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Local connectedness, cyclic element theory and arcwise connectedness in topological spaces

Barbara Lehman
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Local connectedness, cyclic element theory and arcwise connectedness in topological spaces

by

Barbara Lehman

A Dissertation Submitted to the Graduate Faculty in Partial Fulfillment of The Requirements for the Degree of DOCTOR OF PHILOSOPHY

Major: Mathematics

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INTRODUCTION

This paper is divided into three nearly independent chapters. Chapter I is devoted to several conditions which, with appropriate restrictions on a topological space \( M \), are equivalent to local connectedness of \( M \).

In Chapter II, which constitutes the bulk of this paper, we turn to the development and application of cyclic element theory in general connected and locally connected Hausdorff spaces. G.T. Whyburn, in 1926, began the development of cyclic element theory for Peano continua. Over the next several years, the theory evolved as Whyburn, Ayres, Kuratowski and many others developed, refined and applied this theory, which proved to be extremely fruitful in the study of Peano spaces. A comprehensive development of cyclic element theory for metric spaces was presented by Whyburn in Chapter IV of his American Mathematical Society Colloquium Publications volume, Analytic Topology, [11], and an excellent history of the theory is to be found in B.L. McAllister's paper, Cyclic elements in topology, a history, [6].

Over the years, several attempts have been made to develop a cyclic element theory for more general topological spaces. In particular, in 1942, Albert and Youngs, [1], showed that some of the basic theory could be developed for the class of connected and locally connected \( T_1 \)-spaces. In 1968, in [12], Whyburn began the generalization of cyclic
element theory to a class of spaces which he called $H$-spaces which includes the class of connected and locally connected Hausdorff spaces. In 1970, in [8], S.E. Minear began the development of a cyclic element theory for the class of all connected and locally connected spaces and demonstrated that his theory has applications to a class of spaces which includes Whyburn's connected and locally connected $H$-spaces. In particular, Minear showed that, with suitable restrictions on the space $M$, if every cyclic element of $M$ is unicoherent, then $M$ is unicoherent. Such a property is said to be "cyclically extensible". He also showed that if $M$ is unicoherent, then every cyclic element of $M$ is unicoherent; that is, unicoherence is "cyclically reducible". These results were first obtained for Peano continua by Kuratowski in 1929. Minear also showed that the fixed point property, proved by Borsuk in 1932 to be both cyclically extensible and reducible for Peano continua, is both cyclically extensible and reducible in more general spaces.

In Chapter II of this paper, we concern ourselves primarily with connected and locally connected Hausdorff spaces, developing the cyclic element theory initiated by Whyburn in [12] and demonstrating that the theory has many of the applications to connected and locally connected Hausdorff spaces that the classical theory has to Peano spaces. As might be expected, many of our theorems are direct generalizations of theorems that are well known for metric spaces, and a few of
our proofs are identical with those of corresponding theorems in Chapter IV of [11]. However, while much of Chapter II is motivated by the results and techniques found in [11], many of our proofs are of necessity different and many of our results appear to be unknown even for metric spaces. We note also that in several instances our theorems do not depend on the Hausdorff separation axiom and thus several of our results apply to more general spaces.

In Chapter III, we turn to the concept of arcwise connectedness in general Hausdorff spaces, where by an arc we mean a Hausdorff continuum \( A \) with at most two non-cut points, called the end points of \( A \). We define a space \( S \) to be arcwise connected if and only if every two points of \( S \) are the end points of some arc in \( S \). It is well known (The Hahn-Mazurkiewicz Theorem) that a metric space \( M \) is a continuous curve if and only if \( M \) is compact, connected, and locally connected. It is also well known that a metric continuum \( I \) is an arc if and only if \( I \) is homeomorphic to \([0,1]\), and that every connected and locally connected metric space is arcwise connected. It follows, since the product of locally connected continua is a locally connected continuum, that a countable product of nondegenerate metric arcs is arcwise connected. However, examples of locally connected continua that are not arcwise connected have been constructed by S. Mardesic in [4] and [5] and by J.L. Cornette and B. Lehman in [2]. Thus the above argument will not suffice for the
product of nonmetric arcs. In Chapter III, we obtain the stronger result that any product of arcwise connected spaces is arcwise connected.

**Notation.** Throughout this paper we use the following notation. If A is a set, then "x ∈ A" is read "x is an element of A", and "∅" denotes the empty set. If A and B are sets, "A ⊆ B" denotes "A is a subset of B", "A ∪ B" and "A ∩ B" denote respectively the union and intersection of A and B, and "A - B" the complement of B in A. If C is a collection of sets, then " ∪ C " and " ∩ C " denote respectively the union and intersection of all members of C. If M is a topological space and X ⊆ M, we use "∂(X)", "Int X", "Ext X", and "X" to denote respectively the boundary, the interior, the exterior, and the closure of X. However, we will occasionally use "Cl(X)" to denote the closure of X. If X ⊆ M and E ⊆ X, then "∂_X(E)", "Int_X E" and "Ext_X E" will denote respectively the boundary, the interior and the exterior of E in X; and the closure of E in X will be denoted by "Cl_X(E)". We use "iff" to abbreviate the expression "if and only if". We note also that we do not always distinguish between the point p of a space M and the singleton set {p}; for instance, we will often write "M - p" instead of "M - {p}".
CHAPTER I:

SOME CONDITIONS RELATED TO LOCAL CONNECTEDNESS

In this chapter we consider several conditions which may or may not be true of any given topological space $M$. Before stating these conditions, however, we introduce some definitions, the first of which is a modification of a definition found in Kelley, [3].

1.1 Definition. Let $(X, >)$ be a partially ordered set. A net, $\{x_\alpha : \alpha \in (\mathcal{A}, >)\}$ in $X$ is called a decreasing net iff for each $\alpha \in \mathcal{A}$, there is a $\beta \in \mathcal{A}$ such that if $\gamma \in \mathcal{A}$ and $\gamma > \beta$, then $x_\alpha \succ x_\gamma$.

Throughout this paper, "a decreasing net of sets" will mean a net in the set of subsets of a space $M$ and the partial order will be understood to be inclusion ($x_1 \supset x_2$ iff $x_1 \subset x_2$).

1.2 Definition. A net $\{G_\alpha : \alpha \in (\mathcal{A}, >)\}$ is said to be almost distinct iff for each $\alpha \in \mathcal{A}$, there is a $\beta \in \mathcal{A}$ such that if $\gamma \in \mathcal{A}$ and $\gamma > \beta$, then $G_\gamma \neq G_\alpha$.

1.3 Definition. A net $\{G_\alpha : \alpha \in (\mathcal{A}, >)\}$ of sets is said to be almost pairwise disjoint iff for each $\alpha \in \mathcal{A}$ there is a $\beta \in \mathcal{A}$ such that if $\gamma \in \mathcal{A}$ and $\gamma > \beta$, then $G_\gamma \cap G_\alpha = \emptyset$.

It is immediate that if a net of nonempty sets is almost pairwise disjoint, then it is almost distinct.

We consider the following conditions on a topological
space \( M \).

a. \( M \) is locally connected.

b. If \( X \subset M \) and \( C \) is a component of \( M - X \), then
\[ \text{Int } C = C - \overline{X}. \]

c. Every quasicomponent of every open subspace of \( M \) is open.

d. If \( A \) is a closed subset of \( M \), \( \mathcal{C} \) a collection of components of \( M - A \), \( \mathcal{C} = \{ \overline{C} : C \in \mathcal{C} \} \), \( \mathcal{D} = \{ \overline{C} \cap A : C \in \mathcal{C} \} \), and \( p \) a limit point of \( \mathcal{C} \) such that \( p \notin \bigcup \mathcal{C} \), then \( p \in A \) and \( p \notin \overline{\bigcup \mathcal{D}} \).

e. If \( A, B \subset M \) and \( \{ G_\alpha : \alpha \in (\alpha, >) \} \) is a decreasing net of closed subsets of \( M \) each of which separates \( A \) and \( B \) in \( M \), then \( \bigcap_{\alpha} G_\alpha \) separates \( A \) and \( B \) in \( M \).

f. If \( A, B \) are nonempty subsets of \( M \) and \( Y \) is a closed cutting of \( M \) between \( A \) and \( B \), then \( Y \) contains an irreducible closed cutting of \( M \) between \( A \) and \( B \).

g. If \( B, C \) are separated sets in \( M \), \( B \) closed, and \( \mathcal{C} \) is a collection of components of \( M - (B \cup C) \) such that for each \( D \in \mathcal{C} \), \( \emptyset (D) \cap B = \emptyset \), then no point of \( B \) is a limit point of \( \bigcup \mathcal{C} \).

h. For some \( p \) in \( M \), if \( O \) is an open set containing \( p \), then there is an open set \( K \) containing \( p \) such that \( K \subset O \), \( K \) is compact, and \( p \) belongs to a nondegenerate continuum \( D \subset \overline{K} \) such that \( D \) meets \( \emptyset (K) \) and is the limit of an almost pairwise disjoint net of continua each of
which does not contain \( p \), meets \( \mathcal{E}(K) \) and is the closure of a component of \( K \).

It will follow from the theorems in this chapter that \( a - d \) are equivalent; that if \( M \) is regular and connected, then \( a - f \) are equivalent, and that if \( M \) is a connected, locally compact Hausdorff space, then \( a - g \) are equivalent and \( a \) is equivalent to the denial of \( h \).

1.4 Theorem. \( a \) and \( b \) are equivalent.

Proof that \( a \) implies \( b \). Suppose \( M \) is locally connected, \( X \subset M \) and \( C \) is a component of \( M - X \). Let \( p \in C - \overline{X} \). Then \( p \in M - \overline{X} \) and the component \( D \) of \( M - \overline{X} \) such that \( p \in D \) is open, connected and contained in \( M - X \), so \( D = C - X \subset C \). Thus \( p \in \text{Int} \, C \). If \( p \in \text{Int} \, C \), then \( \text{Int} \, C \) is an open set contained in \( M - X \), so \( p \notin \overline{X} \) and thus \( p \in C - \overline{X} \).

Proof that \( b \) implies \( a \). If \( M \) is not locally connected, then there is an open set \( O \) in \( M \) and a component \( C \) of \( O \) such that \( C \) is not open. Then \( C = \text{Int} \, C \). Let \( X = M - O \). Then \( X = \overline{X} \) and \( C - \overline{X} = C - X = C = \text{Int} \, C \).

1.5 Theorem. Let \( K \) be an open subset of \( M \). If \( Q \) is an open quasicomponent of \( K \), then \( Q \) is a component of \( K \).

Proof. By assumption, \( Q \) is open in \( K \), and since \( Q \) is a quasicomponent of \( K \), \( Q \) is closed in \( K \); thus \( (Q, K - Q) \) is a separation of \( K \). If \( Q \) is not connected, there is a separation \((U, V)\) of \( Q \), where \( U \) and \( V \) are open, disjoint and non-empty; then \((U, (K - Q) \cup V)\) is a separation of \( K \) which sepa-
rates $Q$. Since this is a contradiction, $Q$ is connected and is therefore a component.

1.6 **Corollary.** $a$ and $c$ are equivalent.

1.7 **Theorem.** $a$ and $d$ are equivalent.

**Proof that $a$ implies $d$.** Assume $M$ is locally connected. If $p \notin A$, then $p$ belongs to a component $D$ of $M - A$ and $D$ is open. Since $p \notin \bigcup C$, $D \notin C$. Then $D$ meets no member of $C$ and thus $p$ is not a limit point of $\bigcup C$. Therefore, $p \in A$. Now let $O$ be any open set containing $p$. Let $V$ be a connected open set such that $p \in V$ and $V \subseteq O$. Since $p$ is a limit point of $\bigcup C$, $V$ intersects a member $C^*$ of $C$. Since $V$ is connected and $p \in C^*$, $V \cap \mathfrak{G}(C^*) \neq \emptyset$. But for each $C$ in $C$, $\mathfrak{G}(C) \subseteq A$ and it follows that $p \in \overline{\bigcup B}$.

**Proof that $d$ implies $a$.** Suppose $M$ is not locally connected. Let $O$ be an open set such that $O$ contains a point $p$ that is not interior to a component of $O$. Let $A = M - O$. Then $p \notin A$. If $C = \{C : C$ is a component of $O$ and $p \notin C\}$, then $p \notin \bigcup C$ and $p$ is a limit point of $\bigcup C$. Further, $p \in \overline{\bigcup B}$.

1.8 **Theorem.** $a$ implies $e$.

**Proof.** Let $A, B \subset M$ and let $\{G_\alpha : \alpha \in (\mathcal{A}, >)\}$ be a decreasing net of closed sets each of which separates $A$ and $B$ in $M$. For each $\alpha \in \mathcal{A}$, let $\mathcal{V}_\alpha = \{\text{components of } M - G_\alpha\}$. Since $M$ is locally connected, each $\mathcal{V}_\alpha$ is a collection of
open subsets of $M$. Let $\mathcal{F} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$. Define a relation "\sim" on $\mathcal{F}$ by $C \sim D$ iff for some $\alpha \in \mathcal{A}$, there is a component $E$ in $\mathcal{F}_\alpha$ such that both $C$ and $D$ are contained in $E$.

It is immediate that $\sim$ is reflexive and symmetric. If $C_1 \sim C_2$ and $C_2 \sim C_3$, then there exist $\alpha$, $\beta$ in $\mathcal{A}$ and components $D_\alpha$ and $D_\beta$ in $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ respectively, such that $C_1, C_2 \subset D_\alpha$, and $C_2, C_3 \subset D_\beta$. Since the net $\{G_\alpha : \alpha \in (\mathcal{A}, >)\}$ is a decreasing net, there is a $\delta \in \mathcal{A}$ such that $\delta > \alpha$, $\delta > \beta$, and $G_\delta \subset (G_\alpha \cap G_\beta)$. Then $(M - G_\alpha) \cup (M - G_\beta) \subset M - G_\delta$. Since $D_\alpha \cup D_\beta$ is contained in $M - G_\delta$ and is connected, $D_\alpha \cup D_\beta$ is contained in a component $E$ of $M - G_\delta$, and $E \in \mathcal{F}_\delta$. Thus $C_1 \cup C_3 \subset D_\alpha \cup D_\beta \subset E$; so $C_1 \sim C_3$.

Now let $\mathcal{C}$ be the set of $\sim$-equivalence classes in $\mathcal{F}$, and let $\mathcal{C}_A = \{E \in \mathcal{C} :$ some member of $\mathcal{D}$ meets $A\}$. Then no member of $\mathcal{C}_A$ contains a set which meets $B$. For suppose that for some equivalence class $D$ of $\mathcal{C}_A$, $D$ contains a component $C$ which meets $B$. Then for some component $D$ in $\mathcal{F}$, $D$ meets $A$ and $C \sim D$. Then for some $\alpha \in \mathcal{A}$, $C$ and $D$ lie in a component of $M - G_\alpha$. But this is a contradiction since $G_\alpha$ separates $A$ and $B$ in $M$.

Let $U = \bigcup \{ \bigcup \mathcal{C}_A \}$, $V = \bigcup \{ \bigcup \mathcal{C} - \mathcal{C}_A \}$. Then $M - \bigcap_{\alpha \in \mathcal{A}} G_\alpha = U \cup V$, $U$ and $V$ are open, and it is not difficult to show that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

1.9 Corollary. If $M$ satisfies $e$ and $\mathcal{A}$ is a nest of closed subsets of $M$ each of which separates $A$ and $B$ in $M$ ($A, B$ subsets
of $M$, then $\bigcap_{\alpha} \alpha$ separates $A$ and $B$ in $M$.

**Proof.** Define $\alpha > \beta$ iff $\alpha < \beta$, for $\alpha, \beta \in \alpha$. Then $(\alpha, >)$ is a directed set and the net $\{G_\alpha : G_\alpha = \alpha, \alpha \in (\alpha, >)\}$ is a decreasing net of closed sets each of which separates $A$ and $B$ in $M$. Since $M$ satisfies $e$, $\bigcap_{\alpha} G_\alpha = \bigcap \alpha$ separates $A$ and $B$ in $M$.

1.10 **Theorem.** $e$ implies $f$.

**Proof.** Let $A, B$ be subsets of $M$, $Y$ a closed cutting of $M$ between $A$ and $B$. Let $\mathcal{A} = \{G : G$ is a closed cutting of $M$ between $A$ and $B\}$ and partial order $\mathcal{A}$ by inclusion. Since $Y \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Let $\mathcal{C}$ be a maximal chain in $\mathcal{A}$ such that $Y \in \mathcal{C}$. Then $\mathcal{C}$ is a nest of closed sets each of which separates $A$ and $B$ in $M$, so by 1.9, $\bigcap \mathcal{C}$ separates $A$ and $B$ in $M$. Thus $\bigcap \mathcal{C}$ is a closed cutting of $M$ between $A$ and $B$. If $D$ is a closed cutting of $M$ between $A$ and $B$ such that $D \subseteq \bigcap \mathcal{C}$, then for each $G$ in $\mathcal{C}$, $D \subseteq G$, so $\mathcal{C}$ is not a maximal chain. Thus $\bigcap \mathcal{C}$ is an irreducible closed cutting of $M$ between $A$ and $B$, and since $Y \in \mathcal{C}$, $\bigcap \mathcal{C} \subseteq Y$.

1.11 **Theorem.** If $M$ is connected, regular and not locally connected, then there exist subsets $A$ and $B$ of $M$ and a closed cutting $Y$ of $M$ between $A$ and $B$ such that $Y$ contains no irreducible closed cutting of $M$ between $A$ and $B$. Further, $A$ can be chosen to be a single point.

**Proof.** It follows from 1.6 that there is an open set $K_1$ of $M$ and a quasicomponent $Q^*$ of $K_1$ such that $Q^*$ is not
Let $p \in Q^* - \text{Int } Q^*$. Since $M$ is regular, there is an open set $K_2$ of $M$ such that $p \in K_2$ and $\overline{K_2} \subset K$. Let $A = \{p\}$, $B = \text{Ext } K_2$. Let $Y = \partial(K_2)$. Then $Y$ is closed and separates $A$ and $B$ in $M$. Suppose $X \subset Y$ and $X$ is closed and separates $A$ and $B$ in $M$. Let $(U, V)$ be a separation of $M - X$ such that $p \in U$ and $B \subset V$. Then $U \subset K_2$ and $K_2 = U \cup (K_2 \cap V)$. Since $U$ is a neighborhood of $p$, there is a quasicomponent $Q$ of $K_2$ such that $Q \neq Q^*$ and $Q \cap U \neq \emptyset$. There is a separation $(S, T)$ of $K_2$ such that $Q^* \subset S$ and $Q \subset T$. Then $U \cap T \neq \emptyset$. Now $\partial(U \cap T) \neq \emptyset$ and is a subset of $[\partial(U) \cup \partial(T)] \cap K_2 = \partial(U) \subset X$. Also $X = (S \cap X) \cup (T \cap X)$, and no point of $S$ is a limit point of $T$. Therefore, $\partial(U \cap T) \subset T \cap X$, so $T \cap X \neq \emptyset$. Let $U^* = U \cap S$, $V^* = T \cap V$. Then $p \in U^*$, $B \subset V^*$, $U^*$ and $V^*$ are open and disjoint, so $X' = M - (U^* \cup V^*)$ is a closed cutting of $M$ between $A$ and $B$ and is properly contained in $X$. Thus $X$ is not an irreducible closed cutting of $M$ between $A$ and $B$.

1.12 Corollary. If $M$ is connected, regular and satisfies condition $f$, then $M$ satisfies condition $a$. Thus for connected, regular spaces, $a - f$ are equivalent.

Proof. It follows from Theorem 1.11 that if $M$ does not satisfy condition $a$, then $M$ does not satisfy condition $f$.

1.13 Example. There is a connected Hausdorff space which is nonregular, not locally connected, satisfies $f$ and does not satisfy $e$. The points of $M$ are the points of the closed interval $[0, 1]$. Let $N = \{1/n : n$ is a positive integer$\)
The neighborhoods of points other than 0 are the usual neighborhoods, and a base for the neighborhood system at 0 consists of all sets of the form \( \{ y : 0 \leq y < \varepsilon \} - N, \varepsilon > 0 \).

It is well known that the space \( M \) is connected, Hausdorff and nonregular. Further, \( M \) is not locally connected at 0, but is locally connected elsewhere.

Let \( A, B \) be subsets of \( M \) and \( K \) a closed cutting of \( M \) between \( A \) and \( B \). Let \((U,V)\) be a separation of \( M - K \) such that \( A \subseteq U \) and \( B \subseteq V \). We consider two cases.

Case 1: \( 0 \in K \). In this case, \( U \) and \( V \) are each subsets of \( M - \{0\} \), which is homeomorphic to \((0,1]\) and is therefore a connected and locally connected Hausdorff space.

\( K - \{0\} \) is a closed (in \( M - \{0\} \)) cutting of \( M - \{0\} \) between \( A \) and \( B \). Since \( M - \{0\} \) satisfies \( f \), \( K - \{0\} \) contains a subset \( K' \) which is an irreducible closed cutting of \( M - \{0\} \) between \( A \) and \( B \). Then either \( K' \) or \( K' \cup \{0\} \) is an irreducible closed cutting of \( M \) between \( A \) and \( B \).

Case 2: \( 0 \notin K \). In this case assume \( 0 \in U \). Then since \( B \subseteq V \) and \( U \) and \( V \) are open, \( B \) must be bounded away from 0.

For otherwise, 0 is a limit point in \( M \) of some sequence in \( V \). Let \( \alpha = \text{glb } B \). Then \( 0 < \alpha < 1 \). Now for each \( \varepsilon \) such that \( 0 < \varepsilon < 1 - \alpha \), the interval \( (0, \alpha + \varepsilon) \) is a connected subset of \( M \) which meets both \( U \) and \( V \), so for each such \( \varepsilon \), \( K \cap (0, \alpha + \varepsilon) \neq \emptyset \). This implies that there is a \( \gamma \in K \) such that \( \gamma \leq \alpha \).

