Cellularity and negligibility in infinite-dimensional normed linear spaces

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DIMENSIONAL NORMED LINEAR SPACES.

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Cellularity and negligibility
in infinite-dimensional normed linear spaces

by

Robert Willis Neufeld

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major: Mathematics

Approved:

Signature was redacted for privacy.

In Charge of Major Work

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Iowa State University
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I. INTRODUCTION

McCoy [18] introduced the study of cell-like sets in infinite-dimensional normed linear spaces by defining and studying both cellularity and strong cellularity. The concept is considered here in terms of bases of cells. If $E$ is an infinite-dimensional normed linear space, an $E$-cell in a space $X$ is a closed subset $C$ of $X$ which is homeomorphic to the unit ball $B_1$ in $E$ by a homeomorphism which takes the boundary of $B_1$ onto the boundary of $C$.

Strongly cellular sets can be characterized as closed sets $K$ having a countable neighborhood basis of cells, i.e., a countable collection of cells, each of which contains $K$ in its interior and such that every open set containing $K$ contains one of these cells. The following properties of a closed subset $X$ of an infinite-dimensional normed linear space are considered in Chapter III:

a) $K$ has a countable neighborhood basis of cells.

b) $K$ has a neighborhood basis of cells.

c) $K$ has a countable metric basis of cells, i.e., a collection $\{C_i\}_{i=1}^{\infty}$ of cells such that each $C_i$ contains $K$ in its interior and every $\varepsilon$-neighborhood of $K$ contains some $C_i$.

d) $K$ has a metric basis of cells.

It is shown that a) implies b) implies c), that neither of these implications may generally be reversed and that c)
and d) are equivalent. If a closed set satisfies either a) or b), then it is cellular (Definition 3.2). On the other hand, cellular sets need not even satisfy d). Since cells are connected, all of a) through d) require some degree of connectedness, whereas cellular sets may be rather badly disconnected. For compact sets, a) through d) are equivalent, reducing the classification in this case to cellularity and strong cellularity. Strongly cellular sets are compact and connected, but compact connected cellular sets need not be strongly cellular. The diagram below indicates the situation for an arbitrary closed subset of a normed linear space.

\[ \begin{align*}
a) & \iff b) \iff c) \iff d) \\
\text{strongly cellular} & \iff \text{cellular}
\end{align*} \]

Bessaga and Klee [3] call a subset \( A \) of a topological space \( E \) negligible if \( E \) and \( E \setminus A \) are homeomorphic. [1] and [18] summarize and reference numerous negligibility results. C. A. Riley [19] proves negligibility theorems in certain non-locally convex linear topological spaces, including \( L^p \) with \( 0 < p < 1 \). McCoy proved in [18] that cellular sets are negligible. It is shown in Chapter IV that negligibility and cellularity are equivalent for closed subsets of an
infinite-dimensional normed linear space, a result recently obtained independently by McCoy.

The negligibility results of Chapter IV are obtained by rather direct geometric methods and depend on the extent to which certain sets resemble closed convex bodies. The results of Chapter V are obtained by piecing together a family of homeomorphisms. This can be done to show the negligibility of the union of a locally finite collection of disjoint closed sets each of which is negligible in a strong sense, namely by a homeomorphism which can be made the identity outside an arbitrary neighborhood of the set. The method works for sets satisfying a) and, with some restrictions on the space, for sets satisfying b). For arbitrary disjoint collections of compact sets or negligible (cellular) sets, some additional separation (besides local finiteness) appears to be necessary.

Throughout this dissertation, \( E \) will denote a normed linear space and \( 0 \) the origin in \( E \). If \( \| \cdot \| \) is a norm on \( E \), then let \( d(x,y) = \| x - y \| \). As usual, for nonempty subsets \( A \) and \( B \) of a metric space \( (X,d) \),

\[
    d(x,A) = \inf \{ d(x,a) \mid a \in A \}
\]

and

\[
    d(A,B) = \inf \{ d(a,b) \mid a \in A, \ b \in B \}.
\]

\( N_\varepsilon(A) \) represents the set \( \{ x \in X \mid d(x,A) < \varepsilon \} \) for \( \varepsilon > 0 \). In case \( A = \{ x \} \), we write \( N_\varepsilon(x) \) for \( N_\varepsilon(A) \). For \( p \in E \) and \( r \geq 0 \), let

\[
    B_r(p) = \{ x \in E \mid d(x,p) \leq r \}
\]

and

\[
    S_r(p) = \{ x \in E \mid d(x,p) = r \}.
\]

\( P_r(0) \) and \( S_r(0) \) may sometimes be abbreviated to \( B_r \) and \( S_r \), respectively. For a subset \( A \) of any topological space
$X, \text{Int } A, \text{Cl } A$ and $\text{Bd } A$ denote the interior, closure and boundary of $A$, respectively.
II. PRELIMINARIES

In this chapter are a number of definitions and results which will be used for defining homeomorphisms in linear spaces. The definition of shrinkable neighborhood is due to R. T. Ives and may be found in [14]. A similar notion has been used by D. W. Henderson.

Definition 2.1: A neighborhood $U$ of a point $p$ in a $T_2$ linear space is **shrinkable** (at $p$) if and only if $[0,1) \subseteq \mathcal{C}(U-p) \subseteq \text{Int}(U-p)$.

It is easy to see that finite unions and finite intersections of shrinkable neighborhoods are shrinkable. Both infinite unions and infinite intersections of shrinkable neighborhoods may fail to be shrinkable.

A set $U$ in a linear space is called **starshaped from $\emptyset$** if the line segment from $\emptyset$ to $p$ lies in $U$ for each point $p$ of $U$. $U$ is linearly bounded provided it intersects each line in a bounded set. The definition of a Minkowski functional and elementary properties of Minkowski functionals may be found in [10]. Since the Minkowski functional of a shrinkable neighborhood of a point is continuous [14], it is very useful in defining homeomorphisms.

Proposition 2.1: Let $\mu$ be the Minkowski functional of a starshaped neighborhood $U$ of $\emptyset$ in a $T_2$ linear topological space $E$. Then $U$ is linearly bounded if and only if
\[ \mu^{-1}(0) = \emptyset. \]

**Proof:** Let \( U \) be linearly bounded and \( x \in E, x \neq \emptyset \). Then there is a positive number \( a \) such that 
\[ ([0, \infty)x) \cap U \subset [0, 1/a]x. \]
Then if \( 0 < t < a, 1/t > 1/a \), so 
\[ (1/t)x \notin U. \]
Thus \( \mu(x) > 0. \)

Conversely, suppose \( \mu^{-1}(0) = \emptyset \) and let \( x \in E, x \neq \emptyset \). Then \( \mu(x) > 0 \) and \( \mu(-x) > 0 \). Therefore there is a positive number \( t \) such that \( t < \mu(x) \) or \( (1/t)x \notin U \). Also there is \( s > 0 \) such that \( (1/s)(-x) \notin U \). Since \( U \) is starshaped from \( \emptyset \), \( ((-\infty, \infty)x) \cap U \subset [-1/s, 1/t]x. \)

A modification of an argument used by Corson and Klee [5] yields the following proposition.

**Proposition 2.2:** Let \( V_1, V_2, V_3 \) and \( V_4 \) be closed linearly bounded shrinkable neighborhoods of \( \emptyset \) in a \( T_2 \) linear topological space \( E \) with \( V_1 \subset \text{Int}(V_2 \cap V_3) \) and \( V_2 \cup V_3 \subset \text{Int} V_4 \). Then there is a homeomorphism of \( E \) onto itself which takes the pair \((V_2, \text{Bd} V_2)\) onto \((V_3, \text{Bd} V_3)\) and is the identity on \( V_1 \cup (E \setminus \text{Int} V_4) \).

**Proof:** Let \( \mu_1 \) be the Minkowski functional of \( V_1 \) for \( 1 \leq i \leq 4 \). Define \( h(x) = x \) for \( x \in V_1 \cup (E \setminus \text{Int} V_4) \). If \( x \in V_2 \setminus V_1 \), then it has a unique expression as
\[ x = (1 - t_x)x/\mu_1(x) + t_x x/\mu_2(x). \]
(It can be seen that \( t_x = \mu_2(x)(\mu_1(x) - 1)/(\mu_1(x) - \mu_2(x)) \).) In this case, define \( h(x) = (1 - t_x)x/\mu_1(x) + t_x x/\mu_3(x). \) (If \( x \in \text{Bd} V_2 \), \( t_x = 1 \),
so \( \mu_3(h(x)) = 1 \) and \( h(x) \in E_d V_3 \). For \( x \in V_4 \setminus V_2 \),
there is a unique expression \( x = (1 - t_x)x/\mu_2(x) + t_x x/\mu_4(x) \).
Define \( h(x) = (1 - t_x)x/\mu_3(x) + t_x x/\mu_4(x) \).

The continuity of \( h \) follows from that of the \( \mu_i \) and
of scalar multiplication. The inverse of \( h \) exists and is of
the same form, so \( h \) is a homeomorphism.

It may be noted that the preceding argument holds even
though the \( V_i \) are not linearly bounded provided that
\( \mu_1(x) = 0 \) if and only if \( \mu_1(x) = 0 \) for \( i = 2, 3, 4 \). Some
such restriction is clearly necessary, since \( V_2 \) might sepa­
rate \( E \), while \( V_3 \) did not.

