Steady-state flow patterns of rainwater seeping through bedded soil with and without tile drains

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Norris Ledford Powell

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INTRODUCTION

This thesis is presented in three parts. In the first part we solve a boundary value problem of seepage of rainwater through soil bedding underlain by an impermeable barrier at a finite depth. A tile drain is semiembedded in the impermeable barrier directly below the furrows in the bedding. A potential function designated by $\phi$ and a stream function designated by $\psi$ are determined. Using these functions we draw flow nets for several cases to illustrate the flow patterns of rainwater seeping through the bedded soil to the tile drains. We are able to check our work with Kirkham (1947a) for the special case when the soil bedding has zero slope.

In the second part of this thesis we solve the boundary value problem of seepage of rainwater through soil bedding underlain by a barrier at infinite depth. In the same manner as in the first part we are able to draw flow nets to illustrate the flow patterns of rainwater through the bedded soil. We compare our work with that of Warrick (1970) and then extend the work to include a finite depth of water in the drainage furrows and the case of an arbitrarily shaped soil surface.

The above two problems are solved using an analytical mathematical method developed by Kirkham and Powers (1972).
This method can be used to solve drainage problems with nonrectangular flow regions.

In the third part of this thesis we present streamlines of rainwater seeping through bedded soil. We use laboratory models filled with sand. Photographs are presented to illustrate the development of the streamlines and to check the mathematical analysis. Two cases from Powers et al. (1967) and three cases from the first part of this thesis are compared with the laboratory models.

We have two objectives in this thesis. One is to use the modified Gram-Schmidt technique developed by Kirkham and Powers (1972) to solve analytically two boundary value problems of agricultural drainage. The results will be compared, where applicable, with solutions to the same problems solved using different mathematical techniques. The second objective is to compare the streamlines of the flow nets calculated by the mathematical technique of Kirkham and Powers (1972) with laboratory models filled with sand.

Our attention is fixed on seepage through soil bedding. However, soil bedding when considered on a small scale, approximates corn ridges and furrows underlain by a tight plow sole layer. Therefore the work of this thesis, especially that part of it not dealing with tile drains, will be applicable to saturated water flow through ridged and furrowed soil insofar as the ridges and furrows overlie an
impervious plow layer or other barrier.

The flow medium considered is assumed to be homogenous and isotropic. Tile when used is considered to be surrounded by a thin layer of gravel to make the tile completely pervious.
LITERATURE REVIEW

Phillips (1963) reports that the bedding system is one of the oldest of all drainage systems known. It is used on fields which are practically flat and on soils which are slowly permeable and where tile drainage is not feasible.

In parts of southeastern Iowa and northern Missouri there are areas of relatively flat topography and poor internal drainage. Beer et al. (1961) and Beer and Shrader (1961) report the use of surface drainage as a method of removing excess water from the land. They concluded however that bedding is not feasible for removal of the excess water. Beer et al. (1965) found that tiling furnishes a convenient method of disposing of the excess water, but the costs are greater than can probably be justified for ordinary field crops.

Fausey and Schwab (1969), Schwab and Fouss (1967), Schwab and Thiel (1963), Schwab et al. (1972), Schwab et al. (1963) and Schwab et al. (1966) have reported on the effect of surface drainage, subsurface drainage and the combination of both on crop response. Fausey and Schwab (1969) reported that plots having subsurface or combined surface-subsurface drainage had greater yields of soybean (Glycine max.) than plots with surface drainage alone.

King et al. (1959) reported that over a four year period there was a significant crop yield increase by use of a combination of surface and subsurface drainage. This study was on
a Pickford clay in Michigan's upper peninsula where poor drainage was a handicap to crop production.

Whelchel (1963) and Smith and Beville (1964) reported on the drainage systems used in some areas of Florida to drain citrus groves. They reported that in order to lower the water table within 72 hours after a rain so the roots of the trees would not be damaged a system which combined surface and subsurface drainage had to be used. Smith and Beville (1964) reported the tile lines were spaced at 150 feet on slopes from 0.2 to 2 percent with four feet as the minimum desirable depth. They also reported the use of beds twelve inches high and 20 to 30 feet on center. These citrus groves are located mainly on sandy soils with high water tables.

Now that we have established the use of a bedding system drained also by subsurface drainage we will discuss some of the recent literature dealing with the solution of steady-state drainage problems.

Powers (1966) has a good literature review on steady-state drainage theories. We will review some of the latest work which deals directly with our bedding problem.

Kirkham (1947b) reported the hillside drainage difficulties in the Iowan Drift Area would appear to be due to upward movement of water over the lower areas of hillsides, this upward movement resulting from artesian pressure
developed by downward seepage over the upper portion of the hills. Powers (1966) and Powers et al. (1967) solved a problem of the seepage of steady rain through soil bedding. They used a mathematical technique, a modification of the Gram-Schmidt method, which is later reported by Kirkham and Powers (1972) and Kirkham (1972) and can be used to solve flow problems in nonrectangular regions by direct analytical methods. They were able to draw flow nets and present tables which show the quantities and direction of water flowing in the saturated soil bedding. The work of Powers (1966) and Powers et al. (1967) was limited to a constant slope with no water standing in the drainage furrows of the bedded land.

Selim and Kirkham (1972a) extended the work of Powers (1966) and Powers et al. (1967) to consider ponded water in the drainage furrows of the bedded land. They used the same mathematical technique as did Powers (1966) and Powers et al. (1967). Hereafter in this review we will refer to the technique as the modified Gram-Schmidt method.

Selim and Kirkham (1972b) used the modified Gram-Schmidt method to extend their work (1972b) to consider arbitrarily shaped soil surfaces with and without ponded water in the drainage furrows.

Powers (1966), Powers et al. (1967), and Selim and Kirkham (1972a, 1972b) used rectangular coordinates. Van der Ploeg (1972) and Van der Ploeg et al. (1971) used the
modified Gram-Schmidt method in polar coordinates to solve the steady-state well-flow problem for a confined elliptical aquifer and for a horizontal confined aquifer with arbitrary conditions on the outer boundary.

Others have solved similar problems using different mathematical techniques. Klute et al. (1965) calculated equipotential and isobar patterns for the case of flow in a saturated inclined soil slab resting on an impermeable base. Whisler (1969) used an electrical resistance network analog to analyze the steady-state flow for both saturated and unsaturated conditions.

Warrick (1970), using conformal transformation, solved a hillside seepage problem. He had an impermeable barrier located at infinity and he was limited to a constant slope with no water standing in the drainage furrow.

Freeze and Witherspoon (1966) extended an analytical solution by Toth (1962, 1963) to give flow in a region with an irregular upper surface. In Toth (1962, 1963) and in the extension of Toth by Freeze and Witherspoon (1966), head loss above the uppermost horizontal level of the flow medium was neglected. Freeze and Witherspoon (1966, 1967, 1968) used mainly numerical methods to solve several groundwater flow problems. In his latest work, Freeze (1972), utilized numerical solutions to treat the hillslope seepage problem allowing consideration of transient flow through
both the saturated and unsaturated zones of a nonhomogeneous, anisotropic hillside soil in response to time- and space-dependent rainfall inputs.

Kirkham (1947a) gave a theoretical expression for flow into drains for drains half-embedded in an impervious layer. He considered a soil surface of zero slope with ponded water.
PART I. FLOW PATTERNS OF RAINWATER SEEPING THROUGH BEDDED SOIL TO TILE DRAINS

Introduction

Powers (1966) and Powers et al. (1967) solved a problem of seepage of rainwater through soil bedding underlain by an impermeable barrier at a finite depth. They considered the depth of ponded water in the furrows of the bedded land to be negligible and the soil surface had a constant slope. Selim and Kirkham (1972a) extended the work to include a finite depth of water in the furrows. Selim and Kirkham (1972b) also extended the work to include arbitrarily shaped soil surfaces. Here we extend the work of Powers (1966) and Powers et al. (1967) to include a tile drain semiembedded in the impermeable barrier directly below the furrows.

In Fig. 1a and 1b, rain falling at a rate $R$, keeps bedded land water-saturated. Some of the rainwater infiltrates into the soil along streamlines, such as the one $S$ shown. Rainwater not needed to keep the soil water-saturated flows over the surface of the bedding to a drainage furrow (D in Fig. 1a) and is discharged in a direction perpendicular to the plane of the figure. An impermeable barrier at depth $d$ below the furrow bottom prevents deep seepage. The height from the impermeable barrier to the highest point of the bedding is $b$. The semispacing of the bedding is $L$. The depth of water in the drainage furrow is considered to be
Fig. 1 - A two-dimensional drawing of steady rain falling on bedded land overlying an impermeable barrier and drained by a tile drain located directly below the furrow and semiembedded in the barrier. (a) Bedded land with drains. (b) A semisection; the line $A'B'$ represents the level of water in an outflow ditch to which the tile discharges and is the reference level for hydraulic head $\phi$. 
negligible. The radius of the tile is w. The steady rainfall rate R keeps the soil saturated throughout. We will determine the potential function $\phi$, the stream function $\psi$, and the amount of water flowing through the soil. By knowing the potential function and the stream function we will be able to draw flow nets showing the movement of water through the soil profile to tile drains. The quantity of water moving in the soil and the streamlines are important factors to consider in determining the movement of soluble materials through the soil.

Mathematical Analysis

Geometry

Because of symmetry in Fig. 1a we need only consider a semisection OBCDO as shown in Fig. 1b. We choose the origin for our coordinate system at point O and take the reference level for the hydraulic head $\phi$ to be a horizontal line $A'B'$ given in Fig. 1b at $y = h$. The line $A'B'$ represents physically the level of water in a ditch (not shown) into which the tile discharges. Loss of head in the tile as it flows to the ditch is considered negligible.

The flow medium (Fig. 2) is ABCDEA. The drain tile EA is taken to be half embedded in the impervious layer. We indicate a point P in the flow medium by polar coordinates $r, \theta$, with $r$ measured radially outward from O, and $\theta$ measured counterclockwise from the line OAB. We also indicate the
Fig. 2 - Geometry for tile drainage of ridged land; it is assumed the surface DC is kept water-saturated by rain or other water and that drainage can occur at D (with negligible surface water thickness) as well as at the drain tile. An impermeable boundary is at OAB where a drain tile is centered at O. The tile is assumed to be enveloped by a thin layer of highly conductive material.
Reference Level
for $\phi$
point P by cartesian coordinates x, y having origin at the drain tube center O and with x measured to the right and y vertically upward. We denote the maximum value of r by M(= OC).

**Potential function and boundary conditions**

Laplace's equation is assumed valid and in the polar coordinates is, if \( \phi \) is the potential (hydraulic head), given by

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{1}
\]

We will solve equation [1] for \( \phi \) and then find \( \psi \) from the function \( \phi \). Our potential \( \phi \) has units of length. We take \( K \) to be the hydraulic conductivity of units length over time. Our \( \psi \) has the units of \( K \phi \), the velocity potential. We will use \( K \phi \) later in the Cauchy-Riemann relations to get \( \psi \).

Because we take the reference level for head as given by \( y = h \) we have \( \phi = 0 \) for \( r = h \) and \( \theta = \pi/2 \). And because head loss in the tile is neglected, a piezometer in the drain tile would have water standing in it to height \( h \) (\( \phi = 0 \)) above the impermeable layer. The maximum head difference across the system is given by \( H = b - h \). If the drain should just be running full (i.e. at zero back pressure) then we would have \( h = w \) and \( H = b - w \). We assume the surface CD is water-saturated with a thin layer of water on the surface so that the hydraulic head along CD will be given by \( \phi = y - h \). The
system is shown in Fig. 2.

The boundary conditions (indicated by the circled numbers in Fig. 2) are

along AB: \( K(\partial \phi / \partial \theta) = 0 \) or \( \partial \phi / \partial \theta = 0 \)
(for \( \theta = 0 \) and \( w < r < L \)) [BC 1.1]

along BC: \( K(\partial \phi / \partial x) = 0 \) or \( \partial \phi / \partial x = 0 \)
(for \( x = L \) and \( 0 < y < b \)) [BC 1.2]

along CD: \( \phi = y - h \) [BC 1.3]

along DE: \( K(\partial \phi / \partial \theta) = 0 \) or \( \partial \phi / \partial \theta = 0 \)
(for \( \theta = \pi/2 \) and \( w < r < d \)) [BC 1.4]

along EA: \( \phi = 0 \)
(for \( r = w \) and \( 0 < \theta < \pi/2 \)) [BC 1.5]

To satisfy Laplace's equation [1], [BC 1.1], [BC 1.4], and [BC 1.5] we choose \( \phi \) in the normalized form as

\[
\frac{\phi}{H} = A_{N0} \frac{\ln(r/w)}{\ln(M/w)} + \sum_{m=1}^{N} A_{Nm} \frac{(r/M)^{2m} - [w^2/(Mr)]^{2m}}{1 - (w^2/M^2)^{2m}} \cos 2m\theta
\]

[2]

Notice that \( \phi/H \) as given by the right side of [2] satisfies [BC 1.1], [BC 1.4], and [BC 1.5] exactly for any \( A_{Nm} \).

To satisfy [BC 1.2] we first write the potential ratio \( \phi/H \) of [2] in cartesian form, getting (because we have \( r^2 = x^2 + y^2 \) and \( \theta = \cot^{-1}(x/y) \)) the expression
\[
\phi = A_{NO} \frac{\ln[(x^2 + y^2)^{1/2}/w]}{\ln(M/w)}
\]

\[
+ \sum_{m=1}^{N \to \infty} A_{Nm} \left\{ \left[ \frac{(x^2 + y^2)^m/M^{2m}}{1 - (w^2/M^2)^{2m}} \right] - \frac{w^{4m}/[M^{2m}(x^2 + y^2)^m]}{1 - (w^2/M^2)^{2m}} \right\} \cdot \cos[2m(\cot^{-1}x/y)]
\]

From [3] we now find as worked out in Appendix A (see especially equation [A 15]) a dimensionless normal derivative form of \( \phi/H \), for a normal perpendicular to a plane \( x = \text{const} \), as

\[
\frac{\partial(\phi/H)}{\partial(x/L)} = A_{NO} \frac{1}{\ln(M/w)} \frac{L}{r} \cos \theta
\]

\[
+ \sum_{m=1}^{N \to \infty} A_{Nm} \frac{2m}{\gamma_m} \frac{L}{r} [(r/M)^{2m} \cos(2m - 1)\theta
\]

\[
+ (w^2/Mr)^{2m} \cos(2m + 1)\theta]
\]

where

\[
\gamma_m = 1 - (w^2/M^2)^{2m}
\]

In the right side of [4] we infer from Fig. 2, for \( r \) terminating at a point on line BC, the relation \( L/r \leq 1 \).

In [4] the left side must be put equal to zero, for \( x = L \), to satisfy [BC 1.2].

Thus, to satisfy [BC 1.2] we find by use of [BC 1.2] in [4] that our \( A_{Nm} \) must be found to satisfy the relation
\[ 0 = A_{NO} \frac{1}{\ln(w/M)} \frac{L}{r} \cos \theta \]

\[ + \sum_{m=1}^{N \to \infty} A_{Nm} \frac{2m}{\gamma_m} \frac{L}{r} (r/M)^{2m} \cos(2m - 1)\theta \]  

\[ + \left[ \frac{(w^2/Mr)^2m}{1 - (w^2/M^2)^{2m}} \right] \cos(2m + 1)\theta \]

Equation [6], if the \( A_{Nm} \) are properly chosen, will be valid for values of \( r \) and \( \theta \) corresponding to all points lying along BC in Fig. 2.

To satisfy [BC 1.3] we find by use of [BC 1.3] in [2] that our \( A_{Nm} \) must be found to satisfy also the relation

\[ \frac{V - h}{H} = A_{NO} \frac{\ln(r/w)}{\ln(M/w)} \]

\[ + \sum_{m=1}^{N \to \infty} A_{Nm} \frac{(r/M)^{2m} - \left[ \frac{w^2}{(Mr)^2} \right]^{2m}}{1 - (w^2/M^2)^{2m}} \cos 2m\theta \]

for values of \( r \) and \( \theta \) corresponding to all points lying along CD in Fig. 2. These values are denoted by \( R_2 \) and \( \theta_2 \) and are shown in Appendix B, Fig. 26.

In terms of \( R_1 \) and \( \theta_1 \) equation [6] becomes

\[ 0 = A_{NO} \frac{1}{\ln(M/w)} \frac{L}{R_1} \cos \theta_1 \]

\[ + \sum_{m=1}^{N \to \infty} A_{Nm} \frac{2m}{\gamma_m} \frac{L}{R_1} \left( \frac{R_1/M}{2} \right)^{2m} \cos(2m - 1)\theta_1 \]
+ [(w^2/(MR_1)]^{2m} \cos(2m + 1) \theta_1^{}}

Equation [8] is valid over BC of Fig. 2.

In terms of \(R_2\) and \(\theta_2\) of Fig. 26 of Appendix B, equation [7] becomes

\[
\frac{R_2 \sin \theta_2 - h}{H} = A_{NO} \frac{\ln(R_2/w)}{\ln(M/w)} + \sum_{m=1}^{N+\infty} A_{Nm} \frac{(R_2/M)^{2m} - [w^2/(MR_2)]^{2m}}{1 - (w^2/M^2)^{2m}} \cos 2m \theta_2
\]

Equation [9] is valid over CD of Fig. 2.

We may now write [6] and [7] in terms of \(s_1\) and \(s_2\) shown in Fig. 26 of Appendix B. From Appendix B we have the relations

\[
\frac{L}{R_1} \cos \theta_1 = \frac{L^2}{L^2 + s_1^2}
\]

\[
\theta_1 = \arctan \left( \frac{s_1}{L} \right)
\]

\[
R_1 = (L^2 + s_1^2)^{1/2}
\]

We put [10], [11] and [12] in [8] and get

\[
0 = A_{NO} \frac{1}{\ln(M/w)} \frac{L^2}{L^2 + s_1^2} + \sum_{m=1}^{N+\infty} A_{Nm} \frac{2m}{Y_m} \frac{L}{(L^2 + s_1^2)^{1/2}}
\]
\begin{align*}
\left\{ \frac{L^2 + s_1^2}{M} \right\}^{1/2} & \cos[(2m - 1) \arctan \frac{s_1}{L}] \\
+ \left\{ \frac{w^2}{M(L^2 + s_1^2)^{1/2}} \right\}^{2m} \cos[(2m + 1) \arctan \frac{s_1}{L}] \right\} \quad [13]
\end{align*}

valid over BC of Fig. 2, i.e.

\[ 0 < s_1 < b \]

From Appendix B we have

\[ R_2 \sin \theta_2 = \delta - s_2 \sin \alpha \]

[14]

\[ \theta_2(s) \equiv \theta_2 = \arctan \frac{\delta - s_2 \sin \alpha}{\lambda - s_2 \cos \alpha} \]

[15]

\[ R_2(s) \equiv R_2 = \left[ (\delta - s_2 \sin \alpha)^2 + (\lambda - s_2 \cos \alpha)^2 \right]^{1/2} \]

[16]

By use of [14], [15] and [16], we may write [9] as

\[ \frac{\delta - s_2 \sin \alpha - h}{H} = A_{NO} \frac{\ln[R_2(s)/w]}{\ln(M/w)} \]

\[ + \sum_{m=1}^{N \to \infty} A_{Nm} \frac{[R_2(s)/M]^{2m} - \left\{ w^2/[MR_2(s)] \right\}^{2m}}{1 - \left( w^2 / M^2 \right)^{2m}} \cos[2m\theta_2(s)] \]

[17]

valid over CD

In [13] and [17] we define \( f(s) \) by
\[ f(s) = \begin{cases} 
0, & 0 < s < b \\
\delta - \frac{s_2 \sin \alpha - h}{H}, & b < s < b + \sigma 
\end{cases} \]  \[ \text{[18]} \]

In [13] and [17] we define \( u^0(s) \) by

\[ u^0(s) = \begin{cases} 
\frac{1}{\ln(M/w)} \frac{L^2}{L^2 + s^2}, & 0 < s < b \\
\frac{\ln[R_2(s)/w]}{\ln(M/w)}, & b < s < b + \sigma 
\end{cases} \]  \[ \text{[19]} \]

where \( R_2(s) \) is given by [16].