Now if \( A = \{0\} \), then \( \{\gamma\} \) is an irreducible closed cutting
of $M$ between $A$ and $B$ and is contained in $K$. If $A \neq \{0\}$, then as in case 1, $K$ contains an irreducible closed (in $M - \{0\}$) cutting $K'$ of $M - \{0\}$ between $A - \{0\}$ and $B$. Then either $K'$ or $K' \cup \{\gamma\}$ is an irreducible closed cutting of $M$ between $A$ and $B$. It follows that $M$ satisfies $f$.

Now for each integer $n$ such that $n \geq 2$, let $G_n = \{1/m: m \text{ is an integer, } m \geq n\}$. Let $A = \{0\}$, $B = \{1\}$. Then $\{G_n: n \text{ is an integer, } n \geq 2\}$ is a decreasing net of closed subsets of $M$ each of which separates $A$ and $B$ in $M$, but $\bigcap n G_n$ does not separate $A$ and $B$ in $M$. Thus $M$ does not satisfy $e$.

We state the next four theorems without proof, noting that the theorems and the proofs are similar respectively to theorems 1.1, 1.2, 1.9 and 1.10 of Chapter IV of Wilder, [13].

1.14 Theorem. In a compact, regular space, quasicomponents and components are identical.

1.15 Theorem. If $C$ is a compact component of a locally compact, regular space $M$, and $P$ is an open set containing $C$, then $M$ is the union of disjoint open sets $U, V$ such that $C \subseteq U \subseteq P$.

1.16 Lemma. If $M$ is locally compact and regular, and $A, B$ are disjoint closed subsets of $M$ such that $A$ is compact, then there exist disjoint open subsets of $M$ containing $A$ and $B$ respectively.
1.17 Theorem. If M is a connected, locally compact, regular space, K a closed subset of M, and C is a component of M - K such that $\overline{C}$ is compact, then $K \cap 3(C) \neq \emptyset$.

We now show that by strengthening the conditions on M we can weaken condition e.

1.18 Theorem. If M is a connected, locally compact, regular space that is not locally connected, then there are distinct points p and q of M and a decreasing net $\{G_\alpha : \alpha \in (\mathcal{A}, >)\}$ of closed subsets of M each of which separates p and q in M and $\bigcap_{\alpha \in \mathcal{A}} G_\alpha$ does not separate p and q in M.

Before proceeding with the proof of Theorem 1.18, we suggest that it might be helpful to the reader in following the proof to consider the subspace of the xy-plane consisting of the coordinate axes together with all vertical lines $x = 1/n, n$ a positive integer, and to refer to the figure on the next page.

Proof of Theorem 1.18. Let $0^*$ be an open subset of M such that $0^*$ contains a quasicomponent $Q^*$ which is not open, and let $p \in Q^* - \text{Int } Q^*$. Since M is locally compact and regular, there is an open set K such that $p \in K$, $\overline{K} \subseteq 0^*$ and $\overline{K}$ is compact.

Let $\mathcal{A} = \{\langle V, C, (S, T) \rangle : V$ is an open subset of K and $p \in V; C$ is a component of $\overline{K}$ such that $C \cap Q^* = \emptyset$ and $C \cap V \neq \emptyset; (S, T)$ is a separation of $\overline{K}$ such that $p \in S$ and $C \subset T\}$. Since K is open and $p \in Q^* - \text{Int } Q^*$, K meets a
Figure 1. $M = \{(x, y): x = 0 \text{ or } y = 0 \text{ or } x = 1/n, n = 1, 2, \ldots\}$
component \( C \) of \( \mathbb{K} \) such that \( C \cap Q^* = \emptyset \). Since \( \mathbb{K} \) is compact and regular, \( C \) is a quasicomponent of \( \mathbb{K} \), so there is a separation \( (S, T) \) of \( \mathbb{K} \) such that \( p \in S, C \subset T \). Then
\[
\langle k, C, (S, T) \rangle \in \mathcal{A}.
\]
Thus \( \mathcal{A} \neq \emptyset \). Define a relation \( > \) on \( \mathcal{A} \) by \( \langle V_1, C_1, (S_1, T_1) \rangle > \langle V_2, C_2, (S_2, T_2) \rangle \) iff \( V_1 \subset V_2 \), \( C_2 \cap V_1 = \emptyset \) and \( S_1 \subset S_2 \). We omit the straightforward argument that \( > \) is nonempty and that \( (\mathcal{A}, >) \) is a directed set.

For each \( \alpha = \langle V, C, (S, T) \rangle \in \mathcal{A} \), let \( N(\alpha) = C \). Then \( N \) is a net of components of \( \mathbb{K} \) each of which meets \( \mathbb{K} \) and \( \partial(\mathbb{K}) \). If \( W \) is an open set containing \( p \), then \( W \cap \mathbb{K} \) is an open set contained in \( \mathbb{K} \) and there is a component \( C \) of \( \mathbb{K} \) such that \( C \cap Q^* = \emptyset \) and \( C \cap W \cap \mathbb{K} \neq \emptyset \). There is a separation \( (S, T) \) of \( \mathbb{K} \) such that \( p \in S, C \subset T \). Let \( \alpha^* = \langle W \cap \mathbb{K}, C, (S, T) \rangle \).

If \( \alpha = \langle V_1, C_1, (S_1, T_1) \rangle > \alpha^* \), then \( V_1 \subset W \cap \mathbb{K} \) and \( C_1 \cap V_1 \neq \emptyset \), so \( C_1 \cap W \cap \mathbb{K} \neq \emptyset \); thus \( W \) meets \( N(\alpha) \) for all \( \alpha > \alpha^* \), so \( p \in \lim \inf N \).

For each \( \alpha \in \mathcal{A} \), let \( M(\alpha) \) be a point of \( N(\alpha) \cap \partial(\mathbb{K}) \). Then \( M \) is a net in the compact set \( \partial(\mathbb{K}) \), so there is a point \( q \) in \( \partial(\mathbb{K}) \) and a convergent subnet \( M \circ R, R: (\mathcal{B}, >) \to (\mathcal{A}, >) \), such that \( q \) is a limit of \( M \circ R \). Then \( M \circ R \) is a subnet of \( N \) and \( p, q \in \lim \inf N \circ R \).

Suppose \( (U, V) \) is a separation of \( O^* \) such that \( p \in U, q \in V \). There is a \( \beta^* \in \mathcal{B} \) such that if \( \beta \in \mathcal{B} \), \( \beta > \beta^* \), then \( U \cap N \circ R(\beta) \neq \emptyset \) and \( V \cap N \circ R(\beta) \neq \emptyset \). But for all \( \beta \in \mathcal{B} \), \( N \circ R(\beta) \) is connected and contained in \( U \cup V \). It follows that \( q \in Q^* \).
We now define the net \( \{G_\alpha : \alpha \in (\mathcal{A}, >)\} \). Let \( O \) be an open set such that \( p \in O, \overline{O} \subset K \). Then \( q \notin \overline{O} \). For each \( \alpha = \langle V, C, (S, T) \rangle \) in \( \mathcal{A} \), let \( G_\alpha = \partial(0) \cap S \). Then for each \( \alpha \in \mathcal{A} \), \( G_\alpha \) is closed and neither \( p \) nor \( q \) belongs to \( G_\alpha \). Also, if \( \alpha, \beta \in \mathcal{A}, \beta > \alpha \), then \( G_\beta \subset G_\alpha \), so the net \( \{G_\alpha : \alpha \in (\mathcal{A}, >)\} \) is a decreasing net. Since for each \( \langle V, C, (S, T) \rangle \) in \( \mathcal{A} \), \( \partial(S) \cap K = \emptyset \), \( \partial(0 \cap S) = \partial(0) \cap S = G_\alpha \); thus \( ((0 \cap S), \text{Ext}(0 \cap S)) \) is a separation of \( M - G_\alpha \) and \( p \in 0 \cap S, q \in \text{Ext}(0 \cap S) \).

For each \( x \in K \), let \( C_x \) be the component of \( \overline{K} \) such that \( x \in C_x \). If \( x \notin Q^* \), then \( C_x \cap Q^* = \emptyset \) and there is a separation \( (S, T) \) of \( \overline{K} \) such that \( p \in S, C_x \subset T \). Then \( \alpha = \langle K, C_x, (S, T) \rangle \in \mathcal{A} \) and since \( G_\alpha = \partial(0) \cap S \), \( G_\alpha \cap C_x = \emptyset \) so \( x \notin G_\alpha \). Thus \( \bigcup_{\alpha \in \mathcal{A}} G_\alpha \in Q^* \).

Suppose now that \( (U, V) \) is a separation of \( M - \bigcup_{\alpha \in \mathcal{A}} G_\alpha \) and \( p \in U, q \in V \). Since \( p, q \in \liminf N_oR \), there is a \( \beta \in \mathcal{B} \) such that \( N_oR(\beta) \cap U \downarrow \emptyset \) and \( N_oR(\beta) \cap V \uparrow \emptyset \). But \( N_oR(\beta) \) is a component of \( \overline{K} \) which does not meet \( Q^* \) so is contained in \( M - \bigcup_{\alpha \in \mathcal{A}} G_\alpha \). This is a contradiction and it follows that \( \bigcup_{\alpha \in \mathcal{A}} G_\alpha \) does not separate \( p \) and \( q \) in \( M \).

1.19 Corollary. If \( M \) is connected, regular and locally compact, then \( M \) is locally connected iff \( M \) satisfies the following condition:

\[ e' \text{. If } p, q \in M \text{ and } \{G_\alpha : \alpha \in (\mathcal{A}, >)\} \text{ is a decreasing net of closed sets each of which separates} \]
p and q in M, then $\bigcap_{\alpha \in \mathcal{A}} C_\alpha$ separates p and q in M.

1.20 Theorem. If M is a locally compact Hausdorff space, then a and g are equivalent.

Proof that a implies g. Suppose that $p \in B$ and $p$ is a limit point of $\bigcup \mathcal{C}$. Let $0$ be a connected open set such that $p \in 0$, $\overline{0} \subset M - \overline{C}$ and $\overline{0}$ is compact. Then $\overline{0}$ is a continuum and there is a point $y$ in $\bigcup \mathcal{C}$ such that $y \notin 0$.

Let $C_y \in \mathcal{C}$ such that $y \in C_y$. Then $\partial(C_y) \cap B = \emptyset$. Let $N$ be a continuum in $\overline{0}$ such that $N$ is an irreducible continuum from $y$ to $\overline{0} \cap B$. Then $N - (\overline{0} \cap B) = N - B$ is connected and is contained in $\overline{0} - B$, which is a subset of $M - (B \cup C)$.

Since $y \in N - B$, $N - B \subset C_y$. But every point of $N \cap B$ is a limit point of $N - B$ and thus every point of $N \cap B$ is a limit point of $C_y$. This is a contradiction.

Proof that g implies a. We show that every $T_1$-space that is not locally connected does not satisfy condition g.

If a $T_1$-space $M$ is not locally connected, there is an open set $0$ which contains a point $p$ that is not interior to a component of $0$. Let $B = \{p\}$ and $C = M - 0$. Let $\mathcal{C} = \{\text{components of }0 \text{ which do not contain } p\}$. For each $D$ in $\mathcal{C}$, $D$ is closed in $0$ and does not contain $p$, so $\partial(D) \cap B = \emptyset$.

Also, $M - (B \cup C) \subset 0$, so for each $D$ in $\mathcal{C}$, $D$ is a component of $M - (B \cup C)$. Further, $p$ is a limit point of $\bigcup \mathcal{C}$.

Remark. It follows from the above proof, that in the statement of g we can take B or C to be a singleton.
The next theorem is known and follows from results established by E. Michael in [7].

**Theorem.** Every net of closed, connected sets in a compact Hausdorff space has a nonempty, connected limit superior and a convergent subnet which has a connected limit.

1.21 **Theorem.** If M is a connected, locally compact Hausdorff space that is not locally connected, then M satisfies h.

**Proof.** Since M is locally compact and not locally connected, there is an open set G in M such that G is compact and such that some component C* of G is not open. Let \( p \in C^* \setminus \text{Int } C^* \). If 0 is any open set containing \( p \), then there is an open set K such that \( p \in K \) and \( \overline{K} \subset 0 \cap G \). Then \( \overline{K} \) is compact and if \( D^* \) is a component of \( \overline{K} \) such that \( p \in D^* \), then \( p \notin \text{Int } D^* \). Let \((A, \succ)\) be the neighborhoods of \( p \) contained in \( K \) and directed by inclusion; i.e. \( V_1 \succ V_2 \) iff \( V_1 \subset V_2 \). For each \( V \in A \), let \( D_V \) be a component of \( \overline{K} \) such that \( D_V \neq D^* \) and \( D_V \cap V \neq \emptyset \). Let \( x_V \in D_V \cap V \) and let \( C_V \) be the component of \( K \) such that \( x_V \in C_V \). Define \( N(V) = C_V \). Then \( N \) is a net of subcontinua of \( \overline{K} \), so there is a convergent subnet \( N \circ R, R : (\mathcal{O}, \succ) \to (A, \succ) \) and \( \lim N \circ R = D \) is a continuum. Further, \( p \in \liminf N \) so \( p \in D \). Also, since \( A \) for each \( V \in A \), \( N(V) \cap \partial(K) \neq \emptyset \), \( D \cap \partial(K) \neq \emptyset \). It follows that \( D \) is nondegenerate. Since \( D \) is connected, contains \( p \) and is contained in \( \overline{K}, D \subset D^* \). It remains to show that the net \( N \circ R \) is almost pairwise disjoint. Let
\( \beta^* \in (\mathcal{B}, \succ) \). Then \( R(\beta^*) \in (\mathcal{A}, \succ) \), so \( R(\beta^*) \) is open and is contained in \( K \). Now \( N \circ R(\beta^*) \subseteq D_{R}(\beta^*) \) and \( p \not\in D_{R}(\beta^*) \). Since \( D_{R}(\beta^*) \) is closed, \( R(\beta^*) - D_{R}(\beta^*) \in \mathcal{A} \). There is a \( \beta_1 \in \mathcal{B} \) such that if \( \beta \succ \beta_1 \), then \( R(\beta) \subseteq R(\beta^*) - D_{R}(\beta^*) \) and since \( N \circ R(\beta) \subseteq D_{R}(\beta) \) and \( D_{R}(\beta) \) meets \( R(\beta) \), \( D_{R}(\beta) \cap D_{R}(\beta^*) = \emptyset \). It follows that the net \( N \circ K \) is almost pairwise disjoint.
CHAPTER II:

CYCLIC ELEMENT THEORY IN CONNECTED AND LOCALLY
CONNECTED HAUSDORFF SPACES

2.1 Introduction. We begin this chapter by stating several definitions, many of which are to be found in [12] and all of which are generalizations of the corresponding definitions for metric spaces found in Analytic Topology by G.T. Whyburn, [11].

2.1.1 Definitions. Let $M$ be a connected topological space. A point $p$ of $M$ is a cut point of $M$ iff $M - p$ is not connected. A point $p$ of $M$ is an end point of $M$ iff for every open set $O$ containing $p$, there is an open set $V$ containing $p$ such that $V \subseteq O$ and $\partial(V)$ is a singleton. Two points $a$ and $b$ of $M$ are said to be conjugate in $M$ ("a is conjugate to b in $M$") iff no point of $M$ separates $a$ and $b$ in $M$. If $p \in M$, we define $L_p = \{x \in M: x \text{ is conjugate to } p\}$. For $a, b \in M$, $E(a,b)$ denotes the collection of all points of $M$ which separate $a$ and $b$ in $M$. It follows that $a$ and $b$ are conjugate iff $E(a,b) = \emptyset$. There is a natural (linear) order "<" on $E(a,b) \cup \{a, b\}$ defined by $a < x, x < b$ for all $x \in E(a,b)$; $a < b$, and if $x, y \in E(a,b)$ then $x < y$ iff $x \in E(a,y)$. The order $<$ on $E(a,b) \cup \{a, b\}$ is called the cut point order on $E(a,b) \cup \{a, b\}$. A subset $E$ of $M$ is an $E_0$-set of $M$ iff $E$ is nondegenerate, connected, has no cut point of itself, and is
 maximal with respect to these properties. A **cyclic element** of \( M \) is a subset of \( M \) which either consists of a single cut point or end point of \( M \) or is an \( E_0 \)-set of \( M \). An **A-set** of \( M \) is a closed subset of \( M \) such that \( M - A \) is the union of a collection of open sets each bounded by a single point of \( A \). If \( a, b \) are points of \( M \), \( C(a, b) \) denotes the intersection of all A-sets of \( M \) which contain both \( a \) and \( b \), and the set \( C(a, b) \) is called the **cyclic chain in \( M \) from \( a \) to \( b \)**.

**NOTE:** For the rest of this chapter, unless stated otherwise, "\( M \)" denotes a connected and locally connected Hausdorff space.

2.1.2 The following results have been established in [12].

a. If \( a, b \in M \), then \( E(a, b) \cup \{a, b\} \) is closed and compact.

b. If \( A \) is a collection of A-sets of \( M \), then \( \bigcap A = \emptyset \) or is an A-set of \( M \).

c. A nonempty closed set \( A \) is an A-set of \( M \) iff each component of \( M - A \) has exactly one boundary point.

d. If \( A \) is an A-set of \( M \), then if \( Z \) is any connected subset of \( M \), then \( A \cap Z \) is connected (possibly empty); thus every A-set of \( M \) is connected and locally connected.

e. If \( a \) and \( b \) are distinct conjugate points of \( M \), then \( C(a, b) = \{ p \in M : p \text{ is conjugate to both } a \text{ and } b \} \), and in this case, \( C(a, b) \) is an \( E_0 \)-set of \( M \). Further, if \( C \) is an \( E_0 \)-set of \( M \) and \( a, b \) are distinct points of \( C \), then \( a \) and \( b \) are conjugate in \( M \) and \( C = C(a, b) \).

f. Any two \( E_0 \)-sets of \( M \) have at most one common point.

g. For any two points \( a, b \) of \( M \), \( C(a, b) = E(a, b) \cup \{a, b\} \cup C \), where \( C \) is the union of all \( E_0 \)-sets of \( M \) which meet \( E(a, b) \cup \{a, b\} \) in exactly two points.
2.1.3 Theorem. If \( a, b \in M \), then the subspace topology on \( E(a,b) \cup \{a,b\} \) is the order topology relative to the cut point order.

**Proof.** If \( E(a,b) = \emptyset \), then \( E(a,b) \cup \{a,b\} \) is discrete with either topology. Assume, then, that \( E(a,b) \neq \emptyset \). It is well known that in general the order topology on \( E(a,b) \cup \{a,b\} \) is weaker than the subspace topology. (See for instance, Willard, [14, p. 206]). Suppose, then, that \( U^* \) is a nonempty relative open set in \( E(a,b) \cup \{a,b\} \) and \( x \in U^* \). Let \( U \) be a connected open subset of \( M \) such that \( x \in U \) and \( U \cap (E(a,b) \cup \{a,b\}) \subseteq U^* \). We consider cases.

Case 1. \( x \in \{a,b\} \). Suppose \( x = a \). By 2.1.2-d, \( C(a,b) \) is connected, so \( U \cap C(a,b) \neq \{a\} \). Let \( t \in U \cap C(a,b) \), \( t \neq a \). If \( t \in E(a,b) \cup \{a,b\} \), then \([a,t) \subseteq U \), so \( x \in [a,t) \subseteq U^* \). If \( t \notin E(a,b) \cup \{a,b\} \), then by 2.1.2-g, \( t \) belongs to an \( E_0 \)-set \( E \) of \( M \) such that \( E \) meets \( E(a,b) \cup \{a,b\} \) in exactly two points \( w,z \). We may assume that \( a \leq w < z \). Then \([a,z) \subseteq U \), so \( x \in [a,z) \subseteq U^* \). The case for \( x = b \) is similar.

Case 2. \( x \in E(a,b) \). By 2.1.2-g and 2.1.2-d, \( x \in C(a,b) \) and \( C(a,b) \) is a connected and locally connected Hausdorff space. In \( C(a,b) - x \), let \( C_a \) and \( C_b \) be the components containing \( a \) and \( b \) respectively. Then \( x \) is a limit point of \( C_a \) and \( C_b \). Let \( t_a \) and \( t_b \) be points respectively in \( C_a \cap U \) and \( C_b \cap U \). Again we consider cases. If \( t_a, t_b \in E(a,b) \cup \{a,b\} \), then \( t_a < x < t_b \) and \( x \in (t_a, t_b) \subseteq U^* \). If \( t_a \notin E(a,b) \cup \{a,b\} \) and \( t_b \in E(a,b) \cup \{a,b\} \), then by 2.1.2-g, there is an
$E_0$-set $E$ of $M$ such that $t_a \in E$ and $E$ meets $E(a,b) \cup \{a,b\}$ in exactly two points $w$ and $z$. We may assume that $w < z$. Then $a \leq w < z \leq x < t_b$ and $(w,t_b) \subset U$, so $x \in (w,t_b) \subset U^*$. The other cases are similar and the theorem is proved.

2.1.4 Examples. We shall have occasion to refer to the following two examples. Each example is of a nonmetric locally connected Hausdorff continuum. We suggest figures 2 and 3 as representations respectively of the spaces $S^*$ of Example A and $M^*$ of example B.

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**Figure 2.** The space $S^*$

**Figure 3.** The space $M^*$
Example A. The space $S^*$. The point set of $S^*$ consists of all points of the plane with polar coordinates $(r, \theta)$ such that $1 < r < 2$ together with all points of the straight line interval $\{(r, 0) : -1 < r < 1\}$. We define a topology for $S^*$ by defining a base for the neighborhood system at each point $p = (r, \theta)$. 

a. If $r > 1$, then for each $\varepsilon > 0$, we let $N_\varepsilon(p) = \{(r, \theta) : r - \varepsilon < r < r + \varepsilon \text{ and } 1 \leq r \leq 2\}$. 

