} = \frac{3r^2 + 3r + 2n - 2}{2(n-1)} \) as the coefficient of \( k_{3r} \).

Letting \( r = 3 \), it follows that

\[
k_{33} = \frac{n-1}{n+8} \{k_3^2 - \frac{1}{n} k_6 - \frac{9}{n-1} k_{42} - \frac{6n}{(n-1)(n-2)} k_{222}\}.
\]

Now applying (12), several new formulas can be obtained.

\[
k_{34} = \frac{n-1}{n+29} \{k_3 k_4 - \frac{1}{n} k_7 - \frac{3}{n-1} (\frac{4}{3}) k_{52} - \frac{6n}{(n-1)(n-2)} (1,1,2) \frac{1}{2} k_{223}\}
\]

or

\[
k_{34} = \frac{n-1}{n+29} \{k_3 k_4 - \frac{1}{n} k_7 - \frac{12}{n-1} k_{52} - \frac{36n}{(n-1)(n-2)} k_{223}\};
\]

\[
k_{35} = \frac{2(n-1)}{88+2n} \{k_3 k_5 - \frac{1}{n} k_8 - \frac{3}{n-1} (\frac{5}{2}) k_{44} - \frac{3}{n-1} (\frac{5}{4}) k_{62} - \frac{6n}{(n-1)(n-2)} (2,2,1) \frac{1}{2} k_{332}\}
\]

or

\[
k_{35} = \frac{n-1}{n+44} \{k_3 k_5 - \frac{1}{n} k_8 - \frac{30}{n-1} k_{44} - \frac{15}{n-1} k_{62} - \frac{90n}{(n-1)(n-2)} k_{332}\};
\]
\[ k_{36} = \frac{2(n-1)}{124+2n} \left( k_3 k_6 - \frac{1}{n} k_9 - \frac{3}{n-1} (6) k_{72} - \frac{3}{n-1} (6) k_{45} \right) \]

\[ \quad - \frac{n}{(n-1)(n-2)} (2, 2, 2) k_{333} - \frac{6n}{(n-1)(n-2)} (1, 2, 3) k_{234} \]

\[ \quad - \frac{6n}{(n-1)(n-2)} (1, 1, 4) \left( \frac{1}{2} \right) k_{225} \]

or

\[ k_{36} = \frac{n-1}{n+62} \left( k_3 k_6 - \frac{1}{n} k_9 - \frac{18}{n-1} k_{72} - \frac{45}{n-1} k_{45} \right) \]

\[ \quad - \frac{90n}{(n-1)(n-2)} k_{333} - \frac{360n}{(n-1)(n-2)} k_{234} \]

\[ \quad - \frac{90n}{(n-1)(n-2)} k_{225} \] ;

\[ k_{37} = \frac{2(n-1)}{166+2n} \left( k_3 k_7 - \frac{1}{n} k_{10} - \frac{3}{n-1} (7) k_{55} - \frac{3}{n-1} (7) k_{82} \right) \]

\[ \quad - \frac{3}{n-1} (2) k_{46} - \frac{3}{n-1} (4) k_{64} - \frac{6n}{(n-1)(n-2)} (1, 1, 5) \left( \frac{1}{2} \right) k_{226} \]

\[ \quad - \frac{6n}{(n-1)(n-2)} (1, 3, 3) \left( \frac{1}{2} \right) k_{244} \]

\[ \quad - \frac{6n}{(n-1)(n-2)} (1, 4, 2) k_{253} - \frac{6n}{(n-1)(n-2)} (2, 3, 2) \left( \frac{1}{2} \right) k_{343} \]

or
\[ k_{37} = \frac{n-1}{n+83} \left\{ k_3 k_7 - \frac{1}{n} k_{10} - \frac{105}{n-1} k_{55} - \frac{21}{n-1} k_{82} - \frac{168}{n-1} k_{46} \right\} \]

\[ - \frac{126n}{(n-1)(n-2)} k_{226} - \frac{420n}{(n-1)(n-2)} k_{244} - \frac{630n}{(n-1)(n-2)} k_{253} \]

\[ - \frac{630n}{(n-1)(n-2)} k_{343} \}; \]

\[ k_{38} = \frac{2(n-1)}{214+2n} \left\{ k_3 k_8 - \frac{1}{n} k_{11} - \frac{3}{n-1} \binom{8}{4} k_{65} - \frac{3}{n-1} \binom{8}{3} k_{56} \right\} \]

\[ - \frac{3}{n-1} \binom{8}{5} k_{74} - \frac{3}{n-1} \binom{8}{2} k_{47} - \frac{3}{n-1} \binom{8}{7} k_{92} \]

\[ - \frac{6n}{(n-1)(n-2)} \left( \binom{8}{2,2,4} \frac{1}{2} k_{335} - \frac{6n}{(n-1)(n-2)} \binom{8}{2,3,3} \frac{1}{2} k_{344} \right) \]

\[ - \frac{6n}{(n-1)(n-2)} \left( \binom{8}{1,1,6} \frac{1}{2} k_{227} - \frac{6n}{(n-1)(n-2)} \binom{8}{1,3,4} k_{245} \right) \]

\[ - \frac{6n}{(n-1)(n-2)} \binom{8}{1,5,2} k_{263} \}; \]

so

\[ k_{38} = \frac{n-1}{n+107} \left\{ k_3 k_8 - \frac{1}{n} k_{11} - \frac{378}{n-1} k_{56} - \frac{252}{n-1} k_{47} - \frac{24}{n-1} k_{92} \right\} \]

\[ - \frac{1260n}{(n-1)(n-2)} k_{335} - \frac{1680n}{(n-1)(n-2)} k_{344} - \frac{168n}{(n-1)(n-2)} k_{227} \]

\[ - \frac{1680n}{(n-1)(n-2)} k_{245} - \frac{1008n}{(n-1)(n-2)} k_{263} \}; \]
\[
\begin{align*}
k_{39} &= \frac{2(n-1)}{268+2n} \left\{ k_3 k_9 - \frac{1}{n} k_{12} - \frac{3}{n-1} (9) k_{10,2} - \frac{3}{n-1} (7) k_{93} \\
&\quad - \frac{3}{n-1} (9) k_{39} - \frac{3}{n-1} (9) k_{84} - \frac{3}{n-1} (9) k_{48} - \frac{3}{n-1} (9) k_{75} \\
&\quad - \frac{3}{n-1} (9) k_{57} - \frac{3}{n-1} (9) k_{66} - \frac{6n}{(n-1)(n-2)} (9) \frac{k_{255}}{2} \\
&\quad - \frac{6n}{(n-1)(n-2)} (9) k_{246} - \frac{6n}{(n-1)(n-2)} (9) k_{237} \\
&\quad - \frac{6n}{(n-1)(n-2)} (9) k_{228} - \frac{6n}{(n-1)(n-2)} (9) k_{354} \\
&\quad - \frac{6n}{(n-1)(n-2)} (9) k_{363} - \frac{n}{(n-1)(n-2)} (9) k_{444} \right\},
\end{align*}
\]

so

\[
k_{39} = \frac{n-1}{n+134} \left\{ k_3 k_9 - \frac{1}{n} k_{12} - \frac{27}{n-1} k_{10,2} - \frac{135}{n-1} k_{93} - \frac{360}{n-1} k_{84} \\
&\quad - \frac{630}{n-1} k_{75} - \frac{126}{n-1} k_{66} - \frac{3024n}{(n-1)(n-2)} k_{246} \\
&\quad - \frac{1512n}{(n-1)(n-2)} k_{237} - \frac{216n}{(n-1)(n-2)} k_{228} - \frac{7560n}{(n-1)(n-2)} k_{354} \\
&\quad - \frac{2268n}{(n-1)(n-2)} k_{363} - \frac{1680n}{(n-1)(n-2)} k_{444} \right\}.
\]
F. Randomized Sums

In 1950 Tukey showed some interesting properties of the polykays which he later exploited in studying variance components from certain structured populations. His proofs, involving symbolic multiplications of the single \( k \) statistics and the properties of augmented symmetric functions, do not make use of the facts that the polykays are related to the population cumulants by the relations

\[
E(k_p) = \kappa_p
\]

and

\[
E(k_{pqr...}) = \kappa_p \kappa_q \kappa_r ...
\]

and thus do not capitalize on the essential properties of the polykays. Proofs which expose the role of the cumulants will be given here, thus linking the cumulants with the surprising properties of randomized sums.

If, in infinite populations, \( x \) has a distribution with cumulants \( \kappa(x) \) and \( y \) has a distribution with cumulants \( \kappa(y) \) then

\[
1 + \mu_1(x)t + \mu_2(x)\frac{t^2}{2!} + \ldots = \exp\{\kappa_1(x)t + \kappa_2(x)\frac{t^2}{2!} + \ldots\}
\]

or \( M(x;t) = \exp\{\kappa_1(x)t + \kappa_2(x)\frac{t^2}{2!} + \ldots\} \) where \( M(x;t) \) is the moment generating function of \( x \) and

\[
M(y;t) = \exp\{\kappa_1(y)t + \kappa_2(y)\frac{t^2}{2!} + \ldots\} \]
M(x+y;t) = \exp\{\kappa_1(x+y)t + \kappa_2(x+y)\frac{t^2}{2!} + \ldots\} \quad \text{where}
\kappa(x+y) \text{ refers to the cumulants of the distribution of the variable } x+y. \text{ Now if } x \text{ and } y \text{ are independent,}
M(x+y;t) = M(x;t)M(y;t)
so that
\log M(x+y;t) = \log M(x;t) + \log M(y;t).

Upon equating like powers of t, it follows that
\kappa_r(x+y) = \kappa_r(x) + \kappa_r(y).

Now consider two finite populations, \(x_1, x_2, \ldots, x_N\) and \(y_1, y_2, \ldots, y_N\). For each of the permuted orders of the \(y\) population a set of randomized sums is found by finding the sums \(x_i + y_{\pi(i)}, i = 1, 2, \ldots, N\) where \(\pi(i)\) is the \(y\) population subscript occurring at the \(i\)th position in the permuted order considered.

Let \(k_p\) and \(k_{pq}\) denote the \(k\) statistics calculated from the elements of a particular set of sums. Each of the populations, \(x's\), \(y's\), and \(sums\), can now be considered as samples from infinite populations, where, with respect to the population of sums, \(x\) and \(y\) are regarded as independent since they are independent in the finite population of sums.

Then \(E(k_p) = \kappa_p(x+y)\) since the samples of \(x's\) and
y's and hence sums can be taken as random samples from the infinite populations. But since x and y are independent,

\[ \kappa_p(x+y) = \kappa_p(x) + \kappa_p(y) \]

or

\[ E(k_p) = \kappa_p(x) + \kappa_p(y) . \]

The definition of aver \( k_p \) and the Irwin-Kendall result then give

\[ E_N(k_p) = \text{aver}(k_p) = K_p(x) + K_p(y) . \]

This pairing formula for \( k_p \) was first given by Tukey who gave a proof different from that just presented.

It may also be noted that if \( K_p(x+y) \) denotes the \( p \)th \( k \) statistic calculated from the finite population of sums then

\[ E_N(k_p) = \text{aver}(k_p) = K_p(x+y), \]

also by the Irwin-Kendall results.

The other result of Tukey's of particular interest concerns the polykay \( k_{pq} \).

\[ E(k_{pq}) = \kappa_p(x+y)\kappa_q(x+y) \]

but since x and y are independent,

\[ E(k_{pq}) = [\kappa_p(x) + \kappa_p(y)][\kappa_q(x) + \kappa_q(y)] \]

\[ = \kappa_p(x)\kappa_q(x) + \kappa_p(y)\kappa_q(x) + \kappa_p(x)\kappa_q(y) + \kappa_p(y)\kappa_q(y) \]
Then, by the Irwin-Kendall result as well as the independence of $x$ and $y$,

$$\text{aver}(k_{pq}) = E_N(k_{pq}) = K_{pq}(x) + K_p(x)K_q(y) + K_p(y)K_q(x) + K_{pq}(y).$$

This result can be extended easily by this procedure to polykays of higher order. For instance,

$$\text{aver}(k_{pqr}) = K_{pqr}(x) + K_{pq}(x)K_r(y) + K_{pr}(x)K_q(y) + K_{qr}(x)K_p(y)$$

$$+ K_p(x)K_{qr}(y) + K_q(x)K_{pr}(y) + K_r(x)K_{pq}(y) + K_{pqr}(y).$$

It is possible to extend the results beyond Tukey's to selections from more than two sets. If $x_1, x_2, x_3, \ldots$ now represent finite sets of variables, each of size $N$, randomized sums can be formed by constructing all possible sums with one summand from each population. $k_p$ and $k_{pq}$ are calculated as before. Then

$$E(k_p) = E[K_p(x_1 + x_2 + x_3 + \ldots)]$$

$$= \kappa_p(x_1) + \kappa_p(x_2) + \kappa_p(x_3) + \ldots$$

since

$$\kappa_p(x_1 + x_2 + x_3 + \ldots) = \kappa_p(x_1) + \kappa_p(x_2) + \kappa_p(x_3) + \ldots$$

where $x_i$ in the above equation is now some individual value, the result following from the independence of these. So

$$E_N(k_p) = \text{aver}(k_p) = K_p(x_1) + K_p(x_2) + K_p(x_3) + \ldots$$
The corresponding result for \( k_{pq} \) is somewhat more complicated. For infinite populations,

\[
\kappa_p(x_1 + x_2 + x_3 + ...) \kappa_q(x_1 + x_2 + x_3 + ...) = \left[ \kappa_p(x_1) + \kappa_p(x_2) + \kappa_p(x_3) + ... \right] \left[ \kappa_p(x_1) + \kappa_p(x_2) + \kappa_q(x_3) + ... \right]
\]

\[
= \sum \sum \kappa_p(x_i) \kappa_q(x_j).
\]

As a consequence,

\[
\text{aver}(k_{pq}) = \sum_k p_{pq}(x_i) + \sum_{i,j} k_p(x_i) k_q(x_j).
\]

Wishart's results concerning products of \( k \) statistics have been extended by Dwyer and Tracy (1964) and Tracy (1968, 1969) who consider products of polykays. Dwyer and Tracy (1964) give rules for forming \( k_P k_Q \) where \( P \) and \( Q \) are partitions of \( p \) and \( q \), which are generalizations of those given by Fisher and used by Wishart for finding \( k_P k_Q \). Combinatorial proofs are given.

Tables of results for \( k_P k_Q \) are given for \( Q \) of weight 2, 3, or 4 (Dwyer and Tracy, 1964) and 5 (Tracy, 1969).

Tracy (1968) has extended these results to products of more than two polykays and has given rules for the calculation of pattern functions and coefficients in a manner similar to Fisher's.
III. CHAPTER THREE: MULTIVARIATE \( k \) STATISTICS

A. Introduction

The literature of the \( k \) statistics has been primarily concerned with the univariate case. Fisher (1929), however, noted that his ideas could be extended to the multivariate case and worked out expressions for four bivariate \( k \) statistics in terms of bivariate sums. Kendall (1940c) attempted to obtain multivariate formulas from the univariate ones by symbolic operations. His methods however were not entirely clear and he did not publish new multivariate results from them. Cook (1951a,b), using some of Kendall's procedure, obtained formulas for bivariate cumulants and means in terms of each other through weight 6 as well as various formulas for the cumulants of the simultaneous distribution of two or more bivariate \( k \) statistics, up to weight 9. Her formulas are not complete through this weight and she did not provide formulas for bivariate \( k \) statistics. Applications of bivariate cumulants and symmetric sums by Cook and Robson have been mentioned in Section E of Chapter One.

In this chapter the multivariate \( k \) statistics are defined and a complete set of formulas for them through weight 5 is given in terms of multivariate symmetric sums. A symbolic method is developed for accomplishing this and the results are checked. Formulas are given through weight 4 for expressing bivariate bracket functions in terms of multi-
Bivariate polykays are defined and a method for their formulation based on Fisher's pattern functions is developed as well as a symbolic method based on bivariate bracket functions. A complete set of formulas for bivariate polykays through weight 8 is given.

With the exception of four formulas given by Fisher (1929) all the results are new. All have been carefully checked.

B. Basic Results

The function $\sum_{i_1, \ldots, i_v} p_{i_1} y_{i_1} \cdots p_{i_v} y_{i_v}$ in the variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, where the summation is understood to extend over all the terms of the form indicated with the restriction that unequal subscripts in the form must remain unequal in the sum, is called a bivariate symmetric function. If the pairs $p_{1q_1}, \ldots, p_{\lambda q_\lambda}$ are all different and $v = \Sigma q_i$ then the sum is often written $[(p_1 q_1)^{\pi_1} \cdots (p_\lambda q_\lambda)^{\pi_\lambda}]$. $\Sigma_{i_1} p_{y_{i_1}}$ will be denoted by $s_{p, q}$. Evidently $s_{p, q} = [(p, q)]$.

For a bivariate distribution, cumulants are defined by
Consequently

\[ \kappa_{r,s} = \frac{\varepsilon(-1)^{\rho-1} (\rho-1)!}{\pi_1! \pi_2! \ldots} \frac{\pi_1!}{(p_1!)} \frac{\pi_2!}{(p_2!)} \ldots \frac{r!}{(q_1!)} \frac{s!}{(q_2!)} \]

where \( (p_1 q_1) \pi_1 \pi_2 \ldots \) is a partition of the bipartite number \( r,s \), that is, \( \sum_i p_i = r \), \( \sum_i q_i = s \), \( \sum_i \pi_i = \rho \), and the summation is over all such partitions.

If \( (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \) represents a sample from a bivariate population a statistic \( k_{r,s} \) where \( E(k_{r,s}) = \kappa_{r,s} \) is

\[ k_{r,s} = \varepsilon(-1)^{\rho-1} (\rho-1)! \pi_1! \pi_2! \ldots \frac{r!}{(p_1! q_1!)} \frac{s!}{(p_2! q_2!)} \ldots \frac{\pi_1! \pi_2!}{(1 \pi_1 \pi_2 \ldots)} \]

where

\[ <(p_1 q_1) (p_1 q_1) \ldots> = \frac{(p_1 q_1) \pi_1 (p_2 q_2) \pi_2 \ldots}{n(\rho)}, \]

and the summation is over all partitions of \( r,s \).

Fisher (1929) gives a few values of \( k_{r,s} \), namely
The symbolic operator $\Delta$ is the basis for the procedure. $\Delta$ is defined as acting on a $k$ statistic, a product of $k$ statistics, a symmetric sum, a product of symmetric sums, a mean, or product of means, in the following ways.

The action on a single term will be described first. The function (k statistic, mean, etc.) is represented symbolically. As an example, the function $k_a$ is expressed as $k(r^a)$ and $s_a$ is expressed as $s(r^a)$. The symbol $t$ is inserted, the expression in parentheses differentiated with respect to $r$, constants being brought outside, and the result written as a bivariate function. Thus

$$\Delta k_a = \Delta k(r^a) = \frac{\partial}{\partial r} k(r^a t) = k[\frac{\partial}{\partial r}(r^a t)] = ak(r^{a-1} t)$$

$$= ak_{a-1,1}.$$

Often the operator $\Delta$ will be written as $t^\frac{\partial^*}{\partial r}$, the notation serving as a warning that ordinary differentiation is not involved since the action of $\Delta$ on products is more complex. The product is represented as the sum of terms, each term being a product into which the letter $t$ has been inserted in exactly one factor. Thus there is a term in the sum for each factor in the product. Now the "differentiation" is done only on the factors containing $t$ and finally the
result is rewritten. For example,

\[ \Delta(s_3 s_2) = \frac{3^*}{\partial r} \{ s(r^2 t) s(r^2) + s(r^3) s(r^2 t) \} \]

\[ = s \left[ \frac{3}{\partial r} (r^2 t) \right] s(r^2) + s(r^3) s \left[ \frac{3}{\partial r} (r^2 t) \right] \]

\[ = 3s_{2,1} s_{2,0} + 2s_{3,0} s_{1,1} , \]

where

\[ s_r = \sum_i r^i . \]

The process may be continued. Thus if \( \Delta^2 \) denotes \( \Delta(\Delta) \),

\[ \Delta^2(s_3 s_2) = \Delta\{ 3s_{2,1} s_{2,0} + 2s_{3,0} s_{1,1} \} = \frac{3^*}{\partial r} \{ 3s(r^2 t^2) s(r^2) \]

\[ + 3s(r^2 t) s(r^2 t) + 2s(r^3) s(rt) + 2s(r^3) s(r^2 t) \} \]

\[ = 3s \left[ \frac{3}{\partial r} (r^2 t^2) \right] s(r^2) + 3s(r^2 t) s \left[ \frac{3}{\partial r} (r^2 t) \right] \]

\[ + 2s \left[ \frac{3}{\partial r} (r^3 t) \right] s(rt) + 2s(r^3) s \left[ \frac{3}{\partial r} (rt^2) \right] \]

\[ = 6s_{1,2} s_{2,0} + 12s_{2,1} s_{1,1} + 2s_{3,0} s_{0,2} . \]

It is important to note that the operator \( \frac{3^*}{\partial r} \) acts only on the terms receiving a new factor of \( t \).

It can also be noted that \( \Delta^2 \) and \( \frac{t^2 3^*^2}{\partial r^2} \), for example, are not equivalent. Kendall (1940c) used operators denoted by \( \Delta_p \) where \( p \) is an integer; these operators however are illustrated only by examples, two of which, \( \Delta_2(\mu^3 u_1) \) and
$\Delta_2(\mu_2\mu_1^2)$, are neither consistent with each other nor are they equivalent to $\Delta^2(\mu_3\mu_1^2)$ or $\Delta^2(\mu_2\mu_1^2)$. Kendall states that, except for numerical factors, $\Delta_p\Delta_q = \Delta_{p+q}$. Possibly, under a suitable definition this is true, but $\Delta_2 \neq \Delta_1(\Delta_1)$ since the operators yield entirely different expressions, ignoring the numerical factors.

The importance of the operator here introduced lies in the fact that with its repeated use bivariate results can be derived from univariate ones. Consider, for example,

$$k_2 = \frac{1}{n-1}(s_2 - \frac{1}{n}s_1^2).$$

The application of $\Delta = \frac{d\beta^*}{dx}$ to both sides yields

$$k_{1,1} = \frac{1}{n-1}\{s_{1,1} - \frac{1}{n}s_{0,1}s_{1,0}\}.$$ 

Another application of $\Delta$ yields $k_{0,2} = \frac{1}{n-1}\{s_{0,2} - \frac{1}{n}s_{0,1}^2\}$. These results are otherwise known to be correct (Fisher, 1929).

The reasons for the procedure producing correct results will now be explored. In the following $\Delta^P$ denotes $\Delta(\Delta^{P-1})$ and the operator $\Delta$ now denotes the symbolic operation $\frac{d\beta^*}{dx}$. It will be necessary also to consider $\frac{u}{t} \Delta$. In using the symbol the expression following it is to be multiplied by $\frac{u}{t}$ and then the operator $\Delta$ applied. Finally, $\exp \frac{u}{t} \Delta$ denotes $1 + \frac{u}{t} \Delta + \frac{1}{2!}\left(\frac{u}{t}\right)^2 \Delta^2 + \ldots$. All the operators are used on
infinite sums sequentially as shown below. Now

\[
(\exp \frac{u}{t} \Delta)(1 + \mu_1^1 t + \mu_2^1 \frac{t^2}{2!} + \ldots) = \left(1 + \mu_1^1 \frac{t^2}{2!} + \mu_2^1 \frac{t^2}{2!} \Delta^2 + \ldots\right) \left(1 + \mu_1^0 t + \mu_2^0 \frac{t^2}{2!} + \ldots\right)
\]

\[
= \left(1 + \mu_{1,0}^1 t + \mu_{2,0}^1 \frac{t^2}{2!} + \ldots\right) + \Delta \left(\mu_1^0 u + \mu_2^0 \frac{tu}{2!} + \mu_3^0 \frac{t^2u}{3!} + \ldots\right)
\]

\[
+ \Delta^2 \left(\mu_2^1 \frac{u^2}{2!2!} + \mu_3^1 \frac{tu^2}{2!3!} + \mu_4^1 \frac{t^2u^2}{2!4!} + \ldots\right)
\]

\[
+ \Delta^3 \left(\mu_3^0 \frac{u^3}{3!3!} + \mu_4^1 \frac{u^3t}{3!4!} + \mu_5^1 \frac{u^3t^2}{3!5!} + \ldots\right)
\]

\[
= \left(1 + \mu_{1,0}^1 t + \mu_{2,0}^1 \frac{t^2}{2!} + \ldots\right) + \left(\mu_0^0,1 u + \mu_1^0,1 tu + \mu_2^0,1 \frac{t^2u}{2!} + \ldots\right)
\]

\[
+ \left(\mu_0^1,2 \frac{u^2}{2!} + \mu_1^1,2 \frac{tu^2}{1!2!} + \mu_2^1,2 \frac{t^2u^2}{2!2!} + \ldots\right) + \ldots
\]

so the result of operating on \(1 + \mu_1^1 t + \mu_2^1 \frac{t^2}{2!} + \ldots\) by \(\exp \frac{u}{t} \Delta\) is that all terms of the form

\[
\frac{\mu_p^1 q u^p}{p!q!}
\]

are produced. Thus

\[
(\exp \frac{u}{t} \Delta)(1 + \mu_1^1 t + \mu_2^1 \frac{t^2}{2!} + \ldots) = \left(1 + \mu_{1,0}^1 t + \mu_{0,1}^0,1 ut + \ldots + \mu_p^1 \frac{t^p u^q}{p!q!} + \ldots\right)
\]
Now
\[(\exp \frac{u}{t} \Delta)(\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots) = \kappa_{1,0} t + \kappa_{0,1} u + \ldots \]
\[+ \frac{\kappa_{p,q} u^p t^q}{p!q!} + \ldots, \tag{2}\]

in an exactly similar way.

The univariate generating relation is
\[1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \ldots = \exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\}.\]

Application of the operator \(\exp \frac{u}{t} \Delta\) to both sides then yields
\[(\exp \frac{u}{t} \Delta)(1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \ldots) = (\exp \frac{u}{t} \Delta)\exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\},\]
or
\[1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \ldots = (\exp \frac{u}{t} \Delta)\exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\}, \text{ by (1)}. \tag{3}\]

Now
\[\exp\{\exp \frac{u}{t} \Delta(\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots)\}\]
\[= \exp\{\kappa_{1,0} t + \kappa_{0,1} u + \kappa_{1,1} u + \ldots \}
\[+ \frac{\kappa_{p,q} u^p t^q}{p!q!} + \ldots\]
\[= 1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \ldots + \mu_1^t + \frac{u^p t^q}{p!q!} + \ldots \tag{4}\]

by (2) and the definition of the bivariate generating relation. It follows that
\[(\exp \frac{u}{t} \Delta) \exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\}\]

\[= \exp\{\exp \frac{u}{t} \Delta (\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots)\}\]

from (3) and (4).

It may be noted that operators may be devised to produce trivariate moments and cumulants from bivariate ones. For instance \(q \frac{\partial}{\partial x} \kappa(x^a t^b) = a^d \kappa_{a-1, b, 1} \).

Now (5) may be applied to the univariate generating relation to give

\[1 + \mu'_{1,0} t + \mu'_{0,1} u + \ldots + \frac{\mu'_{p,q}}{p! q!} t^p u^q + \ldots\]

\[= (\exp \frac{u}{t} \Delta) \exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots\}\]

\[= \exp\{\exp \frac{u}{t} \Delta (\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \ldots)\}\]

\[= \exp\{\kappa_{1,0} t + \kappa_{0,1} u + \ldots\}\]

which is the bivariate generating relation. In an analogous fashion the trivariate relation may be obtained from the bivariate relation and the quadrivariate from the trivariate, so that in general a hierarchy of operators will produce a general multivariate generating relation from the univariate one.

From Chapter One, an expression for \(\kappa_r\) is found in terms of symmetric sums, or power sums, by employing the definition, \(E(\kappa_r) = \kappa_r\), and the expression for \(\kappa_r\) in terms
of products of means, the latter formula arising from the univariate generating relation. Since \( k_r \) is unique, there is a one-to-one correspondence between expressions for the \( k_r \) in terms of power sums and those for \( \kappa_r \) in terms of products of means.

In a similar way, a correspondence exists between formulas for \( k_{r,s} \) and those for \( \kappa_{r,s} \). Now the bivariate generating relation is obtainable from the univariate one by the exhibited symbolic process. It follows that \( k_{r,s} \) is obtainable from \( k_p \), say, where \( r + s = p \), by the same process for if it were not then the uniqueness of \( k_{r,s} \) and \( k_p \) would be contradicted. In other words, if the symbolic process operating on \( k_p \) did not produce \( k_{r,s} \) then some relation for \( \kappa_{r,s} \) in terms of population means other than that obtainable from the generating relation would hold and that is impossible. It may be noted in deriving specific results that the operator \( \Delta \) alone, rather than \( \frac{u}{t} \Delta \), is used since only the coefficients of terms in the generating relation are needed. All the results to be given here are produced through successive application of the operator \( \Delta \).

In the preceding discussion, \( \Delta \) acted on a cumulant or mean alone. Its action on products, previously described, is a consequence of its action on single factors. The univariate generating relation may be written as

\[
1 + \mu_1 t + \mu_2 \frac{t^2}{21} + \ldots = 1 + \kappa_1 t + (\kappa_1^2 + \kappa_2) \frac{t^2}{21} + (\kappa_1^3 + 3\kappa_1 \kappa_2 + \kappa_3) \frac{t^3}{31} + \ldots
\]
Upon operating by $\frac{\partial}{\partial \tau} A$ on both sides and on equating coefficients, it can be seen for instance that $\Delta \mu_2 = \Delta (\kappa_1^2 + \kappa_2)$ and $\Delta \mu_3 = \Delta (\kappa_1^3 + 3\kappa_1 \kappa_2 + \kappa_3)$. But $\Delta \mu_2 = \mu_1,1$ and $\Delta \mu_3 = 3\mu_2,1$ so

$\Delta \kappa_1^2 = 2\kappa_1,0 \kappa_0,1$ from the first equation and $\Delta \kappa_1 \kappa_2 = \kappa_0,1 \kappa_2,0 + 2\kappa_1,0 \kappa_1,1$ from the second. These examples illustrate that the action of $A$ on products and powers is as was previously described.

The only multivariate $k$ statistic of weight 2 is $k_{1,1}$, a formula for which has been derived symbolically from $k_2$. Those of higher weights are more numerous. In the next three sections all the multivariate $k$ statistics of weights three, four and five will be derived.

C. Multivariate $k$ Statistics of Weight 3

From expression (1),

$$k_{2,1} = \langle (21) \rangle - 2\langle (11)(10) \rangle - \langle (20)(01) \rangle + 2\langle (0) \rangle^2 \langle (01) \rangle$$

$$= \frac{[(21)]}{n} - \frac{2[(11)(10)]}{n(n-1)} - \frac{[(20)(01)]}{n(n-1)} + \frac{2[(0) \rangle^2 \langle (01) \rangle}{n(n-1)(n-2)}$$

$$= \frac{1}{n} s_{2,1} - \frac{2}{n(n-1)}\{s_{1,1}s_{1,0} - s_{2,1}\} - \frac{1}{n(n-1)}\{s_{2,0}s_{0,1} - s_{2,1}\}$$

$$+ \frac{2}{n(n-1)(n-2)}\{s_{1,0}s_{0,1} - s_{2,1}\} - \frac{2}{n(n-1)(n-2)}\{s_{1,0}s_{0,1} - s_{2,1}\}$$
This result was first given by Fisher (1929). An illustration of the symbolic method will be given next. Note first that 

\[ \Delta k_3 = 3k_{2,1} \]. Now

\[ k_3 = \frac{1}{n(3)} \{n^2 s_3 - 3ns_2 s_1 + 2s_1^3 \} \]

so

\[ 3k_{2,1} = \frac{1}{n(3)} \{3n^2 s_2,1 - 6ns_2,1 s_1,0 - 3ns_2,0 s_0,1 + 6s_0,1 s_1,0 \}. \]

When the factor 3 is removed, the two results are easily seen to be equivalent. The remaining k statistic of weight 3, \( k_{1,1,1} \), can be found by applying \( \Delta \) to \( k_{2,1} \) or by a formula analogous to (1). The latter gives

\[ k_{1,1,1} = \frac{[\text{l11}]}{n} - \frac{[(\text{l10})(001)]}{n(n-1)} - \frac{[(\text{101})(010)]}{n(n-1)} \]

\[ - \frac{[(\text{010})(011)]}{n(n-1)} + 2 \frac{[(\text{100})(010)(001)]}{n(n-1)(n-2)} \]

\[ = \frac{s_{1,1,1}^1}{n} - \frac{1}{n(n-1)} \{s_{1,1,0} s_{0,1},0,1 - s_{1,1,1},1 \} \]

\[ - \frac{1}{n(n-1)} \{s_{1,0,1} s_{0,1,0} - s_{1,1,1},1 \} \]
To apply the symbolic method, first write \( k_{2,1} = k(r^2 t) \).

The operator \( \Delta = u \frac{\partial^2}{\partial r^2} \) is applied to give \( \Delta k(r^2 t) \)

\[
\frac{\partial^2}{\partial r^2} k(r^2 u t) = 2k_{1,1,1} .
\]

Now

\[
k_{2,1} = \frac{n^2}{n(3)} \left\{ s_{2,1} - \frac{2}{n} s_{1,0} s_{1,0} - \frac{1}{n} s_{2,0} s_{0,1} + \frac{2}{n^2} s_{1,0} s_{0,0,1} \right\} ,
\]

so the application of \( \Delta \) yields

\[
2k_{1,1,1} = \frac{n^2}{n(n-1)(n-2)} \frac{\partial^2}{\partial r^2} (s(r^2 u t) - \frac{2}{n} s(r u t) s(r) - \frac{2}{n} s(r t) s(r) - \frac{1}{n} s(r^2 u) s(t) + \frac{4}{n} s(r u) s(r) s(t)) .
\]

Hence

\[
k_{1,1,1} = \frac{n}{(n-1)(n-2)} \left\{ s_{1,1,1} - \frac{1}{n} s_{0,1} s_{1,0,0} - \frac{1}{n} s_{0,1} s_{1,0,0} - \frac{1}{n} s_{1,1} s_{0,0,0} \right\} .
\]
as before.