In [13] and [17] we define \( u^m(s) \) by

\[ u^m(s) = \begin{cases} 
\frac{2m}{\gamma_m} \frac{L}{(L^2 + s^2)^{1/2}} \left[ \frac{(L^2 + s^2)^{1/2}}{L^2 + s^2} \right]^{2m} 
\times [R_2(s)/M]^{2m} - \frac{w^2/[MR_2(s)]}{1 - (w^2/M^2)^{2m}} \cos[2m\theta_2(s)], & b < s < b + \sigma
\end{cases} \]  \[ \text{[20]} \]

where \( R_2(s) \) is given by [16] and \( \theta_2(s) \) by [15].

By use of [18], [19] and [20] and Kirkham and Powers (1972, Appendix 2) we can now get the \( A_{Nm} \) of [13] and [17] and we shall find zeroth, first, second, ..., approximations...
fo(s), f1(s), f2(s), etc., of f(s) given by

\[ f_0(s) = A_{00}u_0(s) \]  \hspace{1cm} (N = 0) \hspace{1cm} [21]

\[ f_1(s) = A_{10}u_0(s) + A_{11}u_1(s) \]  \hspace{1cm} (N = 1) \hspace{1cm} [22]

\[ f_2(s) = A_{20}u_0(s) + A_{21}u_1(s) + A_{22}u_2(s) \]  \hspace{1cm} (N = 2) \hspace{1cm} [23]

etc.

A computer subroutine developed by Boast (1969) was used to get the \( A_{Nm} \). These \( A_{Nm} \), for different N, are substituted into equation [2] for points along the boundary BCD of Fig. 2.

As specific example we take in Fig. 3, b = 0.2333, d = 0.1333, L = 1.0 and w = 0.0167. To see how close the approximations are, prepared graphs of the left side versus the right side of equations [6] and [7], for \( N = 0, 1, 5, 10, \) and \( 15 \) are presented in Fig. 3. In Fig. 3, the circled points are for the right side, and the solid line for the left side of equations [6] and [7]. It is evident from Fig. 3 that the second and third boundary conditions become satisfied as N increases. For our work on this problem we use \( N = 10 \) or \( 15 \) as indicated in Table 4 of Appendix C.

Flow Nets

Equation [2] gives the potential function \( \phi \). Using the Cauchy-Riemann relation (see equation [7] Kreyszig (1967) page 545) \( \partial\phi/\partial r = (1/r)(\partial\psi/\partial \theta); \partial\psi/\partial r = -(1/r)\partial\phi/\partial \theta \)
Fig. 3. Approximations of boundary conditions [BC 1.2] and [BC 1.3] using $N = 0$, $N = 1$, $N = 5$, $N = 10$, and $N = 15$ in the right side of equations [6] and [7]. With increasing $N$, the circles should and do approach the solid line.
together with equation [2] we obtain the following expression for the stream function

\[ \psi = \frac{A_{NO}}{\ln(M/w)} + \sum_{m=1}^{\infty} A_{Nm} \frac{(r/M)^{2m} + \frac{[w^2/(Mr)]^{2m}}{1 - (w^2/M^2)^{2m}}}{\sin 2m\theta} \]  \[24\]

where an arbitrary constant may be added.

Instead of using \( \phi \) and \( \psi \) as given by equations [2] and [24], we will obtain flow nets using equipotential lines and streamlines calculated as a function of the total hydraulic-head loss through the medium and a fraction of the total flow through the medium, respectively. We use the expression

\[ \phi' = \frac{\phi(x,y) - \phi(0,h)}{\phi(L,b) - \phi(0,h)} \]  \[25\]

for the equipotential lines and the expression

\[ \psi' = \frac{\psi_{\text{max}} - \psi(x,y)}{\psi_{\text{max}} - \psi(L,b)} \]  \[26\]

for the streamlines where the value \( \psi_{\text{max}} \) is the maximum values of \( \psi \) for a flow net; \( \phi' \) and \( \psi' \) have values between 0 and 1.

To obtain \( \psi_{\text{max}} \), we use equation [24] and determine value of \( \psi \) along boundary CD of Fig. 2. By inspection of these points we find that our \( \psi_{\text{max}} \) is located at point \((0,d)\) of our flow region. For our example of Fig. 3 we find \( \psi_{\text{max}} = 0.0770K \) (where the coefficient 0.0770 has units of length).

A flow net for our example of Fig. 3 \((d = 0.1333, b = \)
0.2333, L = 1.0, w = 0.0167) is presented in Fig. 4a where equipotentials for \( \phi' = 0.2, 0.4, 0.6, 0.8 \) and streamlines for \( \psi' = 0, 0.2, 0.4, 0.6, 0.8, 1.0 \) are drawn. The equipotential for \( \phi' = 0 \) is the radius of the tile drain and \( \phi' = 1.0 \) is the point \((L,b)\). It is evident from this figure that the equipotentials and the streamlines intersect at right angles. At points along the soil surface the streamlines are not perpendicular to the soil surface, because the soil surface is not an equipotential.

To obtain \( \phi' \) and \( \psi' \) values to plot the flow nets a technique based on the Newton-Raphson method for finding the root of a nonlinear equation (Scarborough, 1962, p. 199-200) is used. The calculations were done by the IBM 360 computer. The computer programs are in Appendix C.

**Amount of water seeping into the soil**

We want to determine the amount of rainfall needed to keep the flow medium saturated; and also the quantity of water discharged through the tile drain. From this information we can determine the quantity of water moving through the soil in relation to the rainfall rate.

The quantity of water passing through the flow medium and entering the tile is

\[
Q = \psi(0,d) - \psi(L,b) 
\]
Fig. 4a - Normalized flow net corresponding to the section OABCDE of Fig. 2 for 
\(d = 0.1333, b = 0.2333, L = 1.0, \) and \(w = 0.0167;\) streamlines (indicated 
by arrows) are drawn such that 0.2 of the total flow of water into the 
soil passes between an adjacent pair of streamlines; between adjacent 
equipotentials 0.2 of the total head is dissipated.
STREAMLINES 
\( \psi' = \text{CONST.} \)

EQUIPOTENTIALS 
\( \phi = \text{CONST.} \)

B5 
\( h = w = 0.0167 \)
Fig. 4b - The stream function $\psi$ as a function of distance $x$ at points along the soil surface of Fig. 4a.
Fig. 4c - The vertical velocity $V_y$ along the soil surface of Fig. 4a as a function of surface position.
VERTICAL VELOCITY, $v_y$

SURFACE POSITION, $x$
We wish to compute the uniform minimum rainfall rate, \( R_{\text{min}} \), needed to keep the soil water-saturated, as is required by our boundary condition \([\text{BC 1.3}]\). By continuity, if all rain falling at a constant rate \( R \), between \( x = L \) and \( x = x \), seeps into the soil, we have

\[
(L - x)R = \psi_{x,y} - \psi_{L,b}, \quad \text{condition 1,} \quad [27b]
\]

where \((x,y)\) represents the coordinates of a point on the soil surface. Analysis of the curves of \( \psi \) versus \( x \) for all cases we have considered shows that for condition 1 to hold the following two conditions must be satisfied.

\[
R_{L,b} \geq (v_y)_{L,b}, \quad \text{condition 2,} \quad [28a]
\]

\[
RL \geq Q, \quad \text{condition 3,} \quad [28b]
\]

In condition 2 we have \((v_y)_{L,b} = -(\partial \psi / \partial x)_{L,b}\) for computation.

While using condition 1 we have applied conditions 2 and 3 to all our curves of \( v_y \) versus \( x \) and \( \psi \) versus \( x \). Also we have selected the greatest value, either \( R_{L,b} \) or \( R \), given by \( R_{L,b} = (v_y)_{L,b} \) and \( R = L/Q \). This selected value is denoted \( R_{\text{min}} \) and \( R_{\text{min}} \) is the value needed to just keep the flow medium water-saturated.

We calculate an example for getting \( R_{\text{min}} \). For the case of Fig. 4a we find, by evaluating the right side of
[28a], the result \( R_{L,b} \geq 0.1754K \). From equation [27a] we compute for Fig. 4a the result \( Q/L = 0.0834K \). Of the values 0.1754K and 0.0834K we see that the greater is 0.1754K which gives \( R_{\text{min}} = 0.1754K \). One may verify from Fig. 4a and 4b that condition 1 is satisfied when we take \( R = R_{\text{min}} = 0.1754K \).

The percent of rainwater, \( W_p \), falling on the soil surface and passing through the flow medium is given by

\[
W_p = \frac{Q}{(R_{\text{min}} L)} \times 100
\]  

[28c]

For our example illustrated in Fig. 4a \( W_p = \frac{0.0843KL}{(0.1754KL)} \times 100 = 48.0\% \). The rest of the rainfall, 52.0\%, is surface runoff.

By inspection of the curve in Fig. 4c we can determine if we have any water seeping upward out of the soil along the slope of the soil surface. This happens at any place along the soil slope where \( \psi/\psi_x \) changes sign from negative to positive. We calculate for our example in Fig. 4a that water emerges at the soil surface between \( x = 0.515 \) and \( 0.625 \). To determine the amount of water \( Q_u \) that seeps out of the soil between these two points we calculate the amount

\[
Q_u = \psi_x = 0.515 - \psi_x = 0.625
\]  

[28d]

of water seeping out of the soil. For our case in Fig. 4a we have \( Q_u = \psi_x = 0.515 - \psi_x = 0.625 = 0.0187KL - 0.0182KL = 0.0005KL \). This represents \( (0.0005KL/0.1754KL) \times 100 = \).
0.29% of the minimum rainfall $R_{\text{min}}$ falling on the soil surface. We may define $Q_u$ by $Q_u = \psi_{x_1} - \psi_{x_2}$ where $x_1$ is the downslope point on the soil surface where upward flow ceases and $x_2$ is the upslope point on the soil surface where upward flow begins. In the above example $x_1 = 0.515$, $x_2 = 0.625$ and $Q_u = 0.0005KL$. Another point $x_3$ is defined as the point on the soil surface upslope of $x_1$ and $x_2$ and on the same streamline as $x_1$. In our example $x_3 = 0.715$. The water flowing between the streamline connecting $x_1$ and $x_3$, and the soil surface is equal to $Q_u$. This is the net quantity of water which seeps into the soil surface upslope and emerges through the soil surface downslope. The points located on the soil surface at $x_1$ and $x_2$ are critical points. These points are located where water is neither entering nor leaving the soil surface, that is, where we have $v_y = 0$.

Results and Discussion

A variety of soil bedding geometries, cases, A4, B4, ..., (Table 1) were examined. The cases were classified into groups 1, 2, 3, 4, and 5 as shown in Table 1. The values of the parameters $L$, $d$, $b$, $w$, and $H$ defined in Fig. 1b, are given. The bedding slope $c = (b - d)/L$ was calculated from these parameters, and it is also given. Finally, the table gives values of $Q/(KL)$, $R_{\text{min}}$, $W_p$ and $Q_u/(KL)$ for each case. We note the relation $W_p = (Q/L)/R$ in agreement with equation
Table 1. Values of the quantities $R_{\text{min}}/L$, $Q/(KL)$, $W_p$, and $Q_u/(KL)$ for several soil bedding geometries when $h = w$ (in Fig. 2) and $H = b - w$.

<table>
<thead>
<tr>
<th>Soil Bedding Geometries</th>
<th>Case</th>
<th>$d/L$</th>
<th>$b/L$</th>
<th>$w/L$</th>
<th>$H/L$</th>
<th>$c$</th>
<th>$R_{\text{min}}/K$</th>
<th>$Q/(KL)$</th>
<th>$W_p$ (%)</th>
<th>$Q_u/(KL)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A1</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.100</td>
<td>0.300</td>
<td>0.2000</td>
<td>0.3450</td>
<td>0.2017</td>
<td>58.5</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>B1</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.050</td>
<td>0.350</td>
<td>0.2000</td>
<td>0.3605</td>
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</tr>
<tr>
<td>C1</td>
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<td>0.4000</td>
<td>0.020</td>
<td>0.380</td>
<td>0.2000</td>
<td>0.3762</td>
<td>0.1263</td>
<td>33.6</td>
<td>-</td>
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</tr>
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<td>D1</td>
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<td>0.010</td>
<td>0.390</td>
<td>0.2000</td>
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<tr>
<td>E1</td>
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<td>0.4000</td>
<td>0.001</td>
<td>0.399</td>
<td>0.2000</td>
<td>0.3990</td>
<td>0.0645</td>
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<td>-</td>
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</tr>
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<td>0.0100</td>
<td>0.1230</td>
<td>0.0330</td>
<td>0.0780</td>
<td>0.0565</td>
<td>70.7</td>
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</tr>
<tr>
<td>B2</td>
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<td>0.0167</td>
<td>0.1163</td>
<td>0.0330</td>
<td>0.0655</td>
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<td>-</td>
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<td>0.0330</td>
<td>0.1000</td>
<td>0.0330</td>
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<td>-</td>
<td></td>
</tr>
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<td>D2</td>
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<td>0.1880</td>
<td>0.0167</td>
<td>0.1713</td>
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<td>-</td>
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</tr>
<tr>
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<td>0.2330</td>
<td>0.0167</td>
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<td>0.0330</td>
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<tr>
<td>F2</td>
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<td>Group 3</td>
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</tr>
<tr>
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<td>0.3670</td>
<td>0.0100</td>
<td>0.3570</td>
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<td>0.3670</td>
<td>0.0330</td>
<td>0.3340</td>
<td>0.0670</td>
<td>0.1787</td>
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<tr>
<td>D3</td>
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<td>0.0167</td>
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Table 1 - continued

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<tr>
<th>Case</th>
<th>d/L</th>
<th>b/L</th>
<th>w/L</th>
<th>H/L</th>
<th>c</th>
<th>$R_{min}/K$</th>
<th>$Q/(KL)$</th>
<th>$W_p(%)$</th>
<th>$Q_u/(KL)$*</th>
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<td>0.5000</td>
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<td>0.4900</td>
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<td>0.5000</td>
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<td>0.4833</td>
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<td>0.5000</td>
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<td>0.5000</td>
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<td>0.0843</td>
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</table>

* Values of $x_1$, $x_2$, and $x_3$ associated with $Q_u/(KL)$ and defined below equation [28d] are given in Appendix C, Table 6.
Values of $x_1$, $x_2$, and $x_3$ associated with values of $Q_u$ where $Q_u$ exists are given in Appendix C, Table 6. Flow nets for the cases of group 1 are given in Fig. 5; group 2 in Fig. 6; group 3 in Fig. 7; group 4 in Fig. 8; and group 5 in Fig. 9. In addition, cases F2, F3, D4, and E4 ($w = 0.005$) are given in Fig. 10. These many nets are presented to illustrate how the depth of the impermeable barrier, slope, and radius of the tile drain affect the flow net.

Table 1 and the figures bring out the influence of different parameters on $R_{min}$, $Q/L$ and $W_p$. We note the relation $W_p = (Q/L)/R_{min}$.

**Effect of increasing slope of soil bedding** Examination of the values of $Q/L$ for cases A5 through D5 shows that increasing the slope $c$ of the soil bedding while keeping $L$, $d$, and $w$ constant increases the quantity of water flowing through the flow medium to the drain. It also increases the minimum rate of rainfall needed to keep the soil saturated. The percent of the rainfall which flows to the tile drain decrease from 100% to 40.1% as the slope is increased.

**Effect of increasing depth of soil bedding** Examination of the values of $Q/L$ for cases B2, D2, and E2 shows that increasing the depth of $b$ and $d$ of the soil bedding while keeping $L$, $w$, and $c$ constant increases the quantity of water flowing through the flow medium to the drain. It also
Fig. 5 - Flow nets for semisections of soil bedding, cases A1, B1, C1, D1, and E1 of Table 1.
Fig. 6 - Flow nets for semisections of soil bedding, cases A2, B2, C2, D2, and F2 of Table 1.
Fig. 7 - Flow nets for semisections of soil bedding, cases A3, B3, C3, D3, E3, and F3 of Table 1.
Fig. 8 - Flow nets for semisections of soil bedding, cases A4, B4, C4, and D4 of Table 1.
Fig. 9 - Flow nets for semisections of soil bedding, cases A5, B5, C5, D5, E5, and F5 of Table 1.
Fig. 10 - Flow nets for semisections of soil bedding, cases F2, F3, D4, and E4 of Table 1.
increases the amount of rainfall needed to keep the soil saturated. One hundred percent of $(R_{\text{min}} \cdot L)$ flows through the flow medium to the tile drain. For Table 1 we take $h = w$.

**Effect of increasing the radius of the tile drain**

Examination of the values of $Q/L$ for all cases in group 1 and the first three cases of groups 2, 3, and 4 shows that increasing the radius $w$ of the tile drain keeping $L$, $d$, $b$, and $c$ constant increases the quantity of water flowing through the flow medium to the drain. As we increase the radius of the tile drain we also increase the percent of water flowing through the flow medium into the tile drain. The rate $R_{\text{min}}$ for the noted cases may decrease or increase with increasing $w$ because $R_{\text{min}}$ depends on the head difference $H = b - w$ across the system as well as the tile radius $w$. We remember for Table 1 we have $h = w$.

We can check our work with that of Kirkham (1947a) using the following equation from his Fig. 3.

\[
Q_3 = \frac{2\pi K(h + t - w)}{\ln \cot \frac{\pi w}{4h} + \sum_{n=1}^{\infty} \frac{\ln}{\frac{\cosh \frac{\pi n L}{2h}}{\cos \frac{\pi w}{2h}} - \frac{\cosh \frac{\pi n L}{2h}}{\cos \frac{\pi w}{2h}}}}
\]

\[29\]

If $L/2 > h$, $\Sigma \rightarrow 0$

where $t$ is the depth of ponded water on the level surface of
soil and \( h \) is depth of soil overlying the impermeable barrier in Kirkham (1947a). This will correspond to our case if we let the amount of ponded water standing on the surface be zero \((t = 0)\) for Kirkham (1947a) and we let the slope in our case be zero \((c = 0)\).

To check the equation with our work we let \( L = 1.0, \ h = 0.2, \ t = 0, \) and \( w = 0.005 \) for Kirkham (1947a) and we calculate \( Q_3 \) from equation [29] and find it to be \( Q_3 = 0.0779K \). In our equation [27a] we let \( L = 1.0, \ b = d = 0.2, \) and \( w = 0.005 \) and calculate \( Q/L \) to be equal to 0.0779K. This agrees exactly with Kirkham (1947a). The agreement is found to occur for several other cases considered but not reported here.

**Summary and Conclusions**

We have solved the problem of the seepage of rainfall seeping through soil bedding to a tile drain. The tile drain is half-imbedded in an impermeable barrier located at a finite depth below the soil surface. The flow medium is saturated with water and we consider only the steady-state condition. We solved the problem using the modified Gram-Schmidt method developed by Kirkham and Powers (1972). The quantity of water flowing through the flow medium into the tile drain has been calculated. The minimum rainfall rate needed to keep the soil bedding saturated has also been calculated, also, the percent of the rainfall that flows
through the flow medium to the tile drain.

We have determined that by increasing the soil slope, increasing the depth to the impermeable barrier, and/or increasing the radius of the tile drain the quantity of water flowing through the flow medium to the tile drain will be increased. The rainfall rate needed to keep the bedded land saturated will be increased or decreased depending on the bedding geometry.

We found that our work agrees exactly with the work of Kirkham (1947a) where he considered the case of ponded water on a zero slope soil surface.