A base for the neighborhood system at $p$ consists of all sets $N_\varepsilon(p)$ such that $0 < \varepsilon < r - 1$.

b. If $\theta = 0$ and $-1 < r < 1$, then for each $\varepsilon > 0$, let $N_\varepsilon(p) = \{(r, 0) : r - \varepsilon < r < r + \varepsilon \text{ and } -1 < r < 1\}$. 

A base for the neighborhood system at $p$ consists of all sets $N_\varepsilon(p)$.

c. If $r = 1$ we define a base for the neighborhood system at $p$ as follows. For each $q^* = (r^*, \theta^*)$ and for each $\varepsilon > 0$, let $F_\varepsilon(q^*)$ be the closed radial segment through $q^*$ of length $2\varepsilon$. ($F_\varepsilon(q^*) = \{(r, \theta) : r^* - \varepsilon \leq r \leq r^* + \varepsilon\}$), and let $F_0(q^*) = \{q^*\}$. For each $\varepsilon > 0$, let $N_\varepsilon(p) = \{(r, \theta) \in S^* : \theta - \varepsilon < \theta < \theta + \varepsilon\}$. 

A base for the neighborhood system at $p$ consists of all sets of the form $N_\varepsilon(p) - \bigcup_{i=1}^{n} F_{\varepsilon_i}(q_i)$ for some finite collection $\{F_{\varepsilon_i}(q_i) : \varepsilon_i > 0, i = 1, \ldots, n\}$, such that $p \notin F_{\varepsilon_i}(q_i)$, $i = 1, \ldots, n$.

When discussing the space $S^*$ we will let $F^* = \{(r, \theta) \in S^* : -1 < r \leq 1\}$. 

We note that $F^*$ is an $E_o^*$-set of $S^*$.

Example B. The space $M^*$. $M^*$ is the subspace of the space $S^*$ of Example A consisting of all points $(r, \theta)$ of $S^*$ such that $1 \leq r \leq 2$. When discussing the space $M^*$, we will let $E^* = \{(r, \theta) \in M^* : r = 1\}$. Then $E^*$ is an $E_o^*$-set of $M^*$.

2.2 $E_o^*$-sets and the conjugacy relation. We have already remarked that if $E$ is an $E_o^*$-set of $M$ and $p$ and $q$ are distinct points of $E$, then $E = \{x : x$ is conjugate in $M$ to both $p$ and $q\}$. In the classical cyclic element theory for metric spaces as set forth by Whyburn in [11], every $E_o^*$-set $E$ of a semi-locally connected metric continuum $M$ contains a point $p$ such that $p$ is neither a cut point of $M$ nor an end point of $M$ and $E = L_p = \{x : x$ is conjugate to $p$ in $M\}$. That this is not true in general is shown by Example B; for in $M^*$, the set $E^*$ is an $E_o^*$-set of $M^*$ and every point of $M^*$ is either a cut point or an end point of $M^*$. However, for a connected and locally connected Hausdorff space $M$, if an $E_o^*$-set $E$ of $M$ contains a point $p$ which is neither a cut point nor an end point of $M$, then $E = L_p$. In this section we consider this and other properties of $E_o^*$-sets.

2.2.1 Lemma. If $x$ is an end point of $M$, $S$ a nondegenerate connected subset of $M$, and $x \in S$, then $x$ is an end point of $S$.

Proof. Let $S \subset M$ such that $x \in S$ and $S$ is connected and
nondegenerate. Let $O^*$ be any set such that $O^*$ is open in $S$ and $x \in O^*$. We may assume that $O^* \supset S$. Let $O$ be open in $M$ such that $O^* = O \cap S$. There is a set $V$, open in $M$, such that $x \in V$, $V \subset O$, and $\partial(V) = \{t\}$ for some $t \in M$. Then $V^* = V \cap S$ is open in $S$, $x \in V^*$, and $V^* \subset O^*$. Further, since $S$ is connected, the boundary in $S$ of $V^*$ is not empty. Since the boundary in $S$ of $V^*$ is a subset of $\partial(V)$, the boundary in $S$ of $V$ is $\{t\}$.

2.2.2 Lemma. If no point of $M$ is a cut point of $M$, then no point of $M$ is an end point of $M$.

Proof. Suppose that no point of $M$ is a cut point of $M$ and that $p$ is an end point of $M$. Then there is an open set $U$ of $M$ and a point $t$ of $M$ such that $p \in U$ and $\partial(U) = \{t\}$. Then $t$ is a cut point of $M$, contrary to hypothesis. Thus no point of $M$ is an end point of $M$.

2.2.3 Lemma. No $E_0^-$-set of $M$ contains an end point of $M$.

Proof. Let $E$ be an $E_0^-$-set of $M$. Then $E$ is nondegenerate, connected, and contains no cut point of itself. Thus by 2.2.2, no point of $E$ is an end point of $E$, so by 2.2.1, $E$ contains no end point of $M$.

2.2.4 Lemma. If an $E_0^-$-set $E$ of $M$ contains a non-cut point $p$ of $M$, then $E = L_p$.

Proof. Let $E$ be an $E_0^-$-set of $M$ and $p \in E$ such that $p$ is a non-cut point of $M$. It follows from 2.1.2-e, page 22,
that $E \subset L_p$. Let $a$ be a point of $E$ distinct from $p$ and let $x \in L_p$. If $t \in M$ and $(U, V)$ is a separation of $M - t$ such that $a \in U$, then $p \in U$. Since $p$ and $x$ are conjugate in $M$, $x \in U$. Thus $x$ is conjugate in $M$ to both $p$ and $a$, so by 2.1.2-e, $x \in E$. The lemma follows.

The next two results are to be found in [8].

2.2.5 Lemma. If $E_1$ and $E_2$ are distinct $E_0$-sets of $M$ and intersect, their intersection is a cut point of $M$ and $E_1 \cap E_2$ separates $E_1 - E_2$ and $E_2 - E_1$ in $M$.

2.2.6 Lemma. If $a, b \in M$ and $E$ is an $E_0$-set of $M$, then $E$ meets $E(a, b) \cup \{a, b\}$ in at most two points.

2.2.7 Lemma. If $Z$ is a connected subset of $M$ and $p$ and $q$ are conjugate in $Z$, then $p$ and $q$ are conjugate in $M$.

Proof. Let $E_Z(p, q) = \{t \in Z : t$ separates $p$ and $q$ in $Z\}$. Then $E(p, q) \subset E_Z(p, q)$, so if $E_Z(p, q) = \emptyset$, then $E(p, q) = \emptyset$.

2.2.8 Lemma. If $A$ is an $A$-set of $M$ and $C$ is a component of $M - A$, then $\overline{C}$ is an $A$-set of $M$.

Proof. If $D$ is a component of $M - \overline{C}$, then $\partial(D) = \partial(C)$.

2.2.9 Theorem. Of the following two statements, if a connected subset $A$ of $M$ satisfies $a$, then $A$ satisfies $b$:

a. If $E$ is a cyclic element of $M$ and $A \cap E$ is non-degenerate, then $E \subset A$.

b. If $x, y \in A$ and $N \subset M$ is an irreducible continuum
from \(x\) to \(y\), then \(N \subseteq A\).

**Proof.** Suppose \(A\) is connected and satisfies \(a\). Suppose further that \(x,y \in A\), \(N\) is an irreducible continuum from \(x\) to \(y\), and \(t \in N - A\). Since \(x,y \in A\) and \(A\) is connected, \(E(x,y) \cup \{x,y\} \subseteq A\). Thus if an \(E_o\)-set \(E\) meets \(E(x,y) \cup \{x,y\}\) in two points, then \(E \cap A\) is nondegenerate so by assumption \(E \subseteq A\). It follows that \(C(x,y) \subseteq A\). In \(M - C(x,y)\), let \(C_t\) be the component which contains \(t\), and let \(z = \partial(C_t)\). Then \(z \in N\). If \(z \notin \{x,y\}\), then \(x,y\) lie in components \(C_x, C_y\), respectively, of \(M - (C_t \cup z)\). But then \(C_x \cup z\) and \(C_y \cup z\) are \(A\)-sets, so \(x\) and \(y\) belong to \((N \cap (C_x \cup z)) \cup (N \cap (C_y \cup z))\), which is a proper subcontinuum of \(N\). Thus \(z \in \{x,y\}\) and we may assume \(z = x\).

But now in \(M - (C_t \cup z)\) if \(D_y\) is the component containing \(y\), then \(N \cap (D_y \cup z)\) is a proper subcontinuum of \(N\) and contains \(x\) and \(y\). It follows that \(N \subseteq A\).

**2.2.10 Corollary.** If \(E\) is an \(E_o\)-set of \(M\) and \(a,b \in E\), then \(E\) contains every continuum \(N \subseteq M\) such that \(N\) is an irreducible continuum from \(a\) to \(b\).

**2.2.11 Corollary.** If \(a\) and \(b\) are conjugate in \(M\) and \(N \subseteq M\) is an irreducible continuum from \(a\) to \(b\), then every point of \(M\) is conjugate to both \(a\) and \(b\) in \(M\).

**2.2.12 Lemma.** If \(E_1, E_2\) are distinct \(E_o\)-sets and \(N\) is a connected set which meets \(E_1\) and \(E_2\), then \(E_1 \cap E_2 = N\).
Proof. If $E_1 \cap E_2 = \emptyset$, then $E_1 \cap E_2 \subset N$. Suppose, then, that $E_1 \cap E_2 \neq \emptyset$. Whyburn proved in [12] that if an A-set $A$ meets each of two intersecting connected sets $S$ and $T$, then $A \cap S \cap T \neq \emptyset$. It follows, then, that $E_1 \cap E_2 \subset N \neq \emptyset$.

2.2.13 Lemma. If $A$ is an A-set of $M$ and $R$ is a component of $M - A$, then $R$ meets at most one $E_o$-set $E$ which meets $A$.

Proof. Suppose $E_1$, $E_2$ are distinct $E_o$-sets which meet both $R$ and $A$. Let $b = \mathfrak{A}(R)$. Then $b \in E_1 \cap E_2$. But by the above lemma, $E_1 \cap E_2 \subset R$. This is a contradiction since $b \in A$.

2.2.14 Lemma. Let $E$ be an $E_o$-set of $M$ and $C$ a component of $M - E$. If $b \in M$ such that either $b \in E - \overline{C}$ or $b \notin E$ and $\mathfrak{A}(C_b) \neq \mathfrak{A}(C)$, where $C_b$ is the component of $M - E$ containing $b$, then $E \subset C(a,b)$ for all $a$ in $C$.

Proof. If $b \in E - \overline{C}$ and $a \in C$, let $t = \mathfrak{A}(C)$. Then $t \neq b$ and $t \in E(a,b)$. Thus by 2.2.6, $E \cap (E(a,b) \cup \{a,b\}) = \{t,b\}$ and $E \subset C(a,b)$. Suppose, then, that $b \notin E$ and $\mathfrak{A}(C_b) \neq \mathfrak{A}(C)$. Let $z = \mathfrak{A}(C_b)$, $t = \mathfrak{A}(C)$, and $a \in C$. Then $C$ is a component of $M - t$ and is disjoint from $(E - t) \cup C_b$, which is connected. Thus $t \in E(a,b)$. Similarly, $z \in E(a,b)$. Thus $E \cap (E(a,b) \cup \{a,b\}) = \{t,z\}$, so $E \subset C(a,b)$.

2.2.15 Theorem. If $M$ is locally compact and $p \in M$ such that $p$ is neither a cut point nor an end point of $M$, then there
is a point q in M such that q \neq p is conjugate to p in M.

**Proof.** We show that if p is not a cut point and is not conjugate to any other point of M, then p is an end point of M. Let O be any open set such that p \in O and O \neq M. Let V be a connected open set such that p \in V, \overline{V} is compact, and V \subset O. For each x \in \partial(V), let G_x be a connected open set containing x such that p \notin \overline{G}_x. \{G_x : x \in \partial(V)\} covers the compact set \partial(V), so there is a finite subcover G_{x_1}, ..., G_{x_n}. Since M - p is connected and locally compact, for each i = 1, ..., n-1, there is a continuum N_i in M - p such that x_i, x_{i+1} \in N_i. Let N = (M - V) \cup \bigcup_{i=1}^{n-1} N_i \cup \bigcup_{i=1}^{n} \overline{G}_{x_i}. Then N is closed and p \notin N. Since M - V \subset N, M - N \subset V and p \in M - N. Let C be the component of M - N such that p \in C. Now C has a boundary point q in N. By assumption, p and q are not conjugate in M, so there is a point x of M and a separation (U, W) of M - x such that p \in U, q \in W. Since C \cup q is connected and contains both p and q, x \in C; thus x \notin N. Since N is connected and q \in N, N \subset W. Thus U \subset M - N \subset V \subset O and \partial(U) = \{x\}. It follows that p is an end point of M.

The next theorem follows immediately from 2.2.15 and 2.2.5.

**2.2.16 Theorem.** If M is locally compact, then every point p of M belongs to a cyclic element of M, and if p is neither a cut point nor an end point of M, then p belongs to a unique
cyclic element of \( M \) that is an \( E_0 \)-set of \( M \).

We shall have several occasions to refer to the next example.

2.2.17 Example. Let \( T^* \) be the subspace of the plane consisting of all points with rectangular coordinates \((x,y)\) such that either \( x \geq 0 \) and \( y = 0 \) or \( x = 1/n \) for \( n \) a positive integer. Then \( T^* \) is a connected, locally connected metric space and is not locally compact. The point \( p = (0,0) \) is neither a cut point nor an end point of \( T^* \) and is not conjugate to any other point. Thus Theorem 2.2.15 is not true in general if \( M \) is not locally compact.

2.3 A-sets and H-sets. In [11], Whyburn defined an H-set in a metric semi-locally connected continuum \( M \) to be a connected subset of \( M \) which satisfies the following condition:

* If \( p \in H \), then there is a cyclic element \( E \) of \( M \) such that \( p \in E \) and \( E \subseteq H \).

H-sets were shown to have many of the properties of A-sets, and the closure of an H-set was shown to be an A-set. However, in the nonmetric setting, it may be that for some connected set \( H \), every point is contained in a cyclic element \( E \) of \( M \) such that \( E \subseteq H \), \( \overline{H} \) is not an A-set and \( H \) fails to have several of the properties that H-sets were shown in [11] to possess. For example, in the space \( M^* \) of 2.1.3, \( H = \{(r,\theta): 1 \leq r \leq 2, 0 \leq \theta \leq \pi/2\} \) is closed and connec-
ted, every singleton in H is a cyclic element of M*, but H is not an A-set.

2.3.1 Definition. A connected subset H of M is an H-set of M iff H satisfies one of the following conditions:

a. \( H = \{p\} \) for \( p \) a cut point or an end point of M.

b. \( H \) is nondegenerate and if \( a, b \in H \), then \( C(a, b) \subseteq H \).

Remark. Since for any two points \( a \) and \( b \) of an A-set \( A \) of M, \( C(a, b) \) is the intersection of all A-sets of M which contain \( a \) and \( b \), \( C(a, b) \subseteq A \). Thus every nondegenerate A-set is an H-set, as is any A-set which consists of a single cut point or end point of M. It follows that every cyclic element of M is an H-set of M.

We now show that for semi-locally connected metric continua, definition 2.3.1 is equivalent to that given in [11].

2.3.2 Theorem. If M is a metric semi-locally connected continuum, then a connected subset \( H \) of M satisfies a or b of definition 2.3.1 iff \( H \) satisfies the condition *.

Proof. Suppose \( H \) satisfies a or b of 2.3.1 and \( p \in H \).

If \( H = \{p\} \), then \( p \) is a cut point or an end point of M, so \( E = \{p\} \) is a cyclic element of M containing \( p \) and contained in \( H \). If \( H \) is nondegenerate and \( p \) is a cut point or end point of M, then again \( E = \{p\} \) is a cyclic element of M containing \( p \) and contained in \( H \). If \( p \) is neither a cut
point nor an end point of \( M \), then let \( q \in H, q \neq p \). Since \( H \) satisfies \( b \), \( C(p,q) \subset H \), and it follows from Theorem 3.3, page 67, of [11] that the cyclic element \( E \) which contains \( p \) is contained in \( C(p,q) \subset H \). Thus \( H \) satisfies the condition *.

Now suppose \( H \) satisfies the condition *. If \( H = \{ p \} \), then some cyclic element \( E \) contains \( p \) and is contained in \( \{ p \} \). This means that \( E \) is degenerate, so \( p \) is a cut point or an end point of \( M \). If \( H \) is nondegenerate and \( a,b \in H \), then by 6.3, page 72 of [11], \( C(a,b) \subset H \). Thus \( H \) satisfies a or b of definition 2.3.1.

2.3.3 Theorem. If \( H \) is an \( H \)-set of \( M \) and \( E \) is an \( E_o \)-set of \( M \) such that \( H \cap E \) is nondegenerate or contains a non-cut point of \( M \), then \( E \subset H \) and is an \( E_o \)-set of \( H \).

Proof. If \( H \cap E \) is nondegenerate, let \( s,t \) be distinct points of \( H \cap E \). Then \( E = C(s,t) \subset H \). Suppose now that \( H \cap E \) contains a point \( p \) that is a non-cut point of \( M \). Then by 2.2.3, \( p \) is not an end point of \( M \), so \( H \) is nondegenerate. Let \( x \in H \) such that \( x \neq p \). If \( x \in E \), then \( E = C(p,x) \subset H \). If \( x \notin E \), then there is a point \( t \in E \) such that \( t \in E(p,x) \), and \( E \cap (E(p,x) \cup \{ p,x \}) = \{ p,t \} \). Thus \( E \subset C(p,x) \subset H \). In either case, \( E \subset H \). Further, since \( E \) is maximal in \( M \) with respect to the properties of being nondegenerate, connected, and having no cut point of itself, \( M \) is maximal in \( H \) with respect to these properties;
thus $E$ is an $E_0$-set of $H$.

2.3.4 **Corollary.** Every $E_0$-set of an $H$-set of $M$ is an $E_0$-set of $M$.

2.3.5 **Corollary.** A nondegenerate, connected subset $H$ of $M$ is an $H$-set of $M$ iff whenever $E$ is an $E_0$-set of $M$ such that $H \cap E$ is nondegenerate, then $E \subset H$.

**Proof.** Necessity is immediate from 2.3.3. Suppose then that if $E \cap H$ is nondegenerate for an $E_0$-set $E$ of $M$, then $E \subset H$. If $a,b \in H$, then since $H$ is connected, $E(a,b) \cup \{a,b\} \subset H$. Thus if an $E_0$-set $E$ meets $E(a,b) \cup \{a,b\}$ in two points, $H \cap E$ is nondegenerate, so by assumption, $E \subset H$. It follows from 2.1.2-g that $C(a,b) \subset H$. Thus $H$ is an $H$-set of $M$.

2.3.6 **Corollary.** If $M$ is locally compact, $H$ an $H$-set of $M$, and $p \in H$, then there is a cyclic element $E$ of $M$ such that $p \in E$ and $E \subset H$.

2.3.7 **Corollary.** If $H$ is an $H$-set of $M$, $x,y \in H$, and $N$ is an irreducible continuum from $x$ to $y$, then $N \subset H$.

**Proof.** We may assume $x \neq y$. By 2.3.3, $H$ satisfies a of 2.2.9, so by 2.2.9, $N \subset H$.

2.3.8 **Theorem.** If $H$ is an $H$-set of $M$ and $H \subset H_0 \subset \overline{H}$, then $H_0$ is an $H$-set of $M$. Further, if $M$ is locally compact, then every point of $\overline{H} - H$ is either a cut point or an end
point of \( M \).

\textbf{Proof.} If \( H \) is degenerate, the result is immediate. Suppose then that \( H \) is nondegenerate and \( x, y \in H \). Since \( H \cup \{x, y\} \) is connected, \( E(x, y) \cup \{x, y\} \) is contained in \( H \cup \{x, y\} \). If an \( E_0 \)-set \( E \) of \( M \) meets \( E(x, y) \cup \{x, y\} \) in two points, then \( E \) must meet \( H \) in more than one point since \( E \cap (H \cup \{x, y\}) \) is connected. It follows from 2.3.3 that \( E \subset H \cup \{x, y\} \). Thus \( C(x, y) \subset H \cup \{x, y\} \subset H_0 \). Thus \( H_0 \) is an \( H \)-set of \( M \).

Suppose now that \( M \) is locally compact and \( p \in M - H \). Suppose further that \( p \) is neither a cut point nor an end point of \( M \). Then \( p \) belongs to an \( E_0 \)-set \( E \) of \( M \) such that \( E \not\subset H \). By 2.3.3, \( E \cap H \) contains at most one point. If \( E \cap H = \emptyset \), then \( H \) is contained in a component \( C \) of \( M - E \) and for some \( t \in M \), \( t = \partial(C) \). Since \( t \) is a cut point of \( M \), \( t \not\in p \). It follows that \( M - C \) is an open set containing \( p \) and missing \( H \), so \( p \not\in \overline{H} \). Now if \( E \cap H \not= \emptyset \), let \( q = E \cap H \). If \( p \in \overline{H} \), then \( H \cup p \) is connected and thus \( E \cap (H \cup p) = \{p, q\} \) is connected. This is a contradiction, so \( p \not\in \overline{H} \). Thus every point of \( \overline{H} - H \) is a cut point or an end point of \( M \).

2.3.9 \textbf{Theorem.} If \( H \) is an \( H \)-set of \( M \) and \( Z \) is a connected subset of \( M \), then \( H \cap Z \) is connected.

\textbf{Proof.} Suppose \( H \cap Z \not= \emptyset \) and \( (Z_1, Z_2) \) is a separation of \( H \cap Z \). Let \( z_i \in Z_i \), \( i = 1, 2 \). Then \( C(z_1, z_2) \subset H \), so
(C(z_1,z_2) \cap Z_1, C(z_1,z_2) \cap Z_2) is a separation of the connected set C(z_1,z_2) \cap Z. Thus H \cap Z is connected.

2.3.10 Corollary. Every H-set in a connected and locally connected Hausdorff space is a connected and locally connected Hausdorff space.

2.3.11 Corollary. If H is an H-set of M and Z is a locally connected (semi-locally connected) subset of M, then H \cap Z is locally connected (semi-locally connected).