**Proposition 2.3:** [14] Let \( E \) be a \( T_2 \) linear topological
space, \( U \) a shrinkable neighborhood of \( 0 \) in \( E \) and \( S \) a
compact subset of \( E \) which is starshaped from \( 0 \). Then
\( S + U \) is a shrinkable neighborhood of \( 0 \).

The compactness of \( S \) is essential to Klee's proof of
Proposition 2.3. In case \( E \) is a normed space and \( U \) is
taken to be an \( \varepsilon \)-ball, then \( S \) need not be compact.

**Proposition 2.4:** Let \( E \) be a normed linear space, \( S \) star­
shaped from \( 0 \) and \( \varepsilon > 0 \). Then \( S + B_\varepsilon \) is a shrinkable
neighborhood of \( 0 \).

**Proof:** Let \( V = S + B_\varepsilon \). Since \( V \) contains \( 0 + B_\varepsilon = B_\varepsilon \), it
is a neighborhood of \( 0 \). Let \( q \in \text{Cl } V \) and \( 0 < t < 1 \). Then
there is \( p \in V \) such that \( d(p,q) < (1 - t)\varepsilon /t \). Also there
is $p' \in S$ such that $d(p', p) < \varepsilon$. Then
\[
d(tp', tq) = td(p', q) \leq t(d(p', p) + d(p, q))
\]
\[
< t(\varepsilon + (1 - t)\varepsilon/t) = \varepsilon.
\]
Since $S$ is starshaped from $\emptyset$ and $p' \in S$, $tp' \in S$. Therefore, $tq \in V = \text{Int } V$, so $V$ is shrinkable.

In a somewhat different direction, it is shown next that closed starshaped sets in a normed linear space have a "basis" of shrinkable neighborhoods.

**Proposition 2.5** If $S$ is a closed set which is starshaped from $\emptyset$ in a normed linear space $E$ and $U$ is a neighborhood of $S$, then there is a closed shrinkable neighborhood $V$ of $\emptyset$ such that $S \subseteq \text{Int } V$ and $V \subseteq \text{Int } U$.

This is a special case of the next proposition.

**Proposition 2.6** Suppose $E$ is a normed linear space and $S$ is a closed set which is starshaped from each point of a compact convex subset $K$ of $S$. If $U$ is a neighborhood of $S$, then there is a closed set $V$ such that $S \subseteq \text{Int } V \subseteq \text{Int } U$ and $V$ is a shrinkable neighborhood of every point of $K$.

**Proof**: If $x \in S$ and $a \in K$, then $d([a; x], E \setminus \text{Int } U) > 0$. ($[a; x]$ is the segment $\{(1 - t)a + tx | 0 \leq t \leq 1\}$ joining $a$ and $x$.) Since $K$ is compact, $\inf_{a \in K} d([a; x], E \setminus \text{Int } U)$ exists and is positive. Let $\varepsilon_x = (1/2)\inf_{a \in K} d([a; x], E \setminus \text{Int } U)$
and define \( V = \text{Cl} \bigcup_{x \in S} a \in K e_{x}([a; x]) \). Clearly, \( S \subseteq \text{Int} V \).

Suppose now that \( p \in V \setminus \text{Int} U \). Then there is a sequence \( \{p_{n}\}_{n=1}^{\infty} \) in \( \text{Int} V \) which converges to \( p \). For each \( n \), there exist \( x_{n} \in S \), \( a_{n} \in K \) and \( y_{n} \in [a_{n}; x_{n}] \) such that \( d(p_{n}, y_{n}) < \varepsilon \). If \( \varepsilon > 0 \), then for some \( N \),
\[
d(p_{n}, p) < \varepsilon /3.
\]
Since \( p \in E \setminus \text{Int} U \), \( d(y_{N}, p) \geq 2\varepsilon /3 \). Thus
\[
2\varepsilon /3 \leq d(y_{N}, p) < d(y_{N}, p_{N}) + d(p_{N}, p) < \varepsilon x_{N} + \varepsilon /3,
\]
so
\[
\varepsilon x_{N} < \varepsilon /3.
\]
Hence, \( d(y_{N}, p) < 2\varepsilon /3 \). Since \( \varepsilon > 0 \) was arbitrary and the sequence \( \{y_{n}\}_{n=1}^{\infty} \) lies in the closed set \( S \), \( p \) must belong to \( S \). This contradiction implies that \( V \subseteq \text{Int} U \).

Let \( a \in K \), \( p \in V \) and \( 0 < t < 1 \) be fixed. To show that \( V \) is a shrinkable neighborhood of \( a \), it must be shown that \( (1 - t)a + tp \in \text{Int} V \). There exist sequences \( \{p_{n}\}_{n=1}^{\infty} \), \( \{x_{n}\}_{n=1}^{\infty} \), \( \{a_{n}\}_{n=1}^{\infty} \) and \( \{y_{n}\}_{n=1}^{\infty} \) as in the preceding paragraph. For each \( n \), the join of \( x_{n} \) and \( K \) is convex \([10, \text{p. 14}]\) and contains \( y_{n} \), hence it contains the join of \( y_{n} \) and \( K \). It follows that \( \varepsilon x_{n} \leq \varepsilon y_{n} \). Consider two cases in showing that \( V \) is shrinkable.

Case 1: There exists \( \varepsilon > 0 \) such that \( \varepsilon x_{n} > \varepsilon_{0} \) for all \( n \). For some \( N \), \( d(p_{N}, p) < (1 - t)\varepsilon_{0}/t \). Then
\[ d((1 - t)a + ty_N \cdot (1 - t)a + tp) = d(ty_N \cdot tp) = td(y_N, p) \leq t(d(y_N, p_N) + d(p_N, p)) < t(\epsilon \cdot x_N + (1 - t)e_0 / t) \]
\[ < t(\epsilon \cdot x_N + (1 - t)e_0 / t) = \epsilon \cdot x_N \leq \epsilon y_N. \] Therefore
\[ (1 - t)a + tp \in N_{\epsilon y_N([a; y_N])} \subset \text{Int } V. \]

Case 2: If \( \{\epsilon x_n\}_{n=1}^{\infty} \) is not bounded away from zero, it may be assumed without loss of generality that \( \lim_{n \to \infty} \epsilon x_n = 0. \)

Let \( \epsilon > 0 \) be given. There is an integer \( N \) such that, for \( n > N, \epsilon x_n < \epsilon / 2 \) and \( d(p_n, p) < \epsilon / 2. \) If \( n > N, \) then
\[ d(y_n, p) \leq d(y_n, p_n) + d(p_n, p) < \epsilon x_n + \epsilon / 2 < \epsilon. \] Thus, the sequence \( \{y_n\}_{n=1}^{\infty} \) of points of \( S \) converges to \( p, \) so \( p \in S. \) It is clear that \((1 - t)a + tp \in \text{Int } V.\)

In either case, \((1 - 0)a + 0p = a \in \text{Int } V, \) so the proof is complete.

The following definition is due to Prof. D. E. Sanderson.

**Definition 2.2:** \( Q \) is a \( \sigma \)-regular open set in a topological space \( X \) if and only if \( Q = \bigcup_{i=1}^{\infty} G_i \) where \( \text{Cl } G_i \subset Q \) and \( G_i \) is open for each \( i. \)

Since the open set \( Q = \bigcup_{i=1}^{\infty} \text{Cl } G_i, \) the complement of \( Q \) is a closed \( G_\delta \)-set. It will be shown that in a normal space this is a necessary and sufficient condition for \( Q \) to be \( \sigma \)-regular.
Definition 2.3: [16] A set \( A \) will be called a regular 
\( G_\delta \)-set if it is the intersection of a sequence of closed sets 
whose interiors contain \( A \).

Proposition 2.7: An open subset \( Q \) of a topological space \( X \) 
is \( \sigma \)-regular if and only if \( X \setminus Q \) is a regular \( G_\delta \)-set.

Proof: If \( Q = \bigcup_{i=1}^{\infty} G_i \), where each \( G_i \) is an open set whose 
closure is contained in \( Q \), then 
\[ X \setminus Q = X \setminus \bigcup_{i=1}^{\infty} G_i = \bigcap_{i=1}^{\infty} (X \setminus G_i) = \bigcap_{i=1}^{\infty} F_i, \]
where each 
\( F_i = X \setminus G_i \) is closed and 
\[ X \setminus Q \subseteq X \setminus \text{Cl} G_i = \text{Int}(X \setminus G_i) = \text{Int} F_i. \]
It is clear that the implication can be reversed.

Lemma 2.1: A closed \( G_\delta \)-set in a normal space \( X \) is a regular \( G_\delta \)-set.