Flow nets for several soil bedding geometries have been presented. From these flow nets and our mathematical analysis we have determined that in some cases the water which enters the soil at the surface flows to the tile drain and does not resurface downslope. In other cases water enters the soil upslope and resurfaces again downslope.
PART II. FLOW PATTERNS OF RAINWATER SEEPPING
THROUGH BEDDED SOIL OF INFINITE DEPTH

Introduction

Warrick (1970) solved a problem of seepage of rainwater through soil bedding underlain by a barrier at infinite depth. Powers et al. (1967) solved the same problem for the barrier at a finite depth. Both Powers et al. (1967) and Warrick (1970) considered the depth of ponded water in the furrows of the bedded land to be negligible and the soil surface had a constant slope. Selim and Kirkham (1972a) extended the work of Powers et al. to include a finite depth of water in the furrows. Selim and Kirkham (1972b) also extended the work to the case of an arbitrarily shaped soil surface.

Warrick (1970) used conformal transformations to find the hydraulic head $\phi$ and stream functions $\psi$. Here, we solve the problem of Warrick (1970) using the modified Gram-Schmidt method of Powers et al. (1967) and Kirkham and Powers (1972). We also extend the work of Warrick (1970) to include a finite depth of water in the furrows and the case of an arbitrarily shaped soil surface.

The flow medium first considered is illustrated in Fig. 11. Rain, falling at rate $R$, keeps bedded soil water-saturated. Some of the rain infiltrates into the soil along streamlines similar to the one $S$ shown. Rain not needed to water-saturate the soil runs over the surface of the bedding.
Fig. 11 - A two-dimensional drawing of steady rain falling on soil bedding overlying an impermeable barrier at infinity; the bedding slope is constant and no water stands in the furrow.
RAINFALL RATE \( R \)

DRAINAGE FURROW

SOIL SURFACE

LINES OF SYMMETRY

LINES OF SYMMETRY

BARRIER
to a furrow which is sufficiently drained such that ponding is negligible. The flow region extends to an infinite distance perpendicular to the (x,y) plane of Fig. 11 and overlies an impermeable barrier at infinite depth below the soil surface. We consider only a unit thickness of flow medium perpendicular to the (x,y) plane. Because of symmetry we need only consider the part designated by OEABO. The total width of this portion of the flow medium is L.

The second case we shall consider is illustrated in Fig. 12, which is similar to Fig. 11 except water is allowed to stand in the drainage furrow. Because of symmetry we need only consider the part designated by ABCOEA. The total width of the flow medium considered is L. The depth of water in the drainage furrow is h, and the maximum width of this water is d.

The third case we shall consider is illustrated in Fig. 13, which is similar to Fig. 11 and 12 except the soil surface is taken as part of the boundary of an ellipse. We will consider both no water and some water standing in the drainage furrow.

For the three cases shown in Fig. 11, 12 and 13 we seek the potential function $\phi$, and the stream function $\psi$. From these functions we will be able to calculate the amount of water flowing through the soil and draw flow nets. The amount of water flowing through the soil and the streamlines
are important factors governing movement of dissolved materials in the soil.
Fig. 12 - A two-dimensional drawing of steady rain falling on soil bedding overlying an impermeable barrier at infinity; the bedding slope is constant and some water stands in the furrow.
RAINFALL RATE $R$

SOIL SURFACE

LINES OF SYMMETRY

BARRIER
Fig. 13 - A two-dimensional drawing of steady rain falling on soil bedding overlying an impermeable barrier at infinity; the bedding surface has an elliptic shape and some water stands in the furrow.
Mathematical Analysis

Since the flow regions are assumed to be water-saturated and isotropic, we may consider a hydraulic head function \( \phi(x,y) \) and stream function \( \psi(x,y) \). These two orthogonal functions each satisfy Laplace's equation

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad [30]
\]

and

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad [31]
\]

First we solve equation [30] for \( \phi \) and then find the stream function \( \psi \) from the function \( \phi \) using the Cauchy-Riemann relations.

\( \phi \) and \( \psi \) for the flow region of Fig. 11

Taking the x-axis to be the reference level for \( \phi \), the boundary conditions (BC's) in Fig. 11 for equation [30] are

Along OE
\[
K(\partial \phi / \partial x) = 0 \quad \text{or} \quad \partial \phi / \partial x = 0 \quad \text{[BC 2.1]}
\]

Along EA
\[
K(\partial \phi / \partial y) = 0 \quad \text{or} \quad \partial \phi / \partial y = 0 \quad \text{[BC 2.2]}
\]

Along AB
\[
K(\partial \phi / \partial x) = 0 \quad \text{or} \quad \partial \phi / \partial x = 0 \quad \text{[BC 2.3]}
\]
Along OB
\[ \phi = c x \]  
[BC 2.4]
where \( c = b/L \) and \( K \) is the saturated hydraulic conductivity of the flow medium (units of length/time).

For later use in connection with Fig. 11, 12 and 13 we define the functions \( f(x) \), \( F(x) \), \( g(x) \), and \( G(x) \) by

\[
f(x) = cx \quad 0 \leq x \leq L \]  
[32]

\[
F(x) = \begin{cases} 
  \bar{h} & 0 \leq x \leq d \\
  cx & d \leq x \leq L 
\end{cases} \]  
[33]

\[
g(x) = \begin{cases} 
  0 & 0 \leq x \leq t \\
  v(x) & t \leq x \leq L 
\end{cases} \]  
[34]

\[
G(x) = \begin{cases} 
  \bar{h} & 0 \leq x \leq d \\
  v(x) & d \leq x \leq L 
\end{cases} \]  
[35]

For Fig. 11, a solution \( \phi \) of equation [30] which satisfies [BC 2.1], [BC 2.2], and [BC 2.3] is

\[
\phi = \sum_{m=0}^{N \to \infty} A_{Nm} \frac{e^{m \pi y/L}}{e^{m \pi b/L}} \cos \frac{m \pi x}{L} 
\]

\[
m = 0, 1, \ldots, N \quad (N \to \infty) \]  
[36]

where the constants \( A_{Nm} \) are to be evaluated to satisfy
[BC 2.4]. The constants $A_{Nm}$ have dimensions of length.

To satisfy [BC 2.4], the $A_{Nm}$ of equation [36] must satisfy

$$f(x) = \sum_{m=0}^{N\to\infty} A_{Nm} \frac{e^{m\pi f(x)/L}}{e^{m\pi b/L}} \cos \frac{m\pi x}{L}$$  \[37\]

We define $u_m(x)$ as

$$u_m(x) = \frac{e^{m\pi f(x)/L}}{e^{m\pi b/L}} \cos \frac{m\pi x}{L}$$  \[38\]

so that equation [37] becomes

$$f(x) = \sum_{m=0}^{N\to\infty} A_{Nm} u_m(x)$$  \[39\]

The $A_{Nm}$ of equation [39] may be obtained by using the modified Gram-Schmidt process of Kirkham and Powers (1972). It should be noted that in order to determine the $A_{Nm}$, using this technique, the following integrals

$$w_m = \int_0^L f(x) u_m(x) \, dx$$  \[39a\]

$$u_{mn} = \int_0^L u_m(x) u_n(x) \, dx$$  \[39b\]

are required. These integrals, equations [39a] and [39b], were evaluated numerically using Simpson's rule as given by
Hildebrand (1956). In our numerical work in this paper we use \( N = 20 \) in equation [36] and associated equations. It is found that this value of \( N \) satisfies the remaining boundary condition [BC 2.4].

To compute the stream function \( \psi \), we use equation [36] in conjunction with the Cauchy-Riemann conditions \( \partial K_\phi / \partial x = \partial \psi / \partial y; \partial K_\phi / \partial y = -\partial \psi / \partial x \), see Kirkham and Powers (1972) Table 3-1]. The result is

\[
\psi = - \sum_{m=0}^{N+\infty} K A_{Nm} e^{nym/L} \sin \frac{nmx}{L} \quad [40]
\]

Fig. 14 is a normalized flow net for the example of Fig. 11, where equipotentials for \( \phi' = 0.2, 0.4, 0.6, 0.8 \) and streamlines for \( \psi' = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0 \) are drawn, with \( \phi' \) defined by

\[
\phi' = \frac{\phi(x,y) - \phi(0,0)}{\phi(L,b) - \phi(0,0)} \quad [41]
\]

and \( \psi' \) by

\[
\psi' = \frac{\psi_{\text{min}} - \psi(x,y)}{\psi_{\text{min}} - \psi(0,0)} \quad [42]
\]

where the value \( \psi_{\text{min}} \) is the minimum value of \( \psi \) for a flow net. In equation [41], values of \( \phi(x,y) \) are obtained from equation [36] where the \( A_{Nm} \) are given in Appendix C Table 5. In equation [42], values of \( \psi(x,y) \) are obtained from equation [40].
Fig. 14 - Normalized flow net corresponding to the section OEABO of Fig. 11; streamlines (indicated by arrows) are drawn such that 0.2 of the total flow in and out of the soil passes between an adjacent pair of the streamlines; between adjacent equipotentials 0.2 of the total head is dissipated; the streamline (point) $\psi' = 0$ is such that to its left the seepage flow is upward out of the soil and to its right the seepage flow is downward into the soil.
To obtain $\psi_{\text{min}}$, we put $y = cx$ in equation [40] and plot the values of $\psi$ along $y = cx$ and determine the minimum value of $\psi$ from the graph. For Fig. 14, with the geometrical values as indicated, we find $\psi_{\text{min}}/KL = -0.0248$. This gives $\psi' = 0$ in Fig. 14.

It is of interest to know the minimum rainfall rate $R_{\text{min}}$ necessary to keep the soil bedding saturated. If the soil can be kept saturated at the center of the bedding $(L,b)$, the remainder of the bedding will be saturated. To keep the soil saturated at $(L,b)$, the rainfall rate must be at least equal to the vertical velocity at $(L,b)$. This vertical velocity is given by $v = K(\partial \phi / \partial y)$ evaluated at $(L,b)$. Using the Cauchy-Riemann relation, $\partial K \phi / \partial y = -\partial \psi / \partial x$, we can write the vertical velocity at $(L,b)$, which is also $R_{\text{min}}$, as

$$R_{\text{min}} = -\left. \frac{\partial \psi}{\partial x} \right|_{L,b}$$

which for the example of Fig. 14 is

$$R_{\text{min}} = 0.18K$$

The total amount of water falling on the soil (assuming uniform recharge) is given by $LR_{\text{min}}$.

The quantity of water $Q$ moving through the soil (that moves in and then out) is the difference between $\psi_{\text{min}}$ and $\psi$ at $(L,b)$. For our example $\psi$ at $(L,b)$ is zero. Therefore, $Q$ is
The percent of water $W_p$ from the product $LR_{min}$ that actually moves through the soil is given by

$$W_p = \frac{Q}{(L R_{min})} \times 100$$

which for Fig. 14 is $\frac{(0.0248K)}{[(1)(0.18K)]} \times 100 = 13.8\%$. The rest of the rainfall, 86.2\%, is surface runoff.

### $\phi$ and $\psi$ for the flow region of Fig. 12

The boundary conditions (BC's) for Laplace's equation [30] for the flow region of Fig. 12 are

Along OE
$$K(\partial \phi/\partial x) = 0 \quad \text{or} \quad \partial \phi/\partial x = 0 \quad \text{[BC 3.1]}$$

Along EA
$$K(\partial \phi/\partial y) = 0 \quad \text{or} \quad \partial \phi/\partial y = 0 \quad \text{[BC 3.2]}$$

Along AB
$$K(\partial \phi/\partial x) = 0 \quad \text{or} \quad \partial \phi/\partial x = 0 \quad \text{[BC 3.3]}$$

Along OCB
$$\phi = F(x), \quad \text{(see equation [33])} \quad \text{[BC 3.4]}$$

The solution of equation [30] which satisfies [BC 3.1], [BC 3.2], [BC 3.3], and [BC 3.4] may be obtained in the same manner as $\phi$ of equation [36], and is
\[
\phi = \sum_{m=0}^{N \to \infty} B_{Nm} \frac{e^{m \pi y/L}}{e^{m \pi b/L}} \cos \frac{m \pi x}{L} \quad [46]
\]

where the constants \( B_{Nm} \) must satisfy the equation

\[
F(x) = \sum_{m=0}^{N \to \infty} B_{Nm} \frac{e^{m \pi f(x)/L}}{e^{m \pi b/L}} \cos \frac{m \pi x}{L} \quad [47]
\]

The \( B_{Nm} \) of equation [47] are obtained by using the modified Gram-Schmidt process of Kirkham and Powers (1972). This is the same process used to obtain the \( A_{Nm} \) of equation [36]. Here, as for the \( A_{Nm} \), \( N \) was taken equal to 20. In Appendix C the \( B_{Nm} \) for geometries considered are tabulated.

The stream function \( \psi \) associated with \( \phi \) in equation [46] is the same \( \psi \) as given by equation [40], except that the coefficients \( A_{Nm} \) in equation [40] are replaced by \( B_{Nm} \), the values of which we have computed from equation [47]. The \( \psi \) for the flow region of Fig. 12 is given by

\[
\psi = -\sum_{m=0}^{N \to \infty} K B_{Nm} \frac{e^{m \pi y/L}}{e^{m \pi b/L}} \cos \frac{m \pi x}{L} \quad [48]
\]

Flow nets for Fig. 12 can be drawn in the same manner as flow nets for Fig. 11 and one is presented in Fig. 15 Case A7 which is the same as Fig. 14 except for the ponded water. The ponded water makes little difference in the nets of Fig. 14 and Fig. 15 Case A7.
Fig. 15 - Normalized flow nets, cases A7 and B8, corresponding to Fig. 12 and Fig. 13.
\[ \phi = 0.0 \text{(along OC)} \]
\[ \psi' = 0.0 \]
\[ Y = 0.067L \]

Diagram A7

Diagram B8
Using equations [43], [44] and [45], as before, we shall later calculate and record the quantity of water moving through the soil in relation to the rainfall rate.

φ and ψ for the flow region of Fig. 13

The boundary conditions (BC's) for equation [30] for the flow region of Fig. 13 are

Along OE
\[ K(\partial \phi / \partial x) = 0 \quad \text{or} \quad \partial \phi / \partial x = 0 \]  \[ \text{[BC 4.1]} \]

Along EA
\[ K(\partial \phi / \partial y) = 0 \quad \text{or} \quad \partial \phi / \partial y = 0 \]  \[ \text{[BC 4.2]} \]

Along AB
\[ K(\partial \phi / \partial x) = 0 \quad \text{or} \quad \partial \phi / \partial x = 0 \]  \[ \text{[BC 4.3]} \]

Along ODCB
\[ \phi = G(x), \quad \text{(see equation [35])} \]  \[ \text{[BC 4.4]} \]

In our analysis, the upper boundary DCB in Fig. 13 is taken as part of the boundary of an ellipse and is described by the equation
\[ y = v(x) \quad t \leq x \leq L \]  \[ \text{[49]} \]

where \( v(x) \) is the equation of the ellipse, with semimajor axis \( (L - t) \) and semiminor axis \( b \) where \( t \) is the semiwidth of the bottom of the furrow. The remaining part of the upper boundary in Fig. 13, the furrow bottom OD, is given by the equation
\[ y = 0 \quad 0 \leq x \leq t \]  \[ \text{[50]} \]
Three different ellipses will be considered.

A solution \( \phi \) of equation [30] which satisfies [BC 4.1], [BC 4.2], [BC 4.3], and [BC 4.4] may be obtained in the same manner as \( \phi \) of equation [36], and is

\[
\phi = \sum_{m=0}^{N \to \infty} C_{Nm} \frac{e^{m\pi y/L}}{e^{m\pi b/L}} \cos \frac{m\pi x}{L} \tag{51}
\]

where the constants \( C_{Nm} \) must satisfy the equation

\[
G(x) = \sum_{m=0}^{N \to \infty} C_{Nm} \frac{e^{m\pi g(x)/L}}{e^{m\pi b/L}} \cos \frac{m\pi x}{L} \tag{52}
\]

In equation [52] we remember that \( g(x) \) and \( G(x) \) are defined by equations [34] and [35].

The \( C_{Nm} \) of equation [51] are obtained as we earlier obtained the \( A_{Nm} \) and \( B_{Nm} \), that is, by using the modified Gram-Schmidt process of Kirkham and Powers (1972). The \( C_{Nm} \) for \( N = 20 \) are tabulated in Appendix C.

The stream function \( \psi \) associated with \( \phi \) in equation [51] is given by

\[
\psi = -\sum_{m=0}^{N \to \infty} K C_{Nm} \frac{e^{m\pi y/L}}{e^{m\pi b/L}} \cos \frac{m\pi x}{L} \tag{53}
\]

Flow nets for Fig. 13 can be drawn in the same manner as the flow nets for Fig. 11 and 12 and one is presented in Fig. 15, Case B8. Using equations [43], [44], and [45] we can
determine the quantity of water moving through the soil in relation to the rainfall rate. This quantity will be recorded in a later table.

Computer programs used to calculate the $A_{Nm}$, $B_{Nm}$, and $C_{Nm}$ of equations [36], [46], and [51] are found in Appendix C.

Results and Discussion

A variety of soil bedding geometries are examined and are designated as shown in column 1 of Table 2. The values of the parameters $L$, $b$, $d$, and $t$, defined in Fig. 11, 12, and 13, are given in the second to fifth columns of Table 2. The height $h$ of the water in the middle of the drainage furrow and the bedding slope $c = b/L$ were calculated from these parameters, and they are reported in columns six and seven of Table 2. Flow nets for cases A6, B6, and C6 of Table 2 are illustrated in Fig. 16 (case B6 is also illustrated in Fig. 14). The flow net for case A7 of Table 2 is illustrated in Fig. 15. Flow nets for cases A8, B8, and C8 of Table 2 are illustrated in Fig. 17 (case B8 is also illustrated in Fig. 15). Flow nets for cases A9, B9, and C9 of Table 2 are illustrated in Fig. 18. Flow nets for B9, D9, and E9 of Table 2 are presented in Fig. 19. These many flow nets are presented to show how the streamlines change for different bedding geometries and depth of water in the furrows. The minimum rainfall rate $R_{\text{min}}$ to keep the bedding saturated, the
Table 2 - The minimum rainfall rate $R_{\text{min}}$ needed to keep the bedding saturated, and corresponding normalized value of the quantity of water flowing through the soil $Q/L$, and the percent of water $W_p$ falling at the rate $R_{\text{min}}$ and actually seeping into and out of the soil bedding; for several soil bedding geometries.

<table>
<thead>
<tr>
<th>Soil Bedding Geometries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case</td>
</tr>
<tr>
<td>------</td>
</tr>
<tr>
<td><strong>Group 6</strong></td>
</tr>
<tr>
<td>A6</td>
</tr>
<tr>
<td>B6</td>
</tr>
<tr>
<td>C6</td>
</tr>
<tr>
<td><strong>Group 7</strong></td>
</tr>
<tr>
<td>A7</td>
</tr>
<tr>
<td><strong>Group 8</strong></td>
</tr>
<tr>
<td>A8</td>
</tr>
<tr>
<td>B8</td>
</tr>
<tr>
<td>C8</td>
</tr>
<tr>
<td><strong>Group 9</strong></td>
</tr>
<tr>
<td>A9</td>
</tr>
<tr>
<td>B9</td>
</tr>
<tr>
<td>C9</td>
</tr>
<tr>
<td>D9</td>
</tr>
<tr>
<td>E9</td>
</tr>
</tbody>
</table>
Fig. 16 - Flow nets for semisections of soil bedding, cases A6, B6, and C6 of Table 2.
Fig. 17 - Flow nets for semisections of soil bedding, cases A8, B8, and C8 of Table 2.
Fig. 18 - Flow nets for semisections of soil bedding, cases A9, B9, and C9 of Table 2.
Fig. 19 - Flow nets for semisections of soil bedding, cases B9, D9, and E9 of Table 2.
flux $Q/L$ of water flowing through the soil, and the percent of water $W_p$ that falls at the rate $R_{\text{min}}$ and actually seeps into and out of the soil bedding are given in the last three columns of Table 2. The units of measurements for $L$, $b$, $d$, $t$ and $h$ can be feet, etc., with $Q$, $R_{\text{min}}$ and $K$ consistent.