Proof. We omit the easy proof that H \cap Z is locally connected if Z is locally connected. Suppose that Z is semi-locally connected, x \in H \cap Z, and 0 is an open set in H \cap Z such that x \in 0. Let 0^* be an open set in M such that 0 = 0^* \cap (H \cap Z). Then 0^* \cap Z is open in Z and contains x, so there is a set V, open in M, such that x \in V and Z - (Z \cap V) = Z_1 \cup \ldots \cup Z_n, Z_i a component of Z - (Z \cap V), i = 1, \ldots, n; and Z \cap V \subset Z \cap 0^*. Then x \in V \cap H \cap Z \subset 0^* \cap H \cap Z = 0, and (H \cap Z) - (V \cap H \cap Z) = H \cap (Z - (V \cap Z)) = (H \cap Z_1) \cup \ldots \cup (H \cap Z_n). By 2.3.9, H \cap Z_i is connected, i = 1, \ldots, n. Thus H \cap Z is semi-locally connected.

2.3.12 Theorem. If H is an H-set of M, then \overline{H} is an A-set of M.

Proof. Let C be a component of M - \overline{H}, and suppose p, q are distinct points of \partial(C). Then p, q \in \overline{H}, so H_0 = H \cup \{p, q\}
is an $H$-set of $M$. Since $C \cup \{p,q\}$ is connected, 
$H_0 \cap (C \cup \{p,q\}) = \{p,q\}$ is connected. Since this is a 
contradiction and $\vartheta(C) \neq \emptyset$, $\vartheta(C)$ is a singleton.

The proof of the next result is similar to that of 2.3.12.

2.3.13 Corollary. If $H$ is an $H$-set of $M$ and $C$ is a compo­
nent of $M - H$, then $\overline{C} \cap \overline{H}$ is a singleton.

2.3.14 Corollary. If $H$ is an $H$-set of $M$, $C$ a component of 
$M - H$, and $b = \overline{C} \cap \overline{H}$, then $\overline{C} = C \cup b$ and $\overline{C}$ is an $A$-set of $M$.

Proof. If $C$ is degenerate, then $C = \{b\}$ and the result 
follows. If not, then by Theorem 1.4 of Chapter I, $\text{Int } C = 
C - \overline{H} = C - b$, and it follows that $\vartheta(C) = \{b\}$. If $R$ is a 
component of $M - \overline{C} = M - (C \cup b)$, then $\emptyset \neq \vartheta(R) \subset \vartheta(C) = \{b\}$. 
Thus $\overline{C}$ is an $A$-set of $M$.

It was proved in [11] that if $\mathcal{H}$ is a family of 
$H$-sets of a semi-locally connected metric continuum $M$, and 
$\bigcup \mathcal{H}$ is connected, then $\bigcup \mathcal{H}$ is an $H$-set of $M$. The 
argument in [11] that $\bigcup \mathcal{H}$ is an $H$-set depends on the 
fact that any $E_o$-set $E$ of $M$ contains at most a countable 
number of cut points of $M$ and therefore any connected set 
which meets $E$ in more than one point contains a point $p$ of 
$E$ such that $p$ is a non-cut point of $M$. This result does 
not hold in general, as shown by the space $M^*$ of 2.1.3 
where every point of the $E_o$-set $E^*$ is a cut point of $M^*$. 
Neither is it true in general that the connected union of H-sets of M is an H-set of M. To see this, we again consider M*. For each $\theta$ in $[0, \pi/2]$, let $H_\theta = \{(r, \theta): 1 \leq r \leq 2\}$. Then $\mathcal{H} = \{H_\theta : 0 \leq \theta \leq \pi/2\}$ is a collection of H-sets of M*. $\bigcup \mathcal{H}$ is connected, but $\bigcup \mathcal{H}$ is not an H-set of M*. We have, however, the following two results.

2.3.15 Theorem. If $H_1, H_2$ are H-sets of M and $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cup H_2$ is an H-set of M.

Proof. Since $H_1 \cap H_2 \neq \emptyset$, $H_1 \cup H_2$ is connected. If $H_1 \cup H_2$ is degenerate, then $H_1 \cup H_2 = H_1 = H_2$ so is an H-set of M. Suppose, then, that $H_1 \cup H_2$ is nondegenerate and $E$ is an $E_\sigma$-set of M such that $E \cap (H_1 \cup H_2)$ is nondegenerate. Since $E \cap (H_1 \cup H_2)$ is connected, either $E \cap H_1$ or $E \cap H_2$ is nondegenerate, so $E \subset H_1$ or $E \subset H_2$. The theorem now follows from Lemma 2.3.6.

2.3.16 Corollary. The union of two intersecting A-sets of M is an A-set of M.

2.3.17 Theorem. If $\mathcal{H}$ is a family of H-sets of M such that for every two members $A, B$ of $\mathcal{H}$, there is a finite collection $A = H_0, H_1, \ldots, H_n = B$ such that $H_i \cap H_{i+1} \neq \emptyset$, $i = 0, \ldots, n-1$, then $\bigcup \mathcal{H}$ is an H-set of M.

Proof. It is well known that under the hypotheses of the theorem $\bigcup \mathcal{H}$ is connected, (see, for instance, Kelley, [3, p. 60]). If $\bigcup \mathcal{H}$ is degenerate, the result is immediate,
so assume that $\bigcup \mathcal{H}$ is nondegenerate. If $x, y \in \bigcup \mathcal{H}$, and $A, B \in \mathcal{H}$ such that $x \in A$, $y \in B$, let $H_0, H_1, \ldots, H_n$ be members of $\mathcal{H}$ such that $A = H_0$, $B = H_n$, and $H_i \cap H_{i+1} \neq \emptyset$, $i = 0, 1, \ldots, n-1$. It follows from 2.3.15 that $\bigcup H_i$ is an $H$-set of $M$, so $C(x, y) \subseteq \bigcup H_i \subseteq \bigcup \mathcal{H}$.

2.3.18 Theorem. If $\mathcal{H}$ is a family of $H$-sets of $M$ and $\bigcap \mathcal{H}$ is nondegenerate or consists of a single cut point or end point of $M$, then $\bigcap \mathcal{H}$ is an $H$-set of $M$. If $M$ is locally compact, then every intersection of $H$-sets of $M$ is an $H$-set of $M$.

Proof. If $\bigcap \mathcal{H}$ is a cut point or an end point of $M$ then $\bigcap \mathcal{H}$ is an $H$-set of $M$. Suppose that $\bigcap \mathcal{H}$ is nondegenerate. Let $a \in \bigcap \mathcal{H}$, and let $x \in \bigcap \mathcal{H}$ such that $a \neq x$. Then for each $H$ in $\mathcal{H}$, $C(a, x) \subseteq H$, so $C(a, x) \subseteq \bigcap \mathcal{H}$. It follows that $\bigcap \mathcal{H}$ is connected. Similarly, if $x, y \in \bigcap \mathcal{H}$, then $C(x, y) \subseteq \bigcap \mathcal{H}$, so $\bigcap \mathcal{H}$ is an $H$-set of $M$.

Now if $M$ is locally compact and $p \in \bigcap \mathcal{H}$ such that $p$ is neither a cut point nor an end point of $M$, then $p$ belongs to an $E_0$-set $E$ of $M$ and $p \in E \cap H$ for all $H$ in $\mathcal{H}$. Thus for each $H$ in $\mathcal{H}$, $E \subseteq H$, so $E \subseteq \bigcap \mathcal{H}$ and the result now follows from the first part.

We note that it is not true in general that the non-empty intersection of $H$-sets is an $H$-set. For in the space $T^*$ of 2.2.17, for each positive integer $n$, let $H_n = \{(x, 0) : x \leq 1/n\}$, and let $\mathcal{H} = \{H_n : n = 1, 2, \ldots\}$. Then each $H_n$ is an $H$-set in $T^*$, but $\bigcap \mathcal{H} = \{(0, 0)\}$ is not.
2.3.19 **Theorem.** If $H$ is an $H$-set of $M$, $x, y \in H$ and $X \subseteq M$ such that $x$ and $y$ lie in distinct components of $H - X$, then $x$ and $y$ lie in distinct components of $M - X$.

**Proof.** If not, then $x$ and $y$ lie in a component $C$ of $M - X$. Then $C \cap H$ is connected, contains $x$ and $y$ and is contained in $H - X$; so $x$ and $y$ lie in the same component of $H - X$. This is a contradiction.

In [11, p. 67, 3.22] the following result is established: "If two points $x$ and $y$ of an $A$-set $A$ in a metric space $M$ are separated in $A$ by a subset $X$ of $A$, then $x$ and $y$ are also separated in $M$ by $X$." We note that this result is stronger than 2.3.19; we do not know whether it is true in general, even if $M$ is a connected, locally connected Hausdorff space. We have, however, the following two corollaries.

**2.3.20 Corollary.** Every cut point of an $H$-set of $M$ is a cut point of $M$.

**2.3.21 Corollary.** If $H$ is an $H$-set of $M$, $A, B \subseteq H$, and $X$ a closed subset of $M$ such that $X \cap H$ separates $A$ and $B$ in $H$, then $X$ separates $A$ and $B$ in $M$.

**Proof.** Suppose not. Then there is a component $C$ of $M - X$ and points $a, b$ of $A$ and $B$, respectively, such that $a, b \in C$. But then by Theorem 2.3.19, $X \cap H$ cannot separate $a$ and $b$ in $H$. 
2.3.22 Corollary. If M is locally compact, H an H-set of M, and E a cyclic element of H, then E is a cyclic element of M.

Proof. If E is an $E_0$-set or a singleton cut point of H, then the result follows from 2.3.4 or 2.3.20. Suppose, then, that E is a singleton end point of H, $E = \{p\}$. If p is neither a cut point nor an end point of M, then p belongs to an $E_0$-set $K^*$ of M. By 2.3.3, $K^*$ is an $E_0$-set of H. But then p is an end point of H belonging to an $E_0$-set of H and this is a contradiction. Thus p is either a cut point or an end point of M so E is a cyclic element of M.

2.3.23 Corollary. Let H be an H-set of M. Then every non-degenerate H-set $H^*$ of H is an H-set of M. If M is locally compact, then every H-set of H is an H-set of M.

Proof. If $H^*$ is nondegenerate, then so is H. If E is an $E_0$-set of M such that $E \cap H^*$ is nondegenerate, then $E \cap H$ is nondegenerate, so $E \subset H$ and E is an $E_0$-set of H. Thus by 2.3.10 and 2.3.5, $E \subset H^*$. It follows that $H^*$ is an H-set of M. Now if M is locally compact and $H^*$ is degenerate, then by 2.3.22, $H^*$ is a cut point or an end point of M.

2.3.24 Corollary. If A is an A-set of M and B is an A-set of A, then B is an A-set of M.

2.3.25 Corollary. If $a, b \in M$, then $C(a, b)$ contains no proper A-set of itself which contains both a and b; i.e.,
the cyclic chain in $C(a,b)$ from $a$ to $b$ is $C(a,b)$. Further, if $t$ is a cut point of $C(a,b)$, then $t \in E(a,b)$.

**Remark.** If $T^*$ is the space defined in 2.2.17 and $H = \{(x,0) : (x,0) \in T^*\}$, then $H$ is an $H$-set of $T^*$, $\{(0,0)\}$ is a cyclic element of $H$ and is not an $H$-set of $M$. Thus neither 2.3.22 nor 2.3.23 is true in general for degenerate $H$-sets.

2.3.26 **Theorem.** If $H$ is an $H$-set of $M$, and $Z$ is any connected and locally connected subset of $M$, such that $H \cap Z$ is nondegenerate, then $H \cap Z$ is an $H$-set of $Z$.

**Proof.** If $H \cap Z$ is nondegenerate, let $E$ be an $E_o$-set of $Z$ such that $E \cap H \cap Z$ is nondegenerate. Then $E \subset E^*$ for some $E_o$-set $E^*$ of $M$. Then $E^* \subset H$, so $E \subset E^* \cap Z \subset H \cap Z$. Thus $H \cap Z$ is an $H$-set of $Z$.

2.3.27 **Corollary.** If $A$ is an $A$-set of $M$ and $Z$ is a connected and locally connected subset of $M$, such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is an $A$-set of $Z$.

2.3.28 **Theorem.** If $a,b \in M$, then $a$ and $b$ are non-cut points of $C(a,b)$, and if $a$ and $b$ are not conjugate in $M$, then $C(a,b) - a - b$ is connected.

**Proof.** Let $D$ be the component of $C(a,b) - a$ such that $b \in D$. Then $\overline{D} = D \cup a$ is an $A$-set of $C(a,b)$ and contains both $a$ and $b$. Thus $D \cup a = C(a,b)$, so $C(a,b) - a = D$, which is connected. Similarly, $b$ is a non-cut point of
Suppose now that \( a \) and \( b \) are not conjugate and that \((U, V)\) is a separation of \( C(a, b) - a - b \). Since \( C(a, b) - a \) is connected, \( \{a, b\} \) is an irreducible closed cutting of \( C(a, b) \) and therefore \( U \cup \{a, b\} \) is connected. Thus \( E(a, b) \cup \{a, b\} \subset U \cup \{a, b\} \). Now no \( E^- \)-set contains both \( a \) and \( b \), so if some \( E^- \)-set meets \( E(a, b) \cup \{a, b\} \) in two points, it meets \( U \). Since \( E - \{a, b\} \) is connected, \( E \subset U \cup \{a, b\} \). It follows that \( C(a, b) \subset U \cup \{a, b\} \), so \( V = \emptyset \). Thus \( C(a, b) - a - b \) is connected.

2.3.29 Theorem. Let \( A \) be a closed, connected subset of \( M \). Then among the following statements, \( a - c \) are equivalent and \( c \) implies \( d \); if \( M \) is locally compact, then \( a - d \) are equivalent and \( a \) implies \( e \).

a. \( A \) is an \( A \)-set of \( M \).

b. If \( C \) is a component of \( M - A \), then \( \overline{C} \cap A \) is a singleton.

c. If \( E \) is a cyclic element of \( M \) and \( A \cap E \) is nondegenerate, then \( E \subset A \).

d. If \( a, b \in A \), and \( N \) is an irreducible continuum from \( a \) to \( b \), then \( N \subset A \).

e. If \( p \in A \), then either \( p = A \) or there is a cyclic element \( E \) of \( M \) such that \( p \in E \subset A \).

Proof. That \( a \) and \( b \) are equivalent was proved in [12], and that \( a \) implies \( c \) follows from 2.3.3 since every
nondegenerate A-set of M is an H-set of M. Suppose that A satisfies c. If A is degenerate, then A is an A-set of M. If not, then by 2.3.5, A is an H-set and therefore an A-set since A is closed. Thus c implies a. That c implies d is 2.2.9.

Now assume that M is locally compact. We show that d implies b. Let C be a component of M - A and suppose \(3(C)\) contains two points p and q. Since A is closed, C is a connected, locally connected, and locally compact Hausdorff space. Let \(R_p, R_q\) be disjoint open sets containing p and q, respectively, such that \(\overline{R_p}\) and \(\overline{R_q}\) are disjoint continua, and let \(x, y\) be points of \(R_p \cap C\) and \(R_q \cap C\) respectively.

Let \(N_{p,x}\) be an irreducible continuum in \(\overline{R_p}\) from p to x; \(N_{q,y}\) an irreducible continuum in \(\overline{R_q}\) from q to y, and \(N_{x,y}\) an irreducible continuum in C from x to y. Then \(N_{p,x} \cup N_{x,y} \cup N_{q,y}\) is a continuum containing p and q so contains an irreducible continuum N from p to q. By d, \(N \subset A\). But this is impossible since then \(N \subset \overline{R_p} \cup \overline{R_q}\), and these are disjoint closed sets each of which meets N. Thus \(\overline{C} \cap A\) contains at most one point, and since M is connected, \(\overline{C} \cap A\) is a singleton.

It remains to show that if M is locally compact, a implies e. If A is nondegenerate, and p is not a cyclic element, then p belongs to an \(E_o\)-set E and the result follows from 2.3.3.
Remark. It was shown in [11] that if $M$ is a locally connected metric continuum, and $A$ is a subcontinuum of $M$, then all five statements $a$ - $e$ are equivalent. That this is not true in general is shown in the following examples. In the space $M^*$ of 2.1.3, let $A = \{(1,\theta) : 0 \leq \theta \leq \pi/2\}$. Then $A$ is a continuum and satisfies $e$ but not $b$. We also note that if $A$ is not closed, then $b$ does not imply $a$, $c$, or $d$; for in $M^*$, if $A = E^* - (1,\pi/2)$, then $A$ satisfies $b$ but none of $a$, $c$, or $d$. To see that if $A$ is not closed, then $b$ does not imply $e$, consider the space $S^*$ of Example A of 2.1.4 and let $A = F^* - \{(0,0)\}$.

2.4 Nodal sets, nodes and cyclic chains.

2.4.1 Definition. A closed subset $N$ of a space $S$ is called a nodal set of $S$ iff $\emptyset (N)$ is at most a singleton.

The next result follows immediately from Definition 2.4.1.

2.4.2 Lemma. Let $S$ be a $T_1$ topological space and $N \subseteq S$. Then

a. every singleton is a nodal set of $S$ as are $\emptyset$ and $S$;

b. if $N$ is a nodal set of $S$, then $S - N$ is a nodal set of $S$;

c. if $S$ is connected and locally connected, and $N$ is a nonempty nodal set of $S$, then $N$ is connected;
d. if $S$ is connected, $p \in S$, and $(S_1, S_2)$ is a separation of $S - p$, then $S_1 \cup p$ and $S_2 \cup p$ are nodal sets of $S$;

e. if $S$ is connected and locally connected, then
if $N$ is a nodal set of $S$, $N$ is an $A$-set of $S$.

2.4.3 **Lemma.** If $A \subset S$, and $N$ is a nodal subset of $S$, then $N \cap A$ is a nodal set of $A$.

**Proof.** $N \cap A$ is closed in $A$. If $x$ is a point of the boundary in $A$ of $N \cap A$, then $x \in \partial(N)$. Since $\partial(N)$ is at most a singleton, the boundary in $A$ of $N \cap A$ is at most a singleton.

2.4.4 **Definition.** A subset $N$ of a connected space $S$ is called a node of $S$ iff either $N = \{p\}$ for some end point $p$ of $S$ or $N$ is an $E_0$-set of $S$ such that $N$ is a nodal set of $S$.

**Remark.** It is immediate from the definition that if a connected $T_1$-space $S$ has no cut point, then $S$ is a node of itself and the only nodal subsets of $S$ are $\emptyset$, $S$, and the singletons of $S$.

**NOTE:** We again let "$M$" denote a connected and locally connected Hausdorff space.

2.4.5 **Theorem.** Let $N$ be a nondegenerate subset of $M$. If $N = M$, then $N$ is a node of $M$ iff $M$ has no cut point. If
N \in M$, then $N$ is a node of $M$ iff $N$ is an $E_o$-set of $M$ and $N$ contains exactly one cut point of $M$.

**Proof.** The first statement is immediate from the definition. Suppose, then, $N \in M$ and $N$ is a node of $M$. Let $p = \partial(N)$. Then $p$ is a cut point of $M$. Suppose $q \in N - p$ and $U, V$ are disjoint open sets such that $M - q = U \cup V$, $p \in U$. Then $N - q \subseteq U$. Now $q \in \text{Int} N \subseteq U \cup q$, so $q \in \text{Int}(U \cup q)$. It follows, since $U$ is open, that $U \cup q$ is both open and closed in $M$, so $V = \emptyset$. Thus $q$ is not a cut point of $M$.

Now suppose that $N$ is an $E_o$-set of $M$ and $N$ contains exactly one cut point, $p$, of $M$. If $C$ is a component of $M - N$, then $\partial(C)$ is a singleton cut point of $M$ and $\partial(C) \subset N$, so $\partial(C) = p$. Suppose $q$ is a point of $\partial(N)$ distinct from $p$.

Let $V$ be a connected open set such that $q \in V$ and $p \notin V$. Then $V$ meets some component $C$ of $M - N$, and since $q \in V - C$, $\partial(C) \cap V \neq \emptyset$. Since $\partial(C) = p$, this is a contradiction. Thus $\partial(N) = p$ and $N$ is a node of $M$.

**2.4.6 Corollary.** Every node $N$ of $M$ contains a non-cut point of $M$, and if $N$ is nondegenerate, then every point of $N$ distinct from the one boundary point of $N$ is neither a cut point of $M$ nor an end point of $M$.

**2.4.7 Lemma.** If $N$ is a nondegenerate nodal subset of $M$, then either $N$ contains a cut point of itself or $N$ is an $E_o$-set and therefore a node of $M$. 
Proof. Suppose $N$ contains no cut point of itself. If $N = M$, then $M$ contains no cut point, so $N$ is a node of $M$. If $N \neq M$, there is a point $p$ of $M$ such that $\partial(N) = \{p\}$, and $(\text{Int } N, \text{Ext } N)$ is a separation of $M - p$. It follows that if $t \in M - N$, then $p$ separates $t$ and $N - p$; thus $N$ is maximal with respect to being nondegenerate, connected, and having no cut point of itself. Thus $N$ is a node of $M$.

2.4.8 Theorem. If $N_1, N_2$ are distinct, intersecting nodes of $M$, then neither is degenerate and their intersection is a cut point of $M$.

Proof. If $N_1 = \{p\}$, then $p$ is an end point of $M$ and $p \in N_2$. But this implies that $N_2 = \{p\} = N_1$. It follows that neither $N_1$ nor $N_2$ is degenerate and each is an $E_0$-set of $M$. By 2.2.5, $N_1 \cap N_2$ is a cut point of $M$.

2.4.9 Theorem. If $x$ is a non-cut point of $M$ belonging to a node, $N$, of $M$, then $N$ is a node of every $H$-set of $M$ containing $x$.