Proof: Suppose \( A = \bigcap_{i=1}^{\infty} U_i \) is closed, each \( U_i \) is open and 
\( U_{i+1} \subseteq U_i \). Define a sequence \( \{V_i\}_{i=1}^{\infty} \) of open sets as 
follows. Let \( V_1 = U_1 \). Inductively, suppose open sets 
\( V_1, V_2, \ldots, V_n \) have been selected so that for \( 2 \leq i \leq n \), 
\( A \subseteq V_i \subseteq \text{Cl} V_i \subseteq V_{i-1} \cap U_i \). Since \( A \) is a closed subset of 
\( V_n \cap \cap_{i=n+1}^{\infty} U_i \), there is an open set \( V_{n+1} \) such that 
\( A \subseteq V_{n+1} \subseteq \text{Cl} V_{n+1} \subseteq V_n \cap \cap_{i=n+1}^{\infty} U_i \). Then \( A \subseteq \bigcap_{i=1}^{\infty} V_i \subseteq \bigcap_{i=1}^{\infty} U_i = A \), 
so \( \{V_i\}_{i=1}^{\infty} \) is a "\( G_\delta \) sequence for \( A \)". Also, for each \( n \), 
\( V_n \subseteq \text{Cl} V_n \subseteq V_{n-1} \). Thus,
A \subset \bigcap_{i=1}^{n} \text{Cl} V_{i} = \bigcap_{i=2}^{n} \text{Cl} V_{i} \subset \bigcap_{i=1}^{n} V_{i} = A$, so $A$ is a regular $G_{\delta}$-set.

The next proposition is an immediate consequence of Proposition 2.7 and Lemma 2.1.

**Proposition 2.8:** An open $F_{\sigma}$-set in a normal space is $\sigma$-regular.

**Corollary 2.1:** Every open subset of a perfectly normal space is $\sigma$-regular.

**Corollary 2.2:** Every open subset of a metric space is $\sigma$-regular.

**Proposition 2.9:** Let $X$ be a regular topological space. If $E$ is a Lindelöf space, then the open homeomorphic image of $E$ in $X$ is $\sigma$-regular.

**Proof:** Let $Q$ be open in $X$ and $h$ a homeomorphism of $E$ onto $Q$. $Q$ is Lindelöf since $E$ is. For each $x$ in $Q$, let $N_{x}$ be an open neighborhood of $x$ such that $\text{Cl} N_{x} \subset Q$. The collection $\{N_{x} | x \in Q\}$ has a countable subcovering which shows that $Q$ is $\sigma$-regular.

The proof depends on the fact that $Q$ is Lindelöf and holds if $X$ is "hereditarily Lindelöf". It may be noted that the proposition applies in case $E$ is a separable metric space.
Example 2.1: Let $E$ be an infinite-dimensional space and $Q$ a closed half-space of $E$ such that $Q \subseteq \text{Bd } Q$. There is a homeomorphism $h$ from $E$ onto itself which takes the pair $(B_1, S_1)$ onto $(Q, \text{Bd } Q)$ (see Proposition 3.10). Let $j$ be a homeomorphism which shrinks $E$ radially onto Int $B_1$. Then $jh(B_1)$ is not closed in $E$.

M. Brown [4] proved that the monotone union of open $n$-cells is an open $n$-cell. Example 2.1 illustrates one of the difficulties that arises in an attempt to adapt his proof to infinite-dimensional normed linear spaces. According to the next theorem, $E$ can be "renormed" so that the image of the new $B_n$ under $jh$ will be closed in $E$ for each $n$.

The definition of $\sigma$-regular was motivated by Theorem 2.1. In a somewhat different form, the statement and proof of this theorem were due to Prof. D. E. Sanderson.

Theorem 2.1: Let $E$ be a normed linear space, $X$ a topological space, $p$ a fixed point of $E$ and $h$ a homeomorphism of $E$ onto the $\sigma$-regular open set $Q$ in $X$. Then there is a homeomorphism $H$ on $E$ such that $hH(B_n)$ is closed in $X$ for each $n$, $H(p) = p$ and $H(B_\epsilon(p)) \subseteq B_\epsilon(p)$ for all $\epsilon > 0$.

Proof: Let $\{G_i\}_{i=1}^\infty$ be the sequence given by $\sigma$-regularity. Without loss of generality, assume that $G_i \subseteq G_{i+1}$ for each $i$ and that $h(B_1(p))$ is a closed (in $X$) subset of $G_i$. For each positive integer $n$, define
Let \( x \in E \). Because \([p; x]\) is compact and \( \{h^{-1}(G_i)\}_{i=1}^{\infty} \) is an "increasing" open cover of \( E \), it follows that
\[ [p; x] \subset h^{-1}(G_i) \quad \text{for some } i. \]
There is an integer \( n \geq i \) such that \( n \geq d(x, p) \) and
\[ d([p; x], E \setminus h^{-1}(G_n)) \geq d([p; x], E \setminus h^{-1}(G_i)) > 1/2^n. \]
Hence \( x \in B_n'(p) \) and \( E = \bigcup_{j=1}^{\infty} B_j'(p) \).

It will be shown next that \( B_n'(p) \subset \text{Int} \ B_{n+1}'(p) \) for all \( n \). Since \( B_n'(p) \subset B_n(p) \subset \text{Int} \ B_{n+1}(p) \), it suffices to show that \( B_n'(p) \subset B_{n+1}'(p) \). If \( y \in B_n'(p) \), then there exist \( y' \in B_n^*(p) \), \( x \in E \) and \( t, 0 \leq t \leq 1 \), such that
\[ d(y, y') < 1/2^{n+1}, \quad d(y', (1 - t)p + tx) < 1/2^n \]
and
\[ d([p; x], E \setminus h^{-1}(G_n)) > 1/2^n. \]
If \( z \in E \setminus h^{-1}(G_{n+1}) \) and \( 0 \leq s \leq 1 \), then
\[ d(z, (1 - s)p + sy') \geq d(z, (1 - s)p + s((1 - t)p + tx)) \]
and
\[ d((1 - s)p + sy', (1 - s)p + s((1 - t)p + tx)) = d(z, (1 - st)p + (st)x) - d(sy', s((1 - t)p + tx)) = d(z, (1 - st)p + (st)x) - sd(y', (1 - t)p + tx) > 1/2^{n+1} - 1/2^n = 1/2^n. \]
Thus \( B_{n+1}'(p) \supset N_{1/2^{n+1}}([p; y']) \) and \( y \in B_{n+1}'(p) \), since \( d(y, y') < 1/2^{n+1} \).
Since, for any \( n \), \( \text{Bd} \ B_n'(p) \subset B_n'(p) \subset \text{Int} \ B_{n+1}'(p) \), every ray from \( p \) intersects \( \text{Bd} \ B_n'(p) \) and \( \text{Bd} \ B_{n+1}'(p) \) in distinct points.

For each positive integer \( n \), let \( \mu_n \) be the Minkowski functional of \( B_n'(p) \) at \( p \). Since each \( B_n'(p) \) is a shrinkable neighborhood of \( p \), each \( \mu_n \) is continuous \([14]\). By Proposition 2.1, \( \mu_n^{-1}(0) = p \) for each \( n \).

Define \( H(p) = p \) and \( H(x) = d(x,p)x/\mu_1(x) \) if \( 0 < d(x,p) \leq 1 \). If \( x \in B_n(p) \setminus \text{Int} \ B_{n-1}(p) \), then there is a unique \( t_x \), \( 0 < t_x \leq 1 \), such that \( x = (1-t_x)(n-1)x/d(x,p) + t_x n_x/d(x,p) \). For such \( x \), let \( H(x) = (1-t_x)x/\mu_{n-1}(x) + t_x x/\mu_n(x) \). \( H \) is a one-to-one mapping of \( E \) onto \( E \). The continuity of \( H \) and \( H^{-1} \) follow from the continuity of \( d \) and the \( \mu_n \). \( hH(B_n(p)) = h(B_n'(p)) \subset G_n \) and \( \text{Cl} \ G_n \subset Q \), so \( hH(B_n(p)) \) is closed in \( X \).

Since \( B_n'(p) \subset B_n(p) \) and \( H(B_n(p)) = B_n'(p) \) for each \( n \), it is clear (from the construction) that \( d(x,p) \geq d(H(x),p) \) for all \( x \) in \( E \). If \( n \) is a positive integer, then there is a positive integer \( N \) such that \( B_{N}(p) \supset B_n \). Since \( hH(B_N(p)) \) is closed in \( X \) and \( B_n \) is closed, it follows that \( hH(B_n) \) is closed in \( X \).

**Corollary 2.3:** Let \( E \) be an infinite-dimensional normed linear space, \( X \) a topological space and \( h \) a homeomorphism of \( E \) onto a \( \sigma \)-regular open subset of \( X \). If \( C \) is a
closed proper subset of $E$, there is a homeomorphism $J$ on
$E$ such that $hJ(C)$ is closed in $X$.

**Proof:** Let $\epsilon > 0$ and $x$ be chosen so that $B_{\epsilon}(x) \subseteq E \setminus C$. There is a homeomorphism $i$ on $E$ such that
\[ i(B_{\epsilon}(x)) = E \setminus \text{Int } B_{\epsilon}(x) \text{ and } i|S_{\epsilon}(x) = \text{identity}. \]
Let $j$ be a homeomorphism on $E$ which takes $B_{\epsilon}(x)$ onto $B_1$. Then $ji(C) \subseteq j(B_{\epsilon}(x)) = B_1$. Let $H$ be the homeomorphism given by Theorem 2.1 and $J = Hji$. Since $ji(C)$ is a closed subset of $B_1$ and $H(B_1)$ is closed in $X$, $J(C)$ is closed in $X$. 
III. CELLLULARITY

Throughout this chapter, $E$ will denote an arbitrary normed linear space and $K$ will be a closed subset of $E$. The following three definitions are due to McCoy [18].