D. Multivariate k Statistics of Weight 4

The five k statistics of weight four are $k_4$, $k_{3,1}$, $k_{2,2}$, $k_{2,1,1}$ and $k_{1,1,1,1}$. Of these $k_{3,1}$ and $k_{2,2}$ can be found from (1), with the use of tables of symmetric functions, such as those of David, Kendall, and Barton (1966), to simplify the sums. This will be illustrated below.

For trivariate and quadrivariate sums, no suitable tables are available. It is possible however to check $k_{2,1,1,1}$ by using the symbolic process from two different starting points as the following diagram shows:

$$
\begin{align*}
&k_4 \\
\downarrow \\
&k_{3,1} \rightarrow k_{2,1,1} \rightarrow k_{1,1,1,1} \\
\downarrow \\
&k_{2,2}
\end{align*}
$$

Here the arrow indicates derivation by the symbolic process.

From (1),

$$
k_{3,1} = <(311)> - <(30)(01)> - 3<(21)(10)> - 3<(20)(11)>
+ 6<(10)^2(11)> + 6<(20)(10)(01)> - 6<(10)^3(01)>
$$
\[ \frac{1}{n} s_{3,1} - \frac{1}{n(n-1)} \{s_{3,0}s_{0,1} - s_{3,1}\} - \frac{3}{n(n-1)} \{s_{2,1}s_{1,0} - s_{3,1}\} \\
- \frac{3}{n(n-1)} \{s_{2,0}s_{1,1} - s_{3,1}\} + \frac{6}{n(n-1)(n-2)} \{2s_{3,1} \}
- 2s_{2,1}s_{1,0} - 2s_{2,0}s_{1,1} + s_{1,1}s_{1,0}^2 \}
+ \frac{6}{n(n-1)(n-2)} \{2s_{3,1} - 3s_{0,1}s_{0,1} - s_{2,1}s_{1,0} - 2s_{0,1}s_{1,1} \}
+ s_{2,1}s_{1,0}s_{0,1} \}
- \frac{6}{n(n-1)(n-2)(n-3)} \{6s_{3,1} \}
+ 2s_{3,0}s_{0,1} + 6s_{2,1}s_{1,0} + 3s_{2,0}s_{1,1} - 3s_{2,0}s_{1,0}s_{0,1} \\
- 3s_{1,1}s_{1,0}^2 + s_{1,0}s_{0,1}^3 \}.
\]

This is equivalent to

\[ k_{3,1} = \frac{n}{(n-1)(n-2)(n-3)} \{(n+1)s_{3,1} - \frac{n+1}{n} s_{3,0}s_{0,1} \\
- \frac{3(n-1)}{n} s_{1,1}s_{2,0} - \frac{3(n+1)}{n} s_{2,1}s_{1,0} \\
+ \frac{6}{n} s_{1,1}s_{1,0}^2 + \frac{6}{n} s_{2,0}s_{1,0}s_{0,1} - \frac{6}{n^2} s_{0,1}s_{1,0}^3 \}.
\]

The result can be confirmed by the symbolic process.

Thus

\[ k_4 = \frac{1}{n(4)} \{(n^3+n^2)s_4 - 4(n^2+n)s_3s_1 - 3(n^2-n)s_2^2 \\
+ 12ns_2s_1^2 - 6s_1^4 \} \]
and

\[ \Delta = \frac{t^3}{\theta} \]
can be applied to give

\[
4k_{3,1} = \frac{1}{n(n-1)} \left\{ 4(n^3 + n^2)s_3,1 - 12(n^2 + n)s_2,1s_1,0 - 4(n^2 + n)s_3,0s_0,1
\]

\[-12(n^2 - n)s_1,1s_2,0 + 24ns_1,1s_1,0^2 + 24ns_2,0s_0,1s_1,0^2
\]

\[-24s_0,1s_1,0^3 \right\}.
\]

The two expressions for \( k_{3,1} \) are obviously equivalent.

Again from (1),

\[
k_{2,2} = <(22)> - 2<(21)(01)> + 8<(10)(11)(01)> - 6<(10)^2(01)^2>
\]

\[-2<(12)(10)> + 2<(10)^2(02)> - 2<(11)^2> + 2<(20)(01)^2>
\]

\[-<(20)(02)>.\]

By tables of symmetric functions,

\[
k_{2,2} = \frac{s_{2,2}}{n} - \frac{2}{n(n-1)} \left\{ s_2,1s_0,0 - s_{2,2} \right\}
\]

\[+ \frac{8}{n(n-1)(n-2)} \left\{ 2s_2,2s_0,0 - s_{2,1}s_0,0 - s_{2,1}s_1,0 - s_{1,0}s_1,0 \right\}
\]

\[+ s_{1,1}s_0,0 + s_1,0s_0,1 \right\} - \frac{6}{n(n-1)(n-2)(n-3)} \left\{ -6s_2,2 + 4s_2,1s_0,1 + s_{2,2}s_0,0 - s_{2,2}s_1,0 - s_{1,0}s_1,0 \right\}
\]

\[+ s_{2,0}s_0,2 + 4s_1,2s_1,0 + 2s_1,1s_0,1 - s_0,2s_0,1 - 4s_1,0s_1,0s_0,1
\]

\[-s_{1,0}s_0,2 + s_{1,0}s_0,1 - s_{1,0}s_0,1 + s_{1,0}s_0,2 \right\} - \frac{2}{n(n-1)} \left\{ -s_{2,2} + s_{1,2}s_1,0 \right\}.
\[
+ \frac{2}{n(n-1)(n-2)} \{2s_{2,2}, 2s_{2,0}s_{0,2}, -2s_{1,2}s_{1,0} + s_{1,0} + 2s_{0,2} \} \\
- \frac{2}{n(n-1)} \{ -s_{2,2} + s_{1,1} \} \frac{2}{n(n-2)} \{2s_{2,2}, -2s_{2,1}s_{0,1} \} \\
- s_{2,0}s_{0,2}s_{0,1}^2 \frac{1}{n(n-1)} \{ -s_{2,2} + s_{0,2}s_{0,2} \},
\]
or
\[
k_{2,2} = \frac{n}{(n-1)(n-2)(n-3)} \{(n+1)s_{2,2} - \frac{2(n+1)}{n}s_{2,1}s_{0,1} \\
- \frac{2(n+1)}{n} s_{1,2}s_{1,0} - \frac{(n-1)}{n} s_{2,0}s_{0,2} - \frac{2(n-1)}{n} s_{1,1}^2 \\
+ \frac{8}{n} s_{1,1}s_{0,1}s_{1,0} + \frac{2}{n} s_{0,2}s_{1,0}^2 + \frac{2}{n} s_{2,0}s_{0,1}^2 \\
- \frac{6}{n^2} s_{1,0}^2 s_{0,1}^2 \}.
\]
The function \( k_{2,2} \) can also be found by applying the operator 
\( \Delta = \frac{\partial^3}{\partial r^3} \) to \( k_{3,1} \) where \( k_{3,1} \) is expressed symbolically
as \( k(r^3t) \). One obtains
\[
3k_{2,2} = \frac{n}{(n-1)(3)} \frac{\partial^3}{\partial r^3} \{(n+1)s(r^3t^2) - \frac{n+1}{n} s(r^3t)s(t) \\
- \frac{3(n-1)}{n} s(rt^2)s(r^2) - \frac{3(n-1)}{n} s(rt)s(r^2t) \\
- \frac{3(n+1)}{n} s(r^2t^2)s(r) - \frac{3(n+1)}{n} s(r^2t)s(rt) \\
+ \frac{6}{n} s(rt^2)s^2(r) + \frac{12}{n} s(rt)s(rt)s(r) \\
+ \frac{6}{n} s(r^2t)s(r)s(t) + \frac{6}{n} s(r^2)s(rt)s(t)
\]
\[ \frac{18}{n^2} s(t)s(rt)s^2(r) \}, \]

or

\[ k_{2,2} = \frac{n}{n-1} \left\{ (n+1)s_{2,2} - \frac{n+1}{n} s_{2,1}s_{0,1} - \frac{n-1}{n} s_{0,2}s_{2,0} \right\} \]

\[ - \frac{2(n-1)}{n} s_{1,1}^2 - \frac{2(n+1)}{n} s_{1,2}s_{1,0} - \frac{n+1}{n} s_{2,1}s_{0,1} \]

\[ + \frac{2}{n} s_{0,2}s_{1,0}^2 + \frac{4}{n} s_{0,1}s_{1,1}s_{1,0} + \frac{4}{n} s_{1,1}s_{1,0}s_{0,1} \]

\[ + \frac{2}{n} s_{2,0}s_{0,1}^2 - \frac{6}{n} s_{0,1}s_{1,0}^2 \}

When similar terms are collected this is seen to be the same result as before. Next an expression for \( k_{2,1,1} \) will be found from those for \( k_{3,1} \) and \( k_{2,2} \). The operator \( \frac{\partial^2}{\partial r^3} \) can be applied to \( k_{3,1} = k(r^3t) \) to give

\[ 3k_{2,1,1} = \frac{n}{n-1} \left( \frac{\partial^2}{\partial r^3} \right) [(n+1)s(r^3t) - \frac{n+1}{n} s(r^3t)s(t) \]

\[ - \frac{3(n-1)}{n} (s(r^2t)s(r^2)+s(rt)s(r^2u)) \]

\[ - \frac{3(n+1)}{n} (s(r^2ut)s(r)+s(r^2t)s(ru)) \]

\[ + \frac{6}{n}(s(r^2t)s(r)s(r)+2s(rt)s(ru)s(r)) \]

\[ + \frac{6}{n}(s(r^2u)s(r)s(t)+s(r^2)s(ru)s(t)) \]

\[ - \frac{18}{n^2}(s(t)s(ru)s^2(r)) \]. \]
So

\[ k_{2,1,1} = \frac{n}{(n-1)(n-2)(n-3)} (n+1)s_{2,1,1} - \frac{n+1}{n} s_{2,1,0}s_{0,0,1} \]

\[ - \frac{(n-1)}{n} s_{0,1,1}s_{2,0,0} - \frac{2(n-1)}{n} s_{1,0,1}s_{1,1,0} \]

\[ - \frac{2(n+1)}{n} s_{1,1,1}s_{1,0,0} - \frac{n+1}{n} s_{2,0,1}s_{0,1,0} \]

\[ + \frac{2}{n} s_{0,1,1}s_{1,0,0} + \frac{4}{n} s_{1,0,1}s_{0,1,0} \]

\[ + \frac{4}{n} s_{1,1,0}s_{1,0,0} + \frac{2}{n} s_{2,0,0}s_{0,1,0} \]

\[ - \frac{6}{n^2} s_{0,0,1}s_{0,1,0} \]

Again from \( k_{2,2} \), an expression for \( k_{2,1,1} \) can be obtained by applying \( \frac{\partial^3}{\partial t^3} \) to \( k_{2,2} = k(r^2t^2) \) to give

\[ 2k_{2,1,1} = \frac{\partial^3}{\partial t^3} \frac{n}{(n-1)(3)} (n+1)s(r^2t^2u) - \frac{2(n+1)}{n} s(r^2tu)s(t) \]

\[ + s(r^2t)s(tu) - \frac{2(n+1)}{n} s(r)s(r^2u) \]

\[ - \frac{(n-1)}{n} s(r^2)s(t^2u) - \frac{2(n-1)}{n} 2s(rtu)s(rt) \]

\[ + \frac{8}{n}(s(rtu)s(t)s(r) + s(rt)s(tu)s(r)) + \frac{2}{n} s(t^2u)s^2(r) \]

\[ + \frac{2}{n} s(r^2)s(tu)s(t) - \frac{12}{n^2} s^2(r)s(tu)s(t), \]

or

\[ k_{2,1,1} = \frac{n}{(n-1)(3)} (n+1)s_{2,1,1} - \frac{n+1}{n} s_{2,0,1}s_{0,1,0} \]
\[- \frac{n+1}{n} s_{1,0,0}^0,0,1,1 - \frac{2(n+1)}{n} s_{1,0,0}^0,1,1,1 \]
\[- \frac{(n-1)}{n} s_{2,0,0}^0,0,1,1 - \frac{2(n-1)}{n} s_{1,0,0}^0,0,1,0 \]
\[+ \frac{4}{n} s_{1,0,0}^1,0,1,0 + \frac{4}{n} s_{1,0,0}^1,0,1,0 \]
\[+ \frac{2}{n} s_{0,1,1}^0,0,1,0 + \frac{2}{n} s_{2,0,0}^0,0,0,1 \]
\[- \frac{6}{n^2} s_{1,0,0}^0,0,1,0 \]

which is the result previously obtained. Finally, \( k_{1,1,1,1} \) can be obtained from \( k_{2,1,1} \) by use of the operator \( \frac{3*}{\partial r} \) on \( k_{2,1,1} = k(r^2tv) \). Thus

\[2k_{1,1,1,1} = \frac{3*}{\partial r} \frac{n}{n-1}(n+1)s(r^2utv) - \frac{n+1}{n} s(r^2uv)s(t) \]
\[- \frac{n+1}{n} s(r^2utv)s(v) - \frac{2(n+1)}{n} (s(ru)s(rtv)+s(r)s(rutv)) \]
\[- \frac{n-1}{n} s(r^2u)s(tv) - \frac{2(n-1)}{n} (s(ruv)s(rt)+s(rv)s(rut)) \]
\[+ \frac{4}{n} (s(ruv)s(t)s(r)+s(rv)s(t)s(ru)) \]
\[+ \frac{4}{n} (s(rut)s(v)s(r) + s(rt)s(v)s(ru)) \]
\[+ \frac{4}{n} s(tv)s(ru)s(r) + \frac{2}{n} s(r^2u)s(v)s(t) \]
\[- \frac{12}{n^2} s(r)s(ru)s(v)s(t) \].
and hence

\[ k_{1,1,1,1} = \frac{n}{(n-1)(n-2)(n-3)} \left\{ (n+1)s_{1,1,1,1} \right\} \]

\[ - \frac{n+1}{n} s_{1,1,0,1}s_{0,0,0,1,0} - \frac{n+1}{n} s_{1,1,0,0}s_{0,0,0,0,1} \]

\[ - \frac{n+1}{n} s_{0,1,0,0}s_{1,0,1,1,1} - \frac{n+1}{n} s_{1,0,0,0}s_{0,1,1,1,1} \]

\[ - \frac{n-1}{n} s_{1,1,0,0}s_{0,0,1,1,1} - \frac{n-1}{n} s_{0,1,0,1}s_{0,1,0,1,1,1} \]

\[ - \frac{n-1}{n} s_{1,0,0,1}s_{0,1,1,1,1} + \frac{2}{n} s_{0,1,0,1}s_{0,1,0,1,0,0} \]

\[ + \frac{2}{n} s_{1,0,0,1}s_{0,1,0,1,0,0} \]

\[ + \frac{2}{n} s_{0,1,1,0}s_{0,1,0,1,0,0} \]

\[ + \frac{2}{n} s_{1,0,1,0}s_{0,1,0,1,0,0} \]

\[ + \frac{2}{n} s_{0,0,1,1}s_{0,1,0,1,0,0} \]

\[ + \frac{2}{n} s_{1,1,0,0}s_{0,1,0,1,0,0} \]

\[ - \frac{6}{n^2} s_{1,0,0,0}s_{0,1,0,1,0,0} \}

This completes the list of multivariate polykays of weight 4. The great advantage of the symbolic method lies in its relative brevity as compared to dealing with sums directly,
as Fisher did. The unavailability of tables of sums of weight 4 or greater renders a direct comparison impossible, but it is obvious that the symbolic method is shorter and that its relative advantage increases sharply with increasing weight.

E. Multivariate k Statistics of Weight 5

The function $k_{4,1}$ is first found by use of (1) and tables of symmetric functions.

From (1),

$$k_{4,1} = \frac{1}{n} \left( \frac{1}{n(n-1)} \right) - \frac{[(40)(01)]}{n(n-1)} - \frac{4[(31)(10)]}{n(n-1)} - \frac{4[(30)(11)]}{n(n-1)}$$

$$- \frac{6[(21)(20)]}{n(n-1)} + \frac{8[(30)(10)(01)]}{n(n-1)(n-2)} + \frac{12[(21)(10)^2]}{n(n-1)(n-2)}$$

$$+ \frac{6[(20)^2(01)]}{n(n-1)(n-2)} + \frac{24[(20)(11)(10)]}{n(n-1)(n-2)} - \frac{36[(20)(10)^2(01)]}{n(n-1)(n-2)(n-3)}$$

$$- \frac{24[(11)(10)^3]}{n(n-1)(n-2)(n-3)} + \frac{24[(10)^4(01)]}{n(n-1)(n-2)(n-3)(n-4)}$$

or

$$k_{4,1} = \frac{1}{n} s_{4,1} - \frac{1}{n(n-1)} \left\{ s_{4,0} s_{0,1} - s_{4,1} \right\}$$

$$- \frac{4}{n(n-1)} \left\{ s_{3,1} s_{1,0} - s_{4,1} \right\} - \frac{4}{n(n-1)} \left\{ s_{3,0} s_{1,1} - s_{4,1} \right\}$$

$$- \frac{6}{n(n-1)} \left\{ s_{2,1} s_{2,0} - s_{4,1} \right\}$$

$$+ \frac{8}{n(n-1)(n-2)} \left\{ s_{3,0} s_{0,1} s_{0,1} - s_{3,0} s_{1,1} - s_{3,1} s_{1,0} - s_{4,0} s_{0,1} \right\}$$
\[ + 2s_{4,1} \{ + \frac{12}{n(n-1)(n-2)} \{ s_{2,1}s_{1,0} - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} - s_{2,0}s_{1,0} \} \}
\]

\[ + \frac{6}{n(n-1)(n-2)} \{ s_{2,0}s_{1,0} - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} - s_{2,0}s_{1,0} \} \}
\]

\[ + \frac{24}{n(n-1)(n-2)} \{ s_{2,0}s_{1,0} - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} - s_{2,0}s_{1,0} \} \}
\]

\[ + \frac{36}{n(n-1)(n-2)(n-3)} \{ s_{2,0}s_{1,0} - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} - s_{2,0}s_{1,0} \} \]

\[ - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} + s_{2,0}s_{1,0} - s_{2,0}s_{1,0} + s_{2,0}s_{1,1} - s_{2,0}s_{1,0} \]

\[ + 4s_{3,1}s_{1,0} - s_{3,1}s_{1,0} + 6s_{4,0}s_{1,0} - s_{4,0}s_{1,0} \}
\]

\[ - \frac{24}{n(n-1)(n-2)(n-3)} \{ s_{1,1}s_{1,0} - s_{3,2}s_{1,0} + s_{3,2}s_{1,0} - s_{3,2}s_{1,0} \} \]

\[ + 3s_{2,1}s_{1,0} - s_{3,2}s_{1,0} + s_{3,2}s_{1,0} - s_{3,2}s_{1,0} \]

\[ + \frac{24}{n(n-1)(n-2)(n-3)(n-4)} \{ s_{1,0}s_{1,0} - s_{4,1}s_{1,0} \} \]

\[ - 6s_{2,0}s_{1,0} + 8s_{3,0}s_{1,0} + 6s_{3,2}s_{1,0} - 6s_{3,2}s_{1,0} \]

\[ + 12s_{2,1}s_{1,0} + 8s_{3,0}s_{1,0} - 12s_{2,1}s_{1,0} - 8s_{3,0}s_{1,0} \]
Thus

\[ k_{4,1} = \frac{1}{n(5)} \left\{ (n^4+5n^3)s_{4,1}S - 4(n^3+5n^2)s_{3,1}s_{1,0}S - (n^3+5n^2)s_{4,0}s_{s,1,0} \right. \]

\[-6(n^3-n^2)s_{2,1}s_{2,0}S - 4(n^3-n^2)s_{3,0}s_{s,1,1} \]

\[ +12(n^2+2n)s_{2,1}s_{1,0}S + 8(n^2+2n)s_{3,0}s_{s,0,1}s_{1,0} \]

\[ +24(n^2-n)s_{1,1}s_{2,0}s_{1,0}S + 6(n^2-n)s_{2,0}s_{s,0,1} \]

\[-24ns_{1,1}s_{1,0}S - 36ns_{0,1}s_{1,0}s_{2,0}s_{s,0,1} + 24s_{s,0,1} \} \].

The expression for \( k_{4,1} \) will also be found by the symbolic method. Now

\[ k_5 = \frac{1}{n(5)} \left\{ (n^4+5n^3)s_5 - 5(n^3+5n^2)s_4s_1 - 10(n^3-n^2)s_3s_2 \right. \]

\[ + 20(n^2+2n)s_3s_1^2 + 30(n^2-n)s_2^2s_1 - 60ns_2s_1^3 + 24s_1^5 \} \].

Application of the operator \( t \frac{3}{5} \) to \( k(r^5) \) yields

\[ 5k_{4,1} = \frac{1}{n(5)} \frac{3}{5} \left\{ (n^4+5n^3)s(r^5) - 5(n^3+5n^2)(s(r^4)t)s(r) \right. \]

\[ + s(r^4)s(rt) - 10(n^3-n^2)(s(r^3)t)s(r^2) + s(r^3)s(r^2) \}

\[ + 20(n^2+2n)(s(r^3)t)s^2(r) + 25s(r^3)s(rt)s(r) \]
\[-60n(s(r^2t)s^3(r)+3s(rt)s^2(r)s(r^2))\]
\[+30(n^2-n)(2s(r^2t)s(r^2)s(r)+s^2(r^2)s(rt)\]
\[+120s(rt)s^4(r)\}.

Hence
\[
k_{4,1} = \frac{1}{n(5)} \{(n^4+5n^3)s_{4,1}-4(n^3+5n^2)s_{3,1}s_{1,0}^2\]
\[-(n^3+5n^2)s_{4,0}s_{0,1}^2-6(n^3-n^2)s_{2,1}s_{2,0}^2-4(n^3-n^2)s_{3,0}s_{1,1}^2\]
\[+12(n^2+2n)s_{2,1}s_{1,0}^2+8(n^2+2n)s_{3,0}s_{0,1}s_{1,0}^2\]
\[+24(n^2-n)s_{1,1}s_{2,0}s_{1,0}^2+6(n^2-n)s_{2,0}s_{0,1}^2-24ns_{1,1}s_{1,0}^3\]
\[-36ns_{0,1}s_{1,0}^2s_{2,0}^2+24s_{0,1}s_{1,0}^4\}.

The only two \(k\) statistics of weight 5 which can be found using tables of symmetric sums are \(k_{4,1}\) and \(k_{3,2}\). The remaining \(k\) statistics of this weight, \(k_{3,1,1}', k_{2,1,2}', k_{1,1,1,2}'\) and \(k_{1,1,1,1,1}\) will be found by symbolic operation according to the following scheme, where the arrow again indicates derivation by the symbolic operation. All of the \(k\) statistics, with the exception of \(k_{1,1,1,1,1}'\), can be found from two different starting points, thus providing a check on the work. The relevant diagram is
Now from \( k_{4,1} \), the statistic \( k_{3,2} \) will be found by applying the operator \( \frac{3\times 4}{3x} \) to \( k(r^4t) \). This gives

\[
4k_{3,2} = \frac{1}{n(5)} \frac{3\times 4}{3x} (\frac{n^4+5n^3}{5})s(r^4t^2) - 4(\frac{n^3+5n^2}{n}) (s(r^3t^2)s(r)
\]

\[+s(r^3t)s(rt) - (\frac{n^3+5n^2}{n}) s(r^4t)s(t) - 6(\frac{n^3-n^2}{n}) (s(r^2t^2)s(r^2)
\]

\[+s(r^2t)s(r^2t) - 4(\frac{n^3-n^2}{n}) (s(r^3t)s(rt) + s(r^3)s(r^2)
\]

\[+12(\frac{n^2+2n}{n}) (s(r^2t^2)s^2(r) + 2s(r^2t)s(rt)s(r))
\]

\[+8(n^2+2n)(s(r^3t)s(t)s(r) + s(r^3)s(t)s(rt))
\]

\[+24(n^2-n)(s(r^2t)s(r^2)s(r) + s(rt)s(r^2t)s(r) +
\]

\[+s(rt)s(r^2)s(rt)) + 6(n^2-n)2s(r^2)s(r^2t)s(t)
\]

\[-24n(s(r^2t)s^3(r) + 3s(rt)s^2(r)s(rt))
\]

\[-36n(s(t)\cdot 2s(r)s(rt)s(r^2)+s(t)s^2(r)s(r^2t)
\]

\[+24\cdot 4s(t)s^3(r)s(rt))
\].
So

\[
k_{3,2} = \frac{1}{n^5} \{ (n^4 + 5n^3)s_{3,2} - 3(n^3 + 5n^2)s_{2,2}s_{1,0} - 2(n^3 + 5n^2)s_{3,1}s_{0,1} - 3(n^3 - n^2)s_{1,2}s_{2,0} - 6(n^3 - n^2)s_{2,1}s_{1,1} - (n^3 - n^2)s_{3,0}s_{0,2} + 6(n^2 + 2n)s_{1,2}s_{1,0}^2 + 12(n^2 + 2n)s_{2,1}s_{0,1} + 2(n^2 + 2n)s_{3,0}s_{0,1}^2 + 6(n^2 - n)s_{0,2}s_{2,0}s_{1,0} + 12(n^2 - n)s_{1,1}s_{1,0} + 12(n^2 - n)s_{1,1}s_{0,1}s_{0,1} + 6ns_{0,2}s_{1,0}^3 - 36ns_{1,1}s_{1,0}s_{0,1} - 18ns_{0,1}s_{1,0}s_{0,1}^2 + 24s_{0,1}s_{1,0}^3 \}
\]

This result has been checked by a direct evaluation from (1).

To find \( k_{3,1,1} \), \( k_{4,1} \) can be written as \( k(r^4t) \) and the operator \( \frac{\partial^2}{\partial r} \) can be applied. This gives

\[
4k_{3,1,1} = \frac{\partial^2}{\partial r} \frac{1}{n^5} \{ (n^4 + 5n^3)s(r^4ut) - 4(n^3 + 5n^2)(s(r^3ut)s(r) + s(r^3t)s(ru)) - 6(n^3 - n^2)(s(r^2ut)s(r^2) + s(r^2t)s(r^2u)) + 12(n^2 + 2n)(s(r^2ut)s^2(r) + 2s(r^2t)s(ru)s(r)) + 8(n^2 + 2n)(s(r^3u)s(t)s(r) + s(r^3)s(t)s(ru)) + 24(n^2 - n)(s(r^2)s(r^2u)s(r) + s(r^2)s(r^2u)s(r) + s(r^2)s(r^2u)s(r) + s(r^2)s(r^2u)s(r)) + 6(n^2 - n)^2s(r^2u)s(r^2)s(t) - 24n(s(r^2)s^3(r) + 3s(r^2)s(r^2) - 2 (r)) \}
\]
-36n(2s(t)s(ru)s(r)s(r^2)+s(t)s^2(r)s(r^2u))

+96s(t)s^3(r)s(ru)}

Or

\[ k_{3,1,1} = \frac{1}{n(5)}\{ (n^4+5n^3)s_{3,1,1}+3(n^3+5n^2)s_{2,1,1}s_{1,0,0,0} \]

- (n^3+5n^2)s_{3,0,1,0,0,0} - (n^3+5n^2)s_{3,1,0}s_{0,0,0,0,1,1} \]

-3(n^3-n^2)s_{1,1,1}s_{2,0,0,0} - 3(n^3-n^2)s_{2,0,1}s_{1,1,1,1,0} \]

-3(n^3-n^2)s_{2,1,0}s_{1,0,0,1,1,1} - (n^3-n^2)s_{3,0,0}s_{0,0,1,1,1,1,1} \]

+6(n^2+2n)s_{1,1,1}s_{1,0,0,0,0} + 6(n^2+2n)s_{2,0,1}s_{0,0,1,1,1,1} \]

+6(n^2+2n)s_{2,1,0}s_{0,0,1,1,1,1} + 6(n^2+2n)s_{3,0,0}s_{0,1,1,1,1,1} \]

+6(n^2-n)s_{1,1,1}s_{2,0,0,1,0,0} + 12(n^2-n)s_{1,1,0}s_{1,1,1,1,0,0,0} \]

+6(n^2-n)s_{1,0,1}s_{2,0,0,0,1,0} + 6(n^2-n)s_{1,1,0}s_{1,0,1,0,0,1,0} \]

+6s_{0,1,1}s_{1,0,0,1,0,0} + 3 - 18s_{0,1,0,1,0,0,0} \]

- 18s_{0,0,1,0,1,0,0} - 18s_{0,0,1,0,0,1,0} - 18s_{0,0,1,0,1,0,0} \]

+ 24s_{0,0,1,0,0,1,0} \}.

This can be checked by applying \( u_{t}^{3k} \) to \( k_{3,2} = k(r^3t^2) \). The function \( k_{2,1,2} \) can be obtained from \( k(r^3t^2) \) by applying \( u_{r}^{3k} \). This produces
\[ k_{2,1,2} = \frac{1}{n(5)} \frac{\partial^3}{\partial r^3} \left( (n^4+5n^3) s(r^3 u^2) - 3(n^3+5n^2) (s(r^2 u t^2) s(r) \right. \\
+ s(r^2 t^2) s(ru) - 2(n^3+5n^2) s(r^3 u t) s(t) - 3(n^3-n^2) (s(r u t^2) s(r^2) \\
+ s(r t^2) s(r^2 u) - 6(n^3-n^2) (s(r^2 u t) s(rt) + s(r^2 t) s(rut)) \\
- (n^3-n^2) s(r^3 u) s(t^2) + 6(n^2+2n) (s(r u t^2) s(r^2) s(t) \\
+ 2s(r t^2) s(ru) s(r) + 12(n^2+2n) (s(r^2 u t) s(rt) s(r) \\
+ s(r^2 t) s(t) s(ru) + 2(n^2+2n) s(r^3 u) s^2(t) \\
+ 6(n^2-n) (s(t^2) s(r^2 u) s(r) + s(t^2) s(r^2) s(ru) \\
+ 12(n^2-n) (2s(r u t) s(rt) s(r) + s^2(rt) s(ru) \\
+ 12(n^2-n) (s(r u t) s(r^2) s(t) + s(rt) s(r^2 u) s(t)) \\
- 6n \cdot 3s(t^2) s^2(r) s(ru) - 36n(s(r u t) s^2(r) s(t) \\
+ s(r t) \cdot 2s(r u) s(r) s(t)) - 18n(s^2(t) s(r u) s(r^2) \\
+ s^2(t) s(r) s(r^2 u)) + 24 \cdot 3s^2(t) s^2(r) s(ru) \right) \\
\]

or

\[ k_{2,1,2} = \frac{1}{n(5)} \left( (n^4+5n^3) s_{2,1,2} - 2(n^3+5n^2) s_{1,1,2} s_{1,0,0} \right. \\
- (n^3+5n^2) s_{2,0,2} s_{0,1,0} - 2(n^3+5n^2) s_{2,1,1} s_{0,1,1} \\
- (n^3-n^2) s_{0,1,2} s_{2,0,0} - 2(n^3-n^2) s_{1,0,2} s_{1,1,0} \\
- 4(n^3-n^2) s_{1,1,1} s_{1,0,1} - 2(n^3-n^2) s_{2,0,1} s_{0,1,1} \]
The function \( k_{2,2,1} \) can be produced from \( k_{3,1,1} \) by writing

\[ k_{3,1,1} = k(r^3tv) \]

and applying \( \frac{\partial}{\partial r} \) to check on the previous result. The \( k \) statistic \( k_{2,1,1,1} \) can be found from \( k_{3,1,1} \) by writing \( k_{3,1,1} = k(r^3tv) \) and applying \( \frac{\partial}{\partial r} \). This gives

\[
3k_{2,1,1,1} = \frac{1}{n(5)} \frac{\partial}{\partial r} \left( (n^4 + 5n^3)s(r^3uv) - 3(n^3 + 5n^2)(s(r^2uv)s(r) + s(r^2tv)s(r) + s(r^2u)) - (n^3 + 5n^2)s(r^3uv)s(t) - (n^3 + 5n^2)s(r^3ut)s(v) - 3(n^3 - n^2)(s(rutv)s(r^2) + s(rtv)s(r^2u)) \right)
\]
\[-3(n^3-n^2)\left(s(r^2uv)s(rt)+s(r^2v)s(rut)\right)\]

\[-3(n^3-n^2)\left(s(r^2ut)s(rv)+s(r^2t)s(ruv)\right)\]

\[-(n^3-n^2)\cdot3s(r^2u)s(tv)+6(n^2+2n)\left(s(rutv)s^2(r)\right)\]

\[+2s(rtv)s(ru)s(r)\left)+6(n^2+2n)\left(s(r^2uv)s(t)s(r)\right)\]

\[+s(r^2v)s(t)s(ru)+6(n^2+2n)\left(s(r^2ut)s(v)s(r)\right)\]

\[+s(r^2t)s(v)s(ru)\left)+2(n^2+2n)s(r^3u)s(v)s(t)\right)\]

\[+6(n^2-n)\left(s(tv)s(r^2u)s(r)+s(tv)s(r^2)s(ru)\right)\]

\[+12(n^2-n)\left(s(ruv)s(rt)s(r)+s(rv)s(rut)s(r)\right)\]

\[+s(rv)s(rt)s(ru)\left)+6(n^2-n)\left(s(ruv)s(r^2)s(t)\right)\]

\[+s(rv)s(r^2u)s(t)\left)+6(n^2-n)\left(s(rut)s(r^2)s(v)\right)\]

\[+s(rt)s(r^2u)s(v)\left)-6n\cdot3s(tv)s(ru)s^2(r)\right)\]

\[-18n(s(ruv)s(t)s^2(r)+2s(rv)s(t)s(ru)s(r))\]

\[-18n(s(v)s(t)s(ru)s(r^2)+s(v)s(t)s(r)s(r^2u)\]

\[+s(v)s(t)s(u)s(r^2)+2s(v)s(t)s(r)s(ru))\]

\[-18n(2s(v)s(ru)s(r)s(rt)+s(v)s^2(r)s(rut))\]

\[+72s(v)s(ru)s^2(r)s(t)\right)\right)\],
\[ k_{2,1,1,1} = \frac{1}{n}(n^4+5n^3)s_{2,1,1,1} - 2(n^3+5n^2)s_{1,1,1,1} + 2(n^3+5n^2)s_{2,1,1,0} + 4(n^3+5n^2)s_{1,1,1,0} + 2(n^3+5n^2)s_{1,1,0,1} + 4(n^3+5n^2)s_{1,1,0,0} + 2(n^3+5n^2)s_{2,1,0,1} + 4(n^3+5n^2)s_{2,1,0,0} + 2(n^3+5n^2)s_{2,0,1,1} + 4(n^3+5n^2)s_{2,0,1,0} + 2(n^3+5n^2)s_{2,0,0,1} + 4(n^3+5n^2)s_{2,0,0,0} + 2(n^3+5n^2)s_{1,0,1,1} + 4(n^3+5n^2)s_{1,0,1,0} + 2(n^3+5n^2)s_{1,0,0,1} + 4(n^3+5n^2)s_{1,0,0,0} + 2(n^3+5n^2)s_{0,1,1,1} + 4(n^3+5n^2)s_{0,1,1,0} + 2(n^3+5n^2)s_{0,1,0,1} + 4(n^3+5n^2)s_{0,1,0,0} + 2(n^3+5n^2)s_{0,0,1,1} + 4(n^3+5n^2)s_{0,0,1,0} + 2(n^3+5n^2)s_{0,0,0,1} + 4(n^3+5n^2)s_{0,0,0,0} \]
\[ +4(n^2 - n)s_{1,0,1,0} + 4(n^2 - n)s_{1,0,1,1,0} \]

\[ +2(n^2 - n)s_{0,1,0,1,0} + 4(n^2 - n)s_{1,0,1,0,1} \]

\[ +2(n^2 - n)s_{0,1,0,1,0} + 4(n^2 - n)s_{1,0,1,0,1} \]

\[ +4(n^2 - n)s_{1,0,1,0,1} + 4(n^2 - n)s_{1,0,1,0,1} \]

\[ -6ns_{0,1,0,1,0}^2 - 6ns_{0,1,0,1,0}^2 \]

\[ -12ns_{1,0,1,0,1} - 12ns_{0,1,0,1,0}^2 \]

\[ -6ns_{0,1,0,1,0}^2 \]

\[ -12ns_{0,1,0,1,0}^2 \]

\[ -12ns_{0,1,0,1,0}^2 \]

\[ -6ns_{0,1,0,1,0}^2 \]

\[ +24s_{0,1,0,1,0}^2 \]

This result can be checked by writing \( k_{2,1,2} = k(r^2 t v^2) \) and applying \( \frac{\partial u}{\partial x} \).