Proof. Let $H$ be an $H$-set of $M$ containing $x$. If $N = \{x\}$, then $x$ is an end point of $M$. If $H = \{x\}$, then $N = H$ and $N$ is a node of $H$. If $H$ is nondegenerate, then by 2.2.1 $x$ is an end point of $H$, so again $N$ is a node of $H$. If $N$ is nondegenerate, then $N$ is an $E_0$-set of $M$ and the theorem follows from 2.3.3 and 2.4.3.

2.4.10 Theorem. If $H$ is an $H$-set of $M$ and $C$ is a component
of $M - H$, then $\overline{C}$ is a nodal $A$-set.

Proof. By 2.3.13 and 2.3.14, $\overline{C}$ is an $A$-set and $\partial(\overline{C}) = \partial(C) = \overline{C} \cap \overline{H}$ is a singleton.

2.4.11 Theorem. Let $N$ be a node of $M$ and $C(x,y)$ a cyclic chain in $M$. If $N \cap C(x,y)$ contains a non-cut point of $M$, then one of $x$ and $y$ is a non-cut point of $M$ that belongs to $N$.

Proof. By 2.4.9, $N \subseteq C(x,y)$. If $N = \{p\}$, then $p$ is an end point of $M$, so $p \notin E(x,y)$ and $p$ belongs to no $E_0$-set of $M$. It follows from 2.1.2-g that $p \in \{x,y\}$. Suppose $N$ is nondegenerate and $p \notin \{x,y\}$. Then $N$ meets $E(x,y) \cup \{x,y\}$ in two points. But $N$ contains at most one cut point of $M$, so $N \cap \{x,y\}$ contains at least one non-cut point of $M$.

2.4.12 Theorem. If $a,b$ are non-cut points of $M$ which belong to distinct nodes of $M$, then $C(a,b)$ is a maximal cyclic chain of $M$; that is, if $C(a,b) \subseteq C(x,y)$, then $C(a,b) = C(x,y)$.

Proof. Suppose $C(a,b) \subseteq C(x,y)$ and $N_a, N_b$ are distinct nodes of $M$ containing $a$ and $b$ respectively. By 2.4.9, $N_a \cup N_b \subseteq C(a,b)$. By 2.4.11, $x,y \in N_a \cup N_b$ and it follows that $C(x,y) \subseteq C(a,b)$.

The next result follows from the proof of Theorem 2.4.12.

2.4.13 Corollary. If $a$ and $b$ are non-cut points of $M$ which belong to distinct nodes $N_a$ and $N_b$, respectively, of $M$ and $C(a,b) = C(x,y)$, then $x$ and $y$ are non-cut points of $M$ and
each of $N_a$, $N_b$ contains one of the points $x, y$ and not both.

2.4.14 Theorem. If $C(a,b)$ is a cyclic chain in $M$ and $N$ is a node of $C(a,b)$, then $a \in N$ or $b \in N$.

Proof. It follows from 2.3.25 that in $C(a,b)$, if $C^*(a,b)$ is the cyclic chain from $a$ to $b$, then $C^*(a,b) = C(a,b)$. Now $N$ is a node of $C(a,b)$ such that $N \cap C^*(a,b)$ contains a non-cut point of $C(a,b)$, so by 2.4.11, $a \in N$ or $b \in N$.

2.4.15 Theorem. If $C(a,b)$ is a cyclic chain in $M$, then $C(a,b)$ contains at most two nodes of itself. Also, if $C(a,b)$ has two nodes, then $E(a,b) \neq \emptyset$; and if $M$ is locally compact and $E(a,b) \neq \emptyset$, then $C(a,b)$ has two nodes.

Proof. Suppose that $C(a,b)$ has three nodes, $N_1$, $N_2$, and $N_3$. By 2.3.28, $a$ and $b$ are non-cut points of $C(a,b)$, and by 2.4.14, either $a$ or $b$ must lie in two of the sets $N_1$, $N_2$, $N_3$; but this contradicts 2.4.8.

Now suppose that $C(a,b)$ has two nodes, $N_1$ and $N_2$. Then by 2.4.14, we may assume that $a \in N_1$. Since $a$ is not a cut point of $C(a,b)$, $a \notin N_2$, so $b \in N_2$. Then either $N_2 = \{b\}$, so $b$ is an end point of $C(a,b)$, or $b \in \text{Int}_{C(a,b)}N_2$. In either case $E(a,b) \neq \emptyset$.

If $M$ is locally compact, then $C(a,b)$ is locally compact. Thus if neither $a$ nor $b$ belongs to an $E_o$-set of $C(a,b)$, then by 2.2.16, each is an end point of $C(a,b)$, so $\{a\}$ and $\{b\}$ are distinct nodes of $C(a,b)$. If $a$ belongs to an $E_o$-set $E_l$
of \( C(a,b) \), then \( E_1 \) is an \( E_0 \)-set of \( M \) which meets 
\( E(a,b) \cup \{a,b\} \) in exactly two points, one of which is \( a \).
Since \( b \notin E_1 \), \( E_1 \) meets \( E(a,b) \) in exactly one point. By
2.3.25, the set of cut points of \( C(a,b) \) is identical with 
\( E(a,b) \) and it follows that \( E_1 \) is an \( E_0 \)-set of \( C(a,b) \) con­
taining exactly one cut point of \( C(a,b) \). Thus by 2.4.5,
\( E_1 \) is a node of \( C(a,b) \). Now if \( b \) is an end point of \( C(a,b) \),
then \( N_2 = \{b\} \) is a node of \( C(a,b) \) distinct from \( E_1 \). If not,
then as in the case for \( a \), \( b \) belongs to an \( E_0 \)-set \( E_2 \) of
\( C(a,b) \) and \( E_2 \) is a node of \( C(a,b) \) distinct from \( E_1 \).

2.4.16 Example. In the xy-plane, for each positive integer
\( n \) let \( E_n \) be the closed rectangular region
\[ E_n = \{(x,y) : 1/2n + 1 \leq x \leq 1/2n, -1 \leq y \leq 1\}, \]
and let \( F_n \) be the closed rectangular region
\[ F_n = \{(x,y) : 1-1/2n \leq x \leq 1-1/2n+1, -1 \leq y \leq 1\}. \]
Let \( S \) be the sum of all the sets \( E_n \) and \( F_n \) together with the x-axis.
Then \( S \) is a connected and locally connected metric space and
is not locally compact. We note that \( C((0,0),(5/7,0)) \) has
one node, \( \{(5/7,0)\} \), while \( C((0,0),(1,0)) \) has no node. Thus
the last part of Theorem 2.4.15 does not hold in general if
\( M \) is not locally compact.

Thus far, we have not demonstrated the existence of
nodes in a connected and locally connected Hausdorff space
\( M \). In fact, it may be that \( M \) contains no nodes, even if \( M \)
is locally compact. For instance, the real line is a
locally connected generalized continuum which contains no nodes, and the nonnegative reals constitute a locally connected generalized continuum with exactly one node, \( N = \{0\} \). The next theorem assures us of the existence of nodes in the case that \( M \) is a locally connected Hausdorff continuum.

2.4.17 Theorem. If \( M \) is compact, then every nondegenerate nodal subset of \( M \) contains a node of \( M \).

**Proof.** Let \( N \) be a nondegenerate nodal subset of \( M \). If \( M \) has no cut point, then \( N = M \), and \( N \) is a node. Assume that \( M \) has a cut point. Then \( M \) is not a node and we may assume that \( N \neq M \). Assume, further, that \( N \) contains no nondegenerate node of \( M \). We show that, in this case, \( N \) contains an end point of \( M \).

Let \( p \in M \) such that \( \partial(N) = \{p\} \). Let \( \mathcal{G} = \{(x, C) : C \text{ is a component of } M - x \text{ and } \overline{C} \subset \text{Int } N\} \).

Now, \( (\text{Ext } N, \text{Int } N) \) is a separation of \( M - p \), so there is a component \( D \) of \( M - p \) such that \( D \subset \text{Int } N \). \( \overline{D} = D \cup p \) is a nondegenerate nodal subset of \( M \) and \( \overline{D} \subset N \). By assumption, \( \overline{D} \) is not a node of \( M \), so \( \overline{D} \) has a cut point \( x \). Since \( p \) is not a cut point of \( \overline{D} \), \( x \neq p \). Since \( \overline{D} \) is an A-set in \( M \), \( x \) is a cut point of \( M \). Let \((U, V)\) be a separation of \( M - x \) such that \( p \in U \). Since \( (M - N) \cup p \) is connected and contained in \( M - x \), \( (M - N) \cup p \subset U \). Thus \( V \subset \text{Int } N \). Let \( C \) be a component of \( M - x \) such that \( C \subset V \). Then
Define a relation "\( > \)" on \( P \) by \((x_1, C_1) > (x_2, C_2)\) iff \( C_1 \subseteq C_2 \). Let \((x_1, C_1) \in P\). Then \( C_1 = C_1 \cup x_1 \) is a nodal subset of \( M \) and \( \overline{C_1} \subseteq \text{Int} \, N \). By assumption, \( \overline{C_1} \) is not a node of \( M \), and it follows as in the above paragraph that there is a cut point \( x_2 \) of \( M \) and a component \( C_2 \) of \( M - x_2 \) such that \( x_2 \notin C_1 \) and \( \overline{C_2} \subseteq C_1 \). Then \((x_2, C_2) \in P\), and \((x_2, C_2) > (x_1, C_1)\). Thus the relation \( > \) is nonempty, and it is not difficult to show that \( > \) is a partial order. Further, we have shown that \((P, >)\) has no maximal element.

Let \( \mathcal{M} \) be a maximal chain in \((P, >)\). Since \((P, >)\) has no maximal element, \( \mathcal{M} \) has no maximum element. Let

\[
C^* = \bigcap \{C : \text{for some } x \in M, (x, C) \in \mathcal{M}\}. 
\]

Since \( \{C : \text{for some } x \in M, (x, C) \in \mathcal{M}\} \) is simply ordered by inclusion and \( M \) is compact, \( C^* \) is nonempty and connected. Also, \( C^* = \bigcap \{C : \text{for some } x \in M, (x, C) \in \mathcal{M}\}; \) for if \( t \in C^* \) and \((x_1, C_1) \in \mathcal{M}\), then since \((x_1, C_1)\) is not maximum in \( \mathcal{M} \), there is a member \((x_2, C_2)\) of \( \mathcal{M} \) such that \((x_2, C_2) > (x_1, C_1)\) and \( t \in \overline{C_2} \subseteq C_1 \). Thus \( C^* = \bigcap \{C : \text{for some } x \in M, (x, C) \in \mathcal{M}\} = C^* \). Suppose \( C^* \) contains a cut point \( t \) of \( M \). Let \((U, V)\) be a separation of \( M - t \) such that \( p \in U \). If \((x, C_1) \in \mathcal{M} \), then \( M - C_1 \) is connected, and \( p \in M - C_1 \), so \( M - C_1 \subseteq U \) and \( V \subseteq C_1 \). Thus \( V \subseteq C^* \). Let \( D^* \) be a component of \( M - t \) contained in \( V \) and consider the pair \((t, D^*)\) of \( P \). If \((x, C) \in \mathcal{M}\) then \( D^* = D^* \cup t \subseteq C \), so \((t, D^*) > (x, C)\).
But then \((t,D^*) \in \mathfrak{M}\) and is a maximum element of \(\mathfrak{M}\), and this is a contradiction. Thus no point of \(C^*\) is a cut point of \(M\).

Since \(C^*\) is an intersection of \(A\)-sets of \(M\), \(C^*\) is an \(A\)-set of \(M\). Then every cut point of \(C^*\) is a cut point of \(M\), so \(C^*\) has no cut point of itself. If \(C^*\) is either non-degenerate or contains a point \(p\) which is neither a cut point nor an end point of \(M\), then for some \(E_o\)-set \(E\) of \(M\), \(C^* \subseteq E\) or \(p \in E\). In either case, \(E = C^*\). But by assumption, \(E\) is not a node of \(M\) so contains cut points of \(M\) and this is a contradiction. Thus \(C^*\) consists of a single end point of \(M\).

2.4.18 Corollary. If \(M\) is compact and has a cut point, then \(M\) has at least two nodes.

**Proof.** Let \(p\) be a cut point of \(M\) and \((U,V)\) a separation of \(M - p\). Then \(U \cup p\) and \(V \cup p\) are nodal subsets of \(M\), so there are nodes \(N_1, N_2\) of \(M\) such that \(N_1 \subseteq U \cup p\) and \(N_2 \subseteq V \cup p\). If \(N_1 = \{t\}\), then \(t \notin p\), so \(t \in U\) and \(t \notin N_2\). If \(N_1\) is nondegenerate, then for some \(t \in M\), \(t \in U \cap N_1\), so \(t \notin N_2\). In either case, \(N_1 \nsubseteq N_2\).

2.4.19 Corollary. If \(M\) is compact, \(H\) an \(H\)-set of \(M\), and \(C\) is a component of \(M - H\), then \(C\) contains a point \(a\) that is a non-cut point of \(M\) belonging to a node of \(M\).

**Proof.** Let \(H\) be an \(H\)-set of \(M\), and let \(b = \overline{C} \cap \overline{H}\). By 2.4.10, \(C \cup b\) is a nodal set, so by 2.4.17, \(C \cup b\)
contains a node \( N \) of \( M \). By 2.4.6, \( N \) contains a non-cut point \( a \) of \( M \) and since \( b \) is a cut point of \( M \), \( a \in C \).

2.4.20 Theorem. If \( M \) is compact, then every point of \( M \) belongs to a cyclic chain \( C(a,b) \) of \( M \) where \( a \) and \( b \) are non-cut points of \( M \) which belong to nodes of \( M \), and if \( M \) has a cut point, then \( a \) and \( b \) can be chosen to belong to distinct nodes of \( M \).

Proof. If \( M \) has no cut point, the result is immediate; so assume that \( M \) has a cut point and let \( x \in M \).

If \( x \) belongs to a node \( N_1 \) of \( M \), let \( a \) be any non-cut point of \( M \) belonging to \( N_1 \). Then there is a node \( N_2 \) of \( M \) distinct from \( N_1 \). Let \( b \) be any non-cut point of \( M \) belonging to \( N_2 \). Then \( a \neq b \) and \( N_1 \subset C(a,b) \), so \( x \in C(a,b) \).

Suppose now that \( x \) belongs to no node of \( M \). We consider two cases.

Case 1. \( x \) is a cut point of \( M \). Let \((U,V)\) be a separation of \( M - x \). Then \( U \cup x \) and \( V \cup x \) are nodal subsets of \( M \) and by 2.4.17 contain nodes \( N_1 \) and \( N_2 \) respectively. By 2.4.6, \( N_1 \) and \( N_2 \) contain points \( a \) and \( b \), respectively, such that \( a \) and \( b \) are non-cut points of \( M \). Then \( a \neq b \) and \( a \in C(a,b) \cap U, b \in C(a,b) \cap V \). Since \( C(a,b) \) is connected, \( x \in C(a,b) \).

Case 2. \( x \) is not a cut point of \( M \). Then since \( x \) belongs to no node of \( M \), \( x \) is not an end point of \( M \). Let \( E \) be the unique \( E_0 \)-set containing \( x \). Then \( E \) is not a node, so \( E \)
contains two distinct cut points of \( M \), \( x_1 \) and \( x_2 \). For each \( i = 1,2 \), in \( M - E_i \) let \( C_i \) be the component which does not contain \( E - x_i \). Then \( C_1, C_2 \) are distinct components of \( M - E \) and \( \partial(C_1) \neq \partial(C_2) \). Now for each \( i = 1,2 \), \( C_i \cup x_i \) is a nodal subset of \( M \) so contains a node \( N_i \). Let \( a \) and \( b \) be non-cut points of \( M \) belonging to \( N_1 \) and \( N_2 \), respectively. Then by 2.2.14, \( E \subset C(a,b) \) so \( x \in C(a,b) \).

2.4.21 Theorem. If \( M \) is compact and \( H \) is an \( H \)-set of \( M \), \( C \) a component of \( M - H \) and \( \overline{C} \cap \overline{H} = \{b\} \); then if \( x \in C \), there is a non-cut point \( a \) of \( M \) such that \( a \in C \) and belongs to a node of \( M \) and \( x \in C(a,b) \subset C \cup b \).

Proof. If \( x \) belongs to a node \( N \) of \( M \), then there is a non-cut point \( a \) of \( M \) such that \( a \in N \). By 2.4.9, \( N \subset C(a,b) \subset C \cup b \). Suppose, then, that \( x \) belongs to no node of \( M \). Again we consider two cases.

Case 1. \( x \) is a cut point of \( M \). Let \((U,V)\) be a separation of \( M - x \) such that \( b \in U \). Then \( H \subset U \). Since \( V \cup x \) is connected and contained in \( M - H \), \( V \cup x \subset C \). Let \( a \) be a non-cut point of \( M \) belonging to \( V \cup x \). Then \( C(a,b) \) meets both \( U \) and \( V \), so \( x \in C(a,b) \). Since \( a,b \in C \cup b \), \( C(a,b) \subset C \cup b \).

Case 2. \( x \) is not a cut point of \( M \). Since \( x \) belongs to no node of \( M \), \( x \) is not an end point of \( M \). Now \( x \) belongs to an \( E_0 \)-set \( E \) of \( M \). Then \( E \subset C \cup b \) and \( E \) contains two distinct cut points of \( M \). If \( b \in E \), let \( t \) be a cut point of \( M \).
distinct from $b$. If $b \notin E$, let $C_b$ be the component of $M - E$ containing $b$ and let $t$ be a cut point of $M$ in $E$ such that $t \notin \partial(C_b)$. Let $D$ be a component of $M - t$ such that $b \notin D$. Then $D \subset C$ and $D$ contains a non-cut point $a$ of $M$ belonging to a node of $M$. Now $C(a,b) \subset C \cup b$. Further, $E \subset C(a,b)$ so $x \in C(a,b)$.

2.4.22 Corollary. If $M$ is compact, $A$ an $A$-set of $M$, $C$ a component of $M - A$ and $b = \partial(C)$, then if $x \in C$, there is a non-cut point $a$ of $M$ belonging to a node of $M$ such that $x \in C(a,b) \subset C \cup b$.

2.5 Null families. The following definition is to be found in Wilder, [13, p. 106].

Definition. If $\mathcal{G}$ is a covering of a space $S$, then a point set $E$ of $S$ is said to be of diameter $< \mathcal{G}$ if some element of $\mathcal{G}$ contains $E$.

Notation. If $\mathcal{G}$ is a covering of $S$ and $E \subset S$ is of diameter $< \mathcal{G}$, we will write "diam $E < \mathcal{G}$". If $E$ is not of diameter $< \mathcal{G}$, we will write "diam $E > \mathcal{G}$".

2.5.1 Definition. Let $\mathcal{F}$ be a family of subsets of a topological space $S$. Then $\mathcal{F}$ is called a null family iff for every open cover $\mathcal{G}$ of $S$, all but a finite number of members of $\mathcal{F}$ have diameter $< \mathcal{G}$.

The next theorem is an easy consequence of definition
2.5.1.

2.5.2 **Theorem.** Every subfamily of a null family of a topological space $S$ is a null family.

2.5.3 **Theorem.** If $\mathcal{F}$ is a null family in a regular space $S$ and $\mathcal{F}^* = \{F : F \in \mathcal{F}\}$, then $\mathcal{F}^*$ is a null family.

**Proof.** Let $\mathcal{J}$ be any open cover of $S$. For each $x \in S$, let $G_x \in \mathcal{J}$ such that $x \in G_x$ and let $O_x$ be an open set with $x \in O_x$ and $\overline{O}_x \subset G_x$. Then $\mathcal{O} = \{O_x : x \in S\}$ is a refinement of $\mathcal{J}$ such that the closure of each member of $\mathcal{O}$ is contained in some member of $\mathcal{J}$. Since $\mathcal{F}$ is a null family, for all but a finite number of members $F$ of $\mathcal{F}$, $\text{diam } F < \mathcal{O}$. But if $F \subset O$ for some member $O$ of $\mathcal{O}$, then $F \subset O$ and it follows that $\text{diam } F < \mathcal{O}$. Thus for all but a finite number of members $F$ of $\mathcal{F}$, $\text{diam } F < \mathcal{J}$, so $\mathcal{F}$ is a null family.

2.5.4 **Lemma.** Let $H$ be an $H$-set of $M$, $C$ a collection of components of $M - H$, $p$ a limit point of $\bigcup C$ such that $p \in \bigcup C$ and $\{p_\alpha : \alpha \in (\Lambda, \geq)\}$ a net in $\bigcup C$ converging to $p$. Then $p \notin H$, and if for each $\alpha \in \Lambda$, $C_\alpha \in C$ such that $p_\alpha \in C_\alpha$ and $b_\alpha = \overline{C_\alpha} \cap H$, then $b_\alpha \to p$. Further if $M$ is locally compact, $p \notin \bigcup \{C : C \in C\}$, and $\{q_\alpha : \alpha \in (\Lambda, \geq)\}$ is a net such that $q_\alpha \in \overline{C_\alpha}$ for each $\alpha \in \Lambda$, then $q_\alpha \to p$.

**Proof.** If $p \notin H$, there is a component $C$ of $M - H$ such that $p \in C - H = \text{int } C$ and $C$ meets no member of $C$. Since this is a contradiction, $p \in H$. 

Now let $O$ be any open set such that $p \in O$, and let $V$ be a connected open set such that $p \in V$ and $V \subset O$. There is an $\alpha^* \in A$ such that if $\alpha \geq \alpha^*$, $p_\alpha \in V$. If $\alpha \geq \alpha^*$, then since $p \notin C_\alpha$, $V$ meets $C_\alpha$ and $M - C_\alpha$ so $V$ meets $\partial(C_\alpha) = b_\alpha$. Thus $b_\alpha \to p$.

Assume now that $M$ is locally compact, that if $C \in C$, then $p \notin C$, and that $\{q_\alpha : \alpha \in (A, \geq)\}$ is a net such that for each $\alpha \in A$, $q_\alpha \in \overline{C_\alpha}$. Suppose $q_\alpha \to p$. Then there is an open set $V$ such that $p \in V$ and the net $\{q_\alpha : \alpha \in (A, \geq)\}$ is frequently in $M - V$. Further, we may assume that $V$ is compact.