Upon examining the flow nets we note certain distinctive features. The point $\psi_{\text{min}}$ is always located along the slope of the soil surface between $(d,h)$ and the top of the slope $(L,b)$. It is never located at either extreme position. The position of $\psi_{\text{min}}$ is a critical point. Between this critical point and the bottom of the slope water is seeping out of the soil. Excess water not needed to keep the soil saturated simply runs over the surface of the soil. All of the rain falling on the soil slope at and below the critical point runs over the surface to the furrow. It never enters the soil.

As a check of our work we examine the flow nets. The equipotentials should cross the streamlines at right angles. As an example let us examine Fig. 14 ($L = 1.0$, $b = 0.067$, $d = 0.0$, $h = 0.0$, $c = 0.067$) where equipotentials for $\phi' = 0.2$, 0.4, 0.6, 0.8, and streamlines for $\psi' = 0$, 0.2, 0.4, 0.6, 0.8, 1.0 are drawn. The equipotential for $\phi' = 0$ is the point $(0,0)$ and $\phi' = 1.0$ is $(L,b)$. It is clear from the figure that the equipotentials and the streamlines intersect perpendicularly. At points along the soil surface the
the streamlines are not perpendicular to the soil surface because the soil surface is not an equipotential.

We can also check our work with that of Warrick (1970). He solved this same problem using conformal transformations. His work was limited, however, to a soil surface of constant slope and no water could be allowed to stand in the furrows. We have demonstrated that by using the method of Kirkham and Powers (1972) we can have a soil surface of any shape and we can allow water to stand in the furrow. Our Fig. 14 corresponds to Fig. 2 of Warrick (1970). He found the maximum value of $\psi/\Omega$ to be 0.025 occurring at $x/L = 0.49$. In our work we found the maximum value of $\psi/\Omega$ to be 0.025 occurring at $x/L = 0.48$. If we superimpose our Fig. 14 over the Fig. 2 of Warrick (1970) we find that the streamlines of the two figures agree exactly. The equipotential lines do not, however, agree exactly. The reason for this is that although Warrick (1970) states his equipotential lines are $\phi' = 0.2, 0.4, 0.6, 0.8$, they in fact are not. Because we have a constant slope and our boundary condition is $\phi = cx$ along the slope, $\phi' = 0.2$ must intersect the slope at $x = 0.2$, $\phi' = 0.4$ must intersect the slope at $x = 0.4$, etc. Upon close inspection of Fig. 2 of Warrick (1970) we find this condition, perhaps due to drafting error, is not exactly met. Because his equipotential lines and streamlines cross at right angles we conclude that they are valid equipotentials.
However, it would appear that they should not be labeled as \( \phi' = 0.2, 0.4, 0.6, 0.8 \) equipotentials but instead they should be more accurately designated as \( \phi' = 0.15, 0.45, 0.62, 0.76 \) equipotentials. From our Fig. 14 we note that our equipotentials intersect the soil surface at \( x = 0.2, 0.4, 0.6, 0.8 \). They also cross the streamlines at right angles. We conclude that our equipotentials agree with Warrick's equipotentials approximately.

Table 2 brings out the influence of different parameters on \( R_{\text{min}} \), \( Q/L \), and \( W_p \).

The effect of increasing the slope of the soil bedding with zero depth of water standing in the furrow is demonstrated by the values of \( R_{\text{min}} \), \( Q/L \), and \( W_p \) for cases A6, B6, and C6. \( R_{\text{min}} \) and \( Q/L \) are increased but \( W_p \) decreases.

The effect of maintaining the center of the bedding constant but changing the shape of the soil surface from a constant slope to elliptic shaped (compare A6 with A8, B6 with B8, and C6 with C8) decreases \( R_{\text{min}} \) but greatly increases \( Q/L \), and \( W_p \).

By increasing the height of the center of the bedding for the elliptic shaped soil surface with no water standing in the furrow (A8, B8, and C8) we increase \( R_{\text{min}} \) and \( Q/L \) but decrease \( W_p \). By adding a constant width of water in the furrows (\( D = .15 \) for A9, B9, and C9) \( R_{\text{min}} \), \( Q/L \), and \( W_p \) increase for increasing height of the bedding center.
By maintaining a constant bedding center height and increasing the level of water in the furrow (B8, B9, D9, and E9) \( R_{\text{min}} \), \( Q/L \) and \( W_p \) are decreased.

The percent of water, \( W_p \), flowing through the soil bedding for the bedding with constant slope is comparable to the percent reported by Powers et al. (1967) and Selim and Kirkham (1972a) for their cases with an impermeable barrier at a finite depth below the soil surface. By changing the shape of the soil surface (from constant slope to elliptic shaped) the percent of water, \( W_p \), flowing through the soil is increased 4 to 5 times. In the cases with a constant soil slope we have about 85% of the rain needed to keep the bedding saturated running over the surface of the soil without running through it. With the elliptic shaped soil surface we have about 25 to 50% of the rain needed to keep the bedding saturated running over the soil surface.

Summary and Conclusions

We solved the problem of the seepage of rainfall through soil bedding underlain by an impermeable barrier at infinity. A constant slope soil surface and elliptic shaped soil surface are considered. The problem is solved using the modified Gram-Schmidt method of Kirkham and Powers (1972). The minimum rainfall needed to keep the bedding saturated, the quantity of water flowing through the soil, and the percent of the total minimum rainfall that flows
through the soil are also determined.

It is determined that by changing the shape of the soil surface from constant slope to elliptic shape the percent of water flowing through the soil is greatly increased. This is because the minimum rainfall needed to keep the bedding saturated is decreased while the total amount of water flowing through the soil is increased.

We are able to check our work with that of Warrick (1970) where he solved the problem with a constant slope and no water standing in the furrow. Our work agreed approximately with his, and exactly, if his equipotentials are renumbered.

Flow nets for several soil bedding geometries are presented. From these flow nets we can see that water enters the soil upslope above a critical point and after flowing through the soil resurfaces again downslope and enters surface runoff. This in and out seepage of water illustrates how soluble material may be removed from the soil and added to surface runoff.
PART III. FLOW PATTERNS OF RAINWATER SEEPPING THROUGH BEDDED SOIL: LABORATORY STUDY

Introduction

Powers (1966) and Powers et al. (1967) solved a problem of seepage of rainwater seeping through soil bedding underlain by a horizontal impermeable barrier at a finite depth. In part one of this thesis we solved the same problem with a tile drain half embedded in the impermeable subsoil layer directly below the drainage furrow. The depth of ponded water considered to be standing in the furrows in each case is considered to be negligible. Here we present photographs of steady-state streamlines of rainwater seeping through bedded soil as obtained by use of laboratory models filled with sand. The purpose is to see if the experimental streamlines agree with those calculated from the theory. Photographs are also presented to illustrate the development of the streamlines.

Methods and Materials

Models made of plexiglas were built to simulate soil profiles with sloping surfaces. The plexiglas allowed visual observation of dyed flow lines in the soil profile below the soil surface. Two models were constructed. The two models were exactly the same with the exception of a tile drain located in the corner directly below the low end of one of the models as indicated in Fig. 20. Dimensions
Fig. 20 - A sketch and dimensions of the plexiglas models used in the laboratory. Dimensions of the model indicated on the sketch are inches. The inside width of the model is 1.18 inches (3 cm).
b = 12

L = 45

d = 6

.75 inside dia.

1.0 outside dia.
of the models are as indicated in the figure. The sides of the models were constructed from plexiglas 1/4" thick. The base was 1/2" thick, 48" long, and 8" wide. The tile drain had an inside diameter of 0.75 inches and an outside diameter of approximately one inch. Approximately 50 holes 1/8" diameter were drilled into the side of the drain exposed to sand to allow water to flow into the drain. A three hundred mesh brass screen was placed over the drain to prevent sand from plugging the holes of the drain.

The model without the tile drain was used to demonstrate the streamlines found in Fig. 5B and 5C of Powers et al. (1967). The model with the tile drain was used to demonstrate the streamlines found in Fig. 9 (B5, C5 and D5) of part one of this thesis.

The model was filled with Clayton sand, a commercial inert white silica sand. A thin surface layer of 6 mm solid glass beads and 40 mesh screen was used on the surface of the slope to distribute the recharge water and prevent erosion of the sand downslope.

After the plexiglas models were packed with sand the system was kept saturated with water by recharge from a battery of 30 tubes with stopcocks leading from a water reservoir with a constant level of water. The stopcocks allowed the flow rate of water from each tube to be adjusted as needed. An aqueous solution of Evans blue dye was intro-
duced at points along the slope by using #20 5-inch hypodermic needles and peristaltic (Sigmamotor Co.) or vibrostaltic pumps (Chemical Rubber Co.). We used two peristaltic and four vibrostaltic pumps. The movement of the dye below the sand surface was observed visually and photographed at approximately 15-minute intervals to record the development of the streamlines.

Results and Discussion

Sequence photographs are presented to demonstrate the development of streamlines for five flow nets. The values of the parameters d, b, L, and w for the five flow nets are given in Table 3. Case 1, 2, and 3 are flow nets illustrated in Fig. 9 (B5, C5, and D5) of this thesis. Case 4 and 5 are for the soil bedding cases presented in Powers et al. (1967) (Fig. 5B and 5C).

Sequence photographs for the cases of Table 3 are presented in Fig. 21, 22, 23, 24 and 25. In Fig. 21, 22, and 23 we can see the flow patterns of rain water through sloping soil bedding to tile drains during saturated, steady-state conditions. The boundary conditions for these three figures are the same as those listed in part one of this thesis. In Fig. 21 and 23 dye is introduced at five points in the flow medium. Two needles are placed, one at each end of the slope, to illustrate the streamlines in contact with the impervious boundaries. Three needles are placed at
Table 3. Values of the parameters d, b, L, and w used for the laboratory models; values are in inches.

<table>
<thead>
<tr>
<th>Case</th>
<th>d</th>
<th>b</th>
<th>L</th>
<th>w</th>
<th>Slope (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>7.5</td>
<td>45</td>
<td>.75</td>
<td>1.909</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>9.0</td>
<td>45</td>
<td>.75</td>
<td>3.814</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>10.5</td>
<td>45</td>
<td>.75</td>
<td>6.968</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>9.0</td>
<td>45</td>
<td>.00</td>
<td>3.814</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>10.5</td>
<td>45</td>
<td>.00</td>
<td>6.968</td>
</tr>
</tbody>
</table>
Fig. 21 - Photographic sequence pictures illustrating the development of streamlines in a plexiglas model filled with sand and drained by a drain tube. Parameters correspond with case 1 of Table 3. Times for the pictures are (hours + minutes): $A = 0 + 00$, $B = 0 + 30$, $C = 1 + 00$, $D = 1 + 50$, $E = 2 + 30$, and $F = 4 + 30$. 
Fig. 22 - Photographic sequence pictures illustrating the development of streamlines in a plexiglas model filled with sand and drained by a drain tube. Parameters correspond with case 2 of Table 3. Times for the pictures are (hours + minutes): A = 0 + 00, B = 0 + 30, C = 1 + 00, D = 1 + 30, E = 1 + 45, and F = 2 + 15.
Fig. 23 - Photographic sequence pictures illustrating the development of streamlines in a plexiglas model filled with sand and drained by a drain tube. Parameters correspond with case 3 of Table 3. Time for the pictures are (hours + minutes): A = 0 + 15, B = 0 + 45, C = 1 + 15, D = 1 + 45, E = 2 + 15, and F = 3 + 15.
Fig. 24 - Photographic sequence pictures illustrating the development of streamlines in a plexiglas model filled with sand and drained by surface drainage. Parameters correspond with case 4 of Table 3. Time for the pictures are (hours + minutes): A = 0 + 00, B = 0 + 30, C = 1 + 00, D = 1 + 45, E = 2 + 30, and F = 4 + 45.
Fig. 25 - Photographic sequence pictures illustrating the development of streamlines in a plexiglas model filled with sand and drained by surface drainage. Parameters correspond with case 5 of Table 3. Time for the pictures are (hours + minutes): \( A = 0 + 00, \ B = 0 + 15, \ C = 0 + 45, \ D = 1 + 15, \ E = 1 + 30, \) and \( F = 2 + 15. \)
points along the slope to illustrate other streamlines. In Fig. 22 four needles are placed at points along the slope to illustrate the streamlines. No needles are placed at the left or right boundaries.

It is evident from this work that the most water enters the drain from the area directly above the drain. This is illustrated in Fig. 21 and 23 by the streamline along the boundary directly above the drain. The dye reached the tile drain in less than one minute after it was introduced at the surface. In contrast, it took over five hours for the dye to reach the drain when introduced at the surface at the most distant point from the drain (top of the slope) in Fig. 21. It took approximately 2.5 hours for the dye to reach the drain by following the boundary from the top of the slope in Fig. 23. The reason for the shorter time in this case as compared to Fig. 21 is because of the steeper slope of Fig. 23 and a greater hydraulic head driving the dye with everything else held constant.

Fig. 22 illustrates the streamlines for a slope of 3.8 degrees whereas Fig. 21 and 23 had slopes of 1.9 and 7.0 degrees respectively.

The streamlines in Fig. 21, 22, and 23 correspond approximately to the flow nets presented in B5, C5, and D5 of Fig. 9 in part one of this thesis. The central streamline in Fig. 23 tends to move upward before it moves down to the
drain. In the absence of a drain it would move to the surface.

Fig. 24 and 25 illustrate the flow patterns of rainwater through sloping soil bedding during saturated, steady-state conditions without a tile drain. The boundary conditions for these cases are given by Powers et al. (1967) (page 6226). Flow nets are given by Fig. 5B and 5C (page 6235) of Powers et al. (1967).

In Fig. 24 and 25 three streamlines are illustrated. As is expected the streamline with the shortest distance to travel is the one first completed and the one with the longest distance to travel is the last to be completed on the same flow net.

The effect of slope on the velocity of flow of water through the sand is illustrated by Fig. 24 and 25. For case 4 in Fig. 24 from the time the dye was introduced at the highest point on the slope until it reached the lowest point on the slope by going along the outer boundary 4.5 hours elapsed. Compare this with case 5 of Fig. 25. With a steeper slope and a longer flow path it took only 3.25 hours for the dye to flow along the longest streamline in Fig. 25. The reason for this is we have 1/3 more hydraulic head driving the water in Fig. 25 than we do in Fig. 24.

By comparing Fig. 24 and 25 with the flow nets of Powers et al. (1967) we can see there is good agreement
between the mathematically derived flow net and the laboratory models.

Summary and Conclusions

We illustrated by photographic sequence pictures the development of streamlines in a plexiglas model filled with sand and drained by a tile drain or by surface drainage. As we increased the slope, thus increasing the hydraulic head, we demonstrated that the water moves with a greater velocity along comparable streamlines. We also illustrated that there is satisfactory agreement between the mathematically derived flow nets and the laboratory models for the cases studied. It should be remembered that these models could be ridges and furrows for corn rows or other row crops, especially if the model slopes are increased. The models could also be applied to a soil bedding system where there would be several rows running perpendicular to the slope. On a greatly expanded scale the models could represent the situation of slopes leading to river valleys with the river running perpendicular to the slope.
GENERAL SUMMARY AND CONCLUSIONS

In this thesis we solve two steady-state water-saturated drainage problems. We solve these problems by use of the modified Gram-Schmidt method of Kirkham and Powers (1972).

First we consider the problem of the seepage of rainfall through soil bedding to a tile drain. The tile drain is half-embedded in an impermeable barrier located at a finite depth below the soil surface. The quantity of water flowing through the flow medium into the tile drain is calculated. The minimum rainfall rate needed to keep the soil bedding saturated is also calculated. The percent of the rainfall that flows through the flow medium to the tile drain is determined. By increasing the soil slope, the depth to the impermeable barrier, and/or the radius of the tile drain, with the tile drain running full with zero back pressure, the quantity of water flowing through the flow medium to the tile drain is increased. The rainfall rate needed to keep the bedded soil saturated may or may not be increased. The flow nets illustrate that the water which enters the soil flows to the tile drain and the water does not resurface again downslope in certain cases. In other cases water may enter the soil upslope and resurface again downslope.

Next we consider the problem of the seepage of rainfall through soil bedding underlain by an impermeable barrier at infinity. Constant slope and elliptic shaped soil sur-
faces are considered. We determine that by changing the shape of the soil surface from constant slope to elliptic shape the percent of water flowing through the soil is greatly increased. From the flow nets we find that water enters the soil upslope above a critical point and resurfaces again downslope to add to surface runoff. The flow nets thus illustrate how soluble material that moves with the seepage water may be removed from the soil and added to surface runoff.

In the third part of this thesis we illustrate by use of dye and photographic sequence the development of streamlines in a plexiglas model filled with sand and drained by a tile drain or by surface drainage. Only constant slopes are considered. As the values of the slope are increased, it is demonstrated that the water moves with a greater velocity along comparable streamlines. There is good agreement between the mathematically derived streamlines and those found in the laboratory models.

In our mathematical analysis we find that our solutions agree with solutions of Kirkham (1947a) and Warrick (1970) for special cases.
Literature Cited


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To everyone "¡Muchas gracias!"
Evaluation of $\partial(\phi/H)/\partial(x/L)$ of Equation [4]

In equation [3] of Part I of this thesis we differentiate to find for satisfying [BC 2]

$$\frac{\partial(\phi/H)}{\partial x} = \frac{A_{NO}}{\ln(M/w)} \frac{\partial}{\partial x} \ln\left((x^2 + y^2)^{1/2}/w\right)$$

$$+ \sum_{m=1}^{N=\infty} A_{Nm} \frac{1}{M^{2m} \left[1 - (w^2/M^2)^{2m}\right]}$$

$$\frac{\partial}{\partial x}\left\{ (x^2 + y^2)^m \cos\left[2m \cot^{-1} x/y\right]\right\}$$

$$- \frac{w^{4m}}{M^{2m} \left[1 - (w^2/M^2)^{2m}\right]}$$

$$\frac{\partial}{\partial x}\left\{ (x^2 + y^2)^{-m} \cos\left[2m \cot^{-1} x/y\right]\right\}$$

[A 1]

For reference we need formula 512.5 from Dwight (1961)

$$\frac{\partial}{\partial x} \left( \cot^{-1} \frac{x}{y} \right) = \frac{-1}{y^2 + x^2}, \quad (y = \text{const.})$$

[A 2]

For use in [A 1] we now find some derivatives.