**Definition 3.1:** A closed subset $C$ of $E$ is a **cell in** $E$ if there exists a homeomorphism from the pair $(B_1,S_1)$ onto the pair $(C,Bd C)$.

**Definition 3.2:** If $A$ is a subset of $E$, a **cellular sequence** for $A$ is a decreasing sequence, $\{C_i\}_{i=1}^{\infty}$, of cells in $E$ such that $\bigcap_{i=1}^{\infty} C_i = A$ and $C_{i+1} \subset \text{Int} C_i$ for each $i$. Also $A$ is **cellular** in $E$ if there exists a cellular sequence for $A$.

**Definition 3.3:** A subset $A$ of $E$ is **strongly cellular** in $E$ if there exists a cellular sequence, $\{C_i\}_{i=1}^{\infty}$, for $A$ such that for each open set $U$ in $E$ containing $A$, there exists an integer $n$ such that $C_n \subset U$. Such a cellular sequence will be called a **strongly cellular sequence** for $A$.

**Proposition 3.1:** $K$ is strongly cellular in $E$ if and only if $K$ has a countable neighborhood basis of cells.

**Proof:** Suppose that $K$ has a countable neighborhood basis $\{C_i\}_{i=1}^{\infty}$ of cells, i.e., each $C_i$ contains $K$ in its interior and every neighborhood of $K$ contains some $C_n$. Let
$i_1 = 1$ and (inductively) assume that positive integers $i_1 < i_2 < \ldots < i_n$ have been chosen so that for $1 \leq j < n - 1$, $C_{i_{j+1}} \subseteq \text{Int } C_{i_j}$. Then choose $i_{n+1}$ such that $C_{i_{n+1}} \subseteq \text{Int } C_{i_n}$. Clearly, the sequence $\{C_{i_j}\}_{j=1}^\infty$ forms a strongly cellular sequence for $K$.

The converse is trivial.

**Theorem 3.1:** Any strongly cellular set in $E$ is compact and connected.

If $K$ is strongly cellular, then it obviously has a neighborhood basis of cells (not necessarily countable). The converse is not true, as shown by the next example.

**Example 3.1:** Let $E$ be an infinite-dimensional normed linear space and consider the unit ball $B_1$ in $E$. By Theorem 3.1, $B_1$ is not strongly cellular. Let $U$ be an open set containing $B_1$. For $x \in S_1$ and $0 \leq t \leq 1$, define $h(tx) = tx + (t/2)d(x, E \setminus U)x$. Then $h(B_1)$ is a cell containing $B_1$ in its interior and contained in $U$. This also follows from Lemma 1.1 of [20].

**Proposition 3.2:** If $K$ has a neighborhood basis of cells, then $K$ is connected.

**Proof:** The proof of Theorem 3.1 applies.

**Proposition 3.3:** If $K$ has a neighborhood basis of cells,
then it has a countable metric basis of cells.

Proof: For each positive integer \( n \), let \( C_n \) be a cell (from the basis given by hypothesis) so that \( C_n \subset N_{1/n}(K) \). If \( \varepsilon > 0 \), then there is an integer \( i \) such that \( 1/i < \varepsilon \), so \( C_i \subset N_{1/i}(K) \subset N_{\varepsilon/2}(K) \).

A similar trivial proof establishes the following proposition.

Proposition 3.4: \( K \) has a countable metric basis of cells if and only if it has a metric basis of cells.

The next example shows that the converse of Proposition 3.3 fails.

Example 3.2: Klee in [12] demonstrates that every non-reflexive separable Banach space \( E \) contains a pair of disjoint bounded closed convex sets which cannot be separated by a hyperplane. Let \( A \) and \( B \) be such a pair of sets and let \( K = A \cup B \).

First, \( K \) is not connected. By Proposition 3.2, \( K \) does not have a neighborhood basis of cells.

Suppose next that \( d(A,B) = \varepsilon > 0 \). Then \( B_{\varepsilon/2}(A) \) is a set \( U \) satisfying the conditions of Theorem 3.9 of [10, p. 23]. Thus \( A \) and \( B \) can be strongly separated by a linear functional contrary to the assumption on \( A \) and \( B \). Therefore, \( d(A,B) = 0 \).

Finally, let \( \varepsilon > 0 \) be given. Then \( Cl N_{\varepsilon/2}(A) \) and
Cl \( N_{c/2}(E) \) are closed bounded convex bodies whose intersection has a nonempty interior. It will be shown later (in Proposition 4.4) that \( Cl N_{c/2}(A) \cup Cl N_{c/2}(B) \) is a cell. This is clearly contained in \( N_c(K) \), so \( K \) does have a (countable) metric basis of cells.

**Proposition 3.5:** If \( K \) has a metric basis of cells, then it is not the union of sets \( A \) and \( B \) which are a positive distance apart.

**Proof:** If \( K = A \cup B \), where \( d(A,B) = \epsilon > 0 \), then \( N_{\epsilon/3}(K) \) is the union of the disjoint open sets \( N_{\epsilon/3}(A) \) and \( N_{\epsilon/3}(B) \). By hypothesis, there is a cell \( C \) contained in \( N_{\epsilon/3}(K) \) which contains \( K \). \( C \) must intersect both \( N_{\epsilon/3}(A) \) and \( N_{\epsilon/3}(B) \) which is impossible because \( C \) is connected.

**Proposition 3.6:** If \( K \) is compact and has a metric basis of cells, then \( K \) is strongly cellular.

**Proof:** If \( U \) is an open set containing \( K \), then \( d(K, E \setminus U) > 0 \), so \( U \) contains an \( \epsilon \)-neighborhood of \( K \). Thus, if \( \{C_n\}_{n=1}^{\infty} \) is a countable metric basis of cells for \( K \), then \( U \) contains some \( C_n \). A strongly cellular sequence \( \{C_{n_j}\}_{j=1}^{\infty} \) for \( K \) may be defined inductively by taking \( n_1 = 1 \) and \( C_{n_j} \subseteq \bigcap_{i=1}^{j-1} \text{Int } C_{n_i} \).

**Proposition 3.7:** If \( K \) has a neighborhood basis of cells, then \( K \) is cellular.
Proof: Let \( C_1 \) be a cell contained in \( N_1(K) \). Suppose that cells \( C_1, C_2, \ldots, C_n \) have been selected (from the basis) so that \( C_{i+1} \subseteq (\text{Int } C_i) \cap N_{1/i+1}(K) \) for \( 1 \leq i \leq n - 1 \). Let \( C_{n+1} \) be a cell neighborhood of \( K \) such that \( C_{n+1} \subseteq (\text{Int } C_n) \cap N_{1/n+1}(K) \). It is clear that \( \{C_n\}_{n=1}^{\infty} \) (inductively defined) is a cellular sequence for \( K \).

Theorem 3.2: [18] Any compact subset of an infinite-dimensional normed linear space \( E \) is cellular in \( E \).

Example 3.3: Let \( K \) be a compact disconnected set in an infinite-dimensional \( E \). By Theorem 3.2, \( K \) is cellular. By Proposition 3.5, \( K \) does not have a metric basis of cells.

Example 3.4: Let \( K \) be a homeomorphic image of an \( n \)-sphere in an infinite-dimensional \( E \). McCoy [18] notes that \( K \) is not strongly cellular in \( E \). By Theorem 3.2 and Proposition 3.6, \( K \) is a compact connected cellular set which does not have a metric basis of cells.

Proposition 3.6: If \( K \) is cellular, then it is the only nondegenerate inverse set of a map \( f \) from \( E \) onto \( E \).

Proof: As McCoy [18] notes in the proof of his Theorem 3.1, the inverse of the homeomorphism defined in the proof of his Theorem 2.2 can be extended to the desired mapping by mapping \( K \) onto \( \emptyset \).

D. G. Stewart [21] observes a three-dimensional version
of the following proposition.

**Proposition 3.9**: If $K_i$ is strongly cellular and $K_{i+1} \subset K_i$ for each positive integer $i$, then $K = \bigcap_{i=1}^{\infty} K_i$ is strongly cellular.

**Proof**: Since each $K_i$ is compact, $K$ is compact and cellular. According to Proposition 3.6, it need only be shown that $K$ has a metric basis of cells. It will be shown first that every $\epsilon$-neighborhood of $K$ contains some $K_i$.

Suppose that, for some $\epsilon > 0$, $N_\epsilon(K)$ contains no $K_i$. For each positive integer $i$, let $p_i \in K_i \setminus N_\epsilon(K)$. Then the sequence $\{p_i\}_{i=1}^{\infty}$ in the compact set $K_1$ has a subsequence $\{p_i\}_{j=1}^{\infty}$ which converges to some point $p$ of $K_1$. Clearly, $p$ must be in $\bigcap_{i=1}^{\infty} K_i = K$. But then the subsequence is eventually in $N_\epsilon(p) \subset N_\epsilon(K)$. This is impossible, so $N_\epsilon(K)$ contains some $K_i$.