Let $\alpha^* \in A$ such that if $\alpha \geq \alpha^*$ then $p_\alpha, b_\alpha \in V$. Let $\mathcal{B} = \{\alpha \in A : \alpha \geq \alpha^*$ and $q_\alpha \notin V\}$. By definition of $V$, $\mathcal{B}$ is a cofinal subset of $A$ and for each $\alpha \in \mathcal{B}$, $C_\alpha \notin V$. Now $\{C_\alpha : \alpha \in \mathcal{B}\}$ must be infinite; for otherwise, $p \in \bigcup_{\alpha \in \mathcal{B}} C_\alpha$, so for some $\alpha \in \mathcal{B}$, $p \in \overline{C_\alpha}$ and this is a contradiction.

Now $\{p_\alpha : \alpha \in (\mathcal{B}, \geq)\}$ and $\{b_\alpha : \alpha \in (\mathcal{B}, \geq)\}$ are subnets, respectively, of $\{p_\alpha : \alpha \in (A, \geq)\}$ and $\{b_\alpha : \alpha \in (A, \geq)\}$, so each converges to $p$. For each $\alpha \in \mathcal{B}$, since $C_\alpha \notin V$ and $p_\alpha \in C_\alpha \cap V$, there is a point $y_\alpha \in C_\alpha \cap \partial(V)$, and since for each $\alpha \in \mathcal{B}$, $\overline{C_\alpha} \cap \overline{H} = b_\alpha \subseteq V$, $y_\alpha \notin \overline{H}$. Since $\partial(V)$ is compact, the net $\{y_\alpha : \alpha \in (\mathcal{B}, \geq)\}$ has a convergent subnet $y_\alpha \to y$, $y \in \partial(V)$. Then $y$ is a limit point of $\bigcup_{\alpha \in \mathcal{B}} C_\alpha$. Now if $y \in \overline{C_\alpha}$ for some $\alpha \in \mathcal{B}$, then $y \in C_\alpha - \overline{H} = \text{Int} C_\alpha$. 
Since $V - \overline{C_p}$ is open, $p \in V - \overline{C_p}$, and the net 
\{b_\alpha : \alpha \in (A, \leq)\} converges to $p$; for some
$Y^* \in A$ if $Y \geq Y^*$, $Y \in A$, then $C_\alpha \neq C_{\overline{a}}$. But there is a
$\exists^* \in A$ such that $\exists^* \geq Y^*$ and $y_{\exists^*} \in C_\alpha$, so $C_{\exists^*} = C_{\overline{a}}$. Since
this is a contradiction, $y \notin \bigcup_{\alpha \in A} C_\alpha$. It follows from the
first part of this proof that $y \in \overline{H}$ and $b_\alpha \rightarrow y$. But
$b_\alpha \rightarrow p$. The lemma follows.

2.5.5 Theorem. Let $M$ be locally compact, $H$ an $H$-set of
$M$, $p \in \overline{H}$, and $V$ an open set containing $p$. Let $C_p =$
\{C : C is a component of $M - H$ and $C \cap \overline{H} = \{p\}$\}. Then all
but a finite number of members of $C_p$ are contained in $V$.

Proof. Let $C_p^* = \{C \in C_p : C \notin V\}$ and suppose $C_p^*$
is infinite. Let $G$ be an open set containing $p$ such that
$\overline{G} \subseteq V$ and $\overline{G}$ is compact. Then for each $C$ in $C_p^*$, there is
a point $y_C \in C \cap \partial(G)$ and $\{y_C : C \in C_p^*\}$ is infinite.
There is a point $y$ in $\partial(G)$ such that $y$ is a limit point of
$\{y_C : C \in C_p^*\}$. Then $y$ is a limit point of $\bigcup C_p^*$ and
$y \notin \bigcup C_p^*$. It follows from 2.5.4 that $y$ is a limit point
of $\bigcup (C \cap \overline{H} : C \in C_p^*) = \{p\}$, so $y = p$. This is a
contradiction.

2.5.6 Corollary. If $M$ is locally compact, $H$ an $H$-set of $M$
and $\mathcal{F}$ is any collection of components of $M - H$ with a
common boundary point, then $\mathcal{F}$ is a null family.

2.5.7 Corollary. If $M$ is locally compact and $\mathcal{C}$ is any
collection of $E_o$-sets of $M$ such that $\bigcap C \neq \emptyset$, then $C$ is a null family.

Proof. Since $\bigcap C \neq \emptyset$, there is a point $p \in M$ such that $\bigcap C = \{p\}$. Let $E^* \in C$. For each $E$ in $C$ such that $E \neq E^*$, $E - E^* = E - p$ is connected so is contained in a component $C_E$ of $M - E^*$. Further, $p$ is a boundary point of $C_E$ for each $E$ in $C$, $E \neq E^*$. By 2.5.6, $\{C_E : E \in C, E \neq E^*\}$ is a null family. It follows that $C$ is a null family.

2.5.8 Theorem. If $M$ is compact, $H$ an $H$-set of $M$, and $\mathcal{C} = \{C : C$ is a component of $M - H\}$, then $C$ is a null family.

Proof. Suppose not. Then there is an open cover $\mathcal{U}$ of $M$ and an infinite collection $\mathcal{V}' \subset \mathcal{V}$ such that no member of $\mathcal{V}'$ is contained in a member of $\mathcal{U}$. Then for each $C \in \mathcal{V}'$, $C$ is nondegenerate and $C \cap H$ is degenerate, so there is a point $p_C \in C - H$. $\{p_C : C \in \mathcal{V}'\}$ is infinite and $M$ is compact, so for some $p \in M$, $p$ is a limit point of $\{p_C : C \in \mathcal{V}'\}$.

If $p \notin H$, then $p$ belongs to $C - H = \text{Int} C$ for some component $C$ of $M - H$. But then $\text{Int} C$ is an open set containing $p$ and meeting $\{p_C : C \in \mathcal{V}'\}$ in at most one point and this is a contradiction. Thus $p \in H$.

Since no member of $\mathcal{V}'$ is contained in a member of $\mathcal{U}$ and $\mathcal{U}$ is an open cover of $M$, it follows from 2.5.5 that
only a finite number of members of \( \mathcal{F}' \) have \( p \) as a boundary point. Thus we may assume that for each \( C \in \mathcal{F}' \), \( p \notin C \). Let \( \{ p_\alpha : \alpha \in (\mathcal{A}, \geq) \} \) be a net in \( \{ p_C : C \in \mathcal{F}' \} \) such that \( p_\alpha \to p \). For each \( \alpha \in (\mathcal{A}, \geq) \), let \( C_\alpha \in \mathcal{F}' \) such that \( p_\alpha \in C_\alpha \). Let \( G \in \mathcal{G} \) such that \( p \in G \). Since for each \( \alpha \in \mathcal{A}, C_\alpha \notin G \), there is a point \( q_\alpha \in C_\alpha - G \). Then \( M, H, \mathcal{F}', p, \{ p_\alpha : \alpha \in (\mathcal{A}, \geq) \} \), and \( \{ q_\alpha : \alpha \in (\mathcal{A}, \geq) \} \) satisfy the conditions of Lemma 2.5.4, so \( q_\alpha \to p \) and therefore \( p \in M - G \). Since this is a contradiction, the theorem follows.

2.5.9 Corollary. If \( M \) is compact and \( A \) is an \( A \)-set of \( M \), then \( \mathcal{F} = \{ C : C \text{ is a component of } M - A \} \) is a null family.

2.5.10 Definition. A nondegenerate continuum \( K \) in a topological space \( S \) is a continuum of convergence iff there is a net \( \{ K_\alpha : \alpha \in (\mathcal{A}, \geq) \} \) of continua such that for each \( \alpha \in \mathcal{A}, K \cap K_\alpha = \emptyset \) and \( K = \lim_\alpha K_\alpha \).

2.5.11 Lemma. If \( K \) is a continuum of convergence in a locally compact Hausdorff space \( S \) and \( \{ K_\alpha : \alpha \in (\mathcal{A}, \geq) \} \) is a net of continua such that \( K \cap K_\alpha = \emptyset \) and \( K = \lim_\alpha K_\alpha \), then the net \( \{ K_\alpha : \alpha \in (\mathcal{A}, \geq) \} \) is almost pairwise disjoint and thus almost distinct.

Proof. We need only note that for each \( \alpha \in \mathcal{A}, M - K_\alpha \) is open, contains \( K \) and the net is eventually in \( M - K_\alpha \).

2.5.12 Lemma. If \( K \) is a continuum of convergence in a
connected $T_1$-space $S$, then every two points of $K$ are conjugate in $S$.

**Proof.** Let $\{K_\alpha : \alpha \in (\mathcal{A}, \geq)\}$ be a net of continua such that $K \cap K_\alpha = \emptyset$ for all $\alpha \in \mathcal{A}$ and $K = \lim_{\alpha} K_\alpha$. If $x, y \in K$ and $t \in E(x, y)$, then there is a separation $(U, V)$ of $S - t$ such that $x \in U$, $y \in V$. There is an $\alpha \in \mathcal{A}$ such that $K_\alpha \cap U \neq \emptyset \neq K_\alpha \cap V$, so $t \in K_\alpha$. This is impossible since $K$ is connected and therefore $t \in K$.

**2.5.13 Theorem.** If $K$ is a continuum of convergence in $M$, $\{K_\alpha : \alpha \in (\mathcal{A}, \geq)\}$ a net of continua such that for each $\alpha \in \mathcal{A}$, $K_\alpha \cap K = \emptyset$ and $K = \lim_{\alpha} K_\alpha$, then there is an $E_0$-set $E$ of $M$ such that $K \subset E$ and $K = \lim_{\alpha} (E \cap K_\alpha)$.

**Proof.** Since $K$ is a continuum of convergence, there is an $E_0$-set $E$ of $M$ such that $K \subset E$. If $k \in \lim_{\alpha} \sup (E \cap K_\alpha)$, then $k \in \lim_{\alpha} \sup K_\alpha$, so $\lim_{\alpha} \sup (E \cap K_\alpha) \subset K$. We show that $K \subset \lim_{\alpha} \inf (E \cap K_\alpha)$.

Suppose not, and let $k \in K - \lim_{\alpha} \inf (E \cap K_\alpha)$. Let $y \in K$ such that $y \notin K$. Then there is an open set $O$ such that $k \in O$, $y \notin \overline{O}$ and $\{\alpha \in \mathcal{A} : K_\alpha \cap E \cap O = \emptyset\}$ is cofinal in $\mathcal{A}$. Let $\mathcal{D} = \{(\alpha, V, W) : \alpha \in \mathcal{A}$ and $K_\alpha \cap E \cap O = \emptyset; V$ is open, $k \in V \subset O$, and $K_\alpha \cap V \neq \emptyset; W$ is open, $y \in W$, and $K_\alpha \cap W \neq \emptyset\}$. It is easy to see that $\mathcal{D} \neq \emptyset$.

Define a relation $>$ on $\mathcal{D}$ by $(\alpha_1, V_1, W_1) > (\alpha_2, V_2, W_2)$ iff $\alpha_1 \geq \alpha_2$, $V_1 \subset V_2$, and $W_1 \subset W_2$. Again it is not difficult to show that $>$ is nonempty and directs $\mathcal{D}$. Also,
if we define for $\delta = (\alpha, V, W) \in \mathcal{B}$, $N(\delta) = \alpha$, then

$$\{K_N(\delta) : \delta \in (\mathcal{B}, >)\}$$

is a subnet of $\{K_\alpha : \alpha \in (\mathcal{A}, \geq)\}$

so

$$\lim_{\delta} K_N(\delta) = K.$$

For each $\delta = (\alpha, V, W) \in \mathcal{B}$, let $x_\delta \in K \cap V$, and $y_\delta \in K_\alpha \cap W$. Then $\{x_\delta : \delta \in (\mathcal{B}, >)\}$ and $\{y_\delta : \delta \in (\mathcal{B}, >)\}$ are nets converging respectively to $k$ and $y$, and for each $\delta \in \mathcal{B}$, $x_\delta \notin E$.

For each $\delta \in \mathcal{B}$, let $C_\delta$ be the component of $M - E$ such that $x_\delta \in C_\delta$, and let $b_\delta = \partial(C_\delta)$. Since $k \in E$ and $k = \lim_{\delta} x_\delta$, $k$ is a limit point of $\bigcup \{C_\delta : \delta \in \mathcal{B}\}$ and $k \notin \bigcup \{C_\delta : \delta \in \mathcal{B}\}$. It follows from 2.5.4 that $k = \lim_{\delta} b_\delta$.

If $\{\delta \in \mathcal{B} : K_N(\delta) \notin C\}$ is not bounded in $\mathcal{B}$, then $y$ is a limit point of $\bigcup \{C_\delta : \delta \in \mathcal{B}\}$ and the net $\{b_\delta : \delta \in (\mathcal{B}, >)\}$ converges to $y$. Since $y \notin k$, this is a contradiction. Thus for some $\delta^* \in \mathcal{B}$, if $\delta > \delta^*$, then $K_N(\delta) \notin C_\delta$. Since for each $\delta \in \mathcal{B}$, $x_\delta \in K_N(\delta) \cap C_\delta$, $b_\delta \in K_N(\delta)$ for each $\delta > \delta^*$. But $b_\delta \in K_N(\delta) \cap E$ and this yields a contradiction since the net $\{b_\delta : \delta \in (\mathcal{B}, >)\}$ is eventually in $0$ and for all $\delta \in \mathcal{B}$, $K_N(\delta) \cap E \cap 0 = \emptyset$. The theorem follows.

2.5.14 Corollary. Any continuum of convergence of $M$ is a continuum of convergence of some single $E$-set of $M$.

2.5.15 Corollary. $M$ has no continuum of convergence iff every cyclic element of $M$ has no continuum of convergence.
2.5.16 Theorem. If $M$ is compact and $\mathcal{C} = \{E \subset M : E$ is an $E_0$-set of $M\}$, then $\mathcal{C}$ is a null family.

Proof. If not, then there is an open cover $\mathcal{U}$ of $M$ and an infinite collection $\mathcal{C}' \subset \mathcal{C}$ such that no member of $\mathcal{C}'$ is contained in a member of $\mathcal{U}$. Further, we may assume that $\mathcal{C}'$ is countable and let $\{E_i : i = 1, 2, \ldots\}$ be an enumeration of $\mathcal{C}'$. Since for each $i$, the components of $M - E_i$ form a null family and each component of $M - E_i$ contains at most one $E_0$-set which meets $E_i$, it follows that for each $i$,

$$\mathcal{C} \cap E_i = \{j : E_i \cap E_j \neq \emptyset\}$$

is finite. Thus we may assume that the sequence $\{E_i : i = 1, 2, \ldots\}$ is pairwise disjoint. Since $M$ is compact and for each $i$, $\text{diam } E_i > \mathcal{U}$, some subnet $\{E_i \beta : \beta \in (\mathcal{U}, >)\}$ converges to a nondegenerate limit continuum $K$. But then $K$ is a continuum of convergence of a single $E_0$-set $E$ and $K = \lim_{\beta} \left( K \cap E_i \right)$ and this is impossible.

2.6 Cyclic chain development theorem. In this section we prove a theorem which is analogous to the Cyclic Chain Approximation Theorem [9 Theorem 7.1, p. 73].

Theorem. If $M$ is compact, then there exist a well-ordered set $(\mathcal{A}, \preceq)$, a net $\{p_\alpha : \alpha \in (\mathcal{A}, \preceq)\}$ of non-cut points of $M$ belonging to nodes of $M$, and a net $\{q_\alpha : \alpha \in (\mathcal{A}, \preceq)\}$ in $M$ such that the net of cyclic chains $\{C(p_\alpha, q_\alpha) : \alpha \in (\mathcal{A}, \preceq)\}$ has the following properties:

a. For each $\alpha \in \mathcal{A}$, $\Pi_\alpha = \bigcup_{\gamma < \alpha} C(p_\gamma, q_\gamma)$ is an
H-set of M.

b. For each $\alpha \in \mathcal{A}$, if $\alpha$ is not the first element of $\mathcal{A}$, then $C(p_\alpha, q_\alpha) \cap \overline{H} = \{q_\alpha\}$. 

c. $M = \bigcup_{\alpha \in \mathcal{A}} C(p_\alpha, q_\alpha)$.

d. For every open cover $\mathcal{U}$ of M, there is an $\alpha_0 \in \mathcal{A}$ such that if $\alpha \geq \alpha_0$ and $C$ is a component of $M - H_\alpha$, then $\text{diam } C < \mathcal{U}$.

Proof. If M has no cut point, we let $\alpha = \{1\}$, and let $p_1$ and $q_1$ be any two distinct points of M. Assume, then, that M has a cut point. Our proof has three steps. We first define the well-ordered set $(\mathcal{A}, \geq)$ and the net $\{p_\alpha : \alpha \in (\mathcal{A}, \geq)\}$. Next, we define the net $\{q_\alpha : \alpha \in (\mathcal{A}, \geq)\}$ by induction on $\mathcal{A}$. Finally, we show that the net of cyclic chains $C(p_\alpha, q_\alpha)$ has the properties a - d.

1. $(\mathcal{A}, \geq)$ and the net $\{p_\alpha : \alpha \in (\mathcal{A}, \geq)\}$. Since M has a cut point, M has at least two nodes. Let $\mathcal{N}$ be the set of all nodes of M and let $(\mathcal{A}, \geq)$ be the set of all ordinals whose cardinal is less than that of $\mathcal{N}$. Let $N^*$ be any (fixed) node of M and let $\{N_\alpha : \alpha \in \mathcal{A}\}$ be an indexing of $\mathcal{N} - \{N^*\}$ by $\mathcal{A}$. For each $\alpha \in \mathcal{A}$, let $p_\alpha$ be a non-cut point of M belonging to $N_\alpha$. Then the net $\{p_\alpha : \alpha \in (\mathcal{A}, \geq)\}$ has been defined.

2. The net $\{q_\alpha : \alpha \in (\mathcal{A}, \geq)\}$. Let $q_1$ be a non-cut point of M belonging to $N^*$. It follows from 2.4.11 that if $\delta \in \mathcal{A}$, $\delta > 1$, then $p_\delta \in C(p_1, q_1)$. In $M - C(p_1, q_1)$,
let $C_2$ be the component which contains $p_2$ and define $q_2 = \exists (C_2)$. Then $H_3 = C(p_1,q_1) \cup C(p_2,q_2)$ is an $H$-set of $M$, and it follows from 2.4.11 that if $\delta > 2$, then $p_\delta \notin C(p_1,q_1) \cup C(p_2,q_2)$. Since $C(p_2,q_2) \subset C_2 \cup q_2$, $C(p_2,q_2) \cap C(p_1,q_1) = \{q_2\}$.

Suppose that for some $\beta \in A$, $\beta \geq 2$, we have defined $q_\alpha$ for each $\alpha \in A$, $\alpha < \beta$ in such a way that if $1 < a$, then

1. $\bigcup_{\gamma < \alpha} C(p_\gamma, q_\gamma)$ is an $H$-set of $M$;
2. $C(p_\alpha, q_\alpha) \cap H_\alpha = \{q_\alpha\}$, $(H_\alpha = \bigcup_{\gamma < \alpha} C(p_\gamma, q_\gamma))$;
3. if $\delta \in A$, $\delta > \alpha$, then $p_\delta \notin \bigcup_{\gamma < \alpha} C(p_\gamma, q_\gamma)$.

It follows easily from hypotheses 1 and 3 that $H_\beta$ is an $H$-set of $M$ and does not contain $p_\beta$. In $M - H_\beta$, let $C_\beta$ be the component which contains $p_\beta$, and let $q_\beta$ be the unique point in $C_\beta \cap H_\beta$. Since both $H_\beta \cup q_\beta$ and $C(p_\beta, q_\beta)$ are $H$-sets of $M$ and $q_\beta$ belongs to each, their union, $\bigcup_{\alpha \leq \beta} C(p_\alpha, q_\alpha)$, is by 2.3.15 an $H$-set of $M$. Also, since $C_\beta \cup q_\beta$ is an $A$-set of $M$ containing $p_\beta$ and $q_\beta$, $C(p_\beta, q_\beta) \cap H_\beta \subset C_\beta \cap H_\beta = \{q_\beta\}$. Further, by 2.4.11, if $\delta \in A$, $\delta > \beta$, then $p_\delta \notin \bigcup_{\alpha \leq \beta} C(p_\alpha, q_\alpha)$. Thus for each $\alpha \in A$, $q_\alpha$ is defined.

3. We now show that $\{C(p_\alpha, q_\alpha) : \alpha \in (A, \geq)\}$ satisfies conditions a - d. It follows from the definition of the net $\{q_\alpha : \alpha \in (A, \geq)\}$ that a and b are satisfied. Also, it is not difficult to show that $H = \bigcup_{\alpha \in A} C(p_\alpha, q_\alpha)$
is an H-set of M. Now if C is a component of M - H, then it follows from 2.4.21 that C contains a point p which belongs to a node of M. But H contains every node of M, and it follows that M = H.

It remains to show that d is satisfied. If A is finite, the result is immediate since then A has a maximum. Suppose, then, that A is infinite. Then A has no maximum. Suppose further that there is an open cover \( \mathcal{U} \) of M such that for each \( a \in A \) there is a component \( R_a \) of \( M - H \) such that diam \( R_a > \mathcal{U} \). For each \( a \in A \), let \( a_a \in R_a \). Then \( \{a_a : a \in (A, \geq)\} \) is a net in M so there is a point \( a \in M \) and a subnet \( \{a_\beta^\alpha : \beta \in (B, \gg)\} \) converging to a. Let \( G \in \mathcal{U} \) such that \( a \in G \). Now for all \( \beta \in B \), there is a point \( b_\beta \in R_a - G \) and a subnet \( \{b_\beta^\delta : \delta \in (\delta, \lambda)\} \) converging to \( b \in M \).