Finally, the formula for \( k_{1,1,1,2} \) is used to produce \( k_{1,1,1,1,1} \). Writing \( k_{1,1,1,2} = k(r t w^2) \), the operator \( \frac{\partial^3}{\partial w^3} \).
gives

\[
2k_{1,1,1,1,1} = \frac{1}{\binom{5}{n}} \frac{\partial^5}{\partial w^5} \left( (n^4 + 5n^3) s(rtw^2 x) - (n^3 + 5n^2) s(tvw^2 x) s(r) \right)
\]

\[-(n^3 + 5n^2) s(rw^2 x) s(t) - (n^3 + 5n^2) s(rtw^2 x) s(v)\]

\[-2(n^3 + 5n^2) s(rtvwx) s(w) + s(rtvw) s(wx)\]

\[-(n^3 - n^2) s(vw^2 x) s(rt) - (n^3 - n^2) s(tw^2 x) s(rv)\]

\[-(n^3 - n^2) s(rw^2 x) s(tv) - 2(n^3 - n^2) s(tvwx) s(rw)\]

\[+ s(tvw) s(rwx) - 2(n^3 - n^2) s(rvwx) s(tw) + s(rvwx) s(tx)\]

\[-2(n^3 - n^2) s(rtwx) s(vw) + s(rtw) s(vwx)\]

\[-(n^3 - n^2) s(rtw) s(w^2 x) + 2(n^3 + 2n) s(rvwx) s(t) s(r)\]

\[+ 2(n^2 + 2n) s(tw^2 x) s(v) s(r) + 2(n^3 + 2n) s(rvwx) s(tx) s(v)\]

\[+ 4(n^2 + 2n) \left( s(tvwx) s(w) s(r) + s(tvw) s(wx) s(r) \right)\]

\[+ 4(n^2 + 2n) \left( s(rvwx) s(w) s(t) + s(rvw) s(wx) s(t) \right)\]

\[+ 4(n^2 + 2n) \left( s(rtwx) s(v) s(w) + 2(n^2 + 2n) \cdot 2s(rtw) s(wx) s(w) \right)\]

\[+ 2(n^2 - n) \cdot 2s(wx) s(tv) s(r) + 2(n^2 - n) s(w^2 x) s(rv) s(t)\]

\[+ 2(n^2 - n) s(w^2 x) s(r) s(v) + 4(n^2 - n) s(vwx) s(tw) s(r)\]

\[+ s(vw) s(twx) s(r) + 4(n^2 - n) s(vwx) s(rw) s(t)\]

\[+ s(vw) s(rwx) s(t) + 4(n^2 - n) s(twx) s(rv) s(v)\]

\[+ s(tw) s(rwx) s(v) + 4(n^2 - n) s(vwx) s(rt) s(w)\]

\]
\[\begin{align*}
+s(\text{vw})s(\text{rt})s(\text{wx}) + 4 (n^2-n)(s(\text{twx})s(\text{rv})s(w) + s(\text{tw})s(\text{rv})s(\text{wx})) \\
+ 4 (n^2-n)(s(\text{rwx})s(\text{tv})s(w) + s(\text{rw})s(\text{tv})s(\text{wx})) \\
- 6n s(w^2x)s(t)s(r)s(v) - 12n (s(\text{vwx})s(t)s(w)s(r) \\
+ s(\text{vw})s(t)s(\text{wx})s(r)) - 12n (s(\text{twx})s(v)s(r)s(w) \\
+ s(\text{tw})s(v)s(r)s(wx)) - 12n (s(\text{rwx})s(v)s(t)s(w) \\
+ s(\text{rw})s(v)s(t)s(wx)) - 12n (s(\text{rvwx})s(t)s(w)s(r) \\
- 24 \cdot 2s(\text{w})s(t)s(\text{rv}) - 6n \cdot 2s(\text{w})s(\text{w})s(\text{r})s(\text{tv}) \\
+ 24 \cdot 2s(\text{w})s(\text{w})s(t)s(\text{r})s(\text{v})},
\end{align*}\]

So

\[k_{1,1,1,1,1} = \frac{1}{n(5)} (n^4 + 5n^3)s_{1,1,1,1} \]

\[-(n^3 + 5n^2)s_{0,1,1,1,1}s_{0,0,0,0,0} -(n^3 + 5n^2)s_{1,0,1,1,1}s_{0,1,0,0,0} \]

\[-(n^3 + 5n^2)s_{1,1,0,1,1}s_{0,0,1,0,0} \]

\[-(n^3 + 5n^2)s_{1,1,1,0,1}s_{0,0,0,1,0} -(n^3 + 5n^2)s_{1,1,1,1,0}s_{0,0,0,0,1} \]

\[-(n^3 - n^2)s_{0,0,1,1,1}s_{1,1,0,0,0} -(n^3 - n^2)s_{0,1,0,1,1}s_{1,0,1,0,0} \]

\[-(n^3 - n^2)s_{1,0,0,1,1}s_{0,1,0,1,0} -(n^3 - n^2)s_{1,1,0,1,0}s_{0,1,0,1,0} \]

\[-(n^3 - n^2)s_{0,1,1,1,0}s_{1,1,0,0,1} -(n^3 - n^2)s_{1,0,1,1,0}s_{1,0,0,1,1} \]

\[-(n^3 - n^2)s_{1,1,0,1,0}s_{1,0,0,1,1} -(n^3 - n^2)s_{1,1,1,0,1}s_{1,0,0,1,1} \]
\[(n^2 + 2n)s_{0,0,1,1,1}^0,0,1,0,0,0^1,0,0,0,0\]
\[(n^2 + 2n)s_{0,1,0,1,1}^0,0,1,0,0,0^1,0,0,0,0\]
\[(n^2 + 2n)s_{1,0,0,1,1}^0,0,1,0,0,0^0,1,0,0,0\]
\[2(n^2 + 2n)s_{0,1,1,1,0}^0,0,0,0,1^0,0,0,0,0\]
\[2(n^2 + 2n)s_{0,1,1,1,0}^0,0,0,0,1^0,1,0,0,0\]
\[2(n^2 + 2n)s_{1,0,1,1,0}^0,0,0,0,1^0,0,0,0,0\]
\[2(n^2 + 2n)s_{1,1,1,0,0}^0,0,0,0,1^0,0,0,0,0\]
\[2(n^2 + 2n)s_{0,0,0,1,1}^0,0,1,0,0,0^1,0,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,1,0,0,0^0,1,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,0,0,1^0,0,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,0,0,1^0,1,0,0,0\]
\[2(n^2 + 2n)s_{0,0,0,1,1}^0,0,1,0,0,0^1,0,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,1,0,0,0^0,1,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,0,0,1^0,0,0,0,0\]
\[2(n^2 + 2n)s_{0,0,1,1,0}^0,0,0,0,1^0,1,0,0,0\]
\[+2(n^2-n)S_{0,1,0,0,1s_1,0,0,0,1,0,0}\]
\[-6ns_{0,0,1,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{0,0,1,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{0,0,1,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{0,0,1,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{1,0,0,0,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{1,0,0,0,1s_0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{1,0,0,0,1s_0,0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{1,0,0,0,1s_0,0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{0,0,0,0,1s_0,0,1,0,0,0,1,0,0,0,1,0,0}\]
\[-6ns_{0,0,0,0,1s_0,0,1,0,0,0,1,0,0,0,1,0,0}\]
Symmetric functions of several variables may be expressed in terms of multivariate bracket functions. Since these are "inherited on the average", in Tukey's (1950) phrase, unbiased estimates of population symmetric functions are easily found with their use. In the next section the bracket functions will be used to determine bivariate polykays and for this their expressions in terms of bivariate \( k \) statistics are needed. The catalog presented here, which is complete through weight 4, is not available elsewhere. Univariate results are not included.

1. **Bracket functions of weight 2**

The bivariate bracket functions of weight 2 are

\[
\langle (11) \rangle \quad \text{and} \quad \langle (10)(01) \rangle .
\]

\[
\langle (11) \rangle = \frac{1}{n} \sum x_i y_i = \frac{1}{n} s_{1,1} .
\]

But \( k_{1,1} = \frac{1}{n(2)} \{ n s_{1,1} - s_{1,0} - s_{0,1} \} \) and \( s_{1,0} = nk_{1,0} \) and \( s_{0,1} = nk_{0,1} \) so \( \langle (11) \rangle = \frac{n-1}{n} k_{1,1} + k_{1,0} k_{0,1} . \)
\[<\text{(10)}(01)> = \frac{1}{n(2)} \sum x_i y_j = \frac{1}{n(2)} \{s_{1,0}s_{0,1} - s_{1,1}\}, \quad \text{so}\]

using the previous result, \[<\text{(10)}(01)> = k_{1,0}^0 s_{0,1} - \frac{1}{n} k_{1,1}.\]

2. **Bivariate bracket functions of weight 3**

The bivariate bracket functions of weight 3, excluding univariate results, are \(<(21)>, <(20)(01)>, <(10)^2(01)>,\) and \(<(11)(10)>\). From Section C of this chapter,

\[K_{2,1} = \frac{n}{(n-1)(n-2)} \{s_{2,1} - \frac{2}{n} s_{1,1}s_{1,0} - \frac{1}{n} s_{2,0}^{s_{0,1}}
\]

\[+ \frac{2}{n} s_{1,0}^{s_{0,1}}\}, \quad \text{so}\]

\[s_{2,1} = \frac{(n-1)(n-2)}{n} K_{2,1} + 2(n^{-1}) K_{1,1} k_{1,0} + n k_{1,0}^2 K_{0,1}\]

\[+ (n-1) k_2,0 k_{0,1}^2.\]

Now

\[<\text{(20)}(01)> = \frac{1}{n(2)} \sum x_i^2 y_j = \frac{1}{n(2)} \{s_{2,0}s_{0,1} - s_{2,1}\} \quad \text{so}\]

\[<\text{(20)}(01)> = \frac{n-1}{n} K_{2,0}^0 s_{0,1} + k_{1,0}^{s_{0,1}} k_{0,1}^2 K_{0,1} - \frac{n-2}{n^2} k_{2,1} - \frac{2}{n} k_{1,1}^2 K_{1,0}\]

Now

\[K_{1,2} = \frac{n}{(n-1)(n-2)} \{s_{1,2} - \frac{2}{n} s_{0,1}s_{1,1} - \frac{1}{n} s_{0,2}s_{1,0}
\]

\[+ \frac{2}{n^2} s_{0,1}s_{1,0}^2\}.\]
so
\[
\langle (12) \rangle = \frac{1}{n} s_{1,2} = \frac{(n-1)(n-2)}{n^2} k_{1,2} + \frac{2(n-1)}{n} k_{0,1} k_{1,1}
\]
\[
+ k_{0,1}^2 k_{1,0} + \frac{(n-1)}{n} k_{0,2} k_{1,0}.
\]

Also
\[
\langle (10) \rangle^2 (01) = \frac{1}{n} \sum_{i,j} x_i y_j = \frac{1}{n} \{ s_{1,0}^2 s_{0,1} - s_{2,0} s_{0,1} + 2 s_{1,0} s_{0,1} \}.
\]

Thus
\[
\langle (10) \rangle^2 (01) = k_{1,0}^2 k_{0,1} - \frac{1}{n} k_{2,0} k_{0,1} + \frac{2}{n} k_{1,1} k_{1,0} + \frac{2}{n} k_{2,1}.
\]

Finally
\[
\langle (11) \rangle (10) = \frac{\sum_{i,j} x_i y_j x_j}{n} = \frac{1}{n} \{ s_{1,1} s_{1,0} - s_{2,1} \}
\]
\[
= \frac{n-2}{n} k_{1,1} k_{1,0} + k_{1,1} k_{1,0}^2 k_{0,1} - \frac{n-2}{n^2} k_{2,1}
\]
\[
- \frac{1}{n} k_{2,0} k_{0,1}.
\]

3. Bivariate bracket functions of weight 4

There are 16 bivariate bracket functions of weight 4:
\[
\langle (31) \rangle, \langle (22) \rangle, \langle (30) \rangle (01), \langle (21) \rangle (10), \langle (20) \rangle (02),
\]
\[
\langle (10) \rangle^2 (02), \langle (21) \rangle (01), \langle (20) \rangle (11), \langle (20) \rangle (10) (01),
\]
\[
\langle (11) \rangle (10)^2, \langle (10) \rangle^3 (01), \langle (12) \rangle (10), \langle (11) \rangle^2, \langle (20) \rangle (01)^2,
\]
\[
\langle (11) \rangle (10) (01), \langle (10) \rangle^2 (01)^2.
\]
From the formula for $k_{3,1}$, given in Section D,

$$s_{3,1} = \frac{(n-1)(n-2)(n-3)}{n(n+1)} k_{3,1} + \frac{1}{n} s_{3,0}s_{0,1}$$

$$+ \frac{3(n-1)}{n(n+1)} s_{1,1}s_{2,0} + \frac{3}{n} s_{2,1}s_{1,0} - \frac{6}{n(n+1)} s_{1,1}s_{1,0}^2$$

$$- \frac{6}{n(n+1)} s_{2,0}s_{1,0}s_{0,1} + \frac{6}{n^2(n+1)} s_{0,1}s_{1,0}^3.$$  

Using tables of symmetric functions (David, Kendall, and Barton, 1966) and the previous results, $<\langle 31 \rangle>$ can be written

$$<\langle 31 \rangle> = \frac{1}{n} s_{3,1} = \frac{(n-1)(n-2)(n-3)}{n(n+1)} k_{3,1} + nk_{0,1}k_{1,0}$$

$$+ \frac{3(n-1)}{n} k_{0,1}k_{2,0}k_{1,0} + \frac{3(n-1)}{n} k_{1,1}k_{1,0}^2$$

$$+ \frac{3(n-1)(n-2)}{n^2} k_{2,1}k_{1,0} + \frac{3(n-1)^3}{n^2(n+1)} k_{1,1}k_{2,0}$$

$$+ \frac{(n-1)(n-2)}{n^2} k_{0,1}k_{3,0}.$$  

From the formula for $k_{2,2}$ in Section D,

$$s_{2,2} = \frac{(n-1)(n-2)(n-3)}{n(n+1)} k_{2,2} + \frac{2}{n} s_{2,1}s_{0,1} + \frac{2}{n} s_{1,2}s_{1,0}$$

$$+ \frac{n-1}{n(n+1)} s_{2,0}s_{0,2} + \frac{2(n-1)}{n(n+1)} s_{1,1}^2$$

$$- \frac{8}{n(n+1)} s_{1,1}s_{0,1}s_{1,0} - \frac{2}{n(n+1)} s_{0,2}s_{1,0}^2$$

$$- \frac{2}{n(n+1)} s_{2,0}s_{0,1}^2 + \frac{6}{n^2(n+1)} s_{1,0}s_{0,1}^2.$$


so
\[ <(22)> = \frac{1}{n} s_{2,2} = \frac{(n-1)(n-2)(n-3)}{n^2(n+1)} k_{2,2} \]
\[
+ \frac{2(n-1)(n-2)}{n^2} k_{0,1}k_{2,1} + \frac{2(n-1)(n-2)}{n^2} k_{1,0}k_{1,2}
\]
\[
+ \frac{4(n-1)}{n} k_{1,1}k_{1,0}^2 + k_{1,0}^2 k_{0,1}^2
\]
\[
+ \frac{n-1}{n} k_{1,0}^2 k_{0,2} + \frac{n-1}{n} k_{2,0}k_{0,1}^2
\]
\[
+ \frac{(n-1)^3}{n^2(n+1)} k_{2,0}k_{0,2} + \frac{2(n-1)^3}{n^2(n+1)} k_{1,1}^2
\]
\[< (30)(01) > = \frac{1}{n(n-1)} \sum_{i}^n x_i^3 y_j = \frac{1}{n(2)} \{ s_{3,2}^1 s_{0,1}^0 - s_{3,1}^1 \}
\]
\[
= k_{0,1}k_{1,0}^3 + \frac{3(n-1)}{n} k_{0,1}k_{2,0}k_{1,0}
\]
\[
+ \frac{(n-1)(n-2)}{n^2} k_{0,1}k_{3,0}
\]
\[
- \frac{(n-1)(n-2)(n-3)}{n(n+1)} k_{3,1}^1 - 3(n-1)k_{1,1}k_{1,0}^2
\]
\[
- \frac{3(n-1)(n-2)}{n} k_{2,1}k_{1,0}^1 - \frac{3(n-1)^3}{n(n+1)} k_{1,1}k_{2,0}
\]
\[< (21)(01) > = \frac{1}{n(2)} \{ s_{2,1}^1 s_{0,1}^0 - s_{2,2}^1 \}
\]
\[
= \frac{(n-2)^2}{n^2} k_{0,1}k_{2,1} + \frac{2(n-2)}{n} k_{1,1}k_{1,0}k_{0,1}
\]
\[
+ k_{1,0}^2 k_{0,1}^2 + \frac{n-1}{n} k_{0,1}^2 k_{2,0}
\]
\[
- \frac{(n-2)(n-3)}{n^2(n+1)} k_{2,2}^2 - \frac{2(n-2)}{n^2} k_{1,0}k_{1,2}
\]
\[
- \frac{1}{n} k_{1,0} k_{0,2} - \frac{(n-1)^2}{n^2(n+1)} k_{2,0} k_{0,2} \\
- \frac{2(n-1)^2}{n^2(n+1)} k_{1,1}^2.
\]

\[< (21) (10) > = \frac{1}{n^{(2)}} \{ s_{2,1} s_{1,0} - s_{3,1} \} \]

\[= \frac{(n-2)(n-3)}{n^2} k_{2,1} k_{1,0} + \frac{2n-3}{n} k_{1,1} k_{1,0}^2 \]
\[+ k_{0,1} k_{1,0}^3 + \frac{n-3}{n} k_{1,0} k_{2,0} k_{0,1}^2 \]
\[= \frac{(n-2)(n-3)}{n^2(n+1)} k_{3,1} - \frac{3(n-1)^2}{n^2(n+1)} k_{1,1} k_{2,0} \]
\[= \frac{(n-2)}{n^2} k_{0,1} k_{3,0}. \]

\[< (20) (02) > = \frac{1}{n^{(2)}} \{ s_{2,0} s_{0,2} - s_{2,2} \} \]

\[= \frac{(n-1)(n^2+1)}{n^2(n+1)} k_{2,0} k_{0,2} + \frac{(n-1)}{n} k_{2,0} k_{0,1}^2 \]
\[+ \frac{(n-1)}{n} k_{0,2} k_{1,0}^2 + k_{1,0}^2 k_{0,1}^2 \]
\[= \frac{(n-2)(n-3)}{n^2(n+1)} k_{2,2} - \frac{2(n-2)}{n^2} k_{0,1} k_{2,1} \]
\[= \frac{2(n-2)}{n^2} k_{1,0} k_{1,2} - \frac{4}{n} k_{1,1} k_{1,0} k_{0,1} \]
\[= \frac{2(n-1)^2}{n^2(n+1)} k_{1,1}^2. \]
\[
<(10)^2(02)> = \frac{1}{n(3)} \{ s_1,0^2s_2,0^2s_1,2^2s_1,0^2s_2,0^2s_0,2^2s_2,2 \}
\]

\[
= \frac{(n-1)}{n} k_{1,0} k_{0,2}^2 + n^2 k_{1,0} k_{0,1}^2
\]

\[
- \frac{2(n-2)}{n^2} k_{1,0} k_{1,2} - \frac{4}{n} k_{1,0} k_{0,1} k_{1,1}
\]

\[
- \frac{(n-1)(n^2-2n+3)}{2(n+1)(n-2)} k_{2,0} k_{0,2}^2 - \frac{1}{n} k_{2,0} k_{0,1} k_{0,1}
\]

\[
+ \frac{2(n-3)}{n^2(n+1)} k_{2,2} + \frac{4}{n} k_{0,1} k_{2,1}
\]

\[
+ \frac{4(n-1)^2}{n^2(n+1)(n-2)} k_{1,1}^2
\]

\[
<(20)(11)> = \frac{1}{n(3)} \{ s_2,0^2s_1,1^2s_3,1 \}
\]

\[
= \frac{(n-1)(n^2-2n+3)}{n^2(n+1)} k_{2,0} k_{1,1} + \frac{n-3}{n} k_{1,1} k_{1,0}^2
\]

\[
+ \frac{n-3}{n} k_{2,0} k_{1,0} k_{0,1}
\]

\[
+ k_{1,0} k_{0,1}^3 - \frac{(n-2)(n-3)}{n^2(n+1)} k_{3,1}
\]

\[
- \frac{3(n-2)}{n^2} k_{2,1} k_{1,0} - \frac{(n-2)}{n^2} k_{0,1} k_{3,0}
\]

\[
<(20)(10)(01)> = \frac{1}{n(3)} \{ s_2,0^2s_1,0^2s_0,1^2s_2,0^2s_1,1^2s_2,1^2s_1,0
\]

\[
- s_3,0^2s_0,1^2s_3,1 \}\}.
\]
\langle (20) (10) (01) \rangle = \frac{n^{-3}}{n} k_{1,0} k_{0,1} k_{2,0} k_{1,0} k_{3,0}
\quad - \frac{(n-1)(n-3)}{n^2(n+1)} k_{2,0} k_{1,1} - \frac{3}{n} k_{1,0} k_{2,1,1}
\quad - \frac{n-6}{n^2} k_{1,0} k_{2,1} - \frac{(n-2)}{n^2} k_{0,1} k_{3,0}
\quad + \frac{2(n-3)}{n^2(n+1)} k_{3,1} .

\langle (11) (10)^2 \rangle = \frac{1}{n(3)} \{ s_{1,1} s_{1,1,0}^2 - s_{2,0} s_{1,1,0} s_{1,0} s_{1,1,0} + 2 s_{2,3,1,1} \}
\quad = \frac{n^{-3}}{n} k_{1,0} k_{1,1} + k_{1,0} k_{3,0} - \frac{3}{n} k_{1,0} k_{2,0} k_{0,1}
\quad - \frac{(n-1)(n+3)}{n^2(n+1)} k_{1,1} k_{2,0} - \frac{2(n-3)}{n^2} k_{2,1} k_{1,0}
\quad + \frac{2(n-3)}{n^2(n+1)} k_{3,1} + \frac{2}{n} k_{0,1} k_{3,0} .

\langle (10)^3 (01) \rangle = \frac{1}{n(4)} \{ s_{1,0}^3 s_{0,1} - 3 s_{1,1} s_{1,0} s_{1,1} s_{1,0} - 3 s_{2,0} s_{1,0} s_{0,1} \}
\quad + 3 s_{2,0} s_{1,0} s_{1,0} + 6 s_{2,0} s_{1,0} s_{1,0} + 2 s_{3,0} s_{0,1} s_{1,0} - 6 s_{3,1} \}
\quad = k_{1,0}^3 k_{0,1} - \frac{3}{n} k_{1,0}^2 k_{1,1} - \frac{3}{n} k_{1,0} k_{2,0} k_{0,1}
\quad + \frac{3(n-1)}{n^2(n+1)} k_{2,0} k_{1,1} + \frac{6}{n^2} k_{1,0} k_{2,1}
\quad + \frac{2}{n^2} k_{0,1} k_{3,0} - \frac{6}{n^2(n+1)} k_{3,1} .
\[
< (12) (10) > = \frac{1}{n(2)} \{ s_{1,2} s_{1,0} - s_{2,2} \}
\]
\[
= \frac{(n-2)^2}{n^2} k_{1,0} k_{1,2} + \frac{2(n-2)}{n} k_{1,0} k_{0,1} k_{1,1}
\]
\[
+ k_{1,0}^2 k_{0,1}^2 + \frac{n-1}{n} k_{1,0}^2 k_{0,2}^2 - \frac{(n-2)(n-3)}{n^2(n+1)} k_{2,2}^2
\]
\[
- \frac{2(n-2)}{n^2} k_{0,1} k_{2,1} - \frac{1}{n} k_{2,0} k_{0,1}^2
\]
\[
- \frac{(n-1)^2}{n^2(n+1)} k_{2,0} k_{0,2} - \frac{2(n-1)^2}{n^2(n+1)} k_{1,1}^2
\]
\[
< (11)^2 > = \frac{1}{n(2)} \{ s_{1,1}^2 - s_{2,2} \}
\]
\[
= \frac{(n-1)(n^2-n+2)}{n^2(n+1)} k_{1,1}^2 + \frac{2(n-2)}{n} k_{1,1} k_{1,0} k_{0,1}
\]
\[
+ k_{1,0}^2 k_{0,1}^2 - \frac{(n-2)(n-3)}{n^2(n+1)} k_{2,2}^2
\]
\[
- \frac{2(n-2)}{n^2} k_{0,1} k_{2,1} - \frac{2(n-2)}{n^2} k_{1,0} k_{1,2}
\]
\[
- \frac{1}{n} k_{1,0} k_{0,2} - \frac{1}{n} k_{2,0} k_{0,1}^2 - \frac{(n-1)^2}{n^2(n+1)} k_{2,0} k_{0,2}^2
\]
\[
< (20) (01)^2 > = \frac{1}{n(3)} \{ s_{2,0} s_{0,1}^2 - s_{2,0} s_{0,2} - 2 s_{2,1} s_{0,1} + 2 s_{2,2} \}
\]
\[
= \frac{n-1}{n} k_{2,0} k_{0,1}^2 + k_{1,0}^2 k_{0,1}^2
\]
\[
- \frac{(n-1)(n^2-n+2)}{n^2(n+1)(n-2)} k_{2,0} k_{0,2} - \frac{1}{n} k_{1,0} k_{0,2}^2
\]
\[
- \frac{2(n-2)}{n^2} k_{0,1} k_{2,1} - \frac{4}{n} k_{1,1} k_{1,0} k_{0,1}
\]
\[<(11)(10)(01)> = \frac{1}{n^3} \{ s_{1,1} s_{1,0} s_{0,1} - s_{1,1} \} - s_{1,2} s_{1,0} \]

\[= \frac{n-4}{n} k_{1,1} k_{0,1} - k_{1,0}^2 k_{0,1} \]

\[- \frac{(n-1) (n^2 - 3n + 4)}{n^2 (n+1) (n-2)} k_{1,1}^2 \]

\[- \frac{(n-4)}{n^2} k_{1,0} k_{1,2} - \frac{1}{n} k_{1,0}^2 k_{0,2} \]

\[- \frac{1}{n} k_{0,1}^2 k_{2,0} \]

\[- \frac{n-4}{n^2} k_{0,1} k_{2,1} + \frac{2(n-3)}{n^2 (n+1)} k_{2,2} \]

\[+ \frac{2(n-1)^2}{n^2 (n+1) (n-2)} k_{2,0} k_{0,2} \]

\[<(10)^2 (01)^2> = \frac{1}{n^4} \{ s_{1,0}^2 s_{0,1}^2 - s_{1,0}^2 s_{0,2}^2 - 4 s_{1,1} s_{1,0} s_{0,1} \}

\[- s_{2,0} s_{0,1}^2 + 2 s_{1,1} s_{0,1}^2 + 4 s_{1,2} s_{1,0}^2 + 4 s_{2,0} s_{0,2} \]

\[+ 4 s_{2,1} s_{0,1} - 6 s_{2,2} \} \].
\[(10)^2 (01)^2 = k_{1,0} k_{0,1}^2 - \frac{1}{n} k_{1,0} k_{0,2}^2 - \frac{4}{n} k_{1,1} k_{1,0} k_{0,1}^\prime \]

\[- \frac{1}{n} k_{0,1}^2 k_{2,0} + \frac{2(n-1)}{n^2(n+1)} k_{1,1}^2 \]

\[+ \frac{4}{n^2} k_{1,0} k_{1,2} + \frac{(n-1)}{n^2(n+1)} k_{2,0} k_{0,2} \]

\[+ \frac{4}{n^2} k_{0,1} k_{2,1} - \frac{6}{n^2(n+1)} k_{2,2} \]

G. Bivariate Polykays

Polykays may be defined for multivariate situations in direct analogy to the univariate case. Only the bivariate case will be considered here. The results to be presented can be extended in theory but the practical difficulties which arise increase rapidly as the number of variables increases, as will soon become evident.

Robson (1957) first used the idea of a multivariate polykay. He used an extension of Tukey's symbolic multiplication in forming the polykays but this method is limited in practical applications. A new method for the formation of bivariate polykays (which could be extended to multivariate polykays) based on Fisher's pattern functions will be presented together with a complete list of bivariate polykays through weight 8. All these formulas are new.

From the bivariate generating relation,
where

\[ \Sigma p_i \pi_i = r, \quad \Sigma p'_i \pi'_i = s, \]

so

\[ \kappa_{r,s} = \sum (-1)^{\rho-1}(\rho-1)! \frac{x!}{\pi_1 \pi_2 \ldots \pi_1' \pi_2'} \frac{\mu^i}{(p_1^i)^1 (p_2^i)^2 \ldots} \]

Now consider

\[ \kappa_{u,m} = \sum (-1)^{\rho'-1}(\rho'-1)! \frac{u!}{\alpha_1 \alpha_2 \ldots} \frac{\mu^i}{(q_1^i)^1 (q_2^i)^2 \ldots} \]

If a quantity \( k(r,s)(u,m) \) is wanted whose expectation is

\[ \kappa_{r,s} \kappa_{u,m} \]
\[ k_{r,s}^ru,m = \sum (-l)^{\rho-1}(\rho-1)!) \frac{r!s!}{\pi_1!\pi_2!...} \frac{1}{(p_1)!...(p_1^!)...(p_1^!)} \]

\[ \frac{(-1)^{\rho^'-1}(\rho^'-1)!) u!m!}{\alpha_1!\alpha_2!...} \frac{1}{(q_1)!...(q_1^!)...(q_1^!)} \]

\[ \alpha^j \cdots \alpha^j \cdots \mu^j \cdots \mu^j \cdots \]

\[ \cdot (u_1^j, p_1^j, p_2^j, \cdots \cdots \cdot (q_1^j, q_1^j, q_2^j, q_2^j, \cdots \cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \...
\[ k(1,1)(1,1) = \{(11)\} - \{(10)(01)\} \cap \{(11)\} - \{(10)(01)\} \]
\[ = \{(11)\}^2 - 2\{(10)(01)\}(11) + \{(10)\}^2(01)^2. \]

Now the bracket functions can be expressed in terms of bivariate \( k \) statistics using some of the equivalences in Section F. So

\[ k(1,1)(1,1) = \frac{(n-1)(n^2-n+2)}{n^2(n+1)} k_{1,1}^2 + \frac{2(n-2)}{n} k_{1,1}k_{1,0}k_{0,1} \]
\[ + k_{1,0}^2 k_{0,1}^2 - \frac{(n-2)(n-3)}{n^2(n+1)} k_{2,2} - \frac{2(n-2)}{n^2} k_{0,1}k_{2,1} \]
\[ - \frac{2(n-2)}{n^2} k_{1,0}k_{1,2} - \frac{1}{n} k_{1,0}^2 k_{0,2} - \frac{1}{n} k_{2,0}k_{0,1}^2 \]
\[ - \frac{(n-1)^2}{n^2(n+1)} k_{2,0}k_{0,2} - 2\left\{ \frac{n-4}{n} k_{1,1}k_{1,0}k_{0,1} + k_{1,0}k_{0,1}^2 \right\} \]
\[ - \frac{(n-1)(n^2-3n+4)}{n^2(n+1)(n-2)} k_{1,1}^2 - \frac{n-4}{n^2} k_{1,0}k_{1,2} - \frac{1}{n} k_{1,0}^2 k_{0,2} \]
\[ - \frac{1}{n} k_{0,1}^2 k_{2,0} - \frac{n-4}{n^2} k_{0,1}k_{2,1} + \frac{2(n-3)}{n^2(n+1)} k_{2,2} \]
\[ + \frac{2(n-1)^2}{n^2(n+1)(n-2)} k_{2,0}k_{0,2} \} + k_{1,0}^2 k_{0,1}^2 - \frac{1}{n} k_{1,0}^2 k_{0,2} \]
\[ - \frac{4}{n} k_{1,1}k_{1,0}k_{0,1} - \frac{1}{n} k_{0,1}^2 k_{2,0} + \frac{2(n-1)}{n^2(n+1)} k_{1,1}^2 \]
\[ + \frac{4}{n^2} k_{1,0}k_{1,2} + \frac{n-1}{n^2(n+1)} k_{2,0}k_{0,2} \]
or

\[ k_{1,1}(1,1) = \frac{(n-1)^2}{(n+1)(n-2)} k_{1,1} - \frac{(n-1)}{n(n+1)} k_{2,2} \]

\[ - \frac{(n-1)}{(n+1)(n-2)} k_{2,0} k_{0,2} \cdot \]

Although the algebra involved here is somewhat lengthy, the example is a fairly simple one. As the total weight of the polykay increases, however, the work involved becomes much more extensive and so alternative procedures are desirable.