If $\epsilon > 0$ is given, there exists a positive integer $n$ such that $K_n \subset N_\epsilon(K)$. Since $K_n$ is strongly cellular, there is a cell neighborhood of $K_n$ contained in $N_\epsilon(K)$. Thus $K$ has a metric basis of cells, which completes the proof.

The next example is of a decreasing sequence of sets, each having a neighborhood basis of cells, whose intersection does not even have a metric basis of cells. This points out the essential role of compactness in the preceding proof.
Definition 3.4: [18] A cell in $E$ is tame if there exists a homeomorphism of $E$ onto itself which takes $B_1$ onto the cell.

It follows from McCoy [19] and Sanderson [20] that tame cells are cellular and have neighborhood bases of cells.

The next proposition is a restatement of Theorem 2.4 of Bor-Luh Lin [15]. It is also an easy consequence of results of Klee [13].

Proposition 3.10: A closed half-space in an infinite-dimensional normed linear space $E$ is a tame cell.

Example 3.5: For each integer $i \geq 2$, let $K_i$ contain all points in $E^2$ on rays from $0$ through $B_1(i,0,0,...)$ or from $(0,1,0,0,...)$ through $B_1(i,1,0,0,...)$. ($K_i$ is the union of two "cones".) Each $K_i \supseteq K_{i+1}$ and it will be shown that each $K_i$ has a neighborhood basis of cells. If

$$R_j = \{(x,j,0,0,...) \mid x \geq 0\} \text{ for } j=0,1,$$

then $\bigcap_{i=2}^\infty K_i = R_0 \cup R_1$. Since $d(R_0,R_1) > 0$, this intersection does not have a metric basis of cells. (By Proposition 3.5)

Take $Q$ to be the closed half-space

$$\{(x_1,x_2,x_3,...) \mid x_1 \geq 0\}$$

and, for each $i$, let $P_i$ be the projection parallel to $R_0$ of $Bd Q$ onto $Bd K_i$. $P_i$ can be extended to a homeomorphism $h_i$ of $E$ onto itself by translating lines parallel to $R_0$ into themselves. Using Proposition 3.10, it can be seen that each $K_i$ is a tame cell. As
noted earlier, each \( K_i \) then has a neighborhood basis of cells.

**Definition 3.5:** [18] An open \( E \)-cell in a topological space \( X \) is an open subset of \( X \) which is homeomorphic to \( E \). If a subset \( Q \) of \( E \) is an open \( E \)-cell in \( E \), then \( Q \) will be said to be an open cell in \( E \).

**Definition 3.6:** [18] The space \( E \) has the monotone union property provided the following is true. If \( \{Q_i\}_{i=1}^{\infty} \) is an increasing sequence of open \( E \)-cells in any space \( X \), then \( \bigcup_{i=1}^{\infty} Q_i \) is an open \( E \)-cell in \( X \).

**Theorem 3.3:** [18] If \( E \) is homeomorphic to the countable infinite product of copies of itself, then \( E \) has the monotone union property.

The proof of Proposition 3.11 will be given in Chapter IV.

**Proposition 3.11:** The union of an increasing sequence of open cells in \( E \) is an open cell if and only if the intersection of a decreasing sequence of cellular sets in \( E \) is cellular.

An immediate consequence of Theorem 3.3 and Proposition 3.11 is the following corollary, which provides a partial analogue to Proposition 3.9.

**Corollary 3.1:** If \( E \) is homeomorphic to the countable infinite product of copies of itself, then the intersection of a
decreasing sequence of cellular sets in $E$ is cellular.

Stewart [21] proved that the intersection of a decreasing sequence of cellular subsets of $E^3$ is cellular. Since cellularity and strong cellularity agree in finite-dimensional spaces, this is Proposition 3.9. M. Brown proved in [4] that finite-dimensional spaces have the monotone union property. Stewart's result was evidently obtained from this fact and an observation like Proposition 3.11. It is not yet known whether all normed linear spaces have the monotone union property. Thus Proposition 3.11 lends significance to the question of whether the intersection of a decreasing sequence of cellular sets is cellular.
Definition 4.1: [18] A subset of a homogeneous space $X$ is point-like in $X$ if its complement in $X$ is homeomorphic to the complement of a point in $X$.

Definition 4.2: [3] A subset $A$ of a topological space $X$ is negligible provided the spaces $X$ and $X \setminus A$ are homeomorphic.

If $E$ is an infinite-dimensional normed linear space, then the complement of a point in $E$ is homeomorphic to $E$ [13]. In this case, point-like and negligible are equivalent.

Proposition 4.1: Suppose that $K$ is a negligible subset of a topological space $X$ and $f$ is a homeomorphism of $X$ onto a topological space $Y$. Then $f(K)$ is a negligible subset of $Y$.

Proof: Let $h$ be a homeomorphism of $X \setminus K$ onto $X$ and define $H(p) = fhf^{-1}(p)$ for $p \in Y \setminus f(K)$. Since $f^{-1}(f(K)) = K$ and $h$ maps $X \setminus K$ onto $X$, $H$ is a homeomorphism of $Y \setminus f(K)$ onto $Y$.

Theorem 4.1: [18] Any cellular set in a normed linear space $E$ is point-like in $E$.

McCoy in [18] showed that connected point-like sets need not be cellular and raised the question of which point-like sets in an infinite-dimensional normed linear space are
cellular. For closed sets, the answer is contained in the next theorem that the two concepts are equivalent. In a recent revision of [13], McCoy has also obtained a (different) proof of this theorem.

Throughout the remainder of this chapter, \( E \) will denote an arbitrary infinite-dimensional normed linear space unless noted otherwise.

**Definition 4.3:** [20] If \( X \) is a topological space and \( h \) is a homeomorphism of the pair \((B_3 \setminus \text{Int } B_1, S_3 \cup S_1)\) in \( E \) onto the pair \((A, \text{Bd } A)\) in \( X \), then \( h(S_2) \) is a **bicollared** \( E \)-sphere in \( X \).

**Theorem 4.2:** A closed point-like subset of \( E \) is cellular in \( E \).

**Proof:** If \( K \) is a closed point-like subset of \( E \), then \( E \setminus K \) is an open cell in \( E \). By Corollary 2.2, \( E \setminus K \) is \( \sigma \)-regular in \( E \). Let \( \{G_i\} \) be the sequence of Definition 2.2. It follows from Theorem 2.1 that there is a homeomorphism \( h \) from \( E \) onto \( E \setminus K \) such that each \( h(B_n) \) is a cell which is closed in \( E \setminus K \). Since \( h(S_n) \) is bicollared in \( E \), the infinite-dimensional Schoenflies theorem of Sanderson [20] applies. Hence \( E \setminus h(S_n) \) has two components whose closures are \( E \)-cells. It is clear from the proof of Theorem 2.1 that \( h(P_n) \subseteq C_n \) for each \( n \). Thus \( K \) lies entirely in one component of \( E \setminus h(S_n) \) for each \( n \). Define
C_n, for each n, to be the closure of this component (a closed E-cell). Then K = ∩_{n=1}^{∞} C_n, since E \setminus K = \bigcup_{n=1}^{∞} h(E_n) and h(E_n) \subseteq E \setminus C_n, for all n. \{C_n\}_{n=1}^{∞} is the required cellular sequence for K.

Corollary 4.1 is essentially a restatement of Corollary 3.1, using Theorems 4.1 and 4.2.

Corollary 4.1: If E is homeomorphic to the countable infinite product of copies of itself, then the intersection of a decreasing sequence of negligible sets in E is negligible.

The proof of Proposition 3.11 can now be given.

Proof of Proposition 3.11: The following are equivalent

a) E_i is an open E-cell in E and E_i \subseteq E_{i+1} for each i.

b) E \setminus E_i = K_i is closed and point-like and K_{i+1} \subseteq K_i for each i.

c) K_i is cellular and K_{i+1} \subseteq K_i for each i.

Since \bigcup_{i=1}^{∞} E_i = \bigcup_{i=1}^{∞} (E \setminus K_i) = E \setminus \bigcap_{i=1}^{∞} K_i, it follows that i=1 E_i is an open E-cell if and only if \bigcap_{i=1}^{∞} K_i is closed and point-like or cellular.

Proposition 4.2: A closed bounded shrinkable neighborhood of a point in E is a tame cell.
Proof: It suffices to consider a closed bounded shrinkable neighborhood $U$ of $\varnothing$. There is $N > 1$ such that $U \subset \text{Int} \ B_N$. The result follows from Proposition 2.2 with $V_2 = B_1$, $V_3 = U$ and $V_4 = B_N$.

Proposition 4.3: A closed bounded starshaped set in $E$ is negligible.

Proof: If $K$ is starshaped, then $C_i = \text{Cl} \ N_1/i(K)$ is shrinkable for each $i$, by Proposition 2.4. By Proposition 4.2, each $C_i$ is a cell. It follows that $\{C_i\}_{i=1}^{\infty}$ is a cellular sequence for $K$, so $K$ is negligible by Theorem 4.1.

Corollary 4.2: A compact starshaped set in $E$ is strongly cellular.

Proof: If $K$ is compact, the sequence described in the proof of the proposition is a strongly cellular sequence.