Now for some \( a^* \in A \), \( a, b \in H_{a^*} \), and if \( a > a^* \), \( a \in A \), then \( M - H_a \subset M - H_{a^*} \) and \( R_a \) is contained in some component of \( M - H_{a^*} \). Since only a finite number of components of \( M - H_{a^*} \) have diameter > \( \mathcal{U} \) and every \( R_a \) is contained in such a component for \( a > a^* \), it follows that for some component \( C \) of \( M - H_{a^*} \), \( \{\delta \in (\delta, \gg) : R_{a_\beta^\delta} \subset C\} \) is cofinal in \( (\delta, \gg) \). But then both a and b are limit points of C, so \( a, b \in \overline{C} \cap \overline{H_{a^*}} \). This is a contradiction.

It follows that if \( \mathcal{U} \) is any open cover of M, then for some \( a_\mathcal{U} \in A \), every component of \( M - H_{a_\mathcal{U}} \) has diameter < \( \mathcal{U} \), so if \( a > a_\mathcal{U} \) then every component of \( M - H_a \) has diameter
2.7 Cyclicly extensible and cyclicly reducible properties.

A property $P$ is said to be cyclicly extensible iff whenever each cyclic element of a space has property $P$ then the whole space has property $P$. $P$ is cyclicly reducible iff whenever a space $S$ has property $P$, then every cyclic element of $S$ has property $P$. Several properties are known to be cyclicly extensible and/or reducible for semi-locally connected metric continua, among them the property of having no continuum of convergence, unicoherence and the fixed point property.

We have established in Corollary 2.5.15 that the property of having no continuum of convergence is both cyclicly extensible and reducible for any connected and locally connected Hausdorff space $M$. Minear has established, [8], that unicoherence is both cyclicly extensible and reducible for a class of spaces that includes the class of connected and locally connected Hausdorff spaces and that the fixed point property is cyclicly reducible for all connected and locally connected spaces and is cyclicly extensible for a class of locally connected continua which includes locally connected Hausdorff continua. In this section we consider the cyclic extensibility and reducibility of two classes of properties related to unicoherence.

Notation. Let $A$ be an $A$-set of $M$ and let $S \subseteq A$. Let $S'$ denote the union of all components $C$ of $M - A$ such that
\(\partial(C) \in S\), and let \(S^* = S \cup S'\).

We note that if \(S\) is a subset of an \(A\)-set \(A\) of \(M\), then \(S'\) is an open subset of \(M\). The next lemma is due to Minear, [8, p. 19].

2.7.1 Lemma. Let \(A\) be an \(A\)-set of \(M\) and \(S \subseteq A\). If \(S\) is closed in \(A\), then \(S^*\) is closed in \(M\). If \(S\) is open in \(A\), then \(S^*\) is open in \(M\).

2.7.2 Theorem. If \(A\) is an \(A\)-set in \(M\) and \(A = S \cup T\) is a division of \(A\) into sets, (closed sets), (connected sets), then there is a division \(M = L \cup N\) of \(M\) into sets, (closed sets), (connected sets) such that \(L \cap N = S \cap T\), \(S \subseteq L\), \(T \subseteq N\). Further, if \(M\) is a continuum and \(S\) and \(T\) are continua, then \(L\) and \(N\) are continua.

Proof. Let \(L = S^*\), \(N = T \cup (A - S)^*\). Then \(M = L \cup N\), \(S \subseteq L\), and \(T \subseteq N\), so \(S \cap T \subseteq L \cap N\). If \(x \in L \cap N\), then if \(x \notin A\) there is a component \(C\) of \(M - A\) such that \(x \in C\).

Since \(x \in L\), \(\partial(C) \in S\). But since \(x \in N\), \(\partial(C) \in A - S\). It follows that \(x \in A\). Since \(x \in L\), \(x \in S\), and since \(x \in N\), \(x \in T\). Thus \(L \cap N = S \cap T\).

It is immediate that if \(S\) and \(T\) are connected then \(L\) and \(N\) are connected.

Now if \(S\) and \(T\) are closed, then by 2.7.1, \(L\) is closed. Now, \(M - N = (A - T)^* \cup (S \cap T)'\), and by 2.7.1 and the definition of \((S \cap T)'\), each of \((A - T)^*\) and \((S \cap T)'\) is open in \(M\) and it follows that \(N\) is closed.
It is now immediate that if \( M, S, \) and \( T \) are continua, then \( L \) and \( N \) are continua.

2.7.3 Theorem. Let \( A \) be an \( A \)-set of \( M, S \subset A, \) and \( C \) a collection of components of \( M - A \) such that if \( C \in C, \) then \( \partial(C) \subset S. \) If \( S \) is locally connected, then \( S \cup \bigcup C \) is locally connected.

Proof. Suppose \( S \) is locally connected and let \( K = S \cup \bigcup C. \) Suppose also that \( p \in K \) and \( O^* = 0 \cap K \) is an open set in \( K \) containing \( p, \) \( O \) open in \( M. \) If \( p \in \text{Int}_K S \) or \( p \in C \) for \( C \in C, \) then since \( \text{Int}_K S \) and \( C \) are locally connected, \( K \) is locally connected at \( p. \) Assume then that \( p \in \partial_K(S). \) Since \( O \cap S \) is open in \( S, \) there is an open set \( V \) of \( M \) such that \( p \in V, V \cap S \subset O \cap S \subset O^* \) and \( V \cap S \) is connected.

Now \( V \cap S = V \cap (\text{Int}_K S) \cup (V \cap \partial_K(S)) \) and \( V \cap \partial_K(S) \subset V \cap \partial(A). \) For each \( x \in V \cap \partial_K(S), \) let \( G_x \) be a connected open subset of \( M \) such that \( x \in G_x \subset V \cap O. \) Then for each \( x \) in \( V \cap \partial_K(S), G_x \cap K \subset V \cap O \cap K \subset O^*. \) Let \( G = (V \cap S) \cup \bigcup \{ G_x \cap K \}. \) Then \( G \subset O^*, p \in G, \) and it is easy to show that \( G \) is connected and open in \( K. \)

2.7.4 Corollary. If \( A \) is an \( A \)-set of \( M \) and \( S \) is a locally connected subset of \( A, \) then \( S^* \) is locally connected.

2.7.5 Corollary. If \( A \) is an \( A \)-set of \( M, A = S \cup T \) a division of \( A \) into locally connected sets and \( L = S^*, \)
Let \( M = L \cup N \) be a division of \( M \) into locally connected sets \( L, N \) such that \( L \cap N = S \cap T \).

We now define a class of properties \( \{P_n : n \text{ an integer, } n \geq -1\} \) and show that each property \( P_n \) is both cyclicly extensible and reducible for all connected, locally connected, and locally compact Hausdorff spaces. We note that our definition is analogous to a definition given by Vietoris in 1932, [10, p. 273]. Apparently little is known concerning the properties Vietoris defined. The difference between our definition and that of Vietoris is in our insistence on local connectedness for \( n \geq 0 \).

2.7.6 Definition. A space \( S \) has property \( P_{-1} \) iff \( S \) is nonempty. \( S \) has property \( P_n, n > -1 \), iff \( S \) is locally connected, has property \( P_{n-1} \), and whenever \( S = A \cup B \) is a division of \( S \) into closed sets each having property \( P_{n-1} \), then \( A \cap B \) also has property \( P_{n-1} \).

We note that property \( P_0 \) is connectedness plus local connectedness.

2.7.7 Theorem. If \( M \) is locally compact, then for each nonnegative integer \( n \), if \( M \) has property \( P_{n-1} \), then \( M \) has property \( P_n \) iff every cyclic element of \( M \) has property \( P_n \).

Proof. We state an induction hypothesis \( I(n) \): If \( M \) is locally compact, then,
a. if $M$ has property $P_n$, then every cyclic element of $M$ has property $P_n$;

b. if $M$ has property $P_{n-1}$ and every cyclic element of $M$ has property $P_n$, then $M$ has property $P_n$;

c. if $M$ has property $P_n$, then every $A$-set of $M$ has property $P_n$; and if $A$ is an $A$-set of $M$ and $Z$ is any locally compact subset of $M$ such that $A \cap Z \neq \emptyset$ and $Z$ has property, $P_n$, then $A \cap Z$ has property $P_n$;

d. if $M$ has property $P_n$, $A$ is an $A$-set of $M$, and $A = S \cup T$ is a division of $A$ into closed sets each having property $P_n$, then if $L = S^*$ and $N = T \cup (A - S)^*$, $M = L \cup N$ is a division of $M$ into closed sets each having property $P_n$.

$I(0)$-a, c, d have already been established, and $I(0)$-b is trivial.

Assume that $I(n-1)$ has been established, $n \geq 1$.

Suppose that $M$ has property $P_n$ and that $E$ is an $E_0$-set of $M$. Then $M$ has property $P_{n-1}$, so by $I(n-1)$, $E$ has property $P_{n-1}$.

Suppose $E = S \cup T$ is a division of $E$ into closed sets each having property $P_{n-1}$. Let $L = S^*$, $N = T \cup (E - S)^*$. Then by $I(n-1)$, $M = L \cup N$ is a division of $M$ into closed sets each having property $P_{n-1}$. Since $M$ has property $P_n$, $L \cap N$ has property $P_{n-1}$, and we have established earlier (Theorem \ref{thm:E0-set}) that $L \cap N = S \cap T$. Thus $E$ has property $P_n$, and it follows that every cyclic element of $M$ has property $P_n$.

Thus $I(n-1)$ implies $I(n)$-a.
Now assume that $M$ has property $P_{n-1}$ and that every cyclic element of $M$ has property $P_n$. Suppose $M = S \cup T$ is a division of $M$ into closed sets each having property $P_{n-1}$. Then each of $S$ and $T$ is connected and locally connected.

We show first that $S \cap T$ is also connected and locally connected.

Suppose that $S \cap T$ is not connected and that $S \cap T = W \cup Z$, $W$, $Z$ disjoint closed sets. If $E$ is an $E_o$-set of $M$, then $E = (E \cap S) \cup (E \cap T)$ and since $S$ and $T$ are connected and locally connected, $E \cap S$ and $E \cap T$ are connected and locally connected. Since $E$ has property $P_n$, $E \cap S \cap T$ is connected. Thus $E \cap S \cap T \in W$ or $E \cap S \cap T \in Z$. It follows that no $E_o$-set of $M$ meets both $W$ and $Z$. Let $w \in W$, $z \in Z$. Since no $E_o$-set of $M$ meets both $W$ and $Z$, $E(w,z) \neq \emptyset$. Since $w, z \in S \cap T$, and $S$ and $T$ are each connected, $E(w,z) = S \cap T$.

Let $t_1$ be the last point in $(E(w,z) \cup \{w,z\}) \cap W$ and $t_2$ the first point in $(E(w,z) \cup \{w,z\}) \cap Z$. Then $t_1 \neq t_2$ and $t_1, t_2$ are conjugate in $M$. But then $C(t_1, t_2)$ is an $E_o$-set of $M$ meeting both $W$ and $Z$. Since this is a contradiction, $S \cap T$ is connected. Since $S \cap T$ is closed in $M$, $S \cap T$ is locally compact.

Suppose now that $S \cap T$ is not locally connected. Then there is an open set $O$ of $M$ and a point $p \in S \cap T$ such that $p$ lies on a continuum of convergence $D = \lim_{\alpha} D_\alpha$, where for each $\alpha$, $D_\alpha$ is the closure of a component $C_\alpha$ of $O \cap S \cap T$ and the components of $O \cap S \cap T$ containing $D$ and $D_\alpha$ are
distinct. Then $D$ is a continuum of convergence of $M$ so there is an $E_0$-set $E$ of $M$ such that $D = \lim_{\alpha} E \cap D_\alpha$. This implies that $E \cap S \cap T$ is nondegenerate. Then $E \cap S$ and $E \cap T$ are connected and locally connected and for each $\alpha$, $E \cap D_\alpha$ is contained in a component of $O \cap S \cap T \cap E$ distinct from the component of $O \cap S \cap T \cap E$ containing $E \cap D$. But this implies that $E \cap S \cap T$ is not locally connected. This is a contradiction, since $E$ has property $P_n$. Thus $S \cap T$ is locally connected.

We have established that $S \cap T$ is a connected, locally connected and locally compact Hausdorff space. Let $E$ be an $E_0$-set of $S \cap T$. Then $E \subset E^*$ for some $E_0$-set $E^*$ of $M$.

Since $M$ has property $P_{n-1}$, $I(n-1)$ implies that each of $E^*$, $S \cap E^*$, and $T \cap E^*$ has property $P_{n-1}$. Since $E^*$ has property $P_n$, $E^* \cap S \cap T$ has property $P_{n-1}$. Now $E \subset E^* \cap S \cap T$ and is an $E_0$-set of $E^* \cap S \cap T$, so by $I(n-1)$, $E$ has property $P_{n-1}$. It follows that every cyclic element of $S \cap T$ has property $P_{n-1}$, so again by the induction hypothesis, $S \cap T$ has property $P_{n-1}$. Thus $M$ has property $P_n$ and $I(n)-b$ is established.

Now assume that $M$ has property $P_n$ and that $A$ is an $A$-set of $M$. By $I(n-1)$, $A$ has property $P_{n-1}$ (since $M$ does). If $E$ is a cyclic element of $A$, then $E$ is a cyclic element of $M$ and we have shown that $I(n-1)$ implies $I(n)-a$. Thus $E$ has property $P_n$. Thus every cyclic element of $A$ has property $P_n$ and since $I(n-1)$ implies $I(n)-b$, $A$ has property $P_n$. Now
let Z be any locally compact subset of M such that \( A \cap Z \neq \emptyset \) and Z has property \( P_n \). Since \( A \cap Z \) is an A-set of Z, it follows from what we have just proved that \( A \cap Z \) has property \( P_n \).

Finally, assume that M has property \( P_n \), that A is an A-set of M and that \( A = S \cup T \) is a division of A into closed sets each having property \( P_n \). Let \( L = S^* \), \( N = T \cup (A - S)^* \). Then by I(n-1), L and N each have property \( P_{n-1} \). Let E be an \( E_o \)-set of L. Then either \( E \subset A \) or \( E \subset C \) for some component C of \( M - A \) such that \( C \subset L \). If the latter, then E is an \( E_o \)-set of C and therefore of M, and it follows from what has already been proved that E has property \( P_n \). If \( E \subset A \), let \( E^* \) be the \( E_o \)-set of M such that \( E = E^* \). Then \( E^* \) has property \( P_n \). Further, since S is a locally compact subset of M and has property \( P_n \), it again follows from what has already been proved that \( S \cap E^* \) has property \( P_n \). Now since E is an \( E_o \)-set of \( S \cap E^* \), E has property \( P_n \). Thus every cyclic element of L has property \( P_n \), so L has property \( P_n \). Since the case of N is similar, the theorem is proved.

2.7.8 Corollary. If M is locally compact, then for every nonnegative integer n, if M has property \( P_n \), then every A-set of M has property \( P_n \) and if A is an A-set of M, Z a locally compact subset of M such that \( A \cap Z \neq \emptyset \) and Z has property \( P_n \), then \( A \cap Z \) has property \( P_n \).

2.7.9 Corollary. If M is locally compact then for every
nonnegative integer \( n \), if \( M \) has property \( P_n \) and \( S \cup T = A \) is a division of an \( A \)-set \( A \) of \( M \) into closed sets having property \( P_n \), \( L = S^* \), \( N = T \cup (A - S)^* \), then \( M = L \cup N \) is a division of \( M \) into closed sets having property \( P_n \).

2.7.10 Corollary. If \( M \) is locally compact and \( n \) is an integer, \( n \geq -1 \), then \( M \) has property \( P_n \) iff every cyclic element of \( M \) has property \( P_n \).

**Proof.** If \( n = -1 \) or \( n = 0 \), the result is immediate, so we may assume that \( n > 0 \). If \( M \) has property \( P_n \), then by 2.7.7, every cyclic element of \( M \) has property \( P_n \). Suppose that every cyclic element of \( M \) has property \( P_n \) and \( M \) does not have property \( P_n \). Since \( M \) is connected and locally connected, \( M \) has property \( P_0 \). Let \( n^* \) be the first integer (\( n^* \geq -1 \)) such that \( M \) does not have property \( P_{n^*} \). Then \( 0 < n^* \leq n \) and \( M \) has property \( P_{n^* - 1} \). Since every cyclic element of \( M \) has property \( P_n \), every cyclic element of \( M \) has property \( P_{n^*} \), and it follows from 2.7.7 that \( M \) has property \( P_{n^*} \), contrary to the definition of \( n^* \). The corollary follows.

2.7.11 Corollary. Every dendron has property \( P_n \) for every nonnegative integer \( n \).

The next definition is due to W.R. Transue, [9, p. 2].

**Definition.** If \( S \) is a topological space

1) \( S \) is \((-1)\)-coherent if \( S \) is nonempty.

2) \( S \) is \( k\)-coherent if \( S \) is \((k-1)\)-coherent and
locally \((k-1)\)-coherent and whenever \(S\) is written as the union of two closed \((k-1)\)-coherent subsets \(A\) and \(B\) then \(A \cap B\) is \((k-1)\)-coherent.

3) \(S\) is **locally \(k\)-coherent** at the point \(p\) of \(S\) provided that if \(U\) is an open set containing \(p\), there is a \(k\)-coherent open set \(V\) lying in \(U\) and containing \(p\).

4) \(S\) is **locally \(k\)-coherent** if \(S\) is locally \(k\)-coherent at each of its points.

We note that 0-coherence is connectedness and 1-coherence is unicoherence plus local connectedness. Thus a space may be 0-coherent and not have property \(P_0\). Clearly if a space has property \(P_1\), it is 0-coherent. Also, as we have shown, every dendron has property \(P_n\) for every non-negative integer \(n\), and we shall show that every dendron also is \(k\)-coherent for every nonnegative integer \(k\). We leave undecided the question of the general relationship of the properties \(P_n\) and \(k\)-coherence for \(n, k \geq 1\).

2.7.12 Theorem. If \(M\) is unicoherent and \(H\) is an \(H\)-set of \(M\), then \(H\) is unicoherent.

**Proof.** \(H\) is a connected and locally connected Hausdorff space. If \(E\) is a cyclic element of \(H\), then since \(E\) is a cyclic element of \(M\) and unicoherence is cyclicly reducible, \(E\) is unicoherent. Since unicoherence is cyclicly extensible, \(H\) is unicoherent.
2.7.13 Theorem. If $H$ is an $H$-set of $M$ and $Z$ is locally connected and unicoherent subset of $M$ such that $H \cap Z \neq \emptyset$, then $H \cap Z$ is locally connected and unicoherent.

Proof. By 2.3.11, if $Z$ is locally connected, $H \cap Z$ is locally connected. If $H \cap Z$ is degenerate, there is nothing to prove. If $H \cap Z$ is nondegenerate and $Z$ is unicoherent and locally connected, then by 2.3.26, $H \cap Z$ is an $H$-set in $Z$, so by 2.7.12, $H \cap Z$ is unicoherent.

2.7.14 Theorem. Let $A$ be an $A$-set of $M$. If $M$ is unicoherent, $A = S \cup T$ a division of $A$ into closed unicoherent sets and $L = S^*, N = T \cup (A - S)^*$, then $L$ and $N$ are unicoherent.

Proof. Suppose $L = B \cup C$, $B, C$ closed and connected. Then $B \cap A = B \cap S$ and $C \cap A = C \cap S$ and each is a closed and connected subset of $S$.

If $B \cap S = \emptyset$, then $B \subset D$ for some component $D$ of $M - A$, $D \subset L$, and $C \cap \overline{D} \neq \emptyset$. Then since $\overline{D}$ is an $A$-set of $M$ and is therefore unicoherent by 2.7.12 and $\overline{D} = (\overline{D} \cap B) \cup (\overline{D} \cap C)$ is a division of $\overline{D}$ into closed connected sets, $\overline{D} \cap B \cap C = B \cap C$ is connected.

If both $B$ and $C$ meet $S$, then $S = (B \cap S) \cup (C \cap S)$ is a division of $S$ into nonempty, closed, connected sets, so if either $B$ or $C$ is contained in $S$, then $B \cap C$ is connected.

Suppose, then, that $B$ and $C$ each meets both $S$ and $L - S$. Suppose, further, that $W, Z$ are separated sets such that $B \cap C = W \cup Z$. Since $B \cap C \cap S$ is connected, we may assume
that $B \cap C \cap S \subset Z$. Suppose $w \in W$. Then $w \notin A$, so $w \in D_w$ for some component $D_w$ of $M - A$. Further, 
\[ \exists (C_w) \in B \cap C \cap S \subset Z, \text{ so } \overline{D_w} \cap W \neq \emptyset \neq \overline{D_w} \cap Z. \]
But $\overline{D_w} \cap B$ and $\overline{D_w} \cap C$ are closed and connected subsets of the unicoherent set $\overline{D_w}$ and $\overline{D_w} = (\overline{D_w} \cap B) \cup (\overline{D_w} \cap C)$. It follows that $W = \emptyset$ and $B \cap C$ is connected. Thus $L$ is unicoherent.
Similarly, $N$ is unicoherent.

2.7.15 Theorem. $M$ is 1-coherent iff every cyclic element of $M$ is 1-coherent.

Proof. If $M$ is 1-coherent, then since every cyclic element of $M$ is locally connected and unicoherence is cyclicly reducible, every cyclic element of $M$ is 1-coherent.

If every cyclic element of $M$ is 1-coherent, then every cyclic element of $M$ is unicoherent. Since unicoherence is cyclicly extensible, $M$ is unicoherent. By assumption, $M$ is locally connected, so $M$ is 1-coherent.

2.7.16 Theorem. If $M$ is locally compact, then for each non-negative integer $k$, if $M$ is $(k-1)$-coherent and locally $(k-1)$-coherent, then $M$ is $k$-coherent iff every cyclic element of $M$ is $k$-coherent.