To find one such alternative, consider the simultaneous distribution of two bivariate \( k \) statistics, say \( k_{\alpha,\alpha'} \) and \( k_{\beta,\beta'} \). Denote the \( rs \) cumulant of their joint distribution by \( \kappa[(\alpha\alpha')^r(\beta\beta')^s] \) (which is also often written \( \kappa(\alpha; \beta; \beta'; \beta') \), and the corresponding mean by \( \mu[(\alpha\alpha')^r(\beta\beta')^s] \). These quantities are related by the generating relation

\[ 1 + \mu[(\alpha\alpha')] t_1 + \mu[(\beta\beta')] t_2 + \mu[(\alpha\alpha')(\beta\beta')] t_1 t_2 \]

\[ + \ldots + \mu[(\alpha\alpha')^p(\beta\beta')^q] \frac{t_1 \overline{t_2}}{p!q!} + \ldots \]
\[ = \exp\left\{ \kappa[(\alpha\alpha')] t_1 + \kappa[(\beta\beta')] t_2 + \ldots + \kappa[(\alpha\alpha')^p(\beta\beta')^q] \frac{t_1^p t_2^q}{p!q!} \right\} \]

So

\[ 1 + \mu[(\alpha\alpha')] t_1 + \mu[(\beta\beta')] t_2 + \ldots + \mu[(\alpha\alpha')^p(\beta\beta')^q] \frac{t_1^p t_2^q}{p!q!} \ldots \]

\[ = \exp\{\kappa[(\alpha\alpha')] t_1\} \exp\{\kappa[(\beta\beta')] t_2\} \ldots \]
\[ \ldots \exp\{\kappa[(\alpha\alpha')^p(\beta\beta')^q] \frac{t_1^p t_2^q}{p!q!} \} \ldots \]

\[ = 1 + \kappa[(\alpha\alpha')] t_1 + \kappa[(\beta\beta')] t_2 + \kappa[(\alpha\alpha')] [(\beta\beta')] \]
\[ + \kappa[(\alpha\alpha')(\beta\beta')] t_1 t_2 + \ldots \]

and consequently

\[ \mu[(\alpha\alpha')(\beta\beta')] = \kappa[(\alpha\alpha')] \kappa[(\beta\beta')] + \kappa[(\alpha\alpha')(\beta\beta')] \]

or

\[ \kappa[(\alpha\alpha')] [(\beta\beta')] = \mu[(\alpha\alpha')(\beta\beta')] - \kappa[(\alpha\alpha')(\beta\beta')] \] \hfill (2)

Now the right hand side of (2) may be written as the expectation of a linear combination of \( k \) statistics and so an expression for \( k(\alpha, \alpha')(\beta, \beta') \) can be found.

The rules for the expansion of \( \kappa(\alpha, \alpha'; \beta, \beta') \) follow Fisher's precisely except that there are more partitions to consider, namely those for \( \alpha, \alpha', \beta, \) and \( \beta' \).

As an example, the procedure will be applied to \( k(1,1)(1,1) \).
From (2), \( \kappa_{11}^2 = \kappa_{11}^2 = E(k_{1,1}^2) - \kappa_{1,1}^2 \)

To evaluate \( \kappa_{1,1}^2 \), the following patterns are possible:

\[
\begin{array}{ccc}
(11) & (11) & (22) \\
(11) & (11) & \\
(01) & (10) & (11) \\
(10) & (01) & (11) \\
(11) & (11) & \\
\end{array}
\]

The coefficients are now calculated using the four columns in the pattern; the \( n \) coefficient is calculated exactly as before. Entries of \((00)\) are inadmissible.

This gives

\[
\kappa_{1,1}^2 = \frac{1}{n} \kappa_{2,2}^2 + \frac{1}{n-1} \kappa_{11}^2 + \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2}.
\]

So

\[
(1 + \frac{1}{n-1}) \kappa_{11}^2 = E[k_{11}^2 - \frac{1}{n} k_{2,2}^2 - \frac{1}{n-1} k_{2,0} \kappa_{0,2}]
\]

or

\[
k(1,1)(1,1) = \frac{n-1}{n} \{k_{1,1}^2 - \frac{1}{n} k_{2,2}^2 - \frac{1}{n-1} k_{2,0} \kappa_{0,2}\}.
\]

Now, using the same procedure, it is easy to show that

\[
k_{2,0}(0,2) = k_{2,0} k_{0,2} - \frac{1}{n} k_{2,2}^2 - \frac{2}{n-1} k(1,1)(1,1)
\]

and when this is substituted in (3), the previous result for \( k(1,1)(1,1) \) is obtained.
The process easily becomes a fairly mechanical one and it is not necessary to write any of the cumulants at all; the pattern functions are calculated and the quantities expressed in terms of bivariate polykays.

This has been the process used in deriving all of the following results, although some have been checked using the much lengthier symbolic multiplication. The details of the derivations will be omitted and the results simply stated. No univariate results are included since most of these are available (Wishart, 1952a).

1. **Weight 2**
   \[ k(1,0)(0,1) = k_{1,0}k_{0,1} - \frac{1}{n} k_{1,1} \]

2. **Weight 3**
   \[ k(1,1)(1,0) = k_{1,1}k_{1,0} - \frac{1}{n} k_{2,1} \]
   \[ k(1,1)(0,1) = k_{1,1}k_{0,1} - \frac{1}{n} k_{1,2} \]
   \[ k(2,0)(0,1) = k_{2,0}k_{0,1} - \frac{1}{n} k_{2,1} \]

3. **Weight 4**
   \[ k(1,1)(1,1) = \frac{n-1}{n} \{ k_{1,1}^2 - \frac{1}{n} k_{2,2} - \frac{1}{n-1} k(2,0)(0,2) \} \]
   \[ k(2,0)(0,2) = k_{2,0}k_{0,2} - \frac{1}{n} k_{2,2} - \frac{2}{n-1} k(1,1)(1,1) \]
\[ k(3,0)(0,1) = k_{3,0}k_{0,1} - \frac{1}{n} k_{3,1} \]

\[ k(1,1)(2,0) = \frac{n-1}{n+1} \{ k_{1,1}k_{2,0} - \frac{1}{n} k_{3,1} \} \]

\[ k(2,1)(0,1) = k_{2,1}k_{0,1} - \frac{1}{n} k_{2,2} \]

\[ k(1,2)(0,1) = k_{1,2}k_{0,1} - \frac{1}{n} k_{1,3} \]

4. **Weight 5**

\[ k(4,0)(0,1) = k_{4,0}k_{0,1} - \frac{1}{n} k_{4,1} \]

\[ k(2,0)(0,3) = k_{2,0}k_{0,3} - \frac{1}{n} k_{2,3} - \frac{6}{n-1} k(1,1)(1,2) \]

\[ k(3,0)(1,1) = \frac{n-1}{n+2} \{ k_{3,0}k_{1,1} - \frac{3}{n-1} k(2,0)(2,1) - \frac{1}{n} k_{4,1} \} \]

\[ k(2,1)(0,2) = \frac{n-1}{n+1} \{ k_{2,1}k_{0,2} - \frac{1}{n} k_{2,3} - \frac{4}{n-1} k(1,1)(1,2) \} \]

\[ k(2,1)(2,0) = \frac{n-1}{n+3} \{ k_{2,1}k_{2,0} - \frac{1}{n} k_{4,1} - \frac{2}{n-1} k(3,0)(1,1) \} \]

\[ k(1,2)(2,0) = \frac{n-1}{n+1} \{ k_{1,2}k_{2,0} - \frac{1}{n} k_{3,2} - \frac{4}{n-1} k(1,1)(2,1) \} \]

\[ k(1,1)(1,2) = \frac{n-1}{n+2} \{ k_{1,1}k_{1,2} - \frac{1}{n} k_{2,3} - \frac{1}{n-1} k(2,0)(0,3) \]

\[ -\frac{2}{n-1} k(2,1)(0,2) \} \]

5. **Weight 6**

\[ k(5,0)(0,1) = k_{5,0}k_{0,1} - \frac{1}{n} k_{5,1} \]

\[ k(4,1)(1,0) = k_{4,1}k_{1,0} - \frac{1}{n} k_{5,1} \]
\[ k(4,1)(0,1) = k_4,1k_0,1 - \frac{1}{n} k_4,2 \]
\[ k(3,2)(1,0) = k_4,1k_0,1 - \frac{1}{n} k_4,2 \]
\[ k(3,2)(0,1) = k_3,2k_0,1 - \frac{1}{n} k_3,3 \]
\[ k(4,0)(0,2) = k_4,0k_0,2 - \frac{1}{n} k_4,2 - \frac{8}{n-1} k(3,1)(1,1) \]
\[ - \frac{6}{n-1} k(2,1)(2,1) \]
\[ k(3,1)(2,0) = k_3,1k_2,0 - \frac{1}{n} k_5,1 - \frac{2}{n-1} k(4,0)(1,1) \]
\[ - \frac{12}{n-1} k(3,0)(2,1) \]
\[ k(3,1)(0,2) = \frac{n-1}{n+1}\{k_3,1k_0,2 - \frac{1}{n} k_3,3 - \frac{6}{n-1} k(1,1)(2,2) \}
\[ - \frac{6}{n-1} k(1,2)(2,1) \}
\[ k(2,2)(2,0) = \frac{n-1}{n+3}\{k_2,2k_2,0 - \frac{1}{n} k_4,2 - \frac{2}{n-1} k(3,0)(1,2) \}
\[ - \frac{8}{n-1} k(2,1)(2,1) - \frac{4}{n-1} k(3,1)(1,1) \}
\[ k(4,0)(1,1) = \frac{n-1}{n+3}\{k_4,0k_1,1 - \frac{1}{n} k_5,1 - \frac{4}{n-1} k(3,1)(2,0) \}
\[ - \frac{6}{n-1} k(3,0)(2,1) \}
\[ k(3,1)(1,1) = \frac{n-1}{n+2}\{k_3,1k_1,1 - \frac{1}{n} k_4,2 - \frac{3}{n-1} k(2,2)(2,0) \}
\[ - \frac{3}{n-1} k(2,1)(2,1) - \frac{3}{n-1} k(3,0)(1,2) \} \]
\[ k_{2,2}(1,1) = \frac{n-1}{n+3} (k_{2,2}^{1,1} k_{1,1}^{1,1} - \frac{1}{n} k_{3,3}^{3,3} - \frac{1}{n-1} k_{3,0}^{3,0}(0,3)) \\
- \frac{2}{n-1} k_{3,1}^{3,1}(0,2) - \frac{1}{n-1} k_{1,2}^{1,2}(2,0) \\
- \frac{5}{n-1} k_{2,1}^{2,1}(1,2) \] 

\[ k_{2,1}(3,0) = \frac{n-1}{n+8} (k_{2,1}^{3,0} k_{3,0}^{3,0} - \frac{1}{n} k_{5,1}^{5,1} - \frac{3}{n-1} k_{4,0}^{4,0}(1,1)) \\
- \frac{6}{n-1} k_{3,1}^{3,1}(2,0) \\
- \frac{6n}{(n-1)(n-2)} k_{1,1}^{1,1}(2,0) \] 

\[ k_{1,2}(3,0) = \frac{n-1}{n+2} (k_{1,2}^{3,0} k_{3,0}^{3,0} - \frac{1}{n} k_{4,2}^{4,2} - \frac{3}{n-1} k_{2,2}^{2,2}(2,0)) \\
- \frac{12}{n-1} k_{2,1}^{2,1}(2,1) - \frac{6}{n-1} k_{3,1}^{3,1}(1,1) \\
- \frac{6n}{(n-1)(n-2)} k_{2,0}^{2,0}(1,1) \] 

\[ k_{2,1}(2,1) = \frac{n-1}{n+4} (k_{2,1}^{2} k_{2,1}^{2} - \frac{1}{n} k_{4,2}^{4,2} - \frac{4}{n-1} k_{2,2}^{2,2}(2,0)) \\
- \frac{4}{n-1} k_{3,1}^{3,1}(1,1) - \frac{1}{n-1} k_{4,0}^{4,0}(0,2) \\
- \frac{4}{n-1} k_{3,0}^{3,0}(1,2) \\
- \frac{4n}{(n-1)(n-2)} k_{2,0}^{2,0}(1,1) \] 

\[ k_{2,0}(1,1) = \frac{n-1}{n+2} (k_{2,0}^{1,1} k_{2,0}^{1,1} - \frac{1}{n} k_{4,2}^{4,2} - \frac{4}{n-1} k_{2,2}^{2,2}(2,0)) \\
- \frac{4}{n-1} k_{3,1}^{3,1}(1,1) - \frac{1}{n-1} k_{4,0}^{4,0}(0,2) \\
- \frac{4n}{(n-1)(n-2)} k_{2,0}^{2,0}(1,1) \] 

\[ k_{3,0}(0,3) = k_{3,0}^{0,3} k_{3,0}^{0,3} - \frac{1}{n} k_{3,3}^{3,3} - \frac{9}{n-1} k_{2,2}^{2,2}(1,1) \\
- \frac{9}{n-1} k_{2,1}^{2,1}(1,2) \]
\[ k(2,1)(1,2) = \frac{n-1}{n+7}(k_{2,1}k_{1,2} - \frac{1}{n}k_{3,3} - \frac{2}{n-1}k_{(3,1)(0,2)} - \frac{2}{n-1}k_{(1,3)(2,0)} - \frac{1}{n-1}k_{(3,0)(0,3)} - \frac{5}{n-1}k_{(2,2)(1,1)} - \frac{2n}{(n-1)(n-2)}k_{(1,1)(1,1)(1,1)} - \frac{4n}{(n-1)(n-2)}k_{(2,0)(1,1)(0,2)}) \]

6. **Weight 7**

\[ k(6,0)(0,1) = k_{6,0}k_{0,1} - \frac{1}{n}k_{6,1} \]

\[ k(5,1)(1,0) = k_{5,1}k_{1,0} - \frac{1}{n}k_{6,1} \]

\[ k(5,1)(0,1) = k_{5,1}k_{0,1} - \frac{1}{n}k_{5,2} \]

\[ k(4,2)(1,0) = k_{4,2}k_{1,0} - \frac{1}{n}k_{5,2} \]

\[ k(4,2)(0,1) = k_{4,2}k_{0,1} - \frac{1}{n}k_{4,3} \]

\[ k(3,3)(1,0) = k_{3,3}k_{1,0} - \frac{1}{n}k_{4,3} \]

\[ k(5,0)(0,2) = k_{5,0}k_{0,2} - \frac{1}{n}k_{5,2} - \frac{10}{n-1}k_{(4,1)(1,1)} - \frac{20}{n-1}k_{(3,1)(2,1)} \]
\[ k_{(4,1)}(0,2) = \frac{n-1}{n+1}k_{4,0}k_{0,2} - \frac{1}{n}k_{4,3} - \frac{12}{n-1}k_{(2,2)}(2,1) \]
\[ - \frac{8}{n-1}k_{(3,2)}(1,1) - \frac{8}{n-1}k_{(3,1)}(1,2) \}
\[ k_{(4,1)}(2,0) = \frac{n-1}{n+7}k_{4,1}k_{2,0} - \frac{1}{n}k_{6,1} - \frac{2}{n-1}k_{(5,0)}(1,1) \]
\[ - \frac{8}{n-1}k_{(4,1)}(2,0) - \frac{12}{n-1}k_{(3,1)}(3,0) \}
\[ k_{(3,2)}(2,0) = \frac{n-1}{n+5}k_{3,2}k_{2,0} - \frac{1}{n}k_{5,2} - \frac{2}{n-1}k_{(4,0)}(1,2) \]
\[ - \frac{4}{n-1}k_{(4,1)}(1,1) - \frac{6}{n-1}k_{(3,0)}(2,2) \]
\[- \frac{12}{n-1}k_{(3,1)}(2,1) \}
\[ k_{(3,2)}(0,2) = \frac{n-1}{n+3}k_{3,2}k_{0,2} - \frac{1}{n}k_{3,4} - \frac{12}{n-1}k_{(2,2)}(1,2) \]
\[ - \frac{2}{n-1}k_{(0,3)}(3,1) - \frac{6}{n-1}k_{(2,3)}(1,1) \]
\[- \frac{6}{n-1}k_{(1,3)}(2,1) \}
\[ k_{(5,0)}(1,1) = \frac{n-1}{n+4}k_{5,0}k_{1,1} - \frac{1}{n}k_{6,1} - \frac{10}{n-1}k_{(4,0)}(2,1) \]
\[ - \frac{5}{n-1}k_{(4,1)}(2,0) - \frac{10}{n-1}k_{(3,1)}(3,0) \}
\[ k_{(4,1)}(1,1) = \frac{n-1}{n+4}k_{4,1}k_{1,1} - \frac{1}{n}k_{5,2} - \frac{1}{n-1}k_{(5,0)}(0,2) \]
\[ - \frac{4}{n-1}k_{(4,0)}(1,2) - \frac{4}{n-1}k_{(3,2)}(2,0) \]
\[- \frac{6}{n-1}k_{(3,0)}(2,2) - \frac{10}{n-1}k_{(3,1)}(2,1) \}
\[ k_{(3,2)}(1,1) = \frac{n-1}{n+4}k_{3,2}^{1,1} - \frac{1}{n}k_{4,3} - \frac{1}{n-1}k_{(4,0)}(0,3) \]
\[- \frac{2}{n-1}k_{(4,1)}(0,2) - \frac{3}{n-1}k_{(2,0)}(2,3) \]
\[- \frac{3}{n-1}k_{(3,0)}(1,3) - \frac{9}{n-1}k_{(2,2)}(2,1) \]
\[- \frac{7}{n-1}k_{(3,1)}(1,2) \]
\[ k_{(4,0)}(0,3) = k_{4,0}k_{0,3} - \frac{1}{n}k_{4,3} - \frac{18}{n-1}k_{(2,2)}(2,1) \]
\[- \frac{12}{n-1}k_{(3,2)}(1,1) - \frac{12}{n-1}k_{(3,1)}(1,2) \]
\[- \frac{18n}{(n-1)(n-2)}k_{(2,1)}(1,1) \]
\[ k_{(2,2)}(0,3) = \frac{n-1}{n+8}(k_{2,2}k_{0,3} - \frac{1}{n}k_{2,5} - \frac{3}{n-1}k_{(0,4)}(2,1) \]
\[- \frac{3}{n-1}k_{(1,4)}(1,1) - \frac{3}{n-1}k_{(2,3)}(0,2) \]
\[- \frac{18}{n-1}k_{(1,3)}(1,2) \]
\[- \frac{6n}{(n-1)(n-2)}[4k_{(0,2)}(1,2)(1,1) \]
\[ + k_{(0,3)}(1,1)(1,1) + k_{(0,2)}(0,2)(2,1) \]}
\[ k_{(3,1)}(0,3) = \frac{n-1}{n+2}(k_{3,1}k_{0,3} - \frac{1}{n}k_{3,4} - \frac{3}{n-1}k_{(0,2)}(3,2) \]
\[- \frac{18}{n-1}k_{(2,2)}(1,2) - \frac{9}{n-1}k_{(2,3)}(1,1) \]
\[- \frac{9}{n-1}k_{(1,3)}(2,1) \]
\[ k_{(3,1)}(3,0) = \frac{n-1}{n+17} \{ k_{3,1} k_{3,0} - \frac{1}{n} k_{6,1} - \frac{3}{n-1} k_{5,0}(1,1) + k_{(1,2)}(1,1)(1,1) \} \]

\[ k_{(4,0)}(2,1) = \frac{n-1}{n+13} \{ k_{4,0} k_{2,1} - \frac{1}{n} k_{6,1} - \frac{4}{n-1} k_{5,0}(1,1) - \frac{8}{n-1} k_{(4,1)}(2,0) - \frac{16}{n-1} k_{3,1}(3,0) - \frac{12n}{(n-1)(n-2)} k_{(2,0)}(3,0)(1,1) - \frac{6n}{(n-1)(n-2)} k_{(2,0)}(2,0)(2,1) \} \]

\[ k_{(4,0)}(1,2) = \frac{n-1}{n+3} \{ k_{4,0} k_{1,2} - \frac{1}{n} k_{5,2} - \frac{8}{n-1} k_{(4,1)}(1,1) - \frac{4}{n-1} k_{(3,2)}(2,0) - \frac{6}{n-1} k_{3,0}(2,2) - \frac{20}{n-1} k_{3,1}(2,1) - \frac{24n}{(n-1)(n-2)} k_{(2,0)}(2,1)(1,1) - \frac{12n}{(n-1)(n-2)} k_{(3,0)}(1,1)(1,1) \} \]

\[ - \frac{18n}{(n-1)(n-2)} \{ k_{(0,2)}(1,1)(2,1) + k_{(1,2)}(1,1)(1,1) \} \]
\[ k_{(3,1)}(2,1) = k_{3,1}k_{2,1} - \frac{1}{n} k_{5,2} - \frac{1}{n-1} k_{(5,0)}(0,2) \]

\[ - \frac{5}{n-1} k_{(4,0)}(1,2) - \frac{5}{n-1} k_{(4,1)}(1,1) \]

\[ - \frac{6}{n-1} k_{(3,2)}(2,0) - \frac{9}{n-1} k_{(3,0)}(2,2) \]

\[ - \frac{16}{n-1} k_{(3,1)}(2,1) \]

\[ - \frac{3n}{(n-1)(n-2)} k_{(3,0)}(1,1)(1,1) \]

\[ - \frac{6n}{(n-1)(n-2)} k_{(3,0)}(2,0)(0,2) \]

\[ - \frac{6n}{(n-1)(n-2)} k_{(2,0)}(2,0)(1,2) \]

\[ k_{(3,1)}(1,2) = \frac{n-1}{n+10}(k_{3,1}k_{1,2} - \frac{1}{n} k_{4,3} - \frac{1}{n-1} k_{(4,0)}(0,3) \]

\[ - \frac{2}{n-1} k_{(4,1)}(0,2) - \frac{3}{n-1} k_{(2,0)}(2,3) \]

\[ - \frac{15}{n-1} k_{(2,2)}(2,1) - \frac{3}{n-1} k_{(3,0)}(1,3) \]

\[ - \frac{7}{n-1} k_{(3,2)}(1,1) \]

\[ - \frac{6n}{(n-1)(n-2)} k_{(2,1)}(2,0)(0,2) \]

\[ - \frac{12n}{(n-1)(n-2)} k_{(2,0)}(1,1)(1,2) \]

\[ - \frac{9n}{(n-1)(n-2)} k_{(2,1)}(1,1)(1,1) \]
\[ k_{(2,2)(2,1)} = \frac{n-1}{n+14} k_{2,2} k_{2,1} - \frac{1}{n} k_{4,3} - \frac{1}{n-1} k_{(4,0)(0,3)} \]

\[ - \frac{2}{n-1} k_{(4,1)(0,2)} - \frac{4}{n-1} k_{(2,0)(2,3)} \]

\[ - \frac{4}{n-1} k_{(3,0)(1,3)} - \frac{6}{n-1} k_{(3,2)(1,1)} \]

\[ - \frac{10}{n-1} k_{(3,1)(1,2)} \]

\[ - \frac{2n}{(n-1)(n-2)} k_{(2,0)(2,0)(0,3)} \]

\[ - \frac{8n}{(n-1)(n-2)} k_{(2,1)(2,0)(0,2)} \]

\[ - \frac{12n}{(n-1)(n-2)} k_{(2,0)(1,1)(1,2)} \]

\[ - \frac{8n}{(n-1)(n-2)} k_{(2,1)(1,1)(1,1)} \}

7. **Weight 8**

\[ k_{(7,0)(0,1)} = k_{7,0} k_{0,1} - \frac{1}{n} k_{7,1} \]

\[ k_{(6,1)(0,1)} = k_{6,1} k_{0,1} - \frac{1}{n} k_{6,2} \]

\[ k_{(6,1)(1,0)} = k_{6,1} k_{1,0} - \frac{1}{n} k_{7,1} \]

\[ k_{(5,2)(0,1)} = k_{5,2} k_{0,1} - \frac{1}{n} k_{5,3} \]

\[ k_{(5,2)(1,0)} = k_{5,2} k_{1,0} - \frac{1}{n} k_{6,2} \]

\[ k_{(4,3)(0,1)} = k_{4,3} k_{0,1} - \frac{1}{n} k_{4,4} \]
\[ k(4,3)(1,0) = k_{4,3}k_{1,0} - \frac{1}{n} k_{5,3} \]

\[ k(6,0)(0,2) = k_{6,0}k_{0,2} - \frac{1}{n} k_{6,2} - \frac{12}{n-1} k(5,1)(1,1) \]
\[ \quad - \frac{30}{n-1} k(4,1)(2,1) - \frac{20}{n-1} k(3,1)(3,1) \]

\[ k(5,1)(0,2) = \frac{n-1}{n+1}(k_{5,1}k_{0,2} - \frac{1}{n} k_{5,3} - \frac{10}{n-1} k(4,2)(1,1) \]
\[ \quad - \frac{10}{n-1} k(4,1)(1,2) - \frac{20}{n-1} k(3,2)(2,1) \]
\[ \quad - \frac{20}{n-1} k(3,1)(2,2) \}

\[ k(5,1)(2,0) = \frac{n-1}{n+9}(k_{5,1}k_{2,0} - \frac{1}{n} k_{7,1} - \frac{10}{n-1} k(6,0)(1,1) \]
\[ \quad - \frac{10}{n-1} k(5,0)(2,1) - \frac{20}{n-1} k(4,1)(3,0) \]
\[ \quad - \frac{20}{n-1} k(4,0)(3,1) \}

\[ k(4,2)(0,2) = \frac{n-1}{n+3}(k_{4,2}k_{0,2} - \frac{1}{n} k_{4,4} - \frac{8}{n-1} k(3,3)(1,1) \]
\[ \quad - \frac{8}{n-1} k(3,1)(1,3) - \frac{16}{n-1} k(3,2)(1,2) \]
\[ \quad - \frac{12}{n-1} k(2,1)(2,3) - \frac{2}{n-1} k(4,1)(0,3) \}

\[ k(4,2)(2,0) = \frac{n-1}{n+9}(k_{4,2}k_{2,0} - \frac{1}{n} k_{6,2} - \frac{4}{n-1} k(5,1)(1,1) \]
\[ \quad - \frac{2}{n-1} k(5,0)(1,2) - \frac{16}{n-1} k(4,1)(2,1) \]
\[ \quad - \frac{8}{n-1} k(4,0)(2,2) - \frac{12}{n-1} k(3,1)(3,1) \]
\[-\frac{12}{n-1} k_{(3,2)}(3,0) \}

\[k_{(3,3)}(0,2) = \frac{n-1}{n+5}(k_{3,3}k_{0,2} - \frac{1}{n}k_{3,5} - \frac{2}{n-1}k_{(0,4)}(3,1)\]

\[-\frac{6}{n-1} k_{(2,4)}(1,1) - \frac{6}{n-1} k_{(1,4)}(2,1)\]

\[-\frac{6}{n-1} k_{(0,3)}(3,2) - \frac{18}{n-1} k_{(2,3)}(1,2)\]

\[-\frac{18}{n-1} k_{(1,3)}(2,2)\}

\[k_{(6,0)}(1,1) = \frac{n-1}{n+5}(k_{6,0}k_{1,1} - \frac{1}{n}k_{7,1} - \frac{6}{n-1}k_{(5,1)}(2,0)\]

\[-\frac{15}{n-1} k_{(5,0)}(2,1) - \frac{15}{n-1} k_{(4,1)}(3,0)\]

\[-\frac{20}{n-1} k_{(4,0)}(3,1)\}

\[k_{(5,1)}(1,1) = \frac{n-1}{n+9}(k_{5,1}k_{1,1} - \frac{1}{n}k_{6,2} - \frac{10}{n-1}k_{(4,1)}(2,1)\]

\[-\frac{10}{n-1} k_{(4,0)}(2,2) - \frac{10}{n-1} k_{(3,1)}(3,1)\]

\[-\frac{10}{n-1} k_{(3,2)}(2,0) - \frac{5}{n-1} k_{(5,0)}(1,2)\]

\[-\frac{1}{n-1} k_{(6,0)}(0,2) - \frac{5}{n-1} k_{(4,2)}(2,0)\}

\[k_{(4,2)}(1,1) = \frac{n-1}{n+7}(k_{4,2}k_{1,1} - \frac{1}{n}k_{5,3} - \frac{4}{n-1}k_{(4,0)}(1,3)\]

\[-\frac{9}{n-1} k_{(4,1)}(1,2) - \frac{4}{n-1} k_{(3,3)}(2,0)\]

\[-\frac{6}{n-1} k_{(3,0)}(2,3) - \frac{14}{n-1} k_{(3,2)}(2,1)\]
\[- \frac{12}{n-1} k(3,2)(3,0) \}
\]

\[
k(3,3)(0,2) = \frac{n-1}{n+5} \{ k_3,3^{0,2} - \frac{1}{n} k_3,5 - \frac{2}{n-1} k(0,4)(3,1) \}
\]

\[
- \frac{6}{n-1} k(2,4)(1,1) - \frac{6}{n-1} k(1,4)(2,1)
\]

\[
- \frac{6}{n-1} k(0,3)(3,2) - \frac{18}{n-1} k(2,3)(1,2)
\]

\[
- \frac{18}{n-1} k(1,3)(2,2) \}
\]

\[
k(6,0)(1,1) = \frac{n-1}{n+5} \{ k_6,0^{1,1} - \frac{1}{n} k_7,1 - \frac{6}{n-1} k(5,1)(2,0) \}
\]

\[
- \frac{15}{n-1} k(5,0)(2,1) - \frac{15}{n-1} k(4,1)(3,0)
\]

\[
- \frac{20}{n-1} k(4,0)(3,1) \}
\]

\[
k(5,1)(1,1) = \frac{n-1}{n+9} \{ k_5,1^{1,1} - \frac{1}{n} k_6,2 - \frac{10}{n-1} k(4,1)(2,1) \}
\]

\[
- \frac{10}{n-1} k(4,0)(2,2) - \frac{10}{n-1} k(3,1)(3,1)
\]

\[
- \frac{10}{n-1} k(3,2)(2,0) - \frac{5}{n-1} k(5,0)(1,2)
\]

\[
- \frac{1}{n-1} k(6,0)(0,2) - \frac{5}{n-1} k(4,2)(2,0) \}
\]

\[
k(4,2)(1,1) = \frac{n-1}{n+7} \{ k_4,2^{1,1} - \frac{1}{n} k_5,3 - \frac{4}{n-1} k(4,0)(1,3) \}
\]

\[
- \frac{9}{n-1} k(4,1)(1,2) - \frac{4}{n-1} k(3,3)(2,0)
\]

\[
- \frac{6}{n-1} k(3,0)(2,3) - \frac{14}{n-1} k(3,2)(2,1)
\]
\[ k_{(3,3)}(1,1) = \frac{n-1}{n+5} \left[ k_{3,3} k_{1,1} - \frac{1}{n} k_{4,4} - \frac{10}{n-1} k_{3,1}(1,3) \right] \]

\[ - \frac{12}{n-1} k_{(3,2)}(1,2) - \frac{12}{n-1} k_{(2,1)}(2,3) \]

\[ - \frac{9}{n-1} k_{(2,2)}(2,2) - \frac{1}{n-1} k_{(4,0)}(0,4) \]

\[ - \frac{3}{n-1} k_{(4,1)}(0,3) - \frac{3}{n-1} k_{(4,2)}(0,2) \]

\[ k_{(5,0)}(0,3) = n_5 k_0,3 - \frac{1}{n} n_5,3 - \frac{15}{n-1} k_{(4,2)}(1,1) \]

\[ - \frac{15}{n-1} k_{(4,1)}(1,2) - \frac{30}{n-1} k_{(3,2)}(2,1) \]

\[ - \frac{30}{n-1} k_{(3,1)}(2,2) \]

\[ - \frac{60n}{(n-1)(n-2)} k_{(1,1)}(3,1)(1,1) \]

\[ - \frac{90n}{(n-1)(n-2)} k_{(1,1)}(2,1)(2,1) \]

\[ k_{(4,1)}(3,0) = \frac{n-1}{n+29} \left[ k_{4,1} k_{3,0} - \frac{1}{n} k_{7,1} - \frac{3}{n-1} k_{(6,0)}(1,1) \right] \]

\[ - \frac{12}{n-1} k_{(5,1)}(2,0) - \frac{15}{n-1} k_{(5,0)}(2,1) \]

\[ - \frac{30}{n-1} k_{(4,0)}(3,1) \]

\[ - \frac{72n}{(n-1)(n-2)} k_{(2,1)}(2,0)(3,0) \]
\[ k(4,1)(0,3) = \frac{n-1}{n+2} \left( k_{4,1} k_{0,3} - \frac{1}{n} k_{4,4} - \frac{12}{n-1} k_{3,3}(1,1) \right) \]

\[ - \frac{12}{n-1} k_{(3,1)(1,3)} - \frac{12}{n-1} k_{(3,2)(1,2)} \]

\[ - \frac{18}{n-1} k_{(2,1)(2,3)} - \frac{18}{n-1} k_{(2,2)(2,2)} \]

\[ - \frac{24n}{(n-1)(n-2)} k_{(3,1)(1,1)(0,2)} \]

\[ - \frac{36n}{(n-1)(n-2)} k_{(1,1)(1,1)(2,2)} \]

\[ - \frac{36n}{(n-1)(n-2)} k_{(1,1)(1,2)(2,1)} \]

\[ - \frac{3}{n-1} k_{(4,2)(0,2)} \}

\[ k_{(3,2)(3,0)} = \frac{n-1}{n+17} \left( k_{3,2} k_{3,0} - \frac{1}{n} k_{6,2} - \frac{3}{n-1} k_{5,1}(1,1) \right) \]

\[ - \frac{3}{n-1} k_{(5,0)(1,2)} - \frac{24}{n-1} k_{(4,1)(2,1)} \]

\[ - \frac{12}{n-1} k_{(4,0)(2,2)} - \frac{9}{n-1} k_{(4,2)(2,0)} \]

\[ - \frac{18}{n-1} k_{(3,1)(3,1)} \]

\[ - \frac{6n}{(n-1)(n-2)} k_{(1,1)(1,1)(4,0)} \]
\[ k(3, 2)(0, 3) = \frac{n-1}{n+8} \left\{ k_{3, 2} k_{0, 3} - \frac{1}{n} k_{3, 5} - \frac{3}{n-1} k_{0, 4}(3, 1) \right\} \]

\[ - \frac{9}{n-1} k_{2, 4}(1, 1) - \frac{9}{n-1} k_{1, 4}(2, 1) \]

\[ - \frac{6}{n-1} k_{3, 3}(0, 2) - \frac{27}{n-1} k_{2, 3}(1, 2) \]

\[ - \frac{27}{n-1} k_{1, 3}(2, 2) \]

\[ + \frac{18n}{(n-1)(n-2)} k_{1, 1}(1, 1)(1, 3) \]

\[ + k_{2, 1}(0, 3)(1, 1) + k_{3, 1}(0, 2)(0, 2) \]

\[ + 2k_{1, 1}(1, 2)(1, 2) + 2k_{1, 1}(2, 2)(0, 2) \]

\[ + 2k_{2, 1}(1, 2)(0, 2) \} \]

\[ k(5, 0)(2, 1) = \frac{n-1}{n+19} \left\{ k_{5, 0} k_{2, 1} - \frac{1}{n} k_{7, 1} - \frac{5}{n-1} k_{6, 0}(1, 1) \right\} \]

\[ - \frac{10}{n-1} k_{5, 1}(2, 0) - \frac{25}{n-1} k_{4, 1}(3, 0) \]

\[ - \frac{30}{n-1} k_{4, 0}(3, 1) \]

\[ - \frac{10n}{(n-1)(n-2)} [6k_{2, 1}(2, 0)(3, 0) \]
\[ k_{(5,0)(1,2)} = \frac{n-1}{n+4} \{ k_{5,0}^{1,2} + \frac{1}{n} k_{6,2} + \frac{10}{n-1} k_{(5,1)(1,1)} \}
\]
\[ + \frac{30}{n-1} k_{(4,1)(2,1)} + \frac{10}{n-1} k_{(4,0)(2,2)} \]
\[ - \frac{5}{n-1} k_{(4,2)(2,0)} + \frac{20}{n-1} k_{(3,1)(3,1)} \]
\[ - \frac{10}{n-1} k_{(3,2)(3,0)} \]
\[ - \frac{10n}{(n-1)(n-2)} [2k_{(1,1)(1,1)(4,0)} \]
\[-\frac{12n}{(n-1)(n-2)}[2k(1,2)(3,0)(1,1)]