For the first line of [A 1] we have

$$\frac{\partial}{\partial x} \ln\left((x^2 + y^2)^{1/2}/w\right) = \frac{3}{2} \frac{1}{x^2 + y^2} \ln(x^2 + y^2) = \frac{\partial}{\partial x} \ln w$$

$$= \frac{1}{2} \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r}$$

[A 3]
For the second line of \([A 1]\) we have

\[
\frac{\partial}{\partial x} (x^2 + y^2)^m \cos[2m \cot^{-1} x/y]
\]

\[
= (x^2 + y^2)^m \frac{\partial}{\partial x} \cos[2m \cot^{-1} x/y]
\]

\[
+ \cos[2m \cot^{-1} x/y] \frac{\partial}{\partial x} (x^2 + y^2)^m \quad [A 4]
\]

where in \([A 4]\) we have

\[
\frac{\partial}{\partial x} \cos[2m \cot^{-1} x/y]
\]

\[
= (-) \sin[2m \cot^{-1} x/y] \frac{\partial}{\partial x} (2m \cot^{-1} x/y)
\]

\[
= -\sin[2m \cot^{-1} x/y] 2m \frac{-y}{y^2 + x^2}
\]

\[
= 2m \frac{y}{y^2 + x^2} \sin[2m \cot^{-1} x/y]
\]

\[
= 2m \frac{r \sin \theta}{r^2} \sin 2m \theta
\]

\[
= 2m \frac{\sin \theta}{r} \sin 2m \theta \quad [A 5]
\]

and where in \([A 4]\) we also have
\[ \frac{\partial}{\partial x} (x^2 + y^2)^m = m(x^2 + y^2)^{m-1} 2x \]

\[ = m(r^2)^{m-1} 2r \cos \theta = m r^{2m} r^{-2} 2r \cos \theta \]

\[ = mr^{2m} \frac{2r \cos \theta}{r^2} = 2m \frac{r^{2m} \cos \theta}{r} \quad [A 6] \]


\[ \frac{\partial}{\partial r} (x^2 + y^2)^m \cos \left[ 2m \cot^{-1} x/y \right] \]

\[ = (x^2 + y^2)^m 2m \frac{\sin \theta}{r} \sin 2m\theta \]

\[ + 2mr^{2m} \frac{\cos \theta}{r} \cos \left[ 2m \cot^{-1} x/y \right] \]

\[ = r^{2m} 2m \frac{\sin \theta}{r} \sin 2m\theta \]

\[ + 2mr^{2m} \frac{\cos \theta}{r} \cos \left[ 2m \cot^{-1} x/y \right] \]

\[ = \frac{2mr^{2m}}{r} \left[ \sin \theta \sin 2m\theta + \cos \theta \cos 2m\theta \right] \]

\[ = \frac{2mr^{2m}}{r} \cos(2m-1) \theta = 2mr^{2m-1} \cos(2m-1) \theta \quad [A 7] \]

which is the value needed for the second line of [A 1].

For the third line of [A 1] we write
\[
\frac{\partial}{\partial x} \{(x^2 + y^2)^{-m} \cos[2m \cot^{-1} x/y]\}
= (x^2 + y^2)^{-m} \frac{\partial}{\partial x} \cos[2m \cot^{-1} x/y]
+ \cos 2m[\cot^{-1} x/y] \frac{\partial}{\partial x} (x^2 + y^2)^{-m} \tag{A 8}
\]

where in (A 8) we have

\[
\frac{\partial}{\partial x} \cos[2m \cot^{-1} x/y]
= -\sin[2m \cot^{-1} x/y] \frac{\partial}{\partial x} (2m \cot^{-1} x/y)
= -\sin[2m \cot^{-1} x/y] 2m \frac{-y}{y^2 + x^2} \text{ (by use of (A 2))}
= 2m \sin 2m\theta \frac{y}{r^2}
= 2m \sin 2m\theta \frac{r \sin \theta}{r^2}
= 2m \sin \theta \sin 2m\theta \frac{1}{r}
= 2m \frac{\sin \theta}{r} \sin 2m\theta \tag{A 9}
\]

and where in (A 8) we also have

\[
\frac{\partial}{\partial x} (x^2 + y^2)^{-m}
\]
Putting [A 9] and [A 10] in [A 8] gives

\[
\frac{2m \cos \theta}{r^{2m+1}}
\]

We may write [A 11] as

\[
\frac{\partial}{\partial x} ((x^2 + y^2)^{-m} \cos[2m \cot^{-1} x/y])
\]

\[
= (x^2 + y^2)^{2m} \frac{\sin \theta}{r} \sin 2m \theta + \cos[2m \cot^{-1} x/y] (-2m) \frac{\cos \theta}{r^{2m+1}}
\]

\[
= \frac{2m \sin \theta \sin 2m \theta}{r^{2m+1}} - \frac{2m \cos \theta \cos 2m \theta}{r^{2m+1}}
\]
\[
\frac{2m}{r^{2m+1}} (\sin \theta \sin 2m\theta - \cos \theta \cos 2m\theta)
\]
\[
= -\frac{2m}{r^{2m+1}} [-\cos (\theta + 2m\theta)]
\]
\[
= -\frac{2m}{r^{2m+1}} \cos (2m + 1)\theta \quad [A 12]
\]

which is the derivative needed in the third line of [A 1].

Putting [A 3], [A 7] and [A 12] in [A 1] we get

\[
\frac{\partial (\phi/H)}{\partial x}
\]
\[
= \frac{A_{NO}}{\ln(M/w)} \frac{\cos \theta}{r} + \sum_{m=1}^{N \to \infty} A_{Nm} \left( \frac{1}{M^{2m} [1 - (w^2/M^2)^{2m}]} \cdot 2mr^{2m-1} \cos(2m-1)\theta \right)
\]
\[
- \frac{4m}{M^{2m} [1 - (w^2/M^2)^{2m}]} \frac{(-2m)}{r^{2m+1}} \cos(2m+1)\theta \right) \quad [A 13]
\]

We may write [A 13] as

\[
\frac{\partial (\phi/H)}{\partial x}
\]
\[
= A_{NO} \frac{(1/r) \cos \theta}{\ln(M/w)} + \sum_{m=1}^{N \to \infty} A_{Nm} \left( \frac{1}{M^{2m} [1 - (w^2/M^2)^{2m}]} \cdot 2m r^{2m} \cos(2m-1)\theta \right)
\]
\[ + \frac{(w^2)^{2m}}{M^{2m}} \left[ 1 - \left( \frac{w^2}{M^2} \right)^{2m} \right] \cdot 2m \cdot \frac{1}{r^{2m}} \cos(2m+1) \theta \] 

\[ = A_{N0} \frac{(\cos \theta)/r}{\ln(M/w)} \]

\[ + \sum_{m=1}^{N \rightarrow \infty} A_{Nm} \frac{2m}{\gamma_m} \cdot \left( \frac{r}{M} \right)^{2m} \frac{\cos(2m-1) \theta}{r} \]

\[ + (w^2/Mr)^{2m} \frac{\cos(2m+1) \theta}{r} \]  

where

\[ \gamma_m = 1 - \left( \frac{w^2}{M^2} \right)^{2m} \]

We may make \( x \) in \([A 14]\) dimensionless and hence make the left and right of \([A 14]\) dimensionless (multiply both sides of \([A 14]\) by \( L \) and rearrange to find)

\[ \frac{\partial (\phi/H)}{\partial (x/L)} \]

\[ = A_{N0} \frac{1}{\ln(M/w)} \frac{L}{r} \cos \theta \]

\[ + \sum_{m=1}^{N \rightarrow \infty} A_{Nm} \frac{2m}{\gamma_m} \frac{L}{r} \left[ \left( \frac{r}{M} \right)^{2m} \cos(2m-1) \theta \right] \]

\[ + (w^2/Mr)^{2m} \cos(2m+1) \theta \]  

which is equation [4] of the body of the thesis.
APPENDIX B
Introduction of s

To get the proper values of r and \( \theta \) for use in [6] and [7] some additional lengths, and in particular a length coordinate \( s \), needs to be added to Fig. 2 and these are shown in Fig. 26. The added coordinate is \( s \), measured along BCDEFGJ. Along BF we denote \( s \) by \( s_1 \) and along FG we denote \( s \) by \( s_2 \). The origin for measurement of \( s \) is point B.

When \( r \) extends to the boundary BCDEF as at point C, we denote the value of \( r \) by \( R_1 \) and its angle with OB as \( \theta_1 \). When \( r \) extends to the boundary FGJ as at point G, we denote the value of \( r \) by \( R_2 \) and its angle with OB as \( \theta_2 \).

We shall need \( R_1 \) and \( \theta_1 \) in terms of \( s_1 \); and, shall need also \( R_2 \) and \( \theta_2 \) in terms of \( s_2 \).

From Fig. 26 we have the relation

\[
\tan \theta_1 = s_1/L \tag{B 1}
\]
or

\[
\theta_1 = \arctan(s_1/L) \tag{B 2}
\]

and from Fig. 26 we similarly have

\[
R_1 = (L^2 + s_1^2)^{1/2} \tag{B 3}
\]

Thus [B 1] and [B 3] give us \( \theta_1 \) and \( R_1 \) in terms of \( s_1 \).

To get \( \theta_2 \) and \( R_2 \) in terms of \( s_2 \) we consider a point G. The \( s (= s_2) \) coordinate of point G is given by
Fig. 26 - Additional geometry for Fig. 2 (characteristic points are, in general, not labelled the same as in Fig. 2).
\[ s_2 = b + FG \]
\[ = b + \sigma - JG \quad [B\ 4] \]

From the figure we have

\[ \tan \theta_2 = \frac{AG}{OA} \quad [B\ 5] \]

and have

\[ R_2^2 = \frac{AG^2 + OA^2}{B\ 6} \]

where from the figure we see the relations

\[ AG = d + IG \]
\[ = d + JG \sin \alpha \]
\[ = d + (b + \sigma - s_2) \sin \alpha \quad [B\ 7] \]

and have

\[ OA = JI \]
\[ = L - GE \]
\[ = L - (s_2 - b) \cos \alpha \quad [B\ 8] \]

Putting [B 7] and [B 8] in [B 5] we find

\[ \tan \theta_2 = \frac{d + (b + \sigma - s_2) \sin \alpha}{L - (s_2 - b) \cos \alpha} \quad [B\ 9] \]

or

\[ \theta_2 = \arctan \frac{d + (b + \sigma - s_2) \sin \alpha}{L - (s_2 - b) \cos \alpha} \quad [B\ 10] \]

and putting [B 7] and [B 8] in [B 6] we find

\[ R_2 = \left\{ \left[ d + (b + \sigma - s_2) \sin \alpha \right]^2 + \left[ L - (s_2 - b) \cos \alpha \right]^2 \right\}^{1/2} \quad [B\ 11] \]
In [B 7] we have, from the figure, the relation

$$\sigma = \left[ L^2 + (b - d)^2 \right]^{1/2} \tag{B 12}$$

The values of \( \sin \alpha \) and \( \cos \alpha \) of [B 10] and [B 11] are seen from Fig. 26 to be given by

$$\sin \alpha = \frac{b - d}{\left[ L^2 + (b - d)^2 \right]^{1/2}} \tag{B 13}$$

$$\cos \alpha = \frac{L}{\left[ L^2 + (b - d)^2 \right]^{1/2}} \tag{B 14}$$

Also from Fig. 26 we have

$$M = \left( L^2 + b^2 \right)^{1/2} \tag{B 15}$$

We need to get \( \theta_2 \) and \( R_2 \) of [B 10] and [B 11] in shorter form.

We may write [B 10] as

$$\theta_2 = \arctan \left( \frac{d + (b + \sigma) \sin \alpha - s_2 \sin \alpha}{L + b \cos \alpha - s_2 \cos \alpha} \right) \tag{B 16}$$

And we may write [B 11] as

$$R_2 = \left\{ \left[ d + (b + \sigma) \sin \alpha - s_2 \sin \alpha \right]^2 + \left[ L + b \cos \alpha - s_2 \cos \alpha \right]^2 \right\}^{1/2} \tag{B 17}$$

We define \( \delta \) by

$$\delta = d + (b + \sigma) \sin \alpha \tag{B 18}$$
and \( \lambda \) by

\[
\lambda = L + b \cos \alpha \tag{B 19}
\]

We put [B 18] and [B 19] in [B 16] and get

\[
\theta_2(s) \equiv \theta_2 = \arctan \frac{\delta - s_2 \sin \alpha}{\lambda - s_2 \cos \alpha} \tag{B 20}
\]

We put [B 18] and [B 19] in [B 17] and get

\[
R_2(s) \equiv R_2 = \left[ (\delta - s_2 \sin \alpha)^2 + (\lambda - s_2 \cos \alpha)^2 \right]^{1/2} \tag{B 21}
\]

From [B 20] and [B 21] we find

\[
R_2 \sin \theta_2 = \frac{\delta - s_2 \sin \alpha}{\left[ (\delta - s_2 \sin \alpha)^2 + (\lambda - s_2 \cos \alpha)^2 \right]^{1/2}} \tag{B 22}
\]

which simplifies immediately to

\[
R_2 \sin \theta_2 = \delta - s_2 \sin \alpha \tag{B 22}
\]

We now rewrite a quantity \((L/R_1) \cos \theta_1\) as

\[
\frac{L}{R_1} \cos \theta_1 = \frac{L \cos(\arctan s_1/L)}{(L^2 + s_1^2)^{1/2}} \]

\[
= \frac{L}{L^2/(s_1^2 + L^2)^{1/2}} = \frac{L^2}{L^2 + s_1^2} \tag{B 23}
\]

which simplifies to

\[
\frac{L}{R_1} \cos \theta_1 = \frac{L^2}{L^2 + s_1^2} \tag{B 24}
\]
APPENDIX C
Table 4. Values of the $A_{Nm}$ used for the cases reported in Table 1

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Table 6. Values of $x_1$, $x_2$ and $x_3$ associated with $Q_u/(KL)$ of Table 1

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C PROGRAM NO. 1 FOR TILE DRAIN
PROBLEM OF PART ONE
C ********************************************************
C TO COMPUTE ANM'S FOR ALL CASES REPORTED IN PART ONE
NEED ONLY TO CHANGE VALUES FOR PI, B, D
EL, WW, H, HH, IN BLOCK DATA SUBROUTINE
PI=3.14159
B= BC OF FIG. 2
D= OD OF FIG. 2
L= OB OF FIG. 2
WW= OE OF FIG. 2
H= OB' OF FIG. 2
HH= H OF FIG. 2
C
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION U(21), C(20), D(21), G(21), RJ(210), A(21)
COMMON /SFUM/ Si (257), S2 (257)
COMMON /FFF*/ F1 (257), F2 (257)
COMMON /UMWUMN/ UM1(257,21), nM2 (257, 21)
COMMON /ANGS/ BESRHS, IERTH
COMMON /DEGRAD* PIN180
1000 FORMAT(///50X,3HN =,I3//)
IERTH=0
PI=3.141592653589793
PIN180=180.0D0/PI
CALL STEP
CALL PARM
CALL SUBS
CALL FCT
CALL FJ(BESRHS)
NMAX=16
KA=NMAX
KAM1=NMAX-1
KADIAG=(KA*KAM1)/2
DO 30 M=1,NMAX
NCAPP1=M
CALL SUBW(W,M)
CALL UMN(U,M)
NO=M-1
WRITE(6,1000) NO
CALL ORTH(U,W,C,D,G,RJ,A,NCAPP1,KA,KAM1,KADIAG,BESL
*HS,IER )
IF(IER.EQ.0.AND. IERTH.EQ.0) GO TO 25
CALL ERROR(IER,IERTH)
25 CALL OUTPUT(A,BESLHS,BESRHS,M)
30 CONTINUE
WRITE(7,7000) (A(K), K=1,NMAX)
SUBROUTINE STEP
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK1/ IP1,IP2
COMMON /BLK2/ SL1,SL2,SU1,SU2
COMMON /BLK/ H1,H2,NDIM1,NDIM2,NBIS1,NBIS2
COMMON /OMPAB/ PI,B,D,EL,WW,H,HH
SL1=0.0D0
SU1=B
SL2=B
SU2=B+DSQRT((EL*EL)+((B-D)*(B-D)))
NDIM1=2**IP1+1
NDIM2=2**IP2+1
NBIS1=NDIM1-1
NBIS2=NDIM2-1
H1=(SU1-SL1)/NBIS1
H2=(SU2-SL2)/NBIS2
RETURN
END

SUBROUTINE ERROR(IER,IERTH)
1000 FORMAT(///50X,'** WARNING *****')
1100 FORMAT(/10X,'NO FURTHER COMPUTATIONS ARE CARRIED OUT')
1200 FORMAT(/10X,'BEYOND THE LAST VALUE OF N BECAUSE OF ERRORS.')
1300 FORMAT(/10X,'THE ERROR PARAMETERS ARE:')
2000 FORMAT///30X,5HIER =,I3,30X,7HIERTH =,I3)
WRITE(6,1000)
WRITE(6,1100)
WRITE(6,1200)
WRITE(6,1300)
WRITE(6,2000) IER,IERTH
STOP
END

SUBROUTINE ORTH(U,W,C,D,G,J,A,NCAPP1,KA,KAM1,KADIAG,1BESLHS,IER)
IMPLICIT REAL*8(A-H,O-Z)
REAL*8 J(KADIAG),JTEMP
DIMENSION U(KA),C(KAM1),D(KA),G(KA),A(KA)
IP(NCAPP1-1) 1,2,2
IER = 1
RETURN
IF(NCAP1 - KA) 4, 4, 3
IER = 2
RETURN
IF(KA - 1 - KAM1) 5, 6, 5
IER = 3
RETURN
IF((KA * KAM1) / 2 - KADIAG) 7, 8, 7
IER = 4
RETURN
CONTINUE
IER = 0
NCAP = NCAP1 - 1
NCAPM1 = NCAP - 1
IF(NCAPM1) 10, 20, 30
D(1) = U(1)
G(1) = W
E = G(1) / D(1)
A(1) = E
UANG = U(1)
DANG = D(1)
BESLHS = E * E * DANG
CALL ANGLES(BESLHS, DANG, UANG, W, IER)
RETURN
C(1) = U(1) / D(1)
D(2) = U(2) - C(1) * C(1) * D(1)
G(2) = W - C(1) * G(1)
E = G(2) / D(2)
J(1) = C(1)
A(1) = A(1) - E * J(1)
A(2) = E
UANG = U(2)
DANG = D(2)
BESLHS = BESLHS + E * E * DANG
CALL ANGLES(BESLHS, DANG, UANG, W, IER)
RETURN
C(1) = U(1) / D(1)
NFORJ = 0
DO 120 N = 2, NCAP
CTEMP = U(N)
NM1 = N - 1
DO 110 NN = 1, NM1
NFORJ = NFORJ + 1
CTEMP = CTEMP - U(NN) * J(NFORJ)
110 CONTINUE
C(N) = CTEMP / D(N)
DTEMP = U(NCAPP1)
GTEMP = W
DO 140 N = 1, NCAP
CTEMP = C(N)
140 CONTINUE
DTEMP = DTEMP - CTEMP * CTEMP * D(N)

GTEMP = GTEMP - CTEMP * G(N)
D(NCAPP1) = DTEMP
G(NCAPP1) = GTEMP
E = GTEMP / DTEMP
NSTART = 0
DO 180 N = 1, NCAPM1
   JTEMP = C(N)
   NSTART = NSTART + N
   NFORJ = NSTART
   NP1 = N + 1
   DO 170 NN = NP1, NCAP
      JTEMP = JTEMP - C(NN) * J(NFORJ)
      NFORJ = NFORJ + NN - 1
   J(NFORJ) = JTEMP
   170 A(N) = A(N) - E * JTEMP
   NFORJ = NFORJ + 1
   J(NFORBJ) = C(NCAP)
   A(NCAP) = A(NCAP) - E * J(NFORJ)
   A(NCAPP1) = E
   UANG = U(NCAPP1)
   DANG = D(NCAPP1)
   BESLHS = BESLHS + E * E * DANG
   CALL ANGLES(BESLHS, DANG, UANG, W, IER)
RETURN
END

SUBROUTINE ANGLES(BESLHS, DANG, UANG, W, IER)
IMPLICIT REAL*8(A-H, O-Z)
COMMON /ANGS/ BESLHS, IERTH
COMMON /DEGRAD/ PIN180
1000 FORMAT(' BESSEL'S INEQ.: '<, FPD22.15, '<, FPD22.15, 1
   1, D(N), U(N,N), W(N) = '<, FPD22.15, 0
2000 FORMAT(' IERTH = '<, I2, '<, IER = '<, I2, 1
   1, ANGDES, ANGNEW, ANGTGO = '<, 0P3F15.10)
WHITE(6, 1000) BESLHS, BESRHS, DANG, UANG, W
IERTH = 0
IF(BESLHS.LT.0.D0) GO TO 10
IF(BESLHS.GT.BESRHS) GO TO 20
ANGBES = DARCOS(DSQRT(BESLHS/BESRHS)) * PIN180
IF(DANG.GT.UANG) GO TO 30
IF(DANG.LT.0.D0) GO TO 40
ANGNEW = DARCOS(DSQRT(1.D0 - DANG/UANG)) * PIN180
IF(UANG.LT.0.D0) GO TO 50
ANGTGO = DSQRT(BESRHS * UANG)
IF(DABS(W).GT.ANGTGO) GO TO 60
GO TO 70
10 IERTH = IERTH + 1
ANGTGO = DABCOS(W/ANGTGO)*PIN180

20 IERTH = IERTH + 1
30 IERTH = IERTH + 1
40 IERTH = IERTH + 1
50 IERTH = IERTH + 1
60 IERTH = IERTH + 1
70 WRITE(6,2000) IERTH,IER,ANGBES,ANGNEW,ANGTGO
RETURN
END

C

...........................................