Proof. As in the proof of 2.7.7, we proceed by induction on $k$. Let $H(k)$ be the following statement:

If $M$ is locally compact, then

a. if $M$ is $k$-coherent, then every cyclic element of $M$ is $k$-coherent;
b. if $M$ is $(k-l)$-coherent and locally $(k-l)$-coherent and every cyclic element of $M$ is $k$-coherent, then $M$ is $k$-coherent;

c. if $M$ is $k$-coherent, then every $A$-set of $M$ is $k$-coherent, and if $A$ is an $A$-set of $M$ and $Z$ is any locally compact $k$-coherent subset of $M$ such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is $k$-coherent;

d. if $M$ is $k$-coherent, $A$ an $A$-set of $M$, $A = S \cup T$ a division of $A$ into closed $k$-coherent sets and $L = S^*$, $N = T \cup (A - S)^*$, then $L$ and $N$ are each $k$-coherent.

$H(0)$-a,c,d have been proved elsewhere and $H(0)$-b is trivial. Also, $H(1)$-a and $H(1)$-b are Theorem 2.7.15 and $H(1)$-c and $H(1)$-d follow immediately from 2.7.12-2.7.14.

Suppose then that $H(k-l)$ has been proved for $k \geq 2$. Suppose further that $M$ is $k$-coherent and $E$ is an $E_o$-set of $M$. Since $M$ is $k$-coherent, $M$ is $(k-l)$-coherent, so by $H(k-l)$, $E$ is $(k-l)$-coherent. Now if $x \in E$ and $O$ is an open set in $E$ containing $x$, then there is an open set $O^*$ of $M$ such that $O = O^* \cap E$. Since $M$ is locally $(k-l)$-coherent, there is a set $V^*$, open in $M$, such that $x \in V^* \subseteq O^*$ and $V^*$ is $(k-l)$-coherent. By $H(k-l)$, since $V^*$ is locally compact, $V^* \cap E$ is an open $(k-l)$-coherent subset of $E$. Thus $E$ is locally $(k-l)$-coherent. If $E = S \cup T$ is a division of $E$ into closed $(k-l)$-coherent sets, let $L = S^*$, $N = T \cup (A - S)^*$. The rest of the proof that $H(k-l)$ implies
H(k)-a is similar to the corresponding part of the proof of 2.7.7.

Assume now that M is (k-1)-coherent and locally (k-1)-coherent, that every cyclic element of M is k-coherent, and that M = S u T is a division of M into closed (k-1)-coherent sets. Since k ≥ 2, S and T are each connected and locally connected and S ∩ T is connected. By argument similar to that in the corresponding part of the proof of 2.7.7, S ∩ T is locally connected. The rest of the argument that S ∩ T is (k-1)-coherent is also similar to that in 2.7.7 that S ∩ T has property P_{n-1}.

Suppose that M is k-coherent and A is an A-set of M. Then by H(k-1), A is (k-1)-coherent. That A is locally (k-1)-coherent also follows from H(k-1) and the fact that M is locally (k-1)-coherent. Again the rest of the proof that A is k-coherent is similar to the corresponding part of 2.7.7. A similar statement holds for the second part of H(k)-c.

Finally, suppose that M is k-coherent, A an A-set of M, A = S u T a division of A into closed, k-coherent sets, and L = S*, N = T u (A - S)*. By H(k-1), L and N are (k-1)-coherent. We show that L is locally (k-1)-coherent.

Let x ∈ L. If x ∈ C for C a component of M - A, then C ⊂ L and since C is open in M and M is locally (k-1)-coherent, L is locally (k-1)-coherent at x. If x ∈ Int_L S, then since S is locally (k-1)-coherent, again L is locally
(k-l)-coherent at x. Suppose, then, that x ∈ 3_L(S) and 0* is a set open in L such that x ∈ 0*. Let O be open in M such that 0* = O ∩ L. Since S is locally (k-l)-coherent and x ∈ O ∩ S, there is an open set V of M such that V ∩ S ⊂ O ∩ S, x ∈ V and V ∩ S is (k-l)-coherent. For each y ∈ V ∩ 3_L(S), let W_y be an open (k-l)-coherent subset of M such that y ∈ W_y ⊂ V ∩ O and let 
W* = (V ∩ S) ∪ ∪_{y ∈ V ∩ 3_L(S)} (W_y ∩ L). Then W* is an open connected set in L. Since L is locally connected and locally compact, W* is also locally connected and locally compact. Let E be an E_o-set of W* and let E* be the E_o-set of M such that E ⊂ E*. It follows from what has already been proved, that E* is k-coherent. Now E ⊂ S or E ⊂ C for some component C of M - A, C ⊂ L. If E ⊂ S, then since V ∩ S is (k-l)-coherent, H(k-l) implies that E* ∩ V ∩ S is (k-l)-coherent. Further, E is an E_o-set of E* ∩ V ∩ S, so again by H(k-l), E is (k-l)-coherent. If E ⊂ C, C a component of M - A, then 3(C) ⊂ 3(S) ∩ V and W_3(C) is (k-l)-coherent; so since C is an A-set, W_3(C) ∩ C is (k-l)-coherent. But E is an E_o-set of E* ∩ W_3(C) ∩ C, so H(k-l) implies that E is (k-l)-coherent. It follows that every cyclic element of W* is (k-l)-coherent, so by H(k-l), W* is (k-l)-coherent. Thus L is locally (k-l)-coherent. The proof that N is locally (k-l)-coherent is similar. Again the rest of the proof that H(k-l) implies H(k)-d is similar to the proof in 2.7.7, I(k-l) implies I(k)-d.
2.7.17 **Corollary.** If M is locally compact, then for every nonnegative integer k, if M is k-coherent then every A-set of M is k-coherent. Further, if M is k-coherent, A an A-set of M, and Z is any locally compact k-coherent subset of M such that $A \cap Z \neq \emptyset$, then $A \cap Z$ is k-coherent.

2.7.18 **Corollary.** If M is locally compact, then for every nonnegative integer k, if M is k-coherent, A an A-set of M, $A = S \cup T$ a division of A into closed k-coherent sets, and $L = S^*$, $N = T \cup (A - S)^*$, then L and N are each k-coherent.

2.7.19 **Theorem.** Every dendron is k-coherent for every nonnegative integer k.

**Proof.** Suppose D is a dendron such that for some nonnegative integer k, D is not k-coherent. Let $k^*$ be the first nonnegative integer such that D is not $k^*$-coherent. Then $k^* \geq 2$, and D is $(k^*-1)$-coherent. Let $x \in D$ and let $O$ be an open set containing $x$. There is an open set $V$ containing $x$ such that $V$ is $(k^*-2)$-coherent. Since D is $(k^*-1)$-coherent, $V$ is locally $(k^*-2)$-coherent. Then $V$ is a connected, locally connected and locally compact Hausdorff space that is $(k^*-2)$-coherent and locally $(k^*-2)$-coherent. Since $V$ contains no $E_0$-set, every cyclic element of $V$ is $(k^*-2)$-coherent. Thus $V$ is $(k^*-1)$-coherent; so D is locally $(k^*-1)$-coherent. It follows that D is $k^*$-coherent, contrary to the definition of $k^*$. 
CHAPTER III:

PRODUCTS OF ARCWISE CONNECTED SPACES

3.1 Definitions. By an arc, we mean a Hausdorff continuum \( A \) with at most two non-cut points, called the end points of \( A \). A space \( S \) is said to be arcwise connected iff whenever \( x, y \in S \), then \( x \) and \( y \) are the end points of some arc in \( S \).

NOTE: All spaces considered in this chapter are assumed to be Hausdorff.

3.2 Lemma. If \( A, B \) are arcs in a space \( S \) with end points \( a_1, a_2 \) and \( b_1, b_2 \), respectively, and \( A \cap B \) consists of a single point which is an end point of each of \( A \) and \( B \), then \( A \cup B \) is an arc. Further, the end points of \( A \cup B \) are the end points of \( A \) and \( B \) that are not in \( A \cap B \).

Proof. We may suppose that \( a_1 = b_1 \). Then \( A - a_2 \) is connected and \( a_2 \notin B \). Since \( a_1 \in (A - a_2) \cap B \),
\[
(A \cup B) - a_2 = (A - a_2) \cup B \text{ is connected. Thus } a_2 \text{ is not a cut point of } A \cup B. \text{ Similarly, } b_2 \text{ is not a cut point of } A \cup B. \text{ Further, } ((A - a_1), (B - a_1)) \text{ is a separation of } (A \cup B) - a_1, \text{ so } a_1 \text{ is a cut point of } A \cup B. \text{ If } x \text{ is a cut point of } A, \text{ then } x \notin B. \text{ If } (U, V) \text{ is a separation of } A - x \text{ such that } a_1 \in V, \text{ then } (U, V \cup B) \text{ is a separation of } (A \cup B) - x, \text{ so } x \text{ is a cut point of } A \cup B. \text{ Similarly, every cut point of } B \text{ is a cut point of } A \cup B. \text{ It follows, since } A \cup B \text{ is a Hausdorff continuum, that } A \cup B \text{ is an arc with
end points $a_2$ and $b_2$.

3.3 Lemma. Let $\{X_\alpha : \alpha \in (\mathcal{A}, \succeq)\}$ be a collection of non-degenerate arcs indexed by a well-ordered set $(\mathcal{A}, \succeq)$, and for each $\alpha \in \mathcal{A}$, let $a_\alpha, b_\alpha$ be the end points of $X_\alpha$. Let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ be the product space of the collection, and for each $\alpha \in \mathcal{A}$, let $p_\alpha$ be the projection map of $X$ onto $X_\alpha$.

We note that what follows is analogous to identifying the edges $[(0,0,0), (0,0,1)], [(0,0,1), (0,1,1)]$ and $[(0,1,1), (1,1,1)]$ of the cube $[0,1] \times [0,1] \times [0,1]$.

Let $f, g$ be points of $X$ such that for each $\alpha \in \mathcal{A}$, $f(\alpha) = a_\alpha$ and $g(\alpha) = b_\alpha$. For each $\alpha \in \mathcal{A}$, define the "edge" $A_\alpha \subset X$ and points $f_\alpha$ and $g_\alpha$ of $X$ as follows:

$$
A_\alpha = \{ h \in X : h(\beta) = b_\beta, \beta < \alpha ; h(\beta) = a_\beta, \alpha < \beta \},
$$

$$
f_\alpha(\beta) = \begin{cases} 
  b_\beta, & \beta < \alpha \\
  a_\beta, & \beta \geq \alpha 
\end{cases}
$$

$$
g_\alpha(\beta) = \begin{cases} 
  b_\beta, & \beta \leq \alpha \\
  a_\beta, & \beta > \alpha 
\end{cases}
$$

Then for each $\alpha \in \mathcal{A}$, the following statements are satisfied:

a. $f_1 = f$, (where "1" denotes the first element of $\mathcal{A}$, and "$\alpha + 1$" the successor of $\alpha$ in $\mathcal{A}$), and

$$
\exists_{\alpha} = f_\alpha + 1;
$$

b. $A_\alpha \cap A_\alpha + 1 = g_\alpha$;
c. $A_\alpha$ is an arc homeomorphic to $X_\alpha$ in $X$ and the end points of $A_\alpha$ are $f_\alpha$ and $g_\alpha$;

d. if $\alpha + 1 < \gamma$, $\gamma \in \mathcal{A}$, then $A_\alpha \cap A_\gamma = \emptyset$; and

e. $(\bigcup_{\beta < \alpha} A_\beta) \cup \{f_\beta\}$ is an arc in $X$ with end points $f$ and $f_\alpha$.

Further, $(\bigcup_{\alpha \in \mathcal{A}} A_\alpha) \cup \{g\}$ is an arc in $X$ with end points $f$ and $g$.

**Proof.** It is immediate from the definitions that $a$ and $b$ are satisfied. If we define a function $\theta$ on $X_\alpha$ by

$$[\theta(x)](\delta) = \begin{cases} b_\delta, & \delta < \alpha \\ x, & \delta = \alpha \\ a, & \delta > \alpha, \end{cases}$$

then $\theta$ is a homeomorphism from $X_\alpha$ onto $A_\alpha$ such that $\theta(a_\alpha) = f_\alpha$, and $\theta(b_\alpha) = g_\alpha$, so $c$ is satisfied. Now if $\gamma \in \mathcal{A}$, $\alpha + 1 < \gamma$, and $h \in A_\alpha$, then $h(\alpha + 1) = a_\alpha + 1 \neq b_\alpha + 1$. If $h \in A_\gamma$, then $h(\alpha + 1) = b_\alpha + 1$, and it follows that $A_\alpha \cap A_\gamma = \emptyset$.

We now proceed to prove $e$ by induction on the well-ordered set $\mathcal{A}$. Let $I(\beta)$ be the statement:

$$(\bigcup_{\alpha < \beta} A_\alpha) \cup \{f_\beta\}$$

is an arc in $X$ with end points $f$ and $f_\alpha$.

If $\beta = 1$, then $\bigcup_{\alpha < 1} A_\alpha = \emptyset$, so $(\bigcup_{\alpha < 1} A_\alpha) \cup \{f_1\} = \{f\}$ is an (degenerate) arc with end points $f$ and $f_1$. Suppose that for some $\beta \in \mathcal{A}$, that we have shown that $I(\alpha)$ holds for all $\alpha < \beta$. We consider two cases.
Case 1. $\beta$ has an immediate predecessor $\gamma$ in $A$, so $\beta = \gamma + 1$. By the induction hypothesis, $(\bigcup_{\alpha < \gamma} A_\alpha) \cup \{f_\gamma\}$ is an arc in $X$ with end points $f$ and $f_\gamma$, and we have shown that $A_\gamma$ is an arc in $X$ with end points $f_\gamma$ and $g_\gamma = f_\gamma + 1 = f_\beta$. If $h \in (\bigcup_{\alpha < \gamma} A_\alpha) \cap A_\gamma$, then it must be the case that $\gamma = \delta + 1$ for some $\delta \in A$ and $h \in A_\delta \cap A_\gamma$. This implies that $h = g_\delta = f_\delta + 1 = f_\gamma$. It follows that

$$(\bigcup_{\alpha < \gamma} A_\alpha) \cup \{f_\gamma\} \cap A_\gamma = \{f_\gamma\},$$

so by Lemma 3.2,

$$\bigcup_{\alpha \leq \gamma} A_\alpha = (\bigcup_{\alpha < \beta} A_\alpha) \cup \{f_\beta\}$$

is an arc with end points $f$ and $g_\gamma = f_\beta$.

Case 2. $\beta$ has no immediate predecessor in $A$. The steps in the argument will be as follows:

1) $\bigcup_{\alpha < \beta} A_\alpha$ is connected.

2) If $h \in X$ and $h \notin \bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$, then $h$ is not a limit point of $\bigcup_{\alpha < \beta} A_\alpha$.

3) $f_\beta$ is a limit point of $\bigcup_{\alpha < \beta} A_\alpha$, so $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$ is a continuum and $f_\beta$ is not a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$.

4) $f$ is not a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$, and if $h \in \bigcup_{\alpha < \beta} A_\alpha$ and $h \neq f$, then $h$ is a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$; thus $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$ is an arc with end points $f$ and $f_\beta$.

Proof of 1). By the induction hypothesis, $\bigcup_{\gamma < \beta} A_\gamma = \bigcup_{\gamma < \beta} (\bigcup_{\alpha < \gamma} A_\alpha)$ is the union of connected sets each of which
contains \( f \) and is therefore connected.

Proof of 2). Suppose that \( h \in X - \bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\} \). If for some \( \delta \geq \beta \), \( h(\delta) \neq a_\delta \), then \( P_{\delta}^{-1}(X_\delta - a_\delta) \) is open in \( X \), contains \( h \) and misses \( \bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\} \). Assume, then, that for all \( \delta > \beta \), \( h(\delta) = a_\delta = f_\beta(\delta) \). Let \( \gamma \) be the first member of \( A \) such that \( h(\gamma) \neq f_\beta(\delta) \). Then \( \gamma < \beta \) and if \( \delta < \gamma \), then \( h(\delta) = b_\delta \). If for all \( \varepsilon \) such that \( \gamma < \varepsilon, h(\varepsilon) = a_\varepsilon \), then \( h \in A_\gamma \), contrary to assumption. Thus for some \( \varepsilon \in \mathcal{A} \), \( \gamma < \varepsilon < \beta \), \( h(\varepsilon) \neq a_\varepsilon \). Then \( O = P_{\gamma}^{-1}(X_\gamma - b_\gamma) \cap P_{\varepsilon}^{-1}(X_\varepsilon - a_\varepsilon) \) is open in \( X \), contains \( h \) and misses \( \bigcup_{\alpha < \beta} A_\alpha \). Thus \( h \) is not a limit point of \( \bigcup_{\alpha < \beta} A_\alpha \).

Proof of 3). We consider the net \( \{g_\alpha\}_{\alpha < \beta} \). For all \( \delta \in \mathcal{A} \), the net \( \{P_\delta(g_\alpha)\}_{\alpha < \beta} \) converges to \( P_\delta(f_\beta) \). For if \( \beta \leq \delta \), then for all \( \alpha < \beta \), \( g_\alpha(\delta) = a_\delta = f_\beta(\delta) \); and if \( \delta < \beta \), then since \( \beta \) has no immediate predecessor, there is a \( \gamma \in \mathcal{A} \), such that \( \delta < \gamma < \beta \) and if \( \gamma < \varepsilon < \beta \), then \( P_\delta(g_\varepsilon) = g_\varepsilon(\delta) = b_\delta = f_\beta(\delta) \). It follows that the net \( \{g_\alpha\}_{\alpha < \beta} \) converges in \( X \) to \( f_\beta \). It now follows immediately from 1) that \( f_\beta \) is not a cut point of \( \bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\} \).

Proof of 4). Since \( (\bigcup_{\alpha < \beta} A_\alpha) - f = \bigcup_{\gamma < \beta} [(\bigcup_{\alpha < \gamma} A_\alpha) - f] \) is a union of the connected sets \( (\bigcup_{\alpha < \gamma} A_\alpha) - f \) each of which contains \( f_2 \), \( (\bigcup_{\alpha < \beta} A_\alpha) - f \) is connected.

Now suppose that \( h \in \bigcup_{\alpha < \beta} A_\alpha \), \( h \neq f \). Let \( \alpha^* \) be the first member of \( \mathcal{A} \) such that \( h \in A_{\alpha^*} \), and let \( y_{\alpha^*}, z_{\alpha^*} \) be arcs in \( X_{\alpha^*} \) with end points \( a_{\alpha^*}, h(\alpha^*) \) and \( h(\alpha^*), b_{\alpha^*} \), respectively. (We note that \( Z_{\alpha^*} \) might be degenerate.) For
each $\alpha \in \mathcal{A}$, define $T_\alpha$ and $S_\alpha$ as follows:

$$
T_\alpha = \begin{cases} 
X_\alpha, & \alpha < \alpha^* \\
Y_{\alpha^*}, & \alpha = \alpha^* \\
a_\alpha, & \alpha^* < \alpha
\end{cases}
$$

$$
S_\alpha = \begin{cases} 
b_\alpha, & \alpha < \alpha^* \\
z_{\alpha^*}, & \alpha = \alpha^* \\
X_\alpha, & \alpha^* < \alpha
\end{cases}
$$

Let $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$; $T = \bigcup_{\alpha \in \mathcal{A}} T_\alpha$. Then $S$ and $T$ are closed in $X$ and $S \cap T = \emptyset$. Now $f \in T$, $f_\beta \in S$, and

$$
\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\} \subseteq S \cup T.
$$

It follows that

$$
(\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}) - h \cap S, [(\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}) - h] \cap T
$$

is a separation of $(\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}) - h$, so $h$ is a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$. Thus $I(\beta)$ is established. Statement $e$ follows.

By argument similar to that in the induction step of the proof of $e$, it follows that $\bigcup_{\alpha \in \mathcal{A}} A_\alpha$ is connected; that if $h \in X - \bigcup_{\alpha \in \mathcal{A}} A_\alpha$ and $h \neq g$, then $h$ is not a limit point of $\bigcup_{\alpha \in \mathcal{A}} A_\alpha$; that the net $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ converges in $X$ to $g$, and that every point $h$ of $\bigcup_{\alpha \in \mathcal{A}} A_\alpha$, $h \neq f, g$ is a cut point of $\bigcup_{\alpha \in \mathcal{A}} A_\alpha \cup \{g\}$. Thus $\bigcup_{\alpha \in \mathcal{A}} A_\alpha \cup \{g\}$ is an arc in $X$ with end points $f$ and $g$.

3.4 Theorem. If $\{Y_\alpha : \alpha \in \mathcal{A}\}$ is a collection of arcwise connected spaces, then $Y = \bigcup_{\alpha \in \mathcal{A}} Y_\alpha$ is arcwise connected.
Proof. Let \( f, g \) be points of \( Y \). If \( f = g \), there is nothing to prove, so assume that \( f \neq g \). Let

\[ A^* = \{ \alpha \in A : f(\alpha) \neq g(\alpha) \} \]

and let \( \geq \) be a well-order on \( A^* \). For each \( \alpha \in A^* \), let \( X_\alpha \) be an arc in \( Y_\alpha \) with end points \( a_\alpha = f(\alpha) \) and \( b_\alpha = g(\alpha) \). Let \( f^*, g^* \) be the restrictions to \( A^* \) of \( f \) and \( g \) respectively. Then \( \{ X_\alpha : \alpha \in A^* \} \), \( f^* \) and \( g^* \) satisfy the conditions of Lemma 3.3, so there is an arc \( A \) in \( X^* = \bigcup_{\alpha \in A^*} X_\alpha \) such that the end points of \( A \) are \( f^* \) and \( g^* \). Define a map \( \theta \) from \( X^* \) into \( Y \) by

\[
[\theta(h)](\alpha) = \begin{cases} 
  f(\alpha), & \alpha \in A^* \\
  h(\alpha), & \alpha \in A^*.
\end{cases}
\]

Then \( \theta \) is a homeomorphism of \( X^* \) onto \( \theta(X^*) \) and \( \theta(f^*) = f \), \( \theta(g^*) = g \), so \( \theta(A) \) is an arc in \( Y \) with end points \( f \) and \( g \).
BIBLIOGRAPHY