As Klee notes in [14] that every convex neighborhood is shrinkable, a special instance of Proposition 4.2 is the fact that every closed bounded convex body in $E$ is a tame cell. McCoy [13] describes a cellular sequence for a closed bounded convex set in $E$ and notes that if the set is compact, the sequence is a strongly cellular sequence. These are also special cases of Proposition 4.3 and Corollary 4.2 with the use of Theorem 4.2.
Proposition 4.4: If $C_1$ and $C_2$ are closed bounded convex sets in $E$, then $C_1 \cup C_2$ is negligible.

Proof: The case in which $C_1$ and $C_2$ are disjoint is an easy consequence of results in Chapter V, but for completeness, it is included here without proof.

If $\text{Int}(C_1 \cap C_2) \neq \emptyset$, it will be shown that $C_1 \cup C_2$ is a tame cell, hence is cellular or negligible. For a fixed $p \in \text{Int}(C_1 \cap C_2)$, there exist $\varepsilon > 0$ and $N$ such that $B_\varepsilon(p) \subset \text{Int}(C_1 \cap C_2)$ and $C_1 \cup C_2 \subset \text{Int} B_N(p)$. Letting $V_1 = B_\varepsilon(p)$, $V_2 = C_1 \cup C_2$, $V_3 = B_\varepsilon(p)$ and $V_4 = B_N(p)$ in Proposition 2.2, a space homeomorphism is obtained which takes $C_1 \cup C_2$ onto $B_\varepsilon(p)$.

Finally, if $C_1 \cap C_2 \neq \emptyset$ but $\text{Int}(C_1 \cap C_2) = \emptyset$, then $\text{Cl} N_\varepsilon(C_1)$ and $\text{Cl} N_\varepsilon(C_2)$ satisfy the hypothesis of the preceding paragraph for any $\varepsilon > 0$. Thus \[
\{\text{Cl} N_{1/n}(C_1) \cup \text{Cl} N_{1/n}(C_2)\}_{n=1}^{\infty}
\] is a cellular sequence for $C_1 \cup C_2$.

There are two primary extensions of Proposition 4.4 to finite unions which are considered in the following propositions. Various combinations of these are clearly possible. Proofs of the next two propositions could be given which are similar to the proof of Proposition 4.4.

Proposition 4.5: If $C_1, C_2, \ldots, C_n$ are closed bounded convex bodies in any normed linear space $E$ and $\text{Int}(\bigcap_{i=1}^{n} C_i) \neq \emptyset$, then...
then \( \bigcup_{i=1}^{n} C_i \) is a tame cell, hence is negligible when \( E \) is infinite-dimensional.

**Proof:** Since a finite union of shrinkable neighborhoods of a point is a shrinkable neighborhood, \( \bigcup_{i=1}^{n} C_i \) is a shrinkable neighborhood of a point of \( \text{Int}(\bigcap_{i=1}^{n} C_i) \) and Proposition 4.2 applies.

**Proposition 4.6:** If \( C_1, C_2, \ldots, C_n \) are closed bounded convex sets in \( E \) and \( \bigcap_{i=1}^{n} C_i \neq \emptyset \), then \( \bigcup_{i=1}^{n} C_i \) is negligible.

**Proof:** This is an obvious consequence of Proposition 4.3.

**Definition 4.4:** [9, p. 81] A collection \( \{C_i\}_{i=1}^{n} \) of sets will be called a simple chain provided that \( C_i \cap C_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \) for \( i, j = 1, 2, \ldots, n \).

**Proposition 4.7:** Suppose that \( \{C_i\}_{i=1}^{n} \) is a simple chain of closed bounded sets in an arbitrary normed linear space \( E \). If there are points \( p_1, p_2, \ldots, p_{n-1} \) such that, for \( 1 \leq i \leq n-1 \), each of \( C_i \) and \( C_{i+1} \) is a shrinkable neighborhood of \( p_i \), then \( \bigcup_{i=1}^{n} C_i \) is a tame cell.

**Proof:** For \( i = 1, 2 \), let \( U_i \) be a bounded open set containing \( C_i \) such that \( (\text{Cl} U_i) \cap (\bigcup_{j=i+2}^{n} C_j) = \emptyset \). For
2 < i ≤ n-2, let U_i be a bounded open set whose closure is disjoint from \( \bigcup_{j=i+2}^{n} C_j \cup \bigcup_{j=1}^{i-2} \text{Cl} U_j \). For i = n-1, n, let U_i be a bounded open set containing C_i whose closure is disjoint from \( \bigcup_{j=1}^{i-2} \text{Cl} U_j \). For each i = 1, 2, ..., n-1, by Proposition 2.5, there is a closed bounded neighborhood V_i of C_i which is contained in U_i and is shrinkable from p_i. For each i = 1, 2, ..., n-1, by Proposition 2.2, there is a homeomorphism h_i on E which takes C_i onto C_i \cap C_{i+1} and is the identity on E \ V_i. In fact, it will be shown that h_i is the identity on C_{i+1} \ C_i as well. Suppose that x ∈ C_{i+1} \ C_i. Since the segment \([p_i, x]\) lies in C_{i+1}, it intersects \(\text{Bd} C_i\) and \(\text{Bd}(C_i \cap C_{i+1})\) in the same point. It is clear from the construction in the proof of Proposition 2.2 that h_i(x) = x. (This is true because \(\mu_2(x) = \mu_3(x)\).) As a result, each h_i takes C_i \cup C_{i+1} onto C_{i+1} and is the identity on C_j for j > i + 1.

The homeomorphism \(h_{n-1} \circ h_{n-2} \circ \cdots \circ h_1\) on E maps \(\bigcup_{i=1}^{n} C_i\) onto \(C_n\), which is a tame cell by Proposition 4.2.

**Corollary 4.3:** Suppose that \(C_1, C_2, ..., C_n\) is a simple chain of closed bounded convex bodies in an arbitrary normed linear space E. If \(\text{Int}(C_i \cap C_{i+1}) \neq \emptyset\) for i = 1, 2, ..., n-1, then \(\bigcup_{i=1}^{n} C_i\) is a tame cell.
Proof: This follows immediately from the proposition and the fact that convex neighborhoods are shrinkable.

Proposition 4.8: Suppose that $C_1, C_2, \ldots, C_n$ is a simple chain of closed bounded subsets of $E$ such that each of $C_i, C_{i+1}$ is starshaped from a point $p_i$ of $C_i \cap C_{i+1}$.

Then $\bigcup_{i=1}^{n} C_i$ is negligible.

Lemma 4.1: If a set $S$ in a linear space is starshaped from points $a$ and $b$ of $S$, then it is starshaped from every point of $[a:b]$.

Proof: Let $c = (1 - t) a + t b$ be a point of $[a:b]$ and $d \in S$. Let $0 < t < 1$. Then

$$(1 - t^2) c + t^2 d = (1 - t^2) [(1 - t) a + t b] + t^2 d$$

$$= [(1 - t^2)(1 - t) a + t^2 d] + (1 - t^2) t^2 b$$

$$= (1 - t^2)(1 - t^2) a + t^2 d$$

$$+ (1 - t^2) t^2 b.$$ But this is a point of $S$, since $S$ is starshaped from $b$ and

$$\frac{(1 - t^2)(1 - t)}{1 - t^2} a + \frac{t^2}{1 - t^2} d$$

can be seen to lie on $[a:d]$, hence in $S$.

Proof of Proposition 4.8: Let $U$ be a neighborhood of $\bigcup_{i=1}^{n} C_i$. It will be shown that $UC_i$ has a cell neighborhood contained in $U$ and thus is negligible by Proposition 3.7 and
Theorem 4.1. Using normality as in the proof of Proposition 4.7, a simple chain \( U_1, U_2, \ldots, U_n \) of bounded open sets may be constructed such that, for each \( i \), \( C_i \subset U_i \subset U \). By Proposition 2.5, there is a closed bounded shrinkable neighborhood \( V_1 \) of \( p_1 \) such that \( C_1 \subset \text{Int} V_1 \subset V_1 \subset U_1 \). For \( i > 1 \), \( C_i \) is starshaped from each point of \([p_{i-1}; p_i]\) by Lemma 4.1. Thus, by Proposition 2.6, there is a closed bounded set \( V_i \) such that \( C_i \subset \text{Int} V_i \subset V_i \subset U_i \) and \( V_i \) is a shrinkable neighborhood of each point of \([p_{i-1}; p_i]\). In particular, Proposition 4.7 applies to show that \( \bigcap_{i=1}^{n} V_i \) is a tame cell.

Corollary 4.4: The union of a simple chain of closed bounded convex subsets of \( E \) is negligible in \( E \).

The hypothesis of boundedness in the preceding is needed since even a single closed convex set which is not bounded may separate \( E \), hence not be negligible in \( E \).
V. NEGLIGIBILITY OF UNIONS

Throughout this chapter, $E$ will denote an infinite-dimensional normed linear space.

**Theorem 5.1:** Let $\{K_\alpha\}$ be a locally finite collection of disjoint closed subsets of $E$ such that for each $\alpha$ and any neighborhood $U_\alpha$ of $K_\alpha$ there is a homeomorphism $h_\alpha$ from $E \setminus K_\alpha$ onto $E$ which is the identity on $E \setminus U_\alpha$. Then $\bigcup K_\alpha$ is negligible in $E$.

