Photoelastic waves

Floyd Everett Morris

Iowa State University

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Engineering Mechanics

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Photoelastic waves

by

Floyd Everett Morris

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subject: Engineering Mechanics

Approved:

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Dean Of Graduate College

Iowa State University
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Photoelastic waves

Floyd Everett Morris

Under the supervision of G. A. Nariboli
From the Department of Engineering Mechanics
Iowa State University

With Toupin's elegant dynamical theory of the elastic dielectric as its basis, this work considers the special case of photoelastic waves. Until now, the well-established theory of photoelasticity has been based on Maxwell's equations supplemented by the assumption of a dielectric tensor which depends on initial strains. Any derivation from a more basic system of equations reveals novel features.

After presenting the highly nonlinear coupled system of basic equations, the initial investigation makes a study of 'weak waves' by employing singular surface theory. A unique assumption singles out photoelastic waves from the multiplicity of possible waves. The strain-optic law emerges along with statements about the use of secondary principal directions and the rotation of the light vector with a varying strain field. Modes of propagation are identified and subsequent studies are limited to one such mode.

The second study, of electromagnetic shocks', provides the interesting result that a longitudinal elastic strain must necessarily accompany the passage of an electromagnetic wave. This interrelationship, established for 'weak shocks',
requires that elastic strain be proportional to the square of an electromagnetic variable.

Guided by these results, we continue the study by using the perturbation technique known as asymptotic expansions. Linear photoelasticity appears when the amplitude is 'infinitesimal'; it corresponds to weak waves and has periodic solutions producing interference. But the main feature here is the higher order theory, produced for weakly nonlinear waves. For these waves a fundamental nonlinear governing equation is obtained in which the elastic deformation affects the coefficient of the nonlinear term. A uniformly valid solution of this equation is presented which brings out the amplitude dependence of frequency.

The concluding study determines the entropy change across a shock wave. Derivations of the Hugoniot function and the conservative form of the energy equation for photoelasticity are presented. It is proved that the entropy varies as the fourth power of polarization density.

The results clarify the relative importance of the conservation laws. 'Infinitesimal' photoelasticity is justifiably based on only Maxwell's equations, neglecting the balance laws of mass, linear momentum and energy. The weakest nonlinear theory must include the first two of the above balance laws, due to the accompanying strain, but can still neglect the last. But any higher order nonlinear theory
requires all of the equations to produce a closed system.

The following features of this work are believed to be novel: (i) derivations, in the Eulerian system, of Gibbs equation and of the conservative form of the energy equation, (ii) clarification of the status of the linear theory and implications of constructing higher order theories, (iii) producing one simplest nonlinear model for which a simple solution is provided.
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LIST OF SYMBOLS

$x_i = (x_1, x_2, x_3)$ Fixed rectangular cartesian spatial coordinate system

$i, j, k, \ldots = 1, 2, 3$ Cartesian tensor indices

$a, b, c$ Derivatives of internal energy function used in determination of electromotive intensity

$A, B, C, D, F, G$ Combinations of invariants and derivatives of internal energy function used in determination of stress

$L_{ij} = L_{x}$ Local stress tensor

$t_{ij} = t$ Spatial stress tensor

$m_{ij} = m$ Electromagnetic stress tensor

$e_{ij} = e$ Spatial strain tensor

$I, II, III, IV, V, VI$ Invariants of strain and polarization density

$p$ Density of the deformed medium

$p_0$ Density of the initially strained medium

$p_{00}$ Density in the unstrained natural state

$\lambda, \mu$ First-order elastic constants (elasticities)

$E, \nu$ Young's Modulus and Poisson's ratio

$\alpha, \beta$ Second-order elastic constants (elasticities)

$\alpha_4, \alpha_6, \beta_{14}, \beta_{16}, \beta_{26}$ Coefficients of combinations of invariants in the internal energy functions expansion

$\beta_{46}, \beta_{66}, \gamma_{116}, \gamma_{166}$ Coefficients of combinations of invariants in the internal energy functions expansion

$u_i = (u_1, u_2, u_3)$ Displacement vector

$v_i = (v_1, v_2, v_3)$ Velocity vector
\[ n_i = (n_1, n_2, n_3) \] Unit normal vector

\[ U \] Internal energy function per unit undeformed volume

\[ e = \frac{U}{\rho_{00}} \] Internal energy function per unit undeformed mass

\[ \varepsilon_0 = 8.854 \times 10^{-12} \text{ farad/meter} \] Permittivity of vacuum

\[ \mu_0 = 4\pi \times 10^{-7} \text{ henry/meter} \] Permeability of vacuum

\[ c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \] Speed of light in a vacuum

\[ G_0 \] Speed of weak wave

\[ G \] Speed of shock wave

\[ \chi_{ij} \] Electric susceptibility tensor

\[ \delta_{ij} \] Kronecker delta

\[ K_{ij} = \delta_{ij} + \chi_{ij} \] Dielectric tensor

\[ P_i = \mathcal{P} \] Polarization vector

\[ P_i = \mathcal{P} \] Discontinuity in normal derivative of polarization

\[ Q \] Charge density

\[ J_i = \mathcal{J} \] Current of polarization in a moving medium

\[ \pi_i = \frac{P_i}{\rho} \] Polarization density

\[ \dot{P}_i = \mathcal{P} \] Convected time flux of polarization

\[ \xi_i = \xi \] Electromotive intensity

\[ B_i = B \] Electric field intensity

\[ e_i = e \] Discontinuity in normal derivative of electric field

\[ B_i = B \] Magnetic flux density
\( b_i = b \)  
Discontinuity in normal derivative of magnetic flux density

\( D_i = D \)  
Electric displacement

\( H_i = H \)  
Magnetic field intensity

\( h_i = h \)  
Discontinuity in normal derivative of magnetic field

\( T \)  
Absolute temperature

\( S \)  
Entropy per unit mass

\( f_\alpha \)  
Thermodynamic tensions

\( v_\alpha \)  
Thermodynamic substate parameters

\([Z]\)  
Jump in a field variable \( Z \) across a singular surface

\( C_e \)  
Strain-optic coefficient

\( \xi_i = [u_i,j]n_j \)  
Discontinuity in normal derivative of displacement vector

\( \varepsilon_0, L_0, M_0 \)  
Coefficients in the expansion of electromotive intensity

\( R_0, S_0 \)  
Coefficients in the expansion of stress

\( K_0 \)  
Proportionality constant relating elastic variables to electromagnetic variables

\( \delta_0/4\pi_0 \)  
Coefficient of first nonvanishing term in entropy expansion

\( \varepsilon \)  
Smallness parameter
1

1. INTRODUCTION

1.1. Historical Introduction

Resting between the well developed theories of electromagnetism and elasticity one finds a great variety of less developed fields to be explored. The selection is narrowed, but still enticing, by choosing a particular material to be studied. The choice of a solid dielectric material leaves open such options for consideration as piezoelectricity, the Faraday effect, the dragging of light, and photoelasticity. Photoelasticity has been selected as the object of this investigation.

David Brewster was undoubtedly the most ardent pioneer in this field. But his fascination with "the variety and splendour of the phenomena" [1] of double refraction has been shared as well by modern scientists. Arnold Sommerfeld writes [2], "the interference patterns of crystal plates in polarized light are among the most beautiful and splendidly colored phenomena of nature." Though great value comes by harvesting practical uses from such 'splendour', we must remark here that it is more enjoyable to work in the presence of beauty than otherwise.

Brewster never tired of finding new objects to examine photoelastically. From heated glass to fish eyes, for more than 50 years, his experiments sought answers to his basic

\[1\] Numbers in square brackets refer to literature cited in the Bibliography.
questions of double refraction. He asks, [3]

What is the mechanical condition of crystals that form two images and polarize them in different planes; and what are the mechanical changes which must be induced on uncrystallized bodies in order to communicate to them these remarkable properties...

He never found satisfactory answers. Forty-four years later (1860) he concluded a paper with [4],

How far these results may lead to new views of the structure which produces double refraction, it would be unprofitable to inquire in the present state of knowledge of the atomical constitution of transparent bodies.

However, he did make significant contributions. He reported to the Royal Society, in 1816, that when light passes through a plate of glass stressed transversely to the direction of propagation, the glass behaves like a crystal - the axes of polarization in the glass being along and perpendicular to the direction of stress. This marked the beginning of photoelasticity. He concludes [3],

The experiments furnish us with a method of rendering visible, and even measuring the mechanical changes which take place during the compression, dilatation or bending of transparent bodies. The tints produced are correct measures of the compressing and dilating forces, and by employing transparent gums, of different elasticities, we may ascertain the changes which take place in bodies.

He thus outlined the basic photoelastic method.

In 1841, F. E. Neumann presented his systematic study of the strain-optical effect to the Berlin Academy [5]. The work can be summarized by two laws:
I. The directions of polarization in any given wave front are parallel to the directions of principal strain in that wave front.

II. The difference of the velocities of propagation of two oppositely polarized waves is proportional to the difference of principal stretches in the plane of the wavefront.

Nine years later, J. Clerk Maxwell wrote a paper [6] that was mainly concerned with the then current debate over whether there need be one or two independent material constants to describe an isotropic linearly elastic body. In it he examined several elasticity problems and extended Brewster ideas to confirm his results. The extension is by no means formal;

The phenomena of bent glass seem to prove, that, in homogeneous singly-refracting substances exposed to pressures, the principal axes of pressure coincide with the principal axes of double-refraction; and that the difference of pressures in any two axes is proportional to the difference of the velocities of the oppositely polarized rays whose directions are parallel to the third axis. On this principle I have calculated the phenomena seen by polarized light.

He credits Brewster [6] with originating the photoelastic method, and, unaware of the experiments of Neumann, cites Sir David's work with bent glass as the only available data.

Sir David Brewster has pointed out the method by which polarized light might be made to indicate the strains in elastic solids; and his experiments on bent glass confirm the theories of the bending of beams.

As Timoshenko [7] notes, and near the conclusion of the paper, "Comparisons of the photoelastic color patterns with analytical solutions were made by ... Maxwell." Our feeling
is that the main reliance was upon his analytical solutions and that he calls in photoelastic evidence for secondary support only.

The paper does not develop a theory of photoelasticity. But its inferences and statements about photoelasticity contain remarkable insight. Maxwell's approach, relating forces rather than deformations (Brewster's and Neumann's approach) to the optic patterns, was important in that it poses a different, and perhaps simpler, question: How are the applied loads related to the internal loads (stresses)? Whereas, the deformation approach leads Brewster, for one, into seeking the atomic structure of the material.

The question whether one should find a stress-optic or a strain-optic formulation more elegant was still debated in 1931, when E. G. Coker remarked [8] that the present accuracy of experiments doesn't contradict either form and thus "they have to be treated as equivalent." His colleague L. N. G. Filon later noted departures from these laws for celluloid or bakelite and says [9] "the laws become complicated and are still (1936) not fully understood."

M. M. Frocht summarizes the development of photoelasticity in a recent publication [10]: After Maxwell's paper (1850) it took another 50 years for the first engineering applications to appear. This happened at the turn of the century when the French engineer Mesnager and the British scientists Cocker and Filon became actively interested in photoelasticity. For nearly 35 years the work in photoelasticity was limited to two-
dimensional problems. It was not until 1935 that a quantitative solution of a three-dimensional problem was obtained photoelastically by Oppel.

1.2. Experimental Bases for Three-dimensional Problems

Here our attention will be centered on the three-dimensional problem. In this section we review two current popular methods of experimental three-dimensional photoelasticity.

The most popular method of attack on the three-dimensional problem is called 'stress freezing'. The model is heated to the point at which secondary bonds in the polymeric model material break down. The load is then applied and borne by the primary bonds. When the temperature is reduced, the secondary bonds reform in the deformed configuration. Thus when the model is unloaded, the secondary bonds try to maintain the deformation while the primary bonds would have the model return to its initial shape. This conflict produces a stress pattern similar to that of a loaded, but unheated, model. But now no load is applied and one is free to slice off a section and examine the frozen-in stress pattern.

Such slices will frequently be taken in non-principal directions. Experimentalists [11] assume that

If this slice is observed at normal incidence, the fringe patterns observed... will be due to the stresses in the xy plane (the plane of the slice) and... will not be influenced by the z-components of stress.
Hence, these in-plane principal stresses, called secondary principal stresses, determine the two wave speeds.

We note here that Maxwell only considered principal directions, whereas Neumann allowed secondary principal directions. In the course of our investigation we shall look at both cases.

The second method of three-dimensional stress analysis is that of scattered light. A strong incident beam, usually produced by a laser, is passed through a stressed model. Particles in the model scatter the light which then propagates radially outward from the scattering source. This scattered light is plane polarized. The model contains a very large number of scattering sources, thus the incident ray will scatter from essentially all points in its path. The result is the same as if one had located a polarizer inside the model. The two waves propagating outward, along the directions of secondary principal stresses, develop a phase difference which is detected by inserting an analyzer between the model and the observing instrument.

Here again secondary principal stresses and directions are used to determine the wave speeds. In addition, the light often passes through distances many times greater than the thickness of the slices taken from a frozen stress model. The difference of secondary principal stresses, as well as their orientation, is therefore much more likely to vary.
Interpretation of the resultant fringe pattern becomes crucial.

Frocht [12] assumes, and we shall prove, that for small rotations of principal axes the light vector rotates to remain in the principal plane. Based on this one expects the photoelastic effect to hold even when the directions of principal stresses change along the light path.

1.3. Current Three-dimensional Theories

Various theoretical approaches to formulate a three-dimensional theory of photoelasticity have been made. All use Maxwell's electromagnetic field equations and several show that their results reduce to Neumann's laws as a special case. Neumann hypothesized that the optical effect depends only on strain and used the ideas of crystal optics (especially Fresnel's ellipsoid), and the strain quadric to achieve his results.

Mindlin and Goodman [13], using a generalized form of Neumann's strain-optic law and an order of magnitude argument, deduce (for a wave propagating in the z-direction) that dependence of the electromagnetic field variables on x and y-directions is negligible. Their resultant 'three-dimensional' equations then reduce to

\[
\frac{\partial H_1}{\partial z} - H_2 \frac{\partial \phi}{\partial z} = - \frac{1}{k_2} \frac{\partial H_1}{\partial t}
\]
and
\[ \frac{\partial H_2}{\partial z} + H_1 \frac{\partial \phi}{\partial z} = -\frac{1}{K_\alpha} \frac{\partial H_2}{\partial t} \]

where \( H \) is the magnetic field intensity with components in the directions of secondary principal stresses, \( \phi \) the angle between \( H_1 \) and \( x \), and \( K_\alpha \) the velocities of the two light waves.

O'Rourke [14] combines Maxwell's equations with a constitutive law relating principal stresses \( \sigma_{ij} \), and optic coefficients \( C' \) and \( C'' \), to the dielectric tensor \( K_{ij} \).

\[ D_i = K_{ij} E_j \]

\[ K_{ij} = K_0 \delta_{ij} + C' \sigma_{ij} + C'' \sigma_{kk} \delta_{ij} \quad (k \neq i) \]

and assuming a solution of the form

\[ E = A \exp(iK_0\psi) \]

where \( K_0 \lambda = 2\pi, \lambda \) being the wavelength, and \( \psi \) a surface related to direction cosines \( \lambda_i \) of the wave normal by

\[ \lambda_i = \psi, \{ (\nabla \psi)^2 \}^{-1/2}, \]

results for principal directions, where \( e_{ij} + e_i \), in

\[ e_k \lambda_i^2 \left( \frac{d^2 \psi}{ds^2} \right)^4 - e_1 (e_2 + e_3) \lambda_i^2 + \ldots \text{cyclic} \left( \frac{d^2 \psi}{ds^2} \right)^2 + e_1 e_2 e_3 = 0 \]

If \( \lambda_i = (0,0,1) \frac{d}{ds} + \frac{d}{dz} \) then this reduces to Neumann's law.

Also called the light vector.
Basic to his derivation is the restriction that the wave normal always travels along a principal direction.

Aben [15] also derives three-dimensional equations. In [16] he writes them in the form

\[ \frac{dB_1}{dz} = -i \frac{C'}{2} (\sigma_1 - \sigma_2) B_1 + \frac{d\phi}{dz} B_2 \]

\[ \frac{dB_2}{dz} = \frac{d\phi}{dz} B_1 + i \frac{C'}{2} (\sigma_1 - \sigma_2) B_2 \]

where \( B \) are transformed components of the electric vector, \( C' \) is a photoelastic constant and \( \phi \) is the angle of rotation of principal axes. In another paper [16] he shows that these equations are equivalent to Neumann's equations.

Aben recognizes that the rotation of principal axes is not to be ignored. But rather than the usual approach, of finding what changes rotation produces in the fringe patterns, he seeks [17]

...a method of three-dimensional photoelasticity which would be based on integral optical phenomena and would take into account the rotations of the physical axes.

He asks: When there is rotation via a given light path, what experimental data can be obtained to use directly in finding stresses?

What he finds is that there are always two perpendicular directions of the polarizer for which light emerging from the stressed model is linearly polarized. These
secondary characteristic directions are easily found experimentally and give information about the state of stress in the model. They are dependent on wavelength and thus by using various light colors one can find if rotation is present.

In a more recent publication [18] Aben considers the addition of an external magnetic field to the experimental setup. He derives equations of magnetophotoelasticity.

\[ \frac{dB_1}{dz} = - \frac{iC}{2} (\varepsilon_{11} - \varepsilon_{22}) B_1 - iC(\varepsilon_{12} + i\varepsilon_{12}) B_2 \]

and

\[ \frac{dB_2}{dz} = -iC(\varepsilon_{12} + i\varepsilon_{12}) B_1 + iC(\varepsilon_{11} - \varepsilon_{22}) B_2 \]

where the dielectric tensor \( \varepsilon_{ij} \) is separated into a part \( \varepsilon_{ij}' \) expressing optical anisotropy caused by the photoelastic effect, and a part \( \varepsilon_{ij}'' \) expressing the Faraday effect:

\( \varepsilon_{ij} = \varepsilon_{ij}' + i\varepsilon_{ij}'' \). Of course, to exploit the Faraday effect for photoelasticity one has to use a magnetic material which is capable of exhibiting the Faraday effect. We shall restrict ourselves to a dielectric material.

The most general and elegant approach to photoelasticity we find to be given by Toupin [19]. He extends an earlier paper [20] to develop a theory of the electromagnetic field in a moving, finitely deformed, elastic dielectric. The
theory is derived under the broad assumptions of initial strain, polarization, electric and magnetic fields. The result is a unified mathematical theory of the piezoelectric, photoelastic, and electro- and magneto-optical properties of elastic dielectrics. As a special case, for an isotropic material with initial polarization and magnetic field zero, the usual theory of photoelasticity results.

1.4. Closing Comments

All known theoretical studies of photoelasticity share one common feature: they start with a postulate of the existence of a dielectric tensor (or its equivalent) linearly related to the infinitesimal strain tensor (or stress tensor). This postulate, if it is to be regarded as generally valid, should be a consequence of a more basic theory.

Recently, a number of such theories have evolved, based on the modern principles of continuum mechanics. The work of Eringen and Suhubi [21] on the elastic dielectric is limited to static problems, as was the earlier work by Toupin [20], and is thus of no direct interest to wave propagation studies. The work of direct interest to us is that of Toupin [19]. Our investigation here will be completely based on his equations.

An additional work is that of Brown [22]. This has as its main objective the study of interactions of strain and
magnetization for magnetizable materials. Magnetization here plays the same role as polarization in Toupin's work. For our present purpose, where the objective is to seek a justification for classical photoelasticity, it is Toupin's work that is pertinent.

As a final introductory comment, we caution the reader that our use of the word photoelasticity may differ from the usual usage. In this work, the frequently used term, photoelasticity is given the limited meaning of the study of electromagnetic wave propagation in a hyperelastic dielectric.
2. OBJECTIVES AND RESULTS

As we noted at the end of the last chapter, the present study concerns itself with the dynamical equations of the elastic dielectric as proposed by Toupin. Such a theory is aimed at a unification of the elastic and electromagnetic field theories. Any such theory has to reduce to the component theories as special cases, and hence must exhibit a multiplicity of possible wave phenomena.

Assuming for the present, that one can assign clear meanings to the dominant and subsidiary parts, waves in an elastic dielectric can be loosely expected to be of the following five types.

1. Purely elastic waves
2. Dominantly elastic, subsidiarily electromagnetic waves
3. Equally elastic and electromagnetic waves
4. Dominantly electromagnetic, subsidiarily elastic waves
5. Purely electromagnetic waves

Of course, there remains the problem of proving the existence of such waves.

Clearly, one would not expect the first two types of waves to be photoelastic. McCarthy [23] investigated waves of the third type noted above. His study applied the theory of singular surfaces to the elastic dielectric based on
Toupin's equations. The results did not reduce to the classical photoelastic equations. We shall undertake investigations of the final two wave classes in this study.

The wave we seek in photoelasticity is the commonly understood sinusoidal travelling disturbance specifically described by an expression of the form

\[ A \exp\{i(k_i x_i - \omega t)\}, \omega = kV, k^2 = k_i k_i \]  

(2.1)

where \( A \) is the amplitude, \( k_i \) are the wave numbers (\( 2\pi/k_i = \lambda_i \) the wavelength) in the \( x_i \)-direction, \( \omega \) is the frequency, and \( V \) is the speed of propagation.\(^3\)

A study of waves by use of such expressions is meaningful only for a linear system. In the present study we have thirty-one governing equations which are highly nonlinear and contain strong coupling between the electromagnetic and elastic field variables. Thus we have a long way to go before arriving at equations whose study can be based on such expressions as (2.1). An oscillatory travelling disturbance of the form of (2.1) can only exist in a medium governed by a linear system of equations. These must be derived.