C

SUBROUTINE PARM
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK2/ SL1,SL2,SU1,SU2
COMMON /BLK/ H1,H2,NDIM1,NDIM2,NBIS1,NBIS2
COMMON /UMPAR/ PI,B,D,EL,WW,H,HH
1000 FORMAT(1H1)
2000 FORMAT(///50X,'T I T L E')
3000 FORMAT(6F13.4)
3500 FORMAT(//7X,'FIRST INTERVAL')
3600 FORMAT(//7X,'SECOND INTERVAL')
4000 FORMAT(//10X,
X'THE FOLLOWING VALUES WERE USED FOR THIS PROBLEM :'
5000 FORMAT(/15X,29HLOWER BOUND OF THE INTERVAL =,F5.2)
5100 FORMAT(/15X,29HUPPER BOUND OF THE INTERVAL =,F5.2)
5200 FORMAT(/15X,
X45HSTEP SIZE USED IN THE NUMERICAL INTEGRATION =,D22.15)
5300 FORMAT(/15X,38HNUMBER OF BISECTIONS OF THE INTERVAL =,I4)
WRITE(6,1000)
WRITE(6,2000)
WRITE(6,3000) E,D,EL,WW,H,HH
WRITE(6,4000)
WRITE(6,3500)
WRITE(6,5000) SL1
WRITE(6,5100) SU1
WRITE(6,5200) H1
WRITE(6,5300) NBIS1
WRITE(6,3600)
WRITE(6,5000) SL2
WRITE(6,5100) SU2
WRITE(6,5200) H2
WRITE(6,5300) NBIS2
WRITE(6,1000)
RETURN
END

C

...........................................

C

SUBROUTINE SUBS
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK2/ SL1,SL2,SLU1,SLU2
COMMON /BLK/ H1,H2,NDIM1,NDIM2,NIIS1,NIIS2
COMMON /SFUM/ S1(257),S2(257)

DO 1 J=1,NDIM1
1 S1(J)=SL1+(J-1)*H1

DO 2 J=1,NDIM2
2 S2(J)=SL2+(J-1)*H2

RETURN
END

SUBROUTINE FCT
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK/ H1,H2,NDIM1,NDIM2,NIIS1,NIIS2
COMMON /SFUM/ S1(257),S2(257)
COMMON /FFFW/ F1(257),F2(257)
COMMON /UMPAR/ PI,B,D,EL,WW,H,HH

DO 1 J=1,NDIM1
1 F1(J)=0.0D0

DO 2 J=1,NDIM2
BB=B-D
ELL=DSQRT((EL*EL)+(BB*BB))
2 F2(J)=((D+(B+ELL)*(BB/ELL))-(S2(J)*(BB/ELL)))*H/HH

RETURN
END

SUBROUTINE FF(BESRHS)
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK/ H1,H2,NDIM1,NDIM2,NIIS1,NIIS2
COMMON /FFFW/ F1(257),F2(257)
COMMON /FFWUMN/ Y(257),Z(257)

DO 1 J=1,NDIM1
1 Y(J)=F1(J)*F1(J)
CALL DQSF(H1,Y,Z,NDIM1)
BESRHS=Z(NDIM1)

DO 2 J=1,NDIM2
2 Y(J)=F2(J)*F2(J)
CALL DQSF(H2,Y,Z,NDIM2)
BESRHS = BESRHS + Z(NDIM2)

RETURN
END

SUBROUTINE SUBW(W, M)
IMPLICIT REAL*8(A-H,0-Z)
COMMON /BLK/ H1, H2, NDIM1, NDIM2, NBIS1, NBIS2
COMMON /FPFW/ F1(257), F2(257)
COMMON /UMWUMN/ UM1(257, 21), UM2(257, 21)
COMMON /FPFWUMN/ Y(257), Z(257)
CALL UMFCT(M)
DO 1 J = 1, NDIM1
   1 Y(J) = UM1(J, M) * F1(J)
   CALL DQSF(H1, Y, Z, NDIM1)
   W = Z(NDIM1)
DO 2 J = 1, NDIM2
   2 Y(J) = UM2(J, M) * F2(J)
   CALL DQSF(H2, Y, Z, NDIM2)
   W = W + Z(NDIM2)
RETURN
END

SUBROUTINE UMN(U, M)
IMPLICIT REAL*8(A-H,0-Z)
DIMENSION U(M)
COMMON /BLK/ H1, H2, NDIM1, NDIM2, NBIS1, NBIS2
COMMON /UMWUMN/ UM1(257, 21), UM2(257, 21)
COMMON /FPFWUMN/ Y(257), Z(257)
DO 6 N = 1, M
DO 1 J = 1, NDIM1
   1 Y(J) = UM1(J, M) * UM1(J, N)
   CALL DQSF(H1, Y, Z, NDIM1)
   U(N) = Z(NDIM1)
DO 2 J = 1, NDIM2
   2 Y(J) = UM2(J, M) * UM2(J, N)
   CALL DQSF(H2, Y, Z, NDIM2)
   U(N) = U(N) + Z(NDIM2)
SUBROUTINE UMFCT(M)
IMPLICIT REAL*8(A-H,O-Z)
COMMON /SFUM/ SI(257), S2(257)
COMMON /UMWUMN/ UM1(257,21), UM2(257,21)
COMMON /UMPAR/ PI, B, D, EL, WW, H, HH
COMMON /BLK/ H1, H2, NDIM1, NDIM2, NBIS1, NBIS2
EM=M-1
C1=(EL*EL) + (B*B)
C2=DSQRT(C1)
C3=DLOG(C2/WW)
C0=2.0D0*EM
DO 10 J=1, NDIM1
IF(EM.LE.0.0D0) GO TO 8
C4=DATAN(S1(J)/EL)
C5=((EL*EL) + (S1(J)*S1(J)))
C6=DSQRT(C5)
C7=DCOS(((2.0D0*EM)-1.0D0)*C4)
E7=DCOS(((2.0D0*EM)+1.0D0)*C4)
C8=(WW*WW)/(C2*C6)
C9=C8**C0
C10=1.0D0-(((WW*WW)/C1)**C0)
C11=(C0*EL)/(C10*C6)
C12=((C6/C2)**C0)*C7
C13=C9*E7
GO TO 9
8 UM1(J,M)=(EL*EL)/(C3*((EL*EL)+(S1(J)*S1(J))))
GO TO 10
9 UM1(J,M)=C11*(C12+C13)
10 CONTINUE
D1=(EL*EL) + ((B-D)*(B-D))
D2=DSQRT(D1)
SINA=(B-D)/D2
COSA=EL/D2
DO 20 J=1, NDIM2
D3=B+D2-S2(J)
D4=(D3*SINA)+D
D5=D4*D4
D6=S2(J)-B
D7=D6*COSA
D8=EL-D7
D9=D8*D8
D10=D5+D9
D11=DSQRT(D10)
IF(EM.LE.0.0D0) GO TO 18
D12=DATAN(D4/D8) *C0
D13=DCOS(D12)
D14=C10
D15=(D11/C2)**C0
D16=(WW*WW)/(C2*D11)
D17=D16**C0
D18=(D15-D17)*D13
D19=D18/D14
GO TO 19
18 UM2(J,M)=DLOG(D11/WW)/C3
GO TO 20
19 UM2(J,M)=D19
20 CONTINUE
C
RETURN
END
C
C SUBROUTINE OUTPUT(A, BESLHS, BESRHS,M)
IMPLICIT REAL*8(A-H,0-Z)
REAL*4 XSIZE,YSIZE,XSF,XMIN,YSF,YMIN
DIMENSION A(21), BES(21), BESN(21), XX(65)
COMMON /UMWMN/ UM1(257,21), UM2(257,21)
COMMON /PPP/ F1(257), F2(257)
COMMON /SPUH/ S1(257), S2(257)
COMMON /PFWUMN/ Y(257), Z(257)
COMMON /BLK/ H1, H2, NDIM1, NDIM2, NBIS1, NBIS2
COMMON /UMPAR/ PI, B, D, EL, WW, H, HH
500 FORMAT(/2X,5HA'S ,07D16.7/(' •,6X,07D16.7))
1000 FORMAT(/15X,27HNORMALIZED BESSEL'S INEQ. =,F8.5)
2000 FORMAT(/11X,14HS - COORDINATE,15X,10HVALUE OF F,15X
*, 15HAPP. VALUE
1 OF F/)
3000 FORMAT(F24.6,F25.6,F27.6)
4000 FORMAT(/13X,*N*,12HBESSEL'S LHS,10X,18HNORM. BES
*EL'S LHS, 10X,18HNORM. BESSEL'S LHS/)
5000 FORMAT(35X,I2,15X,F10.5,10X,F10.5)
6000 FORMAT(1H1)
7000 FORMAT(3D25.16)
WRITE (6,500) (A(K),K=1,M)
BES(M)=BESLHS
BESN(M)=BESLHS/BESRHS
WRITE (6,1000) BESN(M)
NO=M-1
IF (MOD(NO,5).EQ.0.OR.NO.EQ.1) GO TO 1
GO TO 50
1 CONTINUE
XX(1)=0.0D0
XX(2)=B
XX(3)=B
XX(4)=DSQRT((EL*EL)+(B-D)*(B-D))+B
Z(1)=0.0D0
Z(2)=0.0D0
Z(3)=(B-H)/HH
Z(4)=(D-H)/HH
XSIZE=10.51
YSIZE=4.01
XSF=0.0
XMIN=0.0
YSF=0.0
YMIN=0.0
ISYM=0
MODE=4
NPTS=-4
CALL GRAPH(NPTS,XX,Z,ISYM,MODE,XSIZE,YSIZE,XSF,XMIN
*YSF,YMIN)
*';';';';';';';')
I=1
M3=16
NPTS1=NBIS1/M3+1
DO 2 J=1,NDIM1,M3
Z(J)=0.0D0
DO 3 K=1,M
5 Z(J)=Z(J)+A(K)*UM1(J,K)
Z(I)=Z(J)
Y(I)=F1(J)
XX(I)=S1(J)
I=I+1
2 CONTINUE
WRITE (6,2000)
WRITE (6,3000) (XX(J),Y(J),Z(J),J=1,NPTS1)
NPTS=-NPTS1
ISYM=1
MODE=7
CALL GRAPH(NPTS,XX,Z,ISYM,MODE,';')
I=1
M3=8
NPTS2=NBIS2/M3+1
DO 4 J=1,NDIM2,M3
Z(J)=0.0D0
DO 5 K=1,M
5 Z(J)=Z(J)+A(K)*UM2(J,K)
Z(I) = Z(J)
Y(I) = F2(J)
XX(I) = S2(J)
I = I + 1

4 CONTINUE
WRITE (6, 3000) (XX(J), Y(J), Z(J), J = 1, NPTS2)
NPTS = -NPTS2
CALL GRAPHS(NPTS, XX, Z, ISYM, MODE, ' ;•)
IF (NO. NE. 20) GO TO 50
WRITE(6, 6000)
WRITE(6, 4000)
DO 6 K = 1, M
KK = K - 1
WRITE(6, 5000) KK, BES(K), BESN(K)
6 CONTINUE
WRITE(6, 7000) (A(K), K = 1, M)
I = 1
DO 7 K = 6, M
XX(I) = 1.0D0 / (I + 4)
Z(I) = BES(K)
I = I + 1
7 CONTINUE
XSIZE = 2.01
YSIZE = 1.01
NPTS = -(M - 5)
ISYM = 13
MODE = 3
XSF = 0.0
YSF = 0.0
XMIN = 0.0
YMIN = 0.0
CALL GRAPHS(NPTS, XX, Z, ISYM, MODE, XSIZE, YSIZE, XSF, XMIN,
YSF, YMIN,
' *• ;• ;• ;• ;• ;• ;• ;• ;• ;• ;• )
XX(1) = 0.0D0
Z(1) = BESRHS
ISYM = 3
MODE = 0
NPTS = 1
CALL GRAPHS(NPTS, XX, Z, ISYM, MODE, ' ;•)
50 CONTINUE
RETURN
END

C
C
C

BLOCK DATA
IMPLICIT REAL*8(A-H,O-Z)
COMMON /BLK1/ IP1, IP2
COMMON /UMPAR/ PI, B, D, EL, WW, H, HH
C ********************************************************
C PROGRAM NO. 2 TO CALCULATE PSI ALONG THE BOUNDARY
C FOR THE TILE DRAIN PROBLEM OF PART ONE
C ********************************************************
DIMENSION ANM(40)
READ (5,1) NAMM,B,D,EL,W,H,HH
1 FORMAT(I5,6F10.4)
MM=NAMM+1
READ(5,2) (ANM(I),I=1,MM)
2 FORMAT(3D25.16)
WRITE (6,1) NAMM,B,D,EL,W,H,HH
WRITE (6,2) (ANM(I),I=1,MM)
WRITE (6,21)
21 FORMAT (M)
X=0.1330
MAXIT=120
DO 14 J=1,MAXIT
PSI=0.0
DO 13 N=1,MM
EM=N-1
C1=(EL*EL)+(B*B)
C2=SQRT(C1)
C3=ALOG(C2/W)
C0=2.0*EM
D1=(EL*EL)+((B-D)*(B-D))
D2=SQRT(D1)
SINA=(B-D)/D2
COSA=EL/D2
D3=B+D2-X
D4=(D3*SINA)+D
D5=D4*D4
D6=X-B
D7=D6*COSA
D8=EL-D7
D9=D8*D8
D10=D5+D9
D11=SQRT(D10)
D12=ATAN(D4/D8)
IF(EM.LE.0.0) GO TO 8
D13=SIN(D12*C0)
D14=1.0-((W*W)/C1)**C0
D15=(D11/C2)**C0
D16=(W*W)/(C2*D11)
D17=D16**C0
D18=(D15+D17)*D13
D19=D18/D14
GO TO 19
8 PSI=PSI+ANM(N)*(D12/C3)
GO TO 13
19 PSI=PSI+ANM(N)*D19
13 CONTINUE
WRITE (6, 30) PSI, X
30 FORMAT ("\/	 PSI=", E15.7, 6X, "X=", F10.6)
X = X + 0.01
IF (X .LE. 1.1300) GO TO 14
X = 1.1335
14 CONTINUE
X = 0.0
MAXIT = 60
DO 50 J = 1, MAXIT
PSI = 0.0
DO 60 N = 1, MM
EM = N - 1
C0 = 2.0 * EM
C1 = (EL * EL) + (B * B)
C2 = SQRT(C1)
B1 = (EL * EL) + (X * X)
B2 = SQRT(B1)
B3 = ATAN(X / EL)
IF (EM .LE. 0.0) GO TO 70
B4 = (B2 / C2) ** C0
B5 = (W * W) / (C2 * B2)
B6 = B5 ** C0
B14 = 1.0 - (((W * W) / C1) ** C0)
B7 = (B4 + B6) / B14
B8 = SIN(C0 * B3)
B9 = B7 * B8
GO TO 80
70 PSI = PSI + ANM(N) * (B3 / C3)
GO TO 60
80 PSI = PSI + ANM(N) * B9
60 CONTINUE
WRITE (6, 30) PSI, X
X = X + 0.01
IF (X .LE. 0.1300) GO TO 50
X = 0.133
50 CONTINUE
STOP
END
C ********************************************************
C PROGRAM NO. 3 TO DRAW FLOWNETS FOR TILE
C DRAIN PROBLEM OF PART ONE
C ********************************************************

DIMENSION XL(5),YL(5),GL(5),DL(5),A(21),XX(101),YY(101)
EXTERNAL FCT
EXTERNAL TFC
EXTERNAL FT1
COMMON B,D,EL,W,HH,PHIOH,PHILB,PHIPRM,PSIMAX,PSIL
*B,PSIPRM,IW,IK
COMMON A
COMMON Y,NORRIS
COMMON /XPHI/ X

110 FORMAT (7F10.6)
500 FORMAT (20A4)
800 FORMAT (4E20.8)
10 FORMAT (3D25.16)
1000 FORMAT (10X,3F20.6)
1500 FORMAT (IHI)
IR=5
IW=6
EPS=0.0001
IEND=10
IK=11
DO 50 L=1,1
READ(IR,110)B,D,EL,W,H,HH,PSIMAX
WRITE(I»,110)B,D,EL,B,H,HH,PSIMAX
READ(IR,500) XL,YL,GL,DL
READ(IR,10) (A(K),K=1,IK)
WRITE (IH,800) (A(K),K=1,IK)
CC=B-D
PHIOH=0.0
PHILB=(B-H)/HH

C GRAPH THE REGION******

XX (1) =0.0
XX (2) =EL
XX (3) =EL
XX (4) =0.0
YY (1) =0.0
YY (2) =0.0
YY (3) =B
YY (4) =D
NPTS=4
XSIZE=10.01
YSIZE=5.01
CALL GRAPH(NPTS,XX,YY,0.4,XSIZE,YSIZE,0.10,0.0,0.10,-0.0,
*XL,YL,GL,DL)

C **************************************** DRAW FLUX LINES ******
JJ=20
PHIPRM=0.2
DO 30 K=1,4
C *********** CALCULATE XMAX,YMAX ***********
  RETI=0.0
  IF((PHIPRM*PHILB) LT. (D-H)/HH) GO TO 4
  YMAX=H+ (PHIPRM*HH*PHILB)
  XMAX=(YMAX-D)/CC
  GO TO 5
4 XMAX=0.0
  X=0.0
  YRI=D-0.0001
  YLI=W+0.0001
  IEND=20
  CALL RTMI(Y,F,FCT,YLI,YRI,EPS,IEND,IER)
  IEND=10
  RETI=IER
  YMAX=Y
5 CONTINUE
C CALCULATE XMIN AND YMIN
  YMIN=0.0
  Y=0.0
  XLI=0.01
  XRI=EL-0.01
  CALL RTMI(X,F,FCT,XLI,XRI,EPS,IEND,IER)
  RETI=IER
  WRITE(6,1000) Y,X,PHIPRM,REI
  XMIN=X
  IF(IER.EQ.2) XMIN=EL
  DX=(XMAX-XMIN)/JJ
  IEND=10
  YLI=0.0
  CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
  REI=IER
  WRITE(6,1000) Y,X,PHIPRM,REI
12 YY(1)=Y
   XX(1)=XMIN
13 WRITE(6,1000) Y,X,PHIPRM,REI
   DO 20 J=2, JJ
     X=X+DX
     YRI=YMAX
     CALL RTMI(Y,F,FCT,YLI,YRI,EPS,IEND,IER)
     RETI=IER
     WRITE(6,1000) Y,X,PHIPRM,REI
     YY(J)=Y
     XX(J)=X
     YLI=Y
20 CONTINUE
   JEND=JJ+1
   YY(JEND)=YMAX
   XX(JEND)=XMAX
   NPTS=JEND
   WRITE(6,120)YMAX,XMAX,REII
120 FORMAT(10X,2F20.6)
   CALL GRAPH(NPTS,XX,YY,0,2,0.0,0.0,0.0,0.0,0.0,0.0,X
*L,YL,GL,DL)
   WRITE(IW,1500)
   PHIprm=PHIprm+0.2
30 CONTINUE