**Remark:** It will be shown that the theorem applies if the $K_\alpha$ are tame cells, strongly cellular sets, strongly negligible sets (Definition 5.1) or closed bounded starshaped sets. If the annulus conjecture [17] holds for $E$ (e.g. if $E$ is homeomorphic to the countable infinite product of copies of itself [6]), the theorem applies to sets having a neighborhood basis of cells.

**Proof:** Since $E$ is collectionwise normal, by a result of Dowker [7], there is a discrete collection $\{G_\alpha\}$ of open sets such that $K_\alpha \subseteq G_\alpha$ for each $\alpha$. Let $C_\alpha$ be a closed set with $K_\alpha \subseteq \text{Int } C_\alpha \subseteq C_\alpha \subseteq G_\alpha$ for each $\alpha$ and let $h_\alpha$ be the homeomorphism given by hypothesis with $U_\alpha = \text{Int } C_\alpha$. Define a map $H$ from $E \setminus \bigcup K_\alpha$ to $E$ by $H|G_\alpha \setminus K_\alpha = h_\alpha$ and $H|E \setminus \bigcup C_\alpha$ is the identity. Clearly $H$ is well-defined, one-to-one and onto. Since the collection $\{C_\alpha\}$ is
discrete, \( \cup C_\alpha \) is closed and \( E \setminus \cup C_\alpha \) is open. Thus \( H \) is continuous [3, p. 83]. Since the map \( H^{-1} \) has a similar description, it is also continuous.

**Definition 5.1:** [1] A subset \( A \) of a topological space \( X \) is **strongly negligible in** \( X \) if, for each open cover \( G \) of \( X \), there is a homeomorphism \( h \) of \( X \) onto \( X \setminus A \) which is limited by \( G \), i.e., for any \( p \in X \) there exists \( g \in G \) such that both \( p \) and \( h(p) \) are elements of \( g \).

As noted in [2], if \( A \) is a strongly negligible subset of \( E \) and \( U \) is an open set containing \( A \), then the homeomorphism of the definition can be made the identity on \( E \setminus U \). Thus Theorem 5.1 applies to strongly negligible sets. In fact, \( \cup K_\alpha \) is strongly negligible in this case.

**Definition 5.2:** [20] If \( C \) is a cell in \( E \), a closed subset \( K \) of \( E \setminus \text{Int } C \) is a **collar** of \( C \) if there exists a homeomorphism \( h \) of the pair \((E_2,E_1)\) onto the pair \((K \cup C,C)\) such that \( h(S_2) = \text{Bd}(K \cup C) \).

The following theorem is due to Sanderson [20] and McCoy [18].

**Theorem 5.2:** A cell in \( E \) is tame if and only if it has a collar.

**Proposition 5.1:** If \( C \) is a tame cell in \( E \) and \( U \) is an open set containing \( C \), then there is a homeomorphism from \( E \setminus C \) onto \( E \) which is the identity on \( E \setminus U \).
Proof: By Lemma 1.1 of [20], there is a collar $C'$ of $C$ contained in $U$. Let $f$ be a homeomorphism on $E$ which takes $(B_1,S_1)$ onto $(C,\text{Bd} \ C)$ and $(B_2,S_2)$ onto $(C \cup C',\text{Bd}(C \cup C'))$. Let $g$ be a homeomorphism of $E \setminus B_1$ onto $E \setminus \{\emptyset\}$ which is the identity on $E \setminus \text{Int} \ B_2$. By Klee [13], there is a homeomorphism $h$ from $E \setminus \{\emptyset\}$ onto $E$ such that $h$ is the identity on $E \setminus \text{Int} \ B_2$. The desired homeomorphism is $H = fghf^{-1}(E \setminus C)$.

Definition 5.3: [18] If $C$ and $C'$ are two closed subsets of $E$ such that $C' \subset \text{Int} \ C$, then $C$ and $C'$ will be said to have annular difference if there exists a homeomorphism $h$ of $B_1 \setminus \text{Int} B_1/2$ onto $C \setminus \text{Int} C'$ such that $h(S_1) = \text{Bd} \ C$ and $h(S_1/2) = \text{Bd} \ C'$.

The annulus conjecture is stated as follows in [17] and in [6] it is shown that this conjecture holds in a space $E$ which is homeomorphic to a countable infinite product of copies of itself.

Annulus Conjecture for $E$: If $C$ is a tame cell in $E$ contained in $\text{Int} B_1$, then there exists a homeomorphism $h$ from $B_1$ onto itself such that $h(B_1/2) = C$ and $h|S_1 = \text{identity}$.

Proposition 5.2: Suppose that the annulus conjecture holds for $E$, that $K$ is a closed subset of $E$ having a neighborhood basis of cells and $U$ is an open set containing $K$. Then there is a homeomorphism of $E$ onto $E \setminus K$ which is the
identity on $E \setminus U$.

**Proof:** (This is a modification of the proof of Theorem 2.2 of [18].) It is clear from the proof of Proposition 3.7 that a cellular sequence $\{C_i\}_{i=1}^{\infty}$ for $K$ can be constructed which lies in $U$. By Theorem 2.1 of [18], it may be assumed that each $C_i$ is tame and each $C_i$ and $C_{i+1}$ have annular difference. Assume, without loss of generality, that $B_{1/2} \subset \operatorname{Int} C_1$. Since $C_1$ and $C_2$ have annular difference, there is a homeomorphism $f$ of $B_1 \setminus \operatorname{Int} B_{1/2}$ onto $C_1 \setminus \operatorname{Int} C_2$ such that $f(S_1) = \partial d C_1$ and $f(S_{1/2}) = \partial d C_2$.

By the annulus conjecture, there is a homeomorphism $f'$ of $B_1 \setminus \operatorname{Int} B_{1/2}$ onto $C_1 \setminus \operatorname{Int} B_{1/2}$ such that $f'(S_1) = \partial d C_1$ and $f'(S_{1/2}) = S_{1/2}$. Define a homeomorphism $g$ of $B_1 \setminus \operatorname{Int} B_{1/2}$ onto itself such that $g|S_1 = f^{-1}f'|S_1$ and $g(S_{1/2}) = S_{1/2}$. Define $h_2 = fgf'^{-1}$, which is a homeomorphism from $C_1 \setminus \operatorname{Int} B_{1/2}$ onto $C_1 \setminus \operatorname{Int} C_2$ such that $h_2(\partial d C_1) = \partial d C_1$, $h_2(S_{1/2}) = \partial d C_2$ and $h_2|\partial d C_1$ is the identity. Then by induction define, for each $n > 2$, $h_n$ from $B_{1/n-1} \setminus \operatorname{Int} B_{1/n}$ onto $C_{n-1} \setminus \operatorname{Int} C_n$ such that $h_n(S_{1/n-1}) = \partial d C_{n-1}$, $h_n(S_{1/n}) = \partial d C_n$ and $h_n|S_{1/n-1} = h_{n-1}|S_{1/n-1}$. A homeomorphism $h$ from $E \setminus \{\emptyset\}$ onto $E \setminus K$ is defined by $h(x) = x$ if $x \in E \setminus \operatorname{Int} C_1$, $h(x) = h_2(x)$ if $x \in C_1 \setminus \operatorname{Int} B_{1/2}$ and $h(x) = h_n(x)$ if $x \in B_{1/n-1} \setminus B_{1/n}$ for $n > 2$. Let $G$ be a homeomorphism of $E$ onto $E \setminus \{\emptyset\}$ such that $G|E \setminus \operatorname{Int} B_{1/2}$ is the identity.
[13]. The desired homeomorphism is \( h_G \).

In case \( K \) is a closed bounded starshaped set, the preceding proposition can be proved without the annulus conjecture.

**Proposition 5.3:** Suppose that \( K \) is a closed bounded starshaped subset of \( E \) and \( U \) is an open set containing \( K \). Then there is a homeomorphism of \( E \) onto \( E \setminus K \) which is the identity on \( E \setminus U \).

**Proof:** The method of proof is to construct a cellular sequence for \( K \) which satisfies the conditions in the proof of Proposition 5.2. Suppose that \( K \) is starshaped from \( p \).

By Proposition 2.5, there is a closed bounded shrinkable neighborhood \( C_1 \) of \( p \) such that \( K \subset \text{Int} \; C_1 \) and \( C_1 \subset (U \cap N_1(K)) \). Inductively, there is a sequence \( \{C_n\}_{n=1}^{\infty} \) of closed bounded shrinkable neighborhoods of \( p \) such that \( K \subset \text{Int} \; C_{n+1} \subset C_{n+1} \subset (N_1/\cap N_1(K) \cap \text{Int} \; C_n) \) for each \( n \).

Since each \( C_n \) is a tame cell (Proposition 4.2), \( \{C_n\}_{n=1}^{\infty} \) is a cellular sequence for \( K \) contained in \( U \).

Let \( n \) be a fixed positive integer. There exist \( \epsilon > 0 \) and \( r \) such that \( B_\epsilon(p) \subset \text{Int} \; C_{n+1} \) and \( C_n \subset \text{Int} \; B_r(p) \).