+2k(1,1)(2,2)(2,0) +2k(1,2)(2,1)(2,0)

+k(2,1)(3,0)(0,2)\]

- \frac{1}{n-1} k(5,0)(0,3) - \frac{2}{n-1} k(5,1)(0,2)

- \frac{1}{n-1} k(4,2)(1,1)\}

\[k(4,1)(2,1) = \frac{n-1}{n+13}[k_{4,1}k_{2,1} + k_{5,2} - \frac{6}{n-1} k(5,1)(1,1)]

- \frac{6}{n-1} k(5,0)(1,2) - \frac{1}{n-1} k(6,0)(0,2)

- \frac{14}{n-1} k(4,0)(2,2) - \frac{8}{n-1} k(4,2)(2,0)

- \frac{16}{n-1} k(3,1)(3,1) - \frac{16}{n-1} k(3,2)(3,0)

- \frac{2n}{(n-1)(n-2)}[4k(1,1)(1,1)(4,0)]

+12k(1,2)(2,0)(3,0) +26k(1,2)(2,0)(3,1)

+3k(2,0)(2,0)(2,2) +4k(4,0)(2,0)(0,2)

+3k(3,0)(3,0)(0,2)\}]

\[k(3,2)(1,2) = \frac{n-1}{n+13}[k_{3,2}k_{1,2} - \frac{1}{n} k_{4,4} - \frac{8}{n-1} k(3,3)(1,1)]

- \frac{14}{n-1} k(3,1)(1,3) - \frac{18}{n-1} k(2,1)(2,3)

- \frac{12}{n-1} k(2,2)(2,2)\]
\[- \frac{2n}{(n-1)(n-2)} \sum_{k=1}^{3} k(1,3)(1,1)(2,0) \]

\[+ 3k(3,0)(1,1)(0,3) + 3k(2,1)(2,0)(0,3) \]

\[+ 6k(3,0)(1,2)(0,2) + 6k(2,0)(2,2)(0,2) \]

\[+ 6k(2,1)(2,1)(0,2) + 2k(3,1)(1,1)(0,2) \]

\[+ 6k(1,2)(1,2)(2,0) - \frac{1}{n-1} k(4,0)(0,4) \]

\[+ \frac{4}{n-1} k(4,2)(0,2) \]

\[k(3,2)(2,1) = \frac{n-1}{n+1} \left[ k_{3,2} k_{2,1} - \frac{1}{n} k_{5,3} - \frac{5}{n-1} k(4,0)(1,3) \right] \]

\[- \frac{7}{n-1} k(4,2)(1,1) - \frac{12}{n-1} k(4,1)(1,2) \]

\[- \frac{6}{n-1} k(3,3)(2,0) - \frac{9}{n-1} k(3,0)(2,3) \]

\[- \frac{25}{n-1} k(3,1)(2,2) \]

\[- \frac{6n}{(n-1)(n-2)} \sum_{k=1}^{2} k(1,1)(1,1)(3,1) \]

\[+ 3k(1,2)(3,0)(1,1) + k(1,3)(2,0)(2,0) \]

\[+ 4k(1,1)(2,2)(2,0) + 5k(1,2)(2,1)(2,0) \]

\[+ 2k(4,0)(1,1)(0,2) + k(2,0)(3,0)(0,3) \]

\[+ 2k(2,1)(3,0)(0,2) + 2k(2,0)(3,1)(0,2) \]

\[+ 3k(1,1)(2,1)(2,1) \]
\[ k(4,0)(0,4) = k_{4,0}k_{0,4} - \frac{1}{n} k_{4,4} - \frac{16}{n-1} k_{3,3}(1,1) \]

\[ = \frac{16}{n-1} k_{3,1}(1,3) - \frac{24}{n-1} k_{3,2}(1,2) \]

\[ - \frac{24}{n-1} k_{2,1}(2,3) - \frac{18}{n-1} k_{2,2}(2,2) \]

\[ - \frac{72n}{(n-1)(n-2)} [k_{1,1}(1,1)(2,2) + 2k_{1,1}(1,2)(2,1)] \]

\[ - \frac{576n(n+1)}{(n-1)(n-2)(n-3)} k_{1,1}(1,1)(1,1)(1,1) \]

\[ k(3,1)(0,4) = k_{3,1}k_{0,4} - \frac{1}{n} k_{3,5} - \frac{4}{n-1} k_{0,4}(3,1) \]

\[ - \frac{12}{n-1} k_{2,4}(1,1) - \frac{12}{n-1} k_{1,4}(2,1) \]

\[ - \frac{4}{n-1} k_{3,3}(0,2) - \frac{12}{n-1} k_{0,3}(3,2) \]

\[ - \frac{48}{n-1} k_{2,3}(1,2) - \frac{48}{n-1} k_{1,3}(2,2) \]

\[ - \frac{36n}{(n-1)(n-2)} [k_{1,1}(1,3)(1,1) + k_{2,1}(0,3)(1,1) + 2k_{1,1}(1,2)(1,2) \]

\[ + k_{1,1}(2,2)(0,2) + k_{2,1}(1,2)(0,2) \]

\[ - \frac{24n(n+1)}{(n-1)(n-2)(n-3)} k_{1,1}(1,1)(1,1)(0,2) \]
\[ k(3,1)(4,0) = \frac{n-1}{n+33} k_{3,1} k_{4,0} - \frac{1}{n} k_{7,1} - \frac{4}{n-1} k_{6,0}(1,1) \]

\[ - \frac{24}{n-1} k_{5,0}(2,1) - \frac{48}{n-1} k_{4,1}(3,0) \]

\[ - \frac{18n}{(n-1)(n-2)} [4k_{2,1}(2,0)(3,0) \]

\[ + 2k_{2,0}(2,0)(3,1) + k_{1,1}(3,0)(3,0) \]

\[ + 2k_{1,1}(2,0)(4,0) \] - \[ \frac{12}{n-1} k_{5,1}(2,0) \]

\[ k_{2,2}(4,0) = \frac{n-1}{n+13} k_{2,2} k_{4,0} - \frac{1}{n} k_{6,2} - \frac{8}{n-1} k_{5,1}(1,1) \]

\[ - \frac{4}{n-1} k_{5,0}(1,2) - \frac{22}{n-1} k_{4,1}(2,1) \]

\[ - \frac{8}{n-1} k_{4,2}(2,0) - \frac{20}{n-1} k_{3,1}(3,1) \]

\[ - \frac{16}{n-1} k_{3,2}(3,0) \]

\[ - \frac{12n}{(n-1)(n-2)} [k_{1,1}(1,1)(4,0) \]

\[ + 2k_{1,2}(2,0)(3,0) + 6k_{1,1}(2,1)(3,0) \]

\[ + 4k_{1,1}(2,0)(3,1) + k_{2,0}(2,0)(2,2) \]

\[ + 4k_{2,1}(2,1)(2,0) \]

\[ - \frac{24n(n+1)}{(n-1)(n-2)(n-3)} k_{1,1}(1,1)(2,0)(2,0) \]
\[ k(3,1)(2,2) = \frac{n-1}{n+28} (k_{3,1} k_{2,2} - \frac{1}{n} k_{5,3} - \frac{k}{n-1} k(4,0)(1,3) \\
- \frac{8}{n-1} k(4,2)(1,1) - \frac{13}{n-1} k(4,1)(1,2) \\
- \frac{6}{n-1} k(3,3)(2,0) - \frac{9}{n-1} k(3,0)(2,3) \\
- \frac{25}{n-1} k(3,2)(2,1) - \frac{1}{n-1} k(5,0)(0,3) \\
- \frac{2}{n-1} k(5,1)(0,2) \\
- \frac{6n}{(n-1)(n-2)} [3k(1,1)(3,1)(1,1) \\
+ 5k(1,2)(3,0)(1,1) + k(1,3)(2,0)(2,0) \\
+ 6k(1,1)(2,1)(2,1) + 2k(2,0)(3,1)(0,2) \\
+ 5k(1,1)(2,2)(2,0) + 6k(1,2)(2,1)(2,0) \\
+ 2k(4,0)(1,1)(0,2) + k(2,0)(3,0)(0,3) \\
+ 3k(2,1)(3,0)(0,2)] \\
- \frac{12n(n+1)}{(n-1)(n-2)(n-3)} [k(1,1)(2,0)(2,0)(0,2) \\
+ 2k(1,1)(1,1)(2,0)(1,1)] \\
\]

\[ k(3,1)(3,1) = \frac{n-1}{n+37} (k_{3,1}^2 - \frac{1}{n} k_{6,2} - \frac{6}{n-1} k(5,1)(1,1) \\
- \frac{6}{n-1} k(5,0)(1,2) - \frac{1}{n-1} k(6,0)(0,2) \]
\[- \frac{24}{n-1} k^{(4,1)}(2,1) - \frac{9}{n-1} k^{(4,2)}(2,0) \]
\[- \frac{18}{n-1} k^{(3,2)}(3,0) \]
\[- \frac{9n}{(n-1)(n-2)} [k^{(1,1)}(1,1)(4,0) \]
\[+ 4k^{(1,2)}(2,0)(3,0) + 4k^{(1,1)}(2,1)(3,0) \]
\[+ 4k^{(1,1)}(2,0)(3,1) + 4k^{(2,0)}(2,0)(2,2) \]
\[+ 5k^{(2,1)}(2,1)(2,0) + k^{(4,0)}(2,0)(0,2) \]
\[+ k^{(3,0)}(3,0)(0,2) \]}
\[- \frac{18n(n+1)}{(n-1)(n-2)(n-3)} k^{(1,1)}(1,1)(2,0)(2,0) \}
\[k^{(3,1)}(1,3) = \frac{n-1}{n+14} \left\{ k^{3,1}_{1,3} - \frac{1}{n} k^{4,4} - \frac{10}{n-1} k^{(3,3)}(1,1) \right\} \]
\[- \frac{21}{n-1} k^{(3,2)}(1,2) - \frac{21}{n-1} k^{(2,1)}(2,3) \]
\[- \frac{36}{n-1} k^{(2,2)}(2,2) \]
\[- \frac{9n}{(n-1)(n-2)} [2k^{(1,3)}(1,1)(2,0) \]
\[+ k^{(3,0)}(1,1)(0,3) + k^{(2,1)}(2,0)(0,3) \]
\[+ k^{(3,0)}(1,2)(0,2) + k^{(2,0)}(2,2)(0,2) \]
\[+ 2k^{(2,1)}(2,1)(0,2) + 2k^{(3,1)}(1,1)(0,2) \]
\[+ 3k^{(1,1)}(1,1)(2,2) + 7k^{(1,1)}(1,2)(2,1) \]
\[ k(2,2)(2,2) = \frac{n-1}{n+1} \left\{ \begin{array}{l}
\frac{1}{2} k_{2,2}^2 - \frac{1}{n} k_{4,4} - \frac{8}{n-1} k_{3,3}(1,1) \\
- \frac{16}{n-1} k_{3,1}(1,3) - \frac{18}{n-1} k_{3,2}(1,2) \\
- \frac{20}{n-1} k_{2,1}(2,3) \\
- \frac{2n}{(n-1)(n-2)} 8k_{(1,3)(1,1)(2,0)} \\
+4k_{(3,0)(1,1)(0,3)} + 8k_{(2,1)(2,0)(0,3)} \\
+8k_{(3,0)(1,2)(0,2)} + 16k_{(2,0)(2,2)(0,2)} \\
+10k_{(2,1)(2,1)(0,2)} + 8k_{(3,1)(1,1)(0,2)} \\
+16k_{(1,1)(1,1)(2,2)} + 8k_{(1,1)(1,2)(2,1)} \\
+20k_{(1,2)(1,2)(2,0)} + k_{(2,0)(2,0)(0,4)} \\
- \frac{4n(n+1)}{(n-1)(n-2)(n-3)} [k_{(2,0)(2,0)(0,2)(0,2)} \\
+8k_{(1,1)(1,1)(2,0)(0,2)} \\
+8k_{(1,1)(1,1)(1,1)(1,1)] - \frac{1}{n-1} k_{(4,0)(0,4)} \\
- \frac{4}{n-1} k_{(4,1)(0,3)} - \frac{4}{n-1} k_{(4,2)(0,2)} \}
\]
IV. CHAPTER FOUR: STATISTICAL STRUCTURES

A. Other Work and Problems Considered

Perhaps it is curious that until recently most of the work done concerning the \( k \) statistics made few assumptions except that all the quantities involved have meaning. In particular, with the exception of a few papers concerning the multivariate case (Kendall, 1940c), (Wishart, 1949), (Cook, 1951a,b), no attention has been given to the structured populations. While it is true that the relative lack of restrictive assumptions introduces great generality into the work, it is equally true that very often structured populations are of interest, especially in applications, as the analysis of variance bears witness.

Hooke (1956a, 1956b) did the initial investigation in this area by considering sampling from a matrix or a two-way crossed classification. His bipoikays (which should not be confused with bivariate polykays) are linear functions of generalized symmetric means and have been described in Chapter One. In his thesis Dayhoff (1964a) considerably extended these functions to generalized polykays and was able to specialize them to structures involving both nested and crossed variables; he presented results for two and three factor structures.

Earlier work concerned with general response structures and the analysis of variance appropriate to general randomized
experiments includes that of Kempthorne (1952), Wilk (1955), Zyskind (1958), Throckmorton (1961), and White (1963). Certain linear functions of the components of variation, called cap sigmas (Σ's), introduced by Wilk (1955) and explicitly defined in general by Zyskind (1958) have been shown to play a central role in the expected mean squares in these analyses. Relevant to this study is the fact that Dayhoff (1964a) has shown the Σ's to be equivalent to the generalized polykays of the second degree.

This chapter is concerned with the study of structures and the formation of the Σ's from them. The Σ's are of uncommon usefulness due to their occurrence in the analysis of variance. It would appear, therefore, that special attention to these particular polykays, rather than a search for example, for a pattern involving a generalized polykay of arbitrary degree from a structure of arbitrary complexity, is quite appropriate. In addition, as the number of factors in a structure increases, it is increasingly difficult to judge general properties of the structure. Non-unitary structures, to be defined later, only begin to appear for example, in structures containing four factors.

Wilk (1955) gave a few restricted rules for the formation of the Σ's, but since Zyskind's definition, general constructive schemes for supplying all the Σ's for any structure have not been available. Two sets of rules for
forming the $\Sigma$ quantities are given, one involving direct products of sets, the other direct products of matrices of a special sort. These rules are somewhat different in the case of non-unitary structures than for unitary structures, which have been the object of most interest heretofore. It is hoped that this work will provide another direction from which the attack on polykays of higher degree from structured populations can proceed.

B. Admissible Structures

A structure is a specification of all the crossing and nesting relationships between the factors under study. The structure, together with the specification of the sampling, if any, of the levels of the factors and the random assignment of combinations of levels of factors to experimental units, or randomization, is called the design of the experiment. It is convenient to use Throckmorton's (1961) diagrams to denote structures as well as a bracket notation (Zyskind, 1958). Throckmorton's diagrams consist of using a small circle or a dot for each factor in the experiment and denoting the direct nesting of factor B in factor A by positioning B below A in the diagram and drawing a line between them. When no such line joins two factors they will be taken as completely crossed. Whether they are specifically included in any diagram or not, the mean will be taken as
nesting all factors, directly or indirectly, and the error will be taken as nested in all factors in the structure. A block structure where $S$, $P$, and $T$ denote sources, plots, and treatments can be diagrammed as in the figure below.

\[ \begin{array}{c}
  \mu \\
  S \quad P \\
  T \\
  \epsilon \\
\end{array} \]

Throckmorton has provided a complete list of all structures including no more than 5 factors.

Zyskind’s bracket notation uses the symbol $A:B$ to denote the nesting of $B$ in $A$ while $(A)(B)$ denotes that $A$ and $B$ are completely crossed. The block structure can be written $(S:P)(T)$ in this notation. Both specifications are useful but neither is adequate to describe incomplete structures.

It will also be convenient to classify the factors in a structure by tiers. A factor will be said to be on the first tier if it nests no other factors; in general a factor is on the $k$th tier if it nests at least one factor on the $k-1$st tier. The letters in the structure can then be partitioned according to tier. Throckmorton’s diagrams allow the positioning of a factor in the $k$-1st tier below that of all the factors on the $k$th tier.
For a structure with at least one factor on the kth tier and no factor on the k+1 th tier a series of k consecutive parentheses or brackets, each containing some letters in the structure or the symbol $\emptyset$ to denote the absence of any letters, will be called an element. The basis element for factor A is formed by placing A alone in the rightmost bracket of the element; the next bracket contains the letters of all factors, if any, which directly nest A, or, if there are none, $\emptyset$; the pattern is continued with the next bracket containing the letters of all factors which nest any of the letters in the previous bracket, or $\emptyset$, until all the brackets are filled. For example, the basis elements for the factors of the structure (S:P)(T) are ($\emptyset$)(S), (S)(P), and ($\emptyset$)(T). The element ($\emptyset$)($\emptyset$) and its counterpart for other structures is called the unit element. It may be noted that only a particular basis is discussed here, that formed from the elements for individual factors.

Reduced population structures or admissible structures (Zyskind, 1958) are obtained from a structure by omitting all, some, or none of the letters, provided that whenever a letter is retained all the letters which nest it must also be retained. All the admissible structures can be obtained for any structure by using a symbolic multiplication on the elements.
**Definition:**

If \((A)(B)...(C)\) and \((D)(E)...(F)\) are elements then the \*(-\) product of these elements, where \(A, B, ..., F\) denote factors or combinations of factors, is \((A)(B)...(C) * (D)(E)\) \...(F) = \((AD)(BE)...(CF)\) subject to the following restrictions:

i) the letters in any one set of parentheses may be written in any order,

ii) \(AA = A; A\emptyset = A\),

iii) a letter may appear in the product only once, in its leftmost possible parentheses.

Thus \((S)(P) * (\emptyset)(T) = (S)(PT)\) and \((\emptyset)(S) * (S)(P) = (S)(P)\). It is easy to verify that the letters of each basis element form an admissible structure as do the \*(-\) products of two or more basis elements. Any admissible structure may be expressed as a \*(-\) product of a finite number of basis elements.

A useful grouping of some of the basis elements is indicated in the following definition. The primary set for each separate parenthesis in the total population structure consists of the basis elements for each letter in that parenthesis and all the distinct \*(-\) products of these basis elements, and the unit element. Some examples of the above concepts follow.

For the block structure \((S:P)(T)\), the basis elements are \((\emptyset)(\emptyset), (\emptyset)(S), (S)(P), \text{ and } (\emptyset)(T)\). The primary
sets are then \{ (\emptyset)(\emptyset), (\emptyset)(S), (S)(P) \} and \{ (\emptyset)(\emptyset), (\emptyset)(T) \}. The structure \( (S:B:P)(G:R)(H:V) \) has the following basis elements: \( (\emptyset)(\emptyset)(\emptyset), (\emptyset)(\emptyset)(S), (\emptyset)(S)(B), (S)(B)(P), (\emptyset)(\emptyset)(G), (\emptyset)(G)(R), (\emptyset)(\emptyset)(H), (\emptyset)(H)(V) \). These are grouped into the following primary sets:

\[
\begin{align*}
\{ (\emptyset)(\emptyset)(\emptyset) \} , & \quad \{ (\emptyset)(\emptyset)(G) \} , \quad \{ (\emptyset)(\emptyset)(H) \} \\
\{ (\emptyset)(S)(B) \} , & \quad \{ (\emptyset)(G)(R) \} , \quad \{ (\emptyset)(H)(V) \} \\
\{ (S)(B)(P) \} , & \quad \{ (S)(Q) \} , \quad \{ (\emptyset)(SP) \} \\
\{ (\emptyset)(P) \} , & \quad \{ (SP)(R) \}
\end{align*}
\]

These primary sets are denoted by \( S_1, S_2, \) and \( S_3 \).

Example:

\( (S:Q)(P) \) and \( (SP:R) \). The primary sets are seen to be

\[
\begin{align*}
\{ (\emptyset)(\emptyset) \} , & \quad \{ (\emptyset)(\emptyset) \} , \quad \{ (\emptyset)(\emptyset) \} \\
\{ (\emptyset)(S) \} , & \quad \{ (\emptyset)(P) \} , \quad \{ (\emptyset)(SP) \} \\
\{ (S)(Q) \} , & \quad \{ (SP)(R) \}
\end{align*}
\]

Example:

\( (S:(R)(C:L))(A:a)(B:b) \). Here the primary sets are the following:
For any population structure, each primary set is a commutative semi-group.

Proof:

Under the binary * multiplication, each primary set is closed, by the definition of primary set. The associativity of the * multiplication may be shown as follows:

\[
\]

Now

\[
\]

Thus the associative law holds and the primary set is a semi-group. Since the letters within each parenthesis may be written in any order, the * multipli-
cation is commutative.

The unit element \((\emptyset)(\emptyset)\ldots(\emptyset)\) serves as an identity in each semi-group.

The direct * product set is formed from two or more primary sets by forming all the possible * products of elements, one element being selected from each primary set.

**Theorem:**

All the admissible structures for a given population structure can be formed from the elements of the direct * product set of the primary sets \(S_i, i = 1,2,\ldots,k\), where the definition of the population structure involves \(k\) sets of parentheses.

**Proof:**

Consider first the case where the parentheses in the definition of the population structure involve unique letters, i.e., a given letter appears in only one set of parentheses. The primary sets will then contain one element corresponding to choosing none, all, or some of the letters in that parenthesis. Since the primary sets can contain no common letters, the products are unique and must exhaust the set of admissible structures. Thus in this case each admissible structure is expressible as a product of \(k\) basis elements, in a unique way.

Now suppose the parentheses can contain the same letter
more than once. The situation is illustrated by the structure
(S:Q)(P) and (SP:L). In this case the basis elements of the
primary sets may be formed by the * product of basis
elements of other primary sets. In the example, (Ø)(S)
appears as a basis element in the primary set of the first
parenthesis and (Ø)(P) appears as a basis element in the
primary set for the second parenthesis. The * product of
these is (Ø)(SP), which appears as a basis element in the
primary set for the third parenthesis. It is thus evident
that in forming the direct * product, the element (Ø)(SP)
is not needed since its use will produce duplication of
admissible structures. If the structure is such that it can
be described in a single series of parentheses, it is called
unitary. Non-unitary structures require the use of the word
"and" in their specification.

It is also unnecessary to form any products where the
set of letters of one or more basis element is a subset of
other elements used in the product, since the letters can
appear only once in the product. Thus if the structure
is unitary, the direct product will give all the admissible
structures but with some duplication of letters unless some
products are not formed. This leads us to the following:

Corollary:

If the population structure contains the same letters in
more than one parenthesis, all the admissible population
structures are formed by finding the direct product of the primary sets with the restriction that a given basis element is multiplied only by factors that either

i) consist of (Ø)'s alone

or

ii) contain at least one letter not appearing in the basis element.

The following examples are given as illustrations of the above results.

**Example:**

For the structure \((S:B:P)(G:R)(H:V)\) there are

\(4 \cdot 3 \cdot 3 = 36\) admissible structures. They are elements of \(S = S_1 \times S_2 \times S_3\) where \(S_1, S_2,\) and \(S_3\) were given when this structure was considered previously. The elements of the direct product are:

\(\emptyset (\emptyset) (\emptyset), (\emptyset) (\emptyset) (H), (\emptyset) (H) (V), (\emptyset) (\emptyset) (G), (\emptyset) (\emptyset) (GH),\)

\((\emptyset) (H) (GV), (\emptyset) (G) (R), (\emptyset) (G) (RH), (\emptyset) (GH) (RV), (\emptyset) (\emptyset) (S),\)

\((\emptyset) (\emptyset) (SH), (\emptyset) (H) (SV), (\emptyset) (\emptyset) (SG), (\emptyset) (\emptyset) (SGH), (\emptyset) (H) (SGV),\)

\((\emptyset) (G) (RS), (\emptyset) (G) (SGH), (\emptyset) (GH) (SRV), (\emptyset) (S) (B), (\emptyset) (S) (BH),\)

\((\emptyset) (SH) (BV), (\emptyset) (S) (BG), (\emptyset) (S) (BGH), (\emptyset) (SH) (BGV),\)

\((\emptyset) (SG) (BR), (\emptyset) (SG) (BRH), (\emptyset) (SGH) (BRV), (S) (B) (P),\)

\((S) (B) (PH), (S) (BH) (PV), (S) (B) (PG), (S) (B) (P) (G), (S) (B) (PGH), (S) (BH) (PGV),\)

\((S) (BG) (BR), (S) (BG) (PRH), (S) (BGH) (PRV).\)
Example:

For the structure \((S:Q)(P)\) and \((SP:R)\), the primary sets, which are denoted by \(S_1, S_2,\) and \(S_3\), have been given previously. The factor \((\emptyset)(SP)\) in \(S_3\) need not be considered at all and it is only necessary to form those products with \((SP)(R)\) according to the rule given in the corollary. This gives the following admissible structures: 
\((\emptyset)(\emptyset), (SP)(R), (\emptyset)(P), (\emptyset)(S), (\emptyset)(SP), (S)(Q),
(SP)(QR), \text{ and } (S)(QP).\)

C. Components, Components of Variation, and \(\bar{E}\)'s

The following definitions (Zyskind, 1958) have become fairly standard and are included here for completeness, since heavy use will be made of the concepts in the remainder of this chapter. A response, from the population of possible responses, may be denoted by the letter \(Y\) with subscripts indicating the factors in the structure on which the response is assumed to depend. In the block example, if the subscripts \(i,j,\) and \(k\) refer to sources, (or blocks), plots, and treatments respectively, then \(Y_{ijk}\) denotes a particular response. Or, by making reference to the admissible structures, this could be written \(Y_{(i)(jk)}\). Now if the convention is adopted that an omitted subscript in a response denotes averaging over the range of the values of that subscript, the
admissible structures yield admissible means. In the block structure these are then $Y(\emptyset)(\emptyset)$, $Y(\emptyset)(i)$, $Y(i)(j)$, $Y(\emptyset)(k)$, $Y(\emptyset)(ik)$, and $Y(i)(jk)$. It is customary to omit the parentheses as well as the letter $\emptyset$, although the expanded notation is useful in forming the means for very complex structures. Clearly the admissible means are the only means making physical sense for any structure.

For each admissible mean a component is formed by a linear combination of the admissible means by omitting from the given admissible (or leading) mean all, some, or none of its rightmost bracket subscripts, the sign of each term being $(-1)^p$ where $p$ is the number of subscripts omitted. Thus $Y(i)(j)$ yields the component $Y(i)(j) - Y(\emptyset)(i)$. The relationship here between the components and the usual main effect and interaction terms in linear models should be noted.

For any structure the sum of all the typical components is known to yield a typical response. For example,

$$
[Y(i)(jk) - Y(i)(j) - Y(\emptyset)(ik) + Y(\emptyset)(i)] + [Y(i)(j) - Y(\emptyset)(i)]
$$

$$
+ [Y(\emptyset)(ik) - Y(\emptyset)(i) - Y(\emptyset)(k) + Y(\emptyset)(\emptyset)]
$$

$$
+ [Y(\emptyset)(i) - Y(\emptyset)(\emptyset)] + [Y(\emptyset)(k) - Y(\emptyset)(\emptyset)]
$$

$$
+ Y(\emptyset)(\emptyset) = Y(i)(jk)
$$
This identity is known as the **population identity**.

A structure is said to be **balanced** when the range of any subscript of the structure is the same for every set of values of the other subscripts. When balance obtains, the components can be shown to obey some very simple and useful relations. Two of these will be quoted here, the proofs being found in Zyskind (1958).

1) The sum of the values of any component over the range of any index appearing in the rightmost bracket of the leading mean is zero.

2) The sum of the squares of any component over the ranges of all the subscripts defining a typical population response is the same linear function of the squares of the individual terms of the component as the component is of the admissible means.

Because 2) is very useful for finding expected mean squares in balanced cases, it will be illustrated by referring to the block example again.

\[
\sum_{i,j,k} (Y(i)(jk) - Y(i)(j) - Y(\emptyset)(ik) + Y(\emptyset)(i))^2 = \\
\sum_{i,j,k} (Y^2(i)(jk) - Y^2(i)(j) - Y^2(\emptyset)(ik) + Y^2(\emptyset)(i)).
\]

The **components of variation** are then defined as the sum over all the population ranges of the indices defining the
component of the square of the component divided by its number of degrees of freedom where the number of degrees of freedom is obtained by finding the product of the population ranges of all the indices not appearing in the rightmost bracket and the diminished ranges (the range minus 1) of all the indices in the rightmost bracket. The components of variation will be denoted by $\sigma^2$ with subscripts indicating the component involved.

The $\Sigma$'s of Wilk (1955) are linear combinations of the components of variation. The following definition is due to Zyskind (1958).

$$\Sigma(A)(B) \ldots (C) = \sum_{S} K(S) \sigma^2(S)$$

where $S$ is a set of elements each of which contains the letters $A, B, \ldots, C$ and such that all excess letters appear in the rightmost bracket. $K(S) = (-1)^p / \pi$ where $p$ is the number of excess letters and $\pi$ is the product of the population ranges of the excess indices. It must be noted that $\Sigma$ refers to a summation while $\Sigma(S)$ denotes a cap sigma. No confusion should result from this notation which is now of fairly long standing.

To illustrate, consider the structure $(S : (R)(C))(T)$. Then

$$\Sigma(\emptyset)(S) = \sigma^2(\emptyset)(S) - \frac{1}{R} \sigma^2(S)(R) - \frac{1}{C} \sigma^2(S)(C)$$

$$+ \frac{1}{RC} \sigma^2(S)(RC)$$
and

\[ \Sigma(s)(RT) = \sigma^2(s)(RT) - \frac{1}{\text{C}} \sigma^2(s)(RCT) \]

where the same symbol has been used for a factor and its population range. A new matrix method for determining all the \( \Sigma \)'s for a wide class of structures will be given in Section D.

Primary sets and the sets used in the formation of the \( \Sigma \)'s are closely related. The theorem to follow exhibits a relationship which has been found.

**Lemma:**

The set consisting of the basis element corresponding to any letter in the structure and the element \((\emptyset)(\emptyset)\ldots(\emptyset)\) together with the operation \(\ast\) form a sub semi-group of the semi-group containing the basis element corresponding to that letter.

**Proof:**

Suppose \((A)(B)\ldots(C)\) is the element corresponding to the letter (or group of letters) \(C\). Then multiplication by the unit element leaves this element unchanged: \((\emptyset)(\emptyset)\ldots(\emptyset) * (A)(B)\ldots(C) = (A)(B)\ldots(C)\). Also \((A)(B)\ldots(C) * (A)(B)\ldots(C) = (A)(B)\ldots(C)\) so the set is closed. The associativity follows from the associativity of the semi-group.
The sub-groups thus formed using each letter in the structure can be used to form the \( \Sigma \)'s as the following theorem shows.

**Theorem:**

All the admissible structures appearing as subscripts of \( \sigma^2 \)'s in the expansion of \( \Sigma^{(A)(B)...(C)} \) (where any of \( A,B,...,C \) can represent a combination of letters as well as a single letter) are elements of the set \( S \) where \( S \) is the direct * product of the sets consisting of

- i) the element \( (A)(B)...(C) \) alone
- ii) all the sub semi-groups corresponding to letters not in \( \{A,B,...,C\} \) such that all the letters nesting the new letters are in \( \{A,B,...,C\} \).