We believe a formal basis for such a derivation can be based on the study of waves first, by singular surface theory and then relating it to an amplitude expansion.

\(^3\)Cartesian tensor notation will be used throughout. Latin indices \((i,j,k,...)\) range over 1,2,3 and denote the \( x,y \) and \( z \)-directions.
Waves predicted by singular surface theory are not easy to observe experimentally. Though with modern experimental techniques it is possible to observe the head of an elastic wavefront, it is not expected that an analogous observation of the head of the wave is possible for the light wave. Still, as this study will show, singular surface theory provides a number of the properties of sinusoidal waves. Further it enables us to produce from the basic equations a linear system which was the starting point of earlier studies.

We rely on singular surface theory since it is a broad concept which includes characteristic theory and shock theory (besides fracture surfaces etc. as explained by Truesdell [24]). The waves we call 'weak waves' are the ones given by characteristic theory. They are waves of infinitesimal amplitudes governed by a linear system of equations for which solutions of the type (2.1) exist. The study of shock waves via singular surface theory provides a recipe for obtaining the correct expansions for small-but-finite amplitude waves studied later in this work.

Thus we find three approaches up Toupin's mountain of thirty-one coupled partial differential equations; two from singular surface theory and the third from an amplitude expansion that is based on the results of the singular surface theory investigations.

Physically, we are analyzing an elastic insulator which
has been somewhat deformed and into which a light wave moves. The wavefront is viewed as a surface across which the electromagnetic field variables may be discontinuous, but across which displacements and strains are continuous.

The initial investigation is of weak waves. It is here that we find assurance that the study leads to the classical theory of photoelasticity. We find the two independent wave speeds, arising from Toupin's equations, whose difference results in a strain-optic law (which reduces to the classical strain-optic law for infinitesimal strains). Further manipulations produce a strain-optic law for secondary principal directions.

Another calculation allows us to assert that the directions of the two light vectors must always be parallel to the secondary principal directions. This result is of interest for three-dimensional problems, with varying strain in the thickness of the model, because it clarifies the part that large rotations of secondary principal axes play in the resultant fringe patterns presented by experimental work.

If we considered these weak waves in the presence of a magnetic field we would find that this imposes an additional constraint on the wave speed equations. Clearly this is an area in which further study is necessary. We would like to be able to draw conclusions based on the continuum mechanics' model of the elastic dielectric but a preliminary investigation indicates the desirability of a more inclusive model.
Being assured by the study of weak waves that Toupin's equations (under the correct assumptions) lead to classical photoelasticity, we proceed to further examine these waves via shock theory which permits nonlinear wave amplitudes. We begin by asking, to which of the wave types (noted on page 13) do photoelastic waves belong. An examination of type 5 allows us to conclude that purely electromagnetic waves in an elastic dielectric are impossible. In other words, some elastic deformation must result as the light wave moves through the material.

Thus we consider waves of type 4. The subsequent calculations reveal that such waves are possible (i.e., all basic equations are satisfied) provided that strain is proportional to the square of polarization i.e.,

\[ e \propto p^2. \]  

(2.2)

As noted by Nye [25] the inverse relation \( e^2 \propto P \) holds for electrostriction which is an example of type 2 waves. This symmetry is pleasing in its elegance.

The relation (2.2), between amplitudes of the basic components allows us to commence the final examination of Toupin's theory. We wish to examine photoelastic waves, defined in the context of type 4 waves which are of a small-but-finite amplitude. A perturbation expansion in terms of an amplitude parameter is employed, in the asymptotic sense.
Further, the similarity hypothesis, that all electromagnetic field variables are within the same order of magnitude, is made. Displacements are assumed to be of higher order in the amplitude parameter (thus of smaller order) which makes the wave dominantly electromagnetic, but accompanied by a small elastic field.

The results confirm the relation (2.2), but more importantly an equation for waves of small-but-finite amplitude is obtained

$$\frac{\partial E}{\partial t} + G(E^2) \frac{\partial E}{\partial z} = 0,$$

where

$$G(E^2) = G_0(1 + 3\alpha_0 E^2).$$

Using the perturbation expansion based on multiple time scales [26], a uniformly valid solution to (2.3) is obtained which clearly exhibits the amplitude dependence of frequency.

Finally, we take up the study of the change in entropy across a weak shock wave. Here derivations of a conservative form of the energy equation and of the Hugoniot function are given. The Hugoniot function is then combined with Gibbs equation to determine the expression that governs the entropy change across a shock. From here we conclude that entropy varies as the fourth power of polarization density. This implies that the energy equation, which can be written in
terms of entropy, may be neglected in the weakest nonlinear theory. It also proves that $\alpha_0$ of Equation (2.3) is necessarily positive.
3. DYNAMICAL THEORY OF THE ELASTIC DIELECTRIC

This chapter is essentially a review of Toupin's work and a collection of basic equations needed in the problem. The only additional feature is an elaboration of the concept of polarization (also drawn from the literature) and the presentation of Gibb's equation. The last is written here in terms of the reference configuration as the current one and is new, though based on approaches in earlier works.

3.1. Polarization

In the microscopic view, the dielectric may be defined as a material devoid of charged particles that are free to move about. However, the presence of an external electric or magnetic field causes some atomic or molecular separation of bound positive and negative charge (or dipole moment alignment in a polar material). A dielectric with such charge separation is said to be polarized. Its particles have induced dipole moments \( p \), where

\[
\mathbf{p} = q \mathbf{s},
\]

(3.1.1)

\( q \) being the charge on the particle and \( s \) the separation between the positive and negative components. With \( N \) particles per unit volume the polarization \( \mathbf{P} \) of the material
is given by

\[ P = \text{Nq} \quad . \tag{3.1.2} \]

Polarization may be related to charge density as follows: Consider the volume element \( d^2 \) shown in Figure 3.1. Suppose that the negative charges are held fixed and that an electric field \( E \), is applied causing all the positive charges to be displaced out of the parallelepiped via the bounding surface element \( d\sigma \). The charge \( Q \)

\[ \tilde{Q} = \int d\tilde{Q} = \int \text{Nq} d\sigma = \int \text{E} \cdot d\sigma \quad , \tag{3.1.3} \]

\( -\tilde{Q} \) remains inside the volume element. By defining a charge density \( Q \) from

\[ -\tilde{Q} = \int_V QdV = -\int_S P \cdot d\sigma = -\int_V \nabla \cdot PdV \quad , \tag{3.1.4} \]

we have that

\[ Q = -\text{div} \; \tilde{P} = -P_{i,j} \quad . \tag{3.1.5} \]
Also via the polarization concept we express what Truesdell [27] has called the current of polarization in a moving medium, \( \mathbf{J} \). \( \mathbf{J} \) is the sum of a conduction current and a convection current. Truesdell shows that the conduction current is analogous to the convected time flux of the polarization denoted by \( \mathbf{\dot{P}} \) and defined by

\[
\mathbf{\dot{P}} = \frac{3\mathbf{P}}{\partial t} + (\nabla \cdot \mathbf{P})\mathbf{v} + \nabla \times (\mathbf{P} \times \mathbf{v}) .
\] (3.1.6)

We note, also from [27], that

\[
\frac{d}{dt} \int \mathbf{P} \cdot d\mathbf{a} = \int \left\{ \frac{3\mathbf{P}}{\partial t} + (\nabla \cdot \mathbf{P}) + \nabla \times (\mathbf{P} \times \mathbf{v}) \right\} \cdot d\mathbf{a}
\]

\[
= \int \mathbf{\dot{P}} \cdot d\mathbf{a}
\]

where \( \frac{d}{dt} \) denotes the material derivative. Thus, as Panofsky and Phillips [28] clearly explain, \( \mathbf{\dot{P}} \) is the time rate change of polarization across a surface, when the surface itself is moving; the terms on the right in (3.1.6) are explained respectively as: charge motion caused by varying polarization, polarization current loss from passage of the surface through the field, and the polarization current loss across the surface described by the boundary of the moving surface. The convection current is given by \( Q\mathbf{v} \).

Hence the current of polarization is given by

\[
\mathbf{J} = \mathbf{\dot{P}} + Q\mathbf{v} ,
\] (3.1.7)

or using (3.1.5) and (3.1.6)
\[ \mathbf{J} = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{y}). \] (3.1.8)

The material velocity given in terms of the displacement vector \( \mathbf{u} \) is

\[ v_i = \frac{\partial u_i}{\partial t} + u_i, j v_j \] (3.1.9)

where the comma denotes the derivative with respect to spacial coordinate \( x_i \). Thus the polarization and displacement vectors, \( \mathbf{P} \) and \( \mathbf{u} \), are sufficient to describe charge density and polarization current.

3.2. Following Toupin's Formulation

From the preceding, we would expect the internal energy per unit mass \( e \), to depend on polarization and displacement for an elastic dielectric material. Toupin shows that for an isotropic dielectric this dependence may take the functional form

\[ e = e \left( \text{invariants of } e_{ij}, \pi_i \right) \] (3.2.1)

where the finite strain tensor \( e_{ij} \) is given by

\[ 2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i} u_{k,j} \] (3.2.2)

and \( \pi_i \), the polarization density is defined by

\[ \pi_i = \frac{P_i}{\rho} \] (3.2.3)
where $\rho$ is the current density of the material.

The requirement that energy be conserved leads to the equation

$$\rho \dot{e} = t_{ij} v_{i,j} + \dot{p}_i \varepsilon_i$$

or

$$\frac{\partial}{\partial \rho_{00}} \dot{U} = t_{ij} v_{i,j} + \rho \varepsilon_i \left( \frac{\partial \pi_i}{\partial t} + \pi_i, j v_j - \pi_j v_i, j \right)$$

where the electromotive intensity $\xi$, is the force on a unit charge which moves with a velocity $\dot{v}$ defined by

$$\xi = E + \dot{v} \times B,$$

the dot (•) indicates material derivative and $U$ is the internal energy (expressed for convenience in units of stress) related to $e$ by

$$U = \rho_{00} e$$

The material is taken to have a natural state where density is denoted $\rho_{00}$ and for which strain and polarization vanish and hence also internal energy.

The angular momentum balance simply requires that the stress tensor be symmetric.

Toupin observes that for the complete set of field equations not to be overdetermined, it is sufficient to require that the angular momentum and energy equations be satisfied identically as a consequence of the constitutive
equations for the stress $t_{ij}$, and electromotive intensity.

Expanding (3.2.5) we have

$$\frac{\rho}{\rho_{00}} \left( \frac{\partial U}{\partial e_{ij}} \frac{\partial U}{\partial e_{ij}} + \frac{\partial U}{\partial \pi_i} \pi_i \right) = (t_{ij} - \rho \varepsilon_i \pi_j) v_{i,j} + \rho \varepsilon_i \pi_i$$

where

$$\varepsilon_{ij} = \frac{1}{2} (\varepsilon_{ij} + \varepsilon_{ji}) - (e_{ki} v_{k,j} + e_{kj} v_{k,i})$$

from which we write

$$\frac{\rho}{\rho_{00}} \left( \frac{\partial U}{\partial e_{ij}} - 2 e_{ik} \frac{\partial U}{\partial e_{kj}} \right) v_{i,j} + \frac{\partial U}{\partial \pi_i} \pi_i = t_{ij} v_{i,j} + \rho \varepsilon_i \pi_i$$

which is satisfied identically when

$$\varepsilon_i = \frac{1}{\rho_{00}} \frac{\partial U}{\partial \pi_i}$$

(3.2.7)

and the local stress $\tau_{ij}$, defined as $t_{ij} - \rho \varepsilon_i \pi_j$, is

$$\tau_{ij} = \frac{\rho}{\rho_{00}} \left( \frac{\partial U}{\partial e_{ij}} - 2 e_{ik} \frac{\partial U}{\partial e_{kj}} \right) = \frac{\rho}{\rho_{00}} a_{ik} \frac{\partial U}{\partial e_{kj}}$$

(3.2.9)

where

$$a_{ik} = (\delta_{ik} - 2 e_{ik})$$

For an isotropic medium $U$ must depend on strain and polarization density only through the six invariants of $\varepsilon$ and $\pi$ which we denote with Roman numerals:
\( I = e_{ii}, \quad II = \frac{1}{2}(e_{ii}e_{jj} - e_{ij}e_{ji}) \)

\[ III = \frac{1}{6}(2e_{ij}e_{jk}e_{ki} - 3e_{ij}e_{ji}e_{kk} + e_{ii}e_{jj}e_{kk}) = \det (e_{ij}) \quad (3.2.10) \]

\( IV = e_{ij}\pi_i\pi_j, \quad V = e_{ik}e_{kj}\pi_i\pi_j, \quad VI = \pi_i\pi_i \)

Thus using

\[ \frac{\partial I}{\partial e_{ij}} = \delta_{ij}, \quad \frac{\partial II}{\partial e_{ij}} = I\delta_{ij} - e_{ij} \]

\[ \frac{\partial III}{\partial e_{ij}} = e_{ik}e_{kj} - Ie_{ij} + II\delta_{ij}, \quad \frac{\partial IV}{\partial e_{ij}} = \pi_i\pi_j \]

\[ \frac{\partial V}{\partial e_{ij}} = (e_{ik}\pi_j + e_{jk}\pi_i)\pi_k, \quad \frac{\partial VI}{\partial e_{ij}} = 0 \quad (3.2.11) \]

\[ \frac{\partial I}{\partial \pi_i} = \frac{\partial II}{\partial \pi_i} = \frac{\partial III}{\partial \pi_i} = 0, \quad \frac{\partial IV}{\partial \pi_i} = 2e_{ij}\pi_j \]

\[ \frac{\partial V}{\partial \pi_i} = 2e_{ij}e_{jk}\pi_k, \quad \frac{\partial VI}{\partial \pi_i} = 2\pi_i \]

and noting that

\[ \frac{\partial U}{\partial e_{ij}} = \sum_{\gamma=1}^{6} U_{\gamma} \frac{\partial I}{\partial e_{ij}}, \quad \frac{\partial U}{\partial \pi_i} = \sum_{\gamma=1}^{6} U_{\gamma} \frac{\partial I}{\partial \pi_i} \quad (3.2.12) \]

one obtains

\[ E_i = (a\delta_{ij} + be_{ij} + ce_{ik}e_{kj})\pi_j \quad (3.2.13) \]

with

\[ a = \frac{2U_{6}}{\rho_{00}}, \quad b = \frac{2U_{4}}{\rho_{00}}, \quad c = \frac{2U_{5}}{\rho_{00}} \quad (3.2.13a) \]
and

\[ t_{ij} = A \delta_{ij} + B e_{ij} + C e_{ik} e_{kj} + D \pi_{ij} \]
\[ + F(e_{ik} \pi_{kj} + e_{jk} \pi_{ki}) + G e_{ik} e_{jl} \pi_{kl} \]  \hspace{1cm} (3.2.14)

with

\[ A = -\frac{\rho}{\rho_{00}} \{ U_{1} + 2U_{2} + (II-2III)U_{3} \} \]
\[ B = -\frac{\rho}{\rho_{00}} \{ -2U_{1} - (1+2I)U_{2} - IU_{3} \} \]  \hspace{1cm} (3.2.14a)
\[ C = \frac{\rho}{\rho_{00}} (2U_{2} + U_{3}) , \hspace{0.5cm} D = \frac{\rho}{\rho_{00}} (U_{4} + 2U_{6}) \]
\[ F = \frac{\rho}{\rho_{00}} U_{5} , \hspace{0.5cm} G = -\frac{2\rho}{\rho_{00}} U_{5} \]

It is apparent from (3.2.14) that the stress tensor is symmetric thus satisfying the requirement of the conservation of angular momentum.

Conservation of linear momentum, the equation of motion, requires that

\[ \rho \dot{v}_{i} = t_{ij, j} + \rho f_{i} \]  \hspace{1cm} (3.2.15)

We assume that the Lorentz force is the only significant body force acting. This is given by

\[ \rho f_{i} = QE_{i} + (J \times B)_{i} \]  \hspace{1cm} (3.2.16)

which we write, following Toupin, as

\[ \rho f_{i} = -P \cdot \lambda E_{i} + \{(E-P) \cdot \lambda y \} \times B \]  \hspace{1cm}
or
\[ \rho f_i = -P_{\ell, i} + \varepsilon_0 (\mathbf{B} \times \mathbf{E})_i, \]  
(3.2.17)

allowing the equation of motion to be expressed by
\[ \rho \dot{V}_i = t_{ij, j} - P_{\ell, i} + \varepsilon_0 e_{ijk} \mathbf{B}_j. \]  
(3.2.18)

To put this equation in the conservative form we make use of the electromagnetic stress tensor \( m_{ij} \) defined by,
\[ m_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2} (\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2) \delta_{ij}. \]  
(3.2.19)

The divergence of \( m_{ij} \) can be shown to be
\[ m_{ij, j} = -P_{\ell, i} + \varepsilon_0 e_{ijk} \mathbf{B}_j + \varepsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})_i. \]  
(3.2.20)

When this is used in the motion equation the conservative form appears as
\[ \frac{\partial}{\partial t} \{ \rho V_i + \varepsilon_0 (\mathbf{E} \times \mathbf{B})_i \} = (t_{ij} + m_{ij} - \rho \dot{V}_i V_j), j. \]  
(3.2.21)

We also have the conservation of mass,
\[ \frac{\partial}{\partial t} + (\rho V)_i = 0, \]  
(3.2.22)

and Maxwell's equations for a dielectric,
\[ \frac{\partial B}{\partial t} + e_{ijk} E_k, j = 0, \quad B_{i, i} = 0 \]  
(3.2.23, 24)
\[ e_{ijk} H_k, j = \frac{\partial D_i}{\partial t}, \quad D_{i, i} = 0 \]  
(3.2.25, 26)
where
\[ D_i = \varepsilon_0 E_i + p_i \quad \text{and} \quad H_i = \frac{1}{\mu_0} B_i + \epsilon_{ijk} J^j P^k . \quad (3.2.27, 28) \]

This derivation has been given here to avoid confusion since we have taken the current configuration of the material as the reference configuration (for the whole of the remaining work), while nearly all of the earlier investigations take the undeformed configuration as their reference.

3.3. Summary of Basic Equations

During the course of this work we found it useful to collect the basic equations in one place. The following compilation is given to aid the reader.

**Maxwell's equations:**

\[
\begin{align*}
\frac{\partial B_i}{\partial t} + (\nabla \times E)_i &= 0 \quad \nabla \cdot B = 0 \\
(\nabla \times H)_i &= \frac{\partial D_i}{\partial t} \quad \nabla \cdot D = 0
\end{align*}
\]

(3.3.1)

are eight equations, two of which are constraints, which using the relations

\[ D_i = \varepsilon_0 E_i + p_i \quad \text{and} \quad H_i = \frac{B_i}{\mu_0} + (\nabla \times P)_i , \quad (3.3.2) \]

one can write in terms of three of the five electromagnetic vectors.
Continuity, or mass conservation:

\[ \frac{\partial \rho}{\partial t} + (\rho \mathbf{v})_i = 0 \]  \hspace{1cm} (3.3.3)

Linear momentum or the equations of motion:

\[ \rho \ddot{\mathbf{v}}_i = t_{ij,j} - \frac{P}{\rho} \delta_{ij} + \delta_{ij} \mathbf{e}_{ij} \mathbf{e}_{ijk}' \mathbf{p}'_k \]  \hspace{1cm} (3.3.4)

where

\[ \epsilon_i = E_i + (\nabla \times \mathbf{B})_i \]

and

\[ \mathbf{p}'_i = \frac{\partial P_i}{\partial t} + P_{ij,j} \mathbf{v}_j - P_{ij} \mathbf{v}_i,j + P_i \mathbf{v}_j,j \]

or

\[ \mathbf{p}' = \frac{\partial P}{\partial t} + (\nabla \cdot \mathbf{P}) \mathbf{v} + \nabla \times (\mathbf{P} \times \mathbf{v}) \]

Constitutive equations:

\[ \mathbf{e}_i = (a \delta_{ij} + b e_{ij} + c e_{ij} e_{k} e_{kj}) \pi_j = \frac{1}{\epsilon_0} \chi_{ij}^{-1} \mathbf{p}_j \]  \hspace{1cm} (3.3.5)

where

\[ a = \frac{2U_6}{P_{00}}, \quad b = \frac{2U_4}{P_{00}}, \quad c = \frac{2U_5}{P_{00}} \]

and

\[ t_{ij} = A \delta_{ij} + B e_{ij} + C e_{ik} e_{kj} + D \pi_i \pi_j + F (e_{ik} \pi_j + e_{jk} \pi_i) \pi_k \]

\[ + G e_{ik} \pi_j e_{jl} \pi_l \]  \hspace{1cm} (3.3.6)

where

\[ A = \frac{\rho}{P_{00}} \{U_1 + IU_2 + (II - 2III)U_3 \} \]
\[ B = \frac{\rho}{\rho_{00}} \{-2U, -(1+2i)U, _{-1}U, _{-3}\} \]

\[ C = \frac{\rho}{\rho_{00}} (2U, _{2}U, _{3}), \quad \overline{D} = \frac{\rho}{\rho_{00}} (U, _{4}U, _{6}), \]

\[ \overline{G} = -2F = -2(\frac{\rho}{\rho_{00}} U, _{5}) \]

**Strain-deformation:**

\[ 2e_{ij} = u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} \quad (3.3.7) \]

We thus tabulate unknowns and equations as:

<table>
<thead>
<tr>
<th>Unknowns</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_i )</td>
<td>3</td>
</tr>
<tr>
<td>( H_i )</td>
<td>3</td>
</tr>
<tr>
<td>( F_i )</td>
<td>3</td>
</tr>
<tr>
<td>( e_{ij} )</td>
<td>9 (6)</td>
</tr>
<tr>
<td>( t_{ij} )</td>
<td>9 (6)</td>
</tr>
<tr>
<td>( u_i )</td>
<td>3</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1</td>
</tr>
</tbody>
</table>

where \( e_{ij} \), \( t_{ij} \), the constitutive equations for \( t_{ij} \), and strain deformation may all be thought of as six unknowns or equations respectively giving the totals as (25).
3.4. Gibbs' Law

Gibbs' law for ideal gases requires that the internal energy of a mixture of gases be equal to the sum of the partial internal energies of the constituent gases taken at the temperature and volume of the mixture. Truesdell [27] expands this concept to solids.