C*****CALCULATION OF STREAMLINES*****

   JJ=20
   PHIprm=0.2
   NSTR=4
   NPTS=JJ
   DO 60 K=1,NSTR
      XLI=0.01
      XRI=EL
      NORRIS=1
      CALL RTMI(X,F,FT1,XLI,XRI,EPS,IEND,IER)
      REI=IER
      NORRIS=0
      XMAX=X
      YMAX= Delta+CC*XMAX
      WRITE(6,1000)YMAXfXMAX, PHIprm,REI
      X=XMAX
      XRI=X
      XX(1)=XMAX
      YY(1)=YMAX
      DY=YMAX/JJ
      Y=YMAX
      DO3 I=2,JJ
      Y=Y-DY
      XLI=0.001
      CALL RTMI(X,F,FT1,XLI,XRI,EPS,IEND,IER)
      REI=IER
      WRITE (6,1000)Y,X,PHIprm,REI
      YY(I)=Y
      XX(I)=X
3 XRI=X
   CALL GRAPH(NPTS,XX,YY,0,2,0.0,0.0,0.0,0.0,0.0,0.0,X
*L,YL,GL,DL)
   WRITE(IW,1500)
   PHIprm=PHIprm+0.2
60 CONTINUE
50 CONTINUE
STOP
END
FUNCTION FCT(X)
DIMENSION A (21)
COMMON B,D,EL,W,H,HH,PHIOH,PHILB,PHIPRM,PSIMAX,PSIL
*B,PSIPRM,IW,IK
COMMON A
COMMON Y,NORRIS
PHI=0.0
DO 5 M=1,IK
EM=M-1
C0=2.0*EM
C1=EL*EL+(B*B)
C2=SQR(T(C1))
C3=ALOG(C2/W)
C4=U.0-(((W*W)/C1)**C0)
C5=(X*X)+(Y*Y)
C6=SQR(T(C5))
C7=C6/W
C8=ALOG(C7)
IF(EM.LE.0.0) GO TO 1
C9=(C6/C2)**C0
C10=((W*W)/(C2*C6))**C0
C11=((C9-C10)/C4)*COS(C0*ATAN(Y/X))
GO TO 3
1 PHI=PHI+((A(M)*C8)/C3)
GO TO 2
3 PHI=PHI+(A(M)*C11)
2 CONTINUE
FCT=(PHI/PHILB)-PHIPRM
5 CONTINUE
RETURN
END

FUNCTION TFC(Y)
DIMENSION A (21)
COMMON B,D,EL,PHIOH,PHILB,PHIPRM,PSIMAX,PSIL
*B,PSIPRM,IW,IK
COMMON A
COMMON /XPHI/ X
PHI=0.0
DO 5 M=1,IK
EM=M-1
C0=2.0*EM
C1=EL*EL+(B*B)
C2=SQR(T(C1))
C3=ALOG(C2/W)
C4 = 1.0 - (((W**W) / C1) ** C0)
C5 = (X**X) + (Y**Y)
C6 = SQRT(C5)
C7 = C6 / W
C8 = ALOG(C7)
IF (EM .LE. 0.0) GO TO 1
C9 = (C6 / C2) ** C0
C10 = (((W**W) / (C2*C6)) ** C0
IF (X > 6.6, 4
4 C11 = (((C9 - C10) / C4) * COS(C0 * ATAN(Y/X)))
GO TO 3
6 C11 = (((C9 - C10) / C4) * ((-1)**(M-1))
GO TO 3
1 PHI = PHI + ((A(M) * C8) / C3)
GO TO 2
3 PHI = PHI + (A(M) * C11)
2 CONTINUE
TFC = (PHI / PHILB) - PHIPRM
5 CONTINUE
RETURN
END

FUNCTION FT1(X)
DIMENSION A(21)
COMMON B, D, EL, W, H, HH, PHIOH, PHILB, PHIPRM, PSIMAX, PSI
*B, PSIPRM, IW, IK
COMMON A
COMMON Y, NORRIS
PSI = 0.0
CC = (B - D) / EL
IF (NORRIS.EQ.1) Y = D + CC*X
DO 5 M = 1, IK
EM = M - 1
C0 = 2.0 + EM
C1 = (EL*EL) + (B*B)
C2 = SQRT(C1)
C3 = ALOG(C2 / W)
C4 = 1.0 - (((W**W) / C1) ** C0)
C5 = (X**X) + (Y**Y)
C6 = SQRT(C5)
C7 = ATAN(Y/X)
IF (EM .LE. 0.0) GO TO 1
C9 = (C6 / C2) ** C0
C10 = (((W**W) / (C2*C6)) ** C0
C11 = (((C9 + C10) / C4) * SIN(C0**C7))
GO TO 3
1 PSI = PSI + ((A(M) * C7) / C3)
GO TO 2
3  \texttt{PSI=PSI+(A(M)*C11)}
2  \texttt{CONTINUE}
   \texttt{FT1=(PSI/PSIMA*X)-PSIPRM}
5  \texttt{CONTINUE}
   \texttt{RETURN}
   \texttt{END}
C

**C PROGRAM NO. 4 TO CALCULATE ANN'S FOR CASES**
**C A6, B6, AND C6 OF PART TWO**

SUBROUTINE FX(IM, NUMBER, X, F)
DIMENSION X(21), F(21)
DOUBLE PRECISION S, B
COMMON S, B, ANM(40)
U(M) = EXP(M*PI*C*(Z/S-1.0)) * COS(M*PI*Z/S)
PI = 3.141593
E = 2.718283
C = B/S
M = IM
DO 6 I = 1, NUMBER
X(I) = (I-1)*S/(NUMBER-1)
Z = X(I)
SUM = 0.0
F(I) = ANM(1)
IF(M.EQ.0) GO TO 6
DO 5 J = 1, M
SUM = SUM + U(J) * ANM(J + 1)
5 CONTINUE
F(I) = SUM + F(I)
6 CONTINUE
7 RETURN
END

C

**C SUBROUTINE ORTH(U, W, C, D, J, A, NCAPP1, KA, KAM1, KADIAG, IER)**
DIMENSION U(KA), C(KAM1), D(KA), J(KA), A(KA)
REAL J(KADIAG), JTEMP
IF(NCAPP1-1) 1, 2, 2
1 IER = 1
RETURN
2 IF(NCAPP1-KA) 4, 4, 3
3 IER = 2
RETURN
4 IF(KA-1-KAM1) 5, 6, 5
5 IER = 3
RETURN
6 IF((KA*KAM1)/2-KADIAG) 7, 8, 7
7 IER = 4
RETURN
8 CONTINUE
IER = 0
NCAP = NCAPP1-1
NCAPM1 = NCAP-1
IF(NCAPM1) 10, 20, 30
10 D(1) = U(1)
G(1) = W  
E = G(1)/D(1)  
A(1) = E  
RETURN

20 C(1) = U(1)/D(1)  
D(2) = U(2) - C(1)*C(1)*D(1)  
G(2) = W - C(1)*G(1)  
E = G(2)/D(2)  
J(1) = C(1)  
A(1) = A(1) - E*J(1)  
A(2) = E  
RETURN

30 C(1) = U(1)/D(1)  
NFORJ = 0  
DO 120 N = 2, NCAP  
CTEMP = U(N)  
NM1 = N - 1  
DO 110 NN = 1, NM1  
NFORJ = NFORJ + 1  
110 CTEMP = CTEMP - U( NN ) * J( NFORJ )  
120 C(N) = CTEMP/D(N)  
DTEMP = U(NCAP+1)  
GTEMP = W  
DO 140 N = 1, NCAP  
CTEMP = C(N)  
DTEMP = DTEMP - CTEMP*CTEMP*D(N)  
140 GTEMP = GTEMP - CTEMP*G(N)  
D(NCAP+1) = DTEMP  
G(NCAP+1) = GTEMP  
E = GTEMP/DTEMP  
NSTART = 0  
DO 180 N = 1, NCAPM1  
JTEMP = C(N)  
NSTART = NSTART + N  
NFORJ = NSTART  
NP1 = N + 1  
DO 170 NN = NP1, NCAP  
JTEMP = JTEMP - C(NN)*J(NFORJ)  
170 NFORJ = NFORJ + NN - 1  
J(NFORJ) = JTEMP  
180 A(N) = A(N) - E*JTEMP  
NFORJ = NFORJ + 1  
J(NFORJ) = C(NCAP)  
A(NCAP) = A(NCAP) - E*J(NFORJ)  
A(NCAP+1) = E  
RETURN
END
DOUBLE PRECISION S,B,PI
DIMENSION U(120),C(40),D(40),G(40),J(780),X(21),F(21)
REAL J
COMMON S,B,ANN(40)
READ(5,1) IEND,NUMBER,S,B
1 FORMAT(2I5,2D20.10)
IEND=IEND+1
PI=3.14159265358979
BN=0.0
WRITE (6,30) S,B
30 FORMAT(1V/////////// S = ',D20.10//////// B = ',D20.10)
TWOP=0.00149633
KA=40
KAM1=KA-1
BN=0.0
KADIAG=KA*KAM1/2
DO 7 NCAPP1=1,IEND
M=NCAPP1-1
CALL INTGRT (1,M,1,W)
DO 2 K=1,NCAPP1
N=K-1
CALL INTGRT (2,M,N,0(K))
2 CONTINUE
WRITE (6,3) M,W,H,M,(U(II),II=1,NCAPPI)
3 FORMAT(1H1//////// ' W(',!2,',!2) = ',E14.7//////// ' U(',!2,
+2,',!2) FOR J= 10, 12//(6(6X,E14.7)))
CALL ORTH(D,W,C,D,G,J,ANN,NCAPP1,KA,KAM1,KADIAG,IER)
IF(IER.EQ.0) GO TO 5
WRITE (6,4) IER
4 FORMAT ('IERR = ',15)
STOP
5 BN=ANN(NCAPP1)**2*D(NCAPP1)+BN
WRITE (6,6) M,M,(ANN(II),II=1,NCAPPI)
6 FORMAT(1H1//////// ' A(',!2,',!2,J) FOR J = 0, 12//(6(6
+X,E14.7)))
CHECK=BN/TWOP
WRITE (6,20) M,D(NCAPP1),BN,CHECK
20 FORMAT(1H1//////// ' D(',!2,',!2) = ',E14.7,6X,' BN=',E14.7,4X,
1' BESSELS CHECK=',E14.7)
CALL FX (M,NUMBER,X,F)
WRITE (6,8) M
8 FORMAT(1H1//////// 20X,' X ',18X,' F',!2,( X )'////////)
WRITE (6,9) (X(I),F(I),I=1,NUMBER)
9 FORMAT(18X,F8.5,14X,F10.6)
7 CONTINUE
WRITE (7,10) (ANN(I),I=1,IEND)
10 FORMAT (4E20.8)
STOP
END
SUBROUTINE INTGRT(INDIC, IM, IN, ANS)
DOUBLE PRECISION COEF1, COEF2, COEF3, COEF4, COEF5, PI, E
*, S, B, C, DSQRT
COMMON S, B, ANM(40)
M=IM
N=IN
PI=3.14159265358979
E=2.71828182845904
C=B/S
IF(INDIC.EQ.2) GO TO 3
IF(M.EQ.0) GO TO 101
GO TO 201
3 IF (M.EQ.0) GO TO 301
IF(M.EQ.N) GO TO 501
GO TO 601
101 ANS= (S**2)/(2.0DO*E**(M*PI*B/S))*C
RETURN
201 COEF1=S**2*(1.0D0-C**2)
COEF2=(M*PI*(C**2+1.0D0))**2
COEF3=E**(M*PI*C)
COEF4=S**2*C
COEF5=DSQRT(COEF2)
ANS= (-1)**(M )* (COEF1/COEF2*(1.0D0-(-1)**(M ))/COEF3)+COEF4/COEF5
1)
ANS= C *ANS
RETURN
301 ANS=S/(E**(2*M*PI*C))
RETURN
501 COEF1=S*(2.0D0*C**2+1.0D0)
COEF2=4.0DO*M*PI*C*(C**2+1.0D0)
COEF3=E**(2.0D0*M*PI*C)
ANS= (COEF1/COEF2)*(1.0D0-1.0D0/COEF3)
RETURN
601 COEF1=E**((M+N)*PI*C)
COEF2=(M+N)*S*C/(2.0DO*PI)
COEF3=(M+N)**2*C**2+(M-N)**2
COEF4=(M+N)**2*(C**2+1.0D0)
ANS= (-1)**(M+N)*COEF2*((1.0D0-(-1.0D0)**(M+N)/COEF1)
*)/COEF3+(1.0D0-
1(-1.0D0)**(M+N)/COEF1)/COEF4)
RETURN
END
C ********************************************************
C PROGRAM NO. 5 CALCULATES PSI ALONG THE SLOPING
C BOUNDARY FOR CASES A6, B6, AND C6
C ********************************************************
DIMENSION ANM(40)
READ(5, 1) NANM, S, B
1 FORMAT(I5, 2F10.4)
MM=NANM+1
C=B/S
READ(5, 2) (ANM(I), I=1, MM)
2 FORMAT(4E20.8)
WRITE (6, 20) S, B, NANM, NANM, (ANM(I), I=1, MM)
20 FORMAT(•1•////////' S= •,E15.7///// » B = ',E15.6///
*/// ///' A('12
1, 'I, J) FOR J = 0, 'I2///(6(E15.7)))
WRITE (6, 21)
21 FORMAT('I1')
E=2.7182818
PI=3.1415927
X=0.47
MAXIT=501
DO 14 J=1, MAXIT
PSI=0.0
DO 13 N=1, MM
M=N-1
COEF1=E**((M*PI*(C*X-B))/S)
COEF2=SIN(M*PI*X/S)
PSI=PSI+ANM(N)*COEF1*COEF2
13 CONTINUE
WRITE (6, 30) PSI, X
30 FORMAT(//' PSI=' ,E15.7,6X,'X=' ,E6.4)
X=X+0.0001
14 CONTINUE
STOP
END
C ********************************************************
C PROGRAM NO. 6 TO DRAW FLOW NETS FOR
C CASES A6, B6, AND C6 OF PART TWO
C ********************************************************

DIMENSION XL(5),YL(5),GL(5),DL(5),A(31),XX(101),YY(101)
EXTERNAL FCT
EXTERNAL TFC
EXTERNAL FT1, FT2
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,PS
*IPRM, IW, IK
COMMON /YPHI/ Y
COMMON /XPHI/ X

C
500 FORMAT(20A4)
800 FORMAT(4E20.8)
10 FORMAT(4E20.8)
1000 FORMAT(10X,3F20.6)
1100 FORMAT(I5, 2F12.5)
1500 FORMAT(1H1)

C ***** PROBLEM PARAMETERS *****
PI=3.14159265
AA=-0.54
SS=1.0
CC=0.100
BB=CC
XPSMIN=0.483
PSIMIN=-0.037101
IR=5
IW=6
EPS=0.0001
IEND=20
IK=IEND+1
READ(IR,500) XL,YL,GL,DL
READ(IR,10) (A(K),K=1,IK)
WRITE(IW,800) (A(K),K=1,IK)
PHIOA=0.0
PHISB=0.100
PSIOA=0.0

C GRAPH THE REGION******
XX(1)=0.0
XX(2)=SS
XX(3)=SS
YY(1)=0.0
YY(2)=BB
YY(3)=AA
NPTS=3
XSIZE=5.01
YSIZE=5.01
CALL GRAPH(NPTS,XX,YY,0.4,XSIZE,YSIZE,0.20,0.0,0.20, 
*-0.6, XL, YL, GL, 

C

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C C********DRAW FLUX LINES *********
JJ=10
PHIPRM=0.2
DO 30 K=1,4
C ********** CALCULATE XMAX,YMAX ***********
YMAX=(PHISB-PHIOA)*PHIPRM+PHIOA
XMAX=YMAX/CC
C
C CALCULATE XMIN AND YMIN
C
C ***** CASE1: POINT HITS BOTTOM *********
YHIN=AA
Y=AA
XLI=0.01
XRI=SS-0.01
CALL RTMI(X,F,FCT,XLI,XRI,EPS,IEND,IER)
IF(IER.EQ.2) GO TO 11
XMIN=X
VPHI=PHIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE(IW,1000) XMAX,YMAX,PHIPRM
GO TO 15
C
C CASE 2: POINT HITS ON THE RIGHT SIDE
11 XMIN=SS
X=SS
YLI=-5.5
YRI=BB-0.001
CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
IF(IER.EQ.2) GO TO 12
YMIN=Y
VPHI=PHIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE(IW,1000) XMAX,YMAX,PHIPRM
GO TO 15
C
C CASE 3 : POINT HITS ON THE LEFT SIDE *********
C
12 XMIN=0.0
X=0.0
XLI=-5.5
YRI=0.001
CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
YMIN=Y
VPHI=PHIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE(IW,1000) XMAX,YMAX,PHIPRM
15 DX=(XMAX-XMIN)/JJ
XX(1)=XMIN
YY(1)=YMIN
YLI=YMIN
X=XMIN+DX
DO 20 J=2,JJ
YRI=CC*X
CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
YY(J)=Y
XX(J)=X
YLI=Y
WRITE(IW,1000) X,Y,F
X=X+DX
20 CONTINUE
JEND=JJ+1
YY(JEND)=YMAX
XX(JEND)=XMAX
WRITE(IW,1100) (L,XX(L),YY(L),L=1,JEND)
NPTS=JEND
CALL GRAPH(NPTS,XX,YY,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0)
WRITE(IW,1500)
PHIPRM=PHIPRM+0.2
30 CONTINUE
C*****CALCULATION OF STREAMLINES*****
JJ=10
PHIPRM=0.2
DO 60 K=1,4
C*******CALCULATIONS OF INTERSECTIONS ON Y=CX**********
XLI=0.01
XRI=XPSMIN
CALL RTMI(X,F,FT1,XLI,XRI,EPS,IEND,IER)
XMIN=X
YMIN=CC*X
VPSI=PSIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPSI
XLI=XPSMIN
XRI=SS-0.01
CALL RTMI(X,F,FT1,XLI,XRI,EPS,IEND,IER)
XMAX=X
YMAX=CC*X
VPSI=PSIPRM+F
WRITE(IW,1000) XMAX,YMAX,VPSI
C***********N.L.POWERL**********
DX=(XMAX-XMIN)/JJ
XX(1)=XMIN
YY(1)=YMIN
X=XMIN+DX
YLI=-0.55
DO 40 J=2,JJ
YRI=CC*X
CALL RTMI(Y,F,FT2,YLI,YRI,EPS,IEND,IER)
FUNCTION TFC(Y)
DIMENSION A(31)
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,PSIPRM,*IW,IK
COMMON /XPHI/ X
PHI=0.0
DO 5 M=1,IK
EM=M-1
ALPHA1=EM*PI/SS
ALPHA2=ALPHA1*(Y-BB)
Q=-19.0
IF(ALPHA2.LE.Q) GO TO 2
COEF1=EXP(ALPHA2)
GO TO 3
2 COEF1=0.0
3 CONTINUE
COEF2=COS(ALPHA1*X)
PHI=PHI+A(M)*COEF1*COEF2
F=((PHI-PHIIOA)/(PHISB-PHIIOA)) - PHIPRM
TFC=F
5 CONTINUE
RETURN
END