According to Proposition 2.2, there is a homeomorphism \( g_1 \) on \( E \) which takes \( (B_\epsilon(p), S_\epsilon(p)) \) onto \( (C_{n+1}, \text{Bd} \; C_{n+1}) \) and is the identity on \( E \setminus \text{Int} \; C_n \). Similarly, there is a homeomorphism \( g_2 \) which takes \( (B_r(p), S_r(p)) \) onto \( (C_n, \text{Bd} \; C_n) \) and
is the identity on \( C_{n+1} \). Then \( g_2g_1 \) is a homeomorphism which maps \( B_r(p) \setminus \text{Int } B_{\varepsilon}(p) \) onto \( C_n \setminus \text{Int } C_{n+1} \) in such a way that \( g_2g_1(S_r(p)) = \text{Bd } C_n \) and \( g_2g_1(S_{\varepsilon}(p)) = \text{Bd } C_{n+1} \). Thus, \( C_n \) and \( C_{n+1} \) have annular difference.

Assume, without loss of generality, that \( B_{1/2} \subset \text{Int } C_1 \). Using Proposition 2.2 again, there is a homeomorphism of \( B_1 \setminus \text{Int } B_{1/2} \) onto \( C_1 \setminus \text{Int } B_{1/2} \) which takes \( S_1 \) onto \( \text{Bd } C_1 \) and \( S_{1/2} \) onto itself.

It is now clear that the proof of Proposition 5.2 may be used to complete the argument.

Proposition 5.2 clearly applies to strongly cellular sets, but it will be shown next that the result for strongly cellular sets does not depend on the annulus conjecture.

In [13], Klee extends some of the results of [11] to arbitrary infinite-dimensional normed linear spaces. A slight modification of the arguments yields the following stronger version of one of the results.

**Proposition 5.4:** If \( K \) is a compact subset of \( \text{Int } B_1 \) in \( E \), then there is a homeomorphism of \( E \) onto \( E \setminus K \) which is the identity on \( E \setminus \text{Int } B_1 \).

**Proof:** In [13], a decreasing sequence \( \{ C_i \}_{i=0}^{\infty} \) of unbounded but linearly bounded closed convex sets with empty intersection is constructed by means of a continuous linear functional \( f \) such that \( C_i \subset f^{-1}([i,\infty)) \) for each \( i \). By
Proposition 3.10, there is a homeomorphism \( h \) on \( E \) which takes \( (B_1, S_1) \) onto \( (f^{-1}([0, \omega)), f^{-1}(0)) \). \( Y = h(K) \) is a compact subset of \( \text{Int } f^{-1}([0, \omega]) \) and \( C_1 \subset f^{-1}([1, \omega]) \), so there exists \( \varepsilon > 0 \) such that \( N_{3\varepsilon}(Y) \subset f^{-1}([0, \omega]) \) and \( N_{2\varepsilon}(C_1) \subset f^{-1}([0, \omega]) \).

The homeomorphism \( T \) is defined as in I.4.1 of [11] beginning with \( C_1 \) instead of \( C_0 \) and using \( f^{-1}([0, \omega]) \) in place of \( B \). \( T \) can be extended to a homeomorphism of \( E \) onto \( E \setminus Y \) which is the identity on \( f^{-1}((\omega, 0]) \). The desired homeomorphism is \( h^{-1}T_h \).

Corollary 5.1: If \( C \) is a cell in \( E \) and \( K \) is a compact subset of \( \text{Int } C \), then there is a homeomorphism of \( E \) onto \( E \setminus K \) which is the identity on \( E \setminus \text{Int } C \).

Proof: It follows easily from Lemma 1.4 of [18] that there is a tame cell \( D \) such that \( K \subset \text{Int } D \subset D \subset C \). If \( f \) is a homeomorphism on \( E \) which takes \( (B_1, S_1) \) onto \( (D, B \setminus D) \), then \( f^{-1}(K) \) satisfies the condition of the proposition.

Corollary 5.2: If \( K \) is strongly cellular in \( E \) and \( U \) is an open set containing \( K \), then there is a homeomorphism of \( E \) onto \( E \setminus K \) which is the identity on \( E \setminus U \).

Proof: Since \( K \) is strongly cellular, \( K \) is compact and \( U \) contains a cell neighborhood of \( K \), so Corollary 5.1 applies.

Proposition 5.5: Let \( K \) be a cellular subset of \( \text{Int } B_1 \) in
Then there is a cellular sequence for \( K \) which is contained in \( \text{Int} \, B_1 \).

**Proof:** By Lemma 1.4 of [12], there is a cell \( C \subseteq \text{Int} \, B_1 \) such that \( K \subseteq \text{Int} \, C \) and \( B_1 \setminus \text{Int} \, C \) is a collar of \( C \).

Let \( \{C_i\}_{i=1}^{\infty} \) be a cellular sequence for \( K \). Since \( \bigcap_{i=1}^{\infty} C_i = K \), \( C_j \cup C \neq E \) for some \( j \). Without loss of generality, suppose \( C_1 \cup C \neq E \). By Theorem 2 of [17], there is a homeomorphism \( h \) on \( E \) such that \( C_1 \subseteq h(B_1) \) and \( h|C \) is the identity. \( \{h^{-1}(C_i)\}_{i=1}^{\infty} \) is a cellular sequence for \( h^{-1}(K) = K \). Since for \( i \geq 2 \),

\[
h^{-1}(C_1) \subseteq h^{-1}(C_2) \subseteq \text{Int} \, h^{-1}(C_1) \subseteq \text{Int} \, B_1,
\]

\( \{h^{-1}(C_i)\}_{i=2}^{\infty} \) is a sequence which completes the proof.

Note that if each \( C_i \) is tame and each \( C_i \) and \( C_{i+1} \) have annular difference, then the same is true about the new sequence constructed in the proof. The proof of the following corollary is similar to the proof of Corollary 5.1 and is omitted.

**Corollary 5.3:** If \( C \) is a cell in \( E \) and \( K \) is a cellular subset of \( \text{Int} \, C \), then there is a cellular sequence for \( K \) which is contained in \( \text{Int} \, C \).

**Proposition 5.6:** Suppose that the annulus conjecture holds for \( E \), \( K \) is closed and negligible in \( E \) and \( C \) is a cell containing \( K \) in its interior. Then there is a homeomorphism
of E onto E \ K which is the identity on E \ Int C.

**Proof:** By Theorem 4.2 and Corollary 5.3, there is a cellular sequence for K which is contained in Int C. The remainder of the proof is the same as the proof of Proposition 5.2.

**Theorem 5.3:** Suppose that the annulus conjecture holds for E and \{K_\alpha\} is a collection of negligible sets in E. If there is a locally finite collection \{C_\alpha\} of disjoint cells such that \(K_\alpha \subset \text{Int }C_\alpha\) for each \(\alpha\), then \(\bigcup K_\alpha\) is negligible.

**Proof:** As in the proof of Theorem 5.1, there is a discrete collection \{G_\alpha\} of open sets such that \(C_\alpha \subset G_\alpha\) for each \(\alpha\). By Proposition 5.6, there is a collection \{h_\alpha\} of maps such that each \(h_\alpha\) is a homeomorphism of E \ K_\alpha onto E which is the identity on E \ Int C_\alpha. Define a map \(H\) from E \ \bigcup K_\alpha to E by \(H|_{G_\alpha \setminus K_\alpha} = h_\alpha\) and H|E \ \bigcup C_\alpha is the identity. The fact that \(H\) is a homeomorphism follows in the same way as in the proof of Theorem 5.1.

Since compact subsets of E are negligible, Theorem 5.3 applies to a collection of compact sets. For compact sets, however, the annulus conjecture is not needed.

**Theorem 5.4:** Suppose that \{K_\alpha\} is a collection of compact subsets of E. If there is a locally finite collection \{C_\alpha\} of disjoint cells such that \(K_\alpha \subset \text{Int }C_\alpha\) for each \(\alpha\), then \(\bigcup K_\alpha\) is negligible.
Proof: The proof is the same as the proof of Theorem 5.3 using Corollary 5.1 in place of Proposition 5.6.

Theorem 5.5: Let \( \{K_\alpha\} \) be a collection of compact subsets of \( E \) whose closed convex hulls form a locally finite disjoint collection. Then \( \bigcup K_\alpha \) is negligible.

Proof: Let \( M_\alpha \) be the closed convex hull of \( K_\alpha \) for each \( \alpha \). Since \( E \) is collectionwise normal, there is a discrete collection \( \{U_\alpha\} \) of open sets such that \( M_\alpha \subseteq U_\alpha \) for each \( \alpha \). By Propositions 2.5 and 4.2, there is a collection \( \{C_\alpha\} \) of cells with \( K_\alpha \subseteq M_\alpha \subseteq \text{Int } C_\alpha \subseteq C_\alpha \subseteq U_\alpha \) for each \( \alpha \). The theorem follows from Theorem 5.4.
VI. LITERATURE CITED


VII. ACKNOWLEDGMENTS

The author thanks Professor D. E. Sanderson for his encouragement, patience and guidance during the preparation of this dissertation.

Research for this dissertation was done in part while the author held a NSF Graduate Summer Traineeship.