**Proof:**

The subscripts appearing in the expansion of \( \Sigma^{(A)(B)...(C)} \) must be such that \( \{A,B,...,C\} \) is a subset of the set of letters appearing in the subscript and all the excess letters must appear in the rightmost bracket. Since the direct product contains the set consisting of \( (A)(B)...(C) \) alone, and since under the * multiplication all letters of all factors are preserved, the resulting product must contain \( \{A,B,...,C\} \) as a subset. Now consider the other factors in the direct product. The unit elements
of course produce no new terms. The other factors are such that any letters not in \{A,B,\ldots,C\} are in the rightmost bracket and all letters nesting these letters are in \{A,B,\ldots,C\} and thus the uncommon letters will appear in the rightmost bracket in the direct product.

Some examples of these results follow.

**Example:**

Suppose \((S;P)(K)\) is the population structure. The admissible structures are found from the basis elements and the above theorem gives the following sets to be used in the relevant direct products.

<table>
<thead>
<tr>
<th>Admissible structures</th>
<th>Sets in the direct product</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\emptyset)(\emptyset))</td>
<td>{(\emptyset)(\emptyset)}, {(\emptyset)(\emptyset)}, {(\emptyset)(\emptyset)} }</td>
</tr>
<tr>
<td>((\emptyset)(K))</td>
<td>{(\emptyset)(K)}, {(\emptyset)(\emptyset)} }</td>
</tr>
<tr>
<td>((\emptyset)(S))</td>
<td>{(\emptyset)(S)}, {(\emptyset)(\emptyset)}, {(\emptyset)(\emptyset)} }</td>
</tr>
<tr>
<td>((S)(P))</td>
<td>{(S)(P)}, {(\emptyset)(\emptyset)} }</td>
</tr>
<tr>
<td>((\emptyset)(SK))</td>
<td>{(\emptyset)(SK)}, {(\emptyset)(\emptyset)} }</td>
</tr>
<tr>
<td>((S)(PK))</td>
<td>{(S)(PK)} }</td>
</tr>
</tbody>
</table>
Thus the $\Sigma$'s are:

$$
\Sigma(\emptyset)(\emptyset) = \sigma^2(\emptyset)(\emptyset) - \frac{1}{K}\sigma^2(\emptyset)(K) - \frac{1}{S}\sigma^2(\emptyset)(S) + \frac{1}{KS}\sigma^2(\emptyset)(SK)
$$

$$
\Sigma(\emptyset)(K) = \sigma^2(\emptyset)(K) - \frac{1}{S}\sigma^2(\emptyset)(SK)
$$

$$
\Sigma(\emptyset)(S) = \sigma^2(\emptyset)(S) - \frac{1}{P}\sigma^2(S)(P) - \frac{1}{K}\sigma^2(S)(SK)
$$

$$
+ \frac{1}{PK}\sigma^2(S)(PK)
$$

$$
\Sigma(S)(P) = \sigma^2(S)(P) - \frac{1}{K}\sigma^2(S)(PK)
$$

$$
\Sigma(\emptyset)(SK) = \sigma^2(\emptyset)(SK) - \frac{1}{P}\sigma^2(S)(PK)
$$

$$
\Sigma(S)(PK) = \sigma^2(S)(PK)
$$

Example:

Consider the population structure $(S:Q)(P)$ and $(SP:R)$.

<table>
<thead>
<tr>
<th>Admissible structure</th>
<th>Sets in direct product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\emptyset)(\emptyset)$</td>
<td>${(\emptyset)(\emptyset)}$, ${(\emptyset)(\emptyset)}$, ${(\emptyset)(\emptyset)}$</td>
</tr>
</tbody>
</table>

The set $\{(\emptyset)(\emptyset)\}$ is not necessary here since its inclusion will produce no new subscript sets. It is omitted in some other sets below for the same reason.

| $(SP)(R)$ | $\{(SP)(R)\}$, $\{(SP)(R)\}$ |
| $(\emptyset)(P)$ | $\{(\emptyset)(P)\}$, $\{(\emptyset)(P)\}$ |
The direct products then give the following $\Sigma$'s:

$$\Sigma(\emptyset)(\emptyset) = \sigma^2(\emptyset)(\emptyset) - \frac{1}{S} \sigma^2(\emptyset)(S) - \frac{1}{P} \sigma^2(\emptyset)(P) + \frac{1}{SP} \sigma^2(\emptyset)(SP)$$

$$\Sigma(SP)(R) = \sigma^2(SP)(R) - \frac{1}{Q} \sigma^2(SP)(RQ)$$

$$\Sigma(\emptyset)(P) = \sigma^2(\emptyset)(P) - \frac{1}{S} \sigma^2(\emptyset)(SP)$$

$$\Sigma(\emptyset)(S) = \sigma^2(\emptyset)(S) - \frac{1}{P} \sigma^2(\emptyset)(SP) - \frac{1}{Q} \sigma^2(\emptyset)(SP) + \frac{1}{PQ} \sigma^2(\emptyset)(SP)$$

$$\Sigma(\emptyset)(SP) = \sigma^2(\emptyset)(SP) - \frac{1}{R} \sigma^2(\emptyset)(SP) - \frac{1}{Q} \sigma^2(\emptyset)(SP) + \frac{1}{RQ} \sigma^2(\emptyset)(SP)$$

$$\Sigma(S)(Q) = \sigma^2(S)(Q) - \frac{1}{P} \sigma^2(S)(SP)$$

$$\Sigma(S)(PQ) = \sigma^2(S)(PQ) - \frac{1}{R} \sigma^2(S)(SP)$$

$$\Sigma(SP)(RQ) = \sigma^2(SP)(RQ)$$
D. Formation of $\Sigma$'s

For any unitary population structure, the **standard order** of the elements (and consequently that of the admissible structures) will be defined as follows. The unit element is written first. Starting at the leftmost bracket in the specification of the population structure, write the basis element of the first occurring letter followed by the basis element of the second letter. The $*$ product of these elements is written next, followed by the basis element for the third letter and the $*$ products of it, in order, with the three previously written elements. The process is continued until the letters used in the population structure are exhausted. This provides the standard order of the admissible structures for the given description of the population structure.

Now suppose that $\Sigma = A \sigma^2$ where $\Sigma$ is the vector of $\Sigma$'s and $\sigma^2$ is the vector of $\sigma^2$'s when both the $\Sigma$'s and $\sigma^2$'s are written in the standard order. The elements of $A$ can be determined either by use of the definition of the $\Sigma$ quantities or the preceding theorem. The matrix $A^{-1}$ exists in general and a descriptive characterization of it was given by Carney (1967). In general subscripts will be attached to the letter $A$ expressing the population structure. In this section rules will be provided whereby $A$ and $A^{-1}$ can be constructed using matrix products for any unitary
structure. The non-unitary case is more complex and will be
treated in Section E. With each letter in the population
structure, say $S$, the matrix $A_{(S)}$ is associated where
\[
A_{(S)} = \begin{pmatrix}
1 & -\frac{1}{s} \\
0 & 1
\end{pmatrix}.
\]

The next theorem indicates the formation of $A_{(B)(C)\ldots(D)}$
for the crossed population structure $(B)(C)\ldots(D)$.

**Theorem:**
\[
A_{(B)(C)\ldots(D)} = A_{(B)} \bigotimes A_{(C)} \bigotimes \ldots \bigotimes A_{(D)}
\]
where $\bigotimes$ is the Kronecker matrix product.

**Proof:**

For a single letter in the population structure, say $(B)$,
\[
E(\emptyset) = \sigma^2(\emptyset) - \frac{1}{B} \sigma^2(B)
\]

\[
E(B) = \sigma^2(B)
\]

so
\[
A_{(B)} = \begin{pmatrix}
1 & -\frac{1}{B} \\
0 & 1
\end{pmatrix}.
\]

As an inductive hypothesis suppose
\[
A_{(B)(C)\ldots(D)} = A_{(B)} \bigotimes A_{(C)} \bigotimes \ldots \bigotimes A_{(D)}
\]
consider the population structure \((B)(C)...(D)(E)\). If the vector \(\Sigma\) for the structure \((B)(C)...(D)\) contained \(n\) components then \(\Sigma\) for the structure \((B)(C)...(D)(E)\) contains \(2n\) components, the first \(n\) being identical to those for \((B)(C)...(D)\) except for a leftmost bracket containing \(\emptyset\).

The matrix \(A_{(B)(C)...(D)(E)}\) can then be partitioned into four submatrices, say

\[
A_{(B)(C)...(D)(E)} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.
\]

Further \(A_1 = A_{(B)(C)...(D)}\) since \(A_1\) relates the components of

\[
\Sigma(B)(C)...(D)^{\dagger}\text{ with } \sigma^2(B)(C)...(D)^{\dagger}
\]

where the notation indicates the additional leftmost (\(\emptyset\)). Also \(A_2\) relates the first \(n\) components of \(\Sigma(B)(C)...(D)(E)\) with the last \(n\) components of \(\sigma^2(B)(C)...(D)(E)\). The subscripts of these components contain the letter \(E\) in the rightmost bracket and in fact are formed by considering the \(*\) product of \((\emptyset)...(E)\) with the basis elements and their products in their standard order for the letters \(B,C,...,D\). Thus \(A_2 = -\frac{1}{E}A_{(B)(C)...(D)}\). Now \(A_3 = 0\) since the subscripts of the last \(n\) basis elements contain the letters of the first \(n\) basis elements as a subset.
Finally, \( A_4 = A(B)(C)\ldots(D) \) since the last \( n \) elements differ from the first \( n \) only by the letter \( E \) in the rightmost bracket and, due to the standard order, otherwise occur in the same order as the first \( n \) elements. Thus the \( \Sigma \)'s for this set must have the same relationship to the corresponding \( \sigma \)'s as the \( \Sigma \)'s and the \( \sigma \)'s in the first set. Thus

\[
A(B)(C)\ldots(D)(E) = \begin{pmatrix}
A(B)(C)\ldots(D) & -\frac{1}{E} A(B)\ldots(D) \\
0 & A(B)\ldots(D)
\end{pmatrix}
\]

\[= A(B)(C)\ldots(D) \otimes A(E),\]

and the theorem is proved.

Note that \( A(B) \otimes A(C) \otimes \ldots \otimes A(E) \) is unambiguous above since it is well known that the Kronecker matrix product is associative.

Now

\[
A^{-1}(B) = \begin{pmatrix}
1 & \frac{1}{E} \\
0 & 1
\end{pmatrix},
\]

and, using the well known fact about the inverse of the Kronecker product,

\[
A^{-1}(B)(C)\ldots(D) = A^{-1}(B) \otimes A^{-1}(C) \otimes \ldots \otimes A^{-1}(D)
\]
making the construction of the matrix \( A^{-1} \) fairly simple.

The preceding theorem concerns purely crossed population structures. The next theorem concerns the pure nesting relationship. The convention will be adopted that although for the structure \((S:T:\ldots:U;V)\) no admissible structure can be formed from the letters \(T,\ldots,U,V\) alone, the matrix \( A(T:\ldots:V) \) will be defined with respect to the pseudo population \((T:\ldots:U;V)\).

**Theorem:**

\[
A(S:T:\ldots:U;V) = \begin{pmatrix}
1 & V_S \\
0 & A(T:\ldots:V)
\end{pmatrix}
\]

where \( V_S \) is the row vector \((\frac{1}{S}, 0, 0, \ldots 0)\) and 0 in the matrix is the column vector \((00\ldots0)'\), and the vectors \( V_S \) and 0 are of the required dimensions to make \( A(S:T:\ldots:U;V) \) square.

**Proof:**

The elements in standard order for the structure \((S:T:\ldots:U)\) are \((\emptyset)(\emptyset)\ldots(\emptyset), (\emptyset)(\emptyset)\ldots(\emptyset)(S), (\emptyset)(\emptyset)\ldots(\emptyset)(S)(T), \ldots, (S)(T)\ldots(U)\). Due to the simple nesting structure each letter occurring in an admissible structure occupies a bracket alone. Thus any particular \( \Sigma \), except for \( \Sigma(S)(T)\ldots(U) \) is a function of two components
of variation: one with the same subscript as the \( \Sigma \) considered with a coefficient of 1 and one with a subscript indicating the next admissible structure in the standard order with coefficient \(-\frac{1}{K}\) if \( K \) is the new letter in that admissible structure. \( \Sigma (S)(T)\ldots(U) = \sigma^2 (S)(T)\ldots(U) \) so \( A(\emptyset) \) must be 1.

It is interesting to note that the nesting in the structure produces a nesting in a sense of the relevant matrices.

The previous two theorems can be combined to give the matrix \( A \) for any unitary population structure. An example follows:

Example:

For the structure \((S:T:U)(R)\),

\[
A(S:T:U)(R) = A(S:T:U) \bigotimes A(R) = \begin{pmatrix}
1 & \frac{1}{S} & 0 & 0 \\
0 & 1 & \frac{1}{T} & 0 \\
0 & 0 & 1 & \frac{1}{U} \\
0 & 0 & 0 & 1
\end{pmatrix} \bigotimes \begin{pmatrix}
1 & \frac{1}{R} \\
0 & 1
\end{pmatrix}
\]
In the preceding theorem, if \( A(S:T:\ldots:V) \) contains \( n \) nested factors then \( A \) is of dimension \((n+1) \times (n+1)\). The 0 vector in this \( A \) matrix is \( nx1 \) and \( V_S \) is a \( lx1 \) row vector.

**Theorem:**

\[
A^{-1}(S:T:\ldots:X) = \begin{pmatrix} 1 & \frac{1}{S} & \frac{1}{ST} & \ldots & \frac{1}{ST\ldots X} \\ \frac{1}{U(S:T:\ldots:X)} & 0 & A^{-1}(T:\ldots:X) \\
\end{pmatrix}
\]

where

\[
U(S:T:\ldots:X) = \frac{1}{S} \frac{1}{ST} \ldots \frac{1}{ST\ldots X}.
\]

**Proof:**

Consider the matrix

\[
B = \begin{pmatrix} 1 & \frac{1}{S} & \frac{1}{ST} & \ldots & \frac{1}{ST\ldots X} \\ \frac{1}{U(S:T:\ldots:X)} & 0 & A^{-1}(T:\ldots:X) \\
\end{pmatrix}
\]
and the product of the $i$th row of $A(S:T:...:X)$ with the $j$th column of $B$. Suppose the $i$th row is of the form

$$(0 \ 0 \ ... \ \frac{1}{X_i} \ 0 \ ... \ 0).$$

The $j$th column has the form

$$(\frac{1}{X_1 \ ... \ X_{j-1}} \ \frac{1}{X_2 \ ... \ X_{j-1}} \ ... \ \frac{1}{X_{j-1}} \ 1 \ 0 \ ... \ 0)^T.$$ 

If $i=j$ the product is 1. If $i<j$ the product is

$$\frac{1}{X_1 \ ... \ X_{j-1}} \cdot 1 - \frac{1}{X_i} \cdot \frac{1}{X_{i+1} \ ... \ X_{j-1}} = 0$$

and if $i>j$ the product is $1 \cdot 0 - \frac{1}{X_i} \cdot 0 = 0$.

Thus $B = A^{-1}(S:T:...:X)$.

The matrix for the structure $(S:T:U)(R)$ was given previously. Now

$$A^{-1}(S:T:U)(R) = A^{-1}(S:T:U) \otimes A^{-1}(R)$$

$$= \begin{pmatrix}
1 & \frac{1}{S} & \frac{1}{ST} & \frac{1}{STU} \\
0 & 1 & \frac{1}{T} & \frac{1}{TU} \\
0 & 0 & 1 & \frac{1}{U} \\
0 & 0 & 0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
1 & \frac{1}{R} \\
0 & 1
\end{pmatrix}$$
or

\[ A^{-1}_{(S:T:U)(R)} = \begin{pmatrix}
1 & \frac{1}{S} & \frac{1}{ST} & \frac{1}{STU} & \frac{1}{R} & \frac{1}{SR} & \frac{1}{STR} & \frac{1}{STUR} \\
0 & 1 & \frac{1}{T} & \frac{1}{TU} & 0 & \frac{1}{R} & \frac{1}{RT} & \frac{1}{TUR} \\
0 & 0 & 1 & \frac{1}{U} & 0 & 0 & \frac{1}{R} & \frac{1}{UR} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{R} \\
0 & 0 & 0 & 0 & 1 & \frac{1}{R} & \frac{1}{ST} & \frac{1}{STU} \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{T} & \frac{1}{TU} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{U} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]

As another example, consider the structure \((S:(R)(C:L))\).

From the previous results it follows that

\[ A_{(S:(R)(C:L))} = \begin{pmatrix}
1 & -\frac{1}{S} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{R} & -\frac{1}{C} & \frac{1}{RC} & 0 & 0 \\
0 & 0 & 1 & 0 & -\frac{1}{C} & 0 & 0 \\
0 & 0 & 0 & 1 & -\frac{1}{R} & -\frac{1}{L} & \frac{1}{RL} \\
0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{L} \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{R} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \]
Let $A^{-1}_{(S: (R)(C:L))}$ be of the form
$\begin{pmatrix}
1 & W_S \\
0 & A^{-1}_{(R)(C:L)}
\end{pmatrix}$.

$A^{-1}_{(R)(C:L)}$ can be obtained from the previous results and is
$\begin{pmatrix}
1 & \frac{1}{R} & \frac{1}{C} & \frac{1}{RC} & \frac{1}{CL} & \frac{1}{RCL} \\
0 & 1 & 0 & \frac{1}{C} & 0 & \frac{1}{CL} \\
0 & 0 & 1 & \frac{1}{R} & 1 & \frac{1}{RL} \\
0 & 0 & 0 & 1 & 0 & \frac{1}{L} \\
0 & 0 & 0 & 0 & 1 & \frac{1}{R} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$

To discover the form of the row vector $W_S$, write

$A_{(S: (R)(C:L))} = \begin{pmatrix}
1 & U_S \\
0 & A_{(R)(C:L)}
\end{pmatrix}$.

Considering the product of $A_{(S: (R)(C:L))}$ and $A^{-1}_{(S: (R)(C:L))}$

$W_S + U_S A^{-1}_{(R)(C:L)} = 0$

or

$W_S = -U_S A^{-1}_{(R)(C:L)} = -\left(\frac{1}{S} 0 0 0 0\right) A^{-1}_{(R)(C:L)}$

so

$W_S = \left(\frac{1}{S} \frac{1}{SR} \frac{1}{SC} \frac{1}{SRC} \frac{1}{SCL} \frac{1}{SRCL}\right)$. 
WS is seen to be the product of the vector $-U_S$ and the vector which is the first row of $A_{(R)(C:L)}^{-1}$. This result is indicative of the result in more general cases. Consider now the structure $(B):(C)(D)\ldots(E)$. As before write

$$A_{(B):(C)(D)\ldots(E)} = \begin{pmatrix} 1 & U_B \\ 0 & A_{(C)(D)\ldots(E)} \end{pmatrix}$$

and

$$A_{(B):(C)(D)\ldots(E)}^{-1} = \begin{pmatrix} 1 & W_B \\ 0 & A_{(C)(D)\ldots(E)}^{-1} \end{pmatrix}.$$

Thus $W_B + U_B A_{(C)(D)\ldots(E)}^{-1} = 0$ and

$$W_B = \begin{pmatrix} \frac{1}{B} & 0 & 0 & \ldots & 0 \end{pmatrix} A_{(C)(D)\ldots(E)}^{-1},$$

or

$$W_B = \begin{pmatrix} \frac{1}{B} & \frac{1}{BC} & \frac{1}{BCD} & \ldots & \frac{1}{BCD} & \ldots & E \end{pmatrix}.$$

The nesting relationship between two factors, say $B$ and $C$, is reflected in a "nesting" of their respective matrices. For the matrix $A_{(B:C)}$ can be visualized as the block diagonal form

$$\begin{pmatrix} A_{(B)} & 0 \\ 0 & A_{(C)} \end{pmatrix}.$$
where $A^*(B)$ represents the matrix $A_{(B)}$ with its last row missing. The block diagonal form has 1's along the main diagonal and so the matrices are nested in a sense. This pattern can be generalized, as in the following theorem, where the matrix $A^*(B)(C)...(D)$ is the matrix $A_{(B)(C)...(D)}$ with its last row missing.

**Theorem:**

$$A_{(B)(C)...(D)} : (E)(F)...(G) = \begin{pmatrix} A^*(B)(C)...(D) & 0 \\ 0 & A_{(E)(F)...(G)} \end{pmatrix}.$$

**Proof:**

The discussion preceding the statement of the theorem shows that it holds for the structure $(B):(E)(F)...(G)$. Consider then as an inductive hypothesis

$$A_{(B)(C)...(X)} : (E)(F)...(G) = \begin{pmatrix} A^*(B)(C)...(X) & 0 \\ 0 & A_{(E)(F)...(G)} \end{pmatrix}.$$  

and consider $A_{(B)(C)...(X)(D)} : (E)(F)...(G)$. Since the letters $B,C,...,X$ are all crossed with $D$, the last admissible structure with letters only in the rightmost bracket is
\((\emptyset)(BC\ldots XD)\). Its \(\Sigma\) expression is dependent on all the admissible structures following it in the standard order since all those letters are nested by those in the first set. The admissible structure immediately preceding this one is \((\emptyset)(C\ldots XD)\) and its \(\Sigma\) expression clearly depends only on
\[
\sigma^2(\emptyset)(C\ldots XD) \quad \text{and} \quad \sigma^2(\emptyset)(BC\ldots XD).
\]
This shows that the block diagonal form remains and thus the proof of the theorem is completed.

The matrix \(A^{-1}(B)(C)\ldots(D):(E)(F)\ldots(G)\) is also of some interest. Let \(A^{-1*}\) denote \(A^{-1}\) with its last row missing. Then
\[
A^{-1}(B)(C)\ldots(D):(E)(F)\ldots(G) = A^{-1*}(B)(C)\ldots(D) = \begin{pmatrix}
A^{-1*}(B)(C)\ldots(D) & 0 \\
0 & A^{-1}(E)(F)\ldots(G)
\end{pmatrix}
\]
where
\[
Q = \begin{pmatrix}
\frac{1}{BC\ldots D} \\
\frac{1}{C\ldots D} \\
\ddots \\
\frac{1}{D}
\end{pmatrix} \cdot \begin{pmatrix}
\frac{1}{E} & \frac{1}{F} & \frac{1}{EF} & \cdots & \frac{1}{EF\ldots G}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{BC\ldots D} \\
\frac{1}{C\ldots D} \\
\ddots \\
\frac{1}{D}
\end{pmatrix}
\]
This result can be verified by multiplication.

For any structure the components can also be formed in a manner completely analogous to that already given for the \( \Sigma \)'s. The components are linear functions of admissible means formed by omitting some, none, or all of the letters in the rightmost bracket with coefficient \((-1)^p\) where \(p\) is the number of letters omitted.

Let \( Y_L \) denote the vector of leading means, and \( Y_C \) a vector of components. The convention is adopted here that the components are written in the reverse order to that used for the formation of the \( \Sigma \)'s. (This is due to the deletion of letters in the rightmost bracket for the formation of components as opposed to their addition in the formation of the \( \Sigma \)'s.) Then the matrix \( \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \) is the appropriate one to associate with any particular individual letter in the structure and if \( Y_C = N Y_L \), where \( N \) carries the appropriate structure as a subscript, then

\[
N(S)(T)(R)\ldots = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \times \ldots,
\]

and

\[
N(S:T:R\ldots) = \begin{pmatrix} 1 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 1 \end{pmatrix}
\]
The $N$ matrix is easily formed for any unitary structure. For example,

$$N_{(B:P)(T)} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. $$

All these statements can be demonstrated by proofs entirely similar to those used in the discussion of the $\Sigma$'s. The population identity in particular is seen to hold by adding the columns in the $N$ matrix.

E. Non-unitary Structures

The only non-unitary structure with four factors is $(S)(P):R$ and $S:Q$. The frequency of occurrence of these structures increases with increasing numbers of factors. Of the 63 five factor structures given by Throckmorton (1961), 15 are non-unitary. Gilbert (1961) has a list of 319 six factor structures and of these 160 are non-unitary. The general description of the analysis of variance proceeds in a standard way (Zyskind, 1958).

One particularly unusual aspect of these structures is that they may be described in the notation adopted here in many different ways. The descriptions "$(S)(P):R$ and $S:Q$" or "$(S:Q)(P)$ and $(S)(P):R$" each specify the structure discussed above. It will be useful to adopt a standard
procedure for describing any structure and the one adopted here is the following.

Choose any letter, say S, at the highest tier and write it and any letters nested only in it directly or indirectly until all the letters nested in S are exhausted or until a letter is reached which is also nested in some other letter. Return to the highest tier and write another letter, if any. If this letter nests another with S this is indicated, followed by any other letter nested in it, and the process is continued until the letters are exhausted or it is not possible to indicate the proper relationship between some new variable and the ones already written. If the latter is the case, "and" is written and the nesting relationship between this new letter and letters previously written is given. This scheme is continued, introducing one new letter at a time, and moving down the tiers of the structure, until the letters are exhausted. This gives a specification of the structure since all the letters and their proper relationships are given. Since the starting point on any tier may be arbitrary in the case of a large number of factors, the process does not yield a unique description.

It is to be noted that the occurrence of any letter implies the occurrence of any letter in which it is nested since otherwise the specification would not be complete. It is also important to note that while the notation
$S:(P)(Q)$ might occur initially, this sort of multiple nesting cannot be used after the first set of letters unless all the letters except one occurring in the new set have been used before. Thus this substructure might have to be denoted by $S:P$ and $S:Q$.

The configuration of letters written before the word "and" is a unitary structure and admits of many descriptions and there are many orders in which the variables it contains may be written. Now the letters which have not been written at this point must be nested, singly or multiply, in letters or combinations of letters already written. The following convention on the description of the unitary structure is then adopted, no cases being known where this is not possible: the order of letters in the unitary structure is selected so that the elements containing the combination of letters nesting the new letter occurs at a point such that at least one of the elements following it has all its excess letters in the rightmost bracket.

The rules supplement those already given and it is most efficient to use them only when complex structures are involved and they are not easily decomposed. For example, the structure diagrammed below is obviously the previously given structure crossed with the letter T and the A matrix can be written down after the rules are applied to the structure $(S:Q)(P)$ and $(S)(P):R$. 

The procedure described gives an order to the letters involved in the structure, although not a unique one. The ordering is usefully noted for the basis elements will be written using that order in the notion of standard order given above.

Now consider the A matrix, using the basis elements and their * products in the specified order. Assuming for the moment that the specification of the structure involves only one connective "and", the matrix A will be written in the partitioned form

\[
\begin{pmatrix}
A_1 & A_4 \\
A_2 & A_3
\end{pmatrix}
\]

The submatrix \( A_1 \) relates the basis elements for the letters occurring before the "and" to each other. The elements occurring after these contain a new letter and so cannot be
a subset of the set of elements occurring before them. Thus $A_2 = 0$.

The second set of letters contains exactly one letter, say $R$, not used previously and some letters, in combinations of which $R$ is nested, which do occur in the first set. Thus the second set of basis elements contains $R$ in the rightmost bracket. In addition to the basis element for $R$, the second set contains only elements which are the products of the basis element for $R$ and elements of the first set which contain letters in addition to those in which $R$ is nested. $A_3$ thus relates the letters which are not in the intersection of the two sets, except $R$ of course.

The only elements in the first set whose $\Sigma$ expression is dependent on elements in the second are those which contain the combination of letters nesting letters appearing in the second. Thus $A_4$ will have the form $\left[ \begin{array}{c} 0 \\ \overline{A}_4 \end{array} \right]$ due to the order adopted. $\overline{A}_4$ must be $\frac{1}{R}A_3$ since the elements used are precisely those in the second set except for the lack of $R$ in the rightmost bracket.

If the structure uses more than one connective "and", the above procedure is followed except that the "first" set referred to above will represent the union of the sets preceding that which is being presently considered. A final possibility must be considered. If at any stage the intersection of the sets exhausts the letters used, except for the
new letter involved, the $A$ submatrix to be written consists of the number 1 alone, reflecting the basis element for the new letter.

The argument just given appears to give the form of the $A$ matrix in general. However, until more is known about non-unitary structures with more than six factors, only a tentative claim of the general validity of this form is made. The result has been verified for all 5 factor structures and for many of the 6 factor structures as well.

Many examples of this procedure will be given, followed by a list of the other non-unitary structures with 5 factors. Together, they will comprise a complete list of non-unitary structures with 5 factors.

**Example:**

The only non-unitary structure with 4 factors has the following structure diagram:

```
        H
       / \  
      S   P
     /\   /\  
    Q  R  Q  R
```

Choosing from the second tier, the factors in standard order are: $S,Q,P,R$. ($Q$ is nested only in $S$, then another
factor on the second tier, P, must be used and R is nested in a combination, namely SP, of the letters previously used.) Thus the specification of the structure is (S:Q)(P) and (S)(P):R. The new letter in the second set is R, and the intersection of the sets consists of the letters S and P.

Hence $A_1 = A_{(S:Q)}(P)$, $A_3 = A_{(Q)}$ and the A matrix is

$$
\begin{pmatrix}
A_{(S:Q)}(P) & 0 \\
\frac{1}{R}A_{(Q)} & 0 \\
0 & A_{(Q)}
\end{pmatrix}.
$$

This can be written

$$
\begin{pmatrix}
1 & \frac{1}{S} & 0 & \frac{1}{P} & \frac{1}{SP} & 0 & 0 & 0 \\
0 & 1 & \frac{1}{Q} & 0 & \frac{1}{P} & \frac{1}{PQ} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{P} & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{S} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{1}{Q} & \frac{1}{R} & \frac{1}{QR} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{R} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{Q} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$
The dashed lines indicate the partitioning. If $P$ were selected from the second tier, the order of the factors would be $P, S, R, Q$ and the specification $(P)(S):R$ and $S:Q$. The matrix is then

$$
\begin{pmatrix}
A(P)(S):R \\
\begin{pmatrix}
0 \\
\frac{1}{Q} A(P:R) \\
0 \\
A(P:R)
\end{pmatrix}
\end{pmatrix},
$$

or

$$
\begin{pmatrix}
1 & \frac{1}{P} & \frac{1}{S} & \frac{1}{SP} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{1}{S} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \frac{1}{P} & 0 & \frac{1}{Q} & \frac{1}{QP} & 0 \\
0 & 0 & 0 & 1 & \frac{1}{R} & 0 & \frac{1}{Q} & \frac{1}{RQ} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{Q} \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{P} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{R} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Since the two 8x8 matrices simply represent a reordering of the basis elements and their * products, they are
equivalent.

**Example:**

As an example of a structure with 5 factors, consider the configuration below and the order $S,T,P,R,Q$.

$\text{Diagram}$

The specification is then $(S \cdot T)(P)$ and $(S)(P):R$ and $P:Q$ for which the $A$ matrix is

$$
\begin{pmatrix}
A_{(S \cdot T)(P)} & 0 & 0 \\
\frac{1}{R} A_{(T)} & \frac{1}{Q} A_{S:(T)(R)} \\
0 & A_{(T)} \\
0 & 0 & A_{S:(T)(R)}
\end{pmatrix}
$$

**Example:**

The structure whose diagram is below can be written, using the order $S,P,Q,R,T$, as $(S)(P):(Q)(R)$ and $P:T$.

$\text{Diagram}$
The $A$ matrix is then
\[
\begin{pmatrix}
A_{(S)}(P):(Q)(R) & 0 \\
-\frac{1}{T}A_{S}(Q)(R) & A_{S}(Q)(R)
\end{pmatrix}
\]

If the order $P,T,S,R,Q$ is chosen, the structure must be described as $(P:T)(S)$ and $(S)(P):R$ and $(S)(P):Q$ and the $A$ matrix is
\[
\begin{pmatrix}
A_{(P:T)}(S) & 0 \\
-\frac{1}{R}A_{(T)} & A_{(T)}(R)
\end{pmatrix}
\]

The two matrices are again equivalent.

Example:
This structure is obviously the 4 factor non-unitary structure described above nesting the factor T. The A matrix can then be described using the nesting relationship and the A matrix for the four factor non-unitary structure or it can equivalently be written:

\[
\begin{pmatrix}
A(S)(P):R & 0 \\
\frac{1}{Q} A_S:R & 0 \\
0 & A_S:R \\
0 & 0 & 1
\end{pmatrix}.
\]

The remaining non-unitary structures with 5 factors are given below with their diagrams and specifications. The A matrices can then be written using the procedures already given. The mean and error will not be shown for these structures.

1) \( S \rightarrow R \rightarrow Q \rightarrow P \rightarrow T \)

\( (S)(P):R \) and \( P:Q \) and \( P:T \) or \( (P:(Q)(T))(S) \) and \( (S)(P):R \)

2) \( S \rightarrow R \rightarrow T \rightarrow P \rightarrow Q \)

\( (S)(T)(P):R \) and \( P:Q \) or \( (P:Q)(S)(T) \) and \( (S)(T)(P):R \)
3)\[ (T)(S)(P):Q \quad \text{and} \quad (S)(P):R \quad \text{or} \quad ((S)(P):R)(T) \quad \text{and} \quad (T)(S)(P):Q \]

4)\[ ((S)(P):R)(T) \quad \text{and} \quad (P)(T):Q \quad \text{or} \quad ((T)(P):Q)(S) \quad \text{and} \quad (S)(P):R \]

5)\[ (P:(Q)(T))(S) \quad \text{and} \quad (S)(P:T):R \]

6) This is the factor \( T \) nesting a structure similar to the 4 factor non-unitary structure

7)\[ (T:S)(P):R \quad \text{and} \quad P:Q \]

8)\[ (P:T:Q)(S) \quad \text{and} \quad (S)(P):R \]
Finally, examples of 6 and 7 factor studies which are non-unitary which have been checked will be given.