FUNCTION FT1(X)
DIMENSION A(31)
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,PSIPRM,*IW,IK
PSI=0.0
CC=BB
Y=CC*X
DO 5 M=1,IK
EM=M-1
ALPHA1=EM*PI/SS
ALPHA2=ALPHA1*(Y-BB)
COEF1=EXP(ALPHA2)
COEF2=SIN(ALPHA1*X)
PSI=PSI+A(M)*COEF1*COEF2
F=((PSI-PSIMIN)/(PSIOA-PSIMIN)) - PSIPRM
FT1=F
5 CONTINUE
WRITE(IW,1000) X,Y,PSI,F
1000 FORMAT(4F15.6)
RETURN
END
C
C
FUNCTION FT2(Y)
DIMENSION A(31)
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,PSIPRM,
*IW,IK
COMMON /XPHI/ X
PSI=0.0
DO 5 M=1,IK
EM=M-1
ALPHA1=EM*PI/SS
ALPHA2=ALPHA1*(Y-BB)
Q=-19.0
IF(ALPHA2.LE.Q) GO TO 2
COEF1=EXP(ALPHA2)
GO TO 3
2 COEF1=0.0
3 CONTINUE
COEF2=SIN(ALPHA1*X)
PSI=PSI+A(M)*COEF1*COEF2
F=((PSI-PSIMIN)/(PSIOA-PSIMIN))-PSIPRM
FT2=F
5 CONTINUE
WRITE(IW,1000) X,Y,PSI,F
1000 FORMAT(4F15.6)
RETURN
END
C ********************************************************
C PROGRAM NO. 7 TO CALCULATE AMN'S AND PSI ALONG THE
C BOUNDARY FOR CASES A8, B8, C8, A9, B9, C9, D9, AND E9
C ********************************************************
DIMENSION U(120), C(40), D(40), G(40), J(780), X(31), P(31
*), R(101), V(201)
1, BO(201), X(M(201), UM(201), FF(201), Y(201), Z(201)
REAL J
COMMON S, B, AMN(40), DD, H, PI, E, A, T
READ (5, 1) IEND, NUMBER, S, B, A, DD, T
1 FORMAT (2I5, 5D8.4)
IEND = IEND + 1
PI = 3.14159265358979
E = 2.7182818
WRITE (6, 30) S, B, A, DD, T
XI = 0.0
P = 0.005
DO 11 IJ = 1, 201
11 XX(IJ) = XI + (IJ - 1) * P
CALL BFCT(BO, XX)
CALL FCT(FF, BO)
CALL FA(TWOPI, FF, Y, Z, P)
WRITE(6, 51) (IJ, XX(IJ), BO(IJ), IJ = 1, 201)
51 FORMAT (1I6, 2F16.8)
WRITE (6, 52) TWOPI
52 FORMAT (' TWOPI = ', D20.10)
KA = 40
KAM1 = KA - 1
BN = 0.0
KADIAG = KA*KAM1/2
DO 7 NCAPP1 = 1, IEND
M = NCAPP1 - 1
CALL WINT(M, W, OH, XX, BO, FF, Y, Z, P)
DO 2 K = 1, NCAPP1
N = K - 1
CALL DINTG(M, N, D(K))
2 CONTINUE
WRITE (6, 3) M, W, M, M, (U(IJ), IJ = 1, NCAPP1)
3 FORMAT (1H1/// ' W(', I2, ') = ', E14.7/// ' U(', I
* ', J) FOR J =
1 0, 'I2/// (6X, E14.7))
CALL ORTH(U, W, C, D, G, J, AMN, NCAPP1, KA, KAM1, KADIAG, IER)
IF (IER .EQ. 0) GO TO 5
WRITE (6, 4) IER
4 FORMAT (' IERR = ', I5)
STOP
5 BN=ANM(NCAPP1)**2*D(NCAPP1)+BN
   WRITE(6,6) M, M, (ANN(II), II=1, NCAPP1)
6 FORMAT('/// A('','I2,','J) FOR J = 0, ',I2//'(6(6X
   *,E14.7))
   CHECK=BN/THOPI
   WRITE (6,20) M, D(NCAPP1), BN, CHECK
20 FORMAT('/// D(',I2,') = ',E14.7,6X,' BN=',E14.7,4X,
   ' BESSELS CHECK=',E14.7)
   CALL FX(M, NUMBER, X, F)
   WRITE (6, 8) B
8 FORMAT(///20X,' X ',18X,'F',I2,'( X )'///)
   WRITE(6,9) (X(I), F(I), I=1, NUMBER)
9 FORMAT(18X,F8.5,14X,P10.6)
   CONTINUE
   WRITE (7,10) (ANM(I), I=1, IEND)
10 FORMAT(4E20.8)
   CALL MPSI(IEND, XPSMIN, PSIMIN)
   WRITE(6,60) XPSMIN, PSIMIN
60 FORMAT(//18X,'XPSMIN=',E12.4,6X,'PSIMIN=',E20.8)
   WRITE (6,33)
33 FORMAT(' NORMAL END OF PROGRAM')
   STOP
   END

SUBROUTINE FX(IM, NUMBER, X, F)
DIMENSION X(31), F(31)
COMMON S, B, ANM(40), DD, H, PI, E, A, T
   U(M)=E**((COEF1/A-1.)*M*PI*B/S)*COS(M*PI*Z/S)
   M=IM
   DO 6 I=1, NUMBER
   X(I)=(I-1)*S/(NUMBER-1)
   Z=X(I)
   SUM=0.0
   F(I)=ANM(1)
   IF (M.EQ.0) GO TO 6
   DO 5 J=1, M
   COEF1=0.0
   COEF2=(A**2-Z**2+2.*Z*S-S**2)
   IF (COEF2.LT.0) GO TO 4
   COEF1=COEF2**.5
4  SUM=SUM+U(J)*ANM(J+1)
   5 CONTINUE
   F(I)=SUM+F(I)
   CONTINUE
   F(I)=SUM+F(I)
   CONTINUE
   RETURN
END
SUBROUTINE ORTH(U,W,C,D,G,J,A,NCP1,KA,KM1,KDIAG,IER)
DIMENSION U(KA),C(KM1),D(KA),G(KA),A( KA)
REAL J(KDIAG),JTEMP
IF(NCP1-1) 1,2,2
1 IER=1
RETURN
2 IF(NCP1-KA) 4,4,3
3 IER=2
RETURN
4 IF(KA-1-KM1) 5,6,5
5 IER=3
RETURN
6 IF((KA*KM1)/2-KDIAG) 7,8,7
7 IER=4
RETURN
8 CONTINUE
IER=0
NCAP = NCP1-1
NCAPM1 = NCAP-1
IF(NCAPM1) 10,20,30
10 D(1) = U(1)
G(1) = H
E = G(1)/D(1)
A(1) = E
RETURN
20 C(1) = U(1)/D(1)
D(2) = U(2)-C(1)*C(1)*D(1)
G(2) = W-C(1)*G(1)
E = G(2)/D(2)
J(1) = C(1)
A(1) = A(1)-E*J(1)
A(2) = E
RETURN
30 C(1) = U(1)/D(1)
NFORJ = 0
DO 120 N = 2,NCAP
CTEMP = U(N)
NM1 = N-1
DO 110 NN = 1,NM1
NFORJ = NFORJ+1
110 CTEMP = CTEMP-U(NN)*J(NFORJ)
120 C(N) = CTEMP/D(N)
DTEMP = U(NCP1)
GTEMP = W
DO 140 N = 1,NCAP
CTEMP = C(N)
DTEMP = DTEMP-CTEMP*CTEMP*D(N)
140 GTEMP = GTEMP-CTEMP*G(N)
D(NCP1) = DTEMP
G(NCAPP1) = GTEMP
E = GTEMP/DTEMP
NSTART = 0
DO 180 N = 1,NCAPM1
JTEMP = C(N)
NSTART = NSTART+N
NFORJ = NSTART
NP1 = N+1
DO 170 NN = NP1,NCAP
JTEMP = JTEMP-C(NN)*J(NFORJ)
NFORJ = NFORJ+NN-1
J(NFORJ) = JTEMP
170
A(N) = A(N)-E*JTEMP
NFORJ = NFORJ+1
J(NFORJ) = C(NCAP)
A(NCAP) = A(NCAP)-E*J(NFORJ)
A(NCAP1) = E
RETURN
END

SUBROUTINE WINT(M,W,UM,XX,BO,FF,Y,Z,P)
DIMENSION UM(201),XX(201),BO(201),FF(201),Y(201),Z(201)
COMMON S,B,ANM(40),DD,H,PI,E,A,T
CALL UMFFC(UM,XX,BO,M)
DO 1 J=1,201
1 Y(J)=UM(J)*FF(J)
CALL QSF(P,Y,Z,201)
W=Z(201)
RETURN
END

SUBROUTINE UINTG(IH,IN,ANS)
DIMENSION R(201),V(201)
COMMON S,B,ANM(40),DD,H,PI,E,A,T
M=IM
N=IN
P=-.005
DO 43 I=1,201
P=P+.005
COEF1=0.0
COEF2=(A*A-P*P+2.*P*S-S*S)
IF(COEF2.LT.0) GO TO 43
COEF1=COEF2*.5
43 R(I)=E**((PI*B/S*(COEF1/A-1.))*(M+N))*COS(M*PI*P/S)*COS(N*PI*P/S)
CALL QSF(.005,R,V,201)
ANS=V(201)
RETURN
END

SUBROUTINE MPSI(IEND,XPSMIN,PSIMIN)
COMMON S,B,ANH(40),DD,H,PI,E,A,T
MN=IEND
PSIM=0.
X=0.
MAXIT=101
DO 14 J=1,MAXIT
  PSI=0.0
  DO 13 N=1,MN
    M=N-1
    COEF1=0.0
    COEF2=(A*A-X*X+2.*X*S-S*S)
    IF(COEF2.LE.0.0) GO TO 4
    COEF1=COEF2**.5
  4    PSI=PSI+ANH(N)*E**(M*PI*B*(COEF1/A-1)/S)*SIN(M*PI*X/S)
  13 CONTINUE
  WRITE (6, 30) PSI,X
  30 FORMAT(' ',PSI=',E15.7,6X,' X=',F6.4)
  IF(PSI.GT.PSIM) GO TO 15
  PSIM=PSI
  XPSM=X
  15 X=X+0.01
  14 CONTINUE
  XPSMIN=XPSM
  PSIMIN=PSIM
  RETURN
END

SUBROUTINE BFCT(BO,XX)
DIMENSION BO(201),XX(201)
COMMON S,B,ANH(40),DD,H,PI,E,A,T
DO 1 J=1,21
  BO(J)=0.0
  DO 2 J=22,201
    COEF1=(XX(J)-S)**2
  2    BO(J)=B*SQRT(A*A-COEF1)/A
RETURN
END

SUBROUTINE FCT(FF,BO)
DIMENSION FF(201), BO(201)
COMMON S, B, ANM(40), DD, H, PI, E, A, T
DO 1 J=1, 201
1 FF(J) = BO(J)
RETURN
END

SUBROUTINE FA(TWOPI, FF, Y, Z, P)
DIMENSION FF(201), Y(201), Z(201)
DO 1 J=1, 201
1 Y(J) = FF(J) * FF(J)
CALL QSF(P, Y, Z, 201)
TWOPI = Z(201)
RETURN
END

SUBROUTINE UMFCT(UM, XX, BO, M)
DIMENSION UM(201), XX(201), BO(201)
COMMON S, B, ANM(40), DD, H, PI, E, A, T
COEF1 = M * PI / S
DO 1 J=1, 201
COEF2 = BO(J) - B
COEF3 = COEF1 * XX(J)
1 UM(J) = EXP(COEF1 * COEF2) * COS(COEF3)
RETURN
END
NPTS=51
XSIZE=5.01
YSIZE=5.01
CALL GRAPH(NPTS,XX,YY,0.4,XSIZE,YSIZE,0.20,0.0,0.20,-0.6,*XL,YL,GL,DL)
C C**********DRAW FLUX LINES **********
JJ=10
PHIPRM=0.2
DO 30 K=1,4
C ********** CALCULATE XMAX,YMAX **********
YMAX=(PHISB-PHIOA)*PHIPRM+PHIOA
COEF1=(.81*YMAX**2.)/(BB**2.)
XHAX=1.0-SQRT(4.0-4.0*(.19+COEF1))/2.
C
C CALCULATE XMIN AND YMIN
C
C ***** CASE1: POINT HITS BOTTOM ********
YMIN=AA
Y=AA
XLI=0.01
XRI=SS-0.01
CALL RTMI(X,F,FCT,XLI,XRI,EPS,IEND,IER)
IF (IER.EQ.2) GO TO 11
XMIN=X
VPHI=PHIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE (IW, 1000) XMAX,YMAX,PHIPRM
GO TO 15
C
C CASE 2: POINT HITS ON THE RIGHT SIDE
11 XMIN=SS
X=SS
YLI=-5.5
YRI=BB-0.001
CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
IF (IER.EQ.2) GO TO 12
YMIN=Y
VPHI=PHIPRM+F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE (IW, 1000) XMAX,YMAX,PHIPRM
GO TO 15
C
C***CASE 3 : POINT HITS ON THE LEFT SIDE *******
C
12 XMIN=0.0
X=0.0
XLI=-5.5
YRI=0.001
CALL RTMI(Y,F,TFC,YLI,YRI,EPS,IEND,IER)
YMIN=Y
VPHI = PHI + F
WRITE(IW,1000) XMIN,YMIN,VPHI
WRITE(IW,1000) XMAX,YMAX,PHIPRM

15 DY = (YMAX-YMIN)/JJ
XX(1) = XMIN
YY(1) = YMIN
XLI = XMIN
Y = YMIN + DY
DO 20 J = 2, JJ
XRI = XMAX
II = 0
70 CALL RTMI(X,F,FCT,XLI,XRI,EPS,IEND,IER)
IF(IER.NE.2) GO TO 75
II = II + 1
IF(II.GT.2) GO TO 75
GO TO 70
75 YY(J) = Y
XX(J) = X
XLI = X
WRITE(IW,1000) X,Y,F
Y = Y + DY
20 CONTINUE
JEND = JJ + 1
YY(JEND) = YMAX
XX(JEND) = XMAX
WRITE(IW,1100) (L,XX(L),YY(L),L=1,JEND)
NPTS = JEND
CALL GRAPH(NPTS,XX,YY,0,2,0,0.0,0.0,0.0,0.0,0.0,0.0,X
*Y,YL,GL,DL)
WRITE(IW,1500)
PHIPRM = PHIPRM + 0.2
30 CONTINUE
C*****CALCULATION OF STREAMLINES*****
JJ = 10
PSIPRM = 0.2
DO 60 K = 1, 4
C********CALCULATIONS OF INTERSECTIONS ON Y = CX**********
XLI = 0.01
XRI = XPSMIN
CALL RTMI(X,F,FT1,XLI,XRI,EPS,IEND,IER)
XMIN = X
YMIN = 0.0
COEF1 = AB*AB - (X-1.)*(X-1.)
IF(COEF1.LE.0.) GO TO 2
YMIN = (BB/AE)*SQRT(COEF1)
2 VPSI = PSIPRM + F
WRITE(IW,1000) XMIN,YMIN,VPSI
XLI = XPSMIN
XRI = XPSMIN
CALL RTMI(X,F,FT1,XLI,XBI,EPS,IEND,IER)
XMAX=X
YMAX=0.0
COEF1=AB*AB-(X-1.)*(X-1.)
IF(COEF1.LE.0.) GO TO 3
YMAX=(BB/AB)*SQRT(COEF1)
3 VPSI=PSIPRM+F
WRITE(IWFLOOO) XMAX,YMAX,VPSI
C************ N. L. POWELL************
DX=(XMAX-XMIN)/JJ
XX(1)=XMIN
YY(1)=YMIN
X=XMIN+DX
YLI=-0.55 DO 40 J=2,JJ
YRI=0.0
COEF1=AB*AB-(X-1.)*(X-1.)
IF(COEF1.LE.0.) GO TO 4
YRI=(BB/AB)*SQRT(COEF1)
U CALL RTMI(Y,F,FT2,YLI,YRI,EPS,IEND,IER)
YY(J)=Y
XX(J)=X
WRITE(IW,1000) X,Y,F
X=X+DX
40 CONTINUE
JEND=JJ+1
YY(JEND)=YMAX
XX(JEND)=XMAX
WRITE(IW,1100) (L,XX(L),YY(L),L=1,JEND)
NPTS=JEND
CALL GRAPH(NPTS,XX,YY,0,2,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,X
*L,YL,GL,DL)
WRITE(IW,1500)
PSIPBM=PSIPRM+0.2
JJ=JJ+10
60 CONTINUE
XMIN=XPSMIN
XX(1)=XMIN
YY(1)=0.0
COEF1=AB*AB-(XMIN-1.)*(XMIN-1.)
IF(COEF1.LE.0.) GO TO 5
YY(1)=(BB/AB)*SQRT(COEF1)
5 NPTS=1
CALL GRAPH(NPTS,XX,YY,11,7,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,X
*L,YL,GL,DL)
STOP
END
FUNCTION FCT(X)
DIMENSION A(21)
COMMON A, BB, SS, PI, PHIOA, PHISB, PHIPRM, PSIMIN, PSIOA, P
* SIPRM, IW, IK, AB
COMMON /YPHI/ Y
PHI=0.0
DO 5 M=1, IK
EM=M-1
ALPHA1=EM*PI/SS
ALPHA2=ALPHA1*(Y-BB)
Q=-19.0
IF(ALPHA2.LE.Q) GO TO 2
COEF1=EXP(ALPHA2)
GO TO 3
2 COEF1=0.0
3 CONTINUE
COEF2=COS(ALPHA1*X)
PHI=PHI+A(M)*COEF1*COEF2
F=((PHI-PHIOA)/(PHISB-PHIOA))-PHIPRM
FCT=F
5 CONTINUE
RETURN
END

FUNCTION TFC(Y)
DIMENSION A(21)
COMMON A, BB, SS, PI, PHIOA, PHISB, PHIPRM, PSIMIN, PSIOA, P
* SIPRM, IW, IK, AB
COMMON /XPHI/ X
PHI=0.0
DO 5 M=1, IK
EM=M-1
ALPHA1=EM*PI/SS
ALPHA2=ALPHA1*(Y-BB)
Q=-19.0
IF(ALPHA2.LE.Q) GO TO 2
COEF1=EXP(ALPHA2)
GO TO 3
2 COEF1=0.0
3 CONTINUE
COEF2=COS(ALPHA1*X)
PHI=PHI+A(M)*COEF1*COEF2
F=((PHI-PHIOA)/(PHISB-PHIOA))-PHIPRM
TFC=F
5 CONTINUE
RETURN
END
FUNCTION FT1(X)
DIMENSION A(21)
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,P
* SIRPM,IW,IK,AB
PSI=0.0
Y=0.0
COEF1=AB*AB-(X-1.)*(X-1.)
IF(COEF1.LE.0.) GO TO 1
Y = (BB/AB)*SQRT(COEF1)
1 DO 5 M=1,IK
   EM=M-1
   ALPHA1=EM*PI/SS
   ALPHA2=ALPHA1*(Y-BB)
   COEF1=EXP(ALPHA2)
   COEF2=SIN(ALPHA1*X)
   PSI=PSI+A(M)*COEF1*COEF2
   F=((PSI-PSIMIN)/(PSIOA-PSIMIN))-PSIPRM
   FT1=F
5 CONTINUE
WRITE(IW,1000) X,Y,PSI,F
1000 FORMAT(4F15.6)
RETURN
END

FUNCTION FT2(Y)
DIMENSION A(21)
COMMON A,BB,SS,PI,PHIOA,PHISB,PHIPRM,PSIMIN,PSIOA,P
* SIRPM,IW,IK,AB
COMMON /XPHI/ X
PSI=0.0
DO 5 M=1,IK
   EM=M-1
   ALPHA1=EM*PI/SS
   ALPHA2=ALPHA1*(Y-BB)
   Q=-19.0
   IF(ALPHA2.LE.Q) GO TO 2
   COEF1=EXP(ALPHA2)
   GO TO 3
2 COEF1=0.0
3 CONTINUE
   COEF2=SIN(ALPHA1*X)
   PSI=PSI+A(M)*COEF1*COEF2
   F=((PSI-PSIMIN)/(PSIOA-PSIMIN))-PSIPRM
   FT2=F
5 CONTINUE
WRITE(IW,1000) X,Y,PSI,F
1000 FORMAT (4F15.6)
RETURN
END