A study of finite amplitude disturbances in plane Poiseuille flow by finite-difference methods

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A STUDY OF FINITE AMPLITUDE DISTURBANCES IN PLANE POISEUILLE FLOW

BY FINITE-DIFFERENCE METHODS

by

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I. INTRODUCTION

One of the fundamental problems in the field of fluid mechanics concerns the stability of parallel laminar motion and its transition to turbulence. Interest in these phenomena dates back to the experimental investigations of pipe flows by O. Reynolds (1883) (ref. 1) in which it was observed that when the Reynolds number exceeds a certain critical value, the laminar flow becomes turbulent. The study of stability and transition has since continued to attract a great deal of attention. However, despite many attempts no adequate mathematical theory exists which can predict satisfactorily the transition process and describe the characteristics of turbulent shear flows.

The basic difficulty in predicting the instability of laminar flows arises from the nonlinearity of the Navier-Stokes equations. Most attempts at predicting instability have been made by linear analysis, which assumes that the amplitude of the disturbance is infinitesimally small. While the mathematical aspects of the linear theory are well in hand, the theory is not powerful enough to predict the critical Reynolds numbers observed in many physical flows. The most notable failures are plane Couette and pipe flows for which linear theory predicts stability for all Reynolds numbers and disturbance frequencies in contrast to the experimental evidences of instability. The results suggest that the observed instabilities are due to nonlinear effects that become important with an increase in the size of the disturbance.

With the availability of high-speed computers it is feasible to study the effects of finite amplitude disturbances by the application of suitable finite-difference methods to the nonlinear equations of motion. The
present study applies these methods to classical plane Poiseuille flow.

The advantage of this simple case is that instabilities have been predicted by both linear analysis and theoretical nonlinear studies. Thus, the finite-difference solutions can be compared to both theoretical and experimental results.

The present problem is formulated in terms of streamwise periodic perturbations about the known laminar solution with the first eigenmode of the well-known Orr-Sommerfeld equation providing the initial values. Calculations are performed with the following three objectives in mind. The first objective is to test the accuracy and numerical stability of finite-difference methods by comparing the long time integration of the linearized equations with the eigenfunctions and eigenvectors of the Orr-Sommerfeld equation. The second objective is to study the effect of finite amplitude disturbances on flow stability by integrating the nonlinear equations for Reynolds numbers both below and above the linear critical value. The third objective is to investigate whether or not a flow that will not return to its laminar state will, after some long time integration, reveal characteristics of two-dimensional turbulence.

The difficulties encountered in the analytic description of turbulence are well known. In the process of statistical averaging of the nonlinear equations the rate of change of any averaged quantity inevitably involves some other averaged quantity. Therefore, no finite set of differential equations for any averaged quantity can be deduced. This is an example of the so-called closure problem of statistical averaging. Thus, although the Navier-Stokes equations for the basic motion are determinate, the averaged
equations are not. In practical applications some closure hypothesis for
the averaged equations, such as Prandtl's mixing length hypothesis (ref. 2),
is required.

An alternative approach to the study of turbulence is the direct
numerical solution of the basic equations of motion by finite-difference
analogs. With regard to three-dimensional flows, however, it appears that
the numerical simulation of transition and turbulence, including motions
with scales down to the viscous cutoff, is beyond the capability of present
computers because of the large amount of storage and the large number of
operations required. As an example, Emmons (ref. 3) estimates that with
present computers the numerical simulation of the simplest turbulent pipe
flow at a moderate Reynolds number of 5000 would require $10^{10}$ grid points
and 100 years of computer time!

An alternative approach, within the capability of current computers,
is to consider two-dimensional turbulent motions (refs. 4-6). There are,
however, a number of fundamental differences between turbulent motion in
two and three dimensions. In particular, the magnification of vorticity by
the stretching of vortex lines, which is a well-known feature of the inertial
effect of three-dimensional turbulence, is absent in two dimensions. In
three-dimensional isotropic turbulence the energy spectrum in the inertial
subrange of wave numbers follows the well-known Kolmogoroff $-5/3$ power law
(ref. 7). In two dimensions, however, Kraichnan (ref. 8) has shown that
while the energy spectrum follows a $-5/3$ power law for the small wave num-
ber, it must follow a $-3$ power law for high wave numbers. This is a conse-
quence of the fact that two-dimensional inviscid flow contains, in addition
to kinetic energy, a second quadratic invariant, the squared vorticity.
The -5/3 power law transfers energy down the wave number scale and the -3 power law transfers mean squared vorticity up the wave number scale. Despite the differences between two- and three-dimensional turbulent motion, the fact remains that two fundamental properties are common to both - non-linearity and randomness. Thus, the two-dimensional simulation should illustrate some of the basic effects of turbulent motion, such as the generation of different scales of motion and the formation of Reynolds stresses which deform the mean flow. It is also assumed that the simulation of two-dimensional flows would at least be a rational prelude to calculations in three dimensions.

The restriction to calculations in only two dimensions is not the only limitation of the finite-difference approach. Consideration must also be given to the accumulation of numerical truncation and round-off errors. In the computation of an initial value problem for transition to turbulence it may not be possible to reach a truly equilibrium turbulent state; consequently, the character of the flow must be determined by the integration over some finite time period. The length of the integration is determined both by the accumulation of numerical errors and the practical limitations on available computer time.
II. PROBLEM FORMULATION

A. Basic Equations

The conservation of momentum of an incompressible fluid in the absence of body forces is expressed in vector form as

$$\frac{\partial \vec{v}}{\partial t} = -\vec{v} \cdot \nabla \vec{v} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{v}$$ (2.1)

where $\nu$, the kinematic viscosity, is assumed to be constant. The conservation of mass is expressed as

$$\nabla \cdot \vec{v} = 0$$ (2.2)

For the case of two-dimensional, plane Poiseuille flow the conservation equations are made dimensionless with respect to the channel half-height, $h$, and the centerline velocity of the undisturbed laminar flow, $U_o$. The coordinate system is shown in figure 1. Equation (2.1) then becomes

$$u_t = -UU_x - VU_y - \frac{P_x}{\rho} + \frac{1}{Re} (U_{xx} + U_{yy})$$ (2.3a)

$$v_t = -UV_x - VV_y - \frac{P_y}{\rho} + \frac{1}{Re} (V_{xx} + V_{yy})$$ (2.3b)

where the subscripts denote differentiation and $Re = U_0 h / \nu$ is the Reynolds number. In addition, the continuity equation becomes

$$U_x + V_y = 0$$ (2.4)

A more compact formulation of the problem is obtained by using the stream function $\psi$ and vorticity $\Omega$ as dependent variables. Defining the vorticity as

$$\Omega = U_y - V_x$$ (2.5)
Figure 1.- Coordinate system for plane Poiseuille flow.
and eliminating the pressure by taking the curl of equation (2.1) leads to the vorticity transport equation

\[ \Omega_t = -U\Omega_x - V\Omega_y + \frac{1}{Re} (\Omega_{xx} + \Omega_{yy}) \quad (2.6) \]

The condition that mass is conserved is satisfied by defining a stream function such that

\[ U = \psi_y \quad \text{(2.7a)} \]
\[ V = -\psi_x \quad \text{(2.7b)} \]

Equations (2.5) and (2.7) then relate vorticity and stream function through the Poisson equation

\[ \Omega = \psi_{xx} + \psi_{yy} \quad \text{(2.8)} \]

B. Perturbation Quantities

It is advantageous to divide the flow variables into two parts, the undisturbed fully developed laminar flow and a perturbation about this flow. Let

\[ \Psi = \Psi_0 + \psi \quad \text{(2.9a)} \]
\[ \Omega = \Omega_0 + \omega \quad \text{(2.9b)} \]
\[ U = U_0 + u \quad \text{(2.9c)} \]
\[ V = V_0 + v \quad \text{(2.9d)} \]

where \( \Psi_0, \Omega_0, U_0, \) and \( V_0 \) are the undisturbed laminar quantities and \( \psi, \omega, u, \) and \( v \) are the perturbation variables. The steady laminar solution for the plane Poiseuille is well known and is given as

\[ \Psi_0 = y - \frac{y^3}{3} \quad \text{(2.10a)} \]
\[ \Omega_0 = -2y \quad \text{(2.10b)} \]
If equations (2.9) are substituted into equations (2.6) and (2.8), and one notes that the undisturbed variables themselves satisfy the equations, the following expressions for the perturbation vorticity and stream function are obtained:

\[ \omega_t = -(U_0 + u) \omega_x - v \omega_y + 2v + \frac{1}{Re} (\omega_{xx} + \omega_{yy}) \]  
\[ \psi_{xx} + \psi_{yy} = \omega \]  

In addition, the perturbation velocities are given by

\[ u = \psi_y \]  
\[ v = -\psi_x \]  

It should be noted that formulating the problem as one for the perturbation of a known flow does not alter its nonlinear character. This splitting has, however, produced expressions for the small perturbation quantities of primary interest. It should also be noted that these perturbations are not to be confused with the variables describing turbulent motion, that is, those quantities describing departure from the mean flow. This point is developed more thoroughly later.

C. Boundary Conditions

The boundary conditions at a rigid, impermeable wall, for a viscous flow, are that there be neither mass flow normal to the wall nor slip along it, so that

\[ U_{wall} = 0 \]  
\[ V_{wall} = 0 \]
In terms of the stream function the first condition becomes

\[ (\psi_y)_{\text{wall}} = 0 \]  \hspace{1cm} (2.16)

The second condition must be combined with an additional qualification for steady flow between parallel plates. Typically, either the volume flux or mean pressure gradient is held constant. The condition of constant volume flux is natural to the vorticity - stream-function formulation and is the one chosen here. Thus, the condition that the volume flux remain unchanged requires that the perturbation stream function have no net change across the channel. The wall values may, without loss of generality, be taken as zero.

\[ \psi_{\text{wall}} = 0 \]  \hspace{1cm} (2.17)

The boundary conditions in the streamwise direction are replaced by the assumption of periodicity in \( x \) over a wavelength \( 2\pi/\alpha \) such that

\[ \psi \left( x + \frac{2\pi}{\alpha}, y, t \right) = \psi(x, y, t) \]  \hspace{1cm} (2.18a)

\[ \omega \left( x + \frac{2\pi}{\alpha}, y, t \right) = \omega(x, y, t) \]  \hspace{1cm} (2.18b)

\[ u \left( x + \frac{2\pi}{\alpha}, y, t \right) = u(x, y, t) \]  \hspace{1cm} (2.18c)

\[ v \left( x + \frac{2\pi}{\alpha}, y, t \right) = v(x, y, t) \]  \hspace{1cm} (2.18d)

The disturbance is in the form of a traveling wave with wave number \( \alpha \). This wave, which either grows or decays with time, can be qualitatively interpreted as a physical disturbance seen by an observer traveling with the disturbance.
D. Mean and Fluctuating Quantities

The particular manner in which the flow is separated into the original Poiseuille solution and a perturbation about it has been chosen to increase the accuracy of the numerical integration, since one achieves more accurate results by computing the small quantities rather than solving for the whole. We can take advantage of the perturbation method because the laminar Poiseuille solution is known. On the other hand, a formulation of the dependent variables, which lends itself more readily to physical interpretation, splits the flow into a mean part and a disturbance or fluctuating part having zero mean. This is the formulation typically used to describe turbulent flows, and results in the concept of the mean motion interacting with the fluctuations through the action of a Reynolds stress.

In this formulation the total motion is described as follows:

\[
\begin{align*}
\psi &= \overline{\psi} + \psi' \\
\Omega &= \overline{\Omega} + \omega' \\
U &= \overline{U} + u' \\
V &= \overline{V} + v'
\end{align*}
\]  

(2.19a)  

(2.19b)  

(2.19c)  

(2.19d)

where the barred quantities describe the mean motion and the primed quantities are the fluctuations about the mean. Since in the mathematical model the motion is periodic in the streamwise direction, the fluctuations are statistically homogeneous in the x-direction. The mean is then defined as the average over a period of the primary disturbance. For example, the mean streamwise velocity is given by

\[
\overline{U}(y,t) = \frac{a}{2\pi} \int_{x_0}^{x_0 + 2\pi/a} U(x,y,t) \, dx
\]  

(2.20)
While by definition the mean values of the fluctuations vanish, the mean values of quadratic functions of the fluctuations do not vanish. Important functions of this type are the Reynolds stresses and the energy of the fluctuations.