Following the method of development of such an equation given by Predebon and Nariboli [29], we begin with Truesdell's form of Gibbs' equation

\[
\frac{dU}{\rho_0} = Tds + f_\alpha dv_\alpha
\]  

(3.4.1)

where \( T \) is absolute temperature, \( S \) is entropy per unit mass, \( f_\alpha \) are thermodynamic tensions, and \( v_\alpha \) are thermodynamic substate parameters. The substates, being generalizations of the constituent gases mentioned above, have for our particular case substate parameters, strains and polarization densities. To determine the thermodynamic tensions \( f_\alpha \), we examine the energy Equation (3.2.7) from which we write:

\[
dU = \frac{\partial U}{\partial e_{kj}} de_{kj} + \frac{\partial U}{\partial \pi_i} d\pi_i
\]  

(3.4.2)

\[
dU = \frac{\rho_0}{\rho} b_{ik} L_{ij} de_{kj} + \rho_0 \varepsilon_i d\pi_i
\]  

(3.4.3)

where

\[ b_{ik} = a_{ik}^{-1} \]
Thus Gibbs' equation becomes

\[ \frac{1}{\rho_0} \frac{dU}{d\rho} = T dS + \frac{1}{\rho} b_{ki} \varepsilon_{ij} d_{ex} + \varepsilon_i d\pi_i \]  

(3.4.4)

This expression is believed to be given here for the first time.
Continuum mechanics is a field theory. Common to any field theory of mechanics are the basic conservation, or balance laws of mass, momentum, angular momentum, energy, charge and magnetic flux. These laws originate in an integral form but are most commonly written as differential equations for regions where the variables change sufficiently smoothly, i.e., the field variables or/and their derivatives are assumed to possess continuity or piecewise continuity in such regions. Field variables, for example density, electric field etc. are functions of the independent variables used to define space and time. When one supplements these conservation laws with constitutive laws, describing how a particular material responds to mutual internal forces, one obtains a mathematical model for that selected material.

The singular surface concept clarifies a wave in a continuum. Waves may be thought of as carriers of messages. Since all waves travel at finite speeds, there is a region where the message has arrived and another where it has not. The common boundary of these regions is called the wavefront.

This boundary may not be sharp, as in the case of waves in a dispersive and/or dissipative medium. However, when it can be regarded as sharp, so that it is idealized by a
mathematical two-dimensional surface in the physical three-dimensional space and moving in time, separating the disturbed and undisturbed regions, it is called a singular surface or characteristic manifold. At least some of the field variables or/and their derivatives are discontinuous across this surface.

The pertinent field variables in the present work are displacements (in terms of which velocity, acceleration, strain and density are defined) and electromagnetic quantities. Due to the physical requirement that the material not fracture or separate, displacement must be continuous. If at least some of the first derivatives of displacement are discontinuous and/or the electromagnetic variables discontinuous, then the wave is called a shock wave; while if some, at least, of the discontinuities are in second or higher derivatives of displacements and/or first or higher derivatives of electromagnetic variables then one has described a weak wave. An important distinction between these waves will be seen to be that: weak waves are linear in amplitude (defined in terms of some discontinuity) while shock waves are nonlinear in amplitude.
4.2. Résumé of Compatibility Conditions

The difference between the two values, at the point of discontinuity, of a discontinuous variable or its derivatives is called a jump. Jumps in an arbitrary field variable \( Z \), are denoted by a square bracket by which we mean the difference between \( Z_1 \) (the value of \( Z \) behind the wavefront) and \( Z_0 \) (the value ahead of the wavefront). Thus

\[
[Z] = Z_1 - Z_0 = A
\]  

(4.2.1)

Compatibility conditions express spatial and time derivatives in terms of normal derivatives and tangential derivatives. Such relationships are mainly the result of Thomas' [30] work. Basic to their development is a lemma by Hadamard quoted from [29] as

The tangential derivative of a discontinuity across a singular surface is the same as the discontinuity in the tangential derivative.

If \((\cdot, \alpha)\) denotes the tangential spacial derivative, this lemma stated mathematically is

\[
[Z, \alpha] = [Z]_{\alpha}.
\]  

(4.2.2)

We now list some known compatibility conditions useful to this work. The discontinuities in the first two normal derivatives are denoted as

\[
[Z, i]_n_i = B, \quad [Z, i_j]_n_i n_j = C
\]  

(4.2.3)
where \( \mathbf{n} \) is the unit normal vector to the wavefront. If the field variables are continuous, i.e., \( A = 0 \), we have for their first derivatives

\[
[Z, i] = B n_i
\] (4.2.4)

\[
[\frac{\partial Z}{\partial t}] = -GB
\]

where \( G \) denotes the wave speed.

If the first derivatives are continuous (\( A = B = 0 \)) we have for the second derivatives the relations

\[
[Z, ij] = C n_i n_j
\]

\[
[\frac{\partial^2 Z}{\partial x_i \partial t}] = -G C n_i
\] (4.2.5)

\[
[\frac{\partial^2 Z}{\partial t^2}] = G^2 C
\]

These and other such relations are available in the literature [24, 27, 30, 31]. We also find useful the relation expressing the jump in the product of two field variables \( X \) and \( Y \):

\[
[XY] = [X][Y] + X_0[Y] + Y_0[X]
\] (4.2.6)

When \( B \) does not vanish, more general expressions for the jumps (4.2.5) are given by:
\[ [z_{ij}] = C n_i n_j + g_{\alpha \beta} B, \alpha (n_j n_i, \beta + n_i n_j, \beta) - B b_{\alpha \beta} x_i, \alpha x_j, \beta \]

\[ \frac{\partial^2 z}{\partial x_i \partial t} = (-GC + \frac{\partial B}{\partial t}) n_i - g_{\alpha \beta} (GB), \alpha x_i, \beta \quad (4.2.7) \]

\[ \frac{\partial^2 z}{\partial t^2} = G^2 C - 2G \frac{\delta G}{\delta t} - B \frac{\delta G}{\delta t} \]

where \( g_{\alpha \beta} = x_i, \alpha x_i, \beta \) is the first fundamental form and \( b_{\alpha \beta} = n_i x_i, \alpha \beta \) is the second fundamental form of the surface defined by the wavefront. The delta time derivative \( \frac{\delta}{\delta t} \), denotes the rate of change of a quantity defined on the wavefront as observed by a rider on the wavefront traveling along the direction of the normal. Hence it contains the ordinary time partial derivative and a convective part which is due to the motion of the surface of the wavefront.

4.3. Analysis of Shock Waves

Consider the surface, called the wavefront, which divides the dielectric material into two regions. If there are discontinuities in field variables, as noted in Section 4.1, the wave is defined as a shock wave. To write mathematical relations involving such field variables in a neighborhood of the wavefront one writes the basic equations in integral form and utilizes Reynold’s transport theorem:
\[ \frac{d}{dt} \int_{V(t)} F \, dv = \int_{V(t)} \left( \frac{dF}{dt} + F_{\mathbf{v}_1, i} \right) dv \]  \hspace{1cm} (4.3.1) \\

\[ = \int_{V(t)} \frac{\partial F}{\partial t} \, dv + \int_{S(t)} F_{\mathbf{v}_n} \, dS . \]  \hspace{1cm} (4.3.2)

Here \( \frac{d}{dt} \) is the material derivative, \( V(t) \) is an arbitrary moving volume and \( v_n = v_1 n_1 \) is the normal speed of the boundary \( S(t) \).

Now consider a surface \( \Sigma(t) \), the wavefront, moving with speed \( G \) in the normal direction \( n \).

\[ G = v_1 n_1 = v_n \]  \hspace{1cm} (4.3.3)

![Figure 4.1. Singular surface](image)

Applying Reynolds transport theorem separately to \( V_1 \) and \( V_2 \) one has:

\[ \frac{d}{dt} \int_{V} F \, dv = \int_{V_0} \frac{\partial F}{\partial t} \, dv + \int_{V_1} \frac{\partial F}{\partial t} \, dv + \int_{S_0(t)} F_0 v_n \, dS_0 \]

\[ + \int_{\Sigma(t)} F_0 (-G) \, d\Sigma + \int_{S_1(t)} F_1 v_n \, dS_1 \]

\[ + \int_{\Sigma(t)} F_1 G \, d\Sigma \]
\[ \frac{d}{dt} \int_{V} \mathbf{F} \cdot d\mathbf{V} = \int_{V} \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{V} + \int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS + \int_{\Sigma} (\mathbf{F}_1 - \mathbf{F}_0) \cdot \mathbf{G} d\Sigma. \]

Now shrink \( V \) to zero in a manner such that \( S_1 \rightarrow \Sigma \) and \( S_2 \rightarrow \Sigma \).

Assume that \( \frac{\partial \mathbf{F}}{\partial t} \) remains bounded so that \( \int_{V} + 0 \). Thus

\[ \frac{d}{dt} \int_{V} \mathbf{F} \cdot d\mathbf{V} + \int_{S_1} \mathbf{F}_1 (-\mathbf{v}_1 \mathbf{n}) dS + \int_{\Sigma} \mathbf{F}_2 \mathbf{G} d\Sigma + \int_{S_0} \mathbf{F}_0 \mathbf{v}_0 \mathbf{n} dS \]

\[ + \int_{\Sigma} \mathbf{F}_0 (-\mathbf{G}) d\Sigma \]

\[ = - \int_{\Sigma} [(\mathbf{v} - \mathbf{G}) \mathbf{F}] d\Sigma. \quad (4.3.4) \]

The square brackets denote (as before) the jump in the field variable.

\[ [\mathbf{F}] = \mathbf{F}_1 - \mathbf{F}_0 \quad (4.3.5) \]

Note here that the field equations can be written in the following form.

\[ \frac{d}{dt} \int_{V} \psi d\mathbf{V} = \int_{\Sigma} \phi \mathbf{n} \cdot d\mathbf{S} + \int_{V} X d\mathbf{V} \quad (4.3.6) \]

Using the above result leads to

\[ [(\mathbf{v} - \mathbf{G}) \psi] = [\phi] \mathbf{n} \quad (4.3.7) \]

Thus, desired relations across the wavefront may be obtained directly from the field equations, written in conservative form, by using the following recipe:
\frac{\partial \psi}{\partial t} + \mathbf{v} \cdot \nabla \psi = -G[c(\psi)] \quad (4.3.8)

\nabla \cdot \psi + [c(\psi) \cdot \mathbf{n}] \quad (4.3.9)

This follows since the conservative form

\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad (4.3.10)

follows from

\frac{d}{dt} \int_{V} u \mathbf{d}V = \int_{S} \mathbf{B} \cdot \mathbf{n} \, ds \quad (4.3.11)

where

\mathbf{B}_{ij} = \mathbf{u}_i \mathbf{v}_j - \mathbf{A}_{ij}

In attempting to reduce the algebraic equations that result from (4.3.8-9), one is forced to make some simplifying assumptions. Thus the circumstances required for the results of Chapter 6 are somewhat more specific than those obtained for weak waves in Chapter 5. However, the shock wave analysis uncovers the nature of the nonlinear relation between the elastic and electromagnetic field variables (this is further discussed in the next section). Also it determines which types of interactions (waves) are physically possible in the sense that the requirements of the basic field equations are met.
4.4. The Relationship Between Weak Waves and Shock Waves

One would like to know how weak waves and shock waves are related. What follows is a heuristic argument that indicates this relationship.

Consider a singular surface moving to the right with jumps in $Z$ (shock wave) and in the first partial of $Z$ (weak wave), as shown in Figure 4.2.

Further consider a field equation

$$
\frac{\partial Z}{\partial t} + \frac{\partial F(Z)}{\partial x} + H(Z, x, t) = 0.
$$

(4.4.1)

Applying shock theory across the wavefront results in

$$
-G[Z] + [F(Z)] = 0,
$$

(4.4.2)

and the weak wave assumptions lead to

$$
-G_0 \left[ \frac{\partial Z}{\partial x} \right] + \frac{dF(Z_0)}{dz} \left[ \frac{\partial Z}{\partial x} \right] = 0.
$$

(4.4.3)

The speed of the shock is $G$ and that of the weak wave is $G_0$. We note that $G_0$ is determined by the state of the field.
variable ahead of the wave and that the equation is linear in amplitude \( \frac{\partial Z}{\partial x} \); while for the shock, \( G \) depends on \( F(Z) \) which includes values behind the wavefront and may not be linear in its amplitude \( Z \).

To see clearly that the weak wave theory does not include nonlinear terms, suppose that equation (4.4.1) is nonlinear. For example consider,

\[
\frac{3Z}{3t} + \frac{3(y^2)}{3x} = 0. \tag{4.4.4}
\]

This field equation generates the shock relation

\[-G[Z] + [Y^2] = 0, \tag{4.4.5}\]

and the weak wave relation

\[-G_0 \left( \frac{\partial Z}{\partial x} \right) + 2Y_0 \left( \frac{\partial Y}{\partial x} \right) = 0 \tag{4.4.6}\]

The shock relation gives a nonlinear relation between \( Z \) and \( Y \)

\[Z = A_0 + \frac{1}{G} Y^2 \tag{4.4.7}\]

where

\[A_0 = Z_0 - \frac{1}{G} Y_0^2, \]

but, the relation between the variables, \( \frac{\partial Z}{\partial x} \) and \( \frac{\partial Y}{\partial x} \), in the weak wave equation is linear

\[\frac{\partial Z}{\partial x} = B_0 + C_0 \frac{\partial Y}{\partial x} \tag{4.4.8}\]
where
\[ B_0 = \left( \frac{\partial^2 Y_0}{\partial x^2} - \frac{2Y_0}{G_0} \frac{\partial^2 Y_0}{\partial x \partial y} \right), \quad C_0 = \frac{2Y_0}{G_0} \]

Thus a nonlinear relation between field variables is only found via shock theory.

Shock theory may also indicate the appropriate choices for jumps in a related investigation of weak waves. Taking the example from this study of photoelasticity, in Chapter 7 we show that strain is proportional to the square of polarization density. Say,

\[ e_{ij} = K_0 P_i P_j \quad (4.4.9) \]

From which

\[ e_{ij,k} = K_0 (P_i, k P_j + P_i, P_j, k) \quad (4.4.10) \]

Now, in the weak wave study the electromagnetic field variables are taken to be continuous across the wavefront and zero ahead,

\[ [P_i] = P_{0i} = 0 \quad (4.4.11) \]

This with equations (4.4.9-10) leads to the requirement, for weak waves, that

\[ [P_i] = [e_{ij}] = [e_{ij,k}] = 0. \quad (4.4.12) \]

Higher derivatives may be discontinuous.
5. WEAK WAVE ANALYSIS

5.1. General Considerations: Notation and Initial Assumptions

In this section we lay the foundation for the application of singular surface theory to the study of weak waves in the elastic dielectric. Notation is introduced in accord with our initial assumptions about the basic electromagnetic and elastic field variables. The continuity of velocity, acceleration, and strain follow from these assumptions and the conservation of mass relation.

An arbitrary field variable $Z_{ij}$... ahead of the oncoming wave will be denoted by $Z_{0ij}$... and behind by $Z_{ij}$... Before any initial deformation of the dielectric it will be written $Z_{00ij}$... $Z_{ij}$... may be a scalar, vector, or tensor. Cartesian tensor analysis will be employed throughout and the repetition of an index will indicate summation over the numbers 1,2,3.

All electromagnetic field variables are taken to be zero initially (ahead of the wavefront).

$$E_{0i} = P_{0i} = H_{0i} = B_{0i} = D_{0i} = 0$$

(5.1.1)

The weak wave hypothesis requires, as noted in Section 4.1, that the electromagnetic variables be continuous across the moving wavefront (denoted by $\Sigma(t)$),
\[ [E_i] = [P_i] = [H_i] = [B_i] = [D_i] = 0 \] (5.1.2)

and that their first spacial derivatives are not continuous.

\[ [E_i,j] \neq 0 \text{ etc.} \] (5.1.3)

We denote jumps in the normal derivatives of electromagnetic variables with small letters:

\[ [E_{i,j}] n_j = e_i, \quad [P_{i,j}] n_j = p_i, \quad [H_{i,j}] n_j = h_i. \] (5.1.4)

Since \( D \) and \( B \) are expressible in terms of \( E, P, \) and \( H \) we need not continue writing them.

Finally, we assume that displacement and its first two spacial derivatives are continuous across \( \Sigma(t) \).

\[ [u_i] = [u_{i,j}] = [u_{i,jk}] = 0 \] (5.1.5)

As a result of (5.1.5) and the continuity equation, velocity and acceleration of material elements of the dielectric are continuous across \( \Sigma(t) \). Velocity jumps are

\[ [v_i] = \frac{\partial u_i}{\partial t} + [v_j u_{i,j}], \] (5.1.6)

and using the compatibility conditions we have

\[ [v_i] = -G[u_{i,j}] + [v_j][u_{i,j}] + v_{0j}[u_{i,j}] + u_{0i,j}[v_j] \]

which by (5.1.5) reduces to

\[ [v_i] = u_{0i,j}[v_j] \]
or

\[ [v_j](\delta_{ij}^{-}u_{0i,j}) = 0 \]  \hspace{1cm} (5.1.7)

The conservation of mass requires that

\[ \rho_0 = \rho_{00} \sqrt{(1-2e_{011})(1-2e_{022})(1-2e_{033})} \]

or

\[ = \rho_{00} (1-u_{01,1})(1-u_{02,2})(1-u_{03,3}) . \]

So

\[ \frac{\rho_0}{\rho_{00}} = |\delta_{ij}^{-}u_{0i,j}| \neq 0 \]  \hspace{1cm} (5.1.8)

Hence, the determinant of the coefficients of \([v_j]\) in (5.1.7) does not vanish which leaves only the trivial solution

\[ [v_j] = 0 . \]  \hspace{1cm} (5.1.9)

Since the material ahead of the wave is at rest, (5.1.9) further requires that

\[ v_i = 0 \]  \hspace{1cm} (5.1.10)

The jump in acceleration is defined by

\[ [a_i] = \left[ \frac{\partial v_i}{\partial t} \right] + [v_i, v_j] . \]  \hspace{1cm} (5.1.11a)

Using the relations given in Section 4.2. This can be expressed as
\[ [a_i] = -G[v_i,j]n_j + [v_i,j][v_j] + v_{0i,j}[v_j] + v_{0j}[v_i,j] \]

which reduces, by (5.1.9-10), to

\[ [a_i] = [v_i,j](-Gn_j) , \quad (5.1.11b) \]

or written in terms of displacements

\[
= -Gn_j \left[ \frac{\partial^2 u_i}{\partial x_j \partial t} + u_{i,kj}v_k + u_{i,k}v_{k,j} \right]
= -Gn_j \left[ \frac{\partial^2 u_i}{\partial x_j \partial t} \right].
\]

By using Equation (4.2.5) we have

\[ [a_i] = +G^2n_j[u_{i,kl}n_kn_l] \]

which vanishes by hypothesis (5.1.5). Thus

\[ [a_i] = a_i = 0 \quad (5.1.12) \]

since the material is initially at rest.

We also note here that strain and its first spacial derivative are continuous across \( \Sigma(t) \) as a consequence of (5.1.5) and the definition of finite strain.

\[ [e_{ij}] = [e_{ij,k}] = 0 \quad (5.1.13) \]
5.2. Applying Compatibility Relations to the Field Equations

Under the conditions listed in the previous section, the application of the compatibility relations to Maxwell's field equations yields the following set of equations:

\[ \mu_0 h_i = e_{ijk} \nabla_j e_k \]  \hspace{1cm} (5.2.1)
\[ h_i n_i = 0 \]  \hspace{1cm} (5.2.2)
\[ -G(\varepsilon_0 e_i + p_i) = e_{ijk} n_j h_k \]  \hspace{1cm} (5.2.3)
\[ (\varepsilon_0 e_i + p_i)n_i = 0 \]  \hspace{1cm} (5.2.4)

It is to be noted that these eight equations are not independent. Equations (5.2.2) and (5.2.4) are constraints on the other six equations.