Example:

A description of this structure is \((S:V)(P)\) and \((S)(P):Q\) and \((S)(P):R\) and \(P:T\).
Example:

The description (S:U)(T)(P) and (S)(T)(P):R and P:Q and P:X yields an A matrix of the form described.
V. CHAPTER FIVE: MOMENT CHARACTERISTICS AND NORMALITY

A. Tests of Normality

A very common presumption for the application of many of the usual statistical tests is that the sample be drawn from a normal parent population. In many practical instances the normality of the parent may be assumed from some theoretical standpoint and in other cases the test itself possesses robustness of an extent which renders the test approximately applicable even if normality is not presumable. In the remaining cases, where reasons for presuming normality are not known or where robustness with respect to normality has not been demonstrated, it is necessary to make a test of the normality presumption before proceeding.

Many writers have considered tests for normality for a single sample. Fisher (1929) considered two ratios for testing normality, namely $k_3k_2^{-3/2}$ and $k_4k_2^{-2}$. Using the distributional properties of the $k$ statistics which he had worked out, Fisher found functions of these ratios, which, to a given order of approximation, were normally distributed variables themselves, provided the sampling had been done from a normal parent. Fisher himself soon rendered this work somewhat obsolete by publishing in 1930 two methods for finding the exact moments of functions of these ratios. He
also discussed the exact moments of some other product type statistics for testing normality. Pearson (1930a,b) used Fisher's methods concerning the distributions of the $k$ statistics to work out approximations for the moments of $k_3^2 k_2^{-3/2}$ and $k_4^2 k_2^{-2}$ to a greater extent than it had been possible to do before Fisher's 1929 paper.

Other tests for normality have been proposed, such as tests utilizing the ratio of the mean deviation to the standard deviation which were discussed by Geary (1935). David, Hartley, and Pearson (1954) used the ratio of the sample range to an estimate of the sample standard deviation, both estimates being based on a single sample. Other work preceding this used ratios derived from independent samples. Other common tests for normality based on a single sample are the chi squared, Cramer - Von Mises (Cramer, 1928) and the Kolmogorov - Smirnov tests (Kolmogorov, 1933).

Anscombe (1961) and Anscombe and Tukey (1963) have applied measures analogous to those proposed by Fisher to the examination of the residuals, in particular those arising in a two way classification table. They found that the variances of their measures were much larger than those for samples from an unstructured population. Shapiro and Wilk (1965) proposed a test for normality for a complete sample which uses a function of the order statistics divided by the usual symmetric estimate of the variance. The test compares
favorably with many of the usual tests, although the percentage points of its distribution for large sample sizes are apparently difficult to work out.

A general goodness of fit test, using equiprobable partitions, has recently been proposed by Kempthorne (1968).

The present chapter is concerned with the assessment and computation of moment characteristics arising in consideration of normality. In particular, the first problem considered is that of dealing with several samples, each of which may be drawn from a normal population. The means of the various populations may be different, but these populations are assumed to have the same variance. Two statistics, previously used in unpublished work of F. J. Anscombe and C. P. Cox, are explored with respect to their first four moment characteristics using techniques associated with the k statistics that have been detailed in the previous chapters. A new statistic is proposed in the case of unequal sample sizes. The final section of the chapter discusses bivariate dispersion and directly utilizes the material on multivariate polykays given in Chapter Three. It is hoped that such analyses may illustrate tractable techniques with multivariate distributions and may lead to a series of tests for bivariate normality. No literature is known concerning the analysis of several samples or the testing of multivariate data for normality.
B. Testing Several Samples for Normality

Suppose a series of \( r \) samples, each of size \( n \), is available from normal distributions with means \( m_1, m_2, \ldots, m_r \) and variances all equal to \( \sigma^2 \). For each sample, \( k_1(i), k_2(i), \ldots, i = 1, 2, \ldots, r \) are calculated and let

\[
\overline{k}_j = \frac{1}{r} \sum_i k_j(i), \quad \overline{g}_1 = \frac{k_3}{k_2^{-3/2}}, \quad \text{and} \quad \overline{g}_2 = \frac{k_4}{k_2}. 
\]

Anscombe and Cox argue the statistical independence of \( \overline{g}_1 \) and \( \overline{k}_2 \) and that of \( \overline{g}_2 \) and \( \overline{k}_2 \). The technique followed here is not based on independence and hence is applicable to a wide range of distributions.

If the populations are normal then \( \mathbb{E}(\overline{g}_1) = 0 \) and so \( \mathbb{V}(\overline{g}_1) = \mathbb{E}(\overline{g}_1^2) \). Now write

\[
\overline{g}_1^2 = \frac{k_2}{k_2^3} \left( 1 + \frac{k_2 - \kappa_2}{\kappa_2} \right)^{-3}. 
\]

Note that in general, \( \mathbb{E}(\overline{k}_2) = \mathbb{E}\left( \frac{1}{r} \sum_i k_2(i) \right) = \frac{1}{r} \sum_i \kappa_2(i) \), but if the populations have common \( \sigma^2 \) then

\[
\kappa_2(i) = \sigma^2, \quad \text{so that} \quad \mathbb{E}(\overline{k}_2) = \sigma^2 = \kappa_2. 
\]

Now if

\[
\left( \frac{\overline{k}_2 - \kappa_2}{\kappa_2} \right)^2 < 1
\]

or, equivalently, for \( 0 < \overline{k}_2 < 2 \kappa_2 \), then
The restriction \( 0 < \bar{\kappa}_2 < 2\kappa_2 \) is understood to obtain in the following discussion.

Application of the expectation operator to both sides of the above equation gives

\[
E(\bar{g}_1^2) = \frac{1}{\kappa_2^2} \left\{ E(\bar{k}_3^2) - \frac{3}{\kappa_2^2} E[\bar{k}_3^2(\bar{k}_2 - \kappa_2)] + \frac{3 \cdot 4}{21} \frac{1}{\kappa_2^2} \bar{k}_3^2 (\bar{k}_2 - \kappa_2)^2 \right. \\
- \left. \frac{3 \cdot 4 \cdot 5}{31} \frac{1}{\kappa_2^3} \bar{k}_3^2 (\bar{k}_2 - \kappa_2)^3 + \ldots \right\}
\]

The question of the validity of expansions such as (1) is discussed by David (1949). The results must be understood to be asymptotic ones since the restriction \( 0 < \bar{\kappa}_2 < 2\kappa_2 \) does not hold with certainty. In addition, since the cumulants of order two or greater are origin invariant (Kendall and Stuart, 1969),

\[
\kappa[\bar{k}_3^2, (\bar{k}_2 - \kappa_2)^P] = \kappa(\bar{k}_3^2, \bar{k}_2^P)
\]

where the notation indicates a cumulant of the joint distribution of \( \bar{k}_3^2 \) and \( \bar{k}_2 \).

Now \( E[\bar{k}_3^2(\bar{k}_2 - \kappa_2)^P] \) may be expressed as a function of these cumulants and so \( E[\bar{k}_3^2(\bar{k}_2 - \kappa_2)^P] = E(\bar{k}_3^2 \bar{k}_2^P) \). This will
be abbreviated as $\mu(\bar{3}^2)$, so

$$E(g_1^2) = \frac{1}{\kappa_2^3} \mu(\bar{3}^2) - \frac{3}{\kappa_2^2} \mu(\bar{3}^2) + \frac{6}{\kappa_2} \mu(\bar{3}^2 \bar{2}^2)$$

$$- \frac{10}{\kappa_2^3} \mu(\bar{3}^2 \bar{2}^3) + \frac{15}{\kappa_2^4} \mu(\bar{3}^2 \bar{2}^4) - \ldots \}.$$

The generating relationship connecting the moments and respective cumulants of the joint distribution of $\bar{k}_3$ and $\bar{k}_2$ is

$$1 + \mu(\bar{3}) t_3 + \mu(\bar{3}^2) t_2 + \mu(\bar{3} \bar{2}) t_3 t_2 + \mu(\bar{3}^2) \frac{t_3^2}{24}$$

$$+ \mu(\bar{3}^2) \frac{t_3^2 t_2}{24} + \mu(\bar{3}^2 \bar{2}^2) \frac{t_3^2 t_2}{2124} + \ldots$$

$$= \exp\{\kappa(\bar{3}) t_3 + \kappa(\bar{2}) t_2 + \kappa(\bar{3} \bar{2}) t_3 t_2 + \kappa(\bar{3}^2) \frac{t_3^2}{24} + \ldots\}$$

$$= \{1 + \kappa(\bar{3}) t_3 + \kappa^2(\bar{3}) \frac{t_3^2}{24} + \ldots\} \{1 + \kappa(\bar{2}) t_2 + \kappa^2(\bar{2}) \frac{t_2^2}{24} + \ldots\}$$

$$\{1 + \kappa(\bar{3} \bar{2}) t_3 t_2 + \kappa^2(\bar{3} \bar{2}) \frac{t_3^2 t_2}{2124} + \ldots\} \{1 + \kappa(\bar{3}^2) \frac{t_3^2}{2124} + \kappa^2(\bar{3}^2) \frac{t_3^4}{21242124} + \ldots\}.$$
\[ \mu (\overline{3^2}) = \kappa^2 (\overline{3}) \kappa (\overline{2}) + 2 \kappa (\overline{3}) \kappa (\overline{2^2}) + 2 \kappa (\overline{3^2}) \]

\[ + \kappa (\overline{3^2}) \kappa (\overline{2}) + \kappa^2 (\overline{3}) \kappa (\overline{2^2}) + \kappa (\overline{3^2}) \kappa (\overline{2^2}) + 2 \kappa (\overline{2}) \kappa (\overline{3^2} \overline{2^2}) + 4 \kappa (\overline{2}) \kappa (\overline{3^2}) \kappa (\overline{2^3}) \]

\[ + \kappa^2 (\overline{3}) \kappa (\overline{2^2}) + 2 \kappa (\overline{2}) \kappa (\overline{3^2}) \kappa (\overline{2^3}) + 2 \kappa (\overline{3^2}) \kappa (\overline{2^3}) \kappa (\overline{3}) \cdot \]

\[ \mu (\overline{3^2} \overline{2^3}) = \kappa (\overline{3^2} \overline{2^3}) + 2 \kappa (\overline{3}) \kappa (\overline{3^2} \overline{2^3}) + \kappa^2 (\overline{3}) \kappa (\overline{2^3}) \]

\[ + \kappa (\overline{3^2}) \kappa (\overline{2^3}) + \kappa^3 (\overline{2}) + 3 \kappa (\overline{3}) \kappa (\overline{3^2} \overline{2^2}) + 3 \kappa (\overline{3^2}) \kappa (\overline{2^3}) \]

\[ + \kappa^2 (\overline{3}) \kappa (\overline{2^2}) + 3 \kappa (\overline{2}) \kappa (\overline{3^2} \overline{2^2}) + 3 \kappa (\overline{3^2}) \kappa (\overline{2^3}) \]

\[ + 6 \kappa (\overline{3}) \kappa (\overline{2^3}) + 3 \kappa (\overline{3^2}) \kappa (\overline{2^3}) \kappa (\overline{2^2}) + 3 \kappa^2 (\overline{3}) \kappa (\overline{2^2}) + 6 \kappa (\overline{3}) \kappa (\overline{3^2}) \kappa (\overline{2^2}) \]

\[ + 6 \kappa (\overline{3}) \kappa (\overline{3^2}) \kappa (\overline{2^3}) \cdot \]

Therefore

\[ E (g_1^2) = \frac{1}{\kappa_2^2} \left\{ \kappa^2 (\overline{3}) + \kappa (\overline{3^2}) - \frac{3}{\kappa_2} [2 \kappa (\overline{3}) \kappa (\overline{3^2}) \right\}

\[ \left. + \kappa^2 (\overline{3}) \kappa (\overline{2}) + \kappa (\overline{3^2} \overline{2^2}) + \kappa (\overline{2}) \kappa (\overline{3^2}) \right\} + \frac{6}{\kappa_2^2} [\kappa^2 (\overline{3}) \kappa (\overline{2^3}) \]

\[ + 2 \kappa (\overline{3}) \kappa (\overline{3^2} \overline{2^2}) + 2 \kappa (\overline{3^2}) \kappa (\overline{2^3}) \kappa (\overline{2}) + 2 \kappa (\overline{2}) \kappa (\overline{3^2} \overline{2^2}) \]

\[ + \kappa (\overline{3^2} \overline{2^2}) + 4 \kappa (\overline{2}) \kappa (\overline{3^2} \overline{2^2}) \kappa (\overline{3}) \]

\[ - \frac{10}{\kappa_2^2} [\kappa (\overline{3^2} \overline{2^3}) + 2 \kappa (\overline{3}) \kappa (\overline{3^2} \overline{2^3}) + \kappa^2 (\overline{3}) \kappa (\overline{2^3}) \]
\[ +\kappa (\overline{3}^2) \kappa (\overline{2}^3) + \kappa (\overline{3}^3) \kappa (\overline{3}^2) + 3 \kappa (\overline{3}) \kappa (\overline{3}^2) \kappa (\overline{2}^2) + \kappa (\overline{3}^2) \kappa (\overline{3}^2) \kappa (\overline{2}^2) \]

\[ + \kappa^2 (\overline{3}) \kappa^3 (\overline{2}) + 6 \kappa (\overline{2}) \kappa^2 (\overline{3} \overline{2}) + 2 \kappa (\overline{3} \overline{2}) \kappa (\overline{3} \overline{2}^2) \]

\[ + 3 \kappa (\overline{3}^2 \overline{2}) \kappa (\overline{2}^2) + 3 \kappa (\overline{3} \overline{2}) \kappa^2 (\overline{2}) + 6 \kappa (\overline{3}) \kappa (\overline{2}) \kappa (\overline{3} \overline{2}^2) \]

\[ + 3 \kappa (\overline{3}^2) \kappa (\overline{2}) \kappa (\overline{2}^2) + 3 \kappa^2 (\overline{3}) \kappa (\overline{2}) \kappa (\overline{2}^2) \]

\[ + 6 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa (\overline{2}^2) + 6 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa^2 (\overline{2}) ] \]

\[ + \frac{15}{4} \kappa (\overline{3}^2 \overline{2}^4) + 2 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}^4) + \kappa (\overline{3}^2) \kappa (\overline{2}^4) + 3 \kappa (\overline{3}^2) \kappa^2 (\overline{2}^2) \]

\[ + 8 \kappa (\overline{3} \overline{2}) \kappa (\overline{3} \overline{2}^3) + 4 \kappa (\overline{3}^2 \overline{2}) \kappa (\overline{2}^3) + \kappa^2 (\overline{3}) \kappa (\overline{2}^4) \]

\[ + \kappa^2 (\overline{3}) \kappa^4 (\overline{2}) + \kappa (\overline{3}^2) \kappa^4 (\overline{2}) + 4 \kappa (\overline{3}^2 \overline{2}) \kappa^3 (\overline{2}) \]

\[ + 12 \kappa^2 (\overline{3} \overline{2}^2) + 6 \kappa (\overline{3}^2 \overline{2}^2) \kappa (\overline{2}^2) + 12 \kappa^2 (\overline{3} \overline{2}) \kappa (\overline{2}^2) \]

\[ + 12 \kappa^2 (\overline{3} \overline{2}) \kappa^2 (\overline{2}) + 6 \kappa (\overline{3}^2 \overline{2}^2) \kappa^2 (\overline{2}) + 4 \kappa (\overline{2}) \kappa (\overline{3}^2 \overline{2}^3) \]

\[ + 8 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa (\overline{2}^3) + 8 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa^3 (\overline{2}) \]

\[ + 12 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}^2) \kappa (\overline{2}^2) + 12 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}^2) \kappa^2 (\overline{2}) \]

\[ + 8 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}^3) \kappa (\overline{2}) + 24 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa (\overline{2}^2) \kappa (\overline{2}) \]

\[ + 24 \kappa (\overline{3}) \kappa (\overline{3} \overline{2}) \kappa^2 (\overline{2}) \kappa (\overline{2}) + 4 \kappa (\overline{3}^2) \kappa (\overline{3} \overline{2}) \kappa (\overline{2}) \]

\[ + 4 \kappa^2 (\overline{3}) \kappa (\overline{2}^3) \kappa (\overline{2}) ] - ... \].

In order to simplify the result further the cumulants involved must be expressed in terms of the cumulants of
the individual $k$ statistics.

First, the generating relation for a univariate variable may be written

$$M(X; t) = E^{xt} = \exp\{\sum \frac{\kappa_i t^i}{i!}\}$$

or

$$1 + \mu_1 t + \mu_2 \frac{t^2}{2!} + \mu_3 \frac{t^3}{3!} + \ldots = \exp\{\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \kappa_3 \frac{t^3}{3!} + \ldots\}.$$  

Now if a sample of $n$ independent $x$'s is drawn, then

$$M(\bar{X}; t) = [M(X; \frac{t}{n})]^n = [\exp\{\sum \frac{\kappa_i t^i}{i! n^{i-1}}\}]^n$$

$$= \exp\{\sum \frac{\kappa_i}{n^{i-1}} \frac{t^i}{i!}\}$$

and so the $r$th cumulant of the distribution of $\bar{X}$ is the $r$th cumulant of $x$ divided by $n^{r-1}$, a result given by Fisher (1929).

The generating relation for the distribution of $k_r$ is generally written

$$1 + \mu(r) + \mu(r^2) \frac{t^2}{2!} + \ldots = \exp\{\kappa(r) t + \kappa(r^2) \frac{t^2}{2!} + \ldots\},$$

or

$$E(e^{t \bar{r}}) = \exp\{\kappa(r) t + \kappa(r^2) \frac{t^2}{2!} + \ldots\}.$$  

Now $k_r = \frac{1}{s} \sum_{j=1}^{s} k_r(j)$, say, so
\[ M(\bar{k}_r; t) = [M(\bar{k}_r(j); \frac{t}{s})]^s \]
\[ = \exp\{\kappa(r) t + \frac{k(r^2)}{s^2} \frac{t^2}{2!} + \frac{\kappa(r^3)}{s^3} \frac{t^3}{3!} + \ldots\}. \]

Therefore \( \kappa(\bar{r}^j) = \frac{1}{s^{j-1}} \kappa(r^j) \), where \( \bar{k}_r \) is based on \( s \) observations.

Consider next the joint distribution of \( k_r \) and \( k_v \).

Now
\[ 1 + k(r) t_r + k(v) t_v + k(rv) t_r t_v + \ldots \]
\[ = \exp\{\kappa(r) t_r + \kappa(v) t_v + \kappa(rv) t_r t_v + \ldots\} \]

is the generating relation. This is equivalent to
\[ E(e^{k_r t_r + k_v t_v}) = \exp\{\kappa(r) t_r + \kappa(v) t_v + \kappa(rv) t_r t_v \}
+ \kappa(r^2) \frac{t_r^2}{2!} + \ldots\}. \]

So for the joint distribution of \( \bar{k}_r \) and \( \bar{k}_v \).
\[ E(e^{\bar{k}_r t_r + \bar{k}_v t_v}) = E e^{k_r(s) t_r + k_v(s) t_v}
= \Sigma [k_r(s) \frac{t_r}{s} + k_v(s) \frac{t_v}{s}] \]
\[ = \{M[k_r, k_v; \frac{t_r}{s}, \frac{t_v}{s}]\} \]

due to the independence of the samples. It follows that
\[ \kappa(r^{-a} v^{-b}) = \frac{1}{s^{a+b-1}} \kappa(r^a v^b). \]
Consequently,

\[ E(\mathbf{g}_1^2) = \frac{1}{3} (\kappa^2 (3) + \frac{1}{r} \kappa (3^2) - \frac{3}{\kappa^2} \frac{r^2}{r^2} (3) \kappa (32) \]

\[ + \frac{2}{r^2} \kappa (3) \kappa (32^2) + \frac{2}{r^2} \kappa (3^2) \kappa (2^2) + \frac{2}{r} \kappa (2) \kappa (3^2 2) \]

\[ + \frac{1}{r^3} \kappa (3^2 2^2) \] \[ - \frac{10}{3} \frac{1}{\kappa^2} \frac{r^2}{r^2} (3^2 3) + \frac{2}{r^3} \kappa (3) \kappa (32^3) \]

\[ + \frac{1}{r^2} \kappa^2 (3) \kappa (2^3) + \frac{1}{r^3} \kappa (3^2) \kappa (2^3) + \frac{1}{r} \kappa (3^2) \kappa^3 (2) \]

\[ + \frac{3}{r^2} \kappa (2) \kappa (3^2) \kappa (2^2) + \kappa^2 (3) \kappa^3 (2) + \frac{6}{r^2} \kappa (2) \kappa^2 (32) \]

\[ + \frac{2}{r^3} \kappa (32) \kappa (32^2) + \frac{3}{r^3} \kappa (3^2 2) \kappa (2^2) + \frac{3}{r^2} \kappa (3^2 2) \kappa^2 (2) \]

\[ + \frac{6}{r^2} \kappa (3) \kappa (2) \kappa (32^2) + \frac{3}{r^2} \kappa (3^2) \kappa (2) \kappa (2^2) \]

\[ + \frac{3}{r^2} \kappa^2 (3) \kappa (2) \kappa (2^2) + \frac{6}{r^2} \kappa (3) \kappa (32) \kappa (2^2) \]

\[ + \frac{6}{r^2} \kappa (3) \kappa (32) \kappa^2 (2) \] \[ + \frac{15}{5} \frac{1}{\kappa^2} \frac{r^2}{r^2} (3^2 4) + \frac{2}{r^4} \kappa (3) \kappa (32^4) \]

\[ + \frac{1}{r^4} \kappa (3^2) \kappa (2^4) + \frac{3}{r^3} \kappa (3^2) \kappa^2 (2^2) + \frac{8}{4} \kappa (32) \kappa (32^3) \]
\[ + \frac{4}{r} \kappa(3^2) \kappa(2^3) + \frac{1}{r^3} \kappa^2(3) \kappa(2^4) + \kappa^2(3) \kappa^4(2) \]

\[ + \frac{1}{r} \kappa(3^2) \kappa^4(2) + \frac{4}{r^2} \kappa(3^2) \kappa^3(2) + \frac{12}{r^4} \kappa^2(3^2^2^2) \]

\[ + \frac{6}{5} \kappa(3^2^2) \kappa(2^2) + \frac{12}{r^3} \kappa^2(32) \kappa(2^2) + \frac{12}{r^2} \kappa^2(32) \kappa^2(2^2) \]

\[ + \frac{6}{3} \kappa(3^2^2^2) \kappa^2(2) + \frac{4}{r^4} \kappa(2) \kappa(3^2^2^3) + \frac{8}{r^3} \kappa(3) \kappa(32) \kappa(2^3) \]

\[ + \frac{8}{r} \kappa(3) \kappa(32) \kappa^3(2) + \frac{12}{r^3} \kappa(3) \kappa(32^2) \kappa(2^2) \]

\[ + \frac{12}{r^2} \kappa(3) \kappa(32^2) \kappa^2(2) + \frac{8}{r^3} \kappa(3) \kappa(32^3) \kappa(2) \]

\[ + \frac{24}{r} \kappa(3) \kappa(32) \kappa(2^2) \kappa(2) + \frac{24}{r^2} \kappa(3) \kappa(32) \kappa^2(2) \kappa(2) \]

\[ + \frac{4}{r^3} \kappa(3^2) \kappa(2^3) \kappa(2) + \frac{4}{r^2} \kappa^2(3) \kappa(2^3) \kappa(2) \] \[ - \ldots \}

It is important to note that the above expression is quite general and is valid for any situation in which the expansion leading to (1) is valid. For a normal population, which is a focus of interest here, \( \kappa(3) = 0 \) and, \( \kappa(2) = \kappa[k_2-k_3] = \mu[k_2-k_3] = 0 \). Also there are no patterns for \( \kappa(32^p) \) since any such pattern would imply that \( 3+2p \) is divisible by \( 2 \), so \( \kappa(32^p) = 0 \) \( \forall p \). So

\[ E(\bar{\sigma_1}^2) = \frac{1}{\kappa_2} \left\{ \frac{1}{r} \kappa(3^2) - \frac{3}{\kappa_2 r^2} \kappa(3^2 2) \right\} \]
Further, in the normal case $\kappa(3^p 2^q)$ is of order $n^{-(p+q-1)}$ and this fact has been used to retain terms through order explain $n^{-3}$. Fisher's rules are applied to each term now and the result is

\[
E(\bar{q}_1^2) = \frac{6n}{(n-1)(n-2)} \frac{1}{r} \left( 1 - \frac{18}{r(n-1)} + \frac{12}{r(n-1)} \right)
\]

\[
+ \frac{288}{r^2(n-1)^2} - \frac{4800}{r^3(n-1)^3} - \frac{80}{r^2(n-1)^2} - \frac{30}{r} \cdot \frac{12}{(n-1)^2}
\]

\[
+ \frac{15 \cdot 5760}{r^4(n-1)^4} + \frac{15}{r^3(n-1)^3} + \frac{45}{r^2(n-1)^2} + \frac{60}{r^3} \cdot \frac{8}{(n-1)^3}
\]

\[
+ \frac{90}{r^3} \cdot \frac{48.2}{(n-1)^3} + \ldots \}
\]

or

\[
E(\bar{q}_1^2) = \frac{6n}{(n-1)(n-2)} \frac{1}{r} \left( 1 - \frac{6}{r(n-1)} + \frac{28}{r^2(n-1)^2} + \ldots \right). \quad (2)
\]

For $r = 1$, this result is equivalent to those given by Fisher (1930), and Pearson (1930a,b). Anscombe and Cox give
\[
\text{Var}(\overline{g_1}) = \frac{6n(n-1)}{r(n-2)(n-1+\frac{2}{r})(n-1+\frac{4}{r})}
\]

and this closed form is equivalent to the formula given above to terms through order \( n^{-3} \). The independence argument here produces the closed form of the result, but that argument cannot be applied in the non-normal case while the procedure detailed above remains valid, under the restrictions pertaining to (1), for more general populations.

When \( r = 1 \), the statistic

\[
\sqrt{\frac{(n-1)(n-2)}{6n}} \frac{k_3}{k_2^{3/2}}
\]

has 0 mean and unit variance, a result obtained by retaining only the first term in (2). This approximation, which appears to be a crude one, is given in many standard texts.

The fourth moment of \( \overline{g_1} \) can be found in an analogous way. Now,

\[
E(\overline{g_1}^4) = E \frac{k_3^4}{k_2^6} = E \frac{k_3^4}{k_2^6} \left\{ 1 + \frac{k_2-k_2}{k_2} \right\}^{-6}
\]

\[
= E \frac{k_3^4}{k_2^6} \left\{ 1 - \frac{6(k_2-k_2)}{k_2} + 21\frac{(k_2-k_2)^2}{k_2} \right. \\
- 56\left(\frac{k_2-k_2}{k_2}\right)^3 + \ldots \}
\]

\[
= \frac{1}{k_2^6} \left\{ \mu(3^4) - \frac{6}{k_2} \mu(3^4_2) + \frac{21}{k_2} \mu(3^4_2)^2 \right. \\
\]
\[ -\frac{56}{3} \mu(\overline{3^4})^3 + \ldots. \]

It is desired to examine the behavior of \( E(\overline{g_1^4}) \) as the size of each sample, \( n \), increases. The leading term, \( \mu(\overline{3^4}) \), is of lowest order and from the generating relation,

\[
\mu(\overline{3^4}) = \kappa(\overline{3^4}) + \kappa^4(\overline{3}) + 4\kappa(\overline{3})\kappa(\overline{3^3}) + 6\kappa^2(\overline{3})\kappa(\overline{3^2}) + 3\kappa^2(\overline{3^2})
\]

\[
= \frac{1}{r^3} \kappa(\overline{3^4}) + \frac{3}{r^2} \kappa^2(\overline{3^2})
\]

for samples from normal populations.

So,

\[
E(\overline{g_1^4}) = \frac{1}{6} \kappa_2 \left\{ \frac{3}{r^2} \kappa^2(\overline{3^2}) + \frac{1}{r^3} \kappa(\overline{3^4}) + \ldots \right\}
\]

\[
= \frac{3}{r^2} \frac{36n^2}{(n-1)^2(n-2)^2} + \frac{1}{r^3} \frac{648n^2(5n-12)}{(n-1)^3(n-2)^3} + \ldots .
\]

The missing terms are all of higher order.

Now let 
\[
x = \sqrt{\frac{r(n-1)(n-2)}{6n}} \frac{k_3}{k_2^{3/2}},
\]

so that

\[
E(x) = 0. \quad \text{Var} \ (x) = \frac{r(n-1)(n-2)}{6n} \text{Var} \ \kappa_3 \kappa_2^{-3/2}
\]

\[
= 1 - \frac{6}{r(n-1)} + \frac{28}{r^2(n-1)^2} - \ldots .
\]
Also \( E(x^3) = 0 \) because of the symmetry of the populations. Further,

\[
E(x^4) = 3 + \frac{18(5n-12)}{r(n-1)(n-2)} + \ldots .
\]

Thus for samples from normal populations \( x \) can be regarded as approximately normal, with respect to its small moments, when \( n \) is large. The results for \( r = 1 \) have been given in papers by Fisher (1929) and Pearson (1930a,b).

It would be possible also by this technique to find \( E(g_1^4) \) for more general populations.

Fisher (1929) proposed the use of \( g_2 = \frac{\kappa_4}{\kappa_2^2} \) as a statistic for testing normality. In the present situation where several samples are involved, \( \bar{g}_2 = \frac{-\kappa_4}{\kappa_2} \) will be considered as an analogous statistic.

Now \( E(\bar{g}_2) = 0 \) if the samples are from normal populations.

Also \( \bar{g}_2^2 = \frac{-\kappa_4^2}{\kappa_2^4} = \frac{-\kappa_4^2}{\kappa_2^4} \{1 + \frac{-\kappa_2}{\kappa_2}\} - 4 \)

\[
= \frac{-\kappa_4^2}{\kappa_2^4} \{1 - 4(\frac{-\kappa_2}{\kappa_2}) + 10(\frac{-\kappa_2}{\kappa_2})^2 - 20(\frac{-\kappa_2}{\kappa_2})^3 \}
\]

\[
+ 35(\frac{-\kappa_2}{\kappa_2})^4 - 56(\frac{-\kappa_2}{\kappa_2})^5 + \ldots \},
\]

so

\[
E(\bar{g}_2^2) = \frac{1}{\kappa_2^4} \{\mu(\bar{q}^2) - \frac{4}{\kappa_2^2} \mu(\bar{q}^2\bar{q}) + \frac{10}{\kappa_2^2} \mu(\bar{q}^2\bar{q}^2) \}.
\]
\[- \frac{20}{3} \mu(\bar{4}^2\bar{2}^3) + \frac{35}{4} \mu(\bar{4}^2\bar{2}^4) \]

\[- \frac{56}{5} \mu(\bar{4}^2\bar{2}^5) + \ldots \],

where the expansion is valid for populations for which all the expectations are finite.

Now from the generating relation,

\[ \mu(\bar{4}^2) = \kappa(\bar{4}^2) + \kappa^2(\bar{4}) \]

\[ \mu(\bar{4}^2\bar{2}) = \kappa(\bar{4}^2\bar{2}) + \kappa(\bar{4}^2) \kappa(\bar{2}) + \kappa^2(\bar{4}) \kappa(\bar{2}) + 2 \kappa(\bar{4} \bar{2}) \kappa(\bar{2}). \]

\[ \mu(\bar{4}^2\bar{2}^2) = \kappa(\bar{4}^2\bar{2}^2) + \kappa(\bar{4}) \kappa(\bar{4} \bar{2}^2) + \kappa(\bar{4}^2) \kappa(\bar{2}^2) \]

\[ + 2 \kappa^2(\bar{4} \bar{2}) + \kappa^2(\bar{4}) \kappa(\bar{2}^2) + \kappa^2(\bar{4}) \kappa^2(\bar{2}) \]

\[ + \kappa(\bar{4}) \kappa^2(\bar{2}) + 2 \kappa(\bar{2}) \kappa(\bar{4}^2\bar{2}) + 4 \kappa(\bar{2}) \kappa(\bar{4} \bar{2}) \kappa(\bar{4}). \]

\[ \mu(\bar{4}^2\bar{2}^3) = \kappa(\bar{4}^2\bar{2}^3) + \kappa(\bar{4}) \kappa(\bar{4} \bar{2}^3) + \kappa(\bar{2}) \kappa(\bar{4}^2\bar{2}^2) \]

\[ + \kappa(\bar{2}) \kappa^2(\bar{4} \bar{2}) + \kappa(\bar{4}^2) \kappa(\bar{2}^3) + \kappa^2(\bar{4}) \kappa(\bar{2}^3) \]

\[ + \kappa(\bar{4}^2) \kappa^3(\bar{2}) + \kappa^2(\bar{4}) \kappa^3(\bar{2}) + 3 \kappa(\bar{2}^2) \kappa(\bar{4}^2\bar{2}) \]

\[ + \kappa(\bar{4} \bar{2}) \kappa(\bar{4} \bar{2}^2) + \kappa(\bar{4}^2) \kappa(\bar{2}^2) \]

\[ + \kappa^2(\bar{4}) \kappa(\bar{2}^2) + \kappa(\bar{4} \bar{2}) \kappa(\bar{4}) \kappa(\bar{2}) \]

\[ + \kappa(\bar{4} \bar{2}) \kappa(\bar{4}) \kappa(\bar{2}^2) + \kappa(\bar{4} \bar{2}) \kappa(\bar{4}) \kappa^2(\bar{2}) \].
\[
\mu(\bar{g}^2) = \kappa(\bar{g}^2) + 6 \kappa^2 (\bar{g}^2) + \kappa (\bar{g}) \kappa (\bar{g}^4) \\
+ 4 \kappa (\bar{g}) \kappa (\bar{g}^2) + \kappa (\bar{g}^2) \kappa (\bar{g}^2) + \kappa^2 (\bar{g}) \kappa (\bar{g}^4) \\
+ \kappa^2 (\bar{g}) \kappa^2 (\bar{g}^2) + 3 \kappa^2 (\bar{g}^2) \kappa (\bar{g}^2) + 3 \kappa (\bar{g}^2) \kappa^2 (\bar{g}^2) \\
+ \kappa (\bar{g}^2) \kappa^4 (\bar{g}) + 8 \kappa (\bar{g}^2) \kappa (\bar{g}^3) + 4 \kappa (\bar{g}^3) \kappa (\bar{g}^3) \\
+ 4 \kappa (\bar{g}^2) \kappa^3 (\bar{g}) + 6 \kappa (\bar{g}^2) \kappa^2 (\bar{g}^2) + 12 \kappa^2 (\bar{g}^2) \kappa (\bar{g}^2) \\
+ 12 \kappa^2 (\bar{g}) \kappa^2 (\bar{g}) + 6 \kappa (\bar{g}^2) \kappa^2 (\bar{g}^2) + 12 \kappa^2 (\bar{g}^2) \kappa (\bar{g}^2) \\
+ 8 \kappa (\bar{g}) \kappa (\bar{g}^3) \kappa (\bar{g}) + 12 \kappa (\bar{g}^2) \kappa (\bar{g}^2) \kappa (\bar{g}^2) \\
+ 12 \kappa (\bar{g}^2) \kappa (\bar{g}^2) \kappa (\bar{g}^2) + 8 \kappa (\bar{g}^2) \kappa (\bar{g}^3) \kappa (\bar{g}^3) \\
+ 6 \kappa (\bar{g}^3) \kappa (\bar{g}^3) + 24 \kappa (\bar{g}) \kappa (\bar{g}^2) \kappa (\bar{g}^2) \kappa (\bar{g}) \\
+ 24 \kappa (\bar{g}) \kappa (\bar{g}^2) \kappa (\bar{g}^2) \kappa (\bar{g}^2) + 12 \kappa (\bar{g}) \kappa (\bar{g}^2) \kappa (\bar{g}^2) \kappa (\bar{g}^2) \\
+ 4 \kappa (\bar{g}^3) \kappa (\bar{g}^3) + 4 \kappa^2 (\bar{g}) \kappa (\bar{g}^2) \kappa (\bar{g}^3) .
\]

So
\[
\mathbb{E}(\bar{g}^2) = \frac{1}{\kappa_2} \left\{ \frac{1}{r} \kappa (\bar{g}^2) - \frac{4}{\kappa_2} \frac{1}{r^2} \kappa (\bar{g}^2) \frac{1}{r^2} \kappa (\bar{g}^2) \right\} \\
+ \frac{10}{\kappa_2} \left[ \frac{1}{r^3} \kappa (\bar{g}^2) \frac{1}{r^2} \kappa (\bar{g}^2) \right] \\
- \frac{20}{3} \left[ \frac{1}{r^4} \kappa (\bar{g}^2) \frac{1}{r^3} \kappa (\bar{g}^2) \right] + \frac{3}{r^3} \kappa (\bar{g}^2) \kappa (\bar{g}^2) \right\} + \frac{35}{4} \left[ \frac{1}{r^5} \kappa (\bar{g}^2) \right].
\]
\[ + \frac{1}{r^4} \kappa(4^2) \kappa(2^4) + \frac{3}{r^3} \kappa(4^2) \kappa^2(2^2) \]
\[ + \frac{4}{r^4} \kappa(4^2) \kappa(2^3) + \frac{6}{r^4} \kappa(4^2) \kappa^2(2^2) \]
\[ - \ldots \].

where enough terms have been retained to give the result through order \( n^{-2} \). This expression is valid for any population for which (3) holds. Again, by applying Fisher's rules and using normality, it follows that

\[ E(\bar{g}_2^2) = \frac{24n(n+1)}{(n-1)(n-2)(n-3)} \frac{1}{r} \left( 1 - \frac{32}{r(n-1)} + \frac{800}{r^2(n-1)^2} \right) \]
\[ + \frac{20}{r(n-1)} - \frac{19200}{r^3(n-1)^3} - \frac{160}{r^2(n-1)^2} - \frac{960}{r^2(n-1)^2} \]
\[ + \frac{470400}{r^4(n-1)^4} + \frac{126000}{r^3(n-1)^3} + \frac{1680}{r^3(n-1)^3} + \frac{420}{r^2(n-1)^2} \]
\[ + \frac{8960}{r^3(n-1)^3} + \frac{33600}{r^3(n-1)^3} + \ldots \}, \]

or

\[ E(\bar{g}_2^2) = \frac{24n(n+1)}{(n-1)(n-2)(n-3)} \frac{1}{r} \left( 1 - \frac{12}{r(n-1)} + \frac{100}{r^2(n-1)^2} - \ldots \right). \]

(4)

For \( r = 1 \), expression (4) becomes equivalent to results given by Fisher (1930) and Pearson (1930a,b). Anscombe and Cox give

\[ \text{Var}(\bar{g}_2) = \frac{24n(n+1)(n-1)^2}{r(n-3)(n-2)(n-1+\frac{2}{r})(n-1+\frac{4}{r})(n-1+\frac{6}{r})}. \]
This result is obtained through the use of independence. The procedure above does not utilize independence. The two results are equivalent to terms of order $n^{-3}$; the terms in (4) are the only ones available for verification.

The fourth moment of $\bar{g}_2$ may also be found.

Now

$$E(\bar{g}_2^4) = \frac{\kappa_4^4}{\kappa_2^4} = E\left\{ \frac{\kappa_4^4}{\kappa_2^4} \{ 1 + \frac{\kappa_2 - \kappa_2}{\kappa_2} \}^{-8} \right\}$$

$$= \frac{1}{8} E\kappa_4^4 \{ 1 - \frac{\kappa_2 - \kappa_2}{\kappa_2} + 36\left( \frac{\kappa_2 - \kappa_2}{\kappa_2} \right)^2 $$

$$- 120 \left( \frac{\kappa_2 - \kappa_2}{\kappa_2} \right)^3 + ... \}$$

$$= \frac{1}{8} \left\{ \mu(4^4) - \frac{8}{\kappa_2} \mu(4^4 \bar{2}) + 36 \mu(4^4 \bar{2}^2) $$

$$- 120 \mu(4^4 \bar{2}^3) + ... \right\},$$

provided the expansion is valid. Retaining the term of smallest order yields

$$E(\bar{g}_2^4) = \frac{1}{8} \left\{ \frac{3}{r^2} \kappa_2^2 (4^2) + ... \right\}, \text{ or, in the case of normality,}$$

$$E(\bar{g}_2^4) = \frac{1}{8} \frac{3}{r^2} \left( \frac{24n(n+1)}{(n-1)(n-2)(n-3)} \right)^2 \kappa_2^8 + ... \right\}.$$

So, for sufficiently large $n$,
\[ \sqrt{\frac{r(n-1)(n-2)(n-3)}{24n(n+1)}} \quad \bar{g}_2 \]

has 0 mean, unit variance, and fourth moment approximately 3.

It appears then that \( \bar{g}_1 \) and \( \bar{g}_2 \) offer measures of normality and for sufficiently large \( n \) can be dealt with as approximately normal variables for samples from normal populations. The exact distribution theory even when \( r = 1 \) is unknown: in that case the above derivations can be specialized to give the moments which are known.

The procedure above can also be applied in the case where the parent distribution is not normal although great difficulties are involved. To consider one example, \( E(\bar{g}_1) \) can be expressed as

\[
E(\bar{g}_1) = \frac{1}{\kappa_2^{3/2}} \left\{ \mu(\bar{3}) - \frac{3}{2\kappa_2} \mu(\bar{3} \bar{2}) + \frac{15}{8\kappa_2} \mu(\bar{3} \bar{2}^2) \right. \\
\left. - \frac{35}{16\kappa_2} \mu(\bar{3} \bar{2}^3) + \ldots \right\}.
\]

Since \( \kappa_2^2 - \kappa_2 \) is being used here, \( \kappa(\bar{2}) = 0 \), and it follows that

\[
E(\bar{g}_1) = \frac{1}{\kappa_2^{3/2}} \left\{ \kappa(\bar{3}) - \frac{3}{2\kappa_2} \kappa(\bar{3} \bar{2}) + \frac{15}{8\kappa_2} \kappa(\bar{3} \bar{2}^2) \right. \\
\left. + \kappa(\bar{3}) \kappa(\bar{2}^2) - \frac{35}{16\kappa_2} \left[ \kappa(\bar{3} \bar{2}^3) + 3\kappa(\bar{3} \bar{2}) \kappa(\bar{2}^2) \right] \right\}
\]
\[
+ \kappa(3)\kappa(\overline{z}^3) + \frac{315}{128\kappa_2} [\kappa(3)\overline{z}^4 + 6\kappa(3)\overline{z}^2\kappa(\overline{z}^2)]
\]

\[
+ 4\kappa(3)\kappa(\overline{z}^3) + \kappa(3)\kappa(\overline{z}^4)] - \ldots \}
\]

\[
= \frac{1}{\kappa_2^{3/2}} \{ \kappa(3) - \frac{3}{2\kappa_2} \kappa(32) + \frac{15}{8\kappa_2^2} \kappa(32^2)
\]

\[
+ \frac{15}{8\kappa_2^2} \kappa(3)\kappa(2^2) - \frac{35}{16\kappa_2^3} \frac{1}{r^3} \kappa(32^3)
\]

\[
- \frac{105}{16\kappa_2^3} \frac{1}{r^2} \kappa(32)\kappa(2^2) + \frac{35}{16\kappa_2^3} \kappa(3)\kappa(2^3)
\]

\[
+ \frac{315}{128r^2\kappa_2^4} \kappa(32^4) + \frac{1890}{128r^3\kappa_2^4} \kappa(32^2)\kappa(2^2)
\]

\[
+ \frac{1260}{128r^3\kappa_2^4} \kappa(32)\kappa(2^3) + \frac{315}{128r^3\kappa_2^4} \kappa(3)\kappa(2^4)\ldots \}
\]

Application of Fisher's rules now yields

\[
E(\overline{g}_1) = \frac{1}{\kappa_2^{3/2}} \{ \kappa_3 - \frac{3}{2\kappa_2} \left[ \frac{1}{n} \kappa_5 + \frac{6}{n(n-1)} \kappa_3\kappa_2 \right]
\]

\[
+ \frac{15}{8\kappa_2^2} \left[ \frac{1}{n^2} \kappa_7 + \frac{16}{n(n-1)} \kappa_5\kappa_2 + \frac{12(2n-3)}{n(n-1)^2} \kappa_4\kappa_3 \right]
\]

\[
+ \frac{48}{(n-1)^2} \kappa_2^2 \kappa_3 + \frac{15}{8\kappa_2^2} \kappa_3 \left[ \frac{1}{n^2} \kappa_4 + \frac{2}{n(n-1)} \kappa_2^2 \right]
\]

\[
- \frac{35}{16r^3\kappa_2^3} \left[ \frac{1}{n^3} \kappa_9 + \frac{30}{n^2(n-1)} \kappa_7\kappa_2 + \frac{2(3ln-53)}{n^2(n-1)^2} \kappa_6\kappa_3 \right]
\]
This expression may be written in a variety of forms; the following, carried to terms in $1/n^2$, is one:

$$E(\bar{g}_1) = \left[1 - \frac{21}{4r(n-1)^2} - \frac{25}{4r^2(n-1)^2}\right] \frac{\kappa_3}{\kappa_2}^{3/2}$$

$$+ \left[- \frac{3}{2rn} + \frac{135}{8r^2n(n-1)}\right] \frac{\kappa_5}{\kappa_2}^{5/2} + \frac{15}{8r^2n^2} \frac{\kappa_7}{\kappa_2}^{7/2}$$

$$+ \left[\frac{15}{8rn} - \frac{1050}{16r^2n(n-1)}\right] \frac{\kappa_4\kappa_3}{\kappa_2}^{7/2}$$

$$- \frac{35}{16r^2n^2} \frac{\kappa_6\kappa_3}{\kappa_2}^{9/2} - \frac{105}{16r^2n^2} \frac{\kappa_5\kappa_4}{\kappa_2}^{9/2} + \ldots$$

Similar calculations may be made in general for the second and fourth moments of $\bar{g}_1$ and $\bar{g}_2$ with correspondingly more complex results. The example above has been done
here simply as an illustration of the technique. Specific distributions would introduce specializations into these general results.

C. Unequal Sample Sizes

Consider now \( r \) samples of unequal sizes. Suppose the \( i \)-th sample is of size \( n_i \), \( i = 1, 2, \ldots, r \) and let

\[
\bar{g}_1^* = \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \sqrt{\frac{(n_i+1)(n_i+3)(n_i-2)}{6n_i(n_i-1)}} k_3(i) k_2^{3/2}(i).
\]

If the populations are normal, \( E(\bar{g}_1^*) = 0 \). Also

\[
\text{Var}(\bar{g}_1^*) = \frac{1}{r} \sum_{i=1}^{r} \frac{(n_i+1)(n_i+3)(n_i-2)}{6n_i(n_i-1)} \text{Var} \frac{k_3(i)}{k_2^{3/2}(i)}
\]

because of the independence of the samples. But

\[
\text{Var} \frac{k_3(i)}{k_2^{3/2}(i)} = \frac{6n_i(n_i-1)}{(n_i+1)(n_i+3)(n_i-2)},
\]

and so

\[
\text{Var}(\bar{g}_1^*) = 1.
\]

Also

\[
E(\bar{g}_1^{*4}) = \frac{1}{r^2} \sum_{i=1}^{r} \frac{(n_i+1)^2(n_i+3)^2(n_i-2)^2}{36n_i^2(n_i-1)^2} E(\frac{k_3(i)}{k_2^{3/2}(i)})^4
\]

\[
+ \sum_{i,j=1}^{r} \frac{6(n_i+1)(n_i+3)(n_i-2)}{6n_i(n_i-1)} \cdot \frac{(n_j+1)(n_j+3)(n_j-2)}{6n_j(n_j-1)}
\]
Hence

\[ E(\overline{g_1^4}) = 3 - \frac{3}{r} + \frac{3}{r^2} \sum \frac{n_i+1)(n_i+3)(n_i^2-27n_i-70)}{(n_i-2)(n_i+5)(n_i+7)(n_i+9)} \]

So \( E(\overline{g_1^4}) \to 3 \) as \( n_i \to \infty \). Thus for fairly large \( n_i \)'s, \( \overline{g_1} \) can be regarded as approximately normal. A similar result can be obtained using \( \overline{g_2} \) for samples of various sizes.

D. Bivariate Dispersion

A single test for bivariate normality for a single bivariate sample is not known. The generalized variance is often taken as a measure of bivariate dispersion. An unbiased estimate of this quantity in the normal case will be found and its variance calculated using the multivariate polykays of Chapter Three.

For a bivariate distribution the generalized variance, \(|\Sigma|\), is defined as

\[ |\Sigma| = \begin{vmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{vmatrix} = \sigma_1^2 \sigma_2^2 (1-\rho) \]

where \( \rho \) is the usual correlation coefficient between the two variables.

If \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) is distributed according to the bivariate
normal distribution with mean $\mu$ and variance-covariance matrix $\Sigma$, the moment generating function of $X$ is

$$M(X;t) = e^{t'\mu + \frac{1}{2} t'\Sigma t} = e^{t_1\mu_1 + t_2\mu_2 + \frac{1}{2} t_1^2 \sigma_1^2 + \rho \sigma_1 \sigma_2 t_1 t_2 + \frac{1}{2} t_2^2 \sigma_2^2}.$$

where

$$t = (t_1, t_2).$$

Now $\log M(X;t) = \kappa_{1,0} t_1 + \kappa_{0,1} t_2 + \kappa_{1,1} t_1 t_2 + \kappa_{2,0} t_1^2 + \kappa_{0,2} t_2^2 + \ldots$ and so

$$\kappa_{1,0} = \mu_1; \quad \kappa_{0,1} = \mu_2; \quad \kappa_{1,1} = \rho \sigma_1 \sigma_2; \quad \kappa_{2,0} = \sigma_1^2; \quad \kappa_{0,2} = \sigma_2^2.$$

Thus $|\Sigma| = \kappa_{2,0} \kappa_{0,2} - \kappa_{1,1}^2$ and hence an unbiased estimate of $|\Sigma|$ is

$$k(2,0)(0,2) - k(1,1)(1,1).$$

Now

$$k(1,1)(1,1) = \frac{n-1}{n} \left( k_{1,1}^2 - \frac{1}{n} k_{2,2} - \frac{1}{n-1} k(2,0)(0,2) \right)$$

and

$$k(2,0)(0,2) = k_{2,0} k_{0,2} - \frac{1}{n} k_{2,2} - \frac{2}{n-1} k(1,1)(1,1),$$

so
Thus
\[ d = \frac{n-1}{n-2} \{k_2,0^2 - k_{1,1}^2\} \]
is an unbiased estimator of \(|\Sigma|\).

Clearly if the generating function is known for other distributions then \(d\) can then also be estimated unbiasedly.

A check on this result is provided by calculating \(\mathbb{E}(d)\) directly. It is customary to denote the \(rs\) cumulant of the joint distribution of \(k_{\alpha,\alpha'}\) and \(k_{\beta,\beta'}\) by
\[ \kappa[(\alpha\alpha')^2(\beta\beta')^s] \quad \text{or by} \quad \kappa(\alpha \alpha \ldots \alpha \beta \beta \ldots \beta) \]
and \(\mathbb{E}(k_{\alpha,\alpha'}^r k_{\beta,\beta'}^s)\) by
\[ \mu[(\alpha\alpha')^r(\beta\beta')^s] \quad \text{or} \quad \mu(\alpha \alpha \ldots \alpha \beta \beta \ldots \beta). \]

Thus \(\mathbb{E}(d) = \frac{n-1}{n-2} \{\mu(2 \ 0) - \mu(1 \ 1)\}\). Now the generating relation is
\[
1+\mu[(2,0)]t_1+\mu[(0,2)]t_2+\mu[(2,0)(0,2)]t_1t_2+\mu[(1,1)]t_3
+ \mu[(1,1)^2]\frac{t_3^2}{2!} + \ldots
\]
\[ = \exp\{\kappa[(2,0)]t_1+\kappa[(0,2)]t_2+\kappa[(2,0)(0,2)]t_1t_2+\ldots\}. \]
From this expression it follows that

\[ \mu(2, 0, 2) = \kappa(2, 0, 2) + \kappa(2, 1, 0, 0). \]

Fisher's rules may now be applied, with bivariate partitions, to \( \kappa(2, 0, 2) \). The only non-zero pattern is

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
2 & 0 \\
0 & 2
\end{pmatrix}
\]

with pattern function \( \frac{2}{n-1} \). Therefore,

\[ \mu(2, 0, 2) = \frac{2}{n-1} \kappa_{1,1}^2 + \kappa_{2,0} \kappa_{0,2}. \]

Also

\[ \kappa_{1,1} = \kappa(1, 1) + \kappa(1, 1). \]

Thus \( \kappa(1, 1) \)

admits two non-zero patterns, namely

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 2
\end{pmatrix}
\]

and \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \),

so

\[ \kappa(1, 1) = \frac{1}{n-1} \kappa_{1,1}^2 + \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2}. \]

Thus

\[ \mu(1, 1) = \frac{n}{n-1} \kappa_{1,1}^2 + \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2}. \]
and
\[ E(d) = \frac{n-1}{n-2} \left\{ \frac{2-n}{n-1} \kappa_{1,1}^2 + \frac{n-2}{n-1} \kappa_{2,0} \kappa_{0,2} \right\} = |E| \]

This result depends on the normality of the distribution. The variance of \( d \) will be found next. Now
\[ E(d^2) = \frac{(n-1)^2}{(n-2)^2} \mathbb{E}\{ \kappa_{2,0} \kappa_{0,2}^2 - 2 \kappa_{2,0} \kappa_{0,2} \kappa_{1,1} + 2 \kappa_{1,1}^2 \} \]
and so the quantities
\[ \mu_{(2,0,0,0)^2}, \mu_{(2,0,1,1)}^2, \text{ and } \mu_{(1,1,1,1,1)} \]
are needed. It follows from the generating relation that
\[ \mu_{(2,0)^2(0,2)^2} = \kappa_{(2,0)^2(0,2)^2} + 2 \kappa_{(2,0)} \kappa_{(0,2)^2(2,0)} \]
\[ + 2 \kappa_{(2,0)^2(0,2)} \kappa_{(0,2)} + \kappa_{(0,2)^2(2,0)} \]
\[ + \kappa_{(0,2)^2(2,0)} \kappa_{(2,0)^2} + \kappa_{(0,2)^2(2,0)} \kappa_{(2,0)^2} \]
\[ + \kappa_{(0,2)^2(2,0)} \kappa_{(2,0)^2} + 4 \kappa_{(2,0)^2(0,2)} \kappa_{(2,0)^2(0,2)} + 2 \kappa_{(2,0)^2(0,2)} \kappa_{(2,0)^2} + \kappa_{(0,2)^2(2,0)} \kappa_{(2,0)^2(0,2)} \]

Also
\[ \mu_{(1,1)^2(2,0)(0,2)} = \kappa_{(1,1)^2(2,0)(0,2)} \]
\[ + 2 \kappa_{(1,1)^2(2,0)(0,2)} \kappa_{(1,1)^2(2,0)(0,2)} + \kappa_{(1,1)^2(2,0)(0,2)} \kappa_{(2,0)^2(0,2)} + \kappa_{(2,0)^2(0,2)} \kappa_{(1,1)^2(2,0)(0,2)} \]
\[ + \kappa_{(2,0)^2(0,2)} \kappa_{(1,1)^2(2,0)(0,2)} + 2 \kappa_{(2,0)^2(0,2)} \kappa_{(1,1)^2(2,0)(0,2)} \]
Cook (1951a) has worked out a large number of the bivariate cumulants, using Fisher's pattern functions. These formulas specialize considerably in the bivariate normal case since only \( \kappa_{2,0,2} \) and \( \kappa_{1,1} \) are non-zero.

Cook's formulas then give

\[
\mu[(1,1)^4] = \kappa[(1,1)^4] + 4\kappa[(1,1)^3]\kappa[(1,1)]
\]

\[
+3\kappa^2[(1,1)^2] + 6\kappa[(1,1)^2]\kappa^2[(1,1)]
\]

\[
+\kappa^4[(1,1)].
\]

\[
\mu[(2,0)^2(0,2)^2] = \frac{32}{(n-1)^3} \kappa_{1,1}^2 \kappa_{2,0,2}^2 + \frac{16}{(n-1)^3} \kappa_{1,1}^4
\]

\[
+2\kappa_{2,0}^2 \left( \frac{8}{(n-1)^2} \kappa_{0,2}^2 \kappa_{1,1}^2 \right) + 2\kappa_{0,2}^2 \cdot \frac{8}{(n-1)^2} \kappa_{2,0}^2 \kappa_{1,1}^2
\]

\[
+\kappa_{0,2}^2 \kappa_{2,0}^2 + \kappa_{2,0}^2 \cdot \frac{2}{n-1} \kappa_{0,2}^2 + \kappa_{0,2}^2 \cdot \frac{2}{n-1} \kappa_{2,0}^2
\]

\[
+4\kappa_{2,0}^2 \kappa_{0,2}^2 \cdot \frac{2}{n-1} \kappa_{1,1}^2 + \frac{4}{(n-1)^2} \kappa_{2,0}^2 \kappa_{0,2}^2
\]
\[
\mu[(1,1)^2(2,0)(0,2)] = \frac{4}{(n-1)^2} \kappa_{1,1}^3 + \frac{4}{(n-1)^2} \kappa_{1,1} \kappa_{2,0} \kappa_{0,2}^2 \kappa_{1,1}^2 + \frac{12}{(n-1)^3} \kappa_{1,1}^4 + \frac{32}{(n-1)^3} \kappa_{1,1} \kappa_{2,0} \kappa_{0,2}^2 \\
+ \kappa_{2,0} \left\{ \frac{6}{(n-1)^2} \kappa_{0,2} \kappa_{1,1}^2 + \frac{2}{(n-1)^2} \kappa_{0,2} \kappa_{2,0}^2 \kappa_{1,1} \right\} \\
+ 2 \kappa_{2,0} \kappa_{1,1} \cdot \frac{2}{n-1} \kappa_{0,2} \kappa_{1,1} + 2 \kappa_{0,2} \kappa_{1,1} \cdot \frac{2}{n-1} \kappa_{2,0} \kappa_{1,1} \\
+ 2 \cdot \frac{2}{n-1} \kappa_{2,0} \kappa_{1,1} \cdot \frac{2}{n-1} \kappa_{0,2} \kappa_{1,1} \\
+ \frac{2}{n-1} \kappa_{1,1}^2 \left\{ \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2} + \frac{1}{n-1} \kappa_{1,1}^2 \right\} \\
+ \kappa_{2,0} \kappa_{0,2} \left\{ \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2} + \frac{1}{n-1} \kappa_{1,1}^2 \right\} \\
+ \kappa_{1,1}^2 \kappa_{2,0} \kappa_{0,2} + \kappa_{0,2} \left\{ \frac{6}{(n-1)^2} \kappa_{2,0} \kappa_{1,1}^2 \\
+ \frac{2}{(n-1)^2} \kappa_{2,0} \kappa_{0,2}^2 \right\}.
\]
So

\[\mu[(1,1)^2 (2,0)(0,2)] = \frac{2(n+1)(n+2)}{(n-1)^3} \kappa_{1,1}^4 + \frac{(n+1)(n^2+5n+10)}{(n-1)^3} \kappa_{1,1}^2 \kappa_{0,2} \kappa_{2,0}^2 + \frac{(n+1)^2}{(n-1)^3} \kappa_{2,0}^2 \kappa_{0,2}^2 .\]

Also

\[\mu[(1,1)^4] = \frac{36}{(n-1)^3} \kappa_{1,1}^2 \kappa_{2,0}^0 \kappa_{0,2}^2 + \frac{6}{(n-1)^3} \kappa_{2,0}^2 \kappa_{0,2}^2 + \frac{6}{(n-1)^2} \kappa_{1,1}^2 \kappa_{2,0} \kappa_{0,2}^2 \}

+ \frac{3}{n-1} \kappa_{2,0}^2 \kappa_{0,2}^2 + \frac{1}{n-1} \kappa_{1,1}^2 \}^2 + \kappa_{1,1}^4

+ 6 \kappa_{1,1}^2 \{ \frac{1}{n-1} \kappa_{2,0} \kappa_{0,2}^0 + \frac{1}{n-1} \kappa_{1,1}^2 \},

or

\[\mu[(1,1)^4] = \frac{n(n+1)(n+2)}{(n-1)^3} \kappa_{1,1}^4 + \frac{6(n+1)(n+2)}{(n-1)^3} \kappa_{1,1}^2 \kappa_{2,0} \kappa_{0,2}^2

+ \frac{3(n+1)}{(n-1)^3} \kappa_{2,0}^2 \kappa_{0,2}^2 .\]
Hence
\[
E(d^2) = \frac{(n-1)^2}{(n-2)^2} \cdot \frac{n(n-2)(n+1)}{(n-1)^3} \kappa_{1,1}^2 + \frac{n(n-2)(n+1)}{(n-1)^3} \kappa_{2,0}^2 \kappa_{0,2}^2
\]
\[
- \frac{2n(n-2)(n+1)}{(n-1)^3} \kappa_{1,1}^2 \kappa_{2,0}^2 \kappa_{0,2}^2 \}
\]

Thus
\[
Var(d) = E(d^2) - [E(d)]^2
\]
\[
= \frac{4n-2}{(n-1)(n-2)} \kappa_{2,0}^2 \kappa_{0,2}^2 + \frac{4n-2}{(n-1)(n-2)} \kappa_{1,1}^4
\]
\[
- \frac{8n-4}{(n-1)(n-2)} \kappa_{1,1}^2 \kappa_{2,0}^2 \kappa_{0,2}^2
\]
or
\[
Var(d) = \frac{4n-2}{(n-1)(n-2)} \{ \kappa_{2,0}^2 \kappa_{0,2}^2 - \kappa_{1,1}^4 \}^2
\]

Therefore
\[
Var(d) = \frac{4n-2}{(n-1)(n-2)} |\Sigma|^2 .
\]

In the special case where \( \rho=0 \) it follows that
\[
Var(d) = \frac{4n-2}{(n-1)(n-2)} \kappa_{2,0}^2 \kappa_{0,2}^2 ,
\]
or
\[
Var(d) = \frac{4n-2}{(n-1)(n-2)} \sigma_1^2 \sigma_2^2 .
\]
If $\rho = 1$ then $|\Sigma| = 0$ and $\text{Var}(d) = 0$. This is correct since it is easy to check that $d$ itself is 0 in this case.

E. Conclusion

The novelty of the present chapter is believed to lie primarily in the techniques used. In particular, the methods used in calculating the moments of $\bar{g}_1$ and $\bar{g}_2$ can also be used for other statistics and other measures similar to $\bar{g}^*$ can be devised and their moments approximated in ways similar to those employed here.

With respect to the generalized variance, similar results can be obtained for more general bivariate surfaces than the normal but manageable results occur only when presumptions are made concerning the cumulants of those surfaces. Such assumptions can rarely be translated, however, into obvious properties of the surfaces involved.
VI. SUMMARY

Many of the problems considered here are the consequences of work which initially took a very different direction: that of determining the moments of the sampling distribution of moments for samples drawn from either a finite or an infinite population. This early work, which had been done by statisticians such as Pearson, Student, Neyman, and others, is briefly related in the first chapter. Fisher's work in particular is described in considerable detail since he in essence removed interest in the original problem by changing the direction of the line of inquiry and since familiarity with his work is necessary for a full understanding of the work subsequently presented here.

In 1929 Fisher introduced new functions of the sample values, the $k$ statistics, and showed how to determine some functions of the moments -- which he called cumulants -- of their joint distribution. Dressel in 1940 introduced somewhat more general functions, which were reintroduced in 1950 by Tukey who called them polykays. Wishart later showed how Fisher's work could be applied to their determination.

The $r$th $k$ statistic, $k_r$, is uniquely determined by the condition that its expected value, over all random samples of a fixed size, is the $r$th population cumulant, $\kappa_r$, so that $E(k_r) = \kappa_r$. The polykay $k_{rs}$ is defined similarly by
E(k_{rs}) = \kappa_r \kappa_s.

A new method of determining the polykays is given in the second chapter. This method is also based on Fisher's work but is thought to be simpler than that previously given by Wishart and in addition it leads efficiently to some heretofore unknown formulas in the univariate case. Several new proofs are given of those properties of polykays related to randomized sums which were first formulated by Tukey. These proofs capitalize on the cumulant properties of the polykays and serve to connect them and the single k statistics in a way not previously realized; they are also considerably simpler than those given by Tukey. The treatment of these matters demonstrates in a logical fashion the connection of Tukey's work with that which had been done on Fisher's k statistics.

With very few exceptions most of the literature of the k statistics has concerned the univariate case. Fisher offered four bivariate formulas and Kendall in 1940 proposed a method for deriving multivariate results from certain univariate ones. Unfortunately, Kendall did not clearly define the procedure and he gave examples which are inconsistent; in addition he did not present any new results. These problems are discussed in detail in Chapter Three and a method for deriving multivariate results from univariate ones is demonstrated. This new method is then used to derive
a complete list of multivariate \( k \) statistics of weight 5 or less. In all cases the formulas can be derived in two different ways, either through the use of symmetric sums and the new symbolic method or by applying the symbolic method from two independent starting points.

Certain bivariate symmetric sums, called bivariate bracket functions, occur in expressions for bivariate \( k \) statistics and in addition are useful in considering problems of unbiased estimation in bivariate populations because of their property of "inheritance of the average", to use Tukey's phrase from the univariate case. These functions of weight 4 or less are all expressed in terms of bivariate \( k \) statistics.

Some multivariate \( k \) statistics were used by Robson (1957) in a paper concerning ratio type estimation. His method for forming multivariate polykays was a generalization of Tukey's symbolic multiplication, a process which becomes tedious even in simple cases. A new method for forming multivariate polykays is also included in Chapter Three which specializes to the procedure used in the univariate case. The results were generally checked using the symbolic multiplication and the formulas for bivariate bracket functions in terms of bivariate \( k \) statistics previously developed. A complete catalog of bivariate polykays of weight 8 or less expressed in terms of bivariate \( k \) statistics is presented.
This is believed to be the only set of formulas presently available, and will be useful in problems of unbiased estimation of functions of bivariate population values.

The analysis of variance has long been a central tool in the analysis of experimental data. The expected mean squares in the analysis of general randomized experiments have been shown by Wilk, Zyskind, Throckmorton and White to be certain linear functions of the components of variation, called Σ's, which in turn have been found by Dayhoff to be equivalent to generalized polykays.

The Σ's are symmetric functions and are inherited on the average, as was shown by Throckmorton. The polykays have been developed for general unstructured populations while the analysis of variance is concerned with populations which are structured. The analysis of the Σ's which is given in Chapter Four is an effort to investigate the relationship between the symmetric functions, which can be expressed in terms of the k statistics, and the population structure involved.

Rules for constructing admissible structures, or reduced population structures, are given and these are followed by theorems which indicate how the Σ's may be formed using Kronecker matrix products and a new kind of "nesting" product, provided the structure is of a sort which is called unitary. The inverses of the matrices involved are of interest since
they arise when variance components are to be estimated and rules are given for their formulation. The matrices studied in this chapter are completely characteristic of the structure involved and thus form one connection between the structure and the symmetric functions. Matrix results, similar to those for unitary structures, are presented for non-unitary structures and a list of all non-unitary structures containing five factors is given.

In Chapter Five, another application of bivariate polykays is given in this instance to the bivariate normal distribution. A large body of statistical tests presumes that the data have been selected from a normal population. Fisher in 1929 used the k statistics and the properties of their joint distribution which he had worked out to approximate the moments of two ratios commonly used in testing a single sample for normality. The problem considered, here, suggested in part by some unpublished work of Anscombe and Cox, concerns a series of samples from populations with possibly different means. It is shown that two statistics used can be regarded as approximately normally distributed if the sampling has been done from normal populations with equal sample sizes, for sufficiently large samples. Another statistic is suggested which can be used with unequal sample sizes.

The moment characteristics of these statistics are investigated using the distributional properties of the k
statistics and the techniques developed in the previous chapters. No other literature concerning this problem is known.

The generalized variance is suggested as a measure of bivariate dispersion in general and an unbiased estimate of it in the bivariate normal case is found using the bivariate \( k \) statistics. It would be very difficult to carry this work much beyond this point, say to finding the third and fourth moments of the estimator using this procedure, as a very large number of pattern functions would be involved. It is nevertheless anticipated that analyses such as the one presented here may lead to a test or a series of tests for bivariate normality.

It is hoped that this work will provide a base from which more investigation of multivariate symmetric functions and their role in the characterization and classification of multivariate distributions in general can proceed. Further exploration of the relationships between the population structure and the symmetric functions occurring in the analysis of variance remains to be done. It is interesting that Fisher, who developed both the analysis of variance for structured populations and the \( k \) statistics, never combined the two ideas.
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