When a disturbance is introduced into the flow, momentum is transferred by the fluctuations, resulting in the formation of a system of stresses known as the Reynolds stresses. These stresses arise naturally in the momentum equations for the mean motion. If one recalls that the mean motion is independent of \( x \) and uses the continuity equation, the averages of equations (2.3) become

\[
\overline{U_t} = -(\overline{u'v'})_y - \frac{\overline{F}}{\rho} + \frac{1}{\text{Re}} \overline{u_y} \tag{2.21a}
\]

and

\[
(\overline{v'^2})_y + \frac{\overline{F}}{\rho} = 0 \tag{2.21b}
\]

The additional terms due to fluctuations are the Reynolds shear stress defined as

\[
-\overline{u'v'} = -\frac{a}{2\pi} \int_{x_0}^{x_0 + \frac{2\pi}{a}} u'v' \, dx \tag{2.22a}
\]

and the Reynolds normal stress defined as

\[
-\overline{v'^2} = -\frac{a}{2\pi} \int_{x_0}^{x_0 + \frac{2\pi}{a}} v'^2 \, dx \tag{2.22b}
\]

Often the growth or decay in time of one component of the fluctuation velocity is sufficient to determine the instability or stability of the flow, but a more significant parameter is probably the growth or decay of the energy of the fluctuations. Since the flow is incompressible the energy of the fluctuations consists of kinetic energy only; thus its mean
value is expressed as

\[ E' = \sqrt{\frac{1}{2\alpha} \int_{-\infty}^{\infty} \left( \frac{2\pi}{\alpha} \right) \left( u'^2 + v'^2 \right) dx } \]  \hspace{1cm} (2.23)

E. The Energy Balance and the Role of the Reynolds Stress

The importance of the Reynolds stress on the stability of plane Poiseuille flow is illustrated by deriving the energy balance for the velocity fluctuations (see ref. 9). Subtracting the equations of the mean motion from equations (2.3) results in the following equations for the fluctuating motion:

\[ u'_t = -\overline{u}u'_x - v'_y - u'_u'_x - v'_u'_y + \overline{u'u'_y} + \overline{v'v'_x} - \frac{P'_x}{\rho} + \frac{1}{Re} (u'_{xx} + u'_{yy}) \]  \hspace{1cm} (2.24a)

\[ v'_t = -\overline{v}v'_x - u'_v'_y - v'_v'_x + \overline{u'v'_x} + \overline{v'v'_y} - \frac{P'_y}{\rho} + \frac{1}{Re} (v'_{xx} + v'_{yy}) \]  \hspace{1cm} (2.24b)

Multiplying equation (2.24a) by \( u' \) and equation (2.24b) by \( v' \) and integrating the sum of these equations across the channel in \( y \) and over a given wavelength in \( x \) gives the energy relation

\[ \frac{\partial E'}{\partial t} = I_1 - I_2 \]  \hspace{1cm} (2.25)

where

\[ E' = \int_{-1}^{1} \int_{x_0}^{x_0} \frac{2\pi}{\alpha} \left( u'^2 + v'^2 \right) dx \] dy

\[ I_1 = \int_{-1}^{1} \frac{\partial \overline{U}(y,t)}{\partial y} \left\{ \int_{x_0}^{x_0} \frac{2\pi}{\alpha} -u'v' dx \right\} dy \]

and

\[ I_2 = \frac{1}{Re} \int_{-1}^{1} \int_{x_0}^{x_0} \frac{2\pi}{\alpha} \omega'^2 dx \] dy.
Thus, the time rate of change of the energy of the fluctuations depends on the balance of the two integrals on the right-hand side of equation (2.25). The first integral $I_1$, called the production integral, represents the exchange of energy between the mean flow and the fluctuations. The second integral $I_2$, called the dissipation integral, gives the rate of energy loss due to viscous dissipation. If the Reynolds stress is of the same sign as $\partial U/\partial y$, $I_1$ is positive and this term tends to transfer energy from the mean flow to the fluctuations. Then, since $I_2$ is always positive, instabilities in the sense of $\partial E'/\partial t > 0$ can occur only if $I_1$ is sufficiently positive. Such instabilities may lead to a fully developed turbulent flow or an unsteady laminar secondary flow. If the flow is stable, the initial disturbance energy will decay to zero as the steady laminar flow reappears.

It is apparent from the energy balance equation that the Reynolds stress plays an important role in the stability of plane Poiseuille flow. Now, the distribution of Reynolds stress across the channel depends on viscosity and its effect on the phase relationship between the $u'$ and $v'$ velocity fluctuations. To illustrate this dependence consider the $u'$ velocity fluctuation as a traveling wave proportional to, say, $\cos \alpha(x - ct)$ where $c$ is the phase velocity of the wave. In an inviscid flow, because of flow continuity, $u'$ and $v'$ differ in phase by precisely $\pi/2$, resulting in a vanishing Reynolds stress. There are two regions of the flow, however, where the presence of viscosity significantly alters this phase difference. The first is the Stokes layer of thickness $(v/\alpha c)^{1/2}$ next to the wall where the displacement effect gives rise to a part of $v'$ proportional to $u'$ itself, thus producing a phase shift. The second is near the point where the phase velocity of the fluctuation is equal to the mean
Figure 2.- Reynolds stress from linear theory; Re = 10,000 (ref. 10).
flow velocity (i.e., $c = \overline{U}$). Surrounding this critical point, $y = y_c$, is a region called the critical layer having a thickness on the order of $v/[a(\partial \overline{U}/\partial y)]$. Within this layer, viscosity also brings about a pronounced phase shift in $u'$ and $v'$, so that their phase difference is no longer $\pi/2$. An example of the Reynolds stress calculated from linear theory is shown in figure 2. Note the "spiked" appearance near the critical point. For more detailed presentations of the above discussion the reader is referred to references 11 through 13.

F. Comparison of the Two Formulations

While the concept of a mean motion and fluctuations about the mean gives some physical insight into the problem of parallel flow stability, this formulation is not desirable for the purpose of finite-difference calculations because both the mean flow and the fluctuations are unknown a priori. Alternatively one could calculate the entire flow but this would be considerably less accurate, especially if the fluctuations were small. The method chosen here is to calculate the perturbations about the original undisturbed laminar flow and then through a harmonic decomposition extract the mean part of the perturbations. Thus the results are presented in the more physically meaningful terms of mean flow and fluctuations about the mean, while the calculations are done in terms of the original laminar flow.
Considerable effort has been devoted to the study of hydrodynamic stability through the use of linear theory. For a comprehensive treatment of linear stability theory see Lin (ref. 12). Basic to linear theory is the assumption that the disturbances are infinitesimally small such that there is no modification of the mean flow by the fluctuations and there are no interactions between the fluctuations themselves. Because the mean flow is unmodified, the distinction between perturbations and fluctuations no longer exists.

In applying the linear theory to the present problem the nonlinear terms in the vorticity transport equation (2.11) are ignored and the following linear equation is obtained:

$$\omega_t = -U_0 \omega_x + 2v + \frac{1}{Re} (\omega_{xx} + \omega_{yy})$$

(3.1)

Because of the nature of the linear partial differential equation (3.1), solution by the standard variable separable procedure is possible. It is assumed that the components of the solution are the real parts of complex functions of the form

$$\psi = \hat{\psi}(y)e^{i\alpha(x-ct)}$$

(3.2a)

$$\omega = \hat{\omega}(y)e^{i\alpha(x-ct)}$$

(3.2b)

$$u = \hat{u}(y)e^{i\alpha(x-ct)}$$

(3.2c)

$$v = \hat{v}(y)e^{i\alpha(x-ct)}$$

(3.2d)

These functions describe traveling waves of complex amplitude, denoted by the circumflex (^), with wave number \( \alpha \) and complex wave speed.
c = c_r + ic_i. The sign of c_i determines the growth or decay of the wave with time.

Assuming the functional form (3.2), the linearized vorticity transport equation (3.1) and the Poisson equation (2.12) combine to form the well-known Orr-Sommerfeld equation for the complex amplitude of the stream-function fluctuation

\[(D^2 - \alpha^2)^2 \hat{\psi} = i\alpha \text{Re}(U_0 - c)(D^2 - \alpha^2)\hat{\psi} - (D^2 U_0)\hat{\psi}\]  

(3.3)

where \(D = d/dy\). The boundary conditions are

\[\hat{\psi} = D\hat{\psi} = 0 \text{ at } y = \pm 1\]  

(3.4)

Since equation (3.3) is symmetrical in y, any solution \(\hat{\psi}(y)\) may be split into a part that is even or symmetric about the channel centerline, \(\hat{\psi}_e\), and a part that is odd or antisymmetric about the centerline, \(\hat{\psi}_o\), so that

\[\hat{\psi} = \hat{\psi}_e + \hat{\psi}_o\]  

(3.5)

It is necessary then to consider only half the channel with the boundary conditions at the centerline given as

\[D\hat{\psi}(0) = D^3\hat{\psi}(0) = 0\]  

(3.6a)

for even functions and

\[\hat{\psi}(0) = D^2\hat{\psi}(0) = 0\]  

(3.6b)

for odd eigenfunctions.

Now, equation (3.3) contains four real parameters \(\alpha, \text{Re}, c_r,\) and \(c_i\) for a given laminar velocity profile \(U_0\). The stability problem is studied by fixing two of these parameters and solving the above complex homogeneous linear differential equation for the eigenfunctions \(\hat{\psi}\) and the remaining two parameters as eigenvalues. There is an infinite number of distinct
eigenfunctions for any given values of the fixed parameters. The stability problem, then, can be stated in the following way: if for a given Re and \( \alpha \) the imaginary part of \( c, c_i \) can be positive, the corresponding eigenfunction grows with time and the motion is unstable with respect to that particular eigenmode. If \( c_i \) is negative, that particular mode of the fluctuations will be damped. If at a given Re the value of \( c_i \) is negative for all values of \( \alpha \), the flow is considered stable, and if \( c_i \) vanishes, there will be a neutral oscillation. It is interesting that in plane Poiseuille flows, which are unstable according to linear theory, only one eigenmode is unstable, and the corresponding eigenfunction is even about the centerline. According to convention, the least stable eigenmode (i.e., the one that grows the fastest or decays the slowest) is termed the first eigenmode.

For the case of plane Poiseuille flow extensive work has been done in mapping the regions of linear instability in the Reynolds number, wave number domain. Lin (ref. 14) calculated the neutral curve using asymptotic expansions. Later Shen (ref. 15) determined the stability characteristics shown in figure 3 by a perturbation from Lin's neutral curve. The first finite-difference calculation of the Orr-Sommerfeld equation was done by Thomas (ref. 16). More recent finite-difference calculations have been made by Nachtshiem (ref. 17) and Lee and Reynolds (ref. 18). The minimum Reynolds number calculated by Thomas was 5780 at \( \alpha = 1.02 \) while that found by Nachtshiem was 5767 at \( \alpha = 1.02 \).
Figure 3.- Stability characteristics of plane Poiseuille flow (ref. 15).
IV. NONLINEAR EFFECTS

The study of hydrodynamic stability has mainly been concerned with linear theory, but more recently greater emphasis has been placed on the study of nonlinear effects associated with finite amplitude disturbances. The importance of studying nonlinear effects is due in a large part to the fact that since turbulence involves nonlinear velocity fluctuations of finite size, linear theory is unable to predict how turbulence occurs as a consequence of instability. In addition, experimental evidence indicates that turbulence can occur in plane Poiseuille flow at a Reynolds number well below the critical value obtained by linear theory. It has therefore been suggested that flow which is stable to infinitesimal disturbances may be unstable to finite disturbances. The flow is then said to exhibit subcritical instability. It is conventional to refer to nonlinear disturbances as existing under supercritical conditions if the Reynolds number is such that the flow is unstable according to linear theory, and as existing under subcritical conditions if the Reynolds number is such that the flow is stable with respect to infinitesimal disturbances.

Landau (ref. 19) in 1944 was one of the first to investigate the physical processes that govern the initiation of turbulence. He suggested that the transition to turbulence is caused by successive unstable fluctuations that do not grow exponentially large but, rather, are braked at some high level by nonlinear effects. More recently, major contributions to nonlinear stability theory have been made by Stuart (refs. 9, 13, 20, 21). Stuart gives the following explanation of the effects on nonlinearity on the stability of parallel flows. When a disturbance of finite amplitude and of a given wave number is introduced into the flow, the mean transport of momentum
by the finite fluctuations becomes appreciable, and the Reynolds stress begins to have an appreciable effect on the mean flow. The resulting distortion of the mean flow in turn alters the rate of energy transfer from the mean flow to the disturbance (recall the energy balance equations). Since this energy transfer is the cause of the growth of the disturbance, the rate of its growth is altered. In addition, the disturbance is also modified by the generation of harmonics of the fundamental disturbance mode. Thus, there is a mutual interaction between the mean motion and the disturbance as well as a distortion of the disturbance itself. The nonlinear effects can be summarized then as distortions of the mean flow, modifications of the fundamental disturbance mode, and the generation of harmonics of the fundamental component.

Meksyn and Stuart (ref. 22) in an earlier investigation of nonlinear disturbances in subcritical plane Poiseuille flows included the distortion of the mean flow as well as the modification of the fundamental disturbance but neglected the generation of harmonics. Their method is based on the simultaneous solution of the Orr-Sommerfeld equation (3.3) (which is linear) and the equation of mean motion (which is nonlinear) integrated across the channel to give

$$\overline{u'v'} = -\left(3\overline{p} \overline{u}\right) + \frac{1}{\text{Re}} \frac{d\overline{U}}{dy}$$  \hspace{1cm} (4.1)

The connection between the two equations is the mean velocity $\overline{U}$. The significant result of this investigation is that the critical Reynolds number decreased as the amplitude of the disturbance increased. The minimum critical Reynolds number was found to be about 2900 in contrast to the critical Reynolds number of about 5800 as given by linear theory.
Recently Stuart (ref. 21) and Watson (ref. 23) developed a more extensive method that includes all three nonlinear effects. It is based on an asymptotic power series expansion of the amplitude of the fundamental disturbance given by linear theory where the terms retained are of order \( c_1^{3/2} \) and terms neglected are of order \( c_1^2 \) and higher. This asymptotic method is strictly valid only for small amplitude disturbances in the regions of the Reynolds number and wave number plane near the neutral curve where \( c_1 \) is small, and allows for the growth of only one harmonic of the fundamental oscillation. The basic equation for the disturbance amplitude \( |A| \) is

\[
\frac{d}{dt} |A|^2 = 2ac_1 |A|^2 + (k_1 + k_2 + k_3)|A|^4
\]

Stuart (ref. 21) has given physical meaning to each term multiplying \( |A|^4 \). The term \( k_1 \) represents the change of the flow of energy to the disturbance due to distortion of the mean flow by Reynolds stress, and is negative for small \( c_1 \); \( k_2 \) represents the generation of the first harmonic and is negative; \( k_3 \) represents the distortion of the fundamental wave.

The range of validity of the Stuart-Watson method is uncertain. The important criterion is that the solution converge to that of linear theory as the amplitude and \( c_1 \) tend to zero. Stuart (ref. 21) argues that it is in the critical layer where the convergence criterion is the most severe and that the solution will converge provided

\[
c_1 << (\alpha Re)^{-1/3}
\]

where the thickness of the critical layer is of order \( (\alpha Re)^{-1/3} \). Stuart (ref. 21) suggests that the above condition indicates the method is valid
in the supercritical range from \( \text{Re} = 5800 \) to \( \text{Re} = 10^4 \) or \( 10^5 \) and in the subcritical range down to Reynolds numbers of 2500.

Reynolds and Potter (ref. 24) have used the Stuart-Watson method to calculate for Poiseuille flow with constant volume flux, and Pekeris and Shkoller (refs. 25, 26) have used the method to calculate for Poiseuille flow with constant pressure gradient. The authors are in qualitative agreement, and predict subcritical instabilities along the upper branch of the linear neutral curve and supercritical equilibrium along the lower branch.

Reynolds and Potter (ref. 24) suggest that for subcritical instabilities the generation of higher harmonics is not important; the important effects are instead the modification of the mean flow by the Reynolds stress and, even more important, the distortion of the fundamental disturbance. Their estimate of the dependence of critical Reynolds number of the fluctuation intensity indicates that a fluctuation of only a few percent is sufficient to cause instability at subcritical Reynolds numbers as low as 1500. Further results for the size of disturbance needed to obtain instability at a given subcritical Reynolds number have been reported by Pekeris and Shkoller (ref. 25).

Eckhaus (ref. 27) has introduced a nonlinear theory that includes the amplitudes of any number of harmonics of the fundamental disturbance. This theory is not restricted to regions of the \( \alpha - \text{Re} \) plane near the neutral curve as is the Stuart-Watson asymptotic method. The method considers the full nonlinear equations of motion and expands the disturbance into a Fourier series in \( x \), which gives a system of coupled partial differential equations for the Fourier amplitudes \( f_n(y,t) \) for wave numbers \( n = 1,2, \ldots \). These amplitudes are further expanded in terms of the eigenfunctions, \( \psi \),
of the linearized problem so that

\[ f_n(y,t) = \sum_{m=1}^{\infty} A_{nm}(t) \hat{\psi}_{nm}(y) \]  

(4.4)

where \( m \) is the eigenmode. Equation (4.4) results in a doubly infinite set of nonlinear ordinary differential equations for the functions \( A_{nm}(t) \). In practical calculations the expansions must be truncated at some finite wave number, say \( N \), and some finite eigenmode, say \( M \), so that we are left with a set of \( N \) times \( M \) equations for the time-dependent amplitude functions.

Pekeris and Shkoller (ref. 28) recently used this method to calculate the effect of finite amplitude disturbances on the stability of plane Poiseuille flow at subcritical Reynolds numbers. The disturbance is expanded in terms of even eigenfunctions of the Orr-Sommerfeld equation. However, the nonlinear equations for the Fourier amplitudes do not admit solutions of a strictly even type and thus their results are invalid. An investigation of equations (5), (6), and (7) of their paper shows that if the mean flow term, \( f_0 \), is odd, then the harmonics with even wave numbers, \( f_2, f_4, \ldots \), must also be odd; furthermore the harmonics with odd wave number, \( f_1, f_3, \ldots \), are even functions.
V. PREVIOUS NUMERICAL WORK

Considerable attention has been given in recent years to solving viscous flow problems by the numerical integration of the Navier-Stokes equations. In an article on the numerical simulation of the Von Kármán vortex street behind a plate, Fromm and Harlow (ref. 29) in 1963 graphically illustrated the application of finite-difference methods. Since then papers on the numerical solution of a large variety of viscous flow problems have appeared in the literature. For a recent paper on the numerical integration of the Navier-Stokes equations see Cheng (ref. 30). Of special interest to the present work is the finite-difference solution of the time-dependent, incompressible Navier-Stokes equations. There are two general formulations which are commonly used in incompressible viscous flow problems. The first uses the so-called primitive variables, velocity and pressure, and is typified by the marker-and-cell method developed by Harlow and Welch (ref. 31). This method employs a staggered mesh, such that the velocities and pressure are defined at different mesh points. The treatment of the boundary conditions at the wall require the establishment of virtual mesh points outside the fluid and a specification, based on mass conservation, of the velocity at these points. In this procedure the momentum equations are solved by explicit finite-difference formulas, which are accurate to second order in space and accurate to first order in time. Conservation of mass is not satisfied exactly, but the solution of a Poisson equation for the pressure is handled in such a way that at each point the divergence of the velocity is given a rate of change that will tend to null the accumulated divergence at that point.
The second formulation, the one used in this study, is in terms of the stream function and vorticity. The mass is conserved implicitly, and it is generally possible to derive a consistent explicit formulation of the boundary values for the vorticity without defining virtual points outside the fluid.

DeSanto and Keller (ref. 32) used the stream function and vorticity formulation to study the growth of a disturbance in a flat-plate boundary layer. The problem is formulated in terms of the original laminar flow and perturbations about it. The perturbation vorticity transport equation is solved using an extension of the alternating-direction method of Peaceman and Rachford (ref. 33) while the Poisson equation for the perturbation stream function is solved by the extrapolated line-SOR iterative procedure. Time oscillatory perturbations of amplitude $10^{-5}$ and $10^{-1}$ (relative to a unit free-stream velocity) are imposed on the laminar Blasius flow. At the downstream boundary it is assumed that the perturbation is locally periodic with the same wave number as the initial perturbation, that is,

$$\psi_{xx} = -a^2 \psi \tag{5.1}$$

The perturbation, which is designed to simulate the well-known experiment of Schubauer and Skramstad (ref. 34), is followed in both time and space to determine whether it amplifies or decays. The initial perturbation is taken to be inside the neutral curve (unstable according to linear theory). Results for the small amplitude case show agreement with the linear theory in that the perturbation initially amplifies but eventually decays as the linear neutral curve is passed. The larger amplitude results, however, do not show a rapid growth followed by an equilibrium flow as observed in experiments. The authors attribute the lack of precision to the coarseness
of the finite-difference net in the $y$-direction. In this study the entire boundary-layer thickness is spanned by only five mesh spacings.

A paper on work similar to that of DeSanto and Keller and more relevant to the present study is that of Dixon and Hellums (ref. 35). These authors study spatially growing or decaying perturbations about the laminar solution in both pipe-Poiseuille and plane-Poiseuille flows. The perturbation vorticity transport equation is solved by an alternating direction method adapted by Aziz and Hellums (ref. 36) as an extension of the Douglas-Rachford (ref. 37) method. The perturbation stream function is found by the well-known method of successive-overrelaxation. Symmetry is assumed on the centerline for the vorticity and stream function disturbances with 10-grid spacings between the wall and the centerline. Thus, only the flow between the wall and the centerline is considered. The time oscillatory perturbation is studied over a flow-field length of 37 radii using 296 grid points with the downstream boundary condition the same as that used by DeSanto and Keller, equation (5.1). The stability of the flow was determined by monitoring the perturbation $u$-velocity, vorticity and stream function at a radius of 0.1 and 0.5. Calculations are presented for perturbation amplitudes of $10^{-5}$ and $10^{-1}$ (relative to a unit centerline velocity) and for Reynolds numbers ranging from 500 to 100,000. Pipe-Poiseuille flow results for the smaller $10^{-5}$ amplitude perturbation show the perturbation decaying as it travels downstream, indicating stability. The results for the larger $10^{-1}$ amplitude show the perturbation decays for sufficiently low Reynolds numbers. The stability of the flow at higher Reynolds numbers is uncertain. The results show that the perturbation is highly distorted from its initial distribution but no clear amplification is evident. The authors claim,
however, that this distortion indicates an unstable flow. For plane-
Poiseuille flow the smaller amplitude results show decay with slight ream-
plification at $Re = 100,000$, and at the larger amplitude there is
distortion and reamplification at $Re = 10,000$. The uncertainty in the
results could be due, in part, to the rapid change in the Reynolds stress
near the wall, shown in figure 3, which suggests that using only 10 mesh
points normal to the flow may cause serious errors. It appears that con-
sidering symmetry in plane-Poiseuille flow is overrestrictive and that both
symmetric and antisymmetric perturbations exist in the nonlinear case as was
previously discussed in Section IV. Finally, the effect of the locally
periodic downstream boundary condition is uncertain. It is expected that in
the nonlinear case the perturbation will not remain strictly at its initial
wave number, but rather generate other harmonics as well as a mean flow com-
ponent. It is suggested that the downstream boundary condition used by
DeSanto and Keller (ref. 32) and Dixon and Hellums (ref. 35) has the effect
of suppressing these higher harmonics and the mean component of the pertur-
bation (recall that the perturbations are about the original laminar flow).
VI. FINITE-DIFFERENCE FORMULATION

Incompressible, time-dependent, viscous flow problems require the simultaneous solution of the vorticity transport equation and a Poisson equation. The general procedure is to calculate at each time level a new value of vorticity from the vorticity transport equation and then solve the Poisson equation for the stream function. These equations are quite different in character; the vorticity transport equation is a nonlinear, parabolic, time-dependent partial differential equation, while the Poisson equation is linear, elliptic and independent of time. Because of this difference in character the numerical solution of the two equations will be discussed separately.

A. Finite-Difference Grid

The finite-difference computational domain is defined by a system of grid points as shown in figure 4. The value of any variable, say \( \psi \), at any grid point is defined such that

\[
\psi_{j,k}^n = \psi(x_j, y_k, t^n)
\]  

(6.1)

where

\[
x_j = (j - 2)\Delta x
\]

\[
y_k = (k - 1)\Delta y - 1
\]

\[
t^n = n \Delta t
\]

and

\[
\Delta x = \frac{2\pi}{\alpha(JM - 2)}
\]

\[
\Delta y = \frac{2}{KM - 1}
\]

The columns \( j = 1 \) and \( j = JM \) are so-called "fringe" points that are used to apply the periodic boundary conditions in the x-direction.
Figure 4.- Finite-difference grid.
B. Vorticity Transport Equation

Finite-difference methods for solving mixed time-dependent parabolic and hyperbolic equations can be either implicit or explicit (see, e.g., Richtmeyer and Morton, ref. 38) or a mixture of the two. An implicit method results in a set of algebraic equations that must be solved simultaneously for the coupled set of unknown, say, \( u^{n+1} \). An explicit method results in a scheme that gives each one of the components in the unknown \( u^{n+1} \) in terms of the known \( u^n, u^{n-1} \), etc.

Typical of implicit methods that have been used on problems of interest here are the alternating direction methods such as Peaceman-Rachford (ref. 33), and Douglas-Rachford (ref. 37). Wilkes and Churchill (ref. 39), Aziz and Hellums (ref. 36), Dixon and Hellums (ref. 35), and Chorin (ref. 40) have used extensions of these alternating direction methods to solve incompressible Navier-Stokes problems. In two dimensions these methods proceed in time by successively taking two time steps of \( 1/2 \Delta t \) each; the first being implicit in one space dimension and the second implicit in the other. The predominant feature is that if the boundary conditions are Dirichlet or Neumann, the resulting simultaneous systems of equations are tridiagonal and these can be solved directly by a recursive algorithm given, for example, by Varga (ref. 41, p. 195). However, if the boundary conditions are periodic, as in the present case, the system is no longer tridiagonal and the inversion is more difficult.

Typical of explicit methods are those of Dufort-Frankel (see Fromm, ref. 42) and Lax-Wendroff (ref. 43). The Dufort-Frankel (or leapfrog) method is a time-centered scheme which possesses neutral stability for transport equations with no viscosity and contains no numerical damping. Fromm
(ref. 44) remarks, however, that the use of this method often requires the addition of an artificial viscosity when the Reynolds number is large. His reason is that there are very adverse phase distortions of high wave number components and the addition of artificial viscosity damps out these components.

The present work uses two methods of the Lax-Wendroff type based on the scheme developed by MacCormack (ref. 45). Both schemes are accurate to second order in time; method I is also accurate to second order in space, while method II is accurate to fourth order in space.

The difference formulations are in conservative form (ref. 38). The advantage of the conservative formulation seems to be that it does not appear to permit the spatial truncation errors to accumulate in a systematic way when summed over the mesh. Improvements in the accuracy related to the conservative (rather than the nonconservative) formulation have been reported by Fromm (ref. 44) and Crowley (ref. 46).

1. Method I

The first method is MacCormack's second-order (in both space and time) predictor-corrector differencing of equation (2.11) which can be written

\[
\tilde{w}_{j,k} = w_{j,k} - \frac{\Delta t}{\Delta x} (F_{j+1,k} - F_{j,k}) - \frac{\Delta t}{\Delta y} (G_{j,k+1} - G_{j,k}) \\
+ \frac{\Delta t}{Re \Delta x^2} (w_{j+1,k} - 2w_{j,k} + w_{j-1,k}) \\
+ \frac{\Delta t}{Re \Delta y^2} (w_{j,k+1} - 2w_{j,k} + w_{j,k-1}) \\
+ 2 \Delta t v^n_{j,k}
\] (6.2a)
\[
\omega_{j,k}^{n+1} = \frac{1}{2} \left[ \omega_{j,k}^{n} + \bar{\omega}_{j,k}^{n} - \frac{\Delta t}{\Delta x} (\bar{F}_{j,k} - \bar{F}_{j-1,k}) - \frac{\Delta t}{\Delta y} (\bar{G}_{j,k} - \bar{G}_{j,k-1}) \right. \\
+ \frac{\Delta t}{\text{Re} \Delta x^2} (\bar{\omega}_{j+1,k} - \bar{\omega}_{j,k} + \bar{\omega}_{j-1,k}) \\
+ \frac{\Delta t}{\text{Re} \Delta y^2} (\bar{\omega}_{j,k+1} - \bar{\omega}_{j,k} + \bar{\omega}_{j,k-1}) + 2 \Delta t \bar{\nu}_{j,k} \left. \right] 
\]

(6.2b)

where

\[
F_{j,k}^n = [(U_o + u)\omega]_{j,k}^n
\]

and

\[
G_{j,k}^n = (v\omega)_{j,k}^n
\]

Notice that the convective terms, \(F\) and \(G\), are differenced forward and backward in space for the predictor and corrector equations, respectively.

Actually, this procedure was programmed to be cyclic so that all four possible combinations of forward-backward differencing were used in four successive time steps. In the intermediate time step, \(F\) and \(G\) are determined by solving Poisson's equation (see section VIC) to find \(u\) and \(v\) from \(\omega\).

Notice also that the diffusion term is explicit and central differenced in both the predictor and corrector.

2. **Stability analysis of method I**

For the finite-difference formulation of an initial value problem to be stable the errors occurring in the numerical procedure must not grow unbounded. A widely used method for examining the stability of finite-difference methods is that introduced by Von Neumann (see ref. 38). Three restrictive conditions must be met to apply the Von Neumann method: The solution must be smooth; boundary conditions must be ignored; and the difference equations must be locally linearized. The latter condition is the
most serious and for the present case its fulfillment requires that the velocities be held fixed.

To apply the stability analysis one formulates the solution to the linearized difference equation as a Fourier series with a general term given by:

\[ \zeta(t)e^{i(k_x \Delta x + k_y \Delta y)} \]

where \( k_x \) and \( k_y \) are the wave numbers. The amplification factor \( \lambda \) is defined as

\[ \lambda = \frac{\sum_{n=1}^{N} C_n e^{\frac{6.3}{\xi^n}}}{\zeta} \]

and for stability

\[ |\lambda| \leq 1 \]

To illustrate the concept, consider the one-dimensional linear equation

\[ \frac{\partial \omega}{\partial t} = -c \frac{\partial \omega}{\partial x} + v \frac{\partial^2 \omega}{\partial x^2} \]

where \( c \) and \( v \) are constants. The finite-difference formulation of equation (6.6) identical to the one used for equation (6.2) is

\[ \check{\omega}_j = \omega_j^n - \alpha(\omega_{j+1}^n - \omega_j^n) + \beta(\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n) \]

\[ \omega_j^{n+1} = \frac{1}{2}[\omega_j^n + \check{\omega}_j - \alpha(\check{\omega}_j - \check{\omega}_{j-1}) + \beta(\check{\omega}_{j+1} - 2\check{\omega}_j + \check{\omega}_{j-1})] \]

where \( \alpha \equiv c \Delta t/\Delta x \) (often referred to as the CFL number) and \( \beta \equiv v \Delta t/\Delta x^2 \). Substituting equation (6.7a) into (6.7b), one obtains
\[\omega_j^{n+1} = \omega_j^n - \frac{a}{2} (\omega_{j+1}^n - \omega_{j-1}^n) + \left(\frac{a^2}{2} + \beta\right)(\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n)\]

\[- \frac{a\beta}{2} (\omega_{j+2}^n - 2\omega_{j+1}^n + 2\omega_{j-1}^n - \omega_{j-2}^n)\]

\[+ \frac{\beta^2}{2} (\omega_{j+2}^n - 4\omega_{j+1}^n + 6\omega_j^n - 4\omega_{j-1}^n + \omega_{j-2}^n)\]  

(6.8)

and the amplification factor is

\[\lambda = 1 + (a^2 + 2\beta)[\cos(k \Delta x) - 1] + 2\beta^2[\cos(k \Delta x) - 1] \]

\[-i\{a \sin(k \Delta x) + a\beta [\sin(2k \Delta x) - 2 \sin(k \Delta x)]\}\]  

(6.9)

It can be shown that \(|\lambda|\) has extrema at \(k \Delta x = 0\) and \(\pi\). At \(k \Delta x = 0\),

\[|\lambda| = 1\] for all \(a\) and \(\beta\).

At \(k \Delta x = \pi\)

\[|\lambda| = |1 - 2a^2 + 4\beta(2\beta^2 - 1)| \leq 1\]  

(6.10)

and this forms the stability boundary of the method.

First consider the case when \(\lambda \leq 1\). The result is

\[2\beta(2\beta^2 - 1) - a^2 \leq 0\]  

(6.11)

For plane Poiseuille flow the velocity vanishes at the wall so that the strongest condition is found by setting \(a\) to zero. This amounts to considering the diffusion term alone and leads to the requirement

\[\beta \leq 1/2\]  

(6.12a)

or

\[\Delta t \leq \frac{\Delta x^2}{2\nu}\]  

(6.12b)

Now consider \(\lambda \geq -1\) for which

\[a^2 + 2\beta(1 - 2\beta) \leq 1\]  

(6.13)

or

\[a^2 \leq 1 - 2\beta(1 - 2\beta)\]
The viscosity coefficient is always positive so that if \( \beta \leq 1/2 \) the effect of diffusion is to lower the stability limit of the convective term. Equations (6.11) and (6.13) are plotted in figure 5 showing the effect of diffusion on the numerical stability. The minimum practical upper bound on \( |a| \) is 0.866 at \( \beta = 0.25 \).

In two dimensions calculations of \( |\lambda| \) have shown that the stability bounds are the same as in one dimension with

\[
\alpha = \frac{|u| \Delta t}{\Delta x} + \frac{|v| \Delta t}{\Delta y} \quad (6.14a)
\]

and

\[
\beta = v \Delta t \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \quad (6.14b)
\]

3. Accuracy analysis of method I

The truncation errors of Lax-Wendroff type schemes contain terms that are both dispersive and dissipative. To obtain these error terms it is convenient to define a "modified differential equation," which is the differential equation actually represented by the difference formulation (see ref. 38, p. 330). In this sense, equation (6.7) actually solves the modified equation

\[
\frac{\partial w}{\partial t} + c \frac{\partial w}{\partial x} - v \frac{\partial^2 w}{\partial x^2} = Qw \quad (6.15)
\]

where \( Q \) is some differential operator. In order to find \( Q \), one first combines equations (6.7) to form the single equation

\[
\omega_{j+1}^{n+1} - \omega_j^n + \frac{a}{2} (\omega_{j+1}^n - \omega_{j-1}^n) - \left( \frac{a^2}{2} + \beta \right) (\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n)
\]

\[
- \frac{\alpha \beta}{2} (\omega_{j+2}^n - 2\omega_{j+1}^n + 2\omega_{j-1}^n - \omega_{j-2}^n)
\]

\[
+ \frac{\beta^2}{2} (\omega_{j+1}^n - 4\omega_{j+1}^n + 6\omega_j^n - 4\omega_{j-1}^n + \omega_{j-2}^n) = 0[(\Delta t)^5] \quad (6.16)
\]
Figure 5.- Numerical stability boundaries.
then expands each term in a Taylor series about \( j \Delta x, n \Delta t \). Time derivatives higher than \( \omega_t \) are eliminated by repetitive use of equation (6.15) itself. The result is

\[
\omega_t + c \omega_x - \nu \omega_{xx} = -\frac{c^2}{6} [\Delta x^2 - c^2 \Delta t^2] \omega_{xxx} - \left\{ \frac{c^2 \Delta t}{8} [\Delta x^2 - c^2 \Delta t^2] \right\}
\]

\[
-\frac{\nu}{12} [\Delta x^2 - 6c^2 \Delta t^2] \omega_{xxxx} + \text{higher order terms}
\]

(6.17)

Note: Throughout this study the higher order terms are assumed to have no effect.

The coefficient of the third derivative represents to the lowest order the dispersive error. Notice that it is due only to the finite differencing of the transport term and is of order \((\Delta x)^2\) and \((\Delta t)^2\). It is instructive to write the dispersion error in terms of the CFL number \(\alpha\) as

\[
-\frac{c \Delta x^3}{6 \Delta t} [1 - \alpha^2] \omega_{xxx}
\]

The error vanishes when \(\alpha = 1\) and is maximum at \(\alpha = 1/\sqrt{3}\). It seems reasonable to pick \(\Delta t\) such that \(\alpha\) is always less than \(1/\sqrt{3}\) in order to avoid the maximum dispersion error. The importance of minimizing the dispersion error is brought out later in the discussion of the results.

The coefficient of the fourth derivative term in equation (6.17) gives to the lowest order the numerical dissipation of the finite-difference formulation. If this term is positive it has the effect of reducing the amplitude of the Fourier component \(e^{i\kappa x}\) by an amount proportional to \(e^{-\text{const.} \kappa^4 \Delta t}\). Thus the amplitudes of the high wave number components are the most severely damped. In the absence of viscosity, numerical damping due to the finite-difference approximation of the transport term can be written in
terms of the CFL number as

\[- \frac{a^2}{8} \frac{\Delta x^4}{\Delta t} [1 - a^2]\]

It is easily shown that the maximum damping caused by this term occurs when \(a^2 = 1/2\).

4. Method II

In order to study the effects of numerical dispersion and dissipation, a method was devised for the vorticity transport equation that had fourth-order accuracy in the space derivatives. Significant improvements in accuracy with fourth-order methods have been reported in references 47 and 48. The present scheme, which is still second order in the time integration, can be written for equations (2.11) as

\[
\bar{\omega}_{j,k} = \omega_{j,k} + \frac{\Delta t}{6} \left[ \frac{\Delta x}{\Delta y} \left( -F_{j+2,k} + 6F_{j+1,k} - 3F_{j,k} - 2F_{j-1,k} \right) \right]
\]

\[- \frac{\Delta t}{6} \left( -G_{j,k+2} + 6G_{j,k+1} - 2G_{j,k-1} \right) \]

\[+ \frac{\Delta t}{12Re} \frac{\Delta x^2}{\Delta y} \left( -\omega_{j+2,k} + 16\omega_{j+1,k} - 30\omega_{j,k} + 16\omega_{j-1,k} - \omega_{j-2,k} \right) \]

\[+ \frac{\Delta t}{12Re} \frac{\Delta y^2}{\Delta y} \left( -\omega_{j,k+2} + 16\omega_{j,k+1} - 30\omega_{j,k} + 16\omega_{j,k-1} - \omega_{j,k-2} \right) \]

\[+ 2 \frac{\Delta t}{\Delta v} \bar{\omega}_{j,k}\]

(6.18a)

followed by
Like method I this is a predictor-corrector method but with the derivatives represented by five-point formulas. The extended number of points presents no problem in the $x$ direction where the boundary conditions are periodic, but modifications are, of course, required at the grid points $k = 2$ and $k = KM - 1$ next to the walls. At these points the transport and diffusion terms are treated according to different rules. The transport terms are simply differenced as in method I. This is justified by the fact that the velocities are nearly zero at the walls and the transport terms are, therefore, small with respect to the diffusion terms. The diffusion terms, on the other hand, are replaced by one-sided differences so that

$$
\frac{\partial^2 \omega_{j,1}}{\partial y^2} = \frac{1}{12 \Delta y^2} \left( 10 \omega_{j,1}^n - 15 \omega_{j,2} - 4 \omega_{j,3} + 14 \omega_{j,4} - 6 \omega_{j,5} + \omega_{j,6} \right) + O[(\Delta y)^4]
$$

(6.19a)

$$
\frac{\partial^2 \omega_{j,KM-1}}{\partial y^2} = \frac{1}{12 \Delta y^2} \left( 10 \omega_{j,KM} - 15 \omega_{j,KM-1} - 4 \omega_{j,KM-2} + 14 \omega_{j,KM-3} - 6 \omega_{j,KM-4} + \omega_{j,KM-5} \right) + O[(\Delta y)^4]
$$

(6.19b)
5. Stability analysis of method II

If equations (6.18) are applied to the one-dimensional model equation (6.6), and the predictor is combined with the corrector, one finds

\[
\omega_{j}^{n+1} = \omega_{j}^{n} - \frac{\alpha}{12} (-\omega_{j+2}^{n} + 8\omega_{j+1}^{n} - 8\omega_{j-1}^{n} + \omega_{j-2}^{n}) \\
+ \frac{\alpha^2}{72} (-2\omega_{j+3}^{n} + 9\omega_{j+2}^{n} + 18\omega_{j+1}^{n} - 50\omega_{j}^{n} + 18\omega_{j-1}^{n} + 9\omega_{j-2}^{n} - 2\omega_{j-3}^{n}) \\
+ \frac{\beta}{12} (-\omega_{j+2}^{n} + 16\omega_{j+1}^{n} - 30\omega_{j}^{n} + 16\omega_{j-1}^{n} - \omega_{j-2}^{n}) \\
+ \frac{\beta^2}{288} (\omega_{j+4}^{n} - 32\omega_{j+3}^{n} + 316\omega_{j+2}^{n} - 992\omega_{j+1}^{n} + 144\omega_{j}^{n} \\
- 992\omega_{j-1}^{n} + 316\omega_{j-2}^{n} - 32\omega_{j-3}^{n} + \omega_{j-4}^{n}) \\
+ \frac{\alpha\beta}{144} (-\omega_{j+4}^{n} + 24\omega_{j+3}^{n} - 158\omega_{j+2}^{n} + 248\omega_{j+1}^{n} - 248\omega_{j}^{n} \\
+ 158\omega_{j-2}^{n} - 24\omega_{j-3}^{n} + \omega_{j-4}^{n}) \quad (6.20)
\]

Substituting in a Fourier component gives the following amplification factor

\[
\lambda = 1 - \frac{\beta}{6} [15 - 16 \cos(k \Delta x) + \cos(2k \Delta x)] \\
+ \frac{\beta^2}{144} [707 - 992 \cos(k \Delta x) + 316 \cos(2k \Delta x) \\
- 32 \cos(3k \Delta x) - \cos(4k \Delta x)] \\
- \frac{\alpha^2}{36} [25 - 18 \cos(k \Delta x) - 9 \cos(2k \Delta x) + 2 \cos(3k \Delta x)] \\
- \frac{\alpha}{144} [8 \sin(k \Delta x) - \sin(2k \Delta x)] - \frac{\alpha^2}{144} [248 \sin(k \Delta x) \\
- 158 \sin(2k \Delta x) + 24 \sin(3k \Delta x) - \sin(4k \Delta x)] \quad (6.21)
\]

The maximum value of \(|\lambda|\) occurs at \(k \Delta x = \pi\) and the resulting stability condition is

\[
|\lambda| = |1 - \frac{16}{3} \beta + \frac{128}{9} \beta^2 - \frac{8}{9} \alpha^2| \leq 1 \quad (6.22)
\]
The case for which \( \lambda \leq 1 \) gives

\[
16\beta^2 - 6\beta - \alpha^2 \leq 0 \quad (6.23)
\]

As with method I the most stringent condition occurs when \( \alpha = 0 \) and is

\[
\beta \leq 3/8 \quad (6.24)
\]

If \( \lambda \geq -1 \) the stability condition is

\[
\alpha^2 \leq \frac{1}{4} (9 - 24\beta + 48\beta^2) \quad (6.25)
\]

The actual stability boundaries and the practical ones \( \alpha^2 \leq 1/2 \) are shown in figure 5.

6. **Accuracy analysis of method II**

The derivation of the modified differential equation for method II is extremely tedious. The algebra was actually carried out by means of an IBM 360/67 digital computer using FORMAC computer language.\(^1\) The modified equation is

\[
\omega_t + c \omega_x - v \omega_{xx} = \frac{c^3}{6} \Delta t^2 \omega_{xxx} - \frac{c^2 \Delta t^2}{8} (4v - c^2 \Delta t) \text{\omega} (IV)
\]

\[
+ \left( \frac{v^2 c \Delta t^2}{2} + \frac{c \Delta x^4}{30} - \frac{v c^3 \Delta t^3}{2} + \frac{c^5 \Delta t^4}{20} \right) \text{\omega} (V)
\]

\[
+ \left( \frac{3v^2 c^2 \Delta t^3}{4} - \frac{v c^4 \Delta t^4}{4} - \frac{v^3 \Delta t^2}{6} - \frac{v \Delta x^4}{90} \right) \text{\omega} (VI)
\]

+ higher order terms

(6.26)

which bears out the order of accuracy claimed for the time and space differencing.

The error in numerical dispersion is still second order but now proportional only to \((\Delta t)^2\). The numerical damping due to the transport terms

\(^1\)The results presented were found by Dr. R. F. Warming of Ames Research Center.
is now of order \((\Delta t)^3\). These are significant improvements in accuracy if \(\Delta t\) is much less than \(\Delta x\). It is also advantageous to have the errors dependent only on \(\Delta t\) in problems of two or three space dimension because, as the spatial dimensions are halved, the computation time increases by the power of the number of dimensions, while if the time step is halved the computational time is only doubled.

C. Poisson Equation

In solving the Poisson equation (2.12) for the stream function, use can be made of the fact that the flow is continuous and periodic in the \(x\)-direction. Thus, it is advantageous to form the Fourier expansion

\[
\psi(x,y) = \sum_{m=-\infty}^{\infty} \tilde{\psi}^m(y)e^{imx} \quad (6.27a)
\]

\[
\omega(x,y) = \sum_{m=-\infty}^{\infty} \tilde{\omega}^m(y)e^{imx} \quad (6.27b)
\]

for

\[0 \leq x \leq \frac{2\pi}{\alpha}\]

where the complex Fourier coefficients are given by

\[
\tilde{\psi}^m(y) = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} \psi(x,y)e^{-imx} \, dx \quad (6.28a)
\]

\[
\tilde{\omega}^m(y) = \frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} \omega(x,y)e^{-imx} \, dx \quad (6.28b)
\]

Substituting equation (6.27) into equation (2.12) and using the orthogonality property

\[
\frac{\alpha}{2\pi} \int_0^{2\pi/\alpha} e^{imx}e^{-imx} \, dx = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (6.29)
\]
results in an infinite set of ordinary differential equations for the
Fourier coefficients

\[
\frac{d^2 \tilde{\psi}^m(y)}{dy^2} - m^2 a^2 \tilde{\psi}^m(y) = \tilde{\alpha}^m(y)
\]  

(6.30)

For numerical calculations one must find for this expansion the finite ana­
log that is compatible with a given set of grid points. Not only does this
analog exist but it gives the exact representation of any function composed
of a number of harmonics equal to half the number of grid points in the
x-direction. Thus, if the number of x grid points is \( M = \text{JM} - 2 \), we have
M complex ordinary differential equations, the solution of which can repre­
sent the first \( M/2 \) harmonics in the harmonic decomposition of any function.
Finally, the second derivatives in equation (6.30) are approximated by
three-point central differencing schemes resulting in the set of linear
algebraic equations

\[
\tilde{\psi}^m_{k+1} - [2 + (\Delta y \omega)^2] \tilde{\psi}^m_k + \tilde{\psi}^m_{k-1} = \Delta y^2 \tilde{\alpha}^m_k
\]

(6.31)

(where \(-(M/2) + 1 \leq m \leq M/2 \) and \( k = 2, 3, \ldots, KM - 1 \)). Note that \( \tilde{\psi}^m_1 \)
and \( \tilde{\psi}^m_{KM} \) are zero since \( \psi \) vanishes at the walls.

The actual computing procedure is carried out as follows. First \( \tilde{\alpha}^m_k \)
is calculated using the fast Fourier transform algorithm of Cooley and Tukey
(ref. 49). Then the M complex tridiagonal difference equations (6.31)
are solved by the recursive Gaussian elimination algorithm (e.g., ref. 41,
p. 195) and \( \tilde{\psi}^{j,k} \) is obtained by the inverse transform. It is not neces­
sary, however, to solve all M matrix equations because the real and
imaginary parts of the Fourier coefficients are symmetric and antisymmetric
about \( m = 0 \), respectively. Thus only \((M/2) + 1 \) complex equations need
be solved. The present method is restricted, however, to \( M = 2^n \) points, where \( n \) is an integer.

This direct method of solution is extremely accurate in the \( x \)-direction and requires, for example, half the storage necessary to perform the standard SOR iterative technique. Most important, it has an overwhelming advantage over iterative relaxation methods in computer running time. For a 200 \( \times \) 32 mesh, tests showed it to be about 10 times faster than the alternating direction method (ref. 41) with the optimum relaxation parameter.

D. Velocities

The disturbance velocities used in the finite-difference forms of the vorticity transport equation, equations (6.2) and (6.18), are obtained from fourth-order, central finite-difference approximations of the stream function as follows:

\[
\begin{align*}
\frac{u_j^n}{\Delta y^2} &= \frac{1}{12} \left[ -\psi_{j+k+2}^n + 8(\psi_{j+k+1}^n - \psi_{j+k-1}^n) + \psi_{j+k-2}^n \right] + O[(\Delta y)^4] \\
\frac{v_j^n}{\Delta x^2} &= \frac{1}{12} \left[ -\psi_{j+2,k}^n + 8(\psi_{j+1,k}^n - \psi_{j-1,k}^n) + \psi_{j-2,k}^n \right] + O[(\Delta x)^4]
\end{align*}
\]

(6.32a, 6.32b)

The \( u \)-velocity at the grid points next to the two walls is obtained by the following fourth-order one-sided differences:

\[
\begin{align*}
\frac{u_j^n}{\Delta y^2} &= \frac{1}{12} \left[ -3\psi_{j,1}^n - 10\psi_{j,2}^n + 18\psi_{j,3}^n - 6\psi_{j,4}^n + \psi_{j,5}^n \right] + O[(\Delta y)^4] \\
\frac{u_j^n}{\Delta y^2} &= \frac{1}{12} \left[ -\psi_{j,KM-4}^n + 6\psi_{j, KM-3}^n - 18\psi_{j, KM-2}^n \\
&\quad + 10\psi_{j, KM-1}^n + 3\psi_{j, KM}^n \right] + O[(\Delta y)^4]
\end{align*}
\]

(6.33a, 6.33b)
E. Boundary Conditions

The periodic boundary conditions for $\omega$ and $\psi$ in the $x$-direction are

$$\begin{align*}
\omega^n_{1,k} &= \omega^n_{JM-1,k} \\
\omega^n_{JM,k} &= \omega^n_{2,k} \\
\psi^n_{1,k} &= \psi^n_{JM-1,k} \\
\psi^n_{JM,k} &= \psi^n_{2,k}
\end{align*}$$

(6.34)

At the walls the stream function vanishes so that

$$\begin{align*}
\psi^n_{j,1} &= 0 \\
\psi^n_{j,KM} &= 0
\end{align*}$$

(6.35)

The no slip condition at the walls is satisfied if the normal derivative of the stream function is zero there. This condition is fulfilled by the vorticity equation in the following way. At the wall the vorticity is given by

$$\omega = \psi_{yy}$$

(6.36)

Finite-difference schemes can be formulated that approximate equation (6.36) and, at the same time, satisfy the condition

$$\psi_y = 0 \quad \text{at wall}$$

(6.37)

The derivation proceeds by expanding $\psi$ at the points near the wall in a Taylor series referenced to the wall location.

$$\begin{align*}
A\psi_{j,2} &= A \left[ \psi_{j,1} + (\Delta y) \frac{\partial \psi_{j,1}}{\partial y} + \frac{(\Delta y)^2}{2} \frac{\partial^2 \psi_{j,1}}{\partial y^2} + \frac{(\Delta y)^3}{6} \frac{\partial^3 \psi_{j,1}}{\partial y^3} \\
&+ \frac{(\Delta y)^4}{24} \frac{\partial^4 \psi_{j,1}}{\partial y^4} + \frac{(\Delta y)^5}{120} \frac{\partial^5 \psi_{j,1}}{\partial y^5} \right] + O[(\Delta y)^6]
\end{align*}$$

(6.38a)
\[ B \psi_{j,3} = B \left[ \psi_{j,1} + 2(\Delta y) \frac{\partial^3 \psi}{\partial y^3} + 2(\Delta y)^2 \frac{\partial^2 \psi}{\partial y^2} + \frac{4}{3} (\Delta y)^3 \frac{\partial \psi}{\partial y} \right] + 0[(\Delta y)^6] \quad (6.38b) \]

\[ C \psi_{j,4} = C \left[ \psi_{j,1} + 3(\Delta y) \frac{\partial^3 \psi}{\partial y^3} + \frac{9}{2} (\Delta y)^2 \frac{\partial^2 \psi}{\partial y^2} + \frac{9}{2} (\Delta y)^3 \frac{\partial \psi}{\partial y} \right] + 0[(\Delta y)^6] \quad (6.38c) \]

\[ D \psi_{j,5} = D \left[ \psi_{j,1} + 4(\Delta y) \frac{\partial^3 \psi}{\partial y^3} + 8(\Delta y)^2 \frac{\partial^2 \psi}{\partial y^2} + \frac{16}{3} (\Delta y)^3 \frac{\partial \psi}{\partial y} \right] + 0[(\Delta y)^6] \quad (6.38a) \]

Setting the sum of equations (6.38a) and (6.38b) equal to zero along with the condition that \( \psi_{j,1} = 0 \) and \( \partial \psi_{j,1}/\partial y = 0 \) gives the second order formula

\[ \omega_{j,1} = \frac{8\psi_{j,1} - \psi_{j,2}}{2(\Delta y)^2} \quad (6.39a) \]

This boundary condition is compatible with method I and has been used, for example, by Dixon (ref. 50). A fourth-order formula is obtained by summing all four equations and setting the result to zero so that

\[ \omega_{j,1} = \frac{576\psi_{j,1,2} - 216\psi_{j,1,3} + 64\psi_{j,1,4} - 9\psi_{j,1,5}}{72(\Delta y)^2} \quad (6.39b) \]

The corresponding formulas for the upper wall are

\[ \omega_{j,1,\text{RM}} = \frac{8\psi_{j,1,\text{RM}-1} - \psi_{j,1,\text{RM}-2}}{2(\Delta y)^2} \quad (6.39c) \]

\[ \omega_{j,1,\text{RM}} = \frac{576\psi_{j,1,\text{RM}-1} - 216\psi_{j,1,\text{RM}-2} + 64\psi_{j,1,\text{RM}-3} - 9\psi_{j,1,\text{RM}-4}}{72(\Delta y)^2} \quad (6.39d) \]
F. Initial Conditions

The initial conditions in the present study are taken to be the least stable eigenfunction of the Orr-Sommerfeld equation (i.e., the first eigenmode). For subcritical Reynolds numbers this is the eigenfunction that decays the slowest and for supercritical Reynolds numbers this is the eigenfunction that amplifies. In either case it is the eigenfunction that is even about the flow centerline. This choice of initial conditions not only provides a rational starting point for the nonlinear calculations of a periodic disturbance, but also provides a means of checking the numerical procedure (i.e., calculating the solution of the linearized equations and comparing with the Orr-Sommerfeld results).

If one chooses to solve the Orr-Sommerfeld equation for the vertical disturbance velocity \( \hat{v} \) (one may also choose the stream function), then \( \hat{u} \) is found from the continuity relation to be

\[
\hat{u} = \frac{i}{\alpha} \frac{d\hat{v}}{dy}
\]  

(6.40)

The stream function is

\[
\hat{\psi} = \frac{i}{\alpha} \hat{v}
\]  

(6.41)

and from equation (2.5) the vorticity is given by

\[
\hat{\omega} = \frac{i}{\alpha} \left( \frac{a^2 \hat{v}}{dy^2} - a^2 \hat{\psi} \right)
\]  

(6.42)

The initial conditions become

\[
\psi = A \cdot \text{Real} [\hat{\psi} e^{i\alpha x}]
\]  

(6.43a)

\[
\omega = A \cdot \text{Real} [\hat{\omega} e^{i\alpha x}]
\]  

(6.43b)

\[
u = A \cdot \text{Real} [\hat{u} e^{i\alpha x}]
\]  

(6.43c)

\[
v = A \cdot \text{Real} [\hat{v} e^{i\alpha x}]
\]  

(6.43d)
The eigenfunction is normalized so that $|\hat{v}|$ is unity at the flow centerline and the magnitude of the disturbances in the nonlinear calculations is determined by multiplying the Orr-Sommerfeld values by the amplitude factor $A$ shown in equations (6.43). Since the laminar velocity at the centerline is also normalized to one, the maximum initial disturbance $v$-velocity is equal to $A$ times the maximum laminar velocity, and the value of the fluctuation intensity at the centerline becomes

$$\frac{(v^2)^{1/2}}{U(0)} = \frac{A}{\sqrt{2}} \quad (6.44)$$

The Orr-Sommerfeld eigenvalues and eigenfunctions were calculated by a numerical integration method developed by Lee and Reynolds (ref. 18). The method uses a fourth-order linear algorithm along with a Kaplan filtering technique to maintain linear independence. The calculations were performed on the IBM 360/67 computer using the ORRSOM (ref. 10) program with 401 equally spaced points between the wall and centerline.

G. Computational Procedure

The calculations were performed on an IBM 360/67 computer over a grid given by $KM = 201$ and $JM = 34$. The wave number $\alpha$ of the fundamental disturbance in all cases was taken to be 1.0, and both methods I and II were used for the computations.

The computational procedure is described by the following steps:

**Step 1.**

Calculate the solution of the Orr-Sommerfeld problem as initial conditions.
Step 2.
Calculate a predicted value for the vorticity from equations (6.2a) or (6.18a).

Step 3.
Calculate the corresponding stream function by the Fast Fourier Transform method.

Step 4.
Apply the wall boundary conditions for the vorticity from equations (6.39) and the periodic boundary conditions from equation (6.34).

Step 5.
Calculate the final corrected value of vorticity for equation (6.2b) or (6.18b).

Step 6.
Calculate the corresponding stream function as in Step 3.

Step 7.
Apply the boundary conditions as in Step 4.

The calculations for one time step are now complete. The procedure continues by repeating Steps 2 through 7 until the desired number of time steps are completed.

The stability of the calculations is insured by adjusting the time step so that the diffusion stability criterion, equation (6.12a) or (6.24), is satisfied. In addition, the restriction on the CFL number was that

\[ \alpha \leq 0.60 \]

The calculated values of vorticity and stream function were output on magnetic tape at every \( N \) time step. These data were later analyzed and
the results displayed on an IBM 2250 cathode-ray tube. The final results are discussed in the next section.
VII. RESULTS AND DISCUSSION

Results are presented for both infinitesimal (linear) and finite amplitude (nonlinear) disturbances. For completeness flows that are stable (Re = 2000 and 5000) and unstable (Re = 10,000) according to linear theory are investigated. A summary of cases is given in table 1.

First of all a study was conducted to test the accuracy and stability of the numerical methods described in section VI. This was done by applying these methods to the linear problem and comparing the solution with that obtained from the Orr-Sommerfeld equation for which very accurate solutions are known. In fact, for simplicity, the solutions calculated by the ORRSOM (ref. 10) program are referred to below as exact, and these "exact" solutions for the eigenvalues and eigenfunctions of the linearized theory are the absolutes about which the accuracies of the numerical integration are measured. Actually, this comparison between the exact and numerical solution was considerably enhanced by displaying the two solutions simultaneously on a cathode-ray tube (CRT) and constructing movies by time lapse photography. This dynamic display demonstrates the accuracy of the calculations for the linear theory and gives a degree of confidence to the validity of the subsequent nonlinear computations.

After the completion of the linear calculations, the nonlinear problem was studied for two disturbance amplitudes, A = 0.01 and A = 0.1. The first objective was to study the effect of finite size on the disturbance growth rate as compared to the growth rate of infinitesimal disturbances. The quantity used for measuring the growth or decay of the initial disturbance in both the linear and nonlinear calculations is the kinetic energy of the
Table 1.- Cases studied.

<table>
<thead>
<tr>
<th>Case</th>
<th>Re</th>
<th>Type</th>
<th>Amplitude</th>
<th>Method</th>
<th>Δt</th>
<th>Total time steps</th>
<th>Total time</th>
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<tbody>
<tr>
<td>1</td>
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<td></td>
<td>II</td>
<td>0.05</td>
<td>500</td>
<td>25</td>
</tr>
<tr>
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<td>5000</td>
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<td></td>
<td>I</td>
<td>0.1</td>
<td>500</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>5000</td>
<td>Linear</td>
<td></td>
<td>II</td>
<td>0.1</td>
<td>500</td>
<td>50</td>
</tr>
<tr>
<td>4</td>
<td>10,000</td>
<td>Linear</td>
<td></td>
<td>I</td>
<td>0.1</td>
<td>500</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>10,000</td>
<td>Linear</td>
<td></td>
<td>II</td>
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<td>500</td>
<td>50</td>
</tr>
<tr>
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<td>1600</td>
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</tr>
<tr>
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<td>5000</td>
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<td>0.1</td>
<td>500</td>
<td>50</td>
</tr>
<tr>
<td>8</td>
<td>5000</td>
<td>Nonlinear</td>
<td>0.1</td>
<td>II</td>
<td>0.03</td>
<td>2000</td>
<td>60</td>
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<tr>
<td>9</td>
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<td>II</td>
<td>0.03</td>
<td>2000</td>
<td>60</td>
</tr>
</tbody>
</table>
fluctuations integrated over the region defined by the primary wavelength $2\pi/a$ and the height of the channel, and is expressed as

$$E' = \int_{-1}^{1} \int_{x_0}^{x_0 + \frac{2\pi}{a}} \frac{1}{2} (u'^2 + v'^2) \, dx \, dy \quad (7.1)$$

The velocity fluctuations are obtained by a harmonic decomposition in $x$ of the calculated perturbations using the method given by Villasenor (ref. 51). The integral over $x$ is found by summing the square of the amplitudes of the various harmonics, and the integration over $y$ is obtained using Simpson's rule.

The second objective of the nonlinear computations is to determine if the introduction of a finite amplitude disturbance and the subsequent nonlinear interactions cause a more complicated flow to develop at a later time. The existence of a more complicated flow may indicate transition to turbulence. This aspect is studied by investigating some of the details of the fluctuating motion, such as the spectral density of the fluctuation energy, the Reynolds stress distribution, and the vorticity fluctuations.

Tests conducted using the ORRSM (ref. 10) program show that 201 grid points across the channel are sufficient for accurately defining the eigenfunctions of the linear problem. It is assumed that this same number of grid points in the $y$ direction is also adequate for the nonlinear calculations. In the streamwise direction, the choice of 32 grid points to define the fundamental wavelength appears to be reasonable for the linear calculations, particularly for the fourth-order method II. Nonlinear aspects of the calculations also have a bearing on the choice of the number of grid points in the $x$-direction. Any nonlinearities will trigger the generation
of x-wise harmonics higher than the fundamental disturbance wave. However, calculations by Reynolds and Potter (ref. 24) indicate that the generation of higher harmonics may be of secondary importance in the nonlinear stability problem (see section IV). Furthermore, Kraichnan's theory (ref. 8) of isotropic two-dimensional turbulence predicts a spectral energy density proportional to the $-3$ power of the wave number (see section I). This indicates that in a developing turbulent flow the amplitudes of the higher harmonics may be quite small compared to the amplitude of the first harmonic. As an example, according to $-3$ power law the energy in wave number 10 is only $10^{-3}$ of the energy of the fundamental disturbance. It is therefore concluded that 32 grid points representing 16 harmonics are sufficient for the present study.

As pointed out in section VI the Fourier transform method used for solving the Poisson equation requires that the fundamental wavelength be defined by $2^n$ points, where $n$ is an integer. Doubling the number of points in the $x$ grid requires not only twice the number of computations per time step, but that the time step be halved to satisfy the numerical stability conditions. Hence, the computation time for a 64-point grid increases over that for a 32-point grid by a factor of four. The typical computation time required to calculate 2000 time steps for the large amplitude nonlinear calculations on a 32-point grid was 7-1/2 hours on an IBM 360/67.

A. Linear Results

Linear calculations are performed by neglecting the nonlinear terms in the vorticity transport equations so that
in equations (6.2) and (6.18). The accuracy in computing the growth or
decay of the linear perturbed velocities is shown in figure 6 by comparing
the computed perturbation energy (normalized with respect to its initial
value) with the exact value as time proceeds. The exact value of this
ratio can be expressed as

\[
\ln \left( \frac{E'}{E_{\text{initial}}} \right) = 2c_1 t
\]

for \( a = 1 \).

The decay of the perturbation energy given by method II for a Reynolds
number of 2000 shows that the accuracy of the numerical method is excellent.
For a Reynolds number of 5000, solutions obtained by both methods I and II
are compared with the exact values. The result from the second-order method
I shows a large decay in energy due to truncation errors. Improved results,
however, are obtained with the more accurate fourth-order method II. Simi­
lar results are seen for a growing perturbation at a Reynolds number of
10,000. For all three Reynolds numbers the finite-difference calculations
give an energy lower than the exact solution. Figure 6 shows clearly the
superiority of method II over method I with regard to energy conservation.
As an additional check on the accuracy of the numerical methods, the distur­
bance energy balance equation (2.25) was evaluated as a function of time.
The error in the balance is defined as

\[
\text{Error} = \left| 1 - \frac{(I_1 - I_2)}{\frac{dE'}{dt}} \right|
\]
Figure 6. - Linear disturbance energy.
where the calculation of $dE'/dt$ is independent of $I_1$ and $I_2$. The percent error averaged over the total computation time is given in Table 2. These errors again show that method II is superior to method I. It is worth noting that the largest errors in the energy balance correspond to the largest error between the exact and finite-difference solutions. This indicates that the energy balance can also be used to check the validity of the nonlinear calculations for which no known solution can be used for comparison.

It is of interest to investigate how much of the numerical error is amplitude error and how much is phase error. The modified equations (6.17) and (6.26) predict that the largest error in both methods will appear in the phase of the vorticity equation. Figures 7 through 10 show the comparisons with the exact solution of the amplitude and phase for both the vorticity and stream function at two different Reynolds numbers. In every case the superiority of method II is demonstrated, particularly in phase accuracy. Note that the improvement shown is due strictly to the fourth-order accuracy of the vorticity equation, since both methods calculate the stream function to the same order.

Figure 11 shows the Reynolds stress displayed on the IBM 2250 cathode-ray tube for $Re = 5,000$ and 10,000. The dotted curves represent the numerical solution and the solid curves the solution computed from ORSOM. The photographs show very little visible error for method II, and where errors can be detected in either method they are largest in the region near the critical point ($y = 0.85$). The same conclusions apply to the vorticity distributions shown for $x = 0$ and $\pi/2$ in figures 12 and 13.
Table 2.- Average error in energy balance for linear cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Re</th>
<th>Method</th>
<th>Error, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2000</td>
<td>II</td>
<td>0.36</td>
</tr>
<tr>
<td>2</td>
<td>5000</td>
<td>I</td>
<td>16.09</td>
</tr>
<tr>
<td>3</td>
<td>5000</td>
<td>II</td>
<td>1.18</td>
</tr>
<tr>
<td>4</td>
<td>10,000</td>
<td>I</td>
<td>11.29</td>
</tr>
<tr>
<td>5</td>
<td>10,000</td>
<td>II</td>
<td>1.05</td>
</tr>
</tbody>
</table>
Figure 7.- Amplitude and phase of linear vorticity for $Re = 5000$. 

---

Figure 1.- Amplitude and phase of linear vorticity for $Re = 5000$. 

- Linear theory (ref. 10) 
- $O$ Method I 
- $\square$ Method II
For $Re = 5000$.

Figure 6: Amplitude and phase of linear streamfunction.
Figure 9. - Amplitude and phase of linear vorticity for $Re = 10,000$. 

![Amplitude and phase of linear vorticity graph]

- Linear theory (ref. 10)
- Method I
- Method II
Figure 10.- Amplitude and phase of linear streamfunction for $Re = 10,000$. 
Figure 11.- Comparisons of present results (dotted curve) with reference 10 (solid curve) for linear Reynolds stress distribution at $t = 50$. 

(a) Method I, $Re = 5,000$  
(b) Method II, $Re = 5,000$  
(c) Method I, $Re = 10,000$  
(d) Method II, $Re = 10,000$
Figure 12. - Comparisons of present results (dotted curve) with reference 10 (solid curve) for linear perturbation vorticity profiles at \( Re = 5000 \) and \( t = 50 \).
Figure 13.- Comparisons of present results (dotted curve) with reference 10 (solid curve) for linear perturbation vorticity profiles at $Re = 10,000$ and $t = 50$. 
A final evaluation of the accuracy of the two methods is presented in figures 14 and 15. Figure 14 shows the perturbation vorticity plotted over one wavelength at the channel centerline. If the disturbance is considered as a wave moving from left to right the results from method I show the dotted curve (numerical solution) clearly lagging the solid curve (exact solution). In addition, the numerical solution has a smaller amplitude as predicted by the modified equation. Thus, the numerical errors have the effect of decreasing the speed of the wave as well as giving it additional damping. The alteration of the wave speed due to numerical dispersion also causes a shift in the location of the critical point (recall that the critical point is defined as that point where the wave velocity equals the laminar parabolic velocity). It is in the critical layer that the largest errors occur in the Reynolds stress (fig. 11). It is suggested that this is due in a large measure to the alteration of the phase relation between the $u$ and $v$ velocity perturbations, since the phase shift is related to the wave speed. The numerical errors in the variation of the perturbation vorticity near the critical point is shown in figure 15. Note that at $Re = 5000$ the amplitude does not even appear to be numerically damped in this region, and it is suggested that this may also be attributed to the shift in the critical point.

This concludes the study regarding the accuracy of the numerical methods. All results presented for the nonlinear cases were calculated by the fourth-order method II because the computation time required by method II is only about 10 percent longer than that required for method I and the results are much more accurate.
Figure 14.- Comparisons of present results (dotted curve) with reference 10 (solid curve) for linear perturbation vorticity at centerline.
Figure 15.- Comparisons of present results (dotted curve) with reference 10 (solid curve) for linear perturbation vorticity at critical point.
B. Nonlinear Results

1. Reliability of nonlinear calculations

Now that it has been shown that the numerical integration technique referred to as method II gives results that agree well with the "exact" solutions of the Orr-Sommerfeld equation, it is possible to investigate the effects of finite amplitude disturbances with some confidence that reliable approximate solutions of the partial differential equations are being computed. To verify the nonlinear computations, the average error in the energy balance was calculated as in the linear cases previously discussed. The average percent error in the energy balance for the nonlinear cases is given in table 3. Note that in all cases these errors are of the same order of magnitude as the corresponding linear calculations. The largest errors occur for $A = 0.1$, cases 8 and 10, indicating that the error in the energy balance increases with an increase in the degree of nonlinearity of the vorticity transport equation. It should be mentioned that stable and accurate numerical solutions of the linear problem do not in themselves conclusively show that the subsequent nonlinear calculations will also be stable and accurate. However, in the results presented in this study no explosive and unbounded growth of the perturbations, which is characteristic of numerical instabilities, is in evidence. In addition, the results of the energy balance check strongly support the reliability of the nonlinear solutions.

2. Flow stability

Having demonstrated the validity of the nonlinear calculations we turn to the question of the effect of finite amplitude disturbances on the stability of plane Poiseuille flow. Recall that the laminar flow is
Table 3.- Average error in energy balance for nonlinear cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Re</th>
<th>Amplitude</th>
<th>Error, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2000</td>
<td>0.1</td>
<td>0.72</td>
</tr>
<tr>
<td>7</td>
<td>5000</td>
<td>0.01</td>
<td>1.74</td>
</tr>
<tr>
<td>8</td>
<td>5000</td>
<td>0.1</td>
<td>2.37</td>
</tr>
<tr>
<td>9</td>
<td>10,000</td>
<td>0.01</td>
<td>0.26</td>
</tr>
<tr>
<td>10</td>
<td>10,000</td>
<td>0.1</td>
<td>4.69</td>
</tr>
</tbody>
</table>
considered to be stable if the imposed disturbance ultimately decays to zero and the original laminar flow once again re-emerges. The effect of finite amplitude on the growth or decay of the initial disturbance is first illustrated by considering the time history of the fluctuation energy as shown in figure 16. The result for case 6 (Re = 2000, A = 0.1) shows that the energy is decaying monotonically with a slight oscillation. Although the decay rate is less than that given by linear theory, the flow appears to be stable. The fluctuation energy for a Reynolds number of 5000, cases 7 and 8, shows that for A = 0.01 the energy also decays monotonically, indicating stability; whereas for the larger amplitude, A = 0.1, nonlinear effects are clearly apparent. The energy begins to increase quite rapidly after about four time units and then begins to oscillate with a decreasing period and amplitude. At t = 56 the energy has returned to approximately its initial value. The results shown in figure 6 for the numerical integration of the linear problem can be used to estimate the amount of decay that can be attributed to numerical errors. The obvious decay of the energy after the second peak (t ~ 35) does not appear to be due entirely to numerical dissipation and therefore suggests that the flow is stable. This conclusion is considered preliminary, however, since the possibility exists that the energy may reamplify or reach an equilibrium state at some later time. The results clearly show that nonlinear calculations do not give the well-defined "yes or no" indication of stability that is characteristic of the linear theory.

Finally, for a Reynolds number of 10,000 the fluctuation energy for A = 0.01 grows monotonically at a faster rate than the linear theory and there is no indication that this flow will return to a laminar state. In
Figure 16. Fluctuation energy.
this sense it can be termed unstable. The fluctuation energy for $A = 0.1$ shows an oscillatory growth that appears to level off somewhat after 50 time units. The overall growth of the energy suggests that this case is also unstable in the sense defined above.

In summary, the time history of the fluctuation energy shows that the plane Poiseuille flows that are stable to two-dimensional infinitesimal disturbances are also stable to finite amplitude disturbances of the same type, although for case 8 the absolute stability of the flow is undetermined. The results also show that for the supercritical cases ($Re = 10,000$) the flows are unstable to finite amplitude disturbances as well as infinitesimal disturbances.

The above results can be compared with the nonlinear asymptotic theory of Stuart (ref. 21) and Watson (ref. 23). The estimate of critical Reynolds number made by Reynolds and Potter (ref. 24) predicts instability for all the nonlinear cases presented in this study; however, that estimate is based on the evaluation of the coefficients of equation (4.2) at the neutral curve ($Re = 5800$), and the validity of extrapolating to $Re = 2000$ is uncertain. For $Re = 5000$, Reynolds and Potter (ref. 24) predict a growth for a disturbance of amplitude 0.01, while Pekeris and Shkoller (ref. 25) predict a decay for this amplitude. Both papers, however, predict a growth in the disturbance for $A = 0.1$. Now, in the Stuart-Watson method stability is determined by the time rate of growth of the disturbance amplitude, so that the flow is stable if $\frac{d |A|^2}{dt} < 0$ and unstable if $\frac{d |A|^2}{dt} > 0$. In addition, the coefficients of equation (4.2) are time independent so that this method cannot predict the oscillations in the amplitude apparent in the present results nor can it describe transition. Therefore, since the present
result for case 8 does show an initial growth in disturbance energy, it may be interpreted as being in agreement with the Stuart-Watson method.

For experimental evidence on the stability of plane Poiseuille flow we must resort to experiments performed in rectangular channels of high aspect ratio (channel width/depth) in which the flow is nearly two-dimensional, at least, in the laminar case. A critical Reynolds number of about 600, obtained from measurements of head loss, has been reported by Schiller (ref. 52) for an aspect ratio of 3.5. Similar measurements by Davies and White (ref. 53) give a critical Reynolds number of about 1000 for aspect ratios from $10^4$ to $165$. Recently, Kao and Park (ref. 54) using actual measurements of the fluctuations found a critical Reynolds number of about 1000 for an aspect ratio of 8. It appears from the experimental evidence obtained from these three investigations that the critical Reynolds number increases with the aspect ratio. This implies, therefore, that three-dimensional effects have a destabilizing influence on the flow. It is also well known that when two-dimensional disturbances of sufficient size are present in boundary-layer flows, spanwise disturbances are also excited, and these play an important role in the subsequent transition to turbulent flow.

Perhaps, then, the present disagreement with experimental evidence is due largely to the disturbances in the present study being strictly two-dimensional, and available experimental data essentially being for three-dimensional disturbances.

Finally, it should be pointed out that in the present study the initial disturbances are of a particular type (i.e., eigenfunctions of the Orr-Sommerfeld equation with $\alpha = 1$), and thus the possibility of their being
unstable to two-dimensional finite disturbances of a different nature cannot be ruled out.

3. Characteristics of the disturbed flow

Having discussed the effect of finite amplitude disturbances on the stability of plane Poiseuille flow, we now consider some of the characteristics of the resulting fluid motion. Since the previously discussed energy results for \( A = 0.01 \) are quite similar to the laminar behavior no further consideration of the smaller amplitude disturbances is given.

Figure 17 compares the mean fluctuation energy profiles at the end of the computation time with the initial profiles for the three Reynolds numbers and \( A = 0.1 \). Note that in all three cases the critical point, which corresponds to the profile peaks, has been shifted toward the flow centerline, indicating an increase in the disturbance wave speed. At \( Re = 2000 \) the energy at all points is less than its initial value while at \( Re = 10,000 \) it is always greater. At \( Re = 5000 \), however, the energy is less than its initial value near the wall and greater toward the center of the channel. Thus, although figure 16 shows that the values of the initial and final energies integrated across the channel are nearly the same, the energy in the center region has grown and the energy near the wall has decayed.

Figure 18 shows, for the three Reynolds numbers, the time histories of the fluctuation energy, the mean flow energy, and the total energy, all normalized with respect to the initial total energy. Associated with the decaying fluctuations at \( Re = 2000 \) is a corresponding increase in the mean flow energy and a decrease in the total energy. Some of the fluctuation energy is being transferred to the mean flow through the action of the Reynolds
Figure 17.- Fluctuation energy profiles.
Figure 18.- Flow energies.
stress and a significant portion of the fluctuation energy is being dissipated by viscosity, which accounts for the decrease in total energy. For Re = 5000 there is an exchange of energy between the fluctuations and the mean motion, with the total energy being nearly constant after 10 time units. The energy exchange between the fluctuations and the mean motion is again shown for Re = 10,000. The large oscillations of the mean and fluctuation energies begin to subside after t = 40 and both appear to be approaching a nearly constant level. In addition, the total energy does not oscillate but shows a gradual rise after t = 10 and seems to level out in the last 10 time units. It appears that the flow may be approaching a quasi-equilibrium state in which there is a small periodic exchange of energy between the mean flow and the fluctuations with the total energy remaining nearly constant. This is in agreement with recent numerical results reported by O'Brien (ref. 55).

The mean velocity profiles at t = 60 are compared in figure 19 with the laminar parabolic profiles for cases 8 and 10. In both cases there is an increase in mean velocity near the wall as well as a change in the mean profile curvature. At Re = 5000 the centerline mean velocity is slightly greater than the laminar value, and at Re = 10,000 it is slightly less. These results show that the nonlinear fluctuations do indeed distort the mean motion.

The mean vorticity profiles are also altered as shown in figure 20 for cases 8 and 10. Since $V_x = 0$, the mean vorticity is also the slope of the mean velocity profile and is proportional to the mean shearing stress. Thus, figure 20 shows that for both Reynolds numbers the mean shearing stress at the wall is larger than the laminar value. The decrease in the
Figure 19. Mean velocity profiles.
Figure 20.- Mean vorticity profiles.
mean vorticity near the laminar critical points \( y = 0.85 \) indicates the change in curvature of the mean velocity profile.

Evidence of the generation of smaller scales of motion (i.e., fluctuations of shorter wavelength) may be obtained by investigating the distribution of the fluctuation energy among the harmonics or modes, \( m \), in the periodic streamwise direction. This is shown in figures 21 and 22 where the spectral densities of the fluctuation energy at the centerline, near the linear critical points, and the average value across the channel is shown for \( Re = 5,000 \) and 10,000, respectively, at \( t = 60 \). Both figures show that by far the largest proportion of fluctuation energy remains in the fundamental harmonic. In addition, the higher harmonics are more highly excited at the critical points than at the center of the channel. Recalling that Kraichnan (ref. 8) predicts a \( 1/m^3 \) law for the distribution of energy in the upper part of the inertial subrange in two-dimensional isotropic turbulence, it is of interest to see how this power law compares with the present results, even though the fluctuating motion is not isotropic and only \( x \)-wise homogeneous. Figure 22 indicates that at \( Re = 10,000 \) the average fluctuation energy and its local value near the critical points show general agreement with the -3 power law for the first six modes and thus can be considered as an indication of a developing two-dimensional turbulent flow. Agreement with the -3 power law has also been reported by O'Brien (ref. 55). Modes 12 and 16 show a much more rapid decrease in energy, which is attributed to both viscous and numerical dissipation. How much to attribute to each has yet to be determined. The energy spectral density for \( Re = 5000 \) shows a steeper decline, probably due to the increased effects of viscosity.
Figure 21.- Energy spectral densities for $Re = 5000$. 

- t = 60
- Average energy
- Energy at centerline
- Energy near critical point ($y = ±.85$)
Figure 22.- Energy spectral densities for $Re = 10,000$. 
The Reynolds shearing stress was displayed on the CRT at every 20 time steps and photographed as motion pictures. At \( \text{Re} = 10,000 \) a large degree of fluctuation in the Reynolds stress with time is apparent from these movies. This fluctuation is believed to be due primarily to a continual change in the phase relationship between the \( u' \) and \( v' \) velocity fluctuations. Figure 23 shows the Reynolds stress at the end of the calculations for all three Reynolds numbers, the dotted line is the present result and the solid line is the Reynolds stress from linear theory. At \( \text{Re} = 2000 \) the Reynolds stress is close to the linear value, while at \( \text{Re} = 5,000 \) and 10,000 it departs significantly from the linear value. The "wavy" appearance of the Reynolds stress distribution at \( \text{Re} = 10,000 \) shows the production of additional shear layers toward mid-channel. Since the distribution of mean fluctuation energy (see fig. 17) is relatively smooth, the waviness cannot be attributed to amplitude variations; rather, it is due to the phase relationship between \( u' \) and \( v' \), illustrating that a more complicated flow has developed that has some indications of randomness in phase. Note that at \( \text{Re} = 5000 \) the distribution of Reynolds stress is much smoother, indicating a more orderly flow.

A comparison of the nonlinear perturbation vorticity profiles with linear theory is given in figure 24 for case 8 and in figure 25 for case 10. The displays are shown at \( x = 0, \pi/2, \pi, \) and \( 3\pi/2 \) with \( t = 60 \). For both cases there appears to be a phase shift of about 180° between the perturbation vorticity in the upper half of the channel and the lower half. At \( \text{Re} = 10,000 \) the profile is not as smooth near the centerline as the \( \text{Re} = 5,000 \) profile, again indicating a more complicated flow.
Figure 23.- Comparisons of nonlinear Reynolds stress (dotted curve) with reference 10 (solid curve) for $A = 0.1$. 

(a) Re = 2,000, t = 48

(b) Re = 5,000, t = 60

(c) Re = 10,000, t = 60
Figure 24.— Comparisons of nonlinear perturbation vorticity profiles (dotted curve) with reference 10 (solid curve) at $t = 50$; $Re = 5000$; and $A = 0.1$. 
Figure 25.- Comparisons of nonlinear perturbation vorticity profiles (dotted curve) with reference 10 (solid curve) at \( t = 60 \); \( Re = 10,000 \); and \( A = 0.1 \).
Figure 26 shows CRT displays of the perturbation vorticity at the centerline and near the linear upper and lower critical points for case 8 at $t = 60$. The solid curves represent the linear disturbance with $c_1$ set to zero (i.e., with the damping term removed). The perturbation vorticity at the centerline is slightly larger than the initial amplitude and there is a considerable difference in phase, showing that nonlinearity changes the wave speed. In fact, the movies of the CRT displays show that the wave speed increases. There being no visible excitation of higher harmonics at the centerline suggests that the smaller wavelength fluctuations there are probably being damped by viscosity. The presence of higher harmonics is evident near the critical points, the major contributions being from the second harmonic. In figure 27 we see evidence of higher harmonics at both the centerline and the critical points for $Re = 10,000$ as well as an increase in amplitude. The vorticity at the centerline, however, is still fairly well ordered, with the major contribution clearly stemming from the fundamental harmonic. There does not appear to be enough evidence in these results to predict a randomness that would indicate a truly turbulent flow in the physical sense.

The amplitude and phase of the first two harmonics of the vorticity fluctuations are shown at $t = 60$ in figure 28 for case 8 and in figure 29 for case 10. Comparison of figures 28 and 29 with the linear solution in figures 7 and 9 shows that the fundamental disturbance mode (mode 1) has been distorted, particularly in its phase distribution. Note that while the mean vorticity and the second mode are odd (or antisymmetric) about the centerline, the first mode is even (or symmetric). An investigation of the higher harmonics has shown that for odd harmonics ($m = 1, 3, 5, \ldots$) the
(a) Centerline.

(b) Upper critical point, \( y = 0.85 \).

(c) Lower critical point, \( y = -0.85 \).

Figure 26.- Comparisons of nonlinear perturbation vorticity (dotted curve) with reference 10 and \( c_4 \) set to zero (solid curve) at \( t = 60 \); \( Re = 5000 \); and \( A = 0.1 \).
Figure 27.- Comparisons of nonlinear perturbation vorticity (dotted curve) with reference 10 and $c_1$ set to zero (solid curve) at $t = 60; \text{Re} = 10,000;\text{ and } A = 0.1$. 

(a) Centerline.

(b) Upper critical point, $y = 0.87$.

(c) Lower critical point, $y = -0.87$. 
Figure 28.- Amplitude and phase of vorticity for $Re = 5000$ and $A = 0.1.$
Figure 29. Amplitude and phase of vorticity for $Re = 10,000$ and $A = 0.1$. 
vorticity is even about the centerline and for even harmonics \( m = 2, 4, 6, \ldots \) it is odd. This alternating of odd and even modes is a consequence of the conservation equations and shows that restricting the solution to only symmetric modes, as is done by Pekeris and Shkoller (ref. 28), is indeed not correct.
VIII. SUMMARY AND CONCLUSIONS

Finite-difference methods have been used to investigate the stability of plane Poiseuille flow to periodic initial disturbances given by solutions of the Orr-Sommerfeld equation. Consideration has been given to both the linear problem for infinitesimal disturbances and the nonlinear problem for finite amplitude disturbances.

Comparisons of the linear calculations with the eigenvalues and eigenfunctions of the Orr-Sommerfeld equation have shown the validity of the numerical method as well as illustrated the superior accuracy of a fourth-order method over that of a second-order method.

Calculations of the disturbance energy growth rate suggest that at the subcritical Reynolds number of 2,000 the flow is stable to the particular type of finite amplitude disturbances considered, and for the supercritical Reynolds number of 10,000 the results indicate instability for finite amplitude as well as infinitesimal disturbances. The results for Re = 5,000 and A = 0.1 are considered preliminary since stability has not been conclusively demonstrated. It is suggested that the disagreement with experimental evidence, which indicates instability at Re = 1,000, may be largely due to the fact that only two-dimensional disturbances are considered.

Investigations of the computed results for finite amplitude disturbances illustrate the distortion of the mean flow, generation of higher harmonics of the fundamental mode, and distortion of the fundamental disturbance mode, all due to nonlinear effects. It has also been demonstrated that symmetry cannot be assumed in the nonlinear stability problem for plane Poiseuille flow.
The results for \( \text{Re} = 10,000 \) give indications of transition to two-dimensional turbulent motion. This is particularly evident in the spectral density of the fluctuation energy where there is general agreement with the \(-3\) power law at the lower wave numbers. Some supporting evidence is given by the Reynolds shearing stress distribution which indicates some randomness in the \( u \) and \( v \) velocity fluctuations. It is apparent that the degree of randomness shown in the present study is not as great as would be expected in truly turbulent flow, nor are the scales of fluctuating motion as small. However, a comparison of the \(-3\) power law decay of the energy density for two-dimensional turbulence and the \(-5/3\) power law for three-dimensional turbulence suggests that in the inertial range the smaller scale motions are less evident in two-dimensional turbulence. It is concluded, therefore, that the absence of truly chaotic motion is mainly due to the restriction of only two-dimensional motion.
IX. NOMENCLATURE

A  disturbance amplitude  
c  complex wave speed  
c_1  imaginary part of c  
c_r  real part of c  
E'  total fluctuation energy  
e'  fluctuation energy  
F  See equation (6.2).  
G  See equation (6.2).  
I_1  production integral  
I_2  dissipation integral  
k  wave number in stability analysis  
m  wave number of fluctuations  
P  pressure  
p'  fluctuating pressure  
Re  Reynolds number  
t  time  
U,V  velocities  
u,v  perturbation velocities  
u',v'  fluctuation velocities  
x  streamwise direction  
y  vertical direction  
α  wave number of fundamental disturbance; also, convective stability parameter  
β  diffusion stability parameter  
Δt  time step size
Δx  x-grid interval
Δy  y-grid interval
ζ  amplification factor
ν  kinematic viscosity
ρ  density
ψ  stream function
ψ  perturbation stream function
ψ'  fluctuating stream function
Ω  vorticity
ω  perturbation vorticity
ω'  fluctuating vorticity

Subscripts:
c  critical point
j  x-grid point
k  y-grid point
o  laminar solution

Superscripts:
n  time location
'  fluctuating variable
(−)  mean variable
(∗)  complex variable
1. Reynolds, O. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. Philosophical Transactions of the Royal Society 174: 935-982. 1883.


XI. ACKNOWLEDGMENTS

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