We next consider the jump in normal derivative of the constitutive equation for electromotive intensity. Since \( v_i, [v_i, j] \) vanish, the jumps in the normal derivations of \( \varepsilon \) and \( \Xi \) are the same, and so

\[ [E_i, \ell] n_\ell = \left[ \frac{a}{\rho} \varepsilon_i \right]_{,\ell} n_\ell + \left[ \frac{b}{\rho} e_{ij} p_j \right]_{,\ell} n_\ell \]
\[ + \left[ \frac{c}{\rho} e_{ik} e_{kj} p_j \right]_{,\ell} n_\ell \]  \hspace{1cm} (5.2.5)

To reduce this expression we examine \([b, \ell] \).

\[ [b, \ell] = \frac{2}{\rho_{00}} \frac{\partial U, \ell}{\partial Y} \frac{\partial I, Y}{\partial x_\ell} \text{ where } = 1, 2, \ldots, 6 \]  \hspace{1cm} (5.2.6)
and $U_4$ is the partial of the internal energy density with respect to the fourth invariant. Since $U$ is a function of invariants of strain and polarization density which are continuous across $\Sigma(t)$, $U_4$ and higher partials with respect to the invariants must also be continuous. Thus

$$[a] = [b] = [c] = 0 . \quad (5.2.7)$$

Further, straightforward calculations show that spacial derivatives of the invariants, under the conditions of continuous $e_{ij}$ and $\pi_1$, are continuous across the wavefront. Thus by (4.2.6) the jump of the product in (5.2.6) vanishes, requiring that

$$[b, \pi] = 0 . \quad (5.2.8)$$

Similarly $[a, \pi]$ and $[c, \pi]$ vanish. Taking note of these last results and using the definitions (5.1.4), the expansion of (5.2.5) can be written in the form

$$e_i = (a_0 \delta_{ij} + b_0 e_{0ij} + c_0 e_{0ik} e_{0kj}) \frac{p_j}{\rho_0} . \quad (5.2.9)$$

Now, making use of the concept of electric susceptibility, denoted by $\chi_{ij}$, we have

$$e_i = \frac{1}{\varepsilon_0} \chi_{0ij}^{-1} p_j \quad \text{or} \quad p_i = \varepsilon_0 \chi_{0ji} e_j \quad (5.2.10)$$

Thus, electric susceptibility is defined in terms of initial strains and derivatives of the internal energy function.
\[ x_{0i0j} = \frac{\varepsilon_0}{\rho_0} (a_0 \delta_{ij} + b_0 \varepsilon_{0ij} + c_0 \varepsilon_{0ik}\varepsilon_{0kj}) \]  

(5.2.11)

We note that \( x_{ij} \) is symmetric. The permittivity of a vacuum \( \varepsilon_0 \), was introduced to make \( x_{ij} \) dimensionless where

\[ \varepsilon_0 = 8.854 \times 10^{-12} \text{ farad/meter} \].

Operating on the constitutive equation for stress as we did for electromotive intensity and evaluating \([A], [A', \lambda], \) etc. as we did for \([b, \gamma, \) reveals that

\[ [t_{ij,j}] = 0. \]  

(5.2.12)

So, for the equation of motion, we have asserted that there is no discontinuity in acceleration or in the divergence of the stress tensor. Remaining are the electromagnetic terms,

\[ \frac{\partial}{\partial t} (E \times B)_i \text{ and } m_{ij,j}, \]

the discontinuities of which also vanish since

\[ [\frac{\partial}{\partial t} (e_{ijk}E_{jB_k})] = -G e_{ijk} \{E_{jB_k}, _l, n_l \} \]

\[ = -G e_{ijk} \{ [e_{jB_k}] + [E_{jB_k}] \}, \]  

(5.2.13)

and

\[ [m_{ij,j}] = \varepsilon_0 [E_iE_j], j + \frac{1}{\mu_0} [B_iB_j], j - \frac{1}{2} \delta_{ij} \varepsilon_0 [E_kE_k]_{ij} \]

\[ + \frac{1}{\mu_0} [B_kB_k], j \]

\[ \text{The rationalized MKS system of units will be used throughout.} \]
both vanish from using (4.2.6) and the assumption that values ahead and jumps in electromagnetic field variables are zero. Thus, the equation of motion is identically satisfied for jumps.

We are left with equations which involve jumps in derivatives of only the electromagnetic field variables. This linearized set of equations is the basis for the results obtained in the remainder of this chapter.

However, we do not mean to imply that deformation and the other elastic variables are not related to the electromagnetic quantities. We only conclude here that the linearization of the basic equations, under the discontinuities assumed for weak waves, does not include the elastic variables. As we shall see in Chapter 6, there exists a definite elastic-electromagnetic variable relationship for photoelastic waves.

5.3. Wave Speeds

First we show that there exist two wave speeds which are dependent on the initial strains. We then proceed to determine that their difference takes a form analogous to the classical
strain-optic law.

The magnetic field vector \( \mathbf{H} \) is called the light vector by some authors [12]. It may be more physically useful to examine this electromagnetic field vector than either \( \mathbf{E} \) or \( \mathbf{P} \) since the two waves that result when a single ray enters a birefringent material are such that their light vectors remain orthogonal. Thus we solve the system of equations (5.2.1,2,3,4,10) in terms of the jump in the normal derivative of \( \mathbf{H} \).

Solving (5.2.3), using (5.2.10), for \( e_i \) we have

\[
e_k = \left( -\frac{e_{ijk} K_{jk}^{-1} n_j}{G \varepsilon_0} \right) h_k
\]

(5.3.1)

where \( K_{ij} = \delta_{ij} + \chi_{ij} \) is known as the dielectric tensor. We note that the constraint (5.2.4) is satisfied identically by taking the inner product of (5.2.3) with the unit normal. Using (5.3.1) in (5.2.1) results in

\[
\left\{ \frac{G^2}{c_0^2} \delta_{ik} + (\delta_{ij} - n_i n_j) K_{jk}^{-1} \right\} h_k = 0,
\]

(5.3.2)

where \( c_0^2 = 1/\mu_0 \varepsilon_0 \) is the square of the velocity of light in a vacuum. These are three homogeneous equations for the three unknowns \( h_k \); however, there is an additional constraint (5.2.2), which also is homogeneous. Thus we have only two independent equations which we write in the variables \( h_1 \) and \( h_2 \). By orienting the axes along principal directions, \( K_{ij} \) (since it is a function of strain and \( \delta_{ij} \)) becomes diagonal.
The two resultant independent equations are

\[
\frac{G^2}{C_0} + k_{11}^{-1} + n_1^2 (k_{33}^{-1} - k_{11}^{-1}) h_1 + n_1 n_2 (k_{33}^{-1} - k_{22}^{-1}) h_2 = 0
\]

(5.3.3)

and

\[
n_1 n_2 (k_{33}^{-1} - k_{11}^{-1}) h_1 + \frac{G^2}{C_0} + k_{22}^{-1} + n_2^2 (k_{33}^{-1} - k_{22}^{-1}) h_2 = 0.
\]

For a nontrivial solution the determinant of the coefficients must vanish:

\[
\begin{vmatrix}
\frac{G^2}{C_0} + k_{11}^{-1} + n_1^2 (k_{33}^{-1} - k_{11}^{-1}) & n_1 n_2 (k_{33}^{-1} - k_{22}^{-1}) \\
n_1 n_2 (k_{33}^{-1} - k_{11}^{-1}) & \frac{G^2}{C_0} + k_{22}^{-1} + n_2^2 (k_{33}^{-1} - k_{22}^{-1})
\end{vmatrix} = 0
\]

(5.3.4)

The two solutions for \( G^2/C_0^2 \) are

\[
2 \frac{G^2}{C_0^2} = P + \sqrt{P^2 - 4Q} \quad (\alpha = 1, 2)
\]

(5.3.5)

where

\[
P = (k_{33}^{-1} - k_{11}^{-1}) (n_1^2 + n_2^2) + k_{11}^{-1} + k_{22}^{-1}
\]

\[
Q = k_{11}^{-1} k_{22}^{-1} + (n_1^2 k_{22}^{-1} + n_2^2 k_{11}^{-1}) (k_{33}^{-1} - k_{11}^{-1}) + n_1^2 n_2^2 (k_{22}^{-1} - k_{11}^{-1})
\]

These two wave speeds \( G_1^2 \) and \( G_2^2 \), which depend on the initial strains via \( K_{ij} \), can interfere constructively and destructively to produce dark and light bands.
5.4. The Strain-optic Law and the Stress-optic Law

In this section we examine the special case of a wave propagating in the third principal direction. Here the wave normal is simply

\[ \mathbf{n} = (0, 0, 1), \quad (5.4.1) \]

and the wave speeds reduce to

\[ G_1^2 = C_0^2 \frac{1}{K_{22} - 1} \quad \text{and} \quad G_2^2 = C_0^2 K_{11}^{-1}. \quad (5.4.2) \]

The index of refraction \( n \), is the ratio of the speed of the wave in a vacuum to its speed in the medium. The difference in squares of indices of refraction for this case, is

\[ n_1^2 - n_2^2 = K_{22} - K_{11} = \chi_{22} - \chi_{11} \quad (5.4.3) \]

To involve strain in this, one uses (5.2.11) written in principal directions.

\[ \chi_{11} = \frac{\rho}{\varepsilon_0 (a_0 + b_0 \varepsilon_{011} + c_0 \varepsilon_{011})^2} \]

\[ \chi_{22} = \frac{\rho}{\varepsilon_0 (a_0 + b_0 \varepsilon_{022} + c_0 \varepsilon_{022})^2} \quad (5.4.4) \]

Thus we can write (5.4.3) in the following way:
which is the form of the strain-optic law suggested by Neumann. The second term is much less important than the first, especially for small strains. Moreover, as the following chapter will demonstrate, large amplitude electromagnetic waves generate strains which may not be negligible in comparison with the second term. Thus the act of observing may result in changing what is observed, especially when one uses powerful electromagnetic waves. In either case (by choice or by physical limitation), if we limit our attention to first order effects, where small initial strains only are considered, (5.4.5) reduces to

\[ \eta_1^2 - \eta_2^2 = \frac{\rho b_0 (e_{011} - e_{022})}{\varepsilon_0 (a_0 + b_0 e_{011} + c_0 e_{011}^2) (a_0 + b_0 e_{022} + c_0 e_{022}^2)} + \frac{\rho c_0 (e_{011}^2 - e_{022}^2)}{\varepsilon_0 (a_0 + b_0 e_{011} + c_0 e_{011}^2) (a_0 + b_0 e_{022} + c_0 e_{022}^2)} \]

(5.4.5)

Here Neumann's second law is very nearly reproduced, the difference being that in place of a constant of proportionality we find a slightly strain dependent coefficient.
Currently, one finds the strain-optic law written in the form
\[ \eta_1 - \eta_2 = C_e (e_{011} - e_{022}), \]  
(5.4.8)

where \( C_e \) is the strain-optic coefficient. A simple calculation shows that (5.4.6) can be written as
\[ \eta_1 - \eta_2 = \frac{\rho b_0}{a_0^2 + a_0 b_0 (e_{011} + e_{022})} \]  
(5.4.9)

by neglecting terms quadratic in strain. We observe that \( C_e \) depends on strain and is not quite an absolute constant; one has \( C_e = C_e (e) \). If further, we let \( e \to 0 \) in \( C_e \), one has \( C_e (0) \) given by
\[ C_e (0) = \frac{\rho_0 b_{00}}{a_{00} \sqrt{2 a_{00} \varepsilon_0 (\rho_{00} + \varepsilon_0 a_{00})}}, \]  
(5.4.10)

where \( a_{00} \) and \( b_{00} \) are evaluated for vanishing initial strains. This may be identified as a constant coefficient. If we expand the internal energy function by its invariants (as given in the Appendix) we have
\[ u = \frac{\lambda + 2\mu}{2} I^2 - 2\mu I_{II} + \lambda I_{III} + \alpha IV + \alpha VI + \beta_{14} I_{IV} + \beta_{16} I_{VI} + \beta_{26} I_{VI + \beta_{46} IV + \beta_{66} IV^2 + \gamma_{116} I_{II}^2 + \gamma_{166} I_{IV}^2 + \ldots \]  

which for small initial strains allows us to calculate

\[ a_0 = \frac{2}{\rho_{00}} (\alpha_6 + \beta_{16} I_0), \quad b_0 = \frac{2}{\rho_{00}} (\alpha_4 + \beta_{14} I_0), \quad c_0 = 0. \]  

The stress-optic coefficient, evaluated for vanishing initial strains, then follows from Equation (5.4.10) as

\[ C_e(0) = \frac{\rho_{00}^2 \alpha_4}{2\alpha_6 \varepsilon_0 \alpha_6 (\rho_{00}^2 + 2\varepsilon_0 \alpha_6)} \]  

Using values for commonly employed photoelastic materials [11], \( C_e \) is in the range of \( 10^{-3} \) to \( 10^{-4} \), we present the approximate relation for \( \alpha_4 \) in terms of \( \alpha_6 \) as

\[ 100 \rho_{00}^2 \alpha_4 = \alpha_6 \varepsilon_0 \alpha_6 (\rho_{00}^2 + 2\varepsilon_0 \alpha_6) \]  

Since \( U \) has the dimensions of stress, both \( \alpha_4 \) and \( \alpha_6 \) are in units of stress divided by polarization density squared.

We conclude this section by writing the stress-optic law. Using the classical linear stress-strain relation (Hooke's Law), for principal directions we find the difference and sum of principal strains to be given by
where $E$ is Young's modulus and $v$ is Poisson's ratio. Making use of these in (5.4.8) we find

$$
\eta_1^2 - \eta_2^2 = \frac{\rho (1+v)}{\varepsilon_0 a_0 (E E_0 + (1-v)(t_{011}+t_{022})-2v t_{033})} (t_{011}-t_{022})
$$

The stress-optic coefficient $C_t$ is related to the strain-optic coefficient by

$$
C_t = \frac{1+v}{E} C_e = \frac{1}{2\mu} C_e
$$

and for common photoelastic materials has an approximate range of from $5 \times 10^{-7}$ to $2 \times 10^{-9}$ [11] with units of the reciprocal of stress.

5.5. Secondary Principal Axes

Since Neumann's time it has been assumed, for waves not traveling along principal directions, that the difference in indices of refraction was proportional to the difference in the secondary principal stresses (or strains). Secondary principal stresses refer to the extremum values in stress in
any plane other than a principal plane. We wish to validate
this hypothesis for weak waves on the basis of Toupin's theory
of the elastic dielectric.

We begin by relaxing the restriction that the coordinates
be aligned with the principal axes. We retain the small
strain assumption and examine a wave propagating in the $x_3$-
direction. The secondary principal strains, denoted $e_{11}'$ and
$e_{22}'$, in the $x_1$-$x_2$ plane are given by

$$e_{(a\alpha)'} = \frac{e_{11} + e_{22} \pm \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}}{2}$$  (5.5.1)

where $\alpha$ takes values 1 or 2 and is not summed. From this
the difference in secondary principal strains is calculated
to be

$$e_{11}' - e_{22}' = \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}.$$  (5.5.2)

We seek to show that

$$\eta_1^2 - \eta_2^2 = K(e)(e_{11}' - e_{22}')$$  (5.5.3)

The calculations are simplified by solving the basic set
of jump equations for $e_\alpha$ rather than $h_\alpha$. For the normal to the
wave $\mathbf{n} = (0,0,1)$ this result is

$$\left(\frac{G^2}{C_0} K_{ij} - \delta_{ij} + n_i n_j\right) e_j = 0,$$  (5.5.4)
subject to the constraint $e_1 K_{13} = 0$ which reduces (5.5.4) to two homogeneous equations in $e_\alpha$. The wave speeds are then determined, as in Section 5.3 from the vanishing of the determinant of the coefficients of $e_\alpha$:

$$
\begin{vmatrix}
\frac{G^2}{C_0} (K_{11} - \frac{K_{13}^2}{K_{33}}) - 1 & \frac{G^2}{C_0} (K_{12} - \frac{K_{13} - K_{23}}{K_{33}}) \\
\frac{G^2}{C_0} (K_{12} - \frac{K_{13} K_{23}}{K_{33}}) & \frac{G^2}{C_0} (K_{22} - \frac{K_{23}^2}{K_{33}}) - 1
\end{vmatrix} = 0 \quad (5.5.5)
$$

from which

$$
\frac{G^2}{C_0} = \frac{R + \sqrt{S}}{T} \quad (5.5.6)
$$

where

$$
R = K_{11} + K_{22} - \frac{K_{13}^2 + K_{23}^2}{K_{33}}
$$

$$
T = 2 (K_{11} K_{22} - K_{12}^2 - \frac{K_{11} K_{23}^2 + K_{22} K_{13}^2}{K_{33}} + 2 \frac{K_{12} K_{23} K_{31}}{K_{33}})
$$

$$
S = R^2 - 2T
$$

The Neumann form of the strain-optic law is thereby found to be

$$
\eta_1^2 - \eta_2^2 = \sqrt{S}, \quad (5.5.7)
$$
where $S$ must be written in terms of initial strains.

Assuming small strains, the electric susceptibility is found by inverting

$$X_{ij} = \frac{\varepsilon_0}{\rho}(a_0 \delta_{ij} + b_0 e_{0ij})$$

which is

$$X_{ij} = \frac{\rho a_0 b_0}{\varepsilon_0 \Delta} \begin{pmatrix}
\frac{a_0}{b_0} + e_{022} + e_{033} & -e_{012} & -e_{013} \\
-e_{012} & \frac{a_0}{b_0} + e_{011} + e_{033} & -e_{023} \\
-e_{013} & -e_{023} & \frac{a_0}{b_0} + e_{011} + e_{022}
\end{pmatrix}$$

where

$$\Delta = a_0^3 + a_0^2 b_0 I + a_0 b_0^2 II + b_0^3 III$$

which to the order of strains allowed here is approximately

$$\Delta \approx a_0^2 (a_0 + b_0 I_0)$$

Retaining only the terms which are linear in strains (5.5.7) can be written as

$$\eta_1^2 - \eta_2^2 = \frac{\rho a_0 b_0}{\varepsilon_0 \Delta} \sqrt{(e_{011} - e_{022})^2 + 4 e_{012}^2}.$$  

(5.5.10)

The neglected terms under the radical are

$$\left(\frac{e_{13}^2 + e_{23}^2}{e_{33}}\right)^2 < \left(\frac{\rho a_0 b_0}{\varepsilon_0 \Delta}\right)^4 \left(e_{013}^2 + e_{023}^2\right)^2 = O(e^4)$$
\[-8 \frac{K_{12}K_{23}K_{33}}{K_{33}} < -8 \left( \frac{\rho a_0 b_0}{\epsilon_0 \Delta} \right)^3 e_{012}e_{023}e_{031} = \mathcal{O}(e^3) \]

\[-2(K_{11}-K_{22}) \left( \frac{K_{23}^2-K_{13}^2}{K_{33}} \right) \]

\[< -2 \left( \frac{\rho a_0 b_0}{\epsilon_0 \Delta} \right)^3 \left[ (e_{022}-e_{011})(e_{023}^2-e_{013}^2) \right] = \mathcal{O}(e^3) \]

where we have used the fact that $K_{33} > 1$.

Thus, according to Toupin's theory, the use of secondary principal directions in a strain-optic law (hence stress-optic law also) is justified for weak waves.

5.6. Rotation of Principal Axes Due to Variations in Strain

Frocht [32] states the stress-optic law in three dimensions in the form of several propositions. He bases the validity of these propositions on experimental results. The proposition we shall analytically investigate here is given as follows: "The vibrations associated with a beam of light travelling through the stressed body are at each point parallel to the directions of the secondary principal stresses, for the given ray." Figure 5.1 shows the rotation of one of the two light vectors following the rotations of the corresponding secondary principal direction.
The wave normal \( \mathbf{n} \) is again taken to designate the \( x_3 \)-direction. The light vector \( \mathbf{H} \) has magnitude \( \sqrt{H_i H_i} \), and direction given by the unit vector \( \mathbf{M} \), which remains perpendicular to \( \mathbf{n} \), i.e.,

\[
\mathbf{n}_i = (0,0,1) \quad \text{and} \quad \mathbf{M}_i = (M_1,M_2,0)
\] (5.6.1)

The unit vector in the direction of secondary principal strains, denoted by \( \mathbf{N} \), is also perpendicular to \( \mathbf{n} \) but \( \mathbf{N} \) and \( \mathbf{M} \) are not otherwise related at the outset, i.e., \( \mathbf{N} \) and \( \mathbf{M} \) may lie in different directions and/or rotate with different angular velocities. We shall show here that they must coincide.

The directions of the secondary principal strains are found by resolving

\[
(e_{ij} - \delta_{ij} \epsilon_{\beta \beta}) N_j = 0 \quad \text{(no sum on} \beta)\] (5.6.2)

for \( N_j \), where the secondary principal strains are given by
(5.5.1). This calculation results in the ratios

\[
\begin{align*}
\frac{N_1}{N_2}^{(1)} &= \frac{-2e_{12}}{e_{11} - e_{22} - \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}} \\
&= \frac{e_{11} - e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}}{2e_{12}}
\end{align*}
\]

(5.6.3)

\[
\begin{align*}
\frac{N_1}{N_2}^{(2)} &= \frac{-2e_{12}}{e_{11} - e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}} \\
&= \frac{e_{11} - e_{22} - \sqrt{(e_{11} - e_{22})^2 + 4e_{12}^2}}{2e_{12}}
\end{align*}
\]

(5.6.4)

To find similar ratios for the two unit vectors \( \mathbf{M}^{(a)} \), which also point in the directions of \( \mathbf{n}^{(a)} \), we use (5.3.2) subject to the constraint (5.2.2), which requires that \( h^3 \) vanish for \( \mathbf{n} = (0,0,1) \). We find the following two independent equations,

\[
\frac{G_2}{C_0} + K_{11}^{(-1)} M_1 + K_{12}^{(-1)} M_2 = 0 
\]

(5.6.5)

and

\[
K_{12}^{(-1)} M_1 + \left( -\frac{G_2}{C_0} + K_{22}^{(-1)} \right) M_2 = 0
\]
from which the two wave speeds are

$$\frac{\alpha}{G^2} = -\left(\frac{1}{K_{11} - K_{22}} \right)^{-1} + \sqrt{\left(\frac{1}{K_{11} - K_{22}} \right)^{-1} + 4\left(\frac{1}{K_{12}} \right)^{-1}} \cdot (5.6.6)$$

Now, solving for the ratios of components of \( M \) in (5.6.5) gives

\[
\begin{align*}
\frac{(1)}{M_1} &= -\frac{2K^{-1}_{12}}{K_{11}^{-1} - K_{22}^{-1} + \sqrt{(K_{11}^{-1} - K_{22}^{-1})^2 + 4(K_{12}^{-1})^2}} \\
\frac{(1)}{M_2} &= \frac{K_{11}^{-1} - K_{22}^{-1} - \sqrt{(K_{11}^{-1} - K_{22}^{-1})^2 + 4(K_{12}^{-1})^2}}{2K_{12}^{-1}} \cdot (5.6.7)
\end{align*}
\]

and

\[
\begin{align*}
\frac{(2)}{M_1} &= -\frac{2K^{-1}_{12}}{K_{11}^{-1} - K_{22}^{-1} - \sqrt{(K_{11}^{-1} - K_{22}^{-1})^2 + 4(K_{12}^{-1})^2}} \\
\frac{(2)}{M_2} &= \frac{K_{11}^{-1} - K_{22}^{-1} + \sqrt{(K_{11}^{-1} - K_{22}^{-1})^2 + 4(K_{12}^{-1})^2}}{2K_{12}^{-1}} \cdot (5.6.8)
\end{align*}
\]

To compare these last two equations with (5.6.3-4) we need to express \( K_{ij}^{-1} \) in terms of the initial strain components. \( \chi_{ij} \) is given by (5.5.8) for small strains and since \( K_{ij} = \delta_{ij} \chi_{ij} \), we need to invert...
\[
K_{ij} = \frac{\rho a_0 b_0}{\varepsilon_0 \Delta} \left( \begin{array}{ccc}
\gamma + e_{022} + e_{033} & -e_{012} & -e_{013} \\
-e_{012} & \gamma + e_{011} + e_{033} & -e_{023} \\
-e_{013} & -e_{023} & \gamma + e_{011} + e_{022}
\end{array} \right),
\]

(5.6.9)

The result is, neglecting \( e^2 \) terms, \( K_{ij}^{-1} \) equals

\[
\frac{\varepsilon_0 \Delta}{\rho a_0 b_0 (\gamma^2 + \gamma 2 \Gamma_0)} \left( \begin{array}{ccc}
\gamma + 2e_{011} + e_{022} + e_{033} & e_{012} & e_{013} \\
e_{012} & \gamma + 2e_{011} + 2e_{022} + e_{033} & e_{023} \\
e_{013} & e_{023} & \gamma + e_{011} + e_{022} + 2e_{033}
\end{array} \right)
\]

(5.6.10)

Where \( \gamma = \varepsilon_0 \Delta / \rho a_0 b_0 + a_0 / b_0 \), and \( \Delta = a_0^3 + a_0 b_0 I_0 + a_0 b_0^2 + b_0^3 \).

When these values are used in (5.6.7, 8) we find that

\[
\frac{(1)}{M_1} = \frac{\varepsilon_0 - e_{022}}{2e_{012}} - \frac{\sqrt{(e_{011} - e_{022})^2 + 4e_{013}^2}}{2e_{012}} = \frac{(2)}{M_2}
\]

(5.6.11)

and

\[
\frac{(2)}{M_1} = \frac{\varepsilon_0 - e_{022} + \sqrt{(e_{011} - e_{022})^2 + 4e_{013}^2}}{2e_{012}} = \frac{(1)}{M_2}
\]

(5.6.11)
Since these ratios are the same for unit vectors parallel to the light vectors and the directions of secondary principal strains (hence stresses) the light vectors must coincide with secondary principal directions. If these directions rotate then so will the light vectors. No "small rotations" restriction is needed.
6. ANALYSIS OF SHOCK WAVES

6.1. Introduction

In the previous chapter we developed the classical photoelastic theory from a linearization of the basic field equations based on the hypothesis that the effect due to the electromagnetic variables dominates that of the elastic variables. In this chapter we examine further the relationship between these two sets of variables.

For weak waves, the postulated discontinuities in displacements were such that the equation of motion was identically satisfied (in the absence of any initial electromagnetic field); in fact, the second derivatives of displacements were taken as continuous across the wavefront. If all the wave speeds of light are to be independent of any accompanying elastic effects, this is a necessary assumption. It gave the 'correct' wave speeds.

However, this does not mean that there is no accompanying elastic deformation. Indeed, the derivative of the equation of motion shows that the third derivatives of elastic displacements must be discontinuous and these discontinuities are determined in terms of those in the first derivatives of the electromagnetic variables. Due to this decoupling at the first stage, the waves do not grow.

Such a wave structure is common to a number of problems
such as the transverse wave of isotropic elasticity, the switch-on, switch-off waves of magnetohydrodynamics and other analogs. This leads us to make the broad statement which follows: If the shock conditions for a system of equations relate all of the field variables linearly (for small jumps) then the weak wave relations are just analogs of these. If the basic equations are nonlinear, the weak wave study will show that such a wave grows. However, if the jumps from shock conditions are nonlinearly related, the study of the corresponding weak wave may show no growth. This would seem to indicate linearity of the problem, but such an assumption is erroneous. It is the shock conditions that exhibit these nonlinear relationships and thus reveal the nonlinearity of the basic problem.

Such a feature shows up in this study. We demonstrate that the passage of an electromagnetic wave generates elastic displacements, but with respect to the amplitude of electromagnetic variables these are of a smaller order of magnitude. Thus, while for 'infinitesimal' waves the accompanying elastic deformation may be negligible, for electromagnetic waves of a small but finite amplitude the elastic deformation is no longer negligible. Further, the theory reveals the correct procedure to produce the known laws of light propagation and the limits of these laws.
6.2. Electromagnetic Shocks

As we noted in Section 6.1, the derivative of the equation of motion requires that the jump in the third derivative of displacement is not continuous across the wavefront for weak waves. For shock waves we again find that the equation of motion requires relations between jumps in the elastic and electromagnetic field variables. Thus some discontinuity in the derivatives of displacements is necessary for shock waves. This means that purely electromagnetic shock waves in an elastic dielectric are not physically possible, and so type 5 waves (as defined in Chapter 2) are excluded from consideration as realizable waves.

We seek the nonlinear relationship between the elastic and electromagnetic variables that shock theory is capable of uncovering. Guided by the weak wave hypotheses used in the previous chapter and the relationship between shocks and weak waves, we begin our shock wave study with the following assumptions: displacement is taken as continuous across $\xi(t)$,

$$[u_1] = 0, \quad (6.2.1)$$

electromagnetic variables are initially zero, but are discontinuous across the wavefront,

$$E_0i = P_0i = H_0i = 0 \quad (6.2.2)$$

$$[E_i] = E_i, \quad [P_i] = P_i, \quad [H_i] = H_i.$$
We further limit our attention to the case where 
\( E = (E_1, 0, 0) \) with the normal \( n = (0, 0, 1) \). This 
corresponds to the mode of the weak wave that was studied 
in Chapter 5. From this assumption the shock relations from 
Maxwell's equations are

\[
-G[B_i] + e_{ijk} n_j [E_k] = 0 \quad (6.2.3)
\]

\[
n_i [B_i] = 0 \quad (6.2.4)
\]

\[
e_{ijk} n_j [H_k] + G[D_i] = 0 \quad (6.2.5)
\]

\[
n_i [D_i] = 0 \quad (6.2.6)
\]

and the definitions (3.3.2) are

\[
[H_i] = \frac{1}{\mu_0} [B_i] + e_{ijk} [v_j P_k] \quad (6.2.7)
\]

\[
[D_i] = \varepsilon_0 [E_i] + [P_i] \quad (6.2.8)
\]

These assumptions and equations yield the following vanishing 
electromagnetic variable components.

From (6.2.4) and (6.2.6) \( B_3 \) and \( D_3 \) vanish. The first 
of (6.2.3) requires that \( B_1 = 0 \), and (6.2.8) gives \( P_3 = 0 \) and 
that \( D_2 = P_2 \). Using this last relation in the second of (6.2.5) 
gives \( H_1 = -G P_2 \). We also have \( H_1 \) expressed by (6.2.7) as, 
\( H_1 = -v_3 P_2 \), since \( B_1 \) vanishes. Since the wave speed \( G \), 
differs from the \( z \)-component of the particle velocity \( v_3 \), 
the necessary conclusion is that \( P_2 \) must vanish. From this
H_1 vanishes and also D_2 from (6.2.8). In summary, we have:

\[ \begin{align*}
\mathbf{E} &= (E_1, 0, 0), \quad \mathbf{B} = (0, B_2, 0) \\
\mathbf{P} &= (P_1, 0, 0), \quad \mathbf{H} = (0, H_2, H_3) \\
\mathbf{D} &= (D_1, 0, 0), \quad \mathbf{V} = (v_1, v_2, v_3)
\end{align*} \]

and the relations,

\[ \begin{align*}
E_1 &= GB_2 \\
H_2 &= G(\varepsilon_0 E_1 + P_1) = \frac{1}{\mu_0} B_2 v_3 P_1 \\
D_1 &= \varepsilon_0 E_1 + P_1 \\
H_3 &= -v_2 P_1
\end{align*} \] (6.2.10)

Before considering the equation of motion we must relate the deformation to the other elastic variables. We denote jumps in the normal derivative of displacement by the letter \( \xi_i \),

\[ [u_{i,j}]n_j = [u_{i,3}] = \xi_i \] (6.2.11)

and take principal directions as the coordinate axes. Thus

\[ u_{0i,j} = \begin{pmatrix}
0_{1,1} & 0 & 0 \\
0 & 0_{2,2} & 0 \\
0 & 0 & 0_{3,3}
\end{pmatrix}. \] (6.2.12)

\[ u_{1,3} = \xi_1, \quad u_{2,3} = \xi_2, \quad u_{3,3} = u_{03,3} + \xi_3 \] (6.2.13)

Strains are defined in terms of their initial values and the \( \xi_i \):
\[
\begin{pmatrix}
    e_{011} & 0 & \frac{1}{2}(1-u_{01,1})\xi_1 \\
    0 & e_{022} & \frac{1}{2}(1-u_{02,2})\xi_2 \\
    \frac{1}{2}(1-u_{01,1})\xi_1 & \frac{1}{2}(1-u_{02,2})\xi_2 & e_{033}+(1-u_{03,3})\xi_3 - \frac{1}{2}\xi^2
\end{pmatrix}
\]

where \( \xi^2 = \xi_i \xi_i \),

and the velocity components arising from,

\[
[v_i] = v_i = \left[ \frac{3u_i}{\bar{v}} + u_{i,j}v_j \right]
\]

are expressed, as in [10], by

\[
v_i = \frac{-G\xi_i}{1-\bar{\xi}_3}
\]

where

\[
\bar{\xi}_i = \frac{\xi_i}{1-u_{0i,i}} \quad \text{(no sum on } i) .
\]

Finally, the shock condition based on the conservation of mass is noted as,

\[
\rho (v_3-G) = -\rho_0 G .
\]

We now consider the equation of motion. Shock conditions arising from the conservative form (3.2.21) are

\[
-G[\rho v_i + \epsilon_0 e_{ijk}E_j B_k] = [t_{i3} + m_{i3} - \rho v_i v_3],
\]

or

\[
[\rho (v_3-G)v_i] - G\epsilon_0 e_{ijk}[E_j B_k] = [t_{i3}] + [m_{i3}] .
\]
Making use of (6.2.9-10),

\[(\mathbf{E} \times \mathbf{B}) = (0, 0, E_1 B_2) = (0, 0, GB_2^2) \quad (6.2.21)\]

and from (3.2.19), the electromagnetic stress tensor, for our purposes here, reduces to:

\[\pi_{i3} = (0, 0, -\frac{\varepsilon_0 E_1^2}{2} - \frac{B_2^2}{2\mu_0}) = -\frac{B^2}{2\mu_0} (0, 0, \frac{G^2}{c_0^2} + 1) \quad (6.2.22)\]

So we are able to write (6.2.20) using (6.2.16,18,21,22) as

\[-\rho_0 [\mathcal{E}_{i1}] - G^2 \varepsilon_0 (0, 0, B_2^2) = [t_{i3}] - \frac{B^2}{2\mu_0} (0, 0, \frac{G^2}{c_0^2} + 1)\]

or

\[\frac{\rho_0 G^2 \xi_1}{1-\xi_3} = [t_{i3}] + \frac{B^2}{2\mu_0} (0, 0, \frac{G^2}{c_0^2} - 1) \quad (6.2.23)\]

By considering the first of these equations and expanding \([t_{13}]\) via (3.2.14) to:

\[[t_{13}] = t_{13} = \{B+C(e_{11}+e_{33})+(F+G e_{11})\pi_1 \, 2\} e_{13}, \quad (6.2.24)\]

since \(t_{013}\) is zero, and using (6.2.14) to express \(e_{13}\), we have

\[\rho_0 \frac{G^2 \xi_1}{1-\xi_3} = \{B+C(e_{11}+e_{33})+(F+G e_{11})\pi_1 \, 2\} \frac{1}{2} (1-u_{01},1) \xi_1 \quad (6.2.25)\]

which is satisfied if \(\xi_1=0\), or if

\[\frac{2\rho_0 G^2}{1-\xi_3} - (1-u_{01},1)^2 \{B+C(e_{11}+e_{33})+(F+G e_{11})\pi_1 \, 2\} = 0, \quad (6.2.26)\]
which in the limit, as strains and polarization \( \to 0 \), requires that \( G \) be the speed of the transverse (principal) elastic wave. This option to satisfy (6.2.25) is thus rejected since in the limit the wave speed \( G \) must reduce to the weak wave speed of Chapter 5.

An entirely similar argument involving the second equation of motion results in requiring \( \xi_2 \) to vanish. Thus we add to the restrictions (6.2.9) this result,

\[
\xi_i = (0,0,\xi_3),
\]

which in turn requires \( H_3 \) to vanish, as is obvious from (6.2.10).

The density relation (6.2.18), can also be expressed in terms of \( \bar{\xi}_3 \) or \( \xi_3 \) by

\[
\rho = \rho_0 \left( 1 - \frac{\xi_3}{1 - u_{03,3}} \right) = \rho_0 \left( 1 - \bar{\xi}_3 \right).
\]

(6.2.28)

Before proceeding to the third equation of motion we evaluate the jump in the normal stress component in the direction of the advancing wave:

\[
t_{33} = \Lambda + Be_{33} + Ce_{33} = \frac{\rho}{\rho_0} \left( 1 - 2e_{33} \right) \{U, \bar{U}, (I-e_{33}) + U, \bar{U}, e_{22} \}
\]

\[
= \frac{\rho_0}{\rho_0} \left( 1 - \bar{\xi}_3 \right)^3 \left( 1 - u_{03,3} \right)^2 \{U, \bar{U}, (e_{011} + e_{022}) + U, \bar{U}, e_{011} e_{022} \}
\]

(6.2.29)

Similarly the initial stress is given by
$$t_{033} = \rho_0^0 (1-u_{03,3})^2 \{ U_{0,1} + U_{0,2} (e_{011} + e_{022}) + U_{0,3} e_{011} e_{022} \}$$  

(6.2.30)

Now, from (6.2.23) we write the third equation of motion

$$\frac{\rho_0 G^2 \xi_3}{1-\xi_3} = [t_{33}] + \frac{B_2}{2 \mu_0} \left( \frac{G^2}{c_0^2} - 1 \right),$$  

(6.2.31)

which by (6.2.10) is written in terms of $\xi_3$ and $\pi$ as

$$\rho_0 G^2 \xi_3 = (1-\xi_3) [t_{33}] + \frac{\rho^2 \mu_0 G^2}{2 (1-\xi_3) (\frac{G^2}{c_0^2} - 1)} \pi_1^2,$$  

(6.2.32)

since $[t_{33}]$ is given in these variables. Thus (6.2.32) is a relation between powers of $\xi_3$ and $\pi_1^2$. This may be expressed by

$$f(\pi_1^2, \pi_1^4, \ldots, \xi_3, \xi_3^2, \ldots) = 0$$  

(6.2.33)

Assuming further that $\xi_3 < \pi_1$ and that $\xi_3$ vanishes with $\pi_1^2$ (which follows if $G$ is not to reduce to the elastic wave speed) we can write

$$\xi_3 = K_0 \pi_1^2 + K_1 \pi_1^4 + \ldots,$$  

(6.2.34)

which is truncated for the present purposes to

$$\xi_3 = K_0 \pi_1^2,$$  

(6.2.35)

since for definiteness we retain terms up to order $\pi^2$ only in the future study.
We examine this relation by considering the vanishing limit of $\xi_3$ and $\pi_1^2$ in the motion equation. First we re-write $[t_{33}]$ in the form of the Taylor series expansion

$$t_{33} - t_{033} = R_0 \xi_3 + S_0 \pi_1^2 + ... \quad (6.2.36)$$

where

$$R_0 = \left. \frac{\partial t_{33}}{\partial \xi_1} \right|_0, \quad S_0 = \left. \frac{\partial t_{33}}{\partial \pi_1^2} \right|_0$$

Now, from (6.2.29, 30) by neglecting higher than first powers of $\xi_3$ and $\pi_1^2$ we have

$$t_{33} - t_{033} = \frac{\rho_0}{\rho_{00}} (1 - 2e_{033}) \{ M_{01} + (e_{011} + e_{022}) M_{02} + e_{011} e_{022} M_{03}$$

$$- 3U_1 e_{011} e_{022} U_2 (3e_{011} e_{022} + 4e_{033}) U_3 \} \xi_3$$

$$+ \frac{\rho_0}{\rho_{00}} (1 - 2e_{033}) \{ Q_{01} + (e_{011} + e_{022}) Q_{02} + e_{011} e_{022} Q_{03} \} \pi_1^2$$

where $M_{0i}$ and $Q_{0i}$ are found from Equation (12.1.7) in the Appendix where $U_{1} - U_0, i = M_{0i} \xi_3 + Q_{0i} \pi_1^2$. By comparison of these last two equations $R_0$ and $S_0$ may be found. Since $[t_{33}]$ is approximated by these first two terms, to this order it is also logical to write $\rho$ as $\rho_0$ and $G$ as $G_0$ (which will indicate the weak wave speed). The equation of motion (6.2.32), dropping squares and products of $\xi_3$ and $\pi_1^2$, is written as

$$\rho_0 G_0^2 \xi_3 = R_0 \xi_3 + S_0 \pi_1^2 + \frac{\rho_0^2 G_0^2 \mu_0 \epsilon_0}{2 \epsilon_0 (G_0^2 \mu_0 \epsilon_0 - 1)} \pi_1^2$$

(6.2.38)
It is of interest to note here that for the purely elastic case, where the terms with \( \pi_1^2 \) drop out, (6.2.38) gives for small but nonvanishing \( \xi_3 \)

\[
\rho_0 G_0^2 = R_0. \tag{6.2.39}
\]

Thus \( \frac{R_0}{\rho_0} \) is the wavespeed of the elastic principal longitudinal weak wave. As will be seen later, \( R_0 \) reduces to \( \lambda + 2\mu \) for the initially unstrained case.

To reduce (6.2.38) further we now turn to the two expressions for \( H_2 \) given in (6.2.10). We have

\[
(G - v_3)P_1 = \left( \frac{1}{\mu_0} - \varepsilon_0 G^2 \right) B_2 \tag{6.2.40}
\]

The constitutive equation for electromotive intensity also gives a relation between \( P_1 \) and \( B_2 \). By utilizing the assumptions of this sections we write it as,

\[
\mathcal{E}_1 = (G - v_3)B_2 = (a + b \epsilon_{11} + c \epsilon_{11}^2) \frac{P_1}{\rho_0}. \tag{6.2.41}
\]

By eliminating \( B_2 \) and requiring \( P_1 \) to be nonzero, these equations require that

\[
\rho_0 \mu_0 (G - v_3)^2 = (1 - G^3 \varepsilon_0 \mu_0) (a + b \epsilon_{11} + c \epsilon_{11}^2). \tag{6.2.42}
\]

We wish to linearize \( \xi \) in \( \xi_3 \) and \( \pi_1^2 \) and use the result here. From

\[
\mathcal{E} = (\varepsilon_0 + L_0 \xi_3 + M_0 \pi_1^2 + \ldots) \pi_1, \tag{6.2.43}
\]
and the expression for $v_3$ in terms of $\xi_3$ (6.2.16), Equation (6.2.42) shows that

$$G^2\varepsilon_0\mu_0 = \frac{\varepsilon_0\xi_0}{\rho_0 + \varepsilon_0\xi_0} + \mathcal{O}(\xi_3, \pi_1^2)$$

or, for vanishing $\xi_3$ and $\pi_1^2$ we have

$$G^2\varepsilon_0\mu_0 = \frac{\varepsilon_0\xi_0}{\rho_0 + \varepsilon_0\xi_0}. \quad (6.2.44)$$

and thus to the same order

$$G^2\varepsilon_0\mu_0 - 1 = \frac{-\rho_0}{\rho_0 + \varepsilon_0\xi_0}. \quad (6.2.45)$$

Using these last two in (6.2.38) gives

$$(\rho_0 G_0^2 - R_0)\xi_3 = \{S_0 + \frac{\rho_0^2\varepsilon_0\xi_0 (\rho_0 + \varepsilon_0\xi_0)^{-1}}{-2\varepsilon_0\rho_0 (\rho_0 + \varepsilon_0\xi_0)^{-1}}\} \pi_1^2$$

or

$$\xi_3 = \{\frac{2S_0 - \rho_0\xi_0}{2(\rho_0 G_0^2 - R_0)}\} \pi_1^2 \quad (6.2.46)$$

We denote the proportionality constant $K_0$ by the bracketed term.

$$K_0 = \frac{2S_0 - \rho_0\xi_0}{2(\rho_0 G_0^2 - R_0)} \quad (6.2.47)$$

Further reduction via the internal energy function given in the Appendix, allows us to write for small initial strains
\[ E_0 = \frac{2}{\rho_0} (\alpha_6 + \beta_{16} I_0 + \gamma_4 e_{011}) \]

\[ R_0 = R_{01} + 2m (e_{011} + e_{022}) - 2(\lambda + 2\mu) (I_0 + e_{033}) \]
\[ + 6\mu (e_{011} + e_{022}) \]

\[ S_0 = \beta_{16} + \gamma_1 I_0 + \beta_{14} e_{011} + \beta_{26} (e_{011} + e_{022}) - \beta_{16} (I_0 + e_{033}) \]

and hence for the initially undeformed state

\[ K_0 = \frac{\beta_{16} - \alpha_6}{\rho_0 G_0^2 - (\lambda + 2\mu)} \]  \hspace{1cm} (6.2.48)

Since \( \frac{\lambda + 2\mu}{2} \) is the square of the ratio of the elastic velocity to the velocity of light, it is negligible. Thus

\[ K_0 = \frac{\beta_{16} - \alpha_6}{\rho_0 G_0^2} . \]

Also, since \( E_0 \) is positive, \( \alpha_6 \) must be positive. Hence, if \( \beta_{16} < \alpha_6 \), \( K_0 \) is negative and so the material is compressed. Conversely, if \( \beta_{16} > \alpha_6 \) then the material expands.

Returning briefly to the wave speed in the form given by (6.2.42), we express \( v_3 \) in terms of \( \varphi_3 \) and solve for \( G^2 \) to find:

\[ \frac{G^2}{c_0^2} = \frac{\rho_0 (1 - \varphi_3) (a + be_{011} + ce_{011})}{\rho_0 + \rho_0 (1 - \varphi_3) (a + be_{011} + ce_{011})^2} \]  \hspace{1cm} (6.2.49)

It is clear that this reduces to the weak wave speed (from 5.4.2,4) where \( \pi \), and hence \( \varphi_3 \), vanishes.
6.3. Summary of Shock Wave Results

The main goal of this chapter has been to uncover the nature of the interaction between elastic and electromagnetic variables in photoelasticity. By studying the wave mode characterized by \( \underline{E} = (E_1, 0, 0) \) and \( \underline{n} = (0, 0, 1) \) we parallel the discussion of Chapter 5 and reduced the number of variables to a manageable size. In this reduction we found that no tangential velocity component can exist for the motion of the material. The third equation of motion led us to a relation between the elastic and electromagnetic variables, \( f(\pi_1^2, \pi_1^4, ..., \xi_3, \xi_3^2, ...) = 0 \), from which we examined the basic part, 

\[
\xi_3 = K_0 \pi_1^2 + ...
\]

of a more general expansion of \( \xi_3 \). We also calculated the speed of the shock wave and saw that it reduced to the weak wave speed as the amplitude goes to zero.

The concluding work centered around an examination of \( K_0 \). It was expressed in terms of the coefficients of an expansion of \( U \), the internal energy. In the preceding chapter we obtained a relation between these coefficients via the strain-optic coefficient. In the following work we shall obtain further such relationships.
7. AMPLITUDE EXPANSIONS

7.1. Introduction

Our study of waves thus far has been based on the theory of singular surfaces. The major mathematical limitation of such a study is that the description obtained is limited to the wavefront itself. In addition, it is not applicable when dissipation and/or dispersion are present. However, it does bring out certain properties of waves; and it is closely related to, and inclusive of, the ideas of characteristic theory.

We believe that it has another value which does not seem to have been well appreciated in the literature. It is only a study based on singular surface theory that points the way to the correct asymptotic expansions in terms of the amplitude of the disturbance.

The well accepted theory of classical photoelasticity is indeed a linear theory. It is on the basis of these linear equations that all known studies of photoelastic waves are made. In the case of such equations one can assume expansions in the form of sinusoidal variations (viz., the assumption that each field variable changes as $A \exp\{i(kz\omega t)\}$). From this, all the properties of photoelasticity e.g. interference etc. follow.

Our objective here is to produce an amplitude expansion which clearly brings out the nature of the linear equations and
throws light on what it is that is neglected in the derivation of linear theory. It is of course, unnecessary to repeat that an assumption of such sinusoidal dependence cannot be the starting point of the study of the basic equations, the obvious reason being their high nonlinearity.

This study will clearly show the close analogy between the known linearized photoelastic waves, here called 'infinitesimal' amplitude waves, and weak waves. Further, it will bring out the similarities between the study of shock waves and waves of 'small-but-finite' amplitude. The final section contains a solution of the small-but-finite amplitude wave equation, also developed herein, which brings out the amplitude dependence of frequency. Waves of infinitesimal amplitude do not have such a property.

7.2. Infinitesimal Waves

In order to produce the equations used in classical photoelasticity [13, 15], we first note that these equations consist of pure electromagnetic field variables with the dielectric tensor dependent on the initial stress or strain; these are essentially Maxwell's equations which are linear. Guided by this we assume that the actual electromagnetic variables (denoted here by $E'$, $B'$, etc.) are given by

$$E' = \varepsilon E, \quad B' = \varepsilon B \text{ etc.}$$  (7.2.1)
However, the displacement vector (in terms of which all elastic and mechanical variables are obtained) is given by

\[ u' = \left( u_0 + \lambda, u_1 v_0 + \lambda_2 v_0 + \dot{\lambda}_3 w \right) \]  

(7.2.2)

with

\[ \lambda_i = \lambda_i(\varepsilon), \quad \frac{\lambda_i(\varepsilon)}{\varepsilon} \to 0 \quad \text{as} \quad \varepsilon \to 0 \]

Such an assumption leads to

\[ \varepsilon' = \varepsilon_0 + \mathcal{O}(\text{max } \lambda_i) \]  

(7.2.3)

where \( \text{max } \lambda_i \) is the maximum of \( \lambda_1, \lambda_2, \lambda_3 \) and \( \ldots \) denotes terms which are of lower order and hence negligible compared to those retained.

Now, if all terms of higher order infinitesimals are formally neglected and set to zero, one obtains

\[ \frac{\partial B_i}{\partial t} + e_{ijk} E_{k,j} \quad B_i,i = 0 \]  

(7.2.4)

\[ e_{ijk} H_{x,j} = \frac{\partial D_i}{\partial t} \quad D_i,i = 0 \]

and

\[ \varepsilon_i = \varepsilon E_i + \mathcal{O}(\varepsilon) \]

\[ E_i = \frac{1}{\varepsilon_0} \chi_{0ij} P_j \]  

(7.2.5)

where

\[ \chi_{ij}^{-1} = \frac{1}{\rho} \left( a \delta_{ij} + b e_{ij} + c e_{ik} e_{kj} \right) \]  

(7.2.6)
This system which forms the basis of most photoelastic studies [33], [13], is also noted by Toupin [19]. However, in Mindlin and Goodman's paper [13] a further assumption is made based on infinitesimal initial strains which reduces $\chi_{0ij}$ to a form linear in these initial strains.

It has to be stressed here that all the terms in the equation of motion are necessarily higher order infinitesimals. Consequently this equation is not considered. All studies are based on Maxwell's laws with a modified dielectric tensor. As our investigation will show, this extremely ad hoc procedure fails to bring out what it is that is really being neglected.

Physically speaking, one of the basic principles used in the formulation of the constitutive laws is the principle of equipresence: "A variable present as an independent variable in one constitutive equation should be so present in all." [27]. Fundamental to photoelasticity is the optical anisotropy of an elastically isotropic material purely as a result of initial strain. This implies that an elastic field does effect the optical properties of a material; and hence, the principle of equipresence must necessarily imply the existence of an elastic field due to optical phenomena. This indeed is known as electrostriction (or its inverse, piezoelectricity where, however, both elastic and optic anisotropy are assumed to start with).
A closer examination of the amplitude expansion brings out this feature: The passage of an electromagnetic wave through an elastic dielectric must produce longitudinal deformation. This longitudinal strain accompanying the electromagnetic wave is proportional to the square of the electromagnetic field variable when its intensity is small but finite.

7.3. Small-but-finite Amplitude Waves

It is rather lengthy and unnecessary to produce here in complete generality the study of an arbitrary wave. Guided by the weak wave studies we find it adequate to focus our attention on a single mode of propagation. We shall see that for such a mode, the study of the shock wave points out the correct form of the amplitude expansion.

Analogous to the study of shock waves in Chapter 6, we consider the passage of a plane electromagnetic wave with its normal in the z-direction, a principal direction. We have,

\[ E = (E(z,t),0,0), \quad B = (0,B(z,t),0), \]
\[ D = (D(z,t),0,0), \quad H = (0,H(z,t),0), \]
\[ P = (P(z,t),0,0), \quad \xi = (\xi(z,t),0,0), \]  
\[ \rho = (\rho(z,t),0,0), \quad \sigma = (\sigma(z,t),0,0), \]  

all of these depending on z and t only. Further, we take

\[ \text{as only one component of each electromagnetic vector survives we shall suppress the subscripts.} \]
\( u = (u_0(x), v_0(y), w_0(z) + w(z,t)) \) \hspace{1cm} (7.3.2)

with \( u_0, x \), \( v_0, y \), and \( w_0, z \) constant, and velocity

\( v = (0, 0, v_3) \)

where

\[ v_3 = \frac{-w_{t_z}}{1 - w_3} \quad \text{and} \quad \bar{w} = \frac{w}{1 - w_0, z} . \] \hspace{1cm} (7.3.3)

\( \bar{w} \) is defined, analogous to \( \bar{\xi}_3 \) in Chapter 6, to facilitate comparisons between the results of the two chapters. From these, the strain and stress tensors are given by

\[
e_{ij} = \begin{pmatrix}
e_{11} & 0 & 0 \\
0 & e_{22} & 0 \\
0 & 0 & e_{33}
\end{pmatrix}
\] \hspace{1cm} (7.3.4)

where

\[ e_{11} = u_0, x - \frac{1}{2} u_0, x^2 , \quad e_{22} = u_0, y - \frac{1}{2} u_0, y^2 \]

\[ e_{33} = e_{033} + (1 - w_0, z)^2 (\bar{w}, z - \frac{1}{2} \bar{w}, z^2) \]

and

\[
t_{ij} = \begin{pmatrix}
t_{11} & 0 & 0 \\
0 & t_{22} & 0 \\
0 & 0 & t_{33}
\end{pmatrix}
\] \hspace{1cm} (7.3.5)

where

\[ t_{33} = A + Be_{33} + Ce_{33}^2 . \]
Also the Maxwell stress tensor is given by
\[
\mathbf{m}_{ij} = \begin{pmatrix}
\mathbf{m}_{11} & 0 & 0 \\
0 & \mathbf{m}_{22} & 0 \\
0 & 0 & \mathbf{m}_{33}
\end{pmatrix}
\]
(7.3.6)

where
\[
\mathbf{m}_{33} = -\frac{1}{2}(\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu} \mathbf{B}^2)
\]

With these assumptions the governing equations are

\[
\frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{E}}{\partial z} = 0
\]
(7.3.7)

\[
\frac{\partial (\varepsilon_0 \mathbf{E} + P)}{\partial t} + \frac{\partial}{\partial z} \left( \frac{1}{\mu_0} \mathbf{B} + \mathbf{P} \right) = 0
\]
(7.3.8)

\[
\varepsilon = \mathbf{E} - \mathbf{v} = \frac{\rho_0}{\varepsilon_0 \chi_{11}}
\]
(7.3.9)

where
\[
\frac{\rho_0}{\varepsilon_0 \chi_{11}} = (a + b \varepsilon_0 \chi_{11} + c \varepsilon_0 \chi_{11} ^2)
\]

Finally, the conservation of mass and linear momentum are written:

\[
\rho = \rho_0 (1 - \mathbf{w}, z)
\]
(7.3.10)

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial z} \mathbf{v} \right) = \frac{\partial \mathbf{m}_{33}}{\partial z} + \frac{\partial \mathbf{m}_{33}}{\partial t} - \frac{3}{\sigma_0} (\varepsilon_0 \mathbf{EB})
\]
(7.3.11)

This is the full system of equations governing this particular mode, and will form the starting point of the
perturbation scheme.

The energy equation is not to be considered as an additional equation here. This is because the entropy change across a weak shock varies as the fourth power of polarization density (which is shown in the following chapter); and it is always possible to write the energy equation in terms of entropy. Thus while the elastic strain is not negligible, entropy change continues to be negligible to the order to which attention is limited here. Thus the wave is isentropic to this order.

7.4. Asymptotic Expansions of Small-but-finite Amplitude Waves

Singular surface theory allowed us to study the linear photoelastic wave. Now, photoelastic waves of small-but-finite amplitude may be viewed as perturbations about these linear waves. The expansion procedure, herein applied [34–37], retains the nonlinearity of such waves and reduces to the linear case as the amplitude becomes infinitesimal. Thus the waves discussed in this section are only weakly nonlinear.

All of the electromagnetic variables are expanded by using a smallness parameter ε, which characterizes the magnitude of their amplitudes. Since the shock wave study revealed that, for a small discontinuity in one of the electromagnetic field variables, their jumps are linearly related; we assume
here that
\[ E = \varepsilon E + \varepsilon^3 E + \ldots \]
\[ B = \varepsilon B + \varepsilon^3 B + \ldots \]  
\[ \pi = \varepsilon \pi + \varepsilon^3 \pi + \ldots \]  
(7.4.1)

The largest term in each of these expansions starts with \( \varepsilon \); this indeed is a similarity hypothesis made on the basis of the above noted study of shock waves. Terms beginning with \( \varepsilon^2 \) are deleted since it was seen that \( E \) is governed by the same equation as \( E \), and that this equation is uncoupled with that of \( E \). \( E \) is the first term that is coupled with \( E \); so without loss of generality we set \( E = 0 \).

It is necessary at this point to clarify the physical basis of the subsequent asymptotic expansions. Even though the initial waveform is of small amplitude (governed by a linear system), the basic nonlinearity of the full governing system must subsequently modify it. So we seek a far-field description, viewed from the vicinity of the wavefront. This is achieved by introducing a stretched time variable \( T \), and a wavefront coordinate \( Z \), by means of the transformations
\[ T = \varepsilon^2 t \quad \text{and} \quad Z = z - G_0 t, \]  
(7.4.2)

where \( G_0 \) is the normal wavespeed predicted by linear theory. In this system the derivatives transform as
The choice of $s^2$ for the stretching of time is again based on the physical assumptions made which imply that non-linearity must play a significant role for large time values. As Cole [38] notes, the choice is made so as to obtain the most physically meaningful result. This will be brought out clearly later.

We note that now all of the field variables are regarded as functions of $Z$ and $T$, which are the far-field, wavefront variables.

Guided by the study of shock waves, where $\xi_3$ was proportional to $\pi^2$, we expect here that the longitudinal strain has an amplitude depending on the square of the amplitude of any electromagnetic field variable (since the latter are related amongst themselves linearly for small amplitudes). Thus we take the expansion for the perturbed displacement to be

$$w = \varepsilon^2 w(Z, T) + \ldots,$$  \hspace{1cm} (7.4.4)

which requires that the velocity be

$$v = -\varepsilon^2 g_0 \frac{2}{3} w + \ldots, \quad \text{where } \overline{w} = \frac{w}{1 - w_0, Z},$$  \hspace{1cm} (7.4.5)

Utilizing the definition of polarization density and the continuity relation (7.3.10).
the expansions provide the following relations between the
coefficients of $\varepsilon$ and of $\varepsilon^3$:

\begin{align*}
P &= \rho_0 \pi, \\
P &= \rho_0 (\pi - \pi \bar{w}_r Z) \tag{7.4.6}
\end{align*}

We also expand the electromotive intensity $\xi$ guided by
(6.2.43) it can be written as

$$\xi = (\xi_0 + L_0 \bar{w}_r Z + M_0 \pi^2 + \ldots) \pi,$$

which to terms cubic in $\varepsilon$ is

$$\xi = \varepsilon \frac{1}{\pi} \xi_0 + \varepsilon^3 \left[ \frac{3}{\pi} \xi_0 + \frac{1}{\pi} \left( L_0 \frac{2}{\bar{w}_r Z} + M_0 \frac{1}{\pi^2} \right) \right] \tag{7.4.7}$$

using these expansions in the basic equations (7.3.7-9) results in:

$$\varepsilon \frac{3}{\pi} \left(-G_0 E + E^2\right) + \varepsilon^3 \left[ \frac{3}{\pi} \left(-G_0 E + E^2\right) + \frac{1}{\pi \frac{3}{\bar{w}_r Z}} \right] + \ldots = 0 \tag{7.4.8}$$

$$\varepsilon \left( \frac{3}{\pi} \mu_0 - G_0 (\xi_0 E + \rho_0 \pi^2) + \varepsilon^3 \left[ \frac{3}{\pi \frac{3}{\bar{w}_r Z}} - G_0 (\xi_0 E + \rho_0 \pi^2) \right] \right)$$
\begin{align*}
+ \frac{3}{\pi} \left( \xi_0 E + \rho_0 \pi^2 \right) + \ldots &= 0 \tag{7.4.9}
\end{align*}

$$\varepsilon (E - E_0 \pi) + \varepsilon^3 (E - E_0 \pi) + \frac{3}{\pi \frac{2}{\bar{w}_r Z}} (G_0 E - B_0 \pi) - M_0 \pi^3 \lambda + \ldots = 0 \tag{7.4.10}$$

To explain the choice of $\varepsilon^2$ in (7.4.2) consider the time
$T$ related to $t$ by some function of $\varepsilon$,

$$T = \alpha(\varepsilon) t.$$
The nonlinear terms in the electromagnetic variables must balance, for large times, with the remaining terms of order $\alpha(\varepsilon)$. This requires that $\alpha(\varepsilon)$ equal $\varepsilon^2$. For example, the second group of terms in (7.4.8) appears as
\[ \varepsilon^3 \frac{3}{2Z} \left( -G_0 B + E \right) + \varepsilon \alpha(\varepsilon) \frac{3B}{\beta T} . \]

Let us choose $\alpha = \varepsilon^n$. With $n > 1$ but less than 2 the second term in the expression is larger than the first since $\alpha \varepsilon > \varepsilon$. This leads to another equation in $B$ which is absurd. If $n > 2$, the first term dominates, and this leads to the same equation for $B$ and $E$ as that for $B$ and $E$ in (7.4.8). So if the smaller terms are to be a correction to $E$ and $B'$ the only choice is $n = 2$.

Finally we expand the equation of motion (7.3.4). The largest terms in powers of $\varepsilon$ are related by
\[ \varepsilon^2 \rho_0 G_0 \frac{2}{Z_2} \frac{\partial^2 W}{\partial Z^2} = \varepsilon^2 \frac{3}{2Z} \left( \frac{2}{Z_2} R_0 \frac{\partial W}{\partial Z} + S_0 \frac{\varepsilon}{2} - \frac{E^2}{2} - \frac{B^2}{2\mu_0} + G_0 E B \right) \]
(7.4.11)

This equation can be integrated with respect to $Z$. The resultant arbitrary function of time must vanish since all field variables are zero for $T + \infty$, i.e., as $Z + \infty$. The result is
\[ \rho_0 G_0 \frac{2}{Z_2} \frac{W}{Z} = R_0 \frac{2}{Z_2} W_Z + S_0 \frac{\varepsilon}{2} - \frac{E^2}{2} - \frac{B^2}{2\mu_0} + G_0 E B \]
(7.4.12)

To express $E$ and $B$ in terms of $\pi$ we examine the largest terms in Equations (7.4.8-10). We find
\[ -G_0 B + E = 0 \quad (7.4.13) \]
\[ -G_0 (\varepsilon_0 E + \rho_0 \pi) + \frac{1}{\mu_0} B = 0 \quad (7.4.14) \]
\[ E - \varepsilon_0 \pi = 0 \quad (7.4.15) \]

where the first two have been integrated with respect to \( Z \) as above. The desired relations are

\[ E = \varepsilon_0 \pi \quad (7.4.16) \]
\[ B = \frac{\varepsilon_0}{G_0} \pi \quad (7.4.17) \]

Using these, the equation of motion (7.4.12) can be written in the form

\[ (\rho_0 G_0^2 - R_0) \frac{2}{w, Z} = \varepsilon_0 \frac{2 S_0}{\varepsilon_0^2} + \frac{\varepsilon_0 \mu_0 G_0^2 - 1}{2 \mu_0 G_0^2} \frac{1}{2} \]

or

\[ \frac{2}{w, Z} = \frac{2 \mu_0 G_0^2 S_0 + \varepsilon_0^2 (\varepsilon_0 \mu_0 G_0^2 - 1)}{2 \mu_0 G_0^2 (\rho_0 G_0^2 - R_0)} \frac{1}{2} \quad (7.4.18) \]

To further reduce this we first determine the wave speed by requiring the vanishing of the determinant of the homogeneous Equations (7.4.13-15) from which one finds that

\[ G_0^2 = \frac{\varepsilon_0}{\mu_0 (\varepsilon_0 \varepsilon_0 + \rho_0)} \quad (7.4.19) \]

which can be rewritten as
\[
\frac{\varepsilon_0^2(\varepsilon_0 \mu_0 G_0^2 - 1)}{\mu_0 G_0^2} = -\rho_0 \varepsilon_0
\]  \hspace{1cm} (7.4.20)

We note in passing that the index of refraction is given from

\[
\eta_0^2 = \frac{1}{G_0^2 \varepsilon_0 \mu_0} = 1 + \frac{\rho_0}{\varepsilon_0 \varepsilon_0}
\]  \hspace{1cm} (7.4.21)

which (since \(\eta_0^2 > 1\), \(\rho_0 > 0\) and \(\varepsilon_0 > 0\)) requires that \(\varepsilon_0\) be positive.

Using (7.4.20) we now rewrite (7.4.18) in the more compact form

\[
\frac{2}{w_z} = \frac{2 S_0 - \rho_0 \varepsilon_0}{2(\rho_0 G_0^2 - R_0)}  \hspace{1cm} (7.4.22)
\]

which reduces to the equivalent shock relation (6.2.46), for small amplitude. The bracketed term, denoted by \(K_0\) (see 6.2.47), is the proportionality constant relating the elastic and electromagnetic variables.

Returning now to Equations (7.4.8-10) we examine the terms of order \(\varepsilon^3\),

\[
\frac{\partial}{\partial z} \left[ (G_0 B - E) \right] = \beta_1 \hspace{1cm} (7.4.23)
\]

\[
\frac{\partial}{\partial z} \left\{ B - \nu_0 G_0 (\varepsilon_0 E + \rho_0 \pi) \right\} = \beta_1 \hspace{1cm} (7.4.24)
\]

\[
E - \varepsilon_0 \pi = n_1 \hspace{1cm} (7.4.25)
\]
where

\[ l_1 = \frac{\partial B}{\partial T} \]

\[ m_1 = -\frac{3}{3T}(\varepsilon_0^E \rho_0 + \frac{1}{\varepsilon_0^E}) \mu_0 \]

\[ n_1 = (L_0 + G_0 B) \varepsilon_0^E + \mu_0^3 \]

From (7.4.23) and (7.4.25) we have

\[ \frac{3}{3T} \frac{\partial B}{\partial Z} = \frac{1}{G_0} (\frac{3E}{3Z} + l_1) \]

and

\[ \pi = \frac{1}{E_0} (E - n_1), \]

which are used in (7.4.24) to obtain an equation in \( E \).

\[ \frac{3}{3Z} \left( \frac{1}{G_0} \mu_0 \varepsilon_0^0 G_0 - \frac{\mu_0^3 G_0}{\varepsilon_0^E} \right) = m_1 - \frac{l_1}{G_0} - \frac{\mu_0^3 G_0}{\varepsilon_0^E} \frac{3n_1}{2} \]

The bracketed term vanishes from (7.4.19) and the right hand side is expressed in \( E \) by using (7.4.16-17) as

\[ \pi = \frac{E}{E_0} \]

\[ B = \frac{E}{G_0}, \]

and the definitions of \( l_1, m_1, \) and \( n_1 \). The result is

\[ \frac{1}{3T} \frac{\partial E}{\partial T} + \frac{1}{2E_0} \left( L_0 \varepsilon_0^0 - \varepsilon_0^0 + \mu_0^3 \right) \frac{1}{3} \frac{\partial E}{\partial Z} = 0 \]
or
\[
\frac{1}{2} \frac{\partial E}{\partial T} + G_0 \alpha_0 \frac{1}{3} \frac{\partial E^3}{\partial z} = 0 \tag{7.4.32}
\]
where
\[
\alpha_0 = \frac{\rho_0 \mu_0 G_0^2}{2 \epsilon_0^4} (-\epsilon_0 K_0 + L_0 K_0 + M_0) \tag{7.4.33a}
\]

In the following chapter we show (see 8.3.13) that the entropy change across a weak shock is given by
\[
S - S_0 = \frac{\delta_0}{4T_0} \pi^4 + \ldots
\]
where
\[
\delta_0 = K_0 L_0 + M_0 - K_0 \epsilon_0 .
\]
Thus,
\[
\alpha_0 = \frac{\rho_0 \mu_0 G_0^2}{2 \epsilon_0^4} \delta_0 . \tag{7.4.33b}
\]

Since entropy must increase across the shock, \(\delta_0\) must be positive. Thus Equation (7.4.34) requires that \(\alpha_0\) be positive as well. Of course, such a conclusion for entropy change is valid for weak shocks only as it was derived on that basis; but this also is our basis here. So this conclusion continues to hold.

The differential equation (7.4.32) new to the literature, governs the behavior of the small-but-finite amplitude electromagnetic wave in an elastic dielectric. The wave generated by a laser provides a physical example. We note that the
relations (7.4.16-17) allow us to replace \( E \) with \( B \) or \( \pi \) without changing the form of the equation. The solution of (7.4.32) is obtained in the following section. Before proceeding with that solution we wish to comment on the general simple wave solution.

Reverting to the original variables Equation (7.4.32) can be written as

\[
\frac{\partial E}{\partial t} + G(E^2) \frac{\partial E}{\partial z} = 0 \tag{7.4.34}
\]

where

\[
G = G_0 (1 + 3\alpha_0 E^2), \quad E = \frac{1}{\varepsilon E}
\]

The characteristics of this equation are given by

\[
\frac{dz}{dt} = G(E^2) . \tag{7.4.35}
\]

On these characteristics \( E \) is constant. Indeed we can write a general simple wave solution which always exists for a wave moving into an ambient undisturbed medium, in the form

\[
z = Gt + f(E) \tag{7.4.36}
\]

for an arbitrary function \( f \) [39, 40]. To verify that this is indeed a solution we differentiate with respect to \( z \) and \( t \) respectively to get

\[
1 = (G' t + f') E_z \tag{7.4.37}
\]

\[
0 = G + (G' t + f') E_t \tag{7.4.38}
\]
where primes denote differentiation with respect to the argument \( E \).

If \((G't+f')\) is not zero, substitution of (7.4.37-38) into (7.4.34) shows that (7.4.35) is a general solution. Further one can see from (7.4.37) that \( f'(0) \) is the initial value of \((E',z)^{-1}\).

The above solution fails to be valid when a shock wave is formed. This occurs when the curve of \( E \) vs. \( z \) becomes vertical, for which

\[
\left( \frac{\partial z}{\partial E} \right)_t = 0 . \tag{7.4.39}
\]

This leads to

\[
G_0' t_c + f_0' = 0 \tag{7.4.40}
\]

where \( t_c \) is the critical time for shock formation. It is noted that the nonvanishing of this was the condition for the existence of the simple wave solution. In particular the critical time is now given by

\[
t_c = -\frac{t_0'}{G_0'} = -\frac{1}{(E'_z)G_0'} . \tag{7.4.41}
\]

For a wave moving into a medium with \( E \) equal to zero ahead, this requires that a positive time for shock formation exists only if \( G_0' \neq 0 \) and \((E'_z)G_0'\) is negative. In our case \( G_0' \) is zero and so \( t_c + \infty \) indicating that the simple wave does not break down due to shock formation.
7.5. A Uniformly Valid Perturbation Solution

In order to consider the effects of nonlinearity we seek a uniformly valid solution of the nonlinear wave Equation (7.4.34). As is well known, nonlinearity generates waves of different wavelengths resulting from an input of a given wavelength and also gives an amplitude dependent frequency. It is our intention here to bring out these two features.

For convenience we transform the time by writing \( t \) for the product of \( G_0 \) and the original time. This allows Equation (7.3.34) to be written as

\[
\frac{\partial E}{\partial t} + \frac{\partial E}{\partial z} = -3a_0 E^2 \frac{\partial E}{\partial z} .
\]

This is to be solved subject to the condition

\[
E(z,0) = \varepsilon \cos k z \tag{7.5.2}
\]

where \( \varepsilon \) denotes the amplitude and \( k \) the wave number of the initial disturbance.

One technique for the solution of such nonlinear problems is the method of multiple time scales.

Following Nayfeh [26] we proceed by using an expansion of the form

\[
E = \varepsilon \{ n_1(t)e^{ikz} + \bar{n}_1(t)e^{-ikz} \} + \varepsilon^3 \{ n_3(t)e^{3ikz} + \bar{n}_3(t)e^{-3ikz} + \zeta(t)e^{ikz} + \bar{\zeta}(t)e^{-ikz} \} + o(\varepsilon^3) \tag{7.5.3}
\]
where the bar denotes the complex conjugate $-\text{c.c.}$. By substituting this into (7.5.1) and equating the coefficients of the various wave numbers terms we have

\[ \frac{d\eta_1}{dt} + i\kappa \eta_1 + \epsilon^2 \left( \frac{dk}{dt} + i\kappa \zeta + 3i\kappa_0 \eta_1^2 \right) = 0 \] (7.5.4)

\[ \frac{d\eta_3}{dt} + 3i\kappa \eta_3 + 3i\kappa_0 \eta_1^3 = 0 \] (7.5.5)

To obtain a uniformly valid approximate solution we use the method of multiple time scales [41]. Thus we assume that

\[ \eta_1(t) = \eta_{10}(T_0, T_2) + \epsilon^2 \eta_{12}(T_0, T_2) + \ldots \] (7.5.6)

where

\[ T_0 = t, \quad T_2 = \epsilon^2 t \] (7.5.7)

and similar expansions for the other time dependent variables in (7.5.4-5).

It is interesting to note that this method transforms an ordinary differential Equation (7.5.4) into a partial differential equation. It is essential that $T_0, T_2, \ldots$ are to be regarded as independent variables. Thus

\[ \frac{d\eta_1}{dt} = \frac{\partial \eta_{10}}{\partial T_0} + \epsilon^2 \left( \frac{\partial \eta_{10}}{\partial T_2} + \frac{\partial \eta_{12}}{\partial T_0} \right) + \mathcal{O}(\epsilon^4) \] (7.5.8)

With these expansions, Equation (7.5.4) is written as
\[ \frac{\partial \eta_{10}}{\partial T_0} + ik\eta_{10} + \varepsilon^2 \left( \frac{\partial \eta_{12}}{\partial T_2} + \frac{\partial \eta_{12}}{\partial T_0} + ik(\eta_{12} + \zeta_0 + 3\alpha_0 \eta_{10}^2 - \eta_{10}) \right) + \frac{\partial \zeta_0}{\partial T_0} \} + \ldots = 0 \]  

(7.5.9)

Since \( \varepsilon^2 \eta_{12} \) type terms are considered as small corrections to \( \eta_{10} \) type terms, we equate the coefficients of \( \varepsilon^0 \) and \( \varepsilon^2 \) to zero. The \( \varepsilon^0 \) equation is

\[ \frac{\partial \eta_{10}}{\partial T_0} + ik\eta_{10} = 0 , \]

(7.5.10)

which has the solution

\[ \eta_{10} = A(T_2)e^{-ikT_0} . \]

(7.5.11)

A similar analysis for the \(-k\) wave number shows that

\[ \eta_{10} = A(T_2)e^{ikT_0} . \]

(7.5.12)

Using these results in the \( \varepsilon \) part of Equation (7.5.9) allows us to write

\[ \frac{\partial}{\partial T_0} (\eta_{12} + \zeta_0) + ik(\eta_{12} + \zeta_0) + \left( \frac{dA}{dT_2} + 3i\alpha_0 A^2 \right) e^{-ikT_0} = 0 \]

(7.5.13)

To avoid secular terms in the solution of this equation we must have

\[ \frac{dA}{dT_2} + 3i\alpha_0 A^2 = 0, \]

(7.5.14)
from which
\[ \tilde{A} \frac{dA}{dT_2} = -3i k |A|^4 \] (7.5.15)

and taking the complex conjugate,
\[ \tilde{A} \frac{d\tilde{A}}{dT_2} = 3i k |A|^4 \] (7.5.16)

The sum of these two vanishes
\[ \frac{d|A|^2}{dT_2} = \tilde{A} \frac{dA}{dT_2} + A \frac{d\tilde{A}}{dT_2} = 0 \] (7.5.17)

Now we let
\[ A(T_2) = r(T_2) e^{i \phi(T_2)} \] (7.5.18)

and so
\[ \tilde{A}(T_2) = r(T_2) e^{-i \phi(T_2)} \] (7.5.19)

Applying these in Equation (7.5.17) results in
\[ \frac{d|A|^2}{dT_2} = 2r \frac{dr}{dT_2} = 0 \] (7.5.20)

Hence \( r(T_2) \) must be constant, say \( r_0 \), and
\[ A(T_2) = r_0 e^{i \phi(T_2)} \] (7.5.21)

Now Equation (7.5.14) reduces to
\[ \frac{d\phi}{dT_2} + 3k r_0^2 = 0 \] (7.5.22)
which has the solution
\[ \phi = -3\alpha_0 r_0^2 k T_2. \]  
(7.5.23)

Substituting back into (7.5.6) and then into (7.5.3) we find
\[ \eta_1 = r_0 \exp\{i k(-T_0 - 3\alpha_0 r_0^2 T_2)\} + O(\varepsilon^2) \]
\[ = r_0 \exp(-ik\omega t), \]  
(7.5.24)

where
\[ \omega = 1 + 3\alpha_0 r_0^2 \varepsilon^2; \]

and
\[ E = \text{Re} \varepsilon\{r_0 \exp(ik(z-\omega t)) + r_0 \exp(ik(-z+\omega t))\} + O(\varepsilon^3) \]
\[ = \varepsilon 2r_0 \cos k(z-\omega t) + O(\varepsilon^3) \]  
(7.5.25)

Employing the initial condition (7.5.2) determines the value of \( r_0 \) to be 1/2. Thus the solution to the nonlinear wave equation is approximately given by
\[ E = \varepsilon \cos k(z-\omega t) \]  
(7.5.26)

where
\[ \omega = 1 + \frac{3}{4} \alpha_0 \varepsilon^2 \]

We see that the frequency is slightly amplitude dependent. The frequency itself is modified due to nonlinearity.

Further terms of the solution can be obtained in a similar manner. They will bring out the effects of higher wave
number terms and further modify the frequency. This is not pursued here since all of the preceding work was done only up to this order.
8. ENTROPY CHANGE ACROSS A SHOCK WAVE

8.1. Introduction

As was demonstrated in Chapter 6, the passage of an electromagnetic wave is necessarily accompanied by a longitudinal strain. This feature of photoelastic waves is essentially nonlinear in character and therefore is not seen in either the linearized theory or in the weak wave study explained at the conclusion of Section 7.4. The longitudinal strain is proportional to the square of polarization density.

In this chapter we wish to prove that the change in entropy across a shock wave is proportional to the fourth power of polarization density. This is the feature of photoelastic waves that allows us to neglect the energy equation in the study of weakly nonlinear waves in the preceding chapter. It further brings out, that in the construction of a higher order theory it is wrong to neglect the energy equation.

The method employed to achieve this result parallels that used in the study of shock waves in analogous fields [29]. First, by combining shock conditions from the basic balance laws one obtains a relation between state variables only. This is known as the Hugoniot function and contains the important feature that it reduces the shock wave to a single parameter family. In our work we have taken the polarization
density \( \pi \), to be this parameter. Thus, the discontinuity in the magnitude of \( \pi \) characterizes the shock.

The Hugoniot function is then combined with Gibb's equation, which states that the differential of the internal energy function is equal to the sum of the products of the thermodynamic tensions and differentials of thermodynamic fluxes. In the problem at hand, the fluxes are entropy, strain, and polarization density with corresponding tensions as temperature, stress, and electromotive intensity. This relation was developed in Section 3.4.

8.2. A Conservative Form of the Energy Equation

To obtain the Hugoniot function we need the energy equation written in a conservative form. This is obtained here by combining the balance equations of mass (3.3.3), linear momentum (3.3.4), and energy (3.2.4), and using relations from Maxwell's equations. In particular, we use the following combination,

\[
(e + \frac{1}{2} v^2)(3.3.3) + v \cdot (3.3.4)+(3.2.4) = 0, \tag{8.2.1}
\]

where \( e \) is the internal energy function per unit mass. The result of this combination is

\[
\rho (e + \frac{1}{2} v^2),_t + \rho v_j (e + \frac{1}{2} v^2),_j = (t_{ij} v_i),_j + \frac{\partial F_i}{\partial t} + E_i (P_i v_j),_j - E_i (P_j v_i),_j. \tag{8.2.2a}
\]
Manipulating Maxwell's equations we find that

\[ E_i \frac{\partial P_i}{\partial t} = -S_{ij},j - \frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \frac{\varepsilon_0}{2} \frac{\partial E^2}{\partial t} + E_i,j P_i v_j - E_i,j P_j v_i \]

(8.2.2b)

where \( S \) is the Poynting vector defined as \( \mathbf{E} \times \mathbf{H} \). Thus Equation (8.2.2) can be written as

\[ \{ \rho (e + \frac{1}{2} V^2) \},_t + \{ \rho v_j (e + \frac{1}{2} V^2) \},_j = (t_{ij},v_i),j - S_{ij},j - \frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} - \frac{\varepsilon_0}{2} \frac{\partial E^2}{\partial t} + (E_i P_j v_j),_j \]

or

\[ \frac{\partial}{\partial t} \{ \rho (e + \frac{1}{2} V^2) \} + \frac{\varepsilon_0}{2} \frac{\partial E^2}{\partial t} + \frac{1}{2\mu_0} \frac{\partial B^2}{\partial t} + \{ \rho v_j (e + \frac{1}{2} V^2) \}
- t_{ij},v_i - E_i P_i v_j + E_i P_j v_i + S_{ij},_j = 0 \]

(8.2.3)

This is the conservative form of the energy equation for the elastic dielectric in an electromagnetic field. While not particularly difficult to derive, we have not found it in the literature.

This form is useful to us as it directly gives the shock condition as

\[ -G[\rho (e + \frac{1}{2} V^2) + \frac{1}{2\mu_0} B^2 + \frac{\varepsilon_0}{2} E^2] + n_j [\rho v_j (e + \frac{1}{2} V^2)]
- t_{ij},v_i - E_i P_i v_j + E_i P_j v_i + S_{ij} = 0 \]
In the following section we shall limit our attention to the case where the wave normal is in the z-direction and the electric field has only the first component non-zero.

8.3. Hugoniot Function and Entropy Change

For the shock wave considered in Section 6.2 the jump conditions from the equation of motion given by (6.2.23) are written here as

\[-G \rho_0 v_3 = t_{33} - t_{033} - \frac{B^2}{2\mu_0} (1 - \varepsilon \mu_0 G^2) \]  

(8.3.1)

and so we can write

\[\frac{\rho_0 v_3^2}{2} = \frac{t_{33} - t_{033}}{2} \frac{v_3}{G} \frac{v_3}{G} + \frac{B^2 (1 - \varepsilon \mu_0 G^2)}{2\mu_0} \]  

(8.3.2)

which will be useful in obtaining the Hugoniot function.

For the case under consideration here the energy Equation (8.2.5) reduces to,

\[\rho (v_3 - G) (\varepsilon + \frac{1}{2} v^2) = \frac{G B^2}{2\mu_0} + \frac{\varepsilon_0 E^2}{2} = [t_{33} v_3 - S_3 + EPv_3]. \]

(8.3.3)

By noting that electromagnetic variables and \( v_3 \) are zero ahead and using the continuity Equation (6.2.18) this further reduces to

\[-\rho_0 G (\varepsilon - \varepsilon_0) - \frac{\rho_0 v_3^2}{2} - \frac{G B^2}{2\mu_0} + \frac{\varepsilon_0 E^2}{2} = t_{33} v_3 - EH + EPv_3 \]

(8.3.4a)
Writing all electromagnetic variables in terms of $B$ allows (8.3.4) to be written as

$$\rho_0 (e-e_0) \frac{\rho_0 v_3^2}{2} = -t_{33} v_3 + \frac{B^2}{2 \mu_0} (1-\varepsilon_0 \mu_0 G^2) \quad (8.3.4b)$$

Now we substitute for $\frac{1}{2} \rho_0 v_3^2$ from (8.3.2) and use the relation $\mu = e \rho_0$ to obtain,

$$\frac{2 \rho_0}{\rho_0} (U-U_0) = (t_{33}+t_{033}) \frac{\xi_3}{1-\xi_3} + \frac{B^2 (1-\varepsilon_0 \mu_0 G^2)}{\mu_0} \frac{1-\frac{1}{2} \xi_3}{1-\xi_3} \quad (8.3.4c)$$

From (6.2.40) and (6.2.28) we can show that

$$(1-\varepsilon_0 \mu_0 G^2) \frac{B}{\mu_0} = \rho_0 G \pi \quad (8.3.5)$$

and so the final form that the Hugoniot function takes here, is written as

$$\frac{2 \rho_0}{\rho_0} (U-U_0) = (t_{33}+t_{033}) \frac{\xi_3}{1-\xi_3} + \rho_0 G \pi (1-\frac{1}{2} \xi_3) \quad (8.3.6)$$

By considering $\pi$ as the shock parameter, and using $(\cdot)$ to denote differentiation with respect to $\pi$, this gives

$$\frac{2 \rho_0}{\rho_0} \dot{\xi} = t_{33} \frac{\xi_3}{1-\xi_3} + (t_{033}+t_{33}) \frac{\xi_3}{(1-\xi_3)^2} - \rho_0 \frac{\xi}{2} \pi \frac{\dot{\xi}_3}{\xi_3}$$

$$+ \rho_0 (1-\frac{1}{2} \xi_3) (\xi+\pi \dot{\pi}) \quad (8.3.7)$$

To write Gibbs' equation, we note that for the case at hand we have the following quantities:
\[
a_{ik} = \delta_{ik} - 2e_{ij} = \begin{pmatrix} 1 - 2e_{011} & 0 \\ 0 & 1 - 2e_{022} \\ 0 & 0 & 1 - 2e_{33} \end{pmatrix}
\]

where
\[
\begin{align*}
\text{Also,} \quad a_{ik}^{b_{kj}} &= \delta_{ij},
\end{align*}
\]

\[
b_{kj} = \begin{pmatrix} \frac{1}{1 - 2e_{011}} & 0 & 0 \\ 0 & \frac{1}{1 - 2e_{022}} & 0 \\ 0 & 0 & \frac{1}{1 - 2e_{33}} \end{pmatrix}
\]

Thus Gibbs' Equation (3.4.4) reduces to
\[
\frac{1}{\rho_{00}} \, d\bar{U} = T \, dS + \frac{b_{33} t_{33}}{\rho} \, \text{de}_{33} + \epsilon \, d\pi.
\]

Taking the derivative with respect to the shock parameter \(\pi\), and multiplying by \(2\rho_0\) gives
\[
\frac{2 \rho_0}{\rho_{00}} \dot{U} = 2 \rho_0 \left( \frac{t_{33}}{1 - \xi_3} \right) \dot{\xi}_3 + \frac{2 \rho_0 t_{33}}{(1-2e_{33})} \dot{e}_{33} + 2 \rho_0 \xi
\]

(8.3.9)

Now, eliminating \( \dot{U} \) by using the Hugoniot function (8.3.7) we have

\[
2 \rho_0 T \dot{S} = \frac{t_{33}}{1-\xi_3} \frac{\xi_3}{(1-\xi_3)^2} \rho_0 \xi (1 + \frac{1}{2} \xi_3) + \rho_0 \dot{\xi} \xi (1 - \frac{1}{2} \xi_3)
\]

\[ - \frac{1}{2} \rho_0 \xi \pi \frac{\xi_3}{\xi_3} \]

(8.3.10)

This is the equation that governs the entropy change across the shock wave.

We now make the assumption of a weak shock wave. Then one obtains

\[ \bar{\xi}_3 = K_0 \pi^2 + \ldots \]

where \( K_0 \) is given by (6.2.47; and from (6.2.43)

\[ \xi = (\xi_0 + L_0 K_0 \pi^2 + M_0 \pi^2 + \ldots) \pi \]

It is seen upon substitution into (8.3.10) that the terms in stress are of order \( \pi^5 \) and contribute smaller order quantities compared to the remaining terms which are of order \( \pi^3 \). Indeed, the expression can be simplified to

\[ T \dot{S} = \pi^3 (L_0 K_0 + M_0 - \xi_0 K_0) + O(\pi^5) \]

(8.3.11)

We see that ahead of the shock, where \( \pi=0 \), \( \dot{S} \) vanishes. Higher derivatives of Equation (8.3.11) show that
\[ S(0) = S''(0) = 0 \]

in addition. We note that \( T \) and the bracketed quantities in (8.3.11) depend on \( S \); however since the first three derivatives of \( S \) vanish ahead, these can be assumed to be evaluated in the state ahead.

The first nonvanishing derivative is \( S_0'' \)

\[ T_0 S_0'' = 6(L_0 K_0 + M_0 - \xi_0 K_0) \]  \hspace{1cm} (8.3.12)

where all quantities with suffix zero are now assumed to be evaluated ahead of the wavefront. This allows us to write

\[ S - S_0 = \frac{\delta_0}{4T_0} \pi^4 + \ldots, \]  \hspace{1cm} (8.3.13)

where

\[ \delta_0 = L_0 K_0 + M_0 - \xi_0 K_0. \]  \hspace{1cm} (8.3.14)

Since entropy must always increase across a shock wave we see from this that \( \delta_0 \) must be positive.

It may be of interest to note the value of \( \delta_0 \) for the unstrained case, in terms of the material coefficients in the polynomial expansion of the internal energy given in the Appendix, it is

\[ \delta_0 = \frac{2}{\rho_0^2} \left( \frac{\beta_{16} - \alpha_6}{\rho_0 c_0^2 - (\lambda + 2\mu)} + \beta_{66} \right) > 0. \]  \hspace{1cm} (8.3.15)

For \( \beta_{66} > 0 \) one can see that this is always satisfied.

Further if \( \beta_{66} < 0 \) it gives a bound for \( \beta_{66} \) in terms of
\( \lambda, \mu, \alpha_6 \) and \( \beta_{16} \).

We can argue that the denominator of the first term in the curly brackets is very nearly \( \rho G_0^2 \) by the following argument.

\[
\frac{1}{\rho_0 G_0^2 - R_0} = \frac{1}{\rho_0 G_0^2 (1 - R_1)} \tag{8.3.16}
\]

where

\[
R_1 = \frac{R_0}{\rho_0 G_0^2}
\]

This is the square of the ratio of the elastic velocity to the velocity of light, which is negligible compared to unity.

Thus (8.3.15) requires that for unstrained media

\[
\frac{2(\beta_{16} - \alpha_6)}{\rho_0 G_0^2} + \beta_{66} > 0 \tag{8.3.17}
\]
9. DISCUSSION OF THE RESULTS

The work begins with a presentation of the basic equations of the Dynamical Theory of a Hyper-elastic Dielectric in the spatial form. Of these, Gibb's equation and the conservative form of the energy equation are new.

Based on singular surface theory, the study of 'weak waves' and shock-waves brings out the structure necessary for the construction of any nonlinear theory. The known theory of photoelasticity is a linear theory based only on Maxwell's equations. It corresponds to the 'weak wave' analysis which neglects all accompanying elastic deformations and entropy changes. The lowest order nonlinear theory must consider the equations of motion and continuity since such a wave contains elastic strains, but it still is isentropic. Any higher order nonlinear theory is accompanied with entropy changes and thus the energy equation must be included.

More problems are raised than answered. Further constraints imposed on photoelastic waves, by including initial electromagnetic fields, should provide directions for further theoretical and experimental work. Considerations of a more generally oriented wave (of curved wavefronts with nonlinearity and other features noted above), may prove useful in studies of Laser beams.

Only a few of these are considered here, but even so, the work provides fairly general guidelines for building a more
comprehensive theory of photoelastic waves. Though not studied here, a theory of the hyper-elastic dielectric which is also magnetizable can be constructed from the combination of Toupin's and Brown's works. This can form the basis of magneto-photo-elastic waves, some simple forms of which are discussed by Aben. Such a study can indeed provide a basis for completely new methods in experimental stress analysis.

Boundary conditions have not been given any consideration. Further, dissipative and/or dispersive phenomena are also neglected. Though the latter may be of secondary importance, the former should be of rather more importance.

However, it is hoped that the method which produces the systematic approximations and manageable theory herein developed (and clarified by studying some simple cases), will retain its generality for whatever basic laws one wishes to consider.
10. BIBLIOGRAPHY


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12. APPENDIX

12.1. A Polynomial Expansion of Internal Energy

Derivatives of the internal energy function with respect to strain and polarization density determine stress and electromotive intensity. These in turn are approximated by the following expansions, given in Chapter 6 by Equations (6.2.36, 43) as

\[ t_{33} = t_{033} + R_0 \xi_3 + S_0 \pi^2 \]  \hspace{1cm} (12.1.1)

\[ \varepsilon = (\varepsilon_0 + L_0 \xi_3 + M_0 \pi^2)\pi, \]  \hspace{1cm} (12.1.2)

for the components of \( t_{ij} \) and \( \varepsilon_i \) that are of principal interest in this study. The expansion of the internal energy \( U \) is in terms of combinations of invariants of \( e_{ij} \) and \( \pi_i \). We seek an expansion general enough to determine \( t_{033}, R_0, S_0, \varepsilon_0, L_0 \) and \( M_0 \) to the order of approximation used in the body of this work.

In addition to (11.1.1-2) we are guided by the general expressions for \( t_{ij} \) and \( \varepsilon_i \) which under the considerations of Section 6.2 reduce to:

\[ t_{33} = \frac{2}{\rho_{00}} (1-2e_{33}) \{ U,_{1}^{+} + U,_{2}(e_{011} + e_{022}) + U,_{3}e_{011}e_{022} \} \]  \hspace{1cm} (12.1.3)

\[ \varepsilon = \frac{2}{\rho_{00}} (U,_{6}^{+} + U,_{4}e_{011} + U,_{5}e_{011}^{2})\pi. \] \hspace{1cm} (12.1.4)
Before proceeding we note the following relations for the wave we studied in Chapter 6.

\[ u_{3,3} = u_{03,3} + \xi_3 \]

\[ e_{33} = e_{033} + (1-u_{03,3})\xi_3 - \frac{1}{2} \xi_3^2 \]

\[ = e_{033} + (1-u_{03,3})^2(\xi_3 - \frac{1}{2} \xi_3^2). \]

\[ \frac{\partial e_{33}}{\partial \xi_3} = (1-u_{03,3})^2(1-\xi_3) \]

\[ I = e_{011}e_{022}e_{33}, \quad I-I_0 = (1-u_{03,3})^2(\xi_3 - \frac{1}{2} \xi_3^2) \]

\[ II = e_{011}e_{022}+(e_{011}+e_{022})e_{33}, \]

\[ II-II_0 = (e_{011}+e_{022})(1-u_{03,3})^2(\xi_3 - \frac{1}{2} \xi_3^2) \]

\[ III = e_{011}e_{022}e_{33}, \quad III-III_0 = e_{011}e_{022}(1-u_{03,3})^2(\xi_3 - \frac{1}{2} \xi_3^2) \]

\[ IV = e_{011} \pi^2, \quad V = e_{011} \pi^2, \quad VI = \pi^2 \]

\[ \rho = \rho_{00}(1-2e_{11})(1-2e_{22})(1-2e_{33})^{1/2} \]

\[ = \rho_{00}(1-u_{1,1})(1-u_{2,2})(1-u_{3,3}) \]

\[ = \rho_{00}(1-u_{01,1})(1-u_{02,2})(1-u_{03,3}-\xi_3) \]

\[ (1-2e_{33}) = (1-u_{03,3}-\xi_3)^2 \]

\[ = (1-2e_{033})(1-\xi_3)^2 \]
Expanding $U_1$ now we have

$$U_1 = U_1\bigg|_0 + U_{11}\bigg|_0 (I-I_0) + U_{12}\bigg|_0 (II-II_0) + \ldots$$  \hspace{1cm} (12.1.5)

where the subscript (0) means evaluated at the initial deformed state. The subscript (00) will mean the initial undeformed state. We shall write (12.1.5) in the following way

$$U_1 = U_{0,1} + U_{0,11}(I-I_0) + \ldots$$  \hspace{1cm} (12.1.6)

where

$$U_{1}\bigg|_0 = U_{0,1} \text{ etc.},$$

in what follows. Using the relations that give the invariants and their derivatives we can express $U_1$ as

$$U_1 = U_{0,1} + \{U_{0,11}+U_{0,12}(e_{011}+e_{022})$$
$$+ U_{0,13} e_{011} e_{022} (1-u_{03,3})^2 \xi_3$$
$$+ (U_{0,14} e_{011} + U_{0,15} e_{011} + U_{0,16} e_{011}) \pi^2 + \ldots$$  \hspace{1cm} (12.1.7)

To express the derivatives in terms of the initial undeformed state we note:

$$f(I_0, II_0, III_0) = f(0,0,0) + \frac{\partial f}{\partial I_0}\bigg|_{I_0=0} (I_0) + \ldots$$  \hspace{1cm} (12.1.8)

where initial strains are assumed small.
Thus,
\[ U_{0,1} = U_{00,1} + U_{00,11} + U_{00,12} + U_{00,13} \]  
(12.1.9)
since
\[ IV_0 = V_0 = VI_0 = 0. \]

Hence we can write \( U_{1} \) in terms of derivatives of \( U \) with reference to the undeformed state of the material.

\[ U_{1} = U_{00,1} + U_{00,11} + U_{00,12} + U_{00,13} \]
\[ + (1-u_{03,3})^2 \xi_3 \{ U_{00,11} + U_{00,12} + U_{00,13} \} \]
\[ + U_{00,13} \]  
(12.1.10)

Denoting the bracketed term in (12.1.3) by
\[ \{ \} = R_0' + R_0'' \xi_3 + S_0' \pi^2 \]  
(12.1.11)

we find that by using (12.1.7) that

\[ R_0' = U_{0,1} + U_{0,2} (e_{011} + e_{022}) + U_{0,3} e_{011} e_{022} \]  
(12.1.12)

\[ R_0'' = \{ U_{0,1} + 2 (e_{011} + e_{022}) U_{0,12} + (e_{011} + e_{022})^2 U_{0,22} \]  
\[ + 2 e_{011} e_{022} U_{0,13} + 2 e_{011} e_{022} (e_{011} + e_{022}) U_{0,23} \]  
\[ + e_{011} e_{022} U_{0,33} \} (1-u_{03,3})^2 \]  
(12.1.13)
Now we consider strains to be small and neglect terms containing strains squared and smaller. This approximation will be denoted by (=).

\[ S_0' = U_{0,16} + e_{011}^{0} + (e_{011} + e_{022})U_{0,14} + (e_{011} + e_{022})U_{0,15} + e_{011} (e_{011} + e_{022})U_{0,26} + e_{022} U_{0,36} + e_{022}^2 U_{0,25} + e_{022}^2 U_{0,34} + e_{022}^3 U_{0,35} \]  

\[ (12.1.14) \]

Returning to (12.1.3) and (12.1.1) and noting that \( P = P_0 (1 - \xi_3) \) and the e_{33} relations we find

\[ R_0' = U_{0,0} + U_{0,2} (e_{011} + e_{022}) \]

\[ R_0'' = (1 - 2e_{033})U_{0,11} + 2(e_{011} + e_{022})U_{0,12} \]  

\[ S_0'' = U_{0,16} + e_{011} U_{0,14} + (e_{011} + e_{022}) U_{0,26} \]  

\[ (12.1.15) \]

Also

\[ \frac{\rho_0}{\rho_{00}} (1 - 2e_{033}) = (1 - u_{01,1})(1 - u_{02,2})(1 - u_{03,3})(1 - 2e_{033}) = 1 - I_0 - 2e_{033} \]  

\[ (12.1.19) \]
Now using (12.1.9) to write all derivations of $U$ in the undeformed state

$$R_0' = U_{00,11}I_0 + U_{00,2}(e_{011}+e_{022}) \quad (12.1.20)$$

where $U_{00,11}=0$ since $I$ does not occur by itself in the expansion for $U$ as we require that there be no stress in the undeformed unpolarized state

$$R_0'' = U_{00,11}+U_{00,11}I_0+2U_{00,12}(e_{011}+e_{022})-2U_{00,11}e_{033} \quad (12.1.21)$$

$$S_0' = U_{00,16}+U_{00,16}I_0+U_{00,14}e_{011}+U_{00,26}(e_{011}+e_{022}) \quad (12.1.22)$$

Thus

$$\mathbf{t}_{033} = U_{00,11}I_0 + U_{00,2}(e_{011}+e_{022}) \quad (12.1.23)$$

$$R_0 = U_{00,11}+U_{00,11}I_0+2U_{00,12}(e_{011}+e_{022})-4U_{00,11}(I_0-e_{033})-3U_{00,2}(e_{011}+e_{022}) \quad (12.1.24)$$

$$S_0 = U_{00,16}+U_{00,16}I_0+U_{00,14}e_{011}+U_{00,26}(e_{011}+e_{022})-U_{00,16}(I_0+2e_{033}) \quad (12.1.25)$$

Similarly we find:

$$\mathbf{E}_0 = \frac{2}{p_0}(U_{00,6}+U_{00,6}I_0+U_{00,4}e_{011}) \quad (12.1.26)$$

$$\mathbf{L}_0 = \frac{2}{p_0}(U_{00,61}+(U_{00,61}+U_{00,62}+U_{00,41})e_{011}+(U_{00,61}+U_{00,62})e_{022}+(U_{00,61}-2U_{00,61})e_{033}) \quad (12.1.27)$$
Thus an expansion of $U$ to determine $t_{33}$ and must include, to the order used in this study, sufficient terms such that none of the following derivatives vanish,

$$U_{11}, U_{2}, U_{111}, U_{12}, U_{4}, U_{14}, U_{6}, U_{61}, U_{611}, U_{62}, U_{64}, U_{66}, U_{661}$$

Thus we find the appropriate expansion to be

$$U = \frac{\lambda+2\mu}{2} I^2 - 2\mu II + \lambda I^3 + mI II + a_4 IV + a_6 VI$$

$$+ \beta_1 I VI + \beta_2 II VI + \beta_4 IV VI$$

$$+ \beta_6 VI^2 + \beta_4 I IV + \gamma_{116} I^2 VI + \gamma_{166} I VI^2 + ...$$

(12.1.29)

This expansion is necessary even in the case of small initial strains. For such an expansion we have

$$t_{033} = (\lambda+2\mu)I_0 - 2\mu(e_{011}+e_{022})$$

(12.1.30)

$$R_0 = \lambda+2\mu + \lambda I_0 + 2m(e_{011}+e_{022}) - 2(\lambda+2\mu)(I_0+e_{033})$$

$$+ 6\mu(e_{011}+e_{022})$$

(12.1.31)
\[ S_0 = 3_16 + \gamma_{116}I_0 + \beta_{14}e_{011} + \beta_{26}(e_{011} + e_{022}) \]
\[ - \beta_{16}(I_0 + 2e_{033}) \]  
\( (12.1.32) \)

\[ E_0 = \frac{2}{\rho_{00}}(\alpha_6 + \beta_{16}I_0 + \alpha_4e_{011}) \]  
\( (12.1.33) \)

\[ L_0 = \frac{2}{\rho_{00}}(\beta_{16} + (\gamma_{116} + \beta_{26} + \beta_{14})e_{011} + (\gamma_{116} + \beta_{26})e_{022} \]
\[ + (\gamma_{116} - 2\beta_{16})e_{033} \]  
\( (12.1.34) \)

\[ M_0 = \frac{2}{\rho_{00}}(\beta_{66} + 2\beta_{46}e_{011} + \gamma_{166}I_0) \]  
\( (12.1.35) \)

For vanishing strains
\[ t_{033} = 0, \quad R_0 = \lambda + 2\mu, \quad S_0 = \beta_{16} \]  
\( (12.1.36) \)

\[ E_0 = \frac{2}{\rho_{00}}\alpha_6, \quad L_0 = \frac{2}{\rho_{00}}\beta_{16}, \quad M_0 = \frac{2}{\rho_{00}}\beta_{66} \]  
\( (12.1.37) \)

In which case
\[ K_0 = \frac{2S_0 - \rho_0 E_0}{2(\rho_0 G_0^2 - R_0)} = \frac{\beta_{16} - \alpha_6}{\rho_0 G_0^2 - (\lambda + 2\mu)} \]  
\( (12.1.38) \)

\[ \delta_0 = K_0(L_0 - E_0) + M_0 = \frac{2}{\rho_{00}} \left[ \frac{(\beta_{16} - \alpha_6)^2}{\rho_0 G_0^2 - (\lambda + 2\mu)} + \beta_{66} \right] > 0 \]  
\( (12.1.39) \)

As a final comment, we observe that that \( R_0 / \rho_0 \) gives the speed of the elastic wave. Thus the form of (12.1.31) indicates a distinct possibility of determining higher
order elasticities simply by measuring the speeds of elastic waves traveling in a stressed